RELATIVELY MAXIMAL COVERING SPACES

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By

MORRIS JACK LIEBOVITZ, B.Sc.

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AUTHOR: Morris Jack Liebovitz, B.Sc. (McMaster University)

SUPERVISOR: Professor B. Banaschewski

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SCOPE AND CONTENTS: This thesis deals with the existence and properties of certain types of covering spaces. It contains the discussion of a generalization of the notion of simple connectedness and several well-known theorems depending on this.

PREFACE

The first part of this thesis consists of a detailed presentation of proofs for theorems given in the paper "Zur Existenz von universellen Uberlagerungen" by B. Banaschewski. This paper determines the existence of a uniquely defined greatest covering space $(E_{\mathcal{V}}, g_{\mathcal{V}}, E)$ of a locally connected topological space E with respect to a covering $\mathcal{V}$ of E by domains.

Next the notion of simple connectedness is generalized to $\mathcal{V}$-simple connectedness with respect to some covering $\mathcal{V}$ of the space by domains. It is shown that the covering space $(E_{\mathcal{V}}, g_{\mathcal{V}}, E)$ is $\tilde{\mathcal{V}}$-simply connected with respect to the covering $\tilde{\mathcal{V}}$ of $E_{\mathcal{V}}$ by the connected components of the sets $g_{\mathcal{V}}^{-1}(V)$, $V \in \mathcal{V}$. Then the analogue of the Principle of Monodromy for simply connected spaces (see Chevalley, page 46, Theorem 2) is extended to $\mathcal{V}$-simply connected spaces.

Section 4 is devoted to showing that any relatively maximal covering space is normal; i.e. for any $a \in E$, the set $g_{\mathcal{V}}^{-1}(a)$ is permuted transitively by the automorphisms of $(E_{\mathcal{V}}, g_{\mathcal{V}}, E)$.

In Section 5 a unique method of making the covering space $(E_{\mathcal{V}}, g_{\mathcal{V}}, E)$ into a covering group is developed, in the case where E is a topological group and $aV \in \mathcal{V}$ for each $V \in \mathcal{V}$ and $a \in E$.

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TABLE OF CONTENTS

1. Definitions

(1) The subset $\{(x,x)/x\in E\}$ of $E\times E$ is called the <u>diagonal</u> of $E\times E$ and will be denoted by Δ.

The mapping $(x,y)\to(y,x)$ of $E\times E$ into itself will be denoted by ∇.

If A, $B\subseteq E\times E$, then $A\circ B=\{(x,y)/(x,a)\in A,(a,y)\in B$ for some $a\}$.

A set $R\subseteq E\times E$ is called an <u>equivalence relation</u> if

(i) $\Delta\subseteq R$ (reflexivity)

(ii) $\nabla R=R$ (symmetry)

(iii) $R\circ R=R$ (transitivity)

The equivalence relation generated by any subset $C\subseteq E\times E$ is the smallest equivalence relation $R\supseteq C$. (This clearly exists and is $\bigcap_{R\supseteq C} R$).

(2) For the equivalence relation R, the <u>slice</u> with respect to $x\in E$ is $R(x)=\{y/(x,y)\in R\}$.

For $A\subseteq E$, $R(A)=\bigcup_{x\in A} R(x)$.

(3) If $A=R(A)$, A is called <u>R-saturated</u>.

(4) If, for any open subset $U\subseteq E$ where E is a topological space, $R(U)$ is open, R is called an <u>open equivalence relation</u>.

(5) Let E be a topological space, R an equivalence relation on E. We define the <u>quotient space</u> $E/R=\{R(x)/x\in E\}$.

The mapping $f:E\to E/R$ by $x\to R(x)$ is called the <u>natural mapping</u> of E onto the quotient set E/R.

The <u>quotient topology</u> on the quotient set E/R is the finest topology making f continuous.

(6) A space E is <u>connected</u> if it is not the union of two disjoint, non-void, open sets.

(7) A <u>domain</u> is an open, connected set.

(8) The <u>connected component</u> of a point of a space E is the largest connected subset of E containing this point.

The connected components of a subset $A \subseteq E$ are the connected components of the points of A relative to the subspace A.

(9) A space E is <u>locally connected</u> if any neighbourhood of any point of E contains a connected neighbourhood of the point.

A space E is locally connected if and only if the connected components of any open set in E are open sets.

(10) Let $f: X \to E$ be a continuous mapping onto E. A subset $A \subseteq E$ is said to be <u>evenly covered</u> by X with respect to f if every connected component of $f^{-1}(A)$ is mapped homeomorphically onto A by f.

(11) Let E be a topological space. A <u>covering space</u> of E is a triple (X,f,E) formed from a connected, locally connected space X and a continuous mapping $f: X \to E$ such that each point of E has a neighbourhood which is evenly covered by X with respect to f.

<u>Remark</u>: A space E cannot have a covering space unless it is connected and locally connected. Conversely, if E

is connected and locally connected it has at least one covering space, i.e. the trivial covering space (E,f,E) where f is the identity mapping.

(12) A collection $\mathscr{V}$ of sets $V \subseteq E$ is a <u>covering</u> of the space E if $E = \bigcup_{V \in \mathscr{V}} V$.

(13) Two covering spaces (X,f,E), (Y,g,E) are said to be <u>isomorphic</u> (denoted by $(X,f,E) \cong (Y,g,E)$) if there exists a homeomorphism $h: X \to Y$ such that $f = g \circ h$.

The isomorphisms of a covering space (X,f,E) with itself are called the <u>automorphisms</u> of (X,f,E). These form a group called the automorphism group of (X,f,E).

If (X,f,E), (Y,g,E) are isomorphic with $h: X \to Y$ a homeomorphism, and $t: Y \to Y$ a homeomorphism, then $t \to h^{-1}th$ is an isomorphism of the one automorphism group to the other.

(14) We can define a <u>quasi-partial ordering</u> in the class of all covering spaces of the space E by:

$(X,f,E) \geq (Y,g,E)$ if and only if there exists a continuous mapping $k: X \to Y$ such that (X,k,Y) is a covering space and $f = g \circ k$.

If $(X,f,E) \geq (Y,g,E)$ and $(Y,g,E) \geq (X,f,E)$, then $(X,f,E) \cong (Y,g,E)$.

(15) A space E is said to be <u>simply connected</u> if it is connected and locally connected and every covering space of E is isomorphic to the trivial covering space.

(16) The covering space (X,f,E) is called a <u>simply connected</u>

covering space if X is simply connected.

(17) Let (X,f,E) be a covering space and $\mathcal{V}$ a covering of E by domains. We say (X,f,E) is *even in $\mathcal{V}$* if any $V \in \mathcal{V}$ is evenly covered by X with respect to f.

(18) E is called *$\mathcal{V}$-simply connected* if and only if the only covering space of E even in $\mathcal{V}$ is the trivial covering space.

(19) A *topological group* G is the composite object formed by a group $\tilde{G}$ and a topological space X which satisfy the following conditions: 1) the set of points of X is the same as the set of elements of $\tilde{G}$; 2) the mapping $(\sigma,\tau) \to \sigma\tau^{-1}$ of $X \times X$ into X is continuous. The group $\tilde{G}$ is called the underlying group of the topological group G, and the space X is called its underlying space.

(20) Let G be a topological group. By a *covering group* of G, we mean a triple (H,f,G) composed of a topological group H and a homomorphism $f: H \to G$ such that (H,f,G) is a covering space.

All other topological considerations mentioned will be as defined by Bourbaki except for topological space, by which we will mean Hausdorff space.

2. Relatively Greatest Covering Spaces

Our aim is to prove the following theorem:

<u>Theorem 1</u>: <u>For any covering $\mathcal{V}$ of E by domains, there exists a covering space $(E_{\mathcal{V}},g_{\mathcal{V}},E)$ unique up to isomorphism, even in $\mathcal{V}$, such that for any covering space (X,f,E) even in $\mathcal{V}$, $(E_{\mathcal{V}},g_{\mathcal{V}},E)\geq(X,f,E)$.</u>

Before we prove this theorem, we require a number of preliminary considerations. The first result needed is

<u>Lemma 1</u>: <u>Let there be given a space E, (O_λ,P_λ), $\lambda=1,\dots,n$ pairs of open subsets and homeomorphisms $h_\lambda:O_\lambda\to P_\lambda$. Then the equivalence relation generated by the graphs $\{(x,h_\lambda(x))/x\in O_\lambda\}$ of the h_λ is an open equivalence relation.</u>

<u>Proof</u>: First we will give an explicit description of the equivalence relation generated by any set $C\subseteq E\times E$. Put $C_*=C\cup\sigma C$, $C_*^k=C_*\circ C_*\circ\dots\circ C_*$, $C_*^0=\Delta$ and $\tilde{C}=\bigcup_{k\geq0}C_*^k$. Now (i) $\Delta\subseteq\tilde{C}$ (ii) $\sigma\tilde{C}=\bigcup_{k\geq0}\sigma(C_*^k)=\bigcup_{k\geq0}(\sigma C_*)^k=\bigcup_{k\geq0}C_*^k=\tilde{C}$ (iii) $\tilde{C}\circ\tilde{C}=(\bigcup_{k\geq0}C_*^k)\circ(\bigcup_{l\geq0}C_*^l)=\bigcup_{k,l\geq0}C_*^k\circ C_*^l=\bigcup_{k,l\geq0}C_*^{k+l}=\tilde{C}$. This shows $\tilde{C}$ is an equivalence relation. Now if R is any equivalence relation such that $R\supseteq C$, then $R\supseteq\sigma C$, hence $R\supseteq C_*$, hence $R\supseteq C_*^k$ for all k, so $\tilde{C}\subseteq R$. Thus $\tilde{C}$ is the smallest equivalence relation containing C. Now, let C be the union of the graphs of the h_λ and $Y\subseteq E$ be an open set. We must show that the $\tilde{C}$-saturation of Y is open.

Now $\tilde{C}(Y)=\bigcup_{y\in Y}\tilde{C}(y)$. But $\tilde{C}(y)=\{x/(x,y)\in\tilde{C}\}=\{x/(x,y)\in\bigcup_{k\geq0}C_*^k\}$ $=\bigcup_{k\geq0}\{x/(x,y)\in C_*^k\}=\bigcup_{k\geq0}C_*^k(y)$. Thus $\tilde{C}(Y)=\bigcup_{y\in Y}\bigcup_{k\geq0}C_*^k(y)=\bigcup_{k\geq0}\bigcup_{y\in Y}C_*^k(y)=$

$\bigcup_{k\geq 0} C_*^k(Y)$. Thus it is sufficient to show that $\bigcup_{k\geq 0} C_*^k(Y)$ is open, or that $C_*^k(Y)$ is open for all k. The proof is by induction on k. For k=o, $C_*^o(Y)=Y$ which is open. Assume $C_*^{k-1}(Y)$ is open. Then $C_*^k(Y)=C_*(C_*^{k-1}(Y))$ since: $x\in C_*^k(Y) \Rightarrow$ there exists $x,y_1,\ldots,y_k$ such that $(x,y_1)\in C_*$, $(y_1,y_2)\in C_*,\ldots,(y_{k-1},y_k)\in C_*$, where $y_k\in Y \Rightarrow y_1\in C_*^{k-1}(Y) \Rightarrow x\in C_*(C_*^{k-1}(Y))$. Thus $C_*^k(Y)\subseteq C_*(C_*^{k-1}(Y))$. $x\in C_*(C_*^{k-1}(Y)) \Rightarrow (x,y)\in C_*, y\in C_*^{k-1}(Y)$. Then there exists $y,y_1,\ldots,y_{k-1}$ such that $(y,y_1)\in C_*,(y_1,y_2)\in C_*,\ldots,(y_{k-2},y_{k-1})\in C_*, y_{k-1}\in Y \Rightarrow x\in C_*^k(Y)$. Therefore $C_*^k(Y)=C_*(C_*^{k-1}(Y))$.

It is now sufficient to show that $C_*(C_*^{k-1}(Y))$ is open. But $C_*^{k-1}(Y)$ is open, say Z. Then

$C_*(Z)=\bigcup_{\lambda=1}^{n}[h_\lambda(Z\cap O_\lambda)\cup h_\lambda^{-1}(Z\cap P_\lambda)]$ which is open.

Thus $\tilde{C}$ is an open equivalence relation and Lemma 1 is proved.

Next, we define a space from which, as will be shown, essentially all covering spaces of E even in $\mathcal{V}$ can be obtained by taking suitable quotients.

Fix $U\in\mathcal{V}$. Then a chain $\mathcal{k}$ in $\mathcal{V}$ is a sequence $V_0,V_1,\ldots,V_n$ such that (i) $V_\lambda\in\mathcal{V}$ for all λ and $V_0=U$. (ii) $V_{\lambda-1}\cap V_\lambda\neq\emptyset$ for $\lambda=1,\ldots,n$.

Define the space $S(\mathcal{k})=\bigcup_{\lambda=0}^{n}(V_\lambda\times\{\lambda\})$, taken as a subspace of $E\times N$ where N is the discrete space of natural numbers.

Let $p(\mathcal{k})$ be the restriction of the projection $E\times N\rightarrow E$ to $S(\mathcal{k})$. Thus $p(\mathcal{k})$ restricted to $V_\lambda\times\{\lambda\}$ is a homeomorphism onto V_λ.

Let $i_\lambda(k)$ be the mapping $V_\lambda \to V_\lambda \times \{\lambda\}$ by $x \to (x,\lambda)$. This is a homeomorphism of V_λ onto $V_\lambda \times \{\lambda\}$ such that $p(k) \circ i_\lambda(k)$ is the identity on V_λ.

* Define an equivalence relation Z on $S(k)$ by choosing a connected component C_λ of $V_{\lambda-1} \cap V_\lambda$ for each $\lambda = 1,\ldots,n$, namely generated by the graphs of the mappings $h_\lambda : C_\lambda \times \{\lambda-1\} \to C_\lambda \times \{\lambda\}$ which map $(x,\lambda-1) \to (x,\lambda)$, $x \in C_\lambda$.

Define the space $S(k,Z)$ as $S(k)/Z$ and let $j(k,Z)$ be the mapping $S(k,Z) \to E$ induced by $p(k)$, i.e. if $j(Z) : S(k) \to S(k,Z)$ is the natural mapping, then $p(k) = j(k,Z) \circ j(Z)$.

The space $S(k,Z)$ has the following properties:

(1) $j(k,Z)$ induces a homeomorphism of each $j(Z)(V_\lambda \times \{\lambda\})$ onto V_λ:

We have the mappings $V_\lambda \xrightarrow{i_\lambda(k)} V_\lambda \times \{\lambda\} \xrightarrow{j(Z)} j(Z)(V_\lambda \times \{\lambda\}) \xrightarrow{j(k,Z)} V_\lambda$. Now $j(k,Z) \circ j(Z) \circ i_\lambda(k) = p(k) \circ i_\lambda(k)$ is the identity on V_λ, thus $j(k,Z)$ is one-one, onto, continuous. Also $j^{-1}(k,Z) = j(Z) \circ i_\lambda(k)$ which is continuous. Thus $j(k,Z)$ is a homeomorphism and the assertion is proved.

(2) $S(k,Z)$ is connected:

From (1), $S(k,Z) = \bigcup_{\lambda=0}^{n} j(Z)(V_\lambda \times \{\lambda\})$ and in this union, the individual terms are connected and have successive non-void intersections.

(3) Each $j(Z)(V_\lambda \times \{\lambda\})$ is open in $S(k,Z)$:

From Lemma 1, Z is an open equivalence relation, thus the natural mapping $j(Z):S(\mathfrak{k})\rightarrow S(\mathfrak{k},Z)$ is an open mapping.

Define S as the topological sum of the spaces $S(\mathfrak{k},Z)$ for all arbitrary sequences $\mathfrak{k}$ and all equivalence relations Z. Let h be the conjunction of the $j(\mathfrak{k},Z)$, i.e. $h:S\rightarrow E$.

For any $V\in\mathcal{V}$, if $V=V_\lambda$ in the sequence $\mathfrak{k}$ and Z is any equivalence relation on $S(\mathfrak{k})$ of the type *, then $j(Z)\circ i_\lambda(\mathfrak{k})$ maps V homeomorphically onto the open set $j(Z)(V_\lambda\times\{\lambda\})$ in S. These sets, for a fixed $V\in\mathcal{V}$ are called the V-replicas in S.

Let $\mathcal{E}$ be the set of equivalence relations R on S such that:

(E1) h is constant on the R-classes, i.e. if $(x,y)\in R$ then $h(x)=h(y)$.

(E2) On $\bigcup j(Z)(V_0\times\{0\}):(x,y)\in R\Leftrightarrow h(x)=h(y)$.

(E3) If V', V'' are any two V-replicas in S and if there exists $a\in V', b\in V''$ with $(a,b)\in R$, then for all $x\in V', y\in V''$ with $h(x)=h(y)$, we have $(x,y)\in R$.

The next result we will need is

<u>Lemma 2</u>: <u>For any $R\in\mathcal{E}$, if $g:S/R\rightarrow E$ is induced by h, then $(S/R,g,E)$ is a covering space even in $\mathcal{V}$. Furthermore, if $R\subseteq R'$, then $(S/R,g,E)\geq(S/R',g',E)$.</u>

<u>Proof</u>: S/R is connected since all $S(\mathfrak{k},Z)$ are and their images in S/R all have points in common by (E2). Let h_R be the natural mapping from S onto S/R. Take a particular V-replica, V' in S. Then $R(V')$ is a union of V-replicas

by (E3). Hence $R(V')$ is open. Thus $h_R(V')=h_R(R(V'))$ which is open in S/R by the definition of quotient space. Thus h_R is an open mapping. This shows that S/R is covered by locally connected open sets, thus S/R is locally connected. It remains to be shown that each $V\in\mathcal{V}$ is evenly covered by S/R with respect to g. This will be achieved by proving that for any $V\in\mathcal{V}$, $g^{-1}(V)$ is the h_R-image of all V-replicas in S. From $h=g\circ h_R$ one has that $h^{-1}(V)=h_R^{-1}(g^{-1}(V))$ for any $V\in\mathcal{V}$, and since h_R maps S onto S/R, $g^{-1}(V)=h_R(h^{-1}(V))$. Hence it is required to prove that $h_R(h^{-1}(V))=\cup h_R(V')$, V' the V-replica in S. Now $x\in h^{-1}(V)$ implies x is in some $S(\mathfrak{k},Z)$. If there exists V_λ in $\mathfrak{k}$ with $V_\lambda=V$, then $h_R(x)\in h_R(j(Z)(V_\lambda\times\{\lambda\}))$. If not, assume $h(x)\in V_m$. Define $\tilde{\mathfrak{k}}=\{\tilde{V}_0,\dots,\tilde{V}_{m+1}\}$ by $\tilde{V}_\lambda=V_\lambda$, $\lambda\le m$, $\tilde{V}_{m+1}=V$. Define the equivalence relation $\tilde{Z}$ by $\tilde{C}_\lambda=C_\lambda$, $\lambda\le m$, C_{m+1} the connected component of $h(x)$ in $V_m\cap V$. Consider $S(\mathfrak{k},Z)$ and $S(\tilde{\mathfrak{k}},\tilde{Z})$. $\tilde{x}=j(\tilde{Z})i_m(\tilde{\mathfrak{k}})(h(x))$ satisfies $h_R(x)=h_R(\tilde{x})$. For, $h_R(V_\lambda')=h_R(\tilde{V}_\lambda')$ (where V_λ', $\tilde{V}_\lambda'$ are the replicas in $S(\mathfrak{k},Z)$, $S(\tilde{\mathfrak{k}},\tilde{Z})$ respectively), $o\le\lambda<m$, holds with $\lambda=o$ by (E2) and implies $h_R(V_{\lambda+1})=h_R(\tilde{V}_{\lambda+1})$ by (E3) since $h_R(y)=h_R(\tilde{y})$ for any $y=j(Z)i_{\lambda+1}(\mathfrak{k})(z)=j(Z)i_\lambda(\mathfrak{k})(z)$ and $\tilde{y}=j(\tilde{Z})i_{\lambda+1}(\tilde{\mathfrak{k}})(z)=j(\tilde{Z})i_\lambda(\tilde{\mathfrak{k}})(z)$ where $z\in C_{\lambda+1}=\tilde{C}_{\lambda+1}$. Hence $h_R(x)\in h_R(j(\tilde{Z})(V_{m+1}\times\{m+1\}))$, thus $g^{-1}(V)=\cup h_R(V')$ where V' ranges over all V-replicas in S. Any two V-replicas in S get mapped to either the same set or disjoint sets by (E3), and since the $h_R(V')$ are open and connected in S/R, they are the connected components of $g^{-1}(V)$. Also, each of these is homeomorphic to V under g

and so $(S/R, g, E)$ is even in $\mathcal{V}$. This proves the first part of the theorem.

Now, let $R \subseteq R'$ and $g: S/R \to E$, $g': S/R' \to E$. We have $h_R: S \to S/R$, $h_{R'}: S \to S/R'$. Take $x \in S/R$ and define the mapping $h_{R'R}: x \to h_{R'}(h_R^{-1}(x))$. Now $h_{R'}$ is constant on the R-classes, thus by section 9, Theorem 1 of Bourbaki's "Topologie Générale", it induces a continuous mapping on S/R, which is $h_{R'R}$. Also $h_{R'}$, h_R are open mappings and $h_{R'} = h_{R'R} \circ h_R$. Now it is left to show that $(S/R, h_{R'R}, S/R')$ is a covering space, and this will follow immediately from the following lemma.

<u>Lemma 3</u>: <u>If (X, f, E), (Y, g, E) are two covering spaces even in $\mathcal{V}$ and there exists a continuous mapping $h: Y \to X$ with $g = f \circ h$, then $(Y, g, E) \geq (X, f, E)$, i.e. (Y, h, X) is a covering space.</u>

<u>Proof</u>: Since (Y, g, E) is a covering space, Y is connected and locally connected. Take $x \in X$, $f(x) \in E$, and $V \in \mathcal{V}$ such that $f(x) \in V$. Take V_* as the connected component of x in $f^{-1}(V)$. If V^* is any connected component of $g^{-1}(V)$ with $V^* \cap h^{-1}(V_*) \neq \emptyset$, then $h(V^*) \subseteq V_*$ since $h(V^*)$ is the continuous mapping of a connected set and so is connected, and it is in one connected component of $f^{-1}(V)$, i.e. in the one it meets. Then the connected components of $h^{-1}(V_*)$ are just connected components of $g^{-1}(V)$. Thus $V^* \subseteq h^{-1}(V_*)$ and $h^{-1}(V_*)$ is a union of these. Suppose there exists $y_1, y_2 \in V^*$ with $h(y_1) = h(y_2)$. Then $f(h(y_1)) = f(h(y_2))$ and so $g(y_1) = g(y_2)$

since $g=f\circ h$. Thus $y_1=y_2$ since g is one-one on any connected component of $g^{-1}(V)$, and so h is one-one on V^*. Take $x\in V_*$, $f(x)\in V$. Then there exists $y\in V^*$ such that $g(y)=f(x)$. Take $h(y)\in V_*$. $f(h(y))=g(y)=f(x)$, but f is one-one on V_*, so $x=h(y)$. Thus h is onto, and since it is open, V^* is mapped homeomorphically to V_* by h. Thus we see that (Y,h,X) is a covering space as required. ∎

The next required result is

<u>Lemma 4</u>: <u>Any covering space (X,f,E) even in $\mathcal{R}$ is isomorphic to some $(S/R,g,E)$. Furthermore, if $(X,f,E)\geq(X',f',E)$, then R, R' can be chosen such that $R\subseteq R'$.</u>

<u>Proof</u>: The idea of the proof is to find a mapping $k:S\to X$, open and continuous, such that $R=\{(x,y)/x,y\in S, k(x)=k(y)\}$ is in $\mathcal{E}$. Now we have $\mathfrak{k}=\{V_0,V_1,\ldots,V_n\}$ and $S(\mathfrak{k},Z)=V_0'\cup\ldots\cup V_n'$. Take the fixed $U\in\mathcal{R}$, and V_0'' one connected component of $f^{-1}(U)$. Assume for each i, $0\leq i\leq\lambda<n$, we are given local inverses g_i of f on each V_i such that the conjunction of the $g_i\circ j(\mathfrak{k},Z)$ is continuous from $V_0'\cup\ldots\cup V_\lambda'$ onto $V_0''\cup\ldots\cup V_\lambda''=g_0(V_0)\cup g_1(V_1)\cup\ldots\cup g_\lambda(V_\lambda)$ where each V_i'', $0\leq i\leq\lambda<n$ is a connected component of $f^{-1}(V_i)$. Let $g_{\lambda+1}$ be the local inverse of f on $V_{\lambda+1}$ which maps $V_{\lambda+1}$ onto that connected component of $f^{-1}(V_{\lambda+1})$ which contains $g_\lambda(C_{\lambda+1})$. Since $C_{\lambda+1}$ is connected and evenly covered (since it lies in an evenly covered set) there can be only one such connected component. Now, on $C_{\lambda+1}$, $g_\lambda=g_{\lambda+1}$. Hence on $C'_{\lambda+1}=V'_\lambda\cap V'_{\lambda+1}$ we have $g_\lambda\circ h=g_{\lambda+1}\circ h$. Thus the conjunction $k:S\to X$ of all the mappings $g_i\circ h$,

$i=o,\ldots,\lambda+1$ is continuous, open, and one-one from $V_0'\cup\ldots\cup V_{\lambda+1}'$ onto $V_0''\cup\ldots\cup V_{\lambda+1}''=g_0(V_0)\cup\ldots\cup g_\lambda(V_\lambda)$. Take the relation $R=\{(x,y)/k(x)=k(y)\}$; it follows that S/R is homeomorphic to X since $k=k_R\circ h_R$ is open. Then $R\in\mathcal{E}$ since:

(E1) For $S(k,Z)=V_0'\cup\ldots\cup V_n'$, k on each $V_i'\subseteq V_0'\cup\ldots\cup V_n'$ is defined by $V_i'\xrightarrow{h}V_i\xrightarrow{g_i}V_i''$ where g_i is a suitable local inverse of f. Thus $k=g_i\circ h$, so $f\circ k=f\circ g_i\circ h=h$ on any V-replica in S.

(E2) True by definition.

(E3) Let V', V'' be any two V-replicas on S and assume that for some $a\in V'$, $b\in V''$, $k(a)=k(b)$. Then it has to be shown that $k(x)=k(y)$ for any $x\in V'$, $y\in V''$ with $h(x)=h(y)$. Now $k=g'\circ h$ on V', $k=g''\circ h$ on V'' with suitable local inverses g', g'' of f on V. Then $k(a)=g'(h(a))$, $k(b)=g''(h(b))$ and $k(a)=k(b)$ means that g', g'' have the same value at $h(a)=h(b)$. But then $g'=g''$ and hence $h(x)=h(y)$ implies $k(x)=g'(h(x))=g''(h(y))=k(y)$.

Now we have $\begin{tikzcd} S \arrow[r, "h_R"] \arrow[dr, "h"'] & S/R \arrow[r, "k_R"] \arrow[d, "g"] & X \arrow[dl, "f"] \\ & E & \end{tikzcd}$ where g is defined by $g\circ h_R=h$.

Take $x\in S/R$, y in S such that $h_R(y)=x$. By definition $g(x)=h(y)=f(k(y))=f(k_R(h_R(y)))=f(k_R(x))$. Thus $g=f\circ k_R$ and the first part of the lemma is proved.

Now let $(X,f,E)\geq(X',f',E)$. Then there exists a continuous mapping $f^*:X\rightarrow X'$ such that (X,f^*,X') is a covering space and $f=f'\circ f^*$. We choose $k:S\rightarrow X$ as above and have for $R=\{(x,y)/k(x)=k(y)\}$ that $(X,f,E)\cong(S/R,g,E)$. Define $R'=\{(x,y)/f^*(k(x))=f^*(k(y))\}$. Thus $R\subseteq R'$ and

$k_{R'}: S/R' \rightarrow X'$ is a homeomorphism. Now take $x \in S/R'$, $y \in S$ with $h_{R'}(y)=x$. Then $g'(x)=h(y)=g(h_R(y))=f(k_R(h_R(y)))= f(k(y))=f'(f^*(k(y)))=f'(k_{R'}(h_{R'}(y)))=f'(k_{R'}(x))$. Thus $g'=f'\circ k_{R'}$ and $(S/R', g', E) \cong (X', f', E)$ proving the lemma.

Finally, we require

Lemma 5: $\bigcap_{R \in \mathcal{E}} R = R_0 \in \mathcal{E}$

Proof: (E1) $(x,y) \in R_0 \Rightarrow (x,y) \in R$ all $R \in \mathcal{E} \Rightarrow h(x)=h(y)$.

(E2) If $h(x)=h(y)$, where $x,y \in \bigcup j(Z)(V_0 \times \{o\})$ then $(x,y) \in R$ for each $R \in \mathcal{E} \Rightarrow (x,y) \in R_0$.

(E3) Suppose there exists $a \in V'$, $b \in V''$ with $(a,b) \in R_0$. Then $(a,b) \in R$ for all $R \in \mathcal{E}$. Thus if $h(x)=h(y)$ for $x \in V'$, $y \in V''$, then $(x,y) \in R$ for all $R \in \mathcal{E} \Rightarrow (x,y) \in R_0$.

This proves the lemma.

Theorem 1 is now proved if we can show that $(S/R_0, g_0, E)$ is the required covering space.

Proof: By Lemma 2, $(S/R_0, g_0, E)$ is a covering space even in $\mathcal{B}$. By Lemma 4, if (X,f,E) is any covering space even in $\mathcal{B}$, then there exists $R \in \mathcal{E}$ such that $(S/R,g,E) \cong (X,f,E)$. By Lemma 2, $(S/R_0, g_0, E) \geq (S/R,g,E)$ since $R_0 \subseteq R$ for all $R \in \mathcal{E}$. Now $(S/R,g,E) \cong (X,f,E)$ means there exists a homeomorphism $t: S/R \rightarrow X$ such that $g=f\circ t$. $(S/R_0, g_0, E) \geq (S/R,g,E)$ means there exists a continuous mapping $t': S/R_0 \rightarrow S/R$ such that $(S/R_0, t', S/R)$ is a covering space and $g_0 = g\circ t'$. We want to show that $(S/R_0, g_0, E) \geq (X,f,E)$, i.e. that there exists a continuous mapping $k': S/R_0 \rightarrow X$ such that $g_0 = f\circ k'$ and $(S/R_0, k', X)$ is a covering space. Choose $k' = t\circ t'$ which is continuous. Then $g_0 = g \circ t' = f \circ t \circ t' = f \circ k'$. Thus by Lemma 3, $(S/R_0, g_0, E) \geq (X, f, E)$.

Also, if there exists a covering space (M,m,E) even in $\mathcal{K}$ such that $(M,m,E) \geq (S/R_0, g_0, E)$, then since $(M,m,E) \leq (S/R_0, g_0, E)$, the two covering spaces are isomorphic. Therefore $(S/R_0, g_0, E)$ is unique up to isomorphism.

From the definition of a quasi-partial ordering in the class of all covering spaces, we see that Theorem 1 determines the existence of a greatest covering space in the class of all covering spaces of E even in $\mathcal{K}$.

3. Relatively Simply Connected Spaces

Our aim is to prove the following theorem:

Theorem 2: For the covering space $(E_{\gamma}, g_{\gamma}, E)$, let $\tilde{\gamma}$ be the covering of E_{γ} by the connected components of the open sets $g_{\gamma}^{-1}(V)$, $V \in \gamma$. Then E_{γ} is $\tilde{\gamma}$-simply connected.

Before we prove this theorem, we require the following lemma:

Lemma 6: Assume that (X, f, E) is a covering space. Let g, g' be continuous mappings of a connected space W into X such that $f \circ g = f \circ g'$. If $g(w_0) = g'(w_0)$ for at least one point w_0, then $g = g'$.

Proof: Let $A = \{w / g(w) = g'(w)\}$. Since $g(w_0) = g'(w_0)$, A is not empty. Consider the mapping $w \xrightarrow{t} (g(w), g'(w))$. This is a continuous mapping and $t^{-1}(\Delta) = A$ proving A is closed. The lemma will be proved if we can show that A is open, since in that case we must have $A = W$. If $w \in A$, then $f(g(w))$ has a neighbourhood V which is evenly covered by X with respect to f. By Lemma 1, Chapter 2, section 6 of Chevalley's "Theory of Lie Groups", the component V' of $g(w) = g'(w)$ in $f^{-1}(V)$ is a neighbourhood of $g(w)$ in X. It follows that there exists a neighbourhood U of w in W such that $g(U) \subseteq V'$, $g'(U) \subseteq V'$. Because f maps V' homeomorphically and $f \circ g = f \circ g'$, $w' \in U$ implies $g(w') = g'(w')$, thus $w' \in A$ whence $U \subseteq A$ proving the lemma.

Now we can proceed with the proof of Theorem 2.

Proof: We must show that if $(Y,f,E_{\mathcal{V}})$ is any covering space even in $\tilde{\mathcal{V}}$, then f is one-one. This will be accomplished if we can show that $(Y,g_{\mathcal{V}}\circ f,E)$ is a covering space even in $\mathcal{V}$. Now Y is connected and locally connected. Thus it remains to show that each $V\in\mathcal{V}$ is evenly covered by Y with respect to $g_{\mathcal{V}}\circ f$, i.e. that any connected component of $f^{-1}(g_{\mathcal{V}}^{-1}(V))$ is mapped homeomorphically to V by $g_{\mathcal{V}}\circ f$. Now $g_{\mathcal{V}}^{-1}(V)=\bigcup V'$ where $V'\in\tilde{\mathcal{V}}$, V' the connected components of $g_{\mathcal{V}}^{-1}(V)$. Also $f^{-1}(g_{\mathcal{V}}^{-1}(V))=\bigcup f^{-1}(V')=\bigcup V''$ where the V'' are the connected components of the $f^{-1}(V')$. Now each V'' is mapped homeomorphically to a V' by f and each V' is mapped homeomorphically to V by $g_{\mathcal{V}}$. Hence each V'' is mapped homeomorphically to V by $g_{\mathcal{V}}\circ f$. Also, the V'' are all connected; those belonging to the same $f^{-1}(V')$ are disjoint and so are those belonging to different $f^{-1}(V')$. Therefore, they are the connected components of $f^{-1}(g_{\mathcal{V}}^{-1}(V))$. Thus $(Y,g_{\mathcal{V}}\circ f,E)$ is a covering space even in $\mathcal{V}$. Now, as in Lemma 4, we have the mapping $k:S\to Y$ which maps each $V_0'\cup\ldots\cup V_n'\subseteq S$ onto some $V_0''\cup\ldots\cup V_n''\subseteq Y$ where V_0'' is a fixed connected component of $f^{-1}(g_{\mathcal{V}}^{-1}(U))$, which is constant on the R_0-classes and induces a continuous mapping $g':E_{\mathcal{V}}\to Y$ such that $g_{\mathcal{V}}=g_{\mathcal{V}}\circ f\circ g'$. Now, let $\tilde{U}\subseteq E_{\mathcal{V}}$ be the image of all first U-replicas in S with respect to the natural mapping $S\to E_{\mathcal{V}}$ and let $U'\subseteq Y$ be that connected component of $f^{-1}(g_{\mathcal{V}}^{-1}(U))$ onto which k maps all first U-replicas in S. Without loss of generality, it may be assumed that U' is a connected component of $f^{-1}(\tilde{U})$, and then g' induces on $\tilde{U}$ the local inverse of f on $\tilde{U}$. Hence $f \circ g'$ is the identity on $\tilde{U}$ and so by Lemma 6, $f \circ g'$ is the identity on $E_{\mathcal{V}}$. Let $y_1, y_2 \in Y$ with $f(y_1)=f(y_2)$. Take $x_1, x_2 \in E_{\mathcal{V}}$ with $g'(x_1)=y_1$, $g'(x_2)=y_2$ (which is possible since g' is onto). Thus $f(g'(x_1))=f(y_1)=f(y_2)=f(g'(x_2))$. But $f \circ g'$ is the identity. Therefore $x_1=x_2$. It then follows that $y_1=y_2$ so f is one-one and the theorem is proved.

In general, if (X,f,E), (Y,g,X) are covering spaces, $(Y,f \circ g,E)$ need not be a covering space, but we have seen that if (X,f,E) is even in $\mathcal{V}$ and (Y,g,X) is even in $\tilde{\mathcal{V}}$ as above, then $(Y,f \circ g,E)$ is a covering space even in $\mathcal{V}$.

<u>Corollary 1</u>: <u>If $E=\bigcup V_\alpha$ where the V_α are simply connected domains of E, then E possesses simply connected covering spaces.</u>

<u>Proof</u>: By Lemma 3, Chapter 2, section 6 of Chevalley's "Theory of Lie Groups", any covering space of a space E covers any simply connected domain evenly. Let $\mathcal{V}$ be a covering of E by simply connected domains. Then any covering space of E is even in $\mathcal{V}$. Hence, if E is also $\mathcal{V}$-simply connected, E is simply connected.

If $\mathcal{V}$ is a covering of E by simply connected domains, $E_{\mathcal{V}}$ is $\tilde{\mathcal{V}}$-simply connected by Theorem 2 and $\tilde{\mathcal{V}}$ is a covering of $E_{\mathcal{V}}$ by simply connected domains since each set in $\tilde{\mathcal{V}}$ is homeomorphic to some set in $\mathcal{V}$. Thus $E_{\mathcal{V}}$ is simply connected and so $(E_{\mathcal{V}}, g_{\mathcal{V}}, E)$ is a simply connected covering space.

<u>Corollary 2</u>: <u>If (X,f,E) is a covering space even in $\mathcal{V}$ which</u>

<u>is $\tilde{\mathcal{V}}$-simply connected with respect to the covering $\tilde{\mathcal{V}}$ of X consisting of all connected components of the sets $f^{-1}(V)$, $V \in \mathcal{V}$, then $(X,f,E) \cong (E_{\mathcal{V}},g_{\mathcal{V}},E)$.</u>

<u>Proof</u>: There exists, by Theorem 1, a continuous mapping $g: E_{\mathcal{V}} \to X$ such that $(E_{\mathcal{V}},g,X)$ is a covering space and $g_{\mathcal{V}} = f \circ g$. Now, for any $\tilde{V} \in \tilde{\mathcal{V}}$, $g^{-1}(\tilde{V}) \subseteq g_{\mathcal{V}}^{-1}(V)$ if $V = f(\tilde{V}) \in \mathcal{V}$. Now, if any connected component V' of $g_{\mathcal{V}}^{-1}(V)$ meets $g^{-1}(\tilde{V})$, then $V' \subseteq g^{-1}(\tilde{V})$ since $g(V')$ is connected and hence can only meet one connected component of $f^{-1}(V)$. Thus $g^{-1}(\tilde{V})$ is the union of such V'; these are then the connected components of $g^{-1}(\tilde{V})$ and each of them is mapped homeomorphically onto $\tilde{V}$ by g since $f \circ g = g_{\mathcal{V}}$. It follows that g is one-one and the corollary is proved.

This shows that, up to isomorphism, there exists for each covering $\mathcal{V}$ of E by domains exactly one covering space of E even in $\mathcal{V}$ and $\tilde{\mathcal{V}}$-simply connected with respect to the covering $\tilde{\mathcal{V}}$ determined by $\mathcal{V}$. This constitutes a new conceptual description of the covering spaces $(E_{\mathcal{V}},g_{\mathcal{V}},E)$.

<u>Lemma 7</u>: <u>Let E be a connected, locally connected space, $\mathcal{V}$ a covering of E by domains and $f_1: X_1 \to E$ a continuous mapping such that X_1 is locally connected and each $V \in \mathcal{V}$ is evenly covered by X_1 with respect to f_1. Let X be any connected component of X_1 and f the restriction of f_1 to X; then (X,f,E) is a covering space even in $\mathcal{V}$.</u>

<u>Proof</u>: We first prove $f(X)=E$. Assume $p \in E$ and let $V \in \mathcal{V}$ be a neighbourhood of p. Let V_α be the connected components

of $f_1^{-1}(V)$. If, for some α, $V_\alpha \cap X \neq \emptyset$, then V_α is entirely contained in X, whence $X \cap f_1^{-1}(V) = \cup V_\alpha$ over all such α. It follows that, if $V \cap f(X) \neq \emptyset$, we have $V \subseteq f(X)$; in particular, if p is adherent to $f(X)$, then p is interior to $f(X)$. Thus $f(X)$ is open and closed in E, whence $f(X)=E$. For any $V \in \mathcal{V}$, the connected components of $f^{-1}(V)$ are the sets V_α where $V_\alpha \cap X \neq \emptyset$, since each V_α is a maximal connected subset of $f_1^{-1}(V)$, and so of $f^{-1}(V)$. It follows that (X,f,E) is a covering space even in $\mathcal{V}$ and the lemma is proved.

<u>Theorem 3</u>: <u>Let W be a $\mathcal{V}$-simply connected space where $\mathcal{V}$ is a covering by domains, and let (X,f,E) be a covering space even in $\mathcal{U}$. Let $g:W \rightarrow E$ be a continuous mapping such that for each $V \in \mathcal{V}$, $g(V) \subseteq U$ for some $U \in \mathcal{U}$. Then, for any $(w_0,x_0) \in W \times X$ such that $g(w_0)=f(x_0)$, there exists a unique continuous mapping $h:W \rightarrow X$ such that $g=f \circ h$ and $h(w_0)=x_0$.</u>

<u>Proof</u>: The restriction of $pr_1:(w,x) \rightarrow w$ of $W \times X$ onto W to $W \otimes X = \{(w,x)/f(x)=g(w)\}$ is a continuous mapping $k:W \otimes X \rightarrow W$. If $w \in W$, take $U \in \mathcal{U}$ with $g(w) \in U$ (a connected neighbourhood of $f(x)=g(w)$) and note the U is evenly covered by X with respect to f. Let U' be a connected component of $f^{-1}(U)$. Let f^* be the local inverse of f on U with $f^*:U \rightarrow U'$. Let $V \in \mathcal{V}$ be a connected neighbourhood of w with $g(V) \subseteq U$. Then the set $F = \{(z,f^*(g(z)))/z \in V\}$ is mapped continuously to V by k. The mapping $z \rightarrow (z,f^*(g(z)))$ maps V continuously onto F and $k(z,f^*(g(z)))=z$. Thus k maps F to V homeomorphically. Also, individual F's are disjoint since they come from

disjoint U''s. Now $k^{-1}(V)=\bigcup F$ since $k(w,x)=w\in V$ means $x\in$ some U' which implies the sets F are the connected components of $k^{-1}(V)$ and so V is evenly covered by $W\otimes X$ with respect to k. Let C be the connected component of (w_0,x_0) in $W\otimes X$ and let k_1 be the restriction of k to C. Then by Lemma 7, (C,k_1,W) is a covering space even in $\mathcal{V}$ and so is trivial. Thus k_1^{-1} exists. We define h by $k_1^{-1}(w)=(w,h(w))$ and note h is unique by Lemma 6.

<u>Theorem 4</u>: <u>Let E be a $\mathcal{V}$-simply connected space where $\mathcal{V}$ is a covering of E by domains, and let $D\subseteq E\times E$ be a connected neighbourhood of the diagonal such that $V\times V\subseteq D$ for each $V\in\mathcal{V}$. Now, let a non-empty set T_x be assigned to each $x\in E$ such that $T_x\cap T_y=\emptyset$ if $x\neq y$, and let a mapping $\varphi_{xy}:T_y\to T_x$, one-one and onto, be assigned to each $(x,y)\in D$ such that</u>

<u>(i) φ_{xx} is the identity for each $x\in E$.</u>

<u>(ii) $\varphi_{zx}=\varphi_{zy}\circ\varphi_{yx}$ for any (z,x), (z,y), $(y,x)\in D$.</u>

<u>Then, given any $x_0\in E$ and $t_0\in T_{x_0}$, there exists a unique mapping $\psi:E\to\bigcup T_x$ such that</u>

<u>(i) $\psi(x)\in T_x$ for each $x\in E$.</u>

<u>(ii) $\psi(x_0)=t_0$.</u>

<u>(iii) $\psi(x)=\varphi_{xy}(\psi(y))$ for any $(x,y)\in D$.</u>

<u>Proof</u>: Set $F=\bigcup_{x\in E}T_x$ and let $p:F\to E$ be defined such that $p(T_x)=\{x\}$. For any $A\subseteq E$, we call a mapping $s:A\to F$ a section on A if

(i) $p(s(a))=a$, $a\in A$ (i.e. $s(a)\in T_a$).

(ii) $\varphi_{a_1a_2}(s(a_2))=s(a_1)$ for all $(a_1,a_2)\in(A\times A)\cap D$.

We define a topology on F by taking the images of sections on open sets U as the generating sets. If $U\times U\subseteq D$, $u_0\in U$, $t_0\in T_{u_0}$, then there exists one and only one section s on U with $s(u_0)=t_0$, since $s(u)=\varphi_{uu_0}(t_0)$ is defined for all $u\in U$ and is clearly such a section, and if $s':U\to F$ is another such section, then $s'(u)=\varphi_{uu_0}(t_0)=s(u)$. Thus for $U\times U\subseteq D$, $p^{-1}(U)=\bigcup s(U)$, where the union is over all sections on U, which is open by definition. It follows now that p is continuous; for, if $W\subseteq E$ is open, then $W=\bigcup U_\alpha$ with $U_\alpha\times U_\alpha\subseteq D$, since any $w\in W$ has a neighbourhood V such that $V\subseteq W$, $V\times V\subseteq D$, and then $p^{-1}(W)=\bigcup p^{-1}(U_\alpha)$ which is open.

Also, if $U\times U\subseteq D$ and s is a section on U, then p induces a homeomorphism on s(U) with U. First, p is continuous and one-one. Also, all open sets are arbitrary unions of finite intersections of images of sections by the way we defined the topology. Now $s(U)\cap\bigcup(\bigcap s_\alpha(V_\alpha))=\bigcup(\bigcap s(U)\cap s_\alpha(V_\alpha))$. Here, if $s(U)\cap s_\alpha(V_\alpha)\neq\emptyset$, then s and s_α have the same value at some point of $U\cap V_\alpha$, hence they coincide on the whole of $U\cap V_\alpha$ and $s(U)\cap s_\alpha(V_\alpha)=s(U\cap V_\alpha)$. Thus $s(U)\cap\bigcup(\bigcap s_\alpha(V_\alpha))=\bigcup(\bigcap s(U\cap V_\alpha))=s(U\cap(\bigcup\bigcap V_\alpha))$. Thus any open set W in s(U) is the image of an open subset of U with respect to the section s, and hence p(W) is open; so p is an open mapping, thus a homeomorphism on s(U) with U.

This implies that F is locally connected since $F=\bigcup s(U)$ over all $U\times U\subseteq D$, and these s(U) are homeomorphic to U, and hence locally connected.

Now for any $U \times U \subseteq D$, $p^{-1}(U) = \bigcup s(U)$. Each $s(U)$ is connected, and distinct $s(U)$ are disjoint, i.e. the $s(U)$ are the connected components of $p^{-1}(U)$ and p, as we have seen, induces a homeomorphism on each of them onto U. Thus the U are evenly covered. Since for any $V \in \mathcal{V}$, $V \times V \subseteq D$, the V are also evenly covered.

Take F_0, the connected component of t_0, and let p_0 be the restriction of p to F_0. Then by Lemma 7, (F_0, p_0, E) is a covering space even in $\mathcal{R}$. This implies that p_0 is one-one so p_0^{-1} exists. Then we define the mapping $\psi = p_0^{-1}$ where p_0^{-1} maps $E \to F$ such that $p_0^{-1}(x_0) = t_0$.

It remains only to prove the uniqueness of the mapping ψ. Let ψ' be any mapping which satisfies the same conditions as ψ (including $\psi'(x_0) = t_0$). Let $A = \{x / \psi'(x) = \psi(x)\}$; we know that A is not empty. Let x be any point of E and let N be a neighbourhood of x such that $N \times N \subseteq D$. Assume that N has a point x_1 in common with A; then $\varphi_{x_1 x}(\psi'(x)) = \psi'(x_1) = \psi(x_1) = \varphi_{x_1 x}(\psi(x))$, whence $\psi(x) = \psi'(x)$. It follows immediately that A is open and closed in E, whence A=E as required.

<u>Corollary</u>: <u>Theorem 4 remains valid if $x \neq y$ does not necessarily imply $T_x \cap T_y = \emptyset$.</u>

<u>Proof</u>: Set $T_x' = \{x\} \times T_x$ and let $\varphi_{xy}' : T_y' \to T_x'$ by $(y,t) \to (x, \varphi_{xy}(t))$ where $t \in T_y$. This satisfies all of the required conditions with T_x' and φ_{xy}' in place of T_x and φ_{xy}. Then consider

$F' = \bigcup_{x \in E} T'_x$ and the mapping $k: F' \to F$ by $(x,t) \to t$. Then $\psi = k \circ \psi'$ where $\psi' : E \to \bigcup_{x \in E} T'_x$ is the mapping with the desired properties and this must be unique by the same argument as above.

4. The Automorphism Groups of Relatively Greatest Covering Spaces

Lemma 8: *The automorphisms of a covering space (X,f,E) form a group.*

Proof: Let σ, τ be automorphisms of the covering space (X,f,E) with $f=f\circ\sigma$, $f=f\circ\tau$. Then $f\circ(\sigma\circ\tau)=(f\circ\sigma)\circ\tau=f\circ\tau=f$. Also $f\circ\varepsilon=f$ where ε is the identity transformation. Finally, if $f=f\circ\sigma$, then $f\circ\sigma^{-1}=(f\circ\sigma)\circ\sigma^{-1}=f\circ(\sigma\circ\sigma^{-1})=f\circ\varepsilon=f$ and the lemma is proved.

Remark 1: If σ, τ are two automorphisms of the covering space (X,f,E) and $\sigma(a)=\tau(a)$ for some $a\in X$, then $\sigma=\tau$ (by Lemma 6).

Remark 2: For any $a\in E$, $a'\in f^{-1}(a)$, the $\sigma(a')$, σ the automorphisms of (X,f,E), form a subset of $f^{-1}(a)$. This may be a proper subset. If it is equal to $f^{-1}(a)$ for each $a\in E$, then (X,f,E) is said to be normal.

Theorem 5: *For a connected, locally connected space E, each of the covering spaces (E_γ, g_γ, E), γ a covering of E by domains, is normal.*

Proof: Take any $a\in E$, and let $a', a''\in g_\gamma^{-1}(a)$. Then by Theorem 3 there exists a continuous mapping $\varphi: E_\gamma \to E_\gamma$ such that $g_\gamma\circ\varphi=g_\gamma$ and $\varphi(a')=a''$. Similarly there exists a continuous mapping $\psi: E_\gamma\to E_\gamma$ such that $g_\gamma\circ\psi=g_\gamma$ and $\psi(a'')=a'$. Thus $g_\gamma\circ\varphi\circ\psi=g_\gamma$ and $\varphi(\psi(a''))=a''$, hence by Lemma 6, $\varphi\circ\psi$ is the identity. Similarly $\psi\circ\varphi$ is the identity. Thus φ,ψ are homeomorphisms of $E_{\mathcal{V}}$ onto itself and inverse to each other such that $g_{\mathcal{V}}\circ\varphi=g_{\mathcal{V}}\circ\psi=g_{\mathcal{V}}$. Thus φ,ψ are automorphisms of the covering space $(E_{\mathcal{V}},g_{\mathcal{V}},E)$ with the desired properties and the theorem is proved.

<u>Corollary</u>: <u>Given any covering space (X,f,E), there exist normal covering spaces $(Y,g,E)\geq(X,f,E)$.</u>

<u>Proof</u>: If (X,f,E) is any covering space, there exists a covering $\mathcal{V}$ of E by domains in which (X,f,E) is even. But $(E_{\mathcal{V}},g_{\mathcal{V}},E)\geq(X,f,E)$ and by Theorem 5, $(E_{\mathcal{V}},g_{\mathcal{V}},E)$ is normal, proving the corollary.

5. Relatively Greatest Covering Groups

Theorem 6: Let $(G_{\mathcal{V}}, g_{\mathcal{V}}, G)$ be the greatest covering space even in $\mathcal{V}$ where G is a topological group and $\mathcal{V}$ is a covering of G by domains. Assume for each $V \in \mathcal{V}$, $aV \in \mathcal{V}$ for all $a \in G$. Then $G_{\mathcal{V}}$ can be made in one and only one way into a topological group such that $(G_{\mathcal{V}}, g_{\mathcal{V}}, G)$ is a covering group.

Proof: Let $\tilde{e} \in g_{\mathcal{V}}^{-1}(e)$, e the unit in G, be fixed throughout the following. This $\tilde{e}$ will turn out to be the unit in the group structure we are going to define on $G_{\mathcal{V}}$.

Now $G_{\mathcal{V}}$ is $\tilde{\mathcal{V}}$-simply connected where $\tilde{\mathcal{V}}$ is the covering of $G_{\mathcal{V}}$ by the connected components of the sets $g_{\mathcal{V}}^{-1}(V)$, $V \in \mathcal{V}$. Let the mapping $T_a : G \to G$ by $x \to ax$, $x \in G$ be the left translation of G onto itself. Then we have $T_a(g_{\mathcal{V}}(\tilde{V})) = T_a(V) = aV \in \mathcal{V}$ by hypothesis. Thus for any given $\tilde{a} \in g_{\mathcal{V}}^{-1}(a)$ there exists, by Theorem 3, a unique, continuous mapping $S_{\tilde{a}} : G_{\mathcal{V}} \to G_{\mathcal{V}}$ such that $T_a \circ g_{\mathcal{V}} = g_{\mathcal{V}} \circ S_{\tilde{a}}$ and $S_{\tilde{a}}(\tilde{e}) = \tilde{a}$.

Now for $T_{a^{-1}} \circ g_{\mathcal{V}} : G_{\mathcal{V}} \to G$, there also exists a continuous mapping $S : G_{\mathcal{V}} \to G_{\mathcal{V}}$ such that $S(\tilde{a}) = \tilde{e}$ and $T_{a^{-1}} \circ g_{\mathcal{V}} = g_{\mathcal{V}} \circ S$. Then $T_a \circ g_{\mathcal{V}} = g_{\mathcal{V}} \circ S_{\tilde{a}} \Rightarrow g_{\mathcal{V}} = T_{a^{-1}} \circ T_a \circ g_{\mathcal{V}} = T_{a^{-1}} \circ g_{\mathcal{V}} \circ S_{\tilde{a}} = g_{\mathcal{V}} \circ S \circ S_{\tilde{a}}$, hence $g_{\mathcal{V}} = g_{\mathcal{V}} \circ S \circ S_{\tilde{a}}$. But $S(S_{\tilde{a}}(\tilde{e})) = S(\tilde{a}) = \tilde{e}$ so $S \circ S_{\tilde{a}}$ leaves one point fixed, thus $S \circ S_{\tilde{a}} = \varepsilon$, the identity transformation of $G_{\mathcal{V}}$. Also, $g_{\mathcal{V}} = T_a \circ T_{a^{-1}} \circ g_{\mathcal{V}} = T_a \circ g_{\mathcal{V}} \circ S = g_{\mathcal{V}} \circ S_{\tilde{a}} \circ S$ and $S_{\tilde{a}}(S(\tilde{a})) = S_{\tilde{a}}(\tilde{e}) = \tilde{a}$, hence also $S_{\tilde{a}} \circ S = \varepsilon$. Thus S, $S_{\tilde{a}}$ are one-one, onto and $S = S_{\tilde{a}}^{-1}$, hence

$S_{\tilde{a}}$ is a homeomorphism.

Actually, $S=S_{S(\tilde{e})}$ since: from $T_{a^{-1}}\circ g_{\gamma}=g_{\gamma}\circ S$ we have $a^{-1}=T_{a^{-1}}(g_{\gamma}(\tilde{e}))=g_{\gamma}(S(\tilde{e}))$, hence $S(\tilde{e})\in g_{\gamma}^{-1}(a^{-1})$. Thus $g_{\gamma}\circ S=T_{a^{-1}}\circ g_{\gamma}=g_{\gamma}\circ S_{S(\tilde{e})}$ and since $S_{S(\tilde{e})}(\tilde{e})=S(\tilde{e})$, we have $S=S_{S(\tilde{e})}$.

Now we wish to show that $\{S_{\tilde{a}}/\tilde{a}\in G_{\gamma}\}$ form a group. We have just seen that for each $S_{\tilde{a}}$, there exists $S_{\tilde{a}}^{-1}$ such that $S_{\tilde{a}}\circ S_{\tilde{a}}^{-1}=\varepsilon=S_{\tilde{a}}^{-1}\circ S_{\tilde{a}}$. Next, take $S_{\tilde{a}}$, $S_{\tilde{c}}$ and show $S_{\tilde{a}}\circ S_{\tilde{c}}=S_{S_{\tilde{a}}(\tilde{c})}$. Since $S_{\tilde{a}}\circ S_{\tilde{c}}(\tilde{e})=S_{\tilde{a}}(\tilde{c})$ and $T_{a}\circ g_{\gamma}=g_{\gamma}\circ S_{\tilde{a}}$, $T_{c}\circ g_{\gamma}=g_{\gamma}\circ S_{\tilde{c}}$ then we have $T_{a}\circ g_{\gamma}\circ S_{\tilde{c}}=g_{\gamma}\circ S_{\tilde{a}}\circ S_{\tilde{c}}$. But $T_{a}\circ g_{\gamma}\circ S_{\tilde{c}}=T_{a}\circ T_{c}\circ g_{\gamma}=T_{ac}\circ g_{\gamma}$. Thus $T_{ac}\circ g_{\gamma}=g_{\gamma}\circ S_{\tilde{a}}\circ S_{\tilde{c}}$. Then $g_{\gamma}(S_{\tilde{a}}(\tilde{c}))=g_{\gamma}(S_{\tilde{a}}(S_{\tilde{c}}(\tilde{e})))=T_{ac}\circ g_{\gamma}(\tilde{e})=T_{ac}(e)=ac$, so $S_{\tilde{a}}(\tilde{c})\in g_{\gamma}^{-1}(ac)$. Thus $T_{ac}\circ g_{\gamma}=g_{\gamma}\circ S_{S_{\tilde{a}}(\tilde{c})}$ so $g_{\gamma}\circ S_{S_{\tilde{a}}(\tilde{c})}=g_{\gamma}\circ S_{\tilde{a}}\circ S_{\tilde{c}}$ and since $S_{S_{\tilde{a}}(\tilde{c})}(\tilde{e})=S_{\tilde{a}}(\tilde{c})$ and $S_{\tilde{a}}(S_{\tilde{c}}(\tilde{e}))=S_{\tilde{a}}(\tilde{c})$, we have $S_{\tilde{a}}\circ S_{\tilde{c}}=S_{S_{\tilde{a}}(\tilde{c})}$ proving that $\{S_{\tilde{a}}/\tilde{a}\in G_{\gamma}\}$ is closed under multiplication. Next, since $S_{\tilde{e}}=\varepsilon$, there exists an identity. Thus we see that the $S_{\tilde{a}}$ form a group of homeomorphisms of G_{γ} onto itself.

Now consider the mapping $\tilde{a}\to S_{\tilde{a}}$ which is a mapping from G_{γ} to $\{S_{\tilde{a}}/\tilde{a}\in G_{\gamma}\}$. If $S_{\tilde{a}}=S_{\tilde{c}}$, then $\tilde{a}=S_{\tilde{a}}(\tilde{e})=S_{\tilde{c}}(\tilde{e})=\tilde{c}$, so the mapping is one-one. We define $\tilde{a}\tilde{b}=S_{\tilde{a}}(\tilde{b})$. This is a law of composition in G_{γ} such that $S_{\tilde{a}\tilde{b}}=S_{S_{\tilde{a}}(\tilde{b})}=S_{\tilde{a}}\circ S_{\tilde{b}}$, hence this law of composition makes G_{γ} into a group, isomorphic to the group of homeomorphisms $\{S_{\tilde{a}}/\tilde{a}\in G_{\gamma}\}$.

Next we wish to show that g_{γ} is a homomorphism. Now $g_{\gamma}(\tilde{a}\tilde{b})=g_{\gamma}(S_{\tilde{a}}(\tilde{b}))=g_{\gamma}(S_{\tilde{a}}(S_{\tilde{b}}(\tilde{e})))=T_{ab}(g_{\gamma}(\tilde{e}))=ab=g_{\gamma}(\tilde{a})g_{\gamma}(\tilde{b})$, showing g_{γ} is a homomorphism.

To show that G_{γ}, with its topology as a covering space of G and the group structure just defined is a topological group, the continuity of the mapping $(\tilde{x},\tilde{y})\to\tilde{x}\tilde{y}^{-1}$ has to be shown. Since $S_{\tilde{c}}:\tilde{x}\to\tilde{c}\tilde{x}$ is a homeomorphism, the sets $\tilde{c}\tilde{V}=S_{\tilde{c}}(\tilde{V})$, $\tilde{V}$ the neighbourhoods of $\tilde{e}$, form the neighbourhoods of $\tilde{c}$ for any $\tilde{c}\in G_{\gamma}$. Thus it has to be shown that for any neighbourhood $\tilde{V}$ of $\tilde{e}$, there exist neighbourhoods $\tilde{U},\tilde{W}$ of $\tilde{e}$ such that $\tilde{a}\tilde{U}(\tilde{b}\tilde{W})^{-1}\subseteq\tilde{a}\tilde{b}^{-1}\tilde{V}$. This condition means that $(\tilde{b}\tilde{U}\tilde{b}^{-1})(\tilde{b}\tilde{W}^{-1}\tilde{b}^{-1})\subseteq\tilde{V}$, and if one can find neighbourhoods $\tilde{X},\tilde{Y}$ of $\tilde{e}$ such that $\tilde{X}^{-1}=\tilde{X}$, $\tilde{X}\tilde{X}\subseteq\tilde{V}$ and $\tilde{b}\tilde{Y}\tilde{b}^{-1}\subseteq\tilde{X}$, then $\tilde{U}=\tilde{W}=\tilde{Y}$ will satisfy this.

Now, first, $\tilde{V}$ may be assumed to be the connected component of $\tilde{e}$ in $g_{\gamma}^{-1}(V)$ with some evenly covered neighbourhood V of e in G. Then there exists an open connected neighbourhood X of e with $X^{-1}=X$ and $XX\subseteq V$; let $\tilde{X}$ be the connected component of $g_{\gamma}^{-1}(X)$ which lies in $\tilde{V}$. If $\tilde{x}\in\tilde{X}$ then $\tilde{x}^{-1}\tilde{X}=S_{\tilde{x}}^{-1}(\tilde{X})$ is connected, meets $\tilde{V}$ (at $\tilde{e}$) and lies in $g_{\gamma}^{-1}(V)$; hence $\tilde{x}^{-1}\tilde{X}\subseteq\tilde{V}$ and thus $\tilde{x}^{-1}\in\tilde{V}$. Since $g_{\gamma}(\tilde{x}^{-1})=x^{-1}\in X$, one even has $\tilde{x}^{-1}\in\tilde{X}$, and thus $\tilde{X}^{-1}\subseteq\tilde{X}$ which gives $\tilde{X}^{-1}=\tilde{X}$. Again, for any $\tilde{x}\in\tilde{X}$, $\tilde{x}\tilde{X}$ is connected, lies in $g_{\gamma}^{-1}(V)$ and meets $\tilde{V}$ (at $\tilde{x}$) such that $\tilde{x}\tilde{X}\subseteq\tilde{V}$, and hence $\tilde{X}\tilde{X}\subseteq\tilde{V}$.

Next, it has to be shown that $G_{\gamma}=\bigcup\tilde{X}^{k}$ (k=1,2,...) where $\tilde{X}^{k}=\{\tilde{x}_1\ldots\tilde{x}_k/\tilde{x}_i\in\tilde{X}\}$. First, this union is open since for any $\tilde{x}_1\ldots\tilde{x}_k$, the set $\tilde{x}_1\ldots\tilde{x}_k\tilde{X}\subseteq\tilde{X}^{k+1}$ is a neighbourhood of $\tilde{x}_1\ldots\tilde{x}_k$. Next, let c belong to the closure of this union. Then, in particular, $c\tilde{X}\cap\tilde{X}^{k}\neq\emptyset$ for some k, i.e. $c\tilde{x}=\tilde{x}_1\ldots\tilde{x}_k$ with $\tilde{x},\tilde{x}_i\in\tilde{X}$, hence $c=\tilde{x}_1\ldots\tilde{x}_k\tilde{x}^{-1}\in\tilde{X}^{k}\tilde{X}^{-1}=\tilde{X}^{k+1}$.

Thus, the union is also closed, and since it is non-void it must be equal to G_φ.

It follows that $\tilde{b}=\tilde{x}_1\ldots\tilde{x}_k$ with suitable $\tilde{x}_i\in\tilde{X}$. Now, since G is a topological group, there exist open connected neighbourhoods $X_1,\ldots,X_k$ of e in G such that $X=X_0\supseteq X_1\supseteq\ldots\supseteq X_k$ and $x_1X_1x_1^{-1}\subseteq X_0,\ldots,x_iX_ix_i^{-1}\subseteq X_{i-1},\ldots,x_kX_kx_k^{-1}\subseteq X_{k-1}$ where $x_i=g_\varphi(\tilde{x}_i)$. Let $\tilde{X}_i$ be the connected component of $g_\varphi^{-1}(X_i)$ which lies in $\tilde{V}$. Then $\tilde{X}=\tilde{X}_0\supseteq\tilde{X}_1\supseteq\ldots\supseteq\tilde{X}_k$ and $\tilde{x}_i^{-1}\in\tilde{X}$, $\tilde{X}\tilde{X}\subseteq\tilde{V}$ implies that $\tilde{X}_i\tilde{x}_i^{-1}\subseteq\tilde{V}$. Since g_φ induces a homeomorphism on $\tilde{V}$ and maps $\tilde{X}_i\tilde{x}_i^{-1}$ onto $X_ix_i^{-1}$, $\tilde{X}_i\tilde{x}_i^{-1}$ is connected and open, hence so is $\tilde{x}_i\tilde{X}_i\tilde{x}_i^{-1}$. From $g_\varphi(\tilde{x}_i\tilde{X}_i\tilde{x}_i^{-1})=x_iX_ix_i^{-1}\subseteq X_{i-1}$ it follows that $\tilde{x}_i\tilde{X}_i\tilde{x}_i^{-1}\subseteq g_\varphi^{-1}(X_{i-1})$, and since $\tilde{x}_i\tilde{X}_i\tilde{x}_i^{-1}$ meets $\tilde{X}_{i-1}$ (at $\tilde{e}$) one has $\tilde{x}_i\tilde{X}_i\tilde{x}_i^{-1}\subseteq\tilde{X}_i$. Combining all these relations for $i=0,1,\ldots,k$ one has $\tilde{X}\supseteq\tilde{x}_1\tilde{X}_1\tilde{x}_1^{-1}\supseteq\tilde{x}_1\tilde{x}_2\tilde{X}_2\tilde{x}_2^{-1}\tilde{x}_1^{-1}\supseteq\ldots\supseteq\tilde{b}\tilde{X}_k\tilde{b}^{-1}$. Hence $\tilde{X}_k$ is a neighbourhood of $\tilde{e}$ such that $\tilde{b}\tilde{X}_k\tilde{b}^{-1}\subseteq\tilde{X}$, and by the remarks at the beginning of this proof, this establishes the continuity of the mapping $(\tilde{x},\tilde{y})\rightarrow\tilde{x}\tilde{y}^{-1}$.

Now we must show that the structure of G_φ as a topological group such that (G_φ,g_φ,G) is a covering group is unique. If there is another group structure on G_φ (whose multiplication may be denoted by $\tilde{x}*\tilde{y}$) such that these conditions are satisfied, let $S^*_{\tilde{a}}$ be the left translation by $\tilde{a}$ in this structure, i.e. $S^*_{\tilde{a}}:\tilde{x}\rightarrow\tilde{a}*\tilde{x}$. Then, for $\tilde{x}\in g_\varphi^{-1}(x)$: $g_\varphi(S^*_{\tilde{a}}(\tilde{x}))=ax=T_a(x)$, hence $g_\varphi\circ S^*_{\tilde{a}}=T_a\circ g_\varphi=g_\varphi\circ S_{\tilde{a}}$. Since also $S^*_{\tilde{a}}(\tilde{e})=\tilde{a}=S_{\tilde{a}}(\tilde{e})$, one has $S^*_{\tilde{a}}=S_{\tilde{a}}$ by Lemma 6. This implies $\tilde{a}*\tilde{x}=\tilde{a}\tilde{x}$ for any $\tilde{a},\tilde{x}\in G_\varphi$, hence the multiplication defined above is unique.

<u>Lemma 9</u>: <u>Let $H \subseteq G$ where H is a discrete, normal subgroup of a connected topological group G. Then ax=xa for any $a \in H$, $x \in G$.</u>

<u>Proof</u>: The mapping $x \to x^{-1}ax$ $(a \in H)$ is continuous and maps G into H. Hence $\{x^{-1}ax / x \in G\}$ is connected and contains a. Since H is discrete $\{x^{-1}ax / x \in G\} = \{a\}$ or ax=xa. Thus H is abelian.

<u>Corollary</u>: <u>The automorphism group of the covering space (G_γ, g_γ, G) is abelian.</u>

<u>Proof</u>: First we must show that the automorphism group $\mathcal{A}$ of the covering space (G_γ, g_γ, G) is isomorphic to $\ker(g_\gamma)$.

Now, the left translation $T_{e'}: G_\gamma \to G_\gamma$ $(e' \in g_\gamma^{-1}(e) = \ker(g_\gamma))$ is a homeomorphism such that $g_\gamma \circ T_{e'} = T_e \circ g_\gamma = g_\gamma$, thus $T_{e'} \in \mathcal{A}$.

Next, if $\sigma \in \mathcal{A}$, then $\sigma = T_{\sigma(\tilde{e})}$ since: $g_\gamma = g_\gamma \circ \sigma$ by definition of σ, and $g_\gamma = g_\gamma \circ T_{\sigma(\tilde{e})}$. Also $T_{\sigma(\tilde{e})}(\tilde{e}) = \sigma(\tilde{e})$, thus $\sigma = T_{\sigma(\tilde{e})}$, so each $\sigma \in \mathcal{A}$ is of this form.

Now the mapping $e' \to T_{e'}$ maps $\ker(g_\gamma)$ onto $\mathcal{A}$; also $T_{e'e''} = T_{e'}T_{e''}$ so the mapping is a homomorphism. Now, if $T_{e'} = \varepsilon$ the identity automorphism, $e' = T_{e'}(\tilde{e}) = \varepsilon(e') = e'$, thus $e' = \tilde{e}$ and so $\mathcal{A} \cong \ker(g_\gamma)$. By Lemma 9, $\ker(g_\gamma)$ is abelian, thus $\mathcal{A}$ is abelian as required.

Finally the following two theorems may be proved:

<u>Theorem 7</u>: <u>If G is a γ-simply connected topological group and h a local homomorphism into a group H, defined on</u>

$\cup VV^{-1}$ ($V \in \mathscr{V}$), then h has an extension h' to a homomorphism of G into H.

Theorem 8: If h is a local homomorphism from a topological group G into a group H defined on a neighbourhood W of e in G, then there exists a covering group ($\tilde{G}$,g,G) and a homomorphism $\tilde{h}:\tilde{G} \to H$ such that $\tilde{h}(s)=h(g(s))$ for s in the connected component of $\tilde{e}$ in $g^{-1}(W)$.

The proof of Theorem 7 is based on Theorem 4 and is essentially the same as that of Theorem 3, Chevalley, page 49. Theorem 8 is obtained by applying Theorem 7 to the covering group ($G_{\mathscr{V}}$, $g_{\mathscr{V}}$, G) where $\mathscr{V} = \{aV/a \in G\}$ and V is a connected neighbourhood of e such that $V^{-1}V \subseteq W$; $\tilde{h}$ is the extension of the local homomorphism $G_{\mathscr{V}} \to H$ given on the connected component of $\tilde{e}$ in $g_{\mathscr{V}}^{-1}(W)$ by $s \to h(g_{\mathscr{V}}(s))$.

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