MODEL THEORY AND VALUATIONS

RELATIONSHIPS BETWEEN MODEL THEORY AND VALUATIONS OF FIELDS

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Abstract

This thesis explores some of the relationships between model theoretic and algebraic properties of fields, focusing on valuations of fields. We first show that the dp-rank of henselian valued fields admitting relative quantifier elimination is equal to the sum of the dp-ranks of the value group and of the residue field. Moreover, we give a characterization of henselianity of valued fields of finite dp-rank in terms of the dp-rank of definable sets. We also obtain partial results generalizing the work of Johnson in classifying fields of finite dp-rank. Finally, we consider fields with the property that the algebraic closure is an immediate extension with respect to every valuation. We show that under certain conditions these fields are dense in their algebraic closure with respect to every valuation and provide an example that demonstrates that this property does not hold in general.

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Contents

Chapter 1

Definitions and Preliminary Results

1.1 Introduction

This thesis is focused on a fundamental question of the model theory of fields: given a field in some language, what is the relationship between the model theoretic and algebraic properties of that field. One of the first significant results in this area is Macintyre's proof that all ω-stable fields are algebraically closed (Macintyre, 1971). Since that time, many other model theoretic classes of fields have been shown to have strong algebraic properties: superstable fields (Cherlin and Shelah, 1980), o-minimal fields (van den Dries, 1998), and more recently, superrosy fields (Krupiński, 2015) and dp-minimal fields (Johnson, 2015).

The two recent examples mentioned above use valuation theory as a tool to bridge the gap between logic and algebra. Valued fields, and in particular henselian valued fields, are very amenable to model theoretic techniques, with important classes admitting partial quantifier elimination and cell decompositions. Moreover, the field theoretic structure of a henselian valued field can be deduced almost entirely from the structure of the residue field and value group, providing a way to reduce a problem into hopefully simpler terms.

Each chapter of this thesis has a different approach to applying valuation theory. In Chapter 2, we consider fields with a fixed valuation and deduce properties about the field using the auxiliary structures provided by the valuation. In Chapter 3, we make model theoretic assumptions about the structure of a field and attempt to construct a valuation using only these assumptions. Finally, in Chapter 4, we consider fields in which all possible valuations share a particular algebraic property.

Chapters 2 and 3 focus on dp-finite fields, that is, fields whose theory has finite dp-rank. These theories fit into the universe of stability theory in a number of ways:

$$\begin{tikzcd}
\text{o-minimal} \arrow[d, Rightarrow] & \text{strongly minimal} \arrow[dl, Rightarrow] \arrow[dd, Rightarrow] \\
\text{dp-minimal} \arrow[d, Rightarrow] & \\
\text{dp-finite} \arrow[d, Rightarrow] & \text{superstable} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
\text{strongly dependent} \arrow[d, Rightarrow] & \text{stable} \arrow[dl, Rightarrow] \\
\text{NIP} &
\end{tikzcd}$$

Dp-rank gives a notion of dimension to type-definable sets in NIP structures, and dp-finite structures are ones in which the set defined by "$x = x$" has finite dp-rank. This definition was motivated by an attempt to find a notion for NIP structures that satisfied an analogous relationship to that of superstable and stable structures. The proposed analogy was strongly dependent theories, in which every complete type has finite dp-rank, but partial types (in particular, "$x = x$") may have a countably infinite rank. There is some subtlety to this particular case, which is discussed in more detail in Section 1.4.

While this notion did not fit the analogy perfectly (among other things, the class of structures which is strongly dependent and stable is slightly larger than the class of superstable theories), they have become an active area of research. The dp-rank in particular, and some variants for structures that are not NIP, has been very useful as a measure of complexity for type-definable sets. Unfortunately, it is only partially successful as a notion of dimension: while it satisfies nice properties (for example, $\text{dp-rk}(X \times Y) = \text{dp-rk}(X) + \text{dp-rk}(Y)$ and $\text{dp-rk}(X \cup Y) = \max\{\text{dp-rk}(X), \text{dp-rk}(Y)\}$) it is not always possible to find sets of every dp-rank. There are theories that contain sets of dp-rank 0 and dp-rank 2, but no sets of dp-rank 1.

The second chapter focuses on the structure of valued fields admitting relative quantifier elimination, a class that includes all strongly dependent henselian valued fields by Halevi and Hasson (2017a). First, we show that in the three-sorted Denef-Pas language, the dp-rank of such fields can be easily calculated as the sum of the dp-rank of the value group and the dp-rank of the residue field. Then, we show that the dp-rank is a very coarse notion of dimension on the home sort: in one variable, a set either has dp-rank 0 or dp-rank equal to the dp-rank of the theory. In fact, for dp-finite valued fields, this property can be used to characterize henselianity.

The next chapter focuses on a particular case of a large open question: can we classify the algebraic structure of all NIP fields? In other words, can we classify the NIP fields in the pure field language, up to elementary equivalence? Conjecturally, these fields are either separably closed, real closed, or have a definable henselian valuation. Johnson (2015) showed that this conjecture holds in the dp-minimal case. We generalize portions of Johnson's argument to the dp-finite case, and show that certain dp-finite fields admit a uniformly definable field topology. The last section of this chapter gives an argument that the dp-finite fields that do not admit a field topology should be stable and algebraically closed. We unfortunately can only say "should be," because the proposed proof that we outline would require a stonger version of some of the results in the rest of the chapter, which we have not yet been able to prove.

In the last chapter, we investigate two different classes of fields, first considered by Hong (2013). These two classes each consist of fields that approximate their algebraic closure in a particular way. We show that the two classes are not equal, but that by restricting them slightly (and uniformly between the two classes), they align.

1.2 Algebra of Valued Fields

This section very quickly covers the basic notions and notations for valued fields that will be used in this thesis. It assumes familiarity with all of the concepts of valuation theory; for a more thorough introduction, refer to Engler and Prestel (2005).

Given a field K (with $0 \neq 1$), we can define a valuation on K in two ways. First, we can define a subring $\mathcal{O} \subseteq K$, called a valuation ring, such that for every $x \in K$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. Alternatively, we can define a surjective map $v : K^\times \twoheadrightarrow \Gamma$ onto some ordered abelian group Γ, called the value group, with the following properties:

1. $v(xy) = v(x) + v(y)$
2. $v(x + y) \geq \min\{v(x), v(y)\}$

We usually write vK for the value group, especially when there are multiple fields or multiple valuations being considered. We then extend v to a map $v : K \to \Gamma \cup \{\infty\}$ by setting $v(0) = \infty$ and $\gamma < \infty$ for all $\gamma \in \Gamma$. These concepts are interdefinable:

- Given a valuation v, we can define a valuation ring $\mathcal{O}_v = \{x \in K : v(x) \geq 0\}$.
- Given a valuation ring $\mathcal{O}$, we can define an ordering on $K^\times/\mathcal{O}^\times$ by $x \cdot \mathcal{O}^\times \geq 1 \cdot \mathcal{O}^\times$ if and only if $x \in \mathcal{O}$. Then the usual quotient map $v_{\mathcal{O}} : K^\times \twoheadrightarrow K^\times/\mathcal{O}^\times$ is a valuation.

However a valuation is defined, there are two additional useful structures associated to it:

- $\mathfrak{m} = \{x \in \mathcal{O} : x^{-1} \notin \mathcal{O}\} = \{x \in K : v(x) > 0\}$, the maximal ideal of $\mathcal{O}$
- $Kv = \mathcal{O}/\mathfrak{m}$, the residue field of K.

When specifying the characteristic of a valued field, it is important to also specify the characteristic of the residue field; we list the characteristic as an ordered pair $(\mathrm{char}(K), \mathrm{char}(Kv))$. The possible combinations of characteristics are $(0, 0)$, $(0, p)$, and (p, p) for all primes $p > 0$; we say that a field has equicharacteristic if $\mathrm{char}(K) = \mathrm{char}(Kv)$, and mixed characteristic otherwise.

We write (K, v) or $(K, \mathcal{O})$ for a valued field. An extension of a valued field is a field extension L/K and a valuation w on L such that $w|_K = v$. When there is no ambiguity, we write v for both the valuation on L and its restriction to K. If $(L, v)/(K, v)$ is a valued field extension then Kv is a subfield of Lv and vK is a subgroup of vL. Moreover, if L/K is algebraic then so is Lv/Kv, and vK and vL have the same divisible hull. In the case where $Lv = Kv$ and $vL = vK$, we say that the extension is immediate.

A particularly useful class of valued fields is the class of henselian valued fields. A valued field (K, v) is called henselian if one of the following equivalent conditions holds:

1. There is a unique valuation w on the algebraic closure of K such that $v = w|_K$
2. There is a unique valuation ring $\mathcal{O}'$ on the algebraic closure of K such that $\mathcal{O} = \mathcal{O}' \cap K$
3. Every polynomial $p(X) = X^n + aX^{n-1} + \sum_{i=0}^{n-2} a_i X^i \in \mathcal{O}[X]$ with $v(a) = 0$ and $v(a_i) > 0$ for all i has a root in K.

It is clear from property (3) above that separably closed fields are henselian, but many other fields are also henselian, including the p-adic numbers. It can be shown that every valued field has a minimal algebraic extension that is henselian, and that this extension is unique up to isomorphism. This extension, denoted (K^h, v^h), is called the henselization of (K, v), and is always an immediate extension of (K, v).

Example 1.2.1. (Valuations on the rationals) Given a rational number $a = \frac{x}{y}$ and a prime p, we define $v_p(a)$ to be the unique integer $n \in \mathbb{Z}$ such that $\frac{x}{y} = p^n \frac{x'}{y'}$ with $\gcd(x', p) = \gcd(y', p) = 1$. The map $v_p : \mathbb{Q} \to \mathbb{Z}$ is a valuation for each prime p, called the p-adic valuation, and every non-trivial valuation on $\mathbb{Q}$ is a p-adic valuation for some p. The p-adic numbers $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ with respect to v_p, and $\mathbb{Q}_p \cap \mathbb{Q}^{\text{alg}}$ is the henselization of $\mathbb{Q}$ with respect to v_p.

Example 1.2.2. (Field of Hahn series) Let k be any field and Γ be any group, and consider the set $k[[t^\Gamma]]$ of functions $f : \Gamma \to k$ such that $\text{supp}(f) = \{\gamma \in \Gamma : f(\gamma) \neq 0\}$ is well-ordered. We think of elements of this set as power series and generally write them as $f = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma$, where $a_\gamma = f(\gamma)$. This set is a field with the usual operations on power series, and the map $v(f) = \min(\text{supp}(f))$ is a valuation on $k[[t^\Gamma]]$. In fact, $k[[t^\Gamma]]$ is henselian.

Example 1.2.3. (Natural valuation on an ordered field) Let $(K, <)$ be an ordered field, and let $\mathcal{O}$ be the convex hull of $\mathbb{Q}$ in K. It is easy to check that $\mathcal{O}$ is a valuation ring on K, and that Kv is isomorphic to an ordered subfield of $\mathbb{R}$.

Definition 1.2.4. An *angular component map* is a function $\mathrm{ac} : K \to Kv$ which satisfies the following:

1. $\mathrm{ac}(0) = 0$
2. For all $x \in \mathcal{O}^\times$, $\mathrm{ac}(x) = x + \mathfrak{m}$
3. For all $x, y \in K$, $\mathrm{ac}(xy) = \mathrm{ac}(x)\,\mathrm{ac}(y)$.

On a Hahn field, the leading term map (the map that returns the first nonzero coefficient of a generalized power series) is an angular component map. Not every valued field admits an angular component map, but every valued field has an elementary extension that does (Pas, 1990, Corollary 1.6).

When an angular component map exists, it can be used to determine whether $v(x+y)$ is greater than $\min\{v(x), v(y)\}$ or equal to it. Appendix A contains a precise description of this relationship, along with several other fundamental facts about the interaction between angular components and valuations.

Definition 1.2.5. A subgroup Δ of an ordered abelian group Γ is said to be *convex* if for every $a \in \Delta$, the interval $[-a, a] = \{x \in \Gamma : -a \leq x \leq a\}$ is a subset of Δ. The convex subgroups of Γ are linearly ordered by inclusion, and this order type is called the *rank* of Γ. In particular, if Γ has no proper non-trivial convex subgroups then Γ has rank 1, and is called *archimedean*.

Given a valued group (K, v) and a convex subgroup $\Delta \leq vK$, we can define $\mathcal{O}_\Delta = \{x \in K : v(x) \geq \delta \text{ for some } \delta \in \Delta\}$. Clearly, $\mathcal{O}_\Delta \supseteq \mathcal{O}$ is a valuation ring; it defines a valuation $w : K^\times \to vK/\Delta$, which is called a coarsening of v. Moreover, v induces a valuation $\overline{v} : Kw \to \Delta$ with $\overline{v}(x + \mathcal{O}_\Delta) = v(x)$.

Definition 1.2.6. Let $(a_\rho)_{\rho<\kappa}$ be a sequence of elements of K indexed by κ, a well-ordered set with no maximum element. We say that (a_ρ) is *pseudo-convergent* if

$$v(a_{\rho_2} - a_{\rho_1}) < v(a_{\rho_3} - a_{\rho_2})$$

for all $\rho_1 < \rho_2 < \rho_3$. In this case, for all $\rho < \kappa$ there exists $\gamma_\rho \in vK$ such that

$$v(a_\rho - a_{\rho'}) = \gamma_\rho$$

for all $\rho' > \rho$. We say that $x \in K$ is a *pseudo-limit* of (a_ρ) if $v(x - a_\rho) = \gamma_\rho$ for all ρ.

Note that pseudo-limits are not unique. The definitions of pseudo-convergence and pseudo-limits are due to Kaplansky (1942). In that paper, Kaplansky also shows that a valued field extension $(L, v)/(K, v)$ is immediate if and only if every element of $L \smallsetminus K$ is the limit of a pseudo-convergent sequence in K (Theorem 1). Moreover, he shows that if $(L, v)/(K, v)$ is any field extension and $x, x' \in L$ are pseudo-limits of the same pseudo-convergent sequence in K then $(K(x), v)/(K, v)$ is immediate and $K(x)$ and $K(x')$ are isomorphic as valued fields (Theorems 2 and 3).

1.3 Model Theory of Valued Fields

Valued fields can be viewed as first order structures in a number of ways: the two that we use in this thesis are as a one-sorted structure in $\mathcal{L}_{\text{div}}$, an expansion of $\mathcal{L}_{\text{ring}} = \{0, 1, +, -, \times\}$ by a binary predicate for the relation $v(x) \leq v(y)$, or as a three-sorted structure with sorts for K, vK, and Kv, and various maps between them.

Definition 1.3.1. The *Denef-Pas Language* for valued fields is the three-sorted language $\mathcal{L}_{\text{Pas}}$ with the following sorts and functions:

- The valued field sort VF has the language of rings $\mathcal{L}_{\text{ring}} = \{0, 1, +, -, \cdot\}$
- The value group sort VG has an expansion of the language of ordered abelian groups $\mathcal{L}_{\text{VG}} = \{0, +, -, <, \infty, \ldots\}$
- The residue field sort RF has an expansion $\mathcal{L}_{\text{RF}}$ of the language of rings
- The only maps between sorts are $v : \text{VF} \to \text{VG}$ and $\text{ac} : \text{VF} \to \text{RF}$.

Calling this "the" Denef-Pas Language is slightly misleading, since the value group and residue field languages are some expansion of the appropriate minimum languages. When we consider a valued field (K, v) as an $\mathcal{L}_{\text{Pas}}$-structure, we always assume that the VF-sort is K, the VG-sort is vK, the RF-sort is Kv, v is the valuation map, and ac is an angular component map.

We say that a theory T in $\mathcal{L}_{\text{Pas}}$ admits *relative quantifier elimination* if it eliminates quantifiers $\forall x$ and $\exists x$, where x is a variable in the valued field sort. In other words, T has relative quantifier elimination if every formula $\phi(x^{\text{VF}}, x^{\text{VG}}, x^{\text{RF}})$ in T is equivalent to one of the form

$$\bigvee_{i=1}^{n} \chi_i(v(f_1(x^{\text{VF}}), \ldots, v(f_m(x^{\text{VF}})), x^{\text{VG}}) \wedge \rho_i(\text{ac}(f_1(x^{\text{VF}})), \ldots, \text{ac}(f_m(x^{\text{VF}})), x^{\text{RF}})$$

where $x^{\text{VF}}, x^{\text{VG}}, x^{\text{RF}}$ are tuples of variables in the sorts $\text{VF}, \text{VG}, \text{RF}$, respectively, χ_i are $\mathcal{L}_{\text{VG}}$-formulas, ρ_i are $\mathcal{L}_{\text{RF}}$ formulas, and f_j are polynomials with integer coefficients. Note that there is no $\mathcal{L}_{\text{ring}}$-formula corresponding to the VF-sort; this is because any such formula would be a boolean combination of statements of the form $g(x^{\text{VF}}) = 0$, which is equivalent to $v(g(x^{\text{VF}})) = \infty$, and so this part of the formula can be moved into the $\mathcal{L}_{\text{VG}}$ portion.

Suppose T is a theory with relative quantifier elimination, and consider the special case of a formula $\phi(x)$ with parameters in some model (K, v) such that x is a singleton in the VF-sort. In this case, $\phi(x)$ is equivalent to a formula of the form

$$\bigvee_{i=1}^{n} \chi_i(v(f_1(x), \ldots, v(f_m(x))) \wedge \rho_i(\mathrm{ac}(f_1(x)), \ldots, \mathrm{ac}(f_m(x)))$$

where χ_i are $\mathcal{L}_{\mathrm{VG}}$-formulas with parameters in vK, ρ_i are $\mathcal{L}_{\mathrm{RF}}$ formulas with parameters in Kv, and f_j are polynomials with coefficients in K. This follows immediately from the general form of relative quantifier elimination by substituting a parameter for every variable except a singleton in the VF-sort.

Many theories of henselian valued fields have relative quantifier elimination, including all theories of henselian valued fields of characteristic $(0, 0)$ (Pas, 1989) and algebraically maximal Kaplansky fields of characteristic (p, p) (Bélair, 1999). Of particular interest in Chapter 2, strongly dependent henselian valued fields also have relative quantifier elimination (Halevi and Hasson, 2017a). In the case where T is a theory of henselian valued fields of characteristic $(0, 0)$, we may assume that the polynomials f_i are all linear by the cell decomposition of Pas (1989). In fact, we prove in Section 2.1 that this is true in any characteristic.

A straightforward but significant consequence of relative quantifier elimination is the Ax-Kochen-Ershov (AKE) principle:

Fact 1.3.2. *Suppose (K, v) and (L, w) are both models of some theory T of henselian valued fields in $\mathcal{L}_{\mathrm{Pas}}$ that admits relative quantifier elimination. Then $(K, v) \equiv (L, w)$ if and only if $vK \equiv wL$ (as $\mathcal{L}_{\mathrm{VG}}$-structures) and $Kv \equiv Lw$ (as $\mathcal{L}_{\mathrm{RF}}$-structures).*

1.4 Inp-patterns and Dp-rank

Throughout the rest of this document, various definitions and properties of dp-rank will be used. This section introduces the concept as well as several equivalent definitions; Appendix B lists and proves many of the basic properties of the rank.

The notion of dp-rank was motivated by Shelah (2009), where he attempted to find a property P which satisfied the following: every theory satisfying P should be NIP, and a theory should be superstable if and only if it is stable and satisfies P. While Shelah was unsuccessful in his attempt to find the property P, various ideas have proved interesting in their own right, in particular the notion of strong dependence, which depends on the notion of ict-patterns.

Definition 1.4.1. Let $\pi(x)$ be a partial type in a theory T with x a tuple of variables. A *randomness pattern*, also known as an *ict (independent contradictory types) pattern*, of depth κ for π is a set of formulas $\{\phi_\alpha(x, y_\alpha) : \alpha < \kappa\}$ and a set of parameters $(b_{\alpha,i})_{\alpha<\kappa, i<\omega}$ such that $|y_\alpha| = |b_{\alpha,i}|$ for all $i < \omega$, and the following set of formulas is consistent for each $\eta : \kappa \to \omega$:

$$\pi(x) \wedge \{\phi_\alpha(x, b_{\alpha,\eta(\alpha)}) : \alpha < \kappa\} \wedge \{\neg\phi_\alpha(x, b_{\alpha,i}) : \alpha < \kappa \text{ and } \eta(\alpha) \neq i < \omega\}.$$

The *dp-rank* of $\pi(x)$ is the supremum of all cardinals κ such that there exists a randomness pattern of depth κ for $\pi(x)$. If no such κ exists, we say $\pi(x)$ has unbounded dp-rank, and write $\text{dp-rk}(\pi) = \infty$.

Some comments on dp-rank:

- The dp-rank of $\pi(x)$ measures the complexity of π: the greater the depth κ of a randomness pattern, the closer π is to satisfying the independence property. In fact, $\text{dp-rk}(\pi) = \infty$ if and only if $\pi(x)$ has the independence property.

- Given an element c in a model of T and a set of parameters A, we say that $\text{dp-rk}(c/A) = \text{dp-rk}(\text{tp}(c/A))$. Note that the dp-rank of c depends on the choice of parameters A.

- If $\pi(x)$ contains only the formula $x = x$, where x is a singleton in the main sort of T, then we often write $\text{dp-rk}(T)$ for $\text{dp-rk}(\pi)$.

- If M is a structure, we often write $\text{dp-rk}(M)$ for $\text{dp-rk}(\text{Th}(M))$.

- We name some important possible values for the dp-rank of a theory:
  1. If dp-rk$(T) = 1$ then we say that T is dp-minimal
  2. If dp-rk(T) is finite then we say that T is dp-finite
  3. If $\pi(x) = \{x = x\}$ has a randomness pattern of every finite depth, but no randomness pattern of depth $\aleph_0$, then we say that T is strongly dependent.
- Note that if dp-rk$(T) = \aleph_0$ then either $\pi(x) = \{x = x\}$ has a randomness pattern of depth $\aleph_0$ or T is strongly dependent. This odd situation actually occurs whenever dp-rk(T) is an infinite cardinal and there are several different ways of handling it: see (Adler, 2007) for one. We will avoid confusion by referring to the depth of an explicit randomness pattern (or inp-pattern, see below) whenever the dp-rank could be ambiguous.
- When $\pi(x)$ is a complete type, its dp-rank is always realized by a randomness pattern. So in a strongly dependent theory, every complete type has finite dp-rank.
- If T is a theory in a multi-sorted language (say $\mathcal{L}_{\text{Pas}}$), there are several valid interpretations of dp-rk(T). In this thesis, we will always use dp-rk(T) to mean dp-rk$(\{x = x\})$, where x is a variable in the dominant sort; in the case of $\mathcal{L}_{\text{Pas}}$, this is the valued field sort VF.

Strong dependence was one of Shelah's candidates for the property P, but it was shown in (Shelah, 2009, Claim 3.3(4)) that the class of superstable theories is strictly contained in the class of strongly stable (that is, stable and strongly dependent) theories.

Usvyatsov (2009) isolated the notion of dp-rank from Shelah's work. The desire was to find a notion of rank that could be used in the same way that weight is used in stable theories, and again the attempt was partially successful. In Adler (2007), it is shown that in stable theories, the weight of a complete type is equal to its dp-rank. However, in Kaplan *et al.* (2013), the authors define a theory in which every type in one variable has dp-rank 0 or 2, and so unlike the case with weight in stable theories, problems cannot in general be reduced to types of rank 1. In Section 2.4, we use properties of henselian valued fields to define theories for every $n < \omega$ such that each type in one variable has dp-rank 0 or n, showing that this problem is more than just an unfortunate occurrence in specially constructed theories.

Dp-rank can also be defined using indiscernible sequences, rather than randomness patterns:

Fact 1.4.2. *(Kaplan* et al.*, 2013, Proposition 2.6) The following are equivalent for a partial type $\pi(x)$ over a set A.*

- *There is a randomess pattern of depth κ for $\pi(x)$*
- *There is an element c that realizes $\pi(x)$ and such that* $\text{dp-rk}(c/A) = \kappa$
- *There exists a set of κ infinite mutually indiscernible sequences over A and a realization c of $\pi(x)$ such that none of the sequences are indiscernible over Ac.*

Moreover, the main results of Kaplan *et al.* (2013) show that dp-rank is subadditive. They split the result into the finite case (Theorem 4.8) and the infinite case (Theorem 4.11); note that the fact below is slightly stronger than what is in the statement of the theorems, but follows directly from the proofs without adjustment.

Fact 1.4.3. *Let M be any first order structure, x and y any tuples in M, and A any subset of M. Then*

$$\text{dp-rk}(x, y/A) \leq \text{dp-rk}(x/Ay) + \text{dp-rk}(y/A).$$

A significant portion of the fundamental results about dp-rank are immediate consequences of the two facts above. We include proofs of a number of them in Appendix B, as they are often stated without proof in the literature.

If we add NIP as an assumption, we can also define dp-rank using a related, but simpler notion in place of randomness patterns.

Definition 1.4.4. Let $\pi(x)$ be a partial type. An *inp (independent partition) pattern* in $\pi(x)$ of depth κ consists of tuples $\{b_{\alpha,i} : \alpha < \kappa, i < \omega\}$, formulas $\{\phi_\alpha(x, y_\alpha) : \alpha < \kappa, |y_\alpha| = |b_{\alpha,i}|\}$, and $k_\alpha < \omega$ such that

- $\{\phi_\alpha(x, b_{\alpha,i})\}_{i<\omega}$ is k_α-inconsistent for each $\alpha < \kappa$
- $\pi(x) \cup \{\phi_\alpha(x, b_{\alpha,\eta(\alpha)})\}_{\alpha<\kappa}$ is consistent for any $\eta : \kappa \to \omega$.

The *burden* of $\pi(x)$, written $\text{bdn}(\pi)$, is the supremum of the depths of all inp-patterns in $\pi(x)$.

In order to simplify the notation, we often write $(\phi_\alpha(x, y_\alpha), b_\alpha, k_\alpha)_{\alpha<\kappa}$ for the above inp-pattern. In this notation, b_α represents the sequence $(b_{\alpha,i})_{i<\omega}$.

As with dp-rank, we write $\mathrm{bdn}(T)$ for the burden of $\{x = x\}$. When working in an NIP theory, dp-rank is equal to burden (Adler, 2007), but it is usually easier to find an inp-pattern than an ict-pattern. By strengthening the assumptions on inp-patterns slightly, we can make it even easier to check whether a given array is an inp-pattern.

Definition 1.4.5. Let $\pi(x)$ be a partial type. An *indiscernible inp-pattern* in $\pi(x)$ of depth κ consists of tuples $\{b_{\alpha,i} : \alpha < \kappa, i < \omega\}$ and formulas $\{\phi_\alpha(x, y_\alpha) : \alpha < \kappa, |y_\alpha| = |b_{\alpha,i}|\}$ such that

- The sequences $(b_{\alpha,i})_{i<\omega}$ are mutually indiscernible
- $\{\phi_\alpha(x, b_{\alpha,i})\}_{i<\omega}$ is inconsistent for each $\alpha < \kappa$
- $\pi(x) \cup \{\phi_\alpha(x, b_{\alpha,0})\}_{\alpha<\kappa}$ is consistent.

As with inp-patterns, we often condense the notation for the above indiscernible inp-pattern to $(\phi_\alpha(x, y_\alpha), b_\alpha)_{\alpha<\kappa}$.

It follows immediately from the definition of indiscernibility that every indiscernible inp-pattern is an inp-pattern. By a common argument using Ramsey theory and compactness, any inp-pattern can be used to generate an indiscernible inp-pattern of the same depth; see Lemma 5.1.3 of Tent and Ziegler (2012) for a more detailed explanation. Thus, the burden of a type is equal to the supremum of the depths of all inp-patterns in $\pi(x)$. We could similarly define a notion of indiscernible randomness pattern, but it is less useful since the added assumption of indiscernibility does not simplify the array that needs to be checked for consistency.

1.5 Examples of Dp-finite Fields

Before discussing the structure of dp-finite fields, it seems helpful to give some examples of what they look like. The majority of the concrete examples are dp-minimal, and were classified in Johnson (2015). They include algebraically closed fields, real closed fields, and finite extensions of the p-adics. Moreover, these fields remain dp-minimal when expanded by an appropriate valuation.

For dp-finite fields that are not dp-minimal, one option is to consider expansions of the above fields. For any valuation v on $\mathbb{C}$, the theory of $(\mathbb{C}, v)$ as an $\mathcal{L}_{\mathrm{Pas}}$-structure has dp-rank at least 2, witnessed by the formulas

$$v(z) = y \qquad \text{and} \qquad \mathrm{ac}(z) = x$$

and any non-constant sequences of parameters $b_i \in v\mathbb{C}$ (for y) and $c_i \in \mathbb{C}v$ (for x). We will show in Section 2.2 that the dp-rank of this structure is actually equal to 2. Note that in the pure valued field language $\mathcal{L}_{\mathrm{div}}$, the structure $(\mathbb{C}, v)$ is dp-minimal, and so the increase in dp-rank comes from the angular component map, not the valuation.

Another way to expand the complex numbers to a structure of dp-rank 2 is to add the projections onto the real and imaginary parts. More precisely, consider $\mathbb{C}$ in the language $\{0, 1, +, \cdot, \pi_0, \pi_1\}$ where $\pi_0(z) = \mathrm{Re}(z)$ and $\pi_1(z) = \mathrm{Im}(z)$. This structure clearly has dp-rank at least 2, witnessed by the formulas

$$\pi_0(z) = x \qquad \text{and} \qquad \pi_1(z) = y$$

and any non-constant sequences of parameters $b_i, c_i \in \mathbb{R}$ for x and y. Moreover, this structure has dp-rank at most 2, since it is interdefinable with $\mathbb{R}^2$, with the structure induced from the $\mathcal{L}_{\mathrm{ring}}$-structure on $\mathbb{R}$.

This example leads to generalizations of dp-rank 4 and 8: the quaternions and octonions with projections onto the $\mathbb{R}$-vector spaces generated by the standard basis elements. These examples are of course not fields, as quaternion multiplication is not commutative and octonion multiplication is neither commutative nor associative. They do demonstrate the potential lack of tameness in NIP theories, even in finite dp-rank. For contrast, consider the recent result of Halevi and Palacín:

Fact 1.5.1. *(Halevi and Palacín, 2017, Proposition 7.2) Every infinite stable division ring of finite dp-rank is an algebraically closed field.*

When we add the assumption of stability, the possibility of skew-fields with finite dp-rank is removed, but the examples of the quaternions and octonions show that these are indeed possible in the general dp-finite case.

In this section, all of our examples of fields have been expansions of dp-minimal structures. But in order to find a classification of dp-finite fields, we should also consider examples of pure fields that are dp-finite but not dp-minimal. Combining results of Chernikov (2014) and Halevi and Hasson (2017b), we know that if Γ is an ordered abelian group with finite spines then the Hahn series field $\mathbb{C}[[t^{\Gamma}]]$ is dp-finite, but we do not have a good bound on the dp-rank.

In Section 2.2, we improve the bound on the dp-rank of henselian valued fields of residue characteristic 0 developed by Chernikov (2014), and in Section 2.4, we will give explicit constructions of fields of dp-rank d for every $d \in \mathbb{N}$.

In a recent pre-print, Halevi *et al.* (2018) suggest a conjectural classification of strongly dependent fields that is equivalent to the conjecture studied in Chapter 3. If the conjecture is true, then their classification would imply that all strongly dependent valued fields are dp-finite.

Chapter 2

Henselian Valued Fields

As noted in Section 1.5, all of the currently known examples of dp-finite fields fall into one of two categories: expansions of dp-minimal fields, and fields elementarily equivalent to Hahn series fields $k[[t^\Gamma]]$, where k is dp-minimal and Γ is dp-finite. This chapter focuses on the second source of examples, as they provide more insight on how to extend Johnson's algebraic classification of dp-minimal fields (Johnson, 2016).

In the first two sections, we improve the bound on the dp-rank of $k[[t^\Gamma]]$ given by Chernikov (2014). In the third, we explore the strong relationship between the dp-rank and the topology of definable sets in dp-finite valued fields, and give a criterion for henselianity in terms of dp-rank. Finally, we combine the results of these sections to produce examples of valued fields with dp-rank d for every $d \in \mathbb{N}$. These examples also have the property that every definable set in one variable either has dp-rank 0 or dp-rank d, a property that was previously not known to occur in any theories of algebraic structures.

2.1 Relative Quantifier Elimination

In Section 2.2, we generalize and improve a result of Chernikov (2014) relating the burden of certain valued fields to the burdens of their value groups and residue fields. In order to obtain the generalization, we need a stronger version of relative quantifier elimination than the one given in Section 1.3.

We begin with a classification of 1-types over any model in $\mathcal{L}_{\mathrm{Pas}}$, due to Delon (1981). Consider an elementary extension $\mathcal{K} \prec \mathcal{M}$ of $\mathcal{L}_{\mathrm{Pas}}$-structures, fix $x \in M \setminus K$, and define

$$I_K(x) = \{\gamma \in vK : \exists k \in K(\gamma = v(x - k))\}.$$

Then $\mathrm{tp}(x/K)$ belongs to one of three families:

1. $I_K(x) = \{v(x - k) : k \in K\}$ and does not have a maximum element. In this case, we say that $\mathrm{tp}(x/K)$ is *immediate*.

2. $I_K(x) = \{v(x - k) : k \in K\}$ and has a maximum element. In this case, we say that $\mathrm{tp}(x/K)$ is *residual*.

3. $I_K(x) \neq \{v(x - k) : k \in K\}$. In this case, we say that $\mathrm{tp}(x/K)$ is *valuational*.

In the first two cases, $\{v(x - k) : k \in K\}$ is a subset of vK. In the third, there is a single element $\gamma_0 \in \{v(x-k) : k \in K\} \setminus vK$, which is a least upper bound for $I_K(x)$.

The stronger form of quantifier elimination we need is a consequence of the following theorem:

Theorem 2.1.1. *Suppose $\mathcal{K}$ is a henselian valued field in $\mathcal{L}_{\mathrm{Pas}}$ such that $\mathrm{Th}(\mathcal{K})$ admits relative quantifier elimination. Let $\mathcal{M}$ be a monster model of $\mathrm{Th}(\mathcal{K})$ and let $x \in M \setminus K$ be an element of the valued field sort.*

1. *If $\mathrm{tp}(x/K)$ is immediate, let (a_ρ, γ_ρ) be a sequence indexed by a well-ordered set such that $a_\rho \in K$, $\gamma_\rho = v(x - a_\rho)$, and (γ_ρ) is cofinal in $I_K(x)$. Then $\mathrm{tp}(x/K)$ is completely determined by (a_ρ, γ_ρ) and by the set of formulas $\{v(x - a_\rho) = \gamma_\rho\}$.*

2. *If $\mathrm{tp}(x/K)$ is residual, then it is completely determined by some constants $a \in K$ and $\gamma \in vK$ such that $v(x - a) = \gamma$ and $\mathrm{ac}(x - a) \notin Kv$, by the formula $v(x - a) = \gamma$, and by the type $\mathrm{tp}(\mathrm{ac}(x - a)/Kv)$.*

3. *If $\mathrm{tp}(x/K)$ is valuational, then it is completely determined by some constant $a \in K$ such that $v(x - a) \notin vK$, by the type $\mathrm{tp}(v(x - a)/vK)$, and by the type $\mathrm{tp}(\mathrm{ac}(x - a)/Kv)$.*

In equicharacteristic 0, this theorem was originally proved by Delon (1981); a more detailed proof can be found in Bélair and Bousquet (2010). Before we can prove the result for any characteristic, we state the following technical lemma.

Lemma 2.1.2. *Suppose $\mathcal{K}$ is a henselian valued field in $\mathcal{L}_{\text{Pas}}$ such that* $\text{Th}(\mathcal{K})$ *admits relative quantifier elimination. Let $\mathcal{M}$ be a monster model of* $\text{Th}(\mathcal{K})$ *and suppose there are $y, y' \in M$ such that the following exist:*

- *A valued field isomorphism $\varphi : K(y) \to K(y')$ with $\varphi|_K = \text{id}_K$ and $\varphi(y) = y'$*
- *An $\mathcal{L}_{\text{VG}}$-automorphism $\alpha : vM \to vM$ with $\alpha|_{vK} = \text{id}_{vK}$ and $\alpha(v(y)) = v(y')$*
- *An $\mathcal{L}_{\text{RF}}$-automorphism $\beta : Mv \to Mv$ with $\beta|_{Kv} = \text{id}_{Kv}$ and $\beta(\text{ac}(y)) = \text{ac}(y')$*

Assume moreover that $vK(y)$ is generated by $vK \cup \{v(y)\}$, that $vK(y')$ is generated by $vK \cup \{v(y')\}$, and that either

- $\text{ac}(y)$ *and* $\text{ac}(y')$ *are both transcendental over Kv, or*
- *$v(y^n) \notin vK$ and $v((y')^n) \notin vK$ for any nonzero $n \in \mathbb{Z}$.*

Then there exists an $\mathcal{L}_{\text{Pas}}$-automorphism σ of $\mathcal{M}$ with $\sigma|_{K(y)} = \varphi$; in particular, this means $\text{tp}(y/K) = \text{tp}(y'/K)$.

Proof. First, note that since φ is a valued field automorphism, by choice of α we have $\alpha(v(x)) = v(\varphi(x))$ for all $x \in K(y)$. We claim that we also have $\beta(\text{ac}(x)) = \text{ac}(\varphi(x))$. To prove this, we will first show that for every polynomial $p(X) \in K[X]$, there exists a polynomial $\overline{p}(X) \in Kv[X]$ such that $\text{ac}(p(y)) = \overline{p}(\text{ac}(y))$. Note that $\overline{p}(X)$ will not in general be the residue polynomial of $p(X)$, but a separate polynomial as described below.

Suppose $v(y^n) \neq v(z)$ for any $z \in K$ and $n \in \mathbb{Z}$ and fix a polynomial $p(X)$. If two terms of $p(y)$, say $z_1 y^{n_1}$ and $z_2 y^{n_2}$ have the same valuation, then we must have $v(z_1/z_2) = v(y^{n_2 - n_1})$, which is impossible. Thus, $p(y)$ must have a term zy^n of least valuation, and so $\text{ac}(p(y)) = \text{ac}(z)\,\text{ac}(y)^n$ by Lemma A.2(1).

On the other hand, suppose $\text{ac}(y)$ is transcendental over K. In this case, we proceed by induction on the degree of p. If $\deg(p) = 0$ then $p(X) = z$ for some $z \in K$ and $\text{ac}(p(y)) = \text{ac}(z)$. If $\deg(p) = n > 0$, then we can write $p(X) = z + Xq(X)$ for some $z \in K$ and some polynomial $q(X)$ of degree less than n. By induction, $\text{ac}(q(y)) = \overline{q}(\text{ac}(y))$ for some polynomial $\overline{q}$. Since $\text{ac}(y)$ is transcendental over K, we must have

$\mathrm{ac}(z) \neq \mathrm{ac}(yq(y)) = \mathrm{ac}(y)\overline{q}(\mathrm{ac}(y))$, and so by Lemma A.2, $\mathrm{ac}(q(y))$ must be one of $\mathrm{ac}(z)$, $\mathrm{ac}(y)\overline{q}(\mathrm{ac}(y))$, or $\mathrm{ac}(z) + \mathrm{ac}(y)\overline{q}(\mathrm{ac}(y))$, depending on the relationship between $v(z)$ and $v(yq(y))$. In any case, $\mathrm{ac}(p(y))$ is a polynomial in $Kv[\mathrm{ac}(y)]$, completing the induction.

Note that the polynomial $\overline{p}(y)$ found above depended only on the valuations of the terms of $p(y)$. Because $\alpha(v(x)) = v(\varphi(x))$ for all $x \in K(y)$, an identical argument shows that $\mathrm{ac}(p(y')) = \overline{p}(\mathrm{ac}(y'))$. Then, given any polynomial $p(y) \in K[y]$, we have

$$\beta(\mathrm{ac}(p(y))) = \beta(\overline{p}(\mathrm{ac}(y))) = \overline{p}(\beta(\mathrm{ac}(y))) = \overline{p}(\mathrm{ac}(y')) = \mathrm{ac}(p(y')) = \mathrm{ac}(\varphi(p(y)))$$

by choice of β. Since every element $x \in K(y)$ can be written as a rational function in y, say $x = x_1/x_2$, we can easily extend this result to the entire field:

$$\beta(\mathrm{ac}(x)) = \beta\left(\mathrm{ac}\left(\frac{x_1}{x_2}\right)\right) = \frac{\beta(\mathrm{ac}(x_1))}{\beta(\mathrm{ac}(x_2))} = \frac{\mathrm{ac}(\varphi(x_1))}{\mathrm{ac}(\varphi(x_2))} = \mathrm{ac}\left(\varphi\left(\frac{x_1}{x_2}\right)\right) = \mathrm{ac}(\varphi(x)).$$

Finally, by the above observations, relative quantifier elimination, and the fact that α and β are elementary maps, it follows that $\varphi : K(y) \to M$ is a partial elementary map, and hence can be extended to an automorphism σ of $\mathcal{M}$. □

Proof. (of Theorem 2.1.1)

Case 1: Suppose $\mathrm{tp}(x/K)$ is immediate. Fix any well-ordered cofinal sequence $(\gamma_\rho)_{\rho<\kappa}$ of $I_K(x)$ and any sequence $(a_\rho)_{\rho<\kappa}$ such that $v(x - a_\rho) = \gamma_\rho$. We claim that the set of pairs (a_ρ, γ_ρ) and the set of formulas $\{v(x-a_\rho) = \gamma_\rho\}$ completely determines $\mathrm{tp}(x/K)$.

Note that by choice of a_ρ and γ_ρ, for $\rho_1 < \rho_2 < \kappa$, we have

$$v(a_{\rho_2} - a_{\rho_1}) = v((a_{\rho_2} - x) + (x - a_{\rho_1})) = \min\{v(a_{\rho_2} - x),\ v(x - a_{\rho_1})\} = v(x - a_{\rho_1}) = \gamma_{\rho_1}$$

since $\gamma_{\rho_1} < \gamma_{\rho_2}$. Thus, for $\rho_1 < \rho_2 < \rho_3 < \kappa$, we have

$$v(a_{\rho_2} - a_{\rho_1}) = \gamma_{\rho_1} < \gamma_{\rho_2} < v(a_{\rho_3} - a_{\rho_2})$$

and so $(a_\rho)_{\rho<\kappa}$ is a pseudo-convergent sequence. Moreover, since $v(x - a_\rho) = \gamma_\rho$ for all $\rho < \kappa$, x is a pseudo-limit of $(a_\rho)_{\rho<\kappa}$.

Suppose $x' \in M$ is another element of immediate type such that $v(x' - a_\rho) = \gamma_\rho$ for all $\rho < \kappa$. Then x' is also a pseudo-limit of $(a_\rho)_{\rho<\kappa}$, and so by Theorems 2 and 3 of Kaplansky (1942), $K(x)$ is an immediate extension of K and there exists a valued field isomorphism $\varphi : K(x) \to K(x')$ fixing K.

Since $K(x)$ is an immediate extension, for any $y \in K(x)$, there must exist $b \in K$ with $\mathrm{ac}(y) = \mathrm{ac}(b)$. Then

$$\mathrm{ac}(\varphi(y)) = \mathrm{ac}(\varphi(b)) = \mathrm{ac}(b) = \mathrm{ac}(y)$$

so by relative quantifier elimination, φ is a partial elementary map and can be extended to an automorphism σ of $\mathcal{M}$. Because $\sigma(x) = \varphi(x) = x'$, this automorphism demonstrates that $\mathrm{tp}(x/K) = \mathrm{tp}(x'/K)$ as desired.

Case 2: Suppose $\mathrm{tp}(x/K)$ is residual; we must first show that there exists $a \in K$ and $\gamma \in vK$ as described in the theorem. Let $\gamma \in vK$ be the largest element of $I_K(x)$, and fix $a \in K$ such that $v(x - a) = \gamma$. If $\mathrm{ac}(x - a) \in Kv$ then there must exist some $b \in K$ with $\mathrm{ac}(x - a) = \mathrm{ac}(b)$ and $v(b) = \gamma$ by Lemma A.1. But then

$$v(x - (a + b)) = v((x - a) - b) > v(x - a) = \gamma,$$

by Lemma A.2(2), contradicting the maximality of γ. Thus, $\mathrm{ac}(x - a) \notin Kv$.

Now, suppose $x' \in M$ is another element of residual type such that $\mathrm{ac}(x' - a) = \gamma$, $\mathrm{ac}(x' - a) \notin Kv$, and $\mathrm{tp}(\mathrm{ac}(x - a)/Kv) = \mathrm{tp}(\mathrm{ac}(x' - a)/Kv)$. We wish to show that $\mathrm{tp}(x/K) = \mathrm{tp}(x'/K)$, which we will do by finding an $\mathcal{L}_{\mathrm{Pas}}$-automorphism of $\mathcal{M}$, fixing $\mathcal{K}$, which maps x to x'. Let $y = x - a$ and $y' = x' - a$; finding such a map that sends y to y' is equivalent.

Since $\mathcal{K} \prec \mathcal{M}$ is an elementary extension, K is algebraically closed in M, and so y and y' must both be transcendental over K. Then there is a field isomorphism $\varphi : K(y) \to K(y')$ that fixes K and sends y to y'. Moreover, φ is a valued field isomorphism since $v(y) = \gamma = v(y')$. Setting $\alpha : vM \to vM$ to be the identity automorphism, we even have $\alpha(v(y)) = v(y')$.

Since $\mathrm{tp}(\mathrm{ac}(x - a)/K) = \mathrm{tp}(\mathrm{ac}(x' - a)/K)$, there is an $\mathcal{L}_{\mathrm{RF}}$-automorphism $\beta : Mv \to Mv$ with $\beta|_{Kv} = \mathrm{id}_{Kv}$ and $\beta(\mathrm{ac}(y)) = \mathrm{ac}(y')$. Finally, $\mathrm{ac}(y)$ and $\mathrm{ac}(y')$ must be transcendental over Kv since $Kv \prec Mv$. Then by Lemma 2.1.2, $\mathrm{tp}(y/K) = \mathrm{tp}(y'/K)$, which means $\mathrm{tp}(x/K) = \mathrm{tp}(x'/K)$.

Case 3: Suppose $\mathrm{tp}(x/K)$ is valuational and fix any $a \in K$ with $v(x - a) \notin vK$.

Suppose $x' \in M$ is another element of valuational type such that $v(x' - a) \notin vK$, $\mathrm{tp}(v(x - a)/vK) = \mathrm{tp}(v(x' - a)/vK)$, and $\mathrm{tp}(\mathrm{ac}(x - a)/Kv) = \mathrm{tp}(\mathrm{ac}(x' - a)/Kv)$. As in Case 2, it suffices to show that $\mathrm{tp}(y/K) = \mathrm{tp}(y'/K)$ for $y = x - a$ and $y' = x' - a$.

Again following Case 2, $\mathcal{K} \prec \mathcal{M}$, which means y and y' are both transcendental over K and there exists a field isomorphism $\varphi : K(y) \to K(y')$. Moreover, we have $vK(y) = vK \oplus \mathbb{Z}v(y)$ and $vK(y') = vK \oplus \mathbb{Z}v(y')$ since $vK \prec vM$ and $v(y), v(y') \notin vK$; in particular, $vK(y)$ is generated by $vK \cup \{y\}$ and $v(y^n) = nv(y) \notin vK$ for any $n \in \mathbb{Z}$, and similarly for y'. It then follows from Corollary 2.2.3 of Engler and Prestel (2005) that φ is a valued field isomorphism.

Finally, by choice of x', there exists an $\mathcal{L}_{\mathrm{VG}}$-automorphism α of vM that fixes vK and such that $\alpha(v(y)) = v(y')$. Similarly, there exists an $\mathcal{L}_{\mathrm{RF}}$-automorphism β of Mv that fixes Kv and such that $\beta(\mathrm{ac}(y)) = \mathrm{ac}(y')$. Thus, by Lemma 2.1.2, $\mathrm{tp}(y/K) = \mathrm{tp}(y'/K)$, so $\mathrm{tp}(x/K) = \mathrm{tp}(x'/K)$. □

As a consequence of the above theorem, we can improve the equivalence of formulas provided by relative quantifier elimination.

Proposition 2.1.3. *Suppose $\mathcal{K}$ is a henselian valued field in $\mathcal{L}_{\mathrm{Pas}}$ such that* $\mathrm{Th}(\mathcal{K})$ *admits relative quantifier elimination. Let $\phi(x)$ be a formula in one valued field sort variable with parameters in K. Then $\phi(x)$ is equivalent to a finite disjunction of formulas of the form*

$$\chi\left(v(x-c^1), \ldots, v(x-c^n), b^{\mathrm{VG}}\right) \wedge \rho\left(\mathrm{ac}(x-c^1), \ldots, \mathrm{ac}(x-c^n), b^{\mathrm{RF}}\right)$$

where $\chi(x,\overline{y})$ is an $\mathcal{L}_{\mathrm{VG}}$-formula, $\rho(x,\overline{y})$ is an $\mathcal{L}_{\mathrm{RF}}$-formula, $c^1, \ldots, c^n$ are singletons in the VF*-sort, b^{VF} is a tuple in the* VG*-sort, and b^{RF} is a tuple in the* RF*-sort.*

Proof. First, note that all of the formulas occurring in the conclusion of Theorem 2.1.1 have the desired form:

- (If x is immediate) A fomula of the form $v(x - a_\rho) = \gamma_\rho$.
- (If x is residual) An element of $\mathrm{tp}(\mathrm{ac}(x-a)/Kv)$, and hence of the form $\rho\left(\mathrm{ac}(x-c), c^{\mathrm{RF}}\right)$.
- (If x is valuational) An element of $\mathrm{tp}(v(x-a), vK)$ or $\mathrm{tp}(\mathrm{ac}(x-a), Kv)$, both of which are clearly in the desired form.

We will refer to these formulas as *good formulas* for the duration of this proof. Note that by a simple rearrangement, the conjunction of a finite set of good formulas is itself a good formula.

Let $\{p_\alpha : \alpha < \kappa\}$ be the set of complete K-types containing $\phi(x)$. By the theorem, for each $\alpha < \kappa$ there is a partial type $\pi_\alpha(x)$ consisting only of good formulas such that $\pi_\alpha \vdash p_\alpha$; in particular, $\pi_\alpha \vdash \phi$. By compactness, this implication only requires a finite subset of $\pi_\alpha(x)$; let $\psi_\alpha(x)$ be the conjuction of this finite set, and note that $\psi_\alpha(x)$ is a good formula by the observation above.

Ranging over α, we have $\phi \vdash \bigvee_{\alpha<\kappa} \psi_\alpha(x)$. Of course, this statement is not first-order, due to the infinite disjunction. Let $\Sigma(x) = \{\neg\psi_\alpha : \alpha < \kappa\}$. Then by contrapositive, $\Sigma \vdash \neg\phi$, so again by compactness, there must be a finite subset $\Sigma_0 = \{\neg\psi_{\alpha_1}, \ldots, \neg\psi_{\alpha_n}\}$ such that $\Sigma_0 \vdash \neg\phi$. Then

$$\mathcal{K} \models \left(\bigwedge_{i=1}^{n} \neg\psi_{\alpha_i}(x)\right) \to \neg\phi(x)$$

so by contrapositive again

$$\mathcal{K} \models \phi(x) \to \bigvee_{i=1}^{n} \psi_{\alpha_i}(x).$$

But we already know that $\psi_\alpha(x) \to \phi(x)$ for all $\alpha < \kappa$, and so

$$\mathcal{K} \models \phi(x) \leftrightarrow \bigvee_{i=1}^{n} \psi_{\alpha_i}(x).$$

Since each $\psi_\alpha(x)$ is a good formula, this shows that $\phi(x)$ is equivalent to a finite disjunction of good formulas, as desired. $\square$

2.2 Calculating Dp-rank

Throughout this section, we assume that $\mathcal{K} = (K, vK, Kv)$ is a sufficiently saturated model of some theory T of henselian valued fields in $\mathcal{L}_{\text{Pas}}$ that admits relative quantifier elimination. For example T might be strongly dependent or a theory of fields of characteristic $(0,0)$. Chernikov (2014) gives a bound for $\text{bdn}(T)$ in terms of $\text{bdn}(T_{\text{VG}})$ and $\text{bdn}(T_{\text{RF}})$ in the characteristic $(0,0)$ case, but the proof in that paper uses a Ramsey theory argument, and so the bound is very imprecise. The goal of this section is to improve Chernikov's bound and extend the result to also apply to theories of any characteristic. First, we repeat two results from that paper that we will use throughout the section.

Fact 2.2.1. *(Chernikov, 2014, Lemma 7.1)*

1. *If $(\phi_{\alpha,0}(x, y_{\alpha,0}) \vee \phi_{\alpha_1}(x, y_{\alpha,1}),\ a_\alpha,\ k_\alpha)_{\alpha<\kappa}$ is an (indiscernible) inp-pattern, then*

$$\left(\phi_{\alpha,f(\alpha)}(x, y_{\alpha,f(\alpha)}), a_\alpha, k_\alpha\right)_{\alpha<\kappa}$$

is also an (indiscernible) inp-pattern for some $f : \kappa \to \{0,1\}$.

2. *Let $(\phi_\alpha(x, y_\alpha), a_\alpha, k_\alpha)_{\alpha<\kappa}$ be an (indiscernible) inp-pattern and assume that*

$$\phi_\alpha(x, a_{\alpha,0}) \leftrightarrow \psi_\alpha(x, b_{\alpha,0})$$

for all $\alpha < \kappa$ and some (mutually indiscernible) $(b_\alpha)_{\alpha<\kappa}$. Then there is an (indiscernible) inp-pattern of the form $(\psi_\alpha(x, z_\alpha), b_\alpha, k_\alpha)_{\alpha<\kappa}$.

Fact 2.2.2. *(Chernikov, 2014, Lemma 7.9) Let $(c_i)_{i\in I}$ be an indiscernible sequence of singletons. Consider the function $(i,j) \mapsto v(c_j - c_i)$ with $i < j$. It satisfies one of the following:*

1. *It is strictly increasing depending only on i (so $(c_i)_{i\in I}$ is pseudo-convegent)*
2. *It is strictly decreasing depending only on j (so $(c_i)_{i\in I}$ taken in the reverse direction is pseudo-convergent)*
3. *It is constant (in this case $(c_i)_{i\in I}$ is referred to as a "fan")*

Because there are definable surjections $v : K^\times \twoheadrightarrow vK$ and $\text{ac} : K \twoheadrightarrow Kv$, we only need to consider inp-patterns where the variable is in the VF-sort. Combining

Proposition 2.1.3 and Fact 2.2.1, we can already focus only on inp-patterns with very tame formulas. In Lemma 2.2.4 below, we follow another argument of Chernikov's to find an equivalent inp-pattern such that the VF-sort component of each parameter sequence is a singleton.

Throughout this section, we will write $(\phi_\alpha(x, y_\alpha, z_\alpha), (b_\alpha, c_\alpha))_{\alpha<\kappa}$ for indiscernible inp-patterns, where for each $\alpha < \kappa$

- x is a singleton in the VF-sort,
- y_α is a tuple of VG-sort and RF-sort variables (we will indicate these subtuples as y_α^{VG} and y_α^{RF} whenever it is useful),
- $b_\alpha = (b_{\alpha,i})_{i<\omega}$ is a sequence of VG-sort and RF-sort parameters corresponding to y_α (we will use $b_{\alpha,i}^{\mathrm{VG}}$ and $b_{\alpha,i}^{\mathrm{RF}}$ to indicate the appropriate subtuples),
- z_α is a tuple of VF-sort variables, and
- $c_\alpha = (c_{\alpha,i})_{i<\omega}$ is a sequence of VF-sort parameters corresonding to z_α.

Lemma 2.2.3. *Assume T and $\mathcal{K}$ are as above, and let $(\phi_\alpha(x, y_\alpha, z_\alpha), (a_\alpha, c_\alpha))_{\alpha<\kappa}$ be an indiscernible inp-pattern with x a singleton in the valued field sort. Fix a realization a of $\{\phi(x, b_{\alpha,0}, c_{\alpha,0}) : \alpha < \kappa\}$. Suppose that for each $\alpha < \kappa$, there exists a formula $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha)$ and a sequence of* VF*-sort parameters $c'_\alpha = (c'_{\alpha,i})_{i<\omega}$ such that:*

- *The array $\{(b_\alpha, c_\alpha, c'_\alpha) : \alpha < \kappa\}$ is mutually indiscernible,*
- *$\mathcal{K} \models \psi(a, b_{\alpha,0}, c_{\alpha,0}, c'_{\alpha,0})$, and*
- *$\Psi_\alpha = \{\psi(x, b_{\alpha,i}, c_{\alpha,i}, c'_{\alpha,i}) : i < \omega\}$ is inconsistent.*

Assume moreover that for each $\alpha < \kappa$ there exist finitely many terms $\{t_\alpha^i : 1 \leq i < n_\alpha\}$ occuring in the VG*-sort and* RF*-sort components of $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha)$ such that*

- *x does not appear in any t_α^i and*
- *every occurrence in ψ_α of a variable from the tuple z_α occurs in some t_α^i.*

Then for each $\alpha < \kappa$ there exists a tuple of VG*-sort and* RF*-sort variables y'_α, a parameter sequence b'_α, and a formula $\phi'_\alpha(x, y'_\alpha, z'_\alpha)$ such that $((\phi'_\alpha(x, y'_\alpha, z'_\alpha), (b'_\alpha, c'_\alpha))_{\alpha<\kappa}$ is an indiscernible inp-pattern of the same depth κ.*

Proof. Fix $\alpha < \kappa$ and let $w_\alpha^1, \dots, w_\alpha^n$ be new variable symbols. For each $j \leq n$ and $i < \omega$, let $d_{\alpha,i}^j = t_\alpha^j(b_{\alpha,i}, c_{\alpha,i}, c'_{\alpha,i})$. Let $y'_\alpha = y_\alpha w_\alpha^1 \dots w_\alpha^n$, and let $b'_{\alpha,i} = b_{\alpha,i} d_{\alpha,i}^1 \dots d_{\alpha,i}^n$ such that the variables w_α^j correspond to the parameters $d_{\alpha,i}^j$.

Let $\phi'_\alpha(x, y'_\alpha, z'_\alpha)$ be the same formula as $\psi(x, y_\alpha, z_\alpha, z'_\alpha)$, but with the new variables $w_\alpha^1, \dots, w_\alpha^n$ replacing each occurrence of the terms $t_\alpha^1, \dots, t_\alpha^n$. By assumption, z_α no longer occurs in the formula after making this substitution, and so we can remove it from the list of variables.

Apply the above process to each $\alpha < \kappa$, and note that

$$\mathcal{K} \models \phi'_\alpha(x, b'_{\alpha,i}, c'_{\alpha,i}) \leftrightarrow \psi_\alpha(x, b_{\alpha,i}, c_{\alpha,i}, c'_{\alpha,i})$$

for each $\alpha < \kappa$ and $i < \omega$. Since each coordinate of $b'_{\alpha,i}$ was built from a term including only parameters from $(b_{\alpha,i}, c_{\alpha,i}, c'_{\alpha,i})$, the array $\{(b'_\alpha, c'_\alpha) : \alpha < \kappa\}$ is mutually indiscernible. Then applying Fact 2.2.1(2), $((\phi'_\alpha(x, y'_\alpha, z'_\alpha), (b'_\alpha, c'_\alpha))_{\alpha<\kappa}$ is an indiscernible inp-pattern of depth κ, as desired. $\square$

Proposition 2.2.4. *Assume T and $\mathcal{K}$ are as above, and let $(\phi_\alpha(x, y_\alpha, z_\alpha), (b_\alpha, c_\alpha))_{\alpha<\kappa}$ be an indiscernible inp-pattern with x a singleton in the valued field sort. Then we can construct a new inp-pattern $(\phi'_\alpha(x, y'_\alpha, z'_\alpha), (b'_\alpha, c'_\alpha))_{\alpha<\kappa}$ of the same depth, such that each formula ϕ'_α has the form*

$$\chi_\alpha\big(v(x - z'_\alpha), (y'_\alpha)^{\mathrm{VG}}\big) \wedge \rho_\alpha\big(\operatorname{ac}(x - z'_\alpha), (y'_\alpha)^{\mathrm{RF}}\big)$$

and such that:

- *χ_α and ρ_α are formulas in $\mathcal{L}_{\mathrm{VG}}$ and $\mathcal{L}_{\mathrm{RF}}$, respectively, and*
- *z'_α is a singleton in the valued field sort.*

Proof. We wish to apply Lemma 2.2.3. Fix some $\alpha < \kappa$. By Proposition 2.1.3 and Fact 2.2.1, we may assume that $\phi_\alpha(x, y_\alpha, z_\alpha)$ has the form

$$\chi_\alpha\big(v(x - z_\alpha^1), \dots, v(x - z_\alpha^n), y_\alpha^{\mathrm{VG}}\big) \wedge \rho_\alpha\big(\operatorname{ac}(x - z_\alpha^1), \dots, \operatorname{ac}(x - z_\alpha^n), y_\alpha^{\mathrm{RF}}\big)$$

where $z_\alpha = (z_\alpha^1, \dots, z_\alpha^n)$ for some $n \in \mathbb{N}$.

If $n = 1$ then we may take $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha) = \phi_\alpha(x, y_\alpha, z'_\alpha)$ (z_α will be an unused variable), $c'_\alpha = c_\alpha$, and the set of terms t_α^j to be the empty set. Otherwise, let $\theta = \phi_\alpha$ and fix a realization a of $\{\phi_\alpha(x, b_{\alpha,0}, c_{\alpha,0}) : \alpha < \kappa\}$.

Let I be the set of indices i such that either $v(x - z^i_\alpha)$ or $\mathrm{ac}(x - z^i_\alpha)$ appears in θ. We proceed recursively, at each step replacing θ with a new formula θ' such that $|I_{\theta'}| = |I_\theta| - 1$. Set $r = \min(I)$ and $s = \max(I)$; we break into cases based on the relationships between a, $c^r_{\alpha,0}$, and $c^s_{\alpha,0}$, following the proof of Lemma 7.12 of Chernikov (2014).

Case 1: If $v(a - c^r_{\alpha,0}) < v(c^s_{\alpha,0} - c^r_{\alpha,0})$, then $v(a - c^r_{\alpha,0}) = v(a - c^s_{\alpha,0})$ and $\mathrm{ac}(a - c^r_{\alpha,0}) = \mathrm{ac}(a - c^s_{\alpha,0})$ by Lemma A.3. Take $\theta'(x, y_\alpha, z_\alpha)$ to be the conjunction of

- θ with each occurrence of $v(x - z^s_\alpha)$ replaced by $v(x - z^r_\alpha)$ and each occurence of $\mathrm{ac}(x - z^s_\alpha)$ replaced by $\mathrm{ac}(x - z^r_\alpha)$
- The formula $v(x - z^r_\alpha) < v(z^s_\alpha - z^r_\alpha)$.

Case 2: If $v(a - c^r_{\alpha,0}) > v(c^s_{\alpha,0} - c^r_{\alpha,0})$ then $v(a - c^s_{\alpha,0}) = v(c^s_{\alpha,0} - c^r_{\alpha,0})$ and $\mathrm{ac}(a - c^s_{\alpha,0}) = \mathrm{ac}(c^s_{\alpha,0} - c^r_{\alpha,0})$. Take $\theta'(x, y_\alpha, z_\alpha)$ to be the conjunction of

- θ with each occurrence of $v(x - z^s_\alpha)$ replaced by $v(z^s_\alpha - z^r_\alpha)$ and each occurence of $\mathrm{ac}(x - z^s_\alpha)$ replaced by $\mathrm{ac}(z^s_\alpha - z^r_\alpha)$
- The formula $v(x - z^r_\alpha) > v(z^s_\alpha - z^r_\alpha)$.

Case 3: If $v(a - c^s_{\alpha,0}) < v(c^s_{\alpha,0} - c^r_{\alpha,0})$, proceed symmetrically to case 1.

Case 4: If $v(a - c^s_{\alpha_0}) > v(c^s_{\alpha,0} - c^r_{\alpha,0})$, proceed symmetrically to case 2.

Case 5: If $v(a - c^1_{\alpha,0}) = v(a - c^n_{\alpha,0}) = v(c^n_{\alpha,0} - c^1_{\alpha,0})$ then by Lemma A.3 again, we must have $\mathrm{ac}(a - c^n_{\alpha,0}) = \mathrm{ac}(a - c^1_{\alpha,0}) - \mathrm{ac}(c^n_{\alpha,0} - c^1_{\alpha,0})$. Take $\theta'(x, y_\alpha, z_\alpha)$ to be the conjunction of

- θ with each occurrence of $v(x - z^s_\alpha)$ replaced by $v(z^s_\alpha - z^r_\alpha)$ and each occurence of $\mathrm{ac}(x - z^s_\alpha)$ replaced by $\mathrm{ac}(z^s_\alpha - z^r_\alpha)$
- The formula $v(x - z^r_\alpha) = v(z^s_\alpha - z^r_\alpha) \wedge \mathrm{ac}(x - z^r_\alpha) \neq \mathrm{ac}(z^s_\alpha - z^r_\alpha)$.

Note that in each case, we have $\mathcal{K} \models \theta'(a, b_{\alpha,0}, c_{\alpha,0})$ by construction, and that $|I_{\theta'}| = |I_\theta| - 1$. If $|I_{\theta'}| = 1$, let r be the single index in $I_{\theta'}$, set $c'_\alpha = (c^r_{\alpha,i})_{i<\omega}$, and set $\psi(x, y_\alpha, z_\alpha, z'_\alpha)$ to be θ' with each occurrence of $v(x - z^r_\alpha)$ replaced by $v(x - z'_\alpha)$ and each occurrence of $\mathrm{ac}(x - z^r_\alpha)$ replaced by $\mathrm{ac}(x - z'_\alpha)$. Otherwise, repeat the process recursively with θ' in place of θ.

Since c'_α is a subtuple of c_α, the array $\{(b_\alpha, c_\alpha, c'_\alpha) : \alpha < \kappa\}$ is mutually indiscernible. By choice of θ' and ψ, any realization of $\Psi_\alpha = \{\psi(x, b_{\alpha,i}, c_{\alpha,i}, c'_{\alpha,i}) : i < \omega\}$ would also be a realization of $\{\phi(x, b_{\alpha,i}, c_{\alpha,i} : i < \omega\}$, and so Ψ_α is inconsistent. Finally, for the set of terms take $\{v(z^i_\alpha - z^j_\alpha) : 1 \le i, j \le n\} \cup \{\mathrm{ac}(z^i_\alpha - z^j_\alpha) : 1 \le i, j \le n\}$.

We can then apply Lemma 2.2.3 to obtain $((\phi'_\alpha(x, y'_\alpha, z'_\alpha), (b'_\alpha, c'_\alpha))_{\alpha<\kappa}$, a new indiscernible inp-pattern of depth κ. By choice of ψ_α and the fact that z'_α is a singleton for all $\alpha < \kappa$, the formulas in the new inp-pattern have the desired form. □

We have just shown that we can replace any inp-pattern with one in which there is only one VF-sort parameter in each row. In the next two propositions, we show that we can find a new inp-pattern in which the VF-sort parameter is constant within each row, and then one in which there is no VF-sort paramter in any row.

Proposition 2.2.5. *Assume T and $\mathcal{K}$ are as above, and let $(\phi_\alpha(x, y_\alpha, z_\alpha), (b_\alpha, c_\alpha))_{\alpha<\kappa}$ be an indiscernible inp-pattern with x a singleton in the valued field sort. Then we can construct a new indiscernible inp-pattern $(\phi'_\alpha(x, y'_\alpha, z'_\alpha), (b'_\alpha, c'_\alpha))_{\alpha<\kappa}$ of the same depth, such that for each $\alpha < \kappa$,*

- *the formula ϕ'_α has the form described in Lemma 2.2.4, and*
- *the* VF*-sort sequence $c'_\alpha = (c'_{\alpha,i})_{i<\omega}$ is a constant sequence.*

Proof. First, by applying Proposition 2.2.4, we may assume that each ϕ_α has the form described in that proposition. We again wish to apply Lemma 2.2.3. Fix $\alpha < \kappa$, and write $b_\alpha = (b_i)_{i<\omega}$ and $c_\alpha = (c_i)_{i<\omega}$. From the application of Proposition 2.2.4, $(c_i)_{i<\omega}$ is a sequence of singletons.

Let c_∞ be an element such that $(c_0, c_1, c_2, \ldots, c_\infty)$ is indiscernible (if $(c_i)_{i<\omega}$ is a reverse pseudo-convergent sequence, instead take c_∞ so that $(c_\infty, c_0, c_1, \ldots)$ is indiscernible) and such that everything remains mutually indiscernible; such an element exists by compactness. Take $c'_\alpha = (c'_i)_{i<\omega}$ to be the constant sequence $c'_i = c_\infty$ for all $i < \omega$.

Let z'_α be a new variable symbol corresponding to c'_α. To find the formula $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha)$ needed for Lemma 2.2.3, we split into cases based on the relationship between $v(a - c_\infty)$ and $v(c_0 - c_\infty)$. As in the lemma, we fix a realization a of $\{\phi_\alpha(x, b_{\alpha,0}, c_{\alpha,0} : \alpha < \kappa\}$ and write $\Psi_\alpha(x) = \{\psi_\alpha(x, b_i, c_i, c'_i) : i < \omega\}$. In each case below, we will clearly have $\mathcal{K} \models \psi_\alpha(a, b_0, c_0, c'_0)$ by choice of ψ_α.

Case 1: If $v(a-c_\infty) < v(c_0-c_\infty)$ then $v(a-c_0) = v(a-c_\infty)$ and $\mathrm{ac}(a-c_0) = \mathrm{ac}(a-b_\infty)$ by Lemma A.3. Consider the following formula $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha)$:

$$v(x - z'_\alpha) < v(z_\alpha - z'_\alpha) \wedge \chi\big(v(x - z'_\alpha), y_\alpha^{\mathrm{VG}}\big) \wedge \rho\big(\,\mathrm{ac}(x - z'_\alpha), y_\alpha^{\mathrm{RF}}\big).$$

Note that any realization of $\Psi_\alpha(x)$ would also be a realization of $\{\phi_\alpha(x, b_i, c_i) : i < \omega\}$, so $\Psi_\alpha(x)$ is inconsistent.

Case 2: If $v(a - c_\infty) > v(c_0 - c_\infty)$ then by Lemma A.3, $v(a - c_0) = v(c_\infty - c_0)$ and $\mathrm{ac}(a - c_0) = \mathrm{ac}(c_\infty - c_0)$, so $K \models \phi(c_\infty, b_0, c_0)$. Then by indiscernibility, c_∞ realizes $\{\phi_\alpha(x, b_i, c_i) : i < \omega\}$, contradicting the inconsistency of that row of the inp-pattern. Thus, case 2 cannot occur.

Case 3: Assume $v(a - c_\infty) = v(c_0 - c_\infty)$. In this case, we need to split into subcases based on the form of the sequence $(c_i)_{i<\omega}$ and the relationship between $\mathrm{ac}(a - c_\infty)$ and $\mathrm{ac}(c_0 - c_\infty)$.

Case 3a: If $\mathrm{ac}(a - c_\infty) \neq \mathrm{ac}(c_0 - c_\infty)$ then $v(a - c_0) = v(a - c_\infty) = v(c_0 - c_\infty)$, so $\mathrm{ac}(a - c_0) = \mathrm{ac}(a - c_\infty) - \mathrm{ac}(c_0 - c_\infty)$. Consider the formula $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha)$:

$$v(x - z'_\alpha) = v(z_\alpha - z'_\alpha) \wedge \mathrm{ac}(x - z'_\alpha) \neq \mathrm{ac}(z_\alpha - z'_\alpha)$$

$$\wedge \chi\big(v(x - z'_\alpha), y_\alpha^{\mathrm{VG}}\big) \wedge \rho\big(\,\mathrm{ac}(x - z'_\alpha) - \mathrm{ac}(z_\alpha - z'_\alpha), y_\alpha^{\mathrm{RF}}\big).$$

As in Case 1, note that any realization of $\Psi_\alpha(x)$ would also be a realization of $\{\phi_\alpha(x, b_i, c_i) : i < \omega\}$, so $\Psi_\alpha(x)$ is inconsistent.

Case 3b: If $(c_i)_{i<\omega}$ or its reversal is pseudo-convergent, let $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha)$ be the formula $v(x - z'_\alpha) = v(z_\alpha - z'_\alpha)$. It is easy to check that c_∞ is a pseudo-limit of the pseudo-convergent sequence, and so $v(c_i - c_\infty) \neq v(c_j - c_\infty)$ whenever $i \neq j$. Thus, for any $d \in K$, it is impossible for $v(d - c_\infty)$ to be equal to both $v(c_i - c_\infty)$ and $v(c_j - c_\infty)$; in other words, $\Psi(x)$ is inconsistent.

Case 3c: Finally, assume $(c_i)_{i<\omega}$ is a fan and $\mathrm{ac}(a - c_\infty) = \mathrm{ac}(c_0 - c_\infty)$. Let $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha)$ be the following formula:

$$v(x - z'_\alpha) = v(z_\alpha - z'_\alpha) \wedge \mathrm{ac}(x - z'_\alpha) = \mathrm{ac}(z_\alpha - z'_\alpha).$$

Since c_∞ will be an element of the fan, $\mathrm{ac}(c_i - c_\infty) \neq \mathrm{ac}(c_j - c_\infty)$ for any $i \neq j$. Thus, for any $d \in K$, it is impossible for $\mathrm{ac}(d - c_\infty)$ to be equal to both $\mathrm{ac}(c_i - c_\infty)$ and $\mathrm{ac}(c_j - c_\infty)$, which means $\Psi_\alpha(x)$ is inconsistent.

As noted above, Case 2 cannot occur. In each other case, we have shown that $\Psi_\alpha(x)$ is inconsistent and chosen $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha)$ so that $K \models \psi_\alpha(a, b_0, c_0, c'_0)$.

Note that $(b_i, c_i)_{i<\omega}$ is indiscernible over c_∞: for any formula θ,

$$\theta(c_{i_1}, \ldots, c_{i_n}, c_\infty) \leftrightarrow \theta(c_{j_1}, \ldots, c_{j_n}, c_\infty),$$

since ∞ is greater than each i_k and j_k (in the reverse pseudo-convergent case, ∞ is less than each i_k and j_k). Thus, the array $\{(b_\alpha, c_\alpha, c'_\alpha) : \alpha < \kappa\}$ is mutually indiscernible.

Finally, the terms $t^1_\alpha = v(z_\alpha - z'_\alpha)$ and $t^2_\alpha = \mathrm{ac}(z_\alpha - z'_\alpha)$ satisfy the remaining conditions of Lemma 2.2.3, and we obtain a new inp-pattern $((\phi'_\alpha(x, y'_\alpha, z'_\alpha), (b'_\alpha, c'_\alpha))_{\alpha<\kappa}$. Note that the VF-sort parameter sequence of each row of the new inp-pattern is c'_α, a constant sequence. Moreover, each ψ_α has the form described in Proposition 2.2.4 by construction, and ϕ'_α inherits this form since it is obtained from ψ_α through a substitution of terms. Thus, the new inp-pattern has the desired form. □

Proposition 2.2.6. *Assume T and $\mathcal{K}$ are as above, and let $(\phi_\alpha(x, y_\alpha, z_\alpha), (b_\alpha, c_\alpha))_{\alpha<\kappa}$ be an indiscernible inp-pattern with x a singleton in the valued field sort. Then we can construct a new indiscernible inp-pattern $(\phi'_\alpha(x, y'_\alpha), (b'_\alpha))_{\alpha<\kappa}$ of the same depth, such that for each $\alpha < \kappa$,*

- *the formula ϕ'_α has the form described in Lemma 2.2.4, and*
- *y'_α has no* VF*-sort component.*

Proof. From the previous propositions, we may assume each $\phi_\alpha(x, y_\alpha)$ has the form

$$\chi_\alpha\big(v(x - z_\alpha), y^{\mathrm{VG}}_\alpha\big) \wedge \rho_\alpha\big(\,\mathrm{ac}(x - z_\alpha), y^{\mathrm{RF}}_\alpha\big)$$

and that for each $\alpha < \kappa$, c_α is a constant sequence. Throughout this proof, we will identify a constant sequence with its value. We will again apply Lemma 2.2.3. Let a be some realization of $\{\phi_\alpha(x, b_{\alpha,0}, c_\alpha) : \alpha < \kappa\}$.

For any $\alpha, \beta < \kappa$ such that $v(a - c_\alpha) < v(a - c_\beta)$, we have $v(a - c_\alpha) = v(c_\beta - c_\alpha)$ and $\mathrm{ac}(a - c_\alpha) = \mathrm{ac}(c_\beta - c_\alpha)$ by Lemma A.3. Then, since $K \models \phi_\alpha(a, b_{\alpha,0}, c_\alpha)$, we have $K \models \phi_\alpha(c_\beta, b_{\alpha,0}, c_\alpha)$. But then by mutual indiscernibility, $K \models \phi_\alpha(c_\beta, b_{\alpha,i}, c_\alpha)$ for all $i < \omega$, contradicting the inconsistency of the row α.

Thus, $v(a-c_\alpha)$ is constant for all $\alpha < \kappa$; in particular, it is equal to $v(a-c_0)$. For each α, let $c'_\alpha = c_\alpha - c_0$, and let $a' = a - b_0$. Since $(b_{\alpha,i}, c_{\alpha,i})_{i<\omega}$ is indiscernible over c_0 for all $\alpha < \kappa$ (including $\alpha = 0$, since $c_0 = c_{0,i}$ for all $i < \omega$), the array obtained by replacing c_α with c'_α is still an inp-pattern, and a' will still be a realization of the first column. To simplify notation, assume that $c_0 = 0$, so $a' = a$ and $c'_\alpha = c_\alpha$.

Now $v(a-c_\alpha) = v(a-c_0) = v(a)$ for all $\alpha < \kappa$, and so $\mathrm{ac}(a-c_\alpha)$ equals either $\mathrm{ac}(a)$ or $\mathrm{ac}(a) - \mathrm{ac}(c_\alpha)$, depending on whether $v(a) < v(c_\alpha)$ or $v(a) = v(c_\alpha)$. We again split into cases in order to define formulas $\psi_\alpha(x, y_\alpha, z_\alpha, z'_\alpha)$ for $\alpha < \kappa$.

Case 1: If $v(a) < v(c_\alpha)$, take $\psi(x, y_\alpha, z_\alpha)$ to be the formula

$$v(x) < v(z_\alpha) \wedge \chi(v(x), y_\alpha^{\mathrm{VG}}) \wedge \rho(\mathrm{ac}(x), y_\alpha^{\mathrm{RF}}).$$

Case 2: If $v(a) = v(c_\alpha)$, take $\psi_\alpha(x, y_\alpha, z_\alpha)$ to be the formula

$$v(x) = v(z_\alpha) \wedge \chi(v(x), y_\alpha^{\mathrm{VG}}) \wedge \rho(\mathrm{ac}(x) - \mathrm{ac}(z_\alpha), y_\alpha^{\mathrm{RF}}).$$

In either case, $\mathcal{K} \models \psi(a, b_{\alpha,0}, c_\alpha)$ and any realization of $\Psi_\alpha = \{\psi(x, b_{\alpha,i}, c_{\alpha,i}) : i < \omega\}$ would also be a realization of $\{\phi(x, b_{\alpha,i}, c_{\alpha,i} : i < \omega\}$, so Ψ_α is inconsistent. Take $v(z_\alpha)$ and $\mathrm{ac}(z_\alpha)$ for the terms t^i_α.

Then, setting z'_α and c'_α to be empty tuples, we may apply Lemma 2.2.3 to obtain $((\phi'_\alpha(x, y'_\alpha), (b'_\alpha))_{\alpha<\kappa}$, a new inp-pattern with no VF-sort parameter sequences, and in which each formula has the desired form by choice of ψ_α. $\square$

Now that we can reduce to inp-patterns with no VF-sort parameters, we can prove the main result of the section.

Theorem 2.2.7. *Suppose T is a theory of henselian valued fields in $\mathcal{L}_{\mathrm{Pas}}$ admitting relative quantifier elimination. Then*

$$\mathrm{bdn}(T) = \mathrm{bdn}(T_{\mathrm{VG}}) + \mathrm{bdn}(T_{\mathrm{RF}}),$$

where T_{VG} and T_{RF} are the induced theories on the value group and residue field, respectively.

Proof. We begin by showing that $\mathrm{bdn}(T) \leq \mathrm{bdn}(T_{\mathrm{VG}}) + \mathrm{bdn}(T_{\mathrm{RF}})$. Suppose that $(\phi_\alpha(x, y_\alpha), b_\alpha)_{\alpha<\kappa}$ is an indiscernible inp-pattern for T. If x is a VG-sort variable then we can obtain a new inp-pattern $(\phi'_\alpha(x', y_\alpha), b_\alpha)_{\alpha<\kappa}$ with x' a VF-sort variable by taking $\phi'_\alpha(x', y_\alpha) = \phi_\alpha(v(x'), y_\alpha)$. A similar substitution with $\mathrm{ac}(x')$ can replace an RF-sort variable with a VF-sort variable.

Thus, we may assume without loss of generality that x is in the valued field sort. By Proposition 2.2.6, we may further assume that for each $\alpha < \kappa$, y_α has no VF-sort component and $\phi_\alpha(x, y_\alpha)$ has the form

$$\chi_\alpha\big(v(x), y_\alpha^{\mathrm{VG}}\big) \wedge \rho_\alpha\big(\,\mathrm{ac}(x), y_\alpha^{\mathrm{RF}}\big)$$

where $\chi_\alpha \in \mathcal{L}_{\mathrm{VG}}$ and $\rho_\alpha \in \mathcal{L}_{\mathrm{RF}}$.

Suppose that for some $\alpha < \kappa$, the sets $X_\alpha(x) = \{\chi_\alpha(v(x), b_{\alpha,i}^{\mathrm{VG}}) : i < \omega\}$ and $P_\alpha(x) = \{\rho_\alpha(\mathrm{ac}(x), b_{\alpha,i}^{\mathrm{RF}}) : i < \omega\}$ are both consistent, say they are realized by elements c and d, respectively. Then by Lemma A.1, there exists an element a with $v(a) = v(c)$ and $\mathrm{ac}(a) = \mathrm{ac}(d)$. But then a would be a realization of $X_\alpha(x) \cup P_\alpha(x)$, and so would also be a realization of $\{\phi_\alpha(x, b_{\alpha,i}) : i < \omega\}$, contradicting the inconsistency of the row.

Thus, we can write $\kappa = |G \cup R|$, where $\alpha \in G$ if $X_\alpha(x)$ is inconsistent, and $\alpha \in R$ if $P_\alpha(x)$ is inconsistent. Then for new variable symbols z and w, $(\chi_\alpha(z, y_\alpha^{\mathrm{VG}}), b_\alpha^{\mathrm{VG}})_{\alpha \in G}$ is an inp-pattern in vK and $(\rho_\alpha(w, y_\alpha^{\mathrm{RF}}), b_\alpha^{\mathrm{RF}})_{\alpha \in R}$ is an inp-pattern in Kv, so

$$\kappa = |G \cup R| \leq |G| + |R| \leq \mathrm{bdn}(T_{\mathrm{VG}}) + \mathrm{bdn}(T_{\mathrm{RF}}).$$

Since $\mathrm{bdn}(T)$ is the supremum of all such κ, we have $\mathrm{bdn}(T) \leq \mathrm{bdn}(T_{\mathrm{VG}}) + \mathrm{bdn}(T_{\mathrm{RF}})$.

For the reverse inequality, let $(\chi_\alpha(z, y_\alpha), b_\alpha, k_\alpha)_{0 \leq \alpha < \kappa}$ and $(\rho_\alpha(w, y_\alpha), b_\alpha, k_\alpha)_{\kappa \leq \alpha < \lambda}$ be inp-patterns for T_{VG} and T_{RF}. For each $0 \leq \alpha < \kappa$, let $\phi_\alpha(x, y_\alpha)$ be the formula $\chi_\alpha(v(x), y_\alpha)$, and for each $\kappa \leq \alpha < \lambda$, let $\phi_\alpha(x, y_\alpha)$ be the formula $\rho_\alpha(\mathrm{ac}(x), y_\alpha)$. We claim that $(\phi_\alpha(x, y_\alpha), b_\alpha, k_\alpha)_{0 \leq \alpha < \lambda}$ is an inp-pattern for K.

First, note that each row is k_α-inconsistent, since we started with inp-patterns for T_{VG} and T_{RF}. Fix any function $\eta : \lambda \to \omega$. If $\gamma \in vK$ and $c \in Kv$ are realizations of $\{\chi_\alpha(z, b_{\alpha,\eta(\alpha)}) : 0 \leq \alpha < \kappa\}$ and $\{\rho_\alpha(w, b_{\alpha,\eta(\alpha)}) : \kappa \leq \alpha < \lambda\}$, respectively, then any element $a \in K$ with $v(a) = \gamma$ and $\mathrm{ac}(a) = c$ will realize $\{\phi_\alpha(x, b_{\alpha,\eta(\alpha)}) : 0 \leq \alpha < \lambda\}$.

Thus, $(\phi_\alpha(x, y_\alpha), b_\alpha, k_\alpha)_{0 \leq \alpha < \lambda}$ is an inp-pattern for T, which means $\lambda \leq \mathrm{bdn}(T)$. Taking the supremum over all such inp-patterns, we get $\mathrm{bdn}(T_{\mathrm{VG}}) + \mathrm{bdn}(T_{\mathrm{RF}}) \leq \mathrm{bdn}(T)$, completing the proof. $\square$

In section 2.4, we use this theorem to find valued fields of dp-rank d for every $d \in \mathbb{N}$. Note that equality only holds in the Denef-Pas language; if T' is the theory of a reduct of a model of T, then we only have the inequality

$$\mathrm{bdn}(T') \leq \mathrm{bdn}(T) = \mathrm{bdn}(T_{\mathrm{VG}}) + \mathrm{bdn}(T_{\mathrm{RF}}).$$

We know that this inequality can be strict in certain reducts. For example, any model K of ACVF will have dp-rank 1 in $\mathcal{L}_{\mathrm{div}}$, but dp-rank 2 in $\mathcal{L}_{\mathrm{Pas}}$.

Question 2.2.8. Is there a valued field K where equality holds in a reduct of the Denef-Pas language? This will certainly happen if the angular component map is definable in $\mathcal{L}_{\mathrm{div}}$ and the residue field is infinite; does this ever happen, and is it the only case where equality holds?

2.3 Definable Sets

Throughout this section, we continue to work in the Denef-Pas language. Recall that in this language, expansions of the value group and residue field are possible, but not expansions of the valued field sort. We begin the section by showing that there is a strong relationship between the valuation topology and the dp-rank of types.

Lemma 2.3.1. *Let (K, v) be a valued field. Then every type-definable set $X \subseteq K$ with interior satisfies* $\text{dp-rk}(X) = \text{dp-rk}(K)$*.*

Proof. If X has interior, then X contains a ball $B = \{x \in K : v(x - a) \geq v(r)\}$ for some $a, r \in K$, $r \neq 0$. Since the map $x \mapsto rx + a$ is a definable bijection, it preserves dp-rank, and so we may assume $B = \{x \in K : v(x) \geq 0\} = \mathcal{O}$ without loss of generality. Then the map $f : K \to B$ given by

$$f(x) = \begin{cases} x & \text{if } x \in B \\ x^{-1} & \text{if } x \notin B \end{cases}$$

is definable and finite-to-one. By Proposition B.3 and the fact that $B \subseteq X \subseteq K$, we have

$$\text{dp-rk}(K) \leq \text{dp-rk}(B) \leq \text{dp-rk}(X) \leq \text{dp-rk}(K)$$

and so $\text{dp-rk}(X) = \text{dp-rk}(K)$ as desired. □

Corollary 2.3.2. *Let (K, v) be a valued field. Then every type-definable set $X \subseteq K^n$ with interior satisfies* $\text{dp-rk}(X) = n \cdot \text{dp-rk}(K)$*.*

Proof. Since X has interior, it contains a product of balls $\mathcal{B} = B_1 \times B_2 \times \ldots \times B_n$. As in the lemma, there is a definable bijection between $\mathcal{B}$ and $\mathcal{O}^n$, and so we may assume $\mathcal{B} = \mathcal{O}^n$. Let $f : K \to B$ be the map

$$f(x) = \begin{cases} x & \text{if } x \in B \\ x^{-1} & \text{if } x \notin B \end{cases}$$

from the lemma. Then $g : K^n \to \mathcal{B}$ given by

$$g(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n)).$$

is definable and finite-to-one, and so $\text{dp-rk}(X) \leq n \cdot \text{dp-rk}(K) \leq \text{dp-rk}(X)$ by Proposition B.3 again. □

In the case where K admits relative quantifier elimination, we can prove the converse of Lemma 2.3.1 for definable sets. In fact, we get something even stronger.

Lemma 2.3.3. *Let (K, v) be an infinite valued field admitting relative quantifier elimination and let X be a definable subset of K. The following are equivalent:*

1. *X has interior*
2. $\text{dp-rk}(X) = \text{dp-rk}(K)$
3. $\text{dp-rk}(X) > 0$

Proof. $(1 \Rightarrow 2)$ is the previous lemma, and $(2 \Rightarrow 3)$ is easy: since K is infinite, $\text{dp-rk}(X) = \text{dp-rk}(K) > 0$ by Proposition B.1.

For $(3 \Rightarrow 1)$, we may assume K is $\aleph_1$-saturated, since dp-rank is preserved by elementary extensions and having interior is definable in $\mathcal{L}_{\text{Pas}}$. Suppose $\text{dp-rk}(X) > 0$ and let S be some finite set of parameters such that X is S-definable. By relative quantifier elimination, we may assume X is defined by

$$\bigvee_{i=1}^{n} \chi_i(v(f_1(x), \dots, v(f_m(x))) \wedge \rho_i(\text{ac}(f_1(x)), \dots, \text{ac}(f_m(x))),$$

where χ_i are $\mathcal{L}_{\text{VG}}$-formulas with parameters in $vK \cap S$, ρ_i are $\mathcal{L}_{\text{RF}}$ formulas with parameters in $Kv \cap S$, and f_j are polynomials with coefficients in $K \cap S$.

Fix $a \in X$ with $\text{dp-rk}(a/S) > 0$; then a is not algebraic over S, so $v(f_i(a)) < \infty$ for all $i \leq m$. Let $K_0 \preceq K$ be a countable submodel containing a and S. By saturation, there exists a nonzero element $\varepsilon \in K$ such that $v(\varepsilon) > v(b)$ for all $b \in K_0$. In particular, this means that $v(f_i(a + \varepsilon)) = v(f_i(a))$ and $\text{ac}(f_i(a + \varepsilon)) = \text{ac}(f_i(a))$ for all $i \leq m$. Moreover, the same will hold for every ε' with $v(\varepsilon') \geq v(\varepsilon)$, and so $a + \varepsilon' \in X$ for every ε' with $v(\varepsilon') \geq \varepsilon$. In other words, the ball with centre a and radius ε is a subset of X, and hence X has interior. $\square$

The remainder of this section is devoted to proving that Lemma 2.3.3 characterizes henselianity for dp-finite fields. Unfortunately, satisfying the conclusion of Lemma 2.3.3 in K is not enough to conclude that K is henselian; we must assume K satisfies the following condition $(*)$:

> For every coarsening w of v (including the case where w is the trivial valuation), every infinite definable subset $X \subseteq Kw$ has interior with respect to the valuation induced by v on Kw.

Our goal for the remainder of the section is to prove the following theorem. Throughout the section, we assume that v is a non-trivial valuation.

Theorem 2.3.4. *Let (K, v) be an dp-finite valued field. Then (K, v) is henselian if and only if it satisfies condition $(*)$.*

In private communication with Yatir Halevi, the author learned that Halevi, Hasson, and Jahnke have independently proved a version of Theorem 2.3.4 using properties of the Zariski topology. Their proof is currently not publicly available. The rest of the section will be dedicated to our proof of the theorem, beginning with some intermediate lemmas and propositions.

Note that in the proof of Lemma 2.3.3, we actually show that all elements of positive dp-rank are in the interior of X. By focusing on these elements, we obtain a version of the lemma with relative quantifier elimination replaced by a weak form of condition $(*)$.

Lemma 2.3.5. *Let (K, v) be a valued field such that every infinite definable subset of K has interior, let S be some parameter set, and let $a \in K$ be a singleton. Then the following are equivalent:*

1. $\text{dp-rk}(a/S) = \text{dp-rk}(K)$
2. *a is an interior point of every S-definable set X containing a*
3. $Y = \{y \in K : y \models \text{tp}(a/S)\}$ *is open*

Proof. $(1 \Rightarrow 2)$ Suppose for contradiction that $\text{dp-rk}(a/S) = \text{dp-rk}(K)$ and a is in the topological boundary of some S-definable set X. Since the boundary of X is S-definable and does not have interior, it must be finite. But then a is algebraic over S, contradicting $\text{dp-rk}(a/S) = \text{dp-rk}(K) > 0$.

For $(2 \Rightarrow 3)$, similar to Lemma 2.3.3, we may assume that K is $(\aleph_1 + |S|^+)$-saturated. Let $K_0 \preceq K$ be a small submodel containing a and S, and pick $\varepsilon \in K$ such that $v(\varepsilon) > v(b)$ for all nonzero $b \in K_0$. Then the ball $B = \{x \in K : v(x - a) > v(\varepsilon)\}$ is contained in all S-definable sets X containing a. Since Y is the intersection of all such sets X, we have $B \subseteq Y$, and hence a is an interior point of Y. But the same argument can be made for any $y \in Y$, which means every point in Y is an interior point, and hence Y is open.

Finally, $(3 \Rightarrow 1)$ is just Lemma 2.3.1 since $\text{dp-rk}(Y) = \text{dp-rk}(a/S)$. □

The next proposition demonstrates why finite dp-rank is a necessary assumption for the theorem. If K has infinite dp-rank, there is no way to distinguish between dp-rk(K) and dp-rk(K^n), and so a subset of K^n will have full dp-rank if and only if some projection onto K has interior, which is significantly weaker than what we need.

Proposition 2.3.6. *Let (K, v) be a dp-finite valued field such that every infinite definable subset of K has interior, let S be some finite set of parameters, and let X be an S-definable subset of K^n. If $(a_1, \ldots, a_n) \in X$ satisfies* $\text{dp-rk}(a_1, \ldots, a_n) = n \cdot \text{dp-rk}(K)$ *then $(a_1, \ldots, a_n)$ is an interior point of X.*

Proof.[1] Without loss of generality, assume K is $\aleph_1$-saturated; we proceed by induction on n. If $n = 1$ the result follows immediately from Lemma 2.3.5.

For $n > 1$, note that by Fact B.4, $(a_2, \ldots, a_n)$ has full dp-rank over $\{a_1\} \cup S$. Then since the set

$$X_1 = \{(x_2, \ldots, x_n) : (a_1, x_2, \ldots, x_n) \in X\}$$

is Sa_1-definable, $(a_2, \ldots, a_n)$ is in the interior of X_1 by induction.

Consider the formula $\phi(x, r)$ given by

$$r \neq 0 \wedge \left(\{x\} \times \prod_{i=2}^{n} B(a_i, r) \subseteq X \right),$$

where $B(a, r) = \{x \in K : v(x - a) > v(r)\}$.

Let R be the set $\{r \in K : K \models \phi(a_1, r)\}$; since $(a_2, \ldots, a_n)$ is in the interior of X_1, R is nonempty. Suppose for contradiction that for all $r \in R$, a_1 is in the boundary of $\phi(K, r)$. Since boundary sets do not have interior, they are finite by assumption; then by compactness, there must be a uniform bound k such that $|\phi(K, r)| < k$ for all $r \in R$.

Let $b_1, \ldots, b_{k+1}$ be any elements satisfying $p(x) = \text{tp}(a_1/Sa_2 \ldots a_n)$; such elements exist since this type is not algebraic. Since $\exists r \phi(x, r)$ is a formula in $p(x)$, we can choose $r_1, \ldots, r_{k+1}$ such that $K \models \phi(b_i, r_i)$. Moreover, each b_i is a boundary point of $\phi(K, r)$ for every $r \in R$. Fix any $r_0 \in K$ with $v(r_0) \geq v(r_i)$ for each $1 \leq i \leq k + 1$. Then $r_0 \in R$ and $K \models \phi(b_i, r_0)$ for each i, contradicting the uniform bound k.

[1] The author would like to thank the external examiner, Dr. Assaf Hasson, for suggesting this approach, which is more intuitive than the original submission.

Thus, there must exist $r_2 \in R$ such that the set defined by $\phi(x, r_2)$ contains an open ball $B(a_1, r_1)$. Then the product $B(a_1, r_1) \times \prod_{i=2}^{n} B(a_i, r_2)$ witnesses that $(a_1, \ldots, a_n)$ is an interior point of X, as desired. □

As a corollary, we get a generalization of Lemma 2.3.5 for dp-finite fields:

Corollary 2.3.7. *Let (K, v) be a dp-finite valued field such that every infinite definable subset of K has interior, let S be some parameter set, and let $a \in K$ be a finite tuple. Then the following are equivalent:*

1. $\text{dp-rk}(a/S) = n \cdot \text{dp-rk}(K)$
2. *a is an interior point of every S-definable set X containing a*
3. *$Y = \{y \in K : y \models \text{tp}(a/S)\}$ is open*

Proof. ($1 \Rightarrow 2$) This is precisely Proposition 2.3.6.

For ($2 \Rightarrow 3$), the proof follows identically to Lemma 2.3.5, replacing the ball $B = \{x \in K : v(x - a) > v(\varepsilon)\}$ with the product $\mathcal{B} = \{x \in K : v(x - a) > v(\varepsilon)\}^n$.

Finally, ($3 \Rightarrow 1$) is just Corollary 2.3.2 since $\text{dp-rk}(Y) = \text{dp-rk}(a/S)$. □

We can also combine Proposition 2.3.6 with Corollary 2.3.2 to get the following equivalence between dp-rank and interior:

Corollary 2.3.8. *Let (K, v) be a dp-finite valued field such that every infinite definable subset of K has interior. Then a definable set $X \subseteq K^n$ has interior if and only if it has full dp-rank.*

This result then gives us a strong result about definable finite-to-one maps.

Corollary 2.3.9. *Let (K, v) be a dp-finite valued field such that every infinite definable subset of K has interior and let $f : K^n \to K^n$ be a finite-to-one definable function. Then for every $X \subseteq K^n$ (not necessarily definable), if X has non-empty interior, so does $f(X)$.*

Proof. If X has non-empty interior, then it contains a product of balls $\mathcal{B}$, which will be definable and have full dp-rank. Since dp-rank is preserved by finite-to-one maps, $f(\mathcal{B})$ also has full dp-rank, and hence $f(\mathcal{B}) \subseteq f(X)$ has interior. □

This corollary is the key step towards showing that dp-minimal valued fields are henselian, in both (Jahnke *et al.*, 2017) and (Johnson, 2016). From this point forward, we follow the method of (Jahnke *et al.*, 2017, Section 4) very closely. Write (K^h, v^h) for the henselization of (K, v). We begin by stating two technical facts.

Fact 2.3.10. *(Follows immediately from Theorem 7.4 of Prestel and Ziegler (1978)) Let $G \in K[x_1, \ldots, x_n]^n$ and let $\mathcal{B} \subseteq (K^h)^n$ be a product of balls. Suppose that the Jacobian $J_G(a)$ is nonzero for some $a \in (\mathcal{B} \cap K^n)$. Then there is an open set $U \subseteq B$ containing a such that $G|_U$ is injective.*

Fact 2.3.11. *(Guingona, 2014, Lemma 3.9) Let K be a field and $a \notin K$ algebraic over K. Let $a = a_1, \ldots, a_n$ be the conjugates of a over K and let*

$$g(X_1, \ldots, X_n, Y) = \prod_{i=1}^{n} \left(Y - \sum_{j=0}^{n-1} a_i^j X_i \right).$$

Then there exist $G_0, \ldots, G_{n-1} \in K[X_1, \ldots, X_n]$ such that

$$g(X_1, \ldots, X_n, Y) = Y^n + \sum_{j=0}^{n-1} G_j(X_1, \ldots, X_n).$$

Moreover,

1. *If $c = (c_1, \ldots, c_n) \in K^n$ and $c_j \neq 0$ for some j then $g(c, Y)$ has no roots in K.*
2. *Writing $G = (G_0, \ldots, G_{n-1})$ and J_G for the Jacobian of G, there is $d = (d_1, \ldots, d_n) \in K^n$ such that $J_G(d) \neq 0$ and $d_j \neq 0$ for some j.*

Proposition 2.3.12. *Suppose that (K, v) is a dp-finite valued field such that every infinite definable subset $X \subseteq K$ has interior and let $a \in K^h$ be such that for any $\gamma \in vK$, there exists $b \in K$ with $v^h(b - a) \geq \gamma$. Then $a \in K$.*

Proof. This is Proposition 4.4 of Jahnke *et al.* (2017) with only one minor adjustment; we repeat the proof here in order to keep the thesis self-contained.

Suppose that a has degree n over K, and let $a = a_1, \ldots, a_n$ be the conjugates of a over K. Let g, G, and d be as in Lemma 2.3.11. Then by Lemma 2.3.10, there is an open set U such that $G|_U$ is injective. Further, by Corollary 2.3.9, $G(U)$ has non-empty interior. Since J_G is continuous we may assume, shrinking U if necessary, that J_G is nonzero on U. Shrinking U again if necessary, we may assume that for all

$(x_1, \ldots, x_n) \in U$ there exists j such that $x_j \neq 0$. Finally, after changing d if necessary, we may assume that $e = G(d)$ lies in the interior of $G(U)$.

Let $V \subseteq G(U)$ be an open neighbourhood of e. Define $f : K^h \setminus \{0\} \to K^h$ by

$$f(y) = -\left(y^n + \sum_{j=0}^{n-2} e_j y^j\right) \cdot y^{-(n-1)}$$

and note that f is continuous on all of $K^h \setminus \{0\}$. Then for every $y \neq 0$ we have

$$y^n + f(y)y^{n-1} + \sum_{j=0}^{n-2} e_j y^j = 0.$$

Define

$$h(x) = \sum_{j=0}^{n-1} d_j x^j.$$

Then $h(a)$ is a root of $g(d, Y)$, so $f(h(a)) = e_{n-1}$. Moreover, since $g(d, Y)$ has no roots in K by Lemma 2.3.11(1), $h(a) \in K^h \setminus K$; in particular, $h(a) \neq 0$. If $b \in K$ is sufficiently close to a then $h(b) \neq 0$ and

$$(e_0, \ldots, e_{n-2}, (f \circ h)(b)) \in V.$$

Thus, there exists $c \in U$ with

$$G(c) = (e_0, \ldots, e_{n-2}, (f \circ h)(b)).$$

Then by our choice of U, $c_j \neq 0$ for some j, and so $g(c, Y)$ has no root in K. On the other hand, $h(b)$ is a root of $g(c, Y)$. By contradiction, we must have had $a \in K$ from the beginning. □

With this result, we can now prove Theorem 2.3.4: a dp-finite valued field is henselian if and only if it satisfies condition $(*)$.

Proof. (of Theorem 2.3.4) Suppose (K, v) is henselian. For every coarsening w of v, write $\overline{v}$ for the valuation on Kw induced by v. Then $(Kw, \overline{v})$ is henselian and is interpretable in (K, v), so is dp-finite. By Lemma 2.3.3, every infinite definable subset of Kw has interior, which means (K, v) satisfies condition $(*)$.

For the reverse direction, we follow the proof of (Jahnke *et al.*, 2017, Proposition 4.5); again, only one minor adjustment is required, but we repeat the proof for the sake of completeness.

Suppose (K, v) satisfies condition $(*)$, and assume for contradiction that (K, v) is not henselian. Then there is a polynomial

$$p(X) = X^n + aX^{n-1} + \sum_{i=1}^{n-2} c_i X^i$$

such that $v(a) = 0$, $v(c_i) > 0$ for all i, and such that p has no root in K. Let (K^h, v^h) be the henselization of (F, v) and choose some $\alpha \in K^h$ such that $p(\alpha) = 0$, $v(\alpha - 0) > 0$, and $v(p'(\alpha)) = 0$. Consider the set

$$S = \{v^h(b - \alpha) \in vK : b \in K \text{ and } v^h(b - \alpha) > 0\}.$$

Let Δ be the convex subgroup of vK generated by S and note that Δ is a proper subgroup of vK; for otherwise, we would have $\alpha \in K$ by Proposition 2.3.12. Following the proof of Claim 5.12.1 of Macpherson *et al.* (2000), S is cofinal in Δ.

Let w be the coarsening of v with value group vK/Δ, let w^h be the corresponding coarsening of v^h, and let $\overline{v}$ be the valuation on Kw induced by v. As noted above, $(Kw, \overline{v})$ is dp-finite since it is interpretable in (K, v). Given any element x in K^h, write $\overline{x}$ for its residue in $K^h w^h$.

We claim that there exists $\beta \in K$ such that $w(p(\beta)) > 0$. By condition $(*)$, every infinite definable subset of $(Kw, \overline{v})$ has nonempty interior, and as in the proof of the corresponding claim in Proposition 4.5 of Jahnke *et al.* (2017), $(F^h w^h, \overline{v}^h)$ is the henselization of $(Kw, \overline{v})$. Moreover, by choice of Δ, the residue $\overline{\alpha}$ of α with respect to w^h is approximated arbitrarily well by elements of Fw, so by Proposition 2.3.12, $\overline{\alpha} \in Kw$. Take any $\beta \in K$ with $\overline{\beta} = \overline{\alpha}$; then since $w(p(\alpha)) > 0$, we also have $w(p(\beta)) > 0$.

Now take

$$J = \{b \in K : v(b - \alpha) > 0\}.$$

Since $\beta - \alpha \in \mathfrak{m}_w \subseteq \mathfrak{m}_v$, we have $\beta \in J$. Moreover, for all $b \in J$, we have $v(b - \alpha) = v(p(b))$ by Claim 5.12.2 of Macpherson *et al.* (2000). However, by definition of Δ, $w(p(b)) = 0$ for any $b \in J$. This contradicts our choice of β, and hence (K, v) must be henselian. $\square$

2.4 New Examples of Dp-finite Valued Fields

Using the results of the previous two sections, we will now construct, for each $d \in \mathbb{N}$, a theory T_d with the following properties:

1. T_d is a complete theory of henselian valued fields,
2. $\text{dp-rk}(T_d) = d$,
3. if $d > 2$ then T_d is not the theory of an expansion of a dp-minimal field, and
4. Every partial type $\pi(x)$ in the home sort of T_d has dp-rank equal to 0 or d

Because these examples do not come from expansions of dp-minimal fields, they are fundamentally different from the examples explored in Section 1.5. Moreover, while theories with property (3) have been known to exist since Kaplan *et al.* (2013), to the author's knowledge the only theories with this property have been purely combinatorial in nature, and do not come from any theory of algebraic structures.

First, we note that (4) follows from (1) and (2). If $d = 0$ or $d = 1$, this is trivial; otherwise, if we work in the Denef-Pas language, (4) is an immediate consequence of Corollary 2.3.5. We could alternatively work in a 1-sorted language $\mathcal{L} = \{0, 1, +, -, \cdot, R, S\}$ where R is interpreted as the valuation ring and S is interpreted as the set

$$\{(x, y) : \mathrm{ac}(x) = \mathrm{ac}(y)\}.$$

Henselian valued fields in this language are interdefinable with those in the Denef-Pas language, and so the results of the previous sections will also hold in this language.

We can choose T_0 to be any theory of finite fields with the trivial valuation, T_1 to be any theory of infinite dp-minimal valued fields in $\mathcal{L}_{\text{div}}$, and T_2 to be any completion of ACVF in $\mathcal{L}_{\text{Pas}}$ or $\mathcal{L}$. By Jahnke *et al.* (2017), T_1 and T_2 will satisfy 1, and by section 2.2, T_2 will satisfy 2. For concreteness, take $T_0 = \mathrm{Th}(\mathbb{Z}/97\mathbb{Z})$, $T_1 = \mathrm{Th}_{\mathcal{L}_{\text{div}}}(\mathbb{Q}_{11})$, and $T_2 = \mathrm{Th}_{\mathcal{L}_{\text{Pas}}}(\mathbb{C})$.

By Section 2.2, in both $\mathcal{L}_{\text{Pas}}$ and $\mathcal{L}$ we have $\text{dp-rk}(\mathbb{C}[[t^\Gamma]]) = 1 + \text{dp-rk}(\Gamma)$ for all ordered abelian groups Γ. Thus, given Γ of dp-rank d, we may take T_{d+1} to be $\mathrm{Th}_{\mathcal{L}_{\text{Pas}}}(\mathbb{C}[[t^\Gamma]])$ or $\mathrm{Th}_{\mathcal{L}}(\mathbb{C}[[t^\Gamma]])$. The dp-rank of an ordered abelian group can be easily calculated by the work of Farré (2017) and Halevi and Hasson (2017b): both sets of authors independently characterized strongly dependent ordered abelian groups, and provided a formula for calculating the dp-rank of such a group.

We will state the result using the notation of Halevi and Hasson (2017b). For a strongly dependent ordered abelian group G, let

$$\mathcal{P}_\infty(G) = \{p \in \mathbb{N} : p \text{ is prime and } [G : pG] = \infty\}.$$

For each $p \in \mathcal{P}_\infty(G)$, let k_p be the length of a maximal chain of definable convex subgroups $H_1 \subsetneq \ldots \subsetneq H_n \subsetneq H_{n+1} = G$ such that for all $i \leq n$,

$$[H_{i+1}/H_i : p(H_{i+1}/H_i)] = \infty.$$

This value is always finite by (Halevi and Hasson, 2017b, Lemma 4.4). If $p \notin \mathcal{P}_\infty(G)$ then no such chain exists, and we set $k_p = 0$. Consider G in the language described in (Halevi and Hasson, 2017b, Lemma 3.4), which we will denote $\mathcal{L}_{\mathrm{HH}}$. Then G is dp-minimal if and only if $\mathcal{P}_\infty(G) = \emptyset$, and dp-rk$(G)$ is equal to

$$1 + \sum_{p \in \mathcal{P}_\infty(G)} k_p$$

otherwise.

Recall that an ordered abelian group is called archimedean if it has no proper convex subgroups; this occurs if and only if it is order isomorphic to a subgroup of $\mathbb{R}$. For each prime p, let B_p be a countable subset of $\mathbb{R}$ such that $B = \bigcup_p B_p$ is linearly independent over $\mathbb{Q}$. For each p, consider

$$G_p = \sum_{b \in B_p} b\mathbb{Z}_{(p)} = \bigoplus_{b \in B_p} b\mathbb{Z}_{(p)}$$

as an ordered subgroup of $\mathbb{R}$. Clearly, this group is archimedean, and by Example 2.10 of Halevi and Hasson (2017b), $[G_p : pG_p] = \aleph_0$, while $[G_p : qG_p] = 1$ for every other prime q.

Now, let P be a finite set of primes, and set $G = \bigoplus_{p \in P} G_p$ as an ordered subgroup of $\mathbb{R}$. This group is clearly still archimedean, and so has no proper convex subgroups. Hence, for each prime p, either $k_p = 1$ (if $p \in \mathcal{P}_\infty(G)$) or $k_p = 0$ (otherwise). Note that for all q, we have

$$G/qG = \left(\bigoplus_{p \in P} G_p\right) \Bigg/ \left(\bigoplus_{p \in P} qG_p\right) \cong \bigoplus_{p \in P} G_p/qG_p.$$

Since $qG_p = G_p$ whenever $q \neq p$, this quotient will be trivial whenever $q \notin P$, and will be equal to G_q/qG_q whenever $q \in P$. We have already noted that $[G_q : qG_q] = \aleph_0$, and so $\mathcal{P}_\infty(G) = P$ and $k_p = 1$ for all $p \in \mathcal{P}_\infty(G)$.

Now, it is easy to calculate dp-rk(G) in the language $\mathcal{L}_{\mathrm{HH}}$: it is

$$\text{dp-rk}(G) = 1 + \sum_{p \in \mathcal{P}_\infty(G)} k_p = 1 + \sum_{p \in \mathcal{P}_\infty(G)} 1 = 1 + |\mathcal{P}_\infty(G)| = 1 + |P|.$$

Now, for each $d \geq 1$, we can take T_{d+2} to be the theory of the Hahn series field $\mathbb{C}[[t^G]]$ for some G constructed as above with $|P| = d$.

Note that $\mathcal{L}_{\mathrm{HH}}$ is a definitional expansion of $\mathcal{L}_{\mathrm{oag}}$, so we have

$$\text{dp-rk}_{\mathcal{L}_{\mathrm{oag}}}(G) = \text{dp-rk}_{\mathcal{L}_{\mathrm{HH}}}(G) = 1 + |P|.$$

However, we will now demonstrate an explicit inp-pattern in $\mathcal{L}_{\mathrm{oag}}$ of depth $1+|P|$ for G for the sake of concreteness.

Enumerate P as $\{p_1, \ldots, p_n\}$, let $q = \prod_{i=1}^n p_i$, and let $q_j = \prod_{i \neq j} p_i$. Consider the following formulas:

- $\phi_0(x, y, z)$ is the formula $y < x < z$
- For $1 \leq i \leq n$, $\phi_i(x, y)$ is the formula $x - y \in p_i G$

For the first row, take as parameters any sequence of pairs $(a_i, b_i)_{i<\omega}$ that satisfy $a_i < b_i < a_{i+1}$, and for each other row, take as parameters any sequence $(c_{i,j})_{j<\omega}$ in $q_i G$ such that $c_{i,j}$ and $c_{i,j'}$ are in different cosets of qG whenever $j \neq j'$. Note that

$$q_i G / qG \cong \bigoplus_{j=1}^{n} (q_i G_{p_j})/(qG_{p_j}) \cong G_{p_i}/p_i G_{p_i}$$

since for $j \neq i$, $qG_{p_j} = q_i G_{p_j} = p_j G_{p_j}$. Thus, $[q_i G : qG] = \aleph_0$ for each i, and so the sequences $(c_{i,j})_{j<\omega}$ exist.

Clearly, these rows are 2-inconsistent. Fix some map $\eta : \{0, \ldots, n\} \to \omega$, and some prime $p \notin P$. Then there exists $r \in \mathbb{N}$ such that $qp^{-r} < b_{\eta(0)} - a_{\eta(0)}$, and $s \in \mathbb{Z}$ such that

$$a_{\eta(0)} < qsp^{-r} + \sum_{i=1}^{n} c_{i,\eta(i)} < b_{\eta(0)}.$$

Set $x_\eta = qsp^{-r} + \sum_{i=1}^n c_{i,\eta(i)}$. By choice of r and s, we have $G \models \phi_0(x_\eta, a_{\eta(0)}, b_{\eta(0)})$. Moreover, for each $i > 0$, we have

$$x_\eta - c_{i,\eta(i)} = qsp^{-r} + \sum_{j \neq i}^{n} c_{j,\eta(j)}$$

which is an element of $p_i G$ by choice of q and $c_{j,\eta(j)}$. Thus, $G \models \phi_i(x_\eta, c_{i,\eta(i)})$, which means we have indeed constructed an inp-pattern.

Note that while this does give us examples of henselian valued fields of dp-rank d for all $d \in \mathbb{N}$ in a natural language, as mentioned in Section 2.2, we do not know the dp-rank of the reduct of T_d to $\mathcal{L}_{\text{div}}$. Suppose $\mathcal{K} \models T_d$, and let (K, v) be the reduct of $\mathcal{K}$ to $\mathcal{L}_{\text{div}}$. It is easy to check that any inp-pattern in $\mathcal{L}_{\text{VG}}$ corresponds to an inp-pattern in $\mathcal{L}_{\text{div}}$ of the same depth, so we have

$$d - 1 \leq \text{dp-rk}(K, v) \leq \text{dp-rk}(\mathcal{K}) = d.$$

In the case where $d = 2$, we know that this inequality is strict, but we do not know anything about $d > 2$. This suggests a particular case of Question 2.2.8 to investigate:

Question 2.4.1. Suppose $\mathcal{K} \models T_d$ for $d > 2$ in $\mathcal{L}_{\text{Pas}}$, and that (K, v) is the reduct of $\mathcal{K}$ to $\mathcal{L}_{\text{div}}$. Is the dp-rank of (K, v) equal to d or to $d - 1$?

Chapter 3

Classification of Dp-finite Fields

This chapter is concerned with the following question: what is the algebraic structure of a dp-finite field, up to elementary equivalence? A well-known conjecture, based on Conjecture 5.34(c) of Shelah (2014), states the following:

Conjecture 3.0.1. *If K is a strongly dependent field, then K is either algebraically closed, real closed, or has a definable henselian valuation.*

As part of his thesis, Johnson (2016) proved that this conjecture holds in the special case of dp-minimal fields:

Fact 3.0.2. *(Follows directly from (Johnson, 2016, Theorem 9.4.18) and (Jahnke and Koenigsmann, 2015, Theorem 5.2)) Let K be a sufficiently saturated dp-minimal field. Then K is either algebraically closed, real closed, or has a definable henselian valuation.*

Johnson's proof is completed by constructing a valuation using nothing but the definable sets in the pure field language, and showing that this valuation must be henselian. He in fact characterizes dp-minimal fields up to elementary equivalence (Johnson, 2016, Theorem 9.7.2); a recent pre-print of Halevi *et al.* (2018) gives an analogous characterization of strongly dependent fields, assuming that Conjecture 3.0.1 holds.

Johnson also published a pre-print of his proof to arXiv.org (Johnson, 2015). The thesis is more recent than the pre-print, and some of its organization and explanation are slightly clearer, although the overall structure and content of the arguments is the same in both. Throughout the remainder of this chapter, we will cite results to the pre-print (Johnson, 2015), rather than the thesis, as we believe that arXiv.org is a more accessible platform for the majority of readers.

This chapter contains our progress towards generalizing Johnson's method from dp-minimal fields to dp-finite fields. The main obstacle to overcome is that unlike in the dp-minimal case, it is possible for a definable set in one variable in a dp-finite structure to be infinite but not have full dp-rank. Johnson uses this equivalence (which does hold in the dp-minimal case) liberally, and working without it makes many arguments significantly more challenging.

The first section of this chapter establishes some preliminary results about V-topologies and the remaining sections provide the progress so far on generalizing Johnson's results to the dp-finite case. They are divided based on the following property:

Definition 3.0.3. Let K be a dp-finite field. We say that a definable subset $X \subseteq K$ is *large* if $\text{dp-rk}(X^c) < \text{dp-rk}(K)$. Note that since $X \cup X^c = K$, if X is large then $\text{dp-rk}(X) = \text{dp-rk}(K)$.

We say that K has the *large sets property (LSP)* if for every definable set $X \subseteq K$, either X or $X^c = K \setminus X$ is large; equivalently, either $\text{dp-rk}(X) < \text{dp-rk}(K)$ or $\text{dp-rk}(X^c) < \text{dp-rk}(K)$.

Note that any field with a definable ordering or valuation will not have LSP, since any interval or ball X will satisfy $\text{dp-rk}(X) = \text{dp-rk}(X^c) = \text{dp-rk}(K)$. On the other hand, in an algebraically closed field, definable sets are either finite or cofinite, and hence every algebraically closed field has LSP in the ring language. Sections 3.2 and 3.3 consider fields without LSP, which we expect to have a definable henselian valuation. Section 3.4 considers fields with LSP, which we expect to be algebraically closed.

3.1 V-Topologies

Following Johnson's method, our aim is to show that if a dp-finite field does not have LSP then it has a definable henselian valuation. We approach this goal through the use of a definable topology with a strong algebraic structure:

Fact 3.1.1. *Let K be a field, and let $\mathcal{N}$ be a filtered family of subsets of K, meaning that for all $U, V \in \mathcal{N}$, there exists $W \in \mathcal{N}$ such that $W \subseteq U \cap V$. Suppose moreover that the following conditions hold:*

(1) $\{0\} = \bigcap_{U \in \mathcal{N}} U$

(2) $\{0\} \notin \mathcal{N}$

(3) For all $U \in \mathcal{N}$ there exists $V \in \mathcal{N}$ such that $V - V \subseteq U$

(4) For all $U \in \mathcal{N}$ and $x \in K$ there exists $V \in \mathcal{N}$ such that $x \cdot V \subseteq U$

(5) For all $U \in \mathcal{N}$ there exists $V \in \mathcal{N}$ such that $V \cdot V \subseteq U$

(6) For all $U \in \mathcal{N}$ there exists $V \in \mathcal{N}$ such that $(1 + V)^{-1} \subseteq (1 + U)$

(7) For all $U \in \mathcal{N}$ there exists $V \in \mathcal{N}$ such that for all $x, y \in K$, if $xy \in V$ then $x \in U$ or $y \in U$

Then $\mathcal{N}$ is a neighbourhood basis of 0 of a topology on K.

Definition 3.1.2. A topology generated by a neighbourhood basis as in the previous proposition is called a V-topology.

For more on V-topologies, see Appendix B of Engler and Prestel (2005). Suppose τ is a V-topology. (1)-(3) ensures that τ is Hausdorff and that τ is a group topology on K: that is, $(x, y) \mapsto x + y$ is continuous with respect to τ. Similarly, (4)-(5) ensure that τ is a ring topology and (6) ensures that τ is a field topology: that is, $(x, y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are continuous, respectively. Finally, (7) ensures the following:

Fact 3.1.3. *A topology is a V-topology if and only if it is induced by a non-trivial valuation or a non-trivial absolute value.*

This result was originally proven in Dürbaum and Kowalski (1953). A proof is also included in Appendix B of Engler and Prestel (2005).

The fact that a field with a V-topology must have a non-trivial valuation or non-trivial absolute value on its own is not surprising: almost all fields admit non-trivial valuations. In order for the valuation to be definable and henselian, the V-topology must satisfy an additional axiom schema:

Definition 3.1.4. Suppose τ is a V-topology generated by some family $\mathcal{N}$. If we also assume the following axiom scheme for each $n \in \mathbb{N}$:

$(8)_n$ There exists $U \in \mathcal{N}$ such that for all $f(X) = X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_0$ with $a_i \in U$, there exists $y \in K$ with $f(y) = 0$.

then we say that τ is a *t-henselian* topology.

As the name implies, t-henselianity is related to henselianity; it can be thought of as a non-uniform version of henslianity. This relationship can be made precise:

Fact 3.1.5. *Let K be a field that is not separably closed.*

1. *If K admits a t-henselian topology then this topology is the only t-henselian topology on K and is first-order definable in the language of rings.*

2. *If K admits a t-henselian topology then every field elementarily equivalent to K admits a t-henselian topology.*

3. *K admits a t-henselian topology if and only if K is elementarily equivalent to a field admitting a non-trivial henselian valuation.*

4. *Suppose K is also not real closed and admits a t-henselian topology. Then K admits a definable henselian valuation inducing the t-henselian topology.*

Proof. 1. Fix any irreducible separable polynomial $f \in K[X]$ with $\deg(f) > 1$ and $a \in K$ satisfying $f'(a) \neq 0$. Consider the family of definable sets

$$U_{f,a} = \{f(x)^{-1} - f(a)^{-1} : x \in K\}.$$

The collection $c \cdot U_{f,a}$ for $c \in K^\times$ is a neighbourhood basis of 0 of the unique t-henselian topology on K. See page 203 of Prestel (1991) for the details.

2. This follows immediately from (1).

3. This follows from Theorem 7.2 of Prestel and Ziegler (1978).

4. This is Theorem 5.2 of Jahnke and Koenigsmann (2015). □

From these results, we can now see the general form of our aspirational proof: suppose K is dp-finite and neither real closed nor separably closed, then construct $\mathcal{N}$, a neighbourhood basis of 0 for a t-henselian topology on K. In practice, it will be more helpful to look at the intersection $I_{\mathcal{N}} = \bigcap_{U\in\mathcal{N}} U$ in some very saturated elementary extension $\mathbb{M}$ of K. If each set in $\mathcal{N}$ is definable over K, then this intersection will be type-definable over K. Moreover, each of the properties (1)-(8) has a corresponding property on $I_{\mathcal{N}}$:

Fact 3.1.6. *(Johnson, 2015, Section 2.2) Let $\mathcal{N}$ be a filtered collection of K-definable sets, and let $\mathbb{M}$ and $I_{\mathcal{N}}$ be as in the previous paragraph. Then each property of $\mathcal{N}$ in Proposition 3.1.1 holds if and only if the corresponding property holds below:*

(1) $I_{\mathcal{N}} \cap K = \{0\}$

(2) $I_{\mathcal{N}} \neq \{0\}$

(3) $I_{\mathcal{N}}$ *is a subgroup of* $(\mathbb{M}, +)$

(4) $I_{\mathcal{N}}$ *is closed under multiplication by elements of* K

(5) $I_{\mathcal{N}}$ *is closed under multiplication (by elements of* $I_{\mathcal{N}}$*)*

(6) $(1 + I_{\mathcal{N}})^{-1} = (1 + I_{\mathcal{N}})$

(7) $\mathbb{M} \setminus I_{\mathcal{N}}$ *is closed under multiplication*

(8) *If* $f(X) = X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_0$ *with each* $a_i \in I_{\mathcal{N}}$ *then* $f(X)$ *has a root in* $\mathbb{M}$

If $I_{\mathcal{N}}$ as above satisfies (1)-(5) and (7) then it also satisfies (6) and is the maximal ideal of a valuation ring on $\mathbb{M}$.

Proposition 3.1.7. *Suppose* $I_{\mathcal{N}} \subseteq \mathbb{M}$ *satisfies conditions (1)-(5) and (7) and define*

$$\mathcal{O} = \{x \in \mathbb{M} : xI_{\mathcal{N}} \subseteq I_{\mathcal{N}}\}.$$

Then $\mathcal{O}$ *is a valuation ring on* $\mathbb{M}$*,* $I_{\mathcal{N}}$ *is the unique maximal ideal of* $\mathcal{O}$*, and* $I_{\mathcal{N}}$ *satisfies condition (6).*

Proof. (repeated from Johnson (2015), Observation 2.4) Clearly $\mathcal{O}$ is closed under multiplication; it is also closed under addition by condition (3). It contains $I_{\mathcal{N}}$ and K by conditions (4) and (5), respectively; conditions (1), (3), and the definition of $\mathcal{O}$ then give that $I_{\mathcal{N}}$ is a proper ideal of $\mathcal{O}$.

By condition (7), if $x, y \in \mathbb{M} \smallsetminus I_{\mathcal{N}}$ then $xy \in \mathbb{M} \smallsetminus I_{\mathcal{N}}$. Thus, if $x \in \mathbb{M} \smallsetminus I_{\mathcal{N}}$ and $xy \in I_{\mathcal{N}}$, we must have $y \in I_{\mathcal{N}}$. In other words, if $x \notin I_{\mathcal{N}}$ then $x^{-1}I_{\mathcal{N}} \subseteq I_{\mathcal{N}}$, which means $x^{-1} \in \mathcal{O}$. Equivalently, $x^{-1} \notin \mathcal{O}$ implies $x \in I_{\mathcal{N}}$. The first condition tells us that $\mathcal{O}$ is a valuation ring, and the second tells us that $I_{\mathcal{N}}$ is its maximal ideal.

Finally, we know that if $\mathfrak{m}$ is the maximal ideal of some valuation ring then $(1+\mathfrak{m})^{-1} = 1+\mathfrak{m}$, and hence condition (6) holds for $I_{\mathcal{N}}$. □

If (8) holds as well, then the corresponding valuation is henselian by Theorem 4.1.3 of Engler and Prestel (2005). In Johnson (2015), rather than proving $\mathcal{N}$ generates a (t-henselian) topology, he shows that $I_{\mathcal{N}}$ satisfies properties (1)-(5) and (7), then shows separately that the corresponding valuation is henselian. In the next two sections, we will also focus on $I_{\mathcal{N}}$, although we have only shown that it satisfies some of the necessary properties.

3.2 The Johnson Topology

As outlined in Section 3.1, our general approach is to define a set which we hope to prove is the maximal ideal of a henselian valuation ring on K; following Johnson (2015), we will write I_K for this set. In this section, we define I_K to be a type-definable set and show that it satisfies conditions (1), (2), and (4) from Proposition 3.1.6: specifically, $I_K \cap K = \{0\}$, $\{0\} \neq I_K$, and I_K is closed under multiplication by elements of K.

Let K be an $\aleph_1$-saturated field of dp-rank $d < \omega$ and let $\mathbb{M} \succ K$ be a monster model. If $X, Y \subseteq \mathbb{M}$, let $X -_\infty Y$ denote

$$\begin{aligned} X -_\infty Y &= \{c \in \mathbb{M} : \exists^\infty y \in Y \ (y + c \in X)\} \\ &= \{c \in \mathbb{M} : X \cap (Y + c) \text{ is infinite}\} \end{aligned}$$

and let

$$I_K = \bigcap \{X -_\infty X : X \text{ is } K\text{-definable and } \operatorname{dp-rk}(X) = d\}.$$

We call sets of the form $X -_\infty X$ where X is K-definable and $\operatorname{dp-rk}(X) = d$ *basic neighbourhoods of 0*. We call elements of I_K *infinitesimals*. As mentioned, our goal is to prove that I_K is the maximal ideal of a valuation ring, and hence elements of I_K will have positive valuation.

In Dolich and Goodrick (2015), Corollary 2.2, it is shown that dp-finite fields have uniform finiteness; that is, for any formula $\phi(x, \overline{y})$, there is a number $n < \omega$ such that $|\phi(K, \overline{b})| > n$ implies $\phi(K, \overline{b})$ is infinite. Thus, $X -_\infty Y$ is definable whenever X and Y are, and I_K is type-definable over K.

It is worth noting here that in attempting to generalize Johnson's work to the dp-finite case, we could also consider sets of the form

$$X -_* Y = \{c \in \mathbb{M} : \operatorname{dp-rk}(X \cap (Y + c)) = d\}.$$

Unfortunately, these sets are not definable, or even type definable. As a result, using $X -_* X$ instead of $X -_\infty X$ to define I_K would result in I_K also not being type definable, preventing the use of compactness in Proposition 3.2.2 to conclude that I_K is infinite. Showing that the sets $X -_* X$ generate the same topology as $X -_\infty X$ seems to be a key step in proving condition (3) from Proposition 3.1.6, but is currently out of reach.

In this section and the next, we will work under the additional assumption that K does not have the large set property (LSP), that is, that there exists a definable set $D \subseteq \mathbb{M}$ with $\text{dp-rk}(D) = \text{dp-rk}(D^c) = d$. In Section 3.4, we will discuss fields with LSP; morally, they should all be algebraically closed.

Lemma 3.2.1. *Suppose K is a field of dp-rank $d < \omega$ and $X, Y \subseteq K$ are definable. If X and Y both have dp-rank equal to d then so does $X -_\infty Y$.*

Proof. Suppose X and Y are A-definable. Take $(x, y) \in X \times Y$ of dp-rank $2d$ over A, and let $c = x - y$. By subadditivity of dp-rank,

$$2d = \text{dp-rk}(x, y/A) = \text{dp-rk}(y, c/A) \leq \text{dp-rk}(y/Ac) + \text{dp-rk}(c/A) \leq d + d.$$

Then equality must hold, and hence $y \notin \text{acl}(Ac)$ and $c \notin \text{acl}(A)$. As $y \in Y \cap (X - c)$ and $Y \cap (X - c)$ is Ac-definable, $Y \cap (X - c)$ is infinite. It follows that $c \in X -_\infty Y$, which is an A-definable set by uniform finiteness in K. Thus

$$d = \text{dp-rk}(c/A) \leq \text{dp-rk}(X -_\infty Y) \leq d,$$

and again equality holds. □

Proposition 3.2.2. *Suppose K is an $\aleph_1$-saturated field of dp-rank $d < \omega$ without LSP. Then the set of basic neighbourhoods of 0, as defined at the beginning of the section, is filtered. Moreover, $I_K \subseteq \mathbb{M}$ satisfies the following:*

1. *I_K is infinite*
2. *I_K is closed under multiplication by elements of K*
3. *$I_K \cap K = \{0\}$*

Proof. To show that the set of basic neighbourhoods is filtered, we first show that they form a consistent type. Suppose X and Y are A-definable and of maximal dp-rank d. As in the previous lemma, fix $(x, y) \in X \times Y$ of dp-rank $2d$ over A and let $c = x - y$. Set $X' = X - c$. Then $y \in Y \cap X'$ and $\text{dp-rk}(y/Ac) = d$, so $\text{dp-rk}(Y \cap X') = d$. Note that $X' -_\infty X' = X -_\infty X$, so we have

$$(Y \cap X') -_\infty (Y \cap X') \subseteq (Y -_\infty Y) \cap (X' -_\infty X') = (Y -_\infty Y) \cap (X -_\infty X).$$

Thus, every finite intersection of basic neighbourhoods contains a basic neighbourhood, which means they form a filtered set.

1. Since the partial type defining I_K is consistent, I_K is nonempty by saturation of $\mathbb{M}$. Moreover, since $(Y \cap X') -_\infty (Y \cap X')$ is infinite for every choice of X and Y, we must have that I_K is infinite.

2. Clearly $0 \cdot I_K = \{0\} \subseteq I_K$. Suppose $a \in K^\times$. Then for any K-definable X of full dp-rank, we have
$$aX -_\infty aX = a(X -_\infty X)$$
and hence $I_K = aI_K$.

3. We first show that there exists $c \in K \smallsetminus I_K$. Since K does not have LSP, there is a definable set $D \subseteq K$ with $\text{dp-rk}(D) = \text{dp-rk}(D^c) = d$. Taking $X = D^c$ and $Y = D$ in the lemma, we have $(x, y) \in X \times Y$ and $c = x - y$, all of maximal dp-rank. By Proposition B.5, we may assume that these elements are all contained in K by saturation of K.

   Set $Y' = (X - c) \cap Y$; then $y \in Y'$ and hence Y' has full dp-rank. Clearly, $Y' + c \subseteq X$, so
$$Y' \cap (Y' + c) \subseteq Y \cap X = D \cap D^c = \emptyset,$$
and hence $c \notin Y' -_\infty Y'$, which means $c \notin I_K$.

   Now, fix any $b \in K^\times$. Since K is a field and $I_K = bc^{-1}I_K$ by (2), we have
$$b \in bc^{-1}(K \smallsetminus I_K) = (bc^{-1}K) \smallsetminus (bc^{-1}I_K) = K \smallsetminus I_K$$
and so $b \notin I_K$. Thus, $I_K \cap K^\times = \emptyset$, which means $I_K \cap K = \{0\}$.

$\square$

3.3 An Intermediate Conjecture

In Section 2.3, we showed that a strengthening of the assumption "every infinite definable set has full dp-rank" could be used to show that a dp-finite valued field is henselian. The fact that this assumption holds for dp-minimal fields is used to great effect by Johnson in his classification of dp-minimal fields, specifically when showing that the topology generated by basic neighbourhoods is a V-topology.

In this section, we consider Johnson's topology from the previous section, but with the additional assumption that every infinite definable set has full dp-rank. We show that with this additional assumption, we can get most of the way towards the following conjecture:

Conjecture 3.3.1. *The following are equivalent:*

1. *Every infinite dp-finite field is either algebraically closed, real closed, or has a definable henselian valuation.*

2. *Suppose K is a dp-finite field in the language $\mathcal{L}_{\text{ring}}$. Then every infinite definable subset X of K has* $\text{dp-rk}(X) = \text{dp-rk}(K)$.

The proof that (1) implies (2) is straightforward: it follows directly from strong minimality, o-minimality, or Section 2.3, depending on which case we are in.

Conversely, suppose (2) holds, and that K is a dp-finite field with LSP. If $X \subseteq K$ is infinite then $\text{dp-rk}(K \setminus X) < \text{dp-rk}(K)$, which means it must be finite by (2). Thus, every subset of K is either finite or cofinite. Moreover, since dp-finite fields have uniform finiteness (Dolich and Goodrick, 2015, Corollary 2.2), this is also true of every field elementarily equivalent to K. Thus, if K is a dp-finite field with LSP, it is strongly minimal, and hence algebraically closed.

So for the remainder of the section, we consider the case where (2) holds, K is a dp-finite field in the language $\mathcal{L}_{\text{ring}}$ without LSP, and $\mathbb{M} \succeq K$ is a monster model. Following Section 3.2, the set I_K is an infinite type-definable subset of $\mathbb{M}$ such that $I_K \cap K = \{0\}$. Note that by (2), we have

$$\begin{aligned} I_K &= \{\varepsilon \in \mathbb{M} : X \cap (X + \varepsilon) \text{ is infinite for all infinite } K\text{-definable sets } X\} \\ &= \{\varepsilon \in \mathbb{M} : X \cap (X + \varepsilon) \text{ has full dp-rank for all infinite } K\text{-definable sets } X\}. \end{aligned}$$

The equivalence of these two definitions allows us to make a number of conclusions that would be impossible otherwise.

First, consider the following condition on definable bijections of $\mathbb{M}$:

Definition 3.3.2. Let $K \preceq \mathbb{M}$ be dp-finite structures and let $f : \mathbb{M} \to \mathbb{M}$ be a definable bijection. We say that f is *K-slight* if, for every K-definable set X with full dp-rank, $X \cap f(X)$ has full dp-rank.

By assumption (2), $f : \mathbb{M} \to \mathbb{M}$ is K-slight if and only if, for every infinite K-definable set X, $X \cap f(X)$ is infinite. Consider the map $f(x) = x + \varepsilon$. If $\varepsilon \in I_K$ then by definition $X \cap (X + \varepsilon)$ is infinite for all infinite K-definable sets X, so f is K-slight. Conversely, if f is K-slight, then $\varepsilon \in I_K$ by definition, and so $f(x) = x + \varepsilon$ is K-slight if and only if $\varepsilon \in I_K$.

Now the rest of Section 3.1 of Johnson (2015) goes through without any changes, so I_K is closed under addition. Moreover, an analogue of Theorem 9.3.9 of Johnson (2016) also follows:

Proposition 3.3.3. *Let K be an $\aleph_1$-saturated field of dp-rank $d < \omega$ without LSP in $\mathcal{L}_{\text{ring}}$, and assume that every infinite definable subset of K has full dp-rank. Then the family*

$$\mathcal{N} = \{X -_\infty X : X \subseteq K \text{ is infinite and } K\text{-definable}\}$$

determines a non-discrete group topology on $(K, +)$. Moreover, the family

$$\mathcal{N}' = \{X - X : X \subseteq K \text{ is infinite and } K\text{-definable}\}$$

determines the same topology, and the same type-definable set I_K.

Proof. By Proposition 3.2.2, $\mathcal{N}$ determines a non-discrete topology such that the map $x \mapsto -x$ is continuous. The fact that I_K is closed under addition then implies that $\mathcal{N}$ determines a group topology.

To show that $\mathbb{N}$ and $\mathcal{N}'$ determine the same topology, we must show that for every infinite definable set X, there exist Y and Z, both infinite and definable, such that $Y - Y \subseteq X -_\infty X$ and $Z -_\infty Z \subseteq X - X$. Set $Z = X$; clearly, $X -_\infty X \subseteq X - X$.

Fix $U = X -_\infty X$. Since I_K is a group, if $x, y \in I_K$ then $x - y \in I_K \subseteq U$. Thus, $I_K - I_K \subseteq U$, and so by compactness, there exists a basic neighbourhood Y of 0 such that $Y - Y \subseteq U = X -_\infty X$. Thus $\mathcal{N}$ and $\mathcal{N}'$ determine the same topology; the fact that they determine the same type-definable set I_K follows from taking intersections. $\square$

Thus I_K can also be written as

$$\begin{aligned} I_K &= \bigcap\{X - X : X \text{ is } K\text{-definable and infinite}\} \\ &= \{\varepsilon \in \mathbb{M} : X \cap (X + \varepsilon) \text{ is non-empty for all infinite } K\text{-definable sets } X\}. \end{aligned}$$

We now have that I_K satisfies properties (1) through (4) of Proposition 3.1.6. Unfortunately, at this stage the argument becomes conjectural. To show (5), we want to follow Section 4.1 of Johnson (2015), which begins with the following property:

Conjecture 3.3.4. *Let K be an $\aleph_1$-saturated field of dp-rank $d < \omega$ without LSP in $\mathcal{L}_{\text{ring}}$, and assume that every infinite definable subset of K has full dp-rank. Let $X \subseteq K$ be a definable subset of K. Then $\partial X \subseteq K$ is finite.*

Johnson's proof of this result relies on his assertion that in a theory T of dp-minimal fields, the number of infinitesimal types over a model of T (that is, completions of the partial type defining I_K) is bounded only by T (Johnson, 2015, Corollary 4.10). We have been unsuccesful in proving this conjecture without a bound on the number of infinitesimal types, although we believe it can be done. With the conjecture resolved, we could follow the same proof as (Johnson, 2015, Proposition 4.13) to show that I_K is closed under multiplication:

Proposition 3.3.5. *Suppose K is an $\aleph_1$-saturated field of dp-rank $d < \omega$ without LSP in $\mathcal{L}_{\text{ring}}$, and assume that every infinite definable subset of K has full dp-rank. If Conjecture 3.3.4 holds then the set of K-infinitesimals I_K is closed under multiplication.*

We would then have that I_K satisfies conditions (1) through (5) of Proposition 3.1.6. The next step in Johnson's proof is to show that I_K is the maximal ideal of a valuation ring; then I_K also satisfies condition (7), and so by Proposition 3.1.7, the Johnson topology is a V-topology. Unfortunately, we do not yet have a proof of this result for the dp-finite case, even with the assumption that Conjecture 3.3.4 holds. However, some of the consequences follow from Proposition 3.3.5, including a partial results towards a proof that K admits a henselian valuation.

Define $R = \{a \in \mathbb{M} : aI_K \subseteq I_K\}$; clearly, $0, 1 \in R$. If $a, b \in R$ then

$$(a + b)I_K = aI_K + bI_K \subseteq I_K + I_K = I_K$$

and $abI_K \subseteq aI_K \subseteq I_K$, so R is a ring. By the definition of R and the fact that I_K is closed under addition, I_K is an ideal of R. Let I_R be a maximal ideal containing I_K.

Then I_R is prime, and so by the Chevalley Extension Theorem (Engler and Prestel, 2005, Theorem 3.1.1), there is a valuation ring $\mathcal{O}$ with maximal ideal $\mathfrak{m}$ such that $R \subseteq \mathcal{O}$ and $\mathfrak{m} \cap R = I_R \supseteq I_K$. Write v for the valuation corresponding to $\mathcal{O}$.

Consider an expansion of $\mathbb{M}$ by a predicate for $\mathcal{O}$; this expansion might not be dp-finite, and so we want to obtain an approximation for $\mathcal{O}$ in $\mathcal{L}_{\text{ring}}$. The expansion also might not be saturated, but by passing to an elementary extension of $(\mathbb{M}, \mathcal{O})$, we may be able to use compactness in this expanded language to find an $\mathcal{L}_{\text{ring}}$-definable set B such that $I_K \subseteq B \subseteq \mathcal{O}$. Assuming such a set B exists, consider the topology τ generated by $\{rB + a : r, a \in \mathbb{M}\}$. If X is open with respect to the valuation topology, it contains a ball, which we can write as $r\mathcal{O} + a$; clearly $rB + a \subseteq r\mathcal{O} + a$, and so τ would be finer than the valuation topology. But importantly, τ would be uniformly definable in $\mathcal{L}_{\text{ring}}$.

In the dp-minimal case, there is no need to distinguish between τ and the valuation topology, because they are the same. However, even though we do not have equality, we can still obtain a partial results towards a proof of the henselianity of the valuation topology.

Consider the proof of Proposition 2.3.6. The only place that the valuation is used is that the set of balls $B(a, r)$ form a uniformly definable basis for the topology. Replacing each instance of $B(a, r)$ with $rB + a$, the proof follows identically. We would then obtain the following generalization of Corollary 2.3.9:

Conjecture 3.3.6. *Let $\mathbb{M}$ be a very saturated field of dp-rank $d < \omega$ without LSP in $\mathcal{L}_{\text{ring}}$, and assume that every infinite definable subset of K has full dp-rank. Let $f : \mathbb{M}^n \to \mathbb{M}^n$ be a finite-to-one definable function. Then for every set $X \subseteq \mathbb{M}^n$ (not necessarily definable), if X has non-empty interior, then so does $f(X)$.*

Proof. (Assuming the existence of a definable set B with $I_K \subseteq B \subseteq \mathcal{O}$.) Since $X \subseteq \mathbb{M}^n$ has interior, it contains a product X' of sets of the form $rB + a$. Then X' is definable, and since dp-rk$(rB + a) = d =$ dp-rk$(\mathbb{M})$ for any $a, r \in \mathbb{M}$ with $r \neq 0$, we have dp-rk$(X') = nd$. Then dp-rk$(f(X')) = nd$, and so $f(X')$ has interior. It follows that $f(X)$ also has interior. □

As observed in Section 2.3, this corollary is the main step in proving that the corresponding topology is t-henselian. However, this is where the fact that τ is not the valuation topology becomes a problem. Whether we follow Jahnke *et al.* (2017) or Johnson (2015), at some point we need to consider a residue field $\mathbb{M}w$ of some coarsening of the valuation induced by $\mathcal{O}$. But since $\mathcal{O}$ is not definable, there is no guarantee that $\mathbb{M}w$ is dp-finite.

The missing steps towards a proof of Conjecture 3.3.1 are summarized below:

Question 3.3.7. Suppose K is an $\aleph_1$-saturated field of dp-rank $d < \omega$ without LSP in $\mathcal{L}_{\text{ring}}$, and assume that every infinite definable subset of K has full dp-rank.

1. Is it true that every definable set in K has finite boundary?
2. If so, let $\mathcal{O}$ be a valuation ring on $\mathbb{M} \succeq K$ obtained via the Chevalley Extension Theorem applied to $R = \{a \in \mathbb{M} : aI_K \subseteq I_K\}$. Does there exist an $\mathcal{L}_{\text{ring}}$-definable set B such that $I_K \subseteq B \subseteq \mathcal{O}$?
3. If so, let w be any coarsening of the valuation induced by $\mathcal{O}$. Is $\mathbb{M}w$ dp-finite as an $\mathcal{L}_{\text{ring}}$-structure?

3.4 The Stable Case

Throughout this section, let K be a sufficiently saturated dp-finite field. A recent result of Halevi and Palacín (Halevi and Palacín, 2017, Proposition 7.2) shows that if K is stable then K is algebraically closed. Conjecturally, if K is unstable as a pure field, then K is either real closed or has a definable henselian valuation.

Recall that a field has LSP if, for every formula ϕ, exactly one of ϕ and $\neg\phi$ has full dp-rank. The motivation for considering this property is clear in the previous sections, specifically in Proposition 3.2.2. We make the following conjecture, which explains the title of this chapter:

Conjecture 3.4.1. *Suppose K is a dp-finite field in $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$. Then K has LSP if and only if K is stable.*

In this section, we will show a weaker version of the conjecture that is dependent on successful completions of the previous sections. More precisely, we assume the following condition $(**)$:

> Every dp-finite field without LSP is either real closed or has a definable henselian valuation.

Theorem 3.4.2. *Assume that every dp-finite field without LSP is either real closed or has a definable henselian valuation. Then a dp-finite field in $\mathcal{L}_{\text{ring}}$ has LSP if and only if it is stable.*

One direction of the theorem follows immediately from the condition $(**)$. If K is stable, then it cannot have a definable ordering, so it cannot be real closed or have a definable valuation. Hence, under the assumption of the theorem, it must have LSP. The remainder of the section is dedicated to a proof of the other direction.

We begin by showing that finite extensions of fields with LSP also have LSP. This is actually the only place where condition $(**)$ is used. We believe that the following lemma is true without that assumption, but we do not have a proof at this time.

Lemma 3.4.3. *Assume that every infinite dp-finite field without LSP is either real closed or has a definable henselian valuation. Let L/K be a finite extension of fields such that K is dp-finite and has LSP. Then L is dp-finite and has LSP in $\mathcal{L}_{\text{ring}}$.*

Proof. Since L is interpretable in K, it must be dp-finite. Suppose it does not have LSP. Then by assumption, it is either real closed or has a definable henselian valuation.

But the real closure of K is only a finite extension of K if K is real closed itself, and so if L is real closed then we must have $K = L$, contradicting the fact that K has LSP. Similarly, if L has a definable henselian valuation ring $\mathcal{O}$ then $K \cap \mathcal{O}$ will definable in K, contradicting the fact that K has LSP. Thus, if K has LSP then L must also have LSP. □

Now we prove that LSP implies algebraically closed. To do this, we follow the proof that superstable fields are algebraically closed (Cherlin and Shelah (1980), based in turn on Macintyre (1971)). We assume throughout that the conclusion of Lemma 3.4.3 holds.

Lemma 3.4.4. *("Surjectivity Theorem") Suppose K is an infinite dp-finite field with LSP satisfying the conclusion of Lemma 3.4.3. Let G be either the additive or multiplicative group of K, and let $h : G \to G$ be a definable endomorphism. If* $\ker(h)$ *is finite then* $\mathrm{im}(h) = G$*.*

Proof. Write $H = \mathrm{im}(h) \leq G$. Since $\ker(h)$ is finite, h is a finite-to-one map, and so $\text{dp-rk}(H) = \text{dp-rk}(G) = \text{dp-rk}(K)$ by Proposition B.3. Suppose there exists some $a \in G \setminus H$, and let A be the coset $a + H$ or aH, depending on whether G is the additive or multiplicative subgroup of K. In either case, A is a subset of $K \setminus H$ and there is a definable bijection from H to A, so

$$\text{dp-rk}(K) = \text{dp-rk}(H) = \text{dp-rk}(A) \leq \text{dp-rk}(K \setminus H) \leq \text{dp-rk}(K).$$

This clearly contradicts the fact that K has LSP, and so we must have $G = H$. □

Definition 3.4.5. Let L/K be a Galois extension of prime degree q.

- We say that L/K is a *Artin-Schreier extension* if $q = \mathrm{char}(K)$ and L is generated over K by a root of $X^q - X - a$ for some $a \in K$.
- We say that L/K is a *Kummer extension* if $q \neq \mathrm{char}(K)$ and L is generated over K by a root of $X^q - a$ for some $a \in K$.

Lemma 3.4.6. *Let K be an infinite dp-finite field with LSP satisfying the conclusion of Lemma 3.4.3. Then K has no Artin-Schreier or Kummer extensions.*

Proof. The fact that K has no Artin-Schreier extensions actually holds for all NIP fields by Kaplan *et al.* (2011), but we prove it directly using Lemma 3.4.4 in order to keep the proof self-contained.

Suppose $q = \mathrm{char}(K)$ and $h : K \to K$ is the map $h(x) = x^q - x$. This is clearly an additive endomorphism with finite kernel, and so must be surjective by Lemma 3.4.4. It follows immediately that every polynomial of the form $X^q - X - a$ has a root in K, and so K is closed under Artin-Schreier extensions.

Suppose $q \neq \mathrm{char}(K)$ and $h : K^\times \to K^\times$ is the map $h(x) = x^q$. This is clearly a multiplicative endomorphism with finite kernel, and so must be surjective by Lemma 3.4.4. It follows immediately that every polynomial of the form $X^q - a$ has a root in K, and so K is closed under Kummer extensions. □

If K contains qth roots of unity for some prime q, Artin-Schreier and Kummer extensions are the only Galois extensions of degree q; see, for example, Exercise 14.7.9 and Proposition 37 of Dummit and Foote (2003).

Fact 3.4.7. *Let L/K be a Galois extension of prime degree q, and suppose $x^q - 1$ splits in K. Then L/K is either an Artin-Schreier extension or a Kummer extension.*

Combining the above lemma and fact, we can now prove the main theorem of this section. The proof below is based heavily on Theorem 1 of Cherlin and Shelah (1980); another version of essentially the same proof can be found in Theorem 3.1 of Poizat (1987).

Theorem 3.4.8. *Suppose K is an infinite dp-finite field with LSP satisfying the conclusion of Lemma 3.4.3. Then K is algebraically closed.*

Proof. Assume for contradiction that there is an infinite dp-finite field with LSP which is not algebraically closed. Since dp-finite fields are perfect (Johnson, 2015, Observation 2.1), any finite algebraic extension of a dp-finite field is contained in a finite Galois extension. Consider all pairs of fields (K, L) such that K is infinite, dp-finite, and LSP, and L is a finite Galois extension of K. Choose such a pair (K, L) with $\deg(L/K) = q$ minimal. We claim that q is prime and $x^q - 1$ splits in K.

Suppose r is a prime factor of q, and let K' be the fixed field of an element of order r in $\mathrm{Gal}(L/K)$. Since K'/K is finite, K' is dp-finite and has LSP by Lemma 3.4.3, and so (K', L) is a pair as above with degree r. Thus, minimality of q implies that $r = q$, and hence q is prime.

Now let L' be the splitting field for $x^q - 1$ over K. Then the degree of L' over K divides $q - 1$, so again by minimality of q we have $K = L'$.

Thus, L/K is either an Artin-Schreier or a Kummer extension, but K has no proper Artin-Schreier or Kummer extensions. By contradiction, K must be algebraically closed. □

Corollary 3.4.9. *Suppose K is an infinite dp-finite field with LSP satisfying the conclusion of Lemma 3.4.3. Then the reduct of K to the language of rings is stable.*

Proof. By the previous theorem, K is algebraically closed. It is a well-known fact that in the language of rings, algebraically closed fields are stable. In fact, they are strongly minimal by Corollary 3.2.9 of Marker (2002). □

Combining this corollary with Lemma 3.4.3 completes the proof of Theorem 3.4.2. The conjecture was stated only for fields in $\mathcal{L}_{\text{ring}}$, but it could have been stated more generally, as the following questions are currently open:

Question 3.4.10. Is there an algebraically closed field K and an expansion $\mathcal{L}$ of $\mathcal{L}_{\text{ring}}$ such that the $\mathcal{L}$-theory of K is stable, dp-finite, and does not have LSP?

Question 3.4.11. Is there an algebraically closed field K and an expansion $\mathcal{L}$ of $\mathcal{L}_{\text{ring}}$ such that the $\mathcal{L}$-theory of K is unstable, dp-finite, and has LSP?

The most natural expansions of the above form involve adding an ordering (either to K itself, to a definable subset of K, or a quotient of K) and a related topology. For example, if T is ACVF, there is an ordering on the value group, and a topology on the home sort. Similarly, if T is the theory of $\mathbb{C}$ with real and imaginary projections, there is an ordering on $\mathbb{R}$ from which we obtain the usual Euclidean topology on $\mathbb{C}$. The ordering clearly makes these structures unstable, and the topology means they do not have LSP, since the ball around 0 and its complement both have full dp-rank. So the questions could be answered by finding a way to add an ordering without a topology or a topology without an ordering, all while maintaining finite dp-rank.

As pointed out by the external examiner, the first question is likely false: one should be able to fuse a trivial theory of an equivalence relation with two infinite classes with an ACF. This would resulting in a structure of Morley rank 1 that is strongly minimal, negatively answering the question.

Chapter 4

IAC and VAC Fields

This chapter does not focus on dp-rank, but instead on the relationship between two algebraic properties of fields.

Definition 4.0.1. We say that a field K is *immediately algebraically closed* (IAC) if, for every non-trivial valuation v on K, Kv is algebraically closed and vK is divisible. Equivalently, K is immediately algebraically closed if its algebraic closure is an immediate extension with respect to every non-trivial valuation.

We say that a field K is *valuationally algebraically closed* (VAC) if, for every non-trivial valuation v on K, K is dense in its algebraic closure with respect to any extension of v to K^{alg}. Explicitly, given any $a \in K^{\mathrm{alg}}$, any valuation v on K^{alg}, and any $\gamma \in vK^{\mathrm{alg}}$, there is an element $b \in K$ with $v(b-a) > \gamma$.

These definitions are given in Hong's doctoral thesis (Hong, 2013), where he suggested VAC in particular may be useful as an intermediate step in proving the stable field conjecture. They were independently considered by Krupiński (2015), who showed that every superrosy field of positive characteristic is IAC.

It is easy to see that every algebraically closed field is both VAC and IAC. In fact, this is also true for every separably closed field (Engler and Prestel, 2005, Proposition 3.2.11) and every pseudo-algebraically closed field (Fried and Jarden, 2008, Proposition 11.5.3).

One curious difference between IAC and VAC is that the definition of IAC can be made without specifying any valuations on K^{alg}, whereas VAC seems to depend on ranging over all valuations of K^{alg}, not just on valuations on K. However, as we will prove in Corollary 4.2.2, once a valuation v on K is fixed, either all extensions of v to K^{alg} result in a dense embedding, or none do. For this reason, we will often say that

K is dense in K^{alg} "with respect to v," rather than the more accurate "with respect to some extension of v to K^{alg}."

Suppose K is VAC and fix a non-trivial valuation v on K^{alg}. Then for every $a \in K^{\mathrm{alg}}$, there is $b \in K$ with $v(a-b) > v(a)$. Clearly, this means that $v(a) = v(b)$, so $vK = vK^{\mathrm{alg}}$. Moreover, if $v(a) = v(b) = 0$ then a and b belong to the same residue class of Kv^{alg}, and so $Kv = Kv^{\mathrm{alg}}$. Thus, every VAC field is IAC.

In his thesis, Hong asked whether the converse also holds; that is, is every IAC field also VAC (Hong, 2013, Question 5.6.8)? In Section 4.1, we give an example that negatively answers the question. We also explore two situations in which the equivalence holds with an additional assumption: if $\mathrm{char}(K) > 0$ and K is Artin-Schreier closed (in Section 4.1) and if every $K' \equiv K$ is also IAC (in Section 4.2).

4.1 Artin-Schreier Extensions

Throughout this section, we assume that K has characteristic $p > 0$. We want to show that every Artin-Schreier closed IAC field is VAC. Recall that a polynomial of the form $X^p - X - a$ is called an Artin-Schrier polynomial and that a field extension L/K is called an Artin-Schreier extension if L is generated over K by the root of an Artin-Schreier polynomial over K. Note that Artin-Schreier extensions are always Galois: they are clearly separable, and if θ is the root of an Artin-Schreier polynomial, then the full set of roots is $\{\theta, \theta+1, \ldots, \theta+p-1\}$.

As observed previously, every VAC field is automatically IAC; we begin this section by showing that every VAC field is closed under certain Artin-Schreier extensions. These extensions are distinguished by their defect with respect to a particular valuation.

Definition 4.1.1. Let N/K be a Galois extension, and fix a valuation v on N. Let $e = [vN : vK]$ and $f = [Nv : Kv]$, and let r be the number of distinct valuations v' on N with $v'|_K = v$. The *defect* of $(N, v)/(K, v)$ is the positive integer

$$d = \frac{[N : K]}{ref}.$$

The extension $(N, v)/(K, v)$ is called a *defect extension* if $d > 1$, and *defectless* if $d = 1$. See Section 3.3 of Engler and Prestel (2005) for more detail.

The defect plays an important role in the Galois theory of valued fields, but in the case of Artin-Schreier extensions of IAC fields, it simply measures whether the valuation extends uniquely. More precisely, if L is a proper Artin-Schreier extension of an IAC field K then $[L : K] = p$ and $e = f = 1$, so $r = 1$ if and only if $d \neq 1$.

Proposition 4.1.2. *Suppose K is VAC, and fix a valuation v on K. Then (K, v) has no Artin-Schreier defect extensions.*

Proof. Suppose $L = K(\theta)$ is an Artin-Schreier defect extension of (K, v); then there is a unique extension of v to L, which we also denote by v. By Lemma 2.30 of Kuhlmann (2010), $v(\theta - c) < 0$ for all $c \in K$, which means K is not dense in $K(d)$. But K is dense in K^{alg}, which means it must be dense in every algebraic extension of K; by contradiction, no such L can exist. $\square$

We will use this proposition along with the following fact to construct an example of a field that is IAC but not VAC. In particular, we construct an IAC field with an Artin-Schreier defect extension.

Fact 4.1.3. *(Quigley, 1962, Theorem 1) Let K be a field and fix $\alpha \in K^{alg}$. Let M be a subfield of K^{alg} such that M is maximal with respect to the properties $K \subseteq M$ and $\alpha \notin M$. Then the following hold:*

1. *There exists a prime p such that $[N : M]$ is a power of p for every finite normal extension N of M.*
2. *Either M is perfect or K^{alg} is a purely inseparable extension of M.*
3. *$[M(\alpha) : M] = p$ and $M(\alpha)$ is a normal extension of M.*
4. *M contains all pth roots of unity.*

Example 4.1.4. Let K be the algebraic closure of $\mathbb{F}_p$, the finite field with p elements. Since the only valuation on $\mathbb{F}_p$ is the trivial valuation, and algebraic extensions do not increase the rank of the value group, the only valuation on K is the trivial valuation. Then by Theorem 2.1.4 of Engler and Prestel (2005), every non-trivial valuation on $K(t)$ is either the degree valuation v_∞ or an f-adic valuation for some irreducible $f \in K[t]$.

Let $\theta \in K(t)^{\text{alg}}$ be a root of the Artin-Schreier polynomial $X^p - X - t^{-1}$ and let v be any extension of v_∞ to $K(t, \theta)$. Note that $v(\theta) = -p^{-1} \notin \mathbb{Z} = v_\infty K(t)$, and so $[K(t, \theta) : K(t)] = p = [vK(t, \theta) : v_\infty K(t)]$. Then, rearranging the formula for defect, we have $drf = 1$; since all of these values are integers, $r = 1$, which means v_∞ extends uniquely from $K(t)$ to $K(t, \theta)$.

By a straightforward Zorn's Lemma argument, there is a subfield M of K^{alg} which contains K and is maximal with respect to the property $\theta \notin M$. By Fact 4.1.3, since $M(\theta)$ is a separable extension of M, M is perfect and $[N : M]$ is a power of p for every finite normal extension N of M. Then if $c \in M^{\text{alg}}$ with $c^q \in M$ for some prime $q \neq p$, we must have $c \in M$; for otherwise, $M(c)$ would be a finite normal extension of M with order divisible by q, and hence not a power of p. On the other hand, if $c \in M^{\text{alg}}$ with $c^p \in M$ then we must have $c \in M$ since M is perfect. Thus, for any non-trivial valuation v on M, the value group is divisible.

Clearly, the degree valuation on $K(t)$ has residue field K, which is algebraically closed; any f-adic valuation has residue field $K[t]/(f)$, which is an algebraic extension of K and hence equal to K. Since M is an algebraic extension of $K(t)$, it follows that the residue field Mv for any valuation v on M will also be equal to K. Thus, M is immediately algebraically closed.

It remains to show that M is not valuationally algebraically closed. Fix any extension v of v_∞ to M. Since $[M(\theta) : M] = p = [K(t,\theta) : K(t)]$, M and $K(t,\theta)$ are linearly disjoint, meaning any $K(t)$-linearly independent subset of $K(t,\theta)$ is also linearly independent over M. Then, following the argument in Example 4.21 of Kuhlmann (2010), the fact that v_∞ extends uniquely from $K(t)$ to $K(t,\theta)$ implies that v extends uniquely form M to $M(\theta)$. Since M is immediately algebraically closed, this extension of v to $M(\theta)$ must be a defect extension, and so by Proposition 4.1.2, M is not valuationally algebraically closed.

It turns out that Artin-Schreier extensions are the only way that an IAC field can fail to be VAC. To prove this, we use the following result of Macintyre, McKenna, and van den Dries:

Fact 4.1.5. *(Macintyre* et al.*, 1983, Lemma 7) Let (K,v) be a perfect henselian field of positive characteristic such that*

(a) Kv is algebraically closed,

(b) vK is divisible, and

(c) K is closed under Artin-Schreier extensions.

Then K is algebraically closed.

Consider a field K with two valuation rings $\mathcal{O}$ and $\mathcal{O}'$. Recall that $\mathcal{O}'$ is called a coarsening of $\mathcal{O}$ if $\mathcal{O} \subseteq \mathcal{O}'$. In this case, $\mathcal{O}$ and $\mathcal{O}'$ are dependent valuations, and so by Theorem 2.3.4 of Engler and Prestel (2005), they induce the same topology. Moreover, the set of coarsenings of $\mathcal{O}$ are linearly ordered by inclusion, since each coarsening is determined by a convex subgroup of the value group. Determining whether an IAC field is VAC depends only on the valuations that have a maximum non-trivial coarsening, as the following lemma shows.

Lemma 4.1.6. *Suppose K is an IAC field and $\mathcal{O}$ is a valuation ring on K^{alg} such that the set of coarsenings of $\mathcal{O}$ has no maximum non-trivial element. Then K is dense in K^{alg} with respect to the topology induced by $\mathcal{O}$.*

Proof. Let v be the valuation induced by $\mathcal{O}$. For each $\gamma \in vK$, let Δ_γ be the smallest convex subgroup of vK containing γ, and let $\mathcal{O}_\gamma$ be the coarsening of $\mathcal{O}$ corresponding to Δ_γ. Then the maximal ideal $\mathfrak{m}_\gamma$ of $\mathcal{O}_\gamma$ is a subset of $U_\gamma = \{a \in K^{\text{alg}} : v(a) > \gamma\}$.

Fix $a \in K^{\text{alg}}$, and note that $a \in \mathcal{O}_\gamma$ for all $\gamma > |v(a)|$. Then, since $(K^{\text{alg}}, \mathcal{O}_\gamma)$ is an immediate extension of $(K, \mathcal{O}_\gamma \cap K)$, for all $\gamma > |v(a)|$ there exists $b_\gamma \in K$ such that $a - b_\gamma \in \mathfrak{m}_\gamma$; in other words, $v(a - b_\gamma) > \gamma$. Thus, for any $a \in K^{\text{alg}}$ and $\gamma \in vK$ there exists $b \in K$ with $v(a - b) > \gamma$, which means K is dense in $(K^{\text{alg}}, \mathcal{O})$. □

Combining this lemma with some facts about Artin-Schreier extensions, we obtain the main result of the section.

Theorem 4.1.7. *Suppose K is field of positive characteristic which is immediately algebraically closed and Artin-Schreier closed. Then K is valuationally algebraically closed.*

Proof. Fix a valuation v on K^{alg} with valuation ring $\mathcal{O}$. By the lemma, if there is no maximal non-trivial valuation ring containing $\mathcal{O}$, then K is dense in K^{alg} with respect to v.

On the other hand, if there is a maximal non-trivial valuation ring containing $\mathcal{O}$, we may assume that this ring is equal to $\mathcal{O}$ since they induce the same topology on K^{alg}. Then by (Engler and Prestel, 2005, Proposition 2.3.5), $\mathcal{O}$ has rank 1, which means its value group vK is order isomorphic to a subgroup of the reals.

Let L be the completion of K (in the sense of Cauchy sequences) with respect to v. As remarked on page 85 of Engler and Prestel (2005), the completion of every rank 1 valued field is henselian. Moreover, since K is IAC and L is an immediate extension of K, we get that $Lv = Kv$ is algebraically closed and $vL = vK$ is divisible. Lastly, L is perfect by Lemma 4.7 and closed under Artin-Schreier extensions by Lemma 4.8 of Kuhlmann (2010).

Thus, we may apply Fact 4.1.5 to obtain that L is algebraically closed. Since K is dense in L, an algebraically closed field, it must also be dense in K^{alg}. Since this holds for any choice of valuation v, K is valuationally algebraically closed. □

Thus, if we can remove the possibility of K having Artin-Schreier extensions, IAC and VAC are equivalent. One such case is that of NIP fields:

Corollary 4.1.8. *Suppose K is an NIP field of positive characteristic. Then K is immediately algebraically closed if and only if it is valuationally algebraically closed.*

Proof. Suppose K is NIP and immediately algebraically closed. By Kaplan *et al.* (2011), every infinite NIP field is Artin-Schreier closed, so by the theorem, K is valuationally algebraically closed. The converse always holds. $\square$

A more general converse to the theorem is still open. Below are two possible extensions of the results of this chapter:

Question 4.1.9. Is it true that every VAC field of positive characteristic is Artin-Schreier closed? If so, it would follow that a positive characteristic field is VAC if and only if it is IAC and Artin-Schreier closed.

Question 4.1.10. Suppose K is an IAC field of positive characteristic such that for any valuation v on K, K has no Artin-Schreier defect extensions. Does it follow that K is VAC?

4.2 Model Theory of IAC Fields

The previous section focused on IAC and VAC fields almost exclusively as algebraic objects. In this section, we consider some basic model theoretic properties of these fields. We begin with the proof that the density of (K, v) in its algebraic closure does not depend on the extension of v to K^{alg}.

Proposition 4.2.1. *The theory of valued fields that are dense in their algebraic closure is axiomatizable in $\mathcal{L}_{\mathrm{div}}$.*

Proof. For each n, let σ_n be the formula

$$\forall y_0, \ldots, y_{n-1}\ \exists d\ \forall a\ \exists x_1, \ldots, x_n\ \phi_n(\overline{y}, d, a, \overline{x}),$$

where ϕ_n states that $f(x) = x^n + y_{n-1}x^{n-1} + \ldots + y_0$ is irreducible and either:

- $f(x)$ is not separable, or
- for each $i \neq j$, $v(f(x_i)) > v(a)$ and $v(x_i - x_j) < v(d)$ (that is, each x_i is approximately equal to a root of $f(x)$, and if d is chosen correctly, each x_i approximates a different root).

Let T be the union of the axioms for valued fields with $\{\sigma_n : n \in \mathbb{N}\}$. We claim that T is the desired axiomatization.

Suppose K is dense in its algebraic closure; if $f(x) = x^n + y_{n-1}x^{n-1} + \ldots + y_0$ is not separable then $(K, v) \models \sigma_n$. Otherwise, let $\{b_1, \ldots, b_n\}$ be the set of roots of f in K^{alg} and choose $d \in K$ so that $v(b_i - b_j) < v(d)$ for all $i \neq j$. Then given any $a \in K$, choose x_i so that $v(x_i - b_i) > \max\{0, v(a), v(d)\}$. Then

$$v(f(x_i)) = v(x_i - b_1) + \ldots + v(x_i - b_n) > v(a)$$

and $v(x_i - b_i), v(x_j - b_j) > v(d) > v(b_i - b_j)$, so

$$v(x_i - x_j) = v\left((x_i - b_i) + (b_i - b_j) + (b_j - x_j)\right) = v(b_i - b_j) < d.$$

Thus, $(K, v) \models \sigma_n$ for all $n \in \mathbb{N}$, and hence $(K, v) \models T$.

Conversely, suppose K is not dense in (K^{alg}, v) for some extension of v to K^{alg}. By Theorem 11.74 of Kuhlmann (2011), the separable closure K^{sep} of K is dense in its perfect hull, which is of course K^{alg}. Thus, since K is not dense in K^{alg}, it cannot be dense in K^{sep}. Fix an element $b \in K^{\text{sep}}$ such that $\sup\{v(x-b) : x \in K\} = \gamma < \infty$.

Choose $y_0, \ldots, y_{n-1}$ so that $f(x) = x^n + y_{n-1}x^{n-1} + \ldots + y_0$ is the minimal polynomial for b over K, any $d \in K$, and $a \in K$ such that

$$v(a) > n \cdot \max\{\gamma, v(d)\}.$$

(Such an a exists because K is dense in K^{alg}.) Since $b \in K^{\text{sep}}$, $f(x)$ is separable; let $b = b_1, \ldots, b_n$ be the set of roots of $f(x)$. We claim that there are no $x_1, \ldots, x_n$ such that $(K, v) \models \phi_n(\overline{y}, d, a, \overline{x})$, and hence $(K, v) \not\models \sigma_n$.

Suppose for contradiction that there are. Then

$$v(f(x_i)) = v(x_i - b_1) + \ldots + v(x_i - b_n) > v(a) > n \cdot \max\{\gamma, v(d)\}.$$

Thus, for each i there exists $\eta(i)$ such that $v(x_i - b_{\eta(i)}) > \max\{\gamma, v(d)\}$. Since this cannot occur for $b_{\eta(i)} = b$ by choice of γ, by the pigeonhole principle there must be some $i \neq j$ such that $\eta(i) = \eta(j) = k$. Then

$$v(x_i - x_j) = v((x_i - b_k) - (b_k - x_j)) \geq \min\{v(x_i - b_k), v(x_j - b_k)\} > v(d)$$

contradicting the assumption that $v(x_i - x_j) < v(d)$ for all $i \neq j$. Thus, $(K, v) \not\models T$, and so T axiomatizes the theory of valued fields that are dense in their algebraic closure. ☐

Corollary 4.2.2. *Let (K, v) be a valued field and v_1, v_2 extensions of v to K^{alg}. Then (K, v) is dense in (K^{alg}, v_1) if and only if it is dense in (K^{alg}, v_2).*

Proof. Since T from the previous proposition depends only on (K, v) and not on the extension of v to K^{alg}, K is dense in (K^{alg}, v_1) if and only if $(K, v) \models T$ if and only if K is dense in (K^{alg}, v_2). ☐

Discussing density in a first order way requires adding the valuation to the language, as in the proposition above. In general, IAC and VAC are not first order properties in the language of rings. For example, $\mathbb{R}$ is both IAC and VAC (Hong, 2013, Example 5.6.7), but the real closure of $\mathbb{R}(t)$ has a valuation with residue field isomorphic to $\mathbb{R}$, and hence is neither IAC nor VAC. One way to interpret this is that $\mathbb{R}$ is only IAC because it is small; we can avoid cases like this by considering the following strengthening of IAC:

Definition 4.2.3. We say that a field K is strongly IAC if every field $K' \equiv K$ (in the language of rings) is IAC.

As mentioned in the previous section, Krupiński has shown that every superrosy field of positive characteristic is IAC, and hence all such fields are strongly IAC. There are other classes of fields, including supersimple fields and stable fields, that we might hope are also all strongly IAC. This result appears easier to prove than the bolder conjectures that supersimple fields are PAC and stable fields are separably closed, and may be valuable as a stepping stone towards the full conjectures.

Theorem 4.2.4. *Let K be a strongly IAC field, and fix a distinguished valuation ring $\mathcal{O}$ of K. Then $(K, \mathcal{O})$ is dense in its algebraic closure.*

Proof. Consider a chain $\mathcal{K} = \mathcal{K}_0 \preceq \mathcal{K}_1 \preceq \dots$ of elementary extensions $\mathcal{K}_n = (K_n, \mathcal{O}_n)$ of $\mathcal{K} = (K, \mathcal{O})$ such that each $\mathcal{K}_{n+1}$ is $|vK_n|^+$-saturated. Then $\mathcal{K}' = \bigcup_n \mathcal{K}_n$ is an elementary extension of $\mathcal{K}$ with valuation ring $\mathcal{O}' = \bigcup_n \mathcal{O}_n$. Moreover, since for each n, $\mathcal{K}_n$ contains a realization of the partial type $\pi(x) = \{v(x) > v(a) : a \in K_{n-1}\}$, there is a proper convex subgroup $\Delta_n < vK_n$ which contains vK_{n-1}.

Suppose $\mathcal{O}'$ has a maximal proper overring $\mathcal{O}''$. Then there exists $x \in K'$ such that for all $y \in K'$, there is $n \in \mathbb{N}$ such that $v(y) < n \cdot v(x)$. But if $x \in K'$ then $x \in K_n$ for some n, and by assumption, there exists $y \in K_{n+1} \subseteq K'$ such that $v(y) > n \cdot v(x)$ for all $n \in \mathbb{N}$, and hence $\mathcal{O}'$ has no maximal proper overring. Applying Lemma 4.1.6, $(K', \mathcal{O}')$ is dense in its algebraic closure, and hence by elementary equivalence, $(K, \mathcal{O})$ is dense in K^{alg}. □

Corollary 4.2.5. *Every strongly IAC field is VAC.*

Proof. By the theorem, if K is strongly IAC then it is dense in its algebraic closure with respect to every valuation, and hence is VAC. □

Corollary 4.2.6. *Every superrosy field of positive characteristic is VAC.*

Proof. If K is superrosy of positive characteristic, then so is every $K' \equiv K$. By Krupiński (2015), every such field is IAC. Thus K is strongly IAC, and so by the previous corollary, K is VAC. □

We conclude with some open questions related to IAC and VAC fields:

Question 4.2.7. (Hong, 2013, Question 5.6.10) Is every infinite stable field VAC? By the results of this section, this is equivalent to asking: is every infinite stable field IAC?

Question 4.2.8. (Krupiński, 2015, Conjecture 2) Is every infinite superrosy field with NIP either algebraically closed or real closed?

Question 4.2.9. (Hong, 2013, Question 5.6.11) Is every infinite stable VAC field separably closed?

The second two questions could be generalized to the following:

Question 4.2.10. Under what additional assumptions is a VAC field PAC? Separably closed? Algebraically closed?

Appendix A

Valuations and Angular Components

This appendix contains several facts about the relationship between valuations and angular components. They should be familiar to anyone with an in-depth understanding of valued fields, but we provide them here for the sake of completeness. They are essential to several results in this thesis, particularly those in Chapter 2.

First, we observe that for an individual element the valuation and angular component of an element are independent of each other, in the following sense.

Lemma A.1. *Suppose (K, v) is a valued field. For every $\gamma \in vK$ and $r \in Kv^\times$, there exists $a \in K$ with $v(a) = \gamma$ and $\mathrm{ac}(a) = r$.*

Proof. Since the valuation and residue maps are surjective, there must exist $b, c \in K$ with $v(b) = \gamma$ and $c + \mathfrak{m} = r$. For the same reason, there must be $d \in K$ with $d + \mathfrak{m} = \mathrm{ac}(b)$. Since $r \neq 0$, we must have $v(c) = 0$; similarly, since $v(b) \neq \infty$, $\mathrm{ac}(b) \neq 0$, and so $v(d) = 0$. Then take $a = bcd^{-1}$. We have

$$v(a) = v(b) + v(c) - v(d) = \gamma + 0 - 0 = \gamma$$

and

$$\mathrm{ac}(a) = \mathrm{ac}(b)\,\mathrm{ac}(c)\,\mathrm{ac}(d)^{-1} = \mathrm{ac}(b) r\,\mathrm{ac}(b)^{-1} = r$$

as desired. ∎

The next two lemmas demonstrate how addition and subraction interact with valuations and angular components.

Lemma A.2. *Suppose (K, v) is a valued field and $a, b, c \in K^\times$.*

1. *If $v(a) < v(b)$ then $v(a+b) = v(a)$ and $\mathrm{ac}(a+b) = \mathrm{ac}(a)$.*
2. *If $v(a) = v(b)$ then $\mathrm{ac}(a) \neq \mathrm{ac}(b)$ if and only if $v(a) = v(a-b)$.*
3. *If $v(a) = v(b)$ and $\mathrm{ac}(a) \neq \mathrm{ac}(b)$ then $\mathrm{ac}(a-b) = \mathrm{ac}(a) - \mathrm{ac}(b)$.*

Proof.

1. From the definition, we have $v(a+b) \geq \min\{v(a), v(b)\} = v(a)$. Suppose $v(a+b) > v(a)$. Then

$$v(a) = v((a+b) - b) \geq \min\{v(a+b), v(-b)\} = \min\{v(a+b), v(b)\} > v(a)$$

which is impossible, so we must have $v(a+b) = v(a)$. Moreover, since angular component maps are multiplicative, we have

$$\mathrm{ac}(a+b) = \mathrm{ac}(a)\,\mathrm{ac}(a^{-1})\,\mathrm{ac}(a+b) = \mathrm{ac}(a)\,\mathrm{ac}(a^{-1}a + a^{-1}b) = \mathrm{ac}(a)\,\mathrm{ac}(1 + a^{-1}b).$$

Note that $v(a) < v(b)$ implies $a^{-1}b \in \mathfrak{m}$, so $\mathrm{ac}(1 + a^{-1}b) = 1 + \mathfrak{m} = \mathrm{ac}(1)$, which means $\mathrm{ac}(a+b) = \mathrm{ac}(a)\,\mathrm{ac}(1) = \mathrm{ac}(a)$.

2. Multiplying a and b by a constant if necessary, we may assume $v(a) = v(b) = 0$. If $\mathrm{ac}(a) = \mathrm{ac}(b)$ then

$$(a-b) + \mathfrak{m} = (a + \mathfrak{m}) - (b + \mathfrak{m}) = \mathrm{ac}(a) - \mathrm{ac}(b) = \mathrm{ac}(b) - \mathrm{ac}(b) = 0 + \mathfrak{m},$$

which means $a - b \in \mathfrak{m}$, and hence $v(a-b) > 0 = v(a)$. Conversely, suppose $\mathrm{ac}(a) \neq \mathrm{ac}(b)$. Then

$$0 + \mathfrak{m} \neq \mathrm{ac}(a) - \mathrm{ac}(b) = (a + \mathfrak{m}) - (b + \mathfrak{m}) = (a - b) + \mathfrak{m}$$

which means $v(a-b) \leq 0$. But $v(a-b) \geq \min\{v(a), v(b)\} = 0$, and so we must have $v(a-b) = 0 = v(a)$.

3. As in (2), we may assume that $v(a) = v(b) = 1$. Since $\mathrm{ac}(a) \neq \mathrm{ac}(b)$, we also have $v(a-b) = v(a) = 0$ by (2). Then

$$\mathrm{ac}(a-b) = (a-b) + \mathfrak{m} = (a + \mathfrak{m}) - (b + \mathfrak{m}) = \mathrm{ac}(a) - \mathrm{ac}(b)$$

as desired. □

Lemma A.3. *Suppose (K, v) is a valued field and $a, b, c \in K^\times$.*

1. *If $v(a-b) < v(c-b)$ then $v(a-b) = v(a-c)$ and $\mathrm{ac}(a-b) = \mathrm{ac}(a-c)$.*
2. *If $v(a-b) = v(a-c)$ then $\mathrm{ac}(a-b) \neq \mathrm{ac}(a-c)$ if and only if $v(a-b) = v(c-b)$.*
3. *If $v(a-b) = v(a-c) = v(c-b)$ then $\mathrm{ac}(a-c) = \mathrm{ac}(a-b) - \mathrm{ac}(c-b)$.*

Proof.

1. We have $v(a-c) = v((a-b)-(c-b)) = \min\{v(a-b), v(c-b)\} = v(a-b)$. Suppose for contradiction that $\mathrm{ac}(a-b) \neq \mathrm{ac}(a-c)$. Then by Lemma A.2(2), $v(c-b) = v((a-b)-(a-c)) = v(a-b)$, contradicting our original assumption. Thus, $\mathrm{ac}(a-b) = \mathrm{ac}(a-c)$.

2. Since $c - b = (a-b) - (a-c)$, this follows immediately from Lemma A.2(2) with a and b replaced by $a-b$ and $a-c$, respectively.

3. Since $a - c = (a-b) - (c-b)$, by Lemma A.2(2) we have $\mathrm{ac}(a-b) \neq \mathrm{ac}(c-b)$. Then by Lemma A.2(3) we have

$$\mathrm{ac}(a-c) = \mathrm{ac}((a-b)-(c-b)) = \mathrm{ac}(a-b) - \mathrm{ac}(c-b)$$

as desired. □

Appendix B

Dp-rank

This appendix contains a number of facts about dp-rank that are used in the thesis, often without citation. None of the results are new, but proofs of the results are generally not included in the literature; we provide them here for the sake of completeness. We begin with a number of consequences of Facts 1.4.2 and 1.4.3 that are fundamental to the usefulness of dp-rank as a notion of dimension.

Proposition B.1. *Let M be a sufficiently saturated model of some theory T, and let X and Y be type-definable with parameters from some small set $A \subseteq M$.*

1. $\text{dp-rk}(X) = \sup_{x \in X}(\text{dp-rk}(x/A))$
2. $\text{dp-rk}(c/A) = 0$ *if and only if c is algebraic over A*
3. $\text{dp-rk}(X) = 0$ *if and only if X is finite*
4. $\text{dp-rk}(X \cup Y) = \max\{\text{dp-rk}(X), \text{dp-rk}(Y)\}$
5. $\text{dp-rk}(X \times Y) = \text{dp-rk}(X) + \text{dp-rk}(Y)$

Proof.

1. By definition, $\text{dp-rk}(X)$ is the supremum of the depths of randomness patterns for X. By Fact 1.4.2, there is a randomness pattern of depth κ if and only if $\text{dp-rk}(x/A) = \kappa$ for some $x \in X$. Thus, $\text{dp-rk}(X) = \sup_{x \in X}(\text{dp-rk}(x/A))$.

2. Suppose $c \in \text{acl}(A)$. If I is indiscernible over A then I is also indiscernible over $\text{acl}(A)$; in particular, I is indiscernible over the finitely many realizations of $\text{tp}(c/A)$. Then by Fact 1.4.2 again, there is no randomness pattern of depth 1 for $\text{tp}(c/A)$. In other words, $\text{dp-rk}(c/A) = 0$.

Conversely, suppose $c \notin \operatorname{acl}(A)$ and enumerate infinitely many elements of $\operatorname{tp}(c/A)$ as $(c_i)_{i<\omega}$. Then the formula "$x = y$" with the sequence $(c_i)_{i<\omega}$ forms a randomness pattern of depth 1 for $\operatorname{tp}(c/A)$, and hence $\text{dp-rk}(c/A) \neq 0$.

3. This follows immediately from (1), (2), and the fact that X is finite if and only if every $x \in X$ is algebraic over A.

4. We have

$$\begin{aligned} \text{dp-rk}(X \cup Y) &= \sup_{x \in X \cup Y} (\text{dp-rk}(x/A)) \\ &= \max \left\{ \sup_{x \in X} (\text{dp-rk}(x/A)),\ \sup_{x \in Y} (\text{dp-rk}(x/A)) \right\} \\ &= \max\{\text{dp-rk}(X), \text{dp-rk}(Y)\}. \end{aligned}$$

5. Suppose $x \in X$ and $y \in Y$. Then $\text{dp-rk}(x, y/A) \leq \text{dp-rk}(x/Ay) + \text{dp-rk}(y/A)$ by Fact 1.4.3; taking suprema, we obtain

$$\sup_{(x,y) \in X \times Y} (\text{dp-rk}(x, y/A)) \leq \sup_{x \in X} (\text{dp-rk}(x/Ay)) + \sup_{y \in Y} (\text{dp-rk}(y/A)).$$

By (1), this is exactly

$$\text{dp-rk}(X \times Y) \leq \text{dp-rk}(X) + \text{dp-rk}(Y).$$

For the reverse inequality, suppose that we have a pair of randomness patterns $(\phi_\alpha(x, z_\alpha), c_\alpha)_{0 \leq \alpha < \kappa}$ and $(\phi_\alpha(y, z_\alpha), c_\alpha)_{\kappa \leq \alpha < \kappa + \lambda}$ for X and Y, respectively. Then, taking $\psi_\alpha(x, y, z_\alpha)$ to be ϕ_α with an extra dummy variable, the array $(\psi_\alpha(x, y, z_\alpha), c_\alpha)_{0 \leq \alpha < \kappa + \lambda}$ is a randomness pattern for $X \times Y$. Thus, we have $\text{dp-rk}(X \times Y) \geq \kappa + \lambda$, and taking suprema again,

$$\text{dp-rk}(X \times Y) \geq \text{dp-rk}(X) + \text{dp-rk}(Y). \qquad \square$$

Dp-rank also behaves nicely with respect to definable functions. Before proving those results, we need a lemma about interaglebraic tuples.

Lemma B.2. *Suppose x and y are tuples that are interalgebraic over some set A, that is $y \in \operatorname{acl}(Ax)$ and $x \in \operatorname{acl}(Ay)$. Then $\text{dp-rk}(x/A) = \text{dp-rk}(y/A)$.*

Proof. From the definition of dp-rank, it is clear that dp-rk$(y/A) \leq$ dp-rk$(x, y/A)$. Then by sub-additivity and Proposition B.1(2),

$$\text{dp-rk}(y/A) \leq \text{dp-rk}(x, y/A) \leq \text{dp-rk}(x/Ay) + \text{dp-rk}(y/A) = \text{dp-rk}(y/A),$$

and so dp-rk$(y/A) =$ dp-rk$(x, y/A)$. But by a symmetric argument, we also have dp-rk$(x/A) =$ dp-rk$(x, y/A)$; thus, equality holds as desired. □

Proposition B.3. *Suppose $f : X \to Y$ is a definable function, with X and Y type-definable. Let A be a set of parameters over which f, X, and Y are all defined.*

1. *If f is surjective then* dp-rk$(X) \geq$ dp-rk(Y)
2. *If f is finite-to-one then* dp-rk$(X) \leq$ dp-rk(Y)
3. *If f is a bijection then* dp-rk$(X) =$ dp-rk(Y)

Proof.

1. We use the mutually indiscernible sequences definition of dp-rank. As in the definition, fix any set of mutually indiscernible sequences over A and $y \in Y$ such that none of the sequences are indiscernible over Ay. Fix any $x \in X$ with $f(x) = y$. Then the sequences are also not indiscernible over Ax, simply by replacing y with $f(x)$ in any formula. Thus,

$$\text{dp-rk}(X) = \sup_{x \in X} \text{dp-rk}(x/A) \geq \sup_{y \in Y} \text{dp-rk}(y/A) = \text{dp-rk}(Y).$$

2. Because f is finite-to-one, every $x \in X$ is interalgebraic with $f(x)$. Thus, by the lemma,

$$\text{dp-rk}(X) = \sup_{x \in X} \text{dp-rk}(x/A) = \sup_{y \in \text{im}(f)} \text{dp-rk}(y/A) \leq \text{dp-rk}(Y).$$

3. This follows immediately from (1) and (2). □

Dp-rank can also be used as a sort of independence notion for tuples in dp-finite structures, by defining a tuple to be independent if it has full dp-rank. This is not a particularly robust notion of independence (among other things, it does not satisfy the exchange property), but it does at least satisfy the hereditary property: if a tuple a is independent then every subtuple of a is independent over the other elements.

Proposition B.4. *Suppose M is a structure of dp-rank $d < \omega$. Suppose $(a_1, \ldots, a_n)$ has dp-rank nd over some parameter set S. Then for all permutations σ of $\{1, \ldots, n\}$ and all $1 \leq m < n$, we have*

$$\text{dp-rk}(a_{\sigma(1)} \ldots a_{\sigma(m)}/a_{\sigma(m+1)} \ldots a_{\sigma(n)}S) = md$$

Proof. Rearranging the tuple if necessary, we may assume σ is the identity map. Then by sub-additivity of dp-rank,

$$\begin{aligned} nd &= \text{dp-rk}(a_1 \ldots a_n/S) \\ &\leq \text{dp-rk}(a_1 \ldots a_m/Sa_{m+1} \ldots a_n) + \text{dp-rk}(a_{m+1} \ldots a_n)/S) \\ &\leq md + (n-m)d \\ &\leq nd \end{aligned}$$

so we must have equality, and the result follows. □

Recall that a partial type $\pi(x)$ has dp-rank r if and only if it contains an element of dp-rank r. We conclude this appendix with a result giving a bound on the saturation of a model M required to deduce that the element witnessing dp-rk(π) is an element of M.

Proposition B.5. *Suppose M is an $\aleph_1$-saturated structure, $S \subseteq M$ is at most countable, and $\pi(x)$ is a type over S of dp-rank $r < \omega$. Then there exists $b \in M$ such that $M \models \pi(b)$ and* dp-rk$(b/S) =$ dp-rk(π).

Proof. Let N be a very saturated elementary extension of M and let $a \in N$ be a realization of $\pi(x)$ such that dp-rk$(a/S) =$ dp-rk(π), and choose a set $\{I_1, \ldots, I_r\}$ of sequences as in the mutually indiscernible sequences definition of dp-rank. That is, the sequences are mutually indiscernible over S, but none of the sequences are indiscernible over Sa. For each $t < r$, let $\phi_t(x, y)$ be a formula with parameters in S such that $M \models \phi_t(a, c_t)$ and $M \models \neg\phi_t(a, d_t)$ for some finite subsequences $c_t, d_t \subseteq I_t$.

Write $a_{i,j}$ for the jth element of the sequence I_i. By saturation of M, there exist $(b_{1,0}, \ldots, b_{r,0})$ in M such that

$$\text{tp}(b_{1,0} \ldots b_{r,0}/Sa) = \text{tp}(a_{1,0} \ldots a_{r,0}/Sa).$$

Then, there exist $(b_{1,1}, \ldots, b_{r,1})$ in M such that

$$\text{tp}(b_{1,0}b_{1,1} \ldots b_{r,0}b_{r,1}/Sa) = \text{tp}(a_{1,0}a_{1,1} \ldots a_{r,0}a_{r,1}/Sa).$$

Continuing in this manner, we obtain sequences $I'_t = (b_{t,i})_{i<\omega}$ in M that we claim are mutually indiscernible over S but that are not indiscernible over Sa.

Suppose $\psi(y)$ is a formula with parameters in Sa, e is a finite subset of $I_1 \cup \ldots \cup I_r$, and e' is the corresponding finite subset of $I'_1 \cup \ldots \cup I'_r$. Then by choice of I'_t,

$$N \models \psi(e) \Leftrightarrow N \models \psi(e').$$

In particular, this means that the sequences must be mutually indiscernible over S, since a witness for $\{I'_1, \ldots, I'_r\}$ failing to be mutually indiscernible would also be such a witness for $\{I_1, \ldots, I_r\}$. Moreover, it means

$$N \models \phi_t(a, c'_t) \wedge \neg\phi_t(a, d'_t)$$

and so none of the sequences are indiscernible over Sa.

Now, consider the partial type

$$\pi'(x) = \pi(x) \cup \{\phi_t(x, c'_t) \wedge \neg\phi_t(a, d'_t) : 0 \leq t \leq r\}.$$

This partial type is realized by a, so it is consistent, and has countably many parameters, namely S and a finite subset of the indiscernible sequences. Thus, by saturation of M, there is some $b \in M$ that realizes $\pi'(x)$. But then b is a realization of $\pi(x)$ such that I'_t is not indiscernible over Sb for any t, and so dp-rk$(b/S) = r =$ dp-rk(π), as desired. $\square$

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