Pricing Asian Options and Basket Options by Monte Carlo Methods

PRICING ASIAN OPTIONS AND BASKET OPTIONS BY MONTE CARLO METHODS

BY

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This thesis is dedicated to my parents

Abstract

In this thesis, we investigate pricing Asian options and basket options under different Monte Carlo methods. It is observed that the prices of Asian options and basket options are based on the combinations of stocks prices, while the stocks follow a geometric Brownian motion (GBM). For the price of Asian options, a benchmark price is computed first. A partial differential equations (PDE) (one dimension in time and one in space) due to Večeř with the constant volatility of Asian call option is numerically solved and gives the option prices which we use as a benchmark. After that, three Monte Carlo methods are used to simulate Asian option prices: naive Monte Carlo, antithetic Monte Carlo and control variate. Comparing them with the benchmark and by evaluating the absolute error, mean square error and computation time, we eventually find that control variate method is the most efficient method for pricing Asian options. Next, to price basket options, we choose two different control variate, a classical one and a novel one. After applying these two control variates, we evaluate the performance by mean square error, length of 95% confidence interval and computation time. Taking all factors into consideration, the new control variate is more useful for pricing basket options.

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Chapter 1

Introduction

Options are a type of derivative security with their prices intrinsically linked to the price of underlying assets. Basically, options grant the holders the right but not the obligation to buy or sell the corresponding underlying asset at an appointed price on or before a certain date. Among these, Asian options are securities whose payoff includes the time average of the underlying stock prices. The prices of Asian options examined in the chapter 2 of this thesis will be calculated using arithmetic mean of the underlying in option's payoff. There are some important differences of properties between arithmetic average price and geometric average price, see Zhang (1998)[1]. The property of distribution of the stock price makes it possible to obtain an analytical formula for the geometric average Asian option price. However, it is impossible to express the prices of Asian options in a closed form formula when option's payoff depends on arithmetic underlying asset prices mean, see Curran (1994)[22]. Therefore, a numerical method has to be used in pricing arithmetic Asian option: Monte Carlo can be such a method. Although the analytical evaluation of a statistic is based on its sampling distribution, nothing can be done when there is no strong theory regarding the statistic. That is why Monte Carlo is significant, because it provides an alternative way to evaluate the statistic in random samples, see Mooney (1997)[2]. In order to obtain a more accurate estimator, a large number of samples of the underlying asset price path has to be simulated. Because improving accuracy of estimator is very time consuming, some variance reduction techniques are used to increase the efficiency of Monte Carlo methods, see Wiklund (2012)[3]. Naive Monte Carlo is the most straightforward way to do the simulation. Improvements for naive Monte Carlo method are antithetic variates and control variates, which can reduce the variance to some extent, see Mehrdoust and Vajargah (2012)[13]. In order to evaluate the performance of different Monte Carlo methods, the Večeř approach of pricing Asian options will be used as a benchmark (in his approach the price of the Asian option is characterized by a simple one-dimensional PDE) applied to both discrete and continuous cases, see Večeř (2001)[10].

Chapter 3 of the thesis deals with pricing of basket options. A basket option is an option on a collection or basket of assets, typically stocks. It gives the holder the right but not the obligation to purchase a prespecified fixed portfolio of stocks at a fixed strike price, see Milevsky and Posner (1998)[24]. Their payoff depends on the arithmetic weighted average of the underlying asset prices and there is no closed form solution for the price of basket options, see Dingeç and Hörmann (2013)[24]. There are some approximations available for their prices, see Ju (2002)[4], Deelstra (2010)[5] and Zhou and Wang (2008)[6]. Apart from the naive Monte Carlo, Dingeç and Hörmann (2013)[24] provides a classical control variate and a novel control variate for pricing arithmetic average basket options. In chapter 3 by choosing a conditional geometric average price as a control variate, we show that the efficiency and accuracy of the simulated basket option prices are greatly improved.

Chapter 2

Asian Option

2.1 Introduction of Asian Option

Asian option payoff includes the time average of the underlying asset price. The average may be over the entire time period between initiation and expiration or may be over some period of the time that begins later than the initiation of the option and ends with the option's expiration. The average may be continuous,

$$\frac{1}{T}\int_0^T S(t)dt, \tag{2.1}$$

or discrete,

$$\frac{1}{m}\sum_{j=1}^{m} S(t_j), \tag{2.2}$$

where $0 < t_1 < t_2 < \cdots < t_m = T$. Based on an average asset price, Asian option payoff is more difficult to manipulate.

To derive the partial differential equations for Asian option price, assume the underlying stock price follows a geometric Brownian motion (GBM)

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t), \tag{2.3}$$

where $\widetilde{W}(t)$, $0 \leq t \leq T$, is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Taking fixed-strike Asian call as an example, we can adapt the arguments to treat Asian put option easily. For a continuously sampled fixed-strike Asian call, the payoff at time T is

$$V(T) = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+, \tag{2.4}$$

and for a discrete sampled fixed-strike Asian call, the payoff at time T is

$$V(T) = \left(\frac{1}{m}\sum_{j=1}^{m} S(t_j) - K\right)^+, \tag{2.5}$$

where the strike price K is a nonnegative constant. The price at time t prior to the expiration time T of this call is given by the risk-neutral pricing formula

$$V(t) = \widetilde{\mathbb{E}}[e^{-r(T-t)}V(T)|\mathcal{F}(t)], 0 \leq t \leq T. \tag{2.6}$$

According to the usual iterated conditioning argument, for all $0 \leq t < s \leq T$,

$$\begin{aligned}
\widetilde{\mathbb{E}}[e^{-rs}V(s)|\mathcal{F}(t)] &= \widetilde{\mathbb{E}}\{\widetilde{\mathbb{E}}[e^{-rT}V(T)|\mathcal{F}(s)]|\mathcal{F}(t)\} && (2.7)\\
&= \widetilde{\mathbb{E}}[e^{-rT}V(T)|\mathcal{F}(t)] && (2.8)\\
&= e^{-rt}V(t). && (2.9)
\end{aligned}$$

Therefore the discounted price process $\{e^{-rt}V(t)\},\ t \geq 0$ is a martingale under $\widetilde{\mathbb{P}}$.

2.2 Partial Differential Equations of Asian Option Prices

2.2.1 Augmentation of the State for Continuously Sampled Asian Fixed Strike Call

Because the Asian option payoff $V(T)$ in (2.4) is path-dependent and $V(T)$ depends on the whole path of the stock, $V(t)$ is not a function of t and $S(t)$ only. Therefore, in order to augment the state $S(t)$, define a second process

$$Z(t) = \int_0^t S(u)du \tag{2.10}$$

The stochastic differential equation for $Z(t)$ is thus

$$dZ(t) = S(t)dt. \tag{2.11}$$

Corollary 2.2.1. *Diffusions are Markov processes.*

Governed by the pair of stochastic differential equations

$$\begin{cases} dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t) \\ dZ(t) = S(t)dt, \end{cases}$$

the two dimensional processes $\{(S(t), Z(t))\},\ t \geq 0$ is a diffusion, so by the above Corollary 2.2.1 the two dimensional process is Markov process. Furthermore, the call payoff $V(T)$ depends only on T and $Z(T)$, by the formula

$$V(T) = \left(\frac{1}{T}Z(T) - K\right)^{+}, \tag{2.12}$$

then the call payoff $V(T)$ is a function of T and $(S(T), Z(T))$. This implies that there must exist some function $v(t, s, z)$ such that Asian call price $V(t)$ in (2.6) is given as

$$\begin{aligned} V(t) = v(t, S(t), Z(t)) &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T)|\mathcal{F}(t)\right] && (2.13)\\ &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\frac{1}{T}Z(T) - K\right)^{+}|\mathcal{F}(t)\right] && (2.14)\end{aligned}$$

The function $v(t, s, z)$ satisfies a partial differential equation (PDE) as stated in the next theorem.

Theorem 2.2.2. *The Asian call price function v(t,s,z) of* (2.13) *satisfies the partial differential equation*

$$v_t(t, s, z) + rsv_s(t, s, z) + sv_z(t, s, z) + \frac{1}{2}\sigma^2 s^2 v_{ss}(t, s, z) = rv(t, s, z), \tag{2.15}$$

where $0 \leq t < T, s \geq 0, z \in (0, +\infty)$, *and the three boundary conditions*

$$\begin{aligned} v(t, 0, z) &= e^{-r(T-t)}(\frac{z}{T} - K)^{+}, 0 \leq t < T, z \in \mathbb{R}, && (2.16)\\ \lim_{z\downarrow 0} v(t, s, z) &= 0, 0 \leq t < T, s \geq 0, && (2.17)\\ v(T, s, z) &= (\frac{z}{T} - K)^{+}, s \geq 0, z \in \mathbb{R} && (2.18)\end{aligned}$$

Proof. Cause under the risk-neutral measure $\widetilde{\mathbb{P}}$, $dtdt = 0, dtd\widetilde{W} = 0$, $dS(t)dZ(t) = dZ(t)dZ(t) = 0$. Taking the differential of the $\widetilde{\mathbb{P}}$ martingale $e^{-rt}V(t) = e^{-rt}v(t, S(t), Z(t))$, we can obtain that

$$\begin{aligned} & d(e^{-rt}v(t, S(t), Z(t))) \\ & = e^{-rt}[-rv + v_t + rSv_s + Sv_z + \frac{1}{2}\sigma^2 S^2 v_{ss}]dt + e^{-rt}\sigma S v_s d\widetilde{W}(t), \end{aligned} \tag{2.19}$$

where $e^{-rt}v(t, S(t), Z(t))$ is a martingale, then the dt term in this formula must be zero, hence

$$\begin{aligned} v_t(t, S(t), Z(t)) + rS(t)v_s(t, S(t), Z(t)) + S(t)v_z(t, S(t), Z(t)) \\ + \frac{1}{2}\sigma^2 S^2(t) v_{ss}(t, S(t), Z(t)) = rv(t, S(t), Z(t)). \end{aligned} \tag{2.20}$$

After replacing $S(t)$ and $Z(t)$ by the dummy variables s and z respectively, we obtain (2.15).

Noting that the stock price $S(t)$ must always be nonnegative, we have $s \geq 0$. For the boundary conditions, if $S(t) = 0$ and $Z(t) = z$ for some value of t, then by Black-Scholes-Merton Equation, $S(u) = 0$ for all $u \in [t, T]$, therefore $Z(T) = Z(t) = z$, and the value of the Asian call at time T is $(\frac{z}{T} - K)^+$. Then discounted from T to t, (2.16) can be given. If we hold s fixed and let $z \to -\infty$, then $Z(T)$ goes to $-\infty$, which means the payoff of the call option on the expiring date will approach zero. So by (2.14) the boundary condition there is (2.17). The last one (2.18) is the payoff of the call. □

However, solving 2.15 may be cumbersome due to the 2-dimensionality in the space of the equation. The reduction of dimensionality is required to simplify PDE. That is why Večeř approach will be introduced next.

2.2.2 The Večeř Dimensionality Reduction Approach for Continuously Sampled Asian Fixed Strike Call

First, considering the case of a continuously sampled Asian call with the average from time $T-c$ to the option's expiration T, the payoff is

$$V(T) = \left(\frac{1}{c}\int_{T-c}^{T} S(t)dt - K\right)^{+}, \tag{2.21}$$

here c is a constant satisfying $0 < c \leq T$ and K is a nonnegative constant. While admitting the possibility that the average may be over less than the entire time period, the case in Theorem 2.2.2 is a special one included in this example. To price this call, a hedging portfolio process with the value at time T

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(u)du - K \tag{2.22}$$

is created. Suppose $\gamma(t)$ is a nonrandom function of time t denoting the number of stocks held in the hedging portfolio. It holds that $d\gamma(t)d\gamma(t) = d\gamma(t)dS(t) = 0$.
To replicate the Asian call with payoff (2.21), set $\gamma(t)$ with $r \neq 0$ as

$$\gamma(t) = \begin{cases} \frac{1}{rc}(1-e^{-rc}), 0 \leq t \leq T-c, \\ \frac{1}{rc}(1-e^{-r(T-t)}), T-c \leq t \leq T, \end{cases} \tag{2.23}$$

and take the initial capital as

$$X(0) = \frac{1}{rc}(1 - e^{-rc})S(0) - e^{-rT}K. \tag{2.24}$$

We adopt a strategy in the time interval $[0, T-c]$ that at time 0, we buy $\frac{1}{rc}(1-e^{-rc})$ shares of the stock, costing $\frac{1}{rc}(1-e^{-rc})S(0)$. But our initial capital is not sufficient to do this, and we must borrow $e^{-rT}K$ from the money market account. For $0 \leq t \leq T-c$, the value of our holdings in the stock is $\frac{1}{rc}(1-e^{-rc})S(t)$ and we owe $e^{-r(T-t)}K$ to the money market account. Therefore,

$$X(t) = \frac{1}{rc}(1 - e^{-rc})S(t) - e^{-r(T-t)}K, \text{for } 0 \leq t \leq T-c. \tag{2.25}$$

For $T-c \leq t \leq T$, we can compute $X(t)$ by integrating the differential of $e^{r(T-t)}X(t)$ from $T-c$ to t, to obtain

$$X(t) = \frac{1}{rc}(1 - e^{-r(T-t)})S(t) + e^{-r(T-t)}\frac{1}{c}\int_{T-c}^{t} S(u)du - e^{-r(T-t)}K, \tag{2.26}$$
$$T-c \leq t \leq T.$$

Therefore, as designed,

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(u)du - K, \tag{2.27}$$

and

$$V(T) = X^{+}(T) = \max\{X(T), 0\}. \tag{2.28}$$

as desired. The price of the Asian call at time t prior to expiration is

$$V(t) = \widetilde{\mathbb{E}}[e^{-r(T-t)}V(T)|\mathcal{F}(t)] = \widetilde{\mathbb{E}}[e^{-r(T-t)}X^{+}(T)|\mathcal{F}(t)]. \tag{2.29}$$

To calculate the right side of (2.29), $Y(t)$ is defined to be

$$Y(t) = \frac{X(t)}{S(t)} = \frac{e^{-rt}X(t)}{e^{-rt}S(t)}. \tag{2.30}$$

Next we compute the differential of $Y(t)$. Since

$$d(e^{-rt}S(t)) = -re^{-rt}S(t)dt + e^{-rt}dS(t) = \sigma e^{-rt}S(t)d\widetilde{W}(t), \tag{2.31}$$

and

$$d(e^{-rt}X(t)) = \gamma(t)\sigma e^{-rt}S(t)d\widetilde{W}(t), \tag{2.32}$$

by Itô's product rule, we can have

$$dY(t) = \sigma[\gamma(t) - Y(t)][d\widetilde{W}(t) - \sigma dt]. \tag{2.33}$$

So the process $Y(t)$ is not a martingale under the risk-neutral measure $\widetilde{\mathbb{P}}$. If we change measure as

$$\widetilde{W}^{S}(t) = \widetilde{W}(t) - \sigma t, \tag{2.34}$$

then we have

$$dY(t) = \sigma[\gamma(t) - Y(t)]d\widetilde{W}^{S}(t), \tag{2.35}$$

which makes $Y(t)$ become a martingale under the probability measure $\widetilde{\mathbb{P}}^S$ and $\widetilde{W}^S(t), 0 \leq t \leq T$ is a Brownian motion. Returning to the option price $V(t)$ of (2.29), we can write it as

$$V(t) = S(t)\widetilde{\mathbb{E}}^S[Y^+(T)|\mathcal{F}(t)], \tag{2.36}$$

where $\widetilde{\mathbb{E}}^S[...|\mathcal{F}(t)]$ denotes conditional expectation under the probability measure $\widetilde{\mathbb{P}}^S$. Because Y is a Markov process under $\widetilde{\mathbb{P}}^S$, there exists a function of t and $Y(t)$ such that

$$g(t, Y(t)) = \widetilde{\mathbb{E}}^S[Y^+(T)|\mathcal{F}(t)]. \tag{2.37}$$

Therefore, at time T,

$$g(T, Y(T)) = \widetilde{\mathbb{E}}^S[Y^+(T)|\mathcal{F}(T)] = Y^+(T). \tag{2.38}$$

The partial differential equation and its boundary conditions are given in the theorem below, which summarizes the findings.

Theorem 2.2.3. (**Večeř**). *For $0 \leq t \leq T$, the price $V(t)$ at time t of the continuously averaged Asian call with payoff* (2.21) *at time T is*

$$V(t) = S(t) g\left(t, \frac{X(t)}{S(t)}\right), \tag{2.39}$$

where g(t,y) satisfies the partial differential equation

$$g_t(t, y) + \frac{1}{2}\sigma^2(\gamma(t) - y)^2 g_{yy}(t, y) = 0, 0 \leq t < T, y \in \mathbb{R}. \tag{2.40}$$

The boundary conditions for $g(t, y)$ *are*

$$g(T, y) = y^+, y \in \mathbb{R}, \tag{2.41}$$

$$\lim_{y \to -\infty} g(t, y) = 0, 0 \le t \le T, \tag{2.42}$$

$$\lim_{y \to \infty} [g(t, y) - y] = 0, 0 \le t \le T. \tag{2.43}$$

$X(t)$ *is given as* (2.25) *and* (2.26) *and* γ *is taken to be* (2.23).

Proof. Actually, $X(t)$ defined in (2.22) can be any real number, and the denominator of $Y(T)$ can only be positive number, so y can be any real number which leads to the third boundary condition (2.43).

The usual iterated conditioning argument shows that the right-hand side of (2.37) is a martingale under $\widetilde{\mathbb{P}}^S$, which means the differential of $g(t, Y(t))$ should contain no dt term. By Itô-Doeblin Formula, this differential is

$$\begin{aligned} dg(t, Y(t)) &= g_t(t, Y(t))dt + g_y(t, Y(t))dY(t) \\ &\quad + \frac{1}{2} g_{yy}(t, Y(t))dY(t)dY(t) \\ &= \left[g_t(t, Y(t)) + \frac{1}{2}\sigma^2(\gamma(t) - Y(t))^2 g_{yy}(t, Y(t)) \right] dt \\ &\quad + \sigma(\gamma(t) - Y(t)) g_y(t, Y(t)) d\widetilde{W}^S(t). \end{aligned} \tag{2.44}$$

Using the fact that $g(t, Y(t))$ is a martingale, we can conclude that $g(t, y)$ satisfies (2.40).

If $0 \le t \le T$ is given, when $Y(t)$ is very negative, the probability of $Y(T)$ being negative is near 1 and hence the probability of $Y^+(T)$ being zero is near 1. Then $g(t, Y(t))$ in (2.37) approaches zero. On the other hand, when $Y(t)$ is large, this causes the probability that $Y(T) > 0$ is near one, followed by the fact that the right-hand side of (2.37) is approximately equal to $\widetilde{\mathbb{E}}^S[Y(T)|\mathcal{F}(t)]$. Because of the fact that under $\widetilde{\mathbb{P}}^S$, $\{Y(t)\},\ t \geq 0$ the process is a martingale, we get the probability that $g(t, Y(t)) = Y(t)$ is 1, leading to (2.43).

□

2.2.3 Discretely Sampled Asian Fixed Strike Call

When it comes to a discretely sampled Asian call with strike price K, the stock under risk neutral measure is still given as (2.3) and denote the option holder's trading strategy by q_t, the number of shares held at time t.

The discrete average Asian option payoff could be achieved by taking a step function approximation of the stock position q_t of its continuous average option counterpart. Taking Asian fixed strike call into account, we take $q(t) = 1 - \frac{t}{T}$ and $X(0) = S(0) - K$. A step function approximation of $1 - \frac{t}{T}$ is

$$q(t) = 1 - \frac{1}{n}\left[n\frac{t}{T}\right], \tag{2.45}$$

where $[\cdot]$ denotes the integer part function.

Thus we get the discrete average Asian fixed strike call payoff

$$\left(\frac{1}{n}\sum_{k=1}^{n} S\left(\left(\frac{k}{n}\right)\cdot T\right) - K\right)^{+}. \tag{2.46}$$

In this case, a simple PDE and its boundary condition of Asian options is achieved by Jan Večeř [10], given as the theorem below.

Theorem 2.2.4. *The PDE of Asian options price in this case is given as*

$$u_t + r(q_t - z)u_z + \frac{1}{2}(q_t - z)^2\sigma^2 u_{zz} = 0, \tag{2.47}$$

with the boundary condition

$$u(T, z) = z^+, \tag{2.48}$$

where $q_t = q(t) = 1 - \frac{1}{n}[n\frac{t}{T}]$.

The price of the Asian option at time 0 is given as

$$V(0, S_0, X_0) = S_0 \cdot u\left(0, \frac{X_0}{S_0}\right), \tag{2.49}$$

in which $X_0 = S_0 - K$.

2.2.4 Numerical Solutions to the PDE(2.47)

Now it is time to numerically compute the solution to the partial differential equations (2.47). We follow Crank-Nicolson (C-N) numerical scheme. To begin with, let us set a uniform grid as

$$z_i = z_0 + i \cdot dz, \text{for } 0 \le i \le m, \tag{2.50}$$

$$t_j = j \cdot dt, \text{for } 0 \le j \le n, \tag{2.51}$$

where $z_0 = -1$, $z_m = 1$, $t_0 = 0$ and $t_n = T$. So i represents the spacing points with one terminal point $z_m = 1$ being the Asian option whose strike price K equal to 0 and the other endpoint $z_0 = -1$ being the Asian option whose strike price K equal to double of the stock price. And j represents the time, $m = \frac{z_m - z_0}{dz}$ and $n = \frac{T}{dt}$. Because the PDE will be evaluated at every mesh points, by using the fact that $dt = \frac{T-0}{n}$,

$$q(t_j) = 1 - \frac{1}{n}\left[n \cdot \frac{j \cdot dt}{T}\right] = 1 - \frac{1}{n}\left[n \cdot \frac{j}{T} \cdot \frac{T}{n}\right] = 1 - \frac{j}{n}. \tag{2.52}$$

Then using a simple notation $u_{i,j} = u(t_j, z_i)$ and $q_j = q(t_j)$, the Crank-Nicolson method transforms each component of the PDE using the following discrete approximations.

$$u_t = \frac{\partial u}{\partial t} \approx \frac{u_{i,j+1} - u_{i,j}}{dt}, \tag{2.53}$$

$$u_z = \frac{\partial u}{\partial z} \approx \frac{1}{2}\left(\frac{u_{i+1,j+1} - u_{i-1,j+1}}{2dz} + \frac{u_{i+1,j} - u_{i-1,j}}{2dz}\right), \tag{2.54}$$

$$u_{zz} = \frac{\partial^2 u}{\partial z^2} \approx \frac{1}{2(dz)^2}\Big((u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}) + (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})\Big). \tag{2.55}$$

Substitute (2.53), (2.54) and (2.55) into the original PDE (2.47) and evaluate the PDE at the point (t_j, z_i) to get

$$\frac{u_{i,j+1} - u_{i,j}}{dt} + r \cdot (q_j - z_i) \cdot \frac{1}{2}\left(\frac{u_{i+1,j+1} - u_{i-1,j+1}}{2dz} + \frac{u_{i+1,j} - u_{i-1,j}}{2dz}\right)$$
$$+\frac{1}{2}(q_j - z_i)^2\sigma^2 \cdot \frac{1}{2(dz)^2}\Big((u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}) + (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})\Big) = 0 \tag{2.56}$$

Then arrange the equations by putting the new time terms $j+1$ on the right and putting the present time terms j on the left side, then a finite discretization scheme for (2.47) is given as

$$\begin{aligned}
&\Big[-\frac{1}{2}r(q_j-z_i)dz+\frac{1}{2}(q_j-z_i)^2\sigma^2\Big]u_{i-1,j}+\Big[-\frac{2(dz)^2}{dt}-(q_j-z_i)^2\sigma^2\Big]u_{i,j}\\
&\quad+\Big[\frac{1}{2}(q_j-z_i)^2\sigma^2+\frac{1}{2}r(q_j-z_i)dz\Big]u_{i+1,j}=\\
&\Big[\frac{1}{2}r(q_j-z_i)dz-\frac{1}{2}(q_j-z_i)^2\sigma^2\Big]u_{i-1,j+1}+\Big[-\frac{2(dz)^2}{dt}+(q_j-z_i)^2\sigma^2\Big]u_{i,j+1}\\
&\quad+\Big[-\frac{1}{2}r(q_j-z_i)dz-\frac{1}{2}(q_j-z_i)^2\sigma^2\Big]u_{i+1,j+1} \qquad (2.57)
\end{aligned}$$

Let us suppose that

$$a_{ij}=\sigma^2(q_j-z_i)^2,\ b_{ij}=r(q_j-z_i)dz \text{ and } \gamma=\frac{(dz)^2}{dt}, \qquad (2.58)$$

then the equation above can be expressed as

$$\begin{aligned}
&\frac{1}{2}(a_{ij}-b_{ij})u_{i-1,j}-(a_{ij}+2\gamma)u_{i,j}+\frac{1}{2}(a_{ij}+b_{ij})u_{i+1,j}=\\
&\qquad -\frac{1}{2}(a_{ij}-b_{ij})u_{i-1,j+1}+(a_{ij}-2\gamma)u_{i,j+1}-\frac{1}{2}(a_{ij}+b_{ij})u_{i+1,j+1}, \qquad (2.59)
\end{aligned}$$

with the boundary condition $u_{i,n}=z_i^+$. For the boundary conditions at z_0 and z_m, we can take

$$u_{0,j}=0 \quad \text{and} \quad u_{m,j}=2u_{m-1,j}-u_{m-2,j},$$

where the second equality comes from the linear interpolation. Therefore, writing out (2.59) for $i=1,2,\dots,m-1$ we obtain a system of $m-1$ linear equations for the $m-1$ unknowns $u_{1,j},u_{2,j},u_{3,j},\dots,u_{m-1,j}$ and the corresponding tridiagonal system of the equations in (2.59) can be shown as

$$
\underbrace{\begin{bmatrix}
-(a_{1j}+2\gamma) & \frac{1}{2}(a_{1j}+b_{1j}) & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\frac{1}{2}(a_{2j}-b_{2j}) & -(a_{2j}+2\gamma) & \frac{1}{2}(a_{2j}+b_{2j}) & \ddots & & & & \vdots \\
0 & \frac{1}{2}(a_{3j}-b_{3j}) & -(a_{3j}+2\gamma) & \frac{1}{2}(a_{3j}+b_{3j}) & \ddots & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & & & \ddots & \frac{1}{2}(a_{m-2\ j}-b_{m-2\ j}) & -(a_{m-2\ j}+2\gamma) & \frac{1}{2}(a_{m-2\ j}+b_{m-2\ j}) \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -b_{m-1\ j} & (b_{m-1\ j}-2\gamma)
\end{bmatrix}}_{(m-1)\times(m-1)}
\cdot
\underbrace{\begin{bmatrix}
u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ \vdots \\ \vdots \\ u_{m-2,j} \\ u_{m-1,j}
\end{bmatrix}}_{(m-1)\times 1}
=
$$

$$
\underbrace{\begin{bmatrix}
(a_{1j}-2\gamma) & -\frac{1}{2}(a_{1j}+b_{1j}) & 0 & \cdots & \cdots & \cdots & 0 \\
-\frac{1}{2}(a_{2j}-b_{2j}) & (a_{2j}-2\gamma) & -\frac{1}{2}(a_{2j}+b_{2j}) & \ddots & & & \vdots \\
0 & -\frac{1}{2}(a_{3j}-b_{3j}) & (a_{3j}-2\gamma) & -\frac{1}{2}(a_{3j}+b_{3j}) & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & 0 \\
\vdots & & & & -\frac{1}{2}(a_{m-2\ j}-b_{m-2\ j}) & (a_{m-2\ j}-2\gamma) & -\frac{1}{2}(a_{m-2\ j}+b_{m-2\ j}) \\
0 & \cdots & \cdots & \cdots & 0 & b_{m-1\ j} & -(b_{m-1\ j}+2\gamma)
\end{bmatrix}}_{(m-1)\times(m-1)}
\cdot
\underbrace{\begin{bmatrix}
u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ \vdots \\ \vdots \\ u_{m-2,j+1} \\ u_{m-1,j+1}
\end{bmatrix}}_{(m-1)\times 1}
\tag{2.60}
$$

for $j = 0, 1, \ldots, n-1$.

That is to say, these equations must be solved at each time level $j = n-1, n-2, \ldots, 0$. According to the boundary condition $u_{i,n} = z_i^+, i = 1, \ldots, m-1$, starting from the time level $j = n-1$, we can solve the tridiagonal system of equations in (2.60) one by one to $u_{i,0}$, and then use the equation in (2.49) to get the results, which has been done in MATLAB.

when $r = 0.15$, $S(0) = 100$ and $T = 1$.		
σ	fixed strike K	price at time 0 of Večeř method
0.05	95	11.094
	100	6.795
	105	2.745
0.10	90	15.399
	100	7.028
	110	1.414
0.20	90	15.642
	100	8.409
	110	3.556
0.30	90	16.513
	100	10.210
	110	5.731

Table 2.1: Results of Večeř method for fixed strike Asian call

As has been stated in [10], C-N method has the highest convergence order in dt. So the numerical results got from this method are stable and convergent.

2.3 Monte Carlo Methods for Asian Option with Constant Volatility

2.3.1 Approximation of Continuous Asian Option

Because Kemna and Vorst (1990) had concluded that no explicit formula for the value of an average-value option can be found in [7], simulation methods are expected to be used in this case. However, there are two kinds of average, where one is from continuous sampling, given as

$$\frac{1}{T}\int_0^T S(t)dt, \tag{2.61}$$

and another is from discrete sampling,

$$\frac{1}{n}\sum_{j=1}^{n} S(t_j), \tag{2.62}$$

where $0 < t_1 < t_2 < \cdots < t_n = T$. To address both of these issues simultaneously and apply simulation method into pricing asian option from different samples uniformly, using a sum to approximate an integral is desired to be introduced. Therefore, in this thesis, we may approximate (2.61) by follows:

$$\frac{1}{n}\sum_{j=1}^{n} S(t_j), \tag{2.63}$$

where $t_0 = 0$, $t_n = T$, $dt = \frac{T}{n}$ and $t_j = j \cdot dt$. When n is large enough, this formula will be a satisfactory approximation of (2.61).

In other words, whatever the average is, pricing Asian option can always be solved by estimation of a discrete arithmetic Asian option.

2.3.2 Naive Monte Carlo Method for Arithmetic Average Asian Option

2.3.2.1 Theoretical Foundation

In Naive Monte Carlo simulation, one typically uses unbiased estimators of an unknown quantity, usually those estimators are just sample averages. Its variance is the common measure of performance of such an estimator, see [8], which is the same as its mean square error for a unbiased estimator. Also, a confidence interval for the target quantity can be created by using the standard error of the estimator. However, there are still a lot of methods we can take to reduce the standard deviations of Naive Monte Carlo in order to improve the convergence of the algorithm, such as antithetic variates and control variates, as in [9] which will be discussed later. Generally, Monte Carlo simulation relies on two major theorems in probability theory, the Law of Large Number (LLN) and The Central Limit Theorem.

Theorem 2.3.1. Law of Large Numbers (LLN).
Sample average converges almost surely to the expected value. If $X_1, X_2, \ldots,$ is an infinite sequence of independent and identically distributed (i.i.d.) Lebesgue integrable random variables with expected value $E(X_1) = E(X_2) = \cdots = \mu$, then

$$\bar{X}_n \xrightarrow{a.s.} \mu \quad when \quad n \to \infty.$$

Theorem 2.3.2. Central Limit Theorem (CLT).
Let $\{X_1, \cdots, X_n\}$ be a random sample of size n, which is a sequence of i.i.d. random

variables drawn from distributions of expected value given by μ *and finite variances given by* σ^2*. Then*

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}X_i-\mu\right)\Rightarrow N(0,\sigma^2),\ \textit{as}\ n\to\infty,$$

where $\Rightarrow$ *denotes weak convergence. Then a* $100(1-\alpha)\%$ *confidence interval for* μ *is*

$$\left(\frac{1}{n}\sum_{i=1}^{n}X_i-\frac{Z_{\frac{\alpha}{2}}\sigma}{\sqrt{n}},\frac{1}{n}\sum_{i=1}^{n}X_i+\frac{Z_{\frac{\alpha}{2}}\sigma}{\sqrt{n}}\right)$$

As $n\to\infty$, the interval length converges to 0, in which the only remained point is the sample average $\frac{1}{n}\sum_{i=1}^{n}X_i$.

2.3.2.2 Application in Asian Option

As has been stated above, the stock price $S(t)$ can be denoted as a geometric Brownian motion given by

$$dS(t)=rS(t)dt+\sigma S(t)d\widetilde{W}(t), \tag{2.64}$$

where $\widetilde{W}(t)$, $0\le t\le T$, is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Then we can write

$$S(t)=S(0)\exp\{\sigma\widetilde{W}(t)+(r-\frac{1}{2})\sigma^2)t\}. \tag{2.65}$$

Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ where T is the expiration time. Therefore, as in [15]

$$S(t_j) = S(t_{j-1}) \exp\{\sigma(\widetilde{W}(t_j) - \widetilde{W}(t_{j-1})) + (r - \frac{1}{2}\sigma^2)(t_j - t_{j-1})\}, \tag{2.66}$$

for $j = 1, \ldots, n$. And $(\widetilde{W}(t_j) - \widetilde{W}(t_{j-1})) \sim N(0, t_j - t_{j-1})$. If $t_j = j \cdot dt$, for $0 \leq j \leq n$, then

$$(\widetilde{W}(t_j) - \widetilde{W}(t_{j-1})) \sim N(0, dt). \tag{2.67}$$

Suppose $Z_j \sim N(0,1)$ and so $\sqrt{dt}Z_j \sim N(0, dt)$. Hence, we can have

$$S(t_j) = S(t_{j-1}) \exp\{\sigma(\sqrt{dt}Z_j) + (r - \frac{1}{2}\sigma^2)dt\}, \text{ for } 1 \leq j \leq n. \tag{2.68}$$

For the Asian call option, the payoff at time T is given as

$$V(T) = \left(\frac{1}{n}\sum_{j=1}^{n} S(t_j) - K\right)^+ \tag{2.69}$$

and the option price at time t can be given as

$$V(t) = \widetilde{E}[e^{-r(T-t)}V(T)|\mathcal{F}(t)], \text{for } 0 \leq t \leq T. \tag{2.70}$$

In this case, the option price at time 0 is expected to be calculated, which is

$$e^{-rT} \cdot \widetilde{E}\left[\left(\frac{1}{n}\sum_{j=1}^{n} S(t_j) - K\right)^+\right]. \tag{2.71}$$

Therefore, our point is to estimate the second term $\widetilde{E}[(\frac{1}{n}\sum_{j=1}^{n} S(t_j) - K)^+]$. And set

$$C = \widetilde{E}\left[\left(\frac{1}{n}\sum_{j=1}^{n} S(t_j) - K\right)^+\right]. \tag{2.72}$$

Choosing arithmetic average price Asian call option, we use the Monte Carlo method with m simulated paths to price the Asian option.

To generate m different simulation paths, let us suppose that Z_{ij} is a random sample from i.i.d. $N(0,1)$, so $Z_{ij} \overset{i.i.d.}{\sim} N(0,1),\ 1 \le i \le m, 1 \le j \le n$, then

$$\vec{Z}_1 = (Z_{11}, Z_{12}, \ldots, Z_{1n}),$$

$$\vec{Z}_2 = (Z_{21}, Z_{22}, \ldots, Z_{2n}),$$

$$\vdots$$

$$\vec{Z}_m = (Z_{m1}, Z_{m2}, \ldots, Z_{mn}).$$

Using $\vec{Z}_i$ in ith simulation to generate the stock price, for the stock price at time t_j of ith simulation path, it follows as

$$S_{Z_{ij}}(t_j) = S(0) \cdot \exp\left\{\sigma\sqrt{dt}\sum_{k=1}^{j} Z_{ik} + (r - \frac{1}{2}\sigma^2) \cdot j \cdot dt\right\}. \tag{2.73}$$

Then the estimation value of C at every different simulation path is given as

$$
\begin{aligned}
H(\vec{Z}_1) &= \left(\frac{1}{n}\sum_{j=1}^{n} S_{Z_{1,j}}(t_j) - K\right)^+, \\
H(\vec{Z}_2) &= \left(\frac{1}{n}\sum_{j=1}^{n} S_{Z_{2,j}}(t_j) - K\right)^+, \\
&\vdots \\
H(\vec{Z}_m) &= \left(\frac{1}{n}\sum_{j=1}^{n} S_{Z_{m,j}}(t_j) - K\right)^+.
\end{aligned}
\tag{2.74}
$$

Because of the fact that every $\vec{Z}_i$ is i.i.d. and every $H(\vec{Z}_i)$ has the same structure, it can be considered that the expected value of $H(\vec{Z}_i)$ satisfies

$$E\{H(\vec{Z}_1)\} = E\{H(\vec{Z}_2)\} = \ldots = E\{H(\vec{Z}_m)\} = C, \tag{2.75}$$

and

$$\mathrm{Var}\{H(\vec{Z}_1)\} = \mathrm{Var}\{H(\vec{Z}_2)\} = \ldots = \mathrm{Var}\{H(\vec{Z}_m)\} = \eta^2, \tag{2.76}$$

which here can be set as η^2.

By LLN and CLT, when $m \to \infty$, the average of $H(\vec{Z}_1)$, $H(\vec{Z}_2)$, $\ldots$, $H(\vec{Z}_m)$ converges to C, which means

$$\frac{1}{m}\sum_{i=1}^{m} H(\vec{Z}_i) \to C = E\left[(\frac{1}{n}\sum_{j=1}^{n} S(t_j) - K)^+\right]. \tag{2.77}$$

Obviously, this is an unbiased estimator of C.

Thus, it is reasonable to estimate arithmetic Asian call option price at time 0 by the formula below

$$e^{-rT} \cdot \frac{1}{m} \sum_{i=1}^{m} H(\vec{Z}_i). \tag{2.78}$$

2.3.2.3 Algorithm of Naive Monte Carlo for Asian Call Option Pricing

The algorithm realized by Matlab is as following:

set $sum = 0$

for $i = 1$ to m

  for $j = 1$ to n

    use $\vec{Z}_i \sim \mathbf{N}(\mathbf{0}, \mathbf{1})$ to generate $S_{Z_{i1}}(t_1)$, $S_{Z_{i2}}(t_2)$, $\cdots$, $S_{Z_{in}}(t_n)$

    set $sum = sum + \max\left\{0, \frac{\sum_{j=1}^{n} S_{Z_{ij}}(t_j)}{n} - K\right\}$

  end

end

set $V(0) = e^{-rT} \cdot \frac{sum}{m}$

The numerical results are presented in the Table 2.2 and followed the table, corresponding analysis will be given then.

2.3.3 Antithetic Variates Method for Average Asian Option

2.3.3.1 Introduction of Antithetic Variates Method

According to CLT 2.3.2, the smaller the variance is, the smaller the length of the $100(1-\alpha)$ confidence interval will be. So variance reduction means a more accurate estimation. Reducing the standard deviation which is the square root of variance can be achieved by combining correlated estimators in [13]. Antithetic variates are based on this consideration to construct a sequence of antithetic variates which are perfectly negatively correlated with the basic random variables. For example, in normal case, an i.i.d. sequence of standard normal samples, $X_1, \ldots, X_n$ is generated first. After that, one constructs $Y_i = -X_i$. The resulting $Y_1, \ldots, Y_n$ will also be an i.i.d. sequence of standard normals with each Y_i being perfectively negatively correlated with X_i. Compared with the Naive Monte Carlo estimators, which calculating an estimator of $E(H(X))$ by using $\frac{1}{n}\sum_{i=1}^{n} H(X_i)$, antithetic variates method computes it by using

$$\frac{1}{2n}\sum_{i=1}^{n}(H(X_i) + H(Y_i)). \tag{2.79}$$

Obviously, this is also a unbiased estimator since X_i and Y_i are identically distributed and

$$\rho(X_i, Y_i) = -1. \tag{2.80}$$

In fact, it can also be shown specifically in our case that the variance of the antithetic variates estimator is always equal to or less than the original estimator, which will be presented next.

2.3.3.2 Application of pricing Asian Option

In the Naive Monte Carlo case, we have got $Z_{ij} \overset{i.i.d.}{\sim} N(0,1)$, for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. When applied in the antithetic variates method, another antithetic variates have to be chosen to reduce the variance. Here we can take $X_{ij} = -Z_{ij}$, for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. Then X_{ij} and Z_{ij} are perfectly negatively correlated and $X_{ij} \overset{i.i.d.}{\sim} N(0,1)$, for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. Now we are able to use the two kinds of random variables to generate two stock price path and combine them together to get the estimator of the Asian stock price at time 0.

According to what has been done in the Naive Monte Carlo section, relying on $\vec{Z}_1$, $\vec{Z}_2$, …, $\vec{Z}_m$, we have got $H(\vec{Z}_1)$, $H(\vec{Z}_2)$, …, $H(\vec{Z}_m)$, which satisfies

$$E\{H(\vec{Z}_i)\} = C, \text{ for } i = 1, \ldots, m, \tag{2.81}$$

and

$$\text{Var}\{H(\vec{Z}_i)\} = \eta^2, \text{ for } i = 1, \ldots, m. \tag{2.82}$$

Let

$$\vec{X}_1 = (X_{11}, X_{12}, \ldots, X_{1n}),$$

$$\vec{X}_2 = (X_{21}, X_{22}, \ldots, X_{2n}),$$

$$\vdots$$

$$\vec{X}_m = (X_{m1}, X_{m2}, \ldots, X_{mn}).$$

Through taking the same strategy as Naive Monte Carlo for $\vec{X}_1$, $\vec{X}_2$, …, $\vec{X}_m$, we can generate another m simulation paths and the estimation value of C at this m simulation paths can be indicated as

$$\begin{aligned} H(\vec{X}_1) &= \left(\frac{1}{n}\sum_{j=1}^{n} S_{X_{1,j}}(t_j) - K\right)^+, \\ H(\vec{X}_2) &= \left(\frac{1}{n}\sum_{j=1}^{n} S_{X_{2,j}}(t_j) - K\right)^+, \\ &\vdots \\ H(\vec{X}_m) &= \left(\frac{1}{n}\sum_{j=1}^{n} S_{X_{m,j}}(t_j) - K\right)^+. \end{aligned} \tag{2.83}$$

It is because $\vec{X}_i$ and $\vec{Z}_i$ are identically distributed that all expectations of $H(\vec{X}_i)$ for $i = 1, \ldots, m$ are also C and all variances of $H(\vec{X}_i)$ for $i = 1, \ldots, m$ are η^2. So

$$E[H(\vec{X}_i)] = C \ for \ i = 1, \ldots, m$$

and

$$\text{Var}[H(\vec{X}_i)] = \eta^2, \ for \ i = 1, \ldots, m.$$

By LLN and CLT, when $m \to \infty$, the average of $H(\vec{Z}_1)$, $H(\vec{Z}_2)$, ..., $H(\vec{Z}_m)$ and $H(\vec{X}_1)$, $H(\vec{X}_2)$, ..., $H(\vec{X}_m)$ converges to C, which means

$$\frac{1}{2m} \cdot \left[\sum_{i=1}^{m} H(\vec{X}_i) + \sum_{i=1}^{m} H(\vec{Z}_i) \right] \to C. \tag{2.84}$$

Obviously, this is also an unbiased estimator for C.

Consequently, by antithetic method, the estimation of Call Asian Option price at time 0 can be expressed as

$$e^{-rT} \cdot \frac{1}{2m} \cdot \left[\sum_{i=1}^{m} H(\vec{X}_i) + \sum_{i=1}^{m} H(\vec{Z}_i) \right]. \tag{2.85}$$

2.3.3.3 Proof of Variance Reduction

Actually, the main reason to do simulation in this way is its realization of variance reduction compared with the Naive Monte Carlo method, proved by the following.

If we do the same number of simulations in Naive Monte Carlo method and Antithetic method simultaneously, supposing $2m$ simulation paths. The simulation result of

Naive Monte Carlo for C is given as

$$\frac{1}{2m}\sum_{i=1}^{2m} H(\vec{Z}_i) = M \tag{2.86}$$

where we set it as M. Because all $\vec{Z}_i$, for $i = 1, \ldots, 2m$ are i.i.d., $H(\vec{Z}_i)$, for $i = 1, \ldots, 2m$ are i.i.d., which has been stated above. The variance of M can be calculated as

$$Var(M) = \eta^2 \times 2m \times \frac{1}{(2m)^2} = \frac{\eta^2}{2m}. \tag{2.87}$$

The simulation result of Antithetic Method for C can be expressed as

$$\frac{1}{2m} \cdot \left[\sum_{i=1}^{m} H(\vec{X}_i) + \sum_{i=1}^{m} H(\vec{Z}_i)\right]. \tag{2.88}$$

If we let

$$A_i = \frac{H(\vec{Z}_i) + H(\vec{X}_i)}{2}, \ \ for \ i = 1, \ldots, m,$$

then the simulation result of Antithetic Method is given as

$$\frac{1}{m}\{A_1 + A_2 + \cdots + A_m\}. \tag{2.89}$$

Due to the fact that all $H(\vec{Z}_i)$ are i.i.d. and all $H(\vec{X}_i)$ are i.i.d., it accounts for all A_i for $i = 1, \ldots, m$ are i.i.d.. Thus, the variance of A_i is

$$Var(A_i) = \frac{1}{4}[\eta^2 + \eta^2 + 2\text{Cov}(H(\vec{Z}_i), H(\vec{X}_i))] = \frac{1}{2}[\eta^2 + \text{Cov}(H(\vec{Z}_i), H(\vec{X}_i))]$$

In this case, when $H(\vec{Z}_i)$ and $H(\vec{X}_i)$ are independent, then $\text{Cov}(H(\vec{Z}_i), H(\vec{X}_i)) = 0$, leading to

$$\text{Var}(\bar{A}) = \frac{1}{2}\eta^2 \times m \times \frac{1}{m^2} = \frac{\eta^2}{2m},$$

which is equal to the variance of M. However, if $\vec{Z}_i$ increases, then the stock price generated by it will increase at the same time, which means $H(\vec{Z}_i)$ will non-decrease. Apart from this, according to the fact that $\vec{X}_i = -\vec{Z}_i$, the values of $\vec{X}_i$ will decrease, leading to the decrease of the stock price generated by it. Thus, $H(\vec{X}_i)$ is non-increasing at the same time. Therefore, we can draw the conclusion that $\text{Cov}(H(\vec{X}_i), H(\vec{Z}_i)) \leq 0$. Hence,

$$\text{Var}(A_i) \leq \frac{1}{2}\eta^2.$$

From this step, we can get that

$$\text{Var}(\bar{A}) \leq \frac{\eta^2}{2m} = \text{Var}(M).$$

In summary, antithetic method can reduce the variance of the estimator.

2.3.3.4 Algorithm of Antithetic Method for Asian Call Option Pricing

The algorithm which has been implemented in MATLAB is as following:

set $sum1 = 0$

for $i = 1$ to m

for j=1 to n

use $\vec{Z}_j \sim \mathbf{N}(\mathbf{0}, \mathbf{1})$ to generate $S1_{Z_{i1}}(t_1)$, $S1_{Z_{i2}}(t_2)$, ..., $S1_{Z_{in}}(t_n)$

set $sum1 = sum1 + \max\left\{0, \frac{\sum_{j=1}^{n} S1_{Z_{ij}}(t_j)}{n} - K\right\}$

end

end

set $sum2 = 0$

for $i = 1$ to m

for j=1 to n

use $\vec{X}_j \sim \mathbf{N}(\mathbf{0}, \mathbf{1})$ to generate $S2_{X_{i1}}(t_1)$, $S2_{X_{i2}}(t_2)$, ..., $S2_{X_{in}}(t_n)$

set $sum2 = sum2 + \max\left\{0, \frac{\sum_{j=1}^{n} S2_{X_{ij}}(t_j)}{n} - K\right\}$

end

end

set $V(0) = e^{-rT} \cdot \frac{1}{2}(\frac{sum1}{m} + \frac{sum2}{m})$

The numerical results are also presented in the Table 2.3 and followed the table, corresponding analysis will be given then.

2.3.4 Control Variates Method for Average Asian Option

2.3.4.1 Introduction of Control Variates

The concept of control variates is that in order to estimate a quantity from data, the information about another can be used to adjust the original estimator. Suppose we want to estimate some mean value $E(h(X)) = \mu_h$ and this has been done by using a Naive Monte Carlo estimator $\bar{X}_h = \frac{1}{n}\sum_{i=1}^{n} h(X_i)$. Suppose that the same data set, $\{X_i, 1 \leq i \leq n\}$, can be used to estimate a known quantity, so that this sample can be used to adjust the naive estimator because the actual estimation error can be computed. If let $E(g(X)) = \mu_g$ which is known, at the same time, we can also use the Naive Monte Carlo estimator $\bar{X}_g = \frac{1}{n}\sum_{i=1}^{n} g(X_i)$. Then $E[\bar{X}_g - E(g(X))] = 0$. If $\bar{X}_g$ is correlated with $\bar{X}_h$, we can improve the estimator of μ_h by using

$$\bar{X}_h + \hat{a}(\bar{X}_g - E(g(X))), \tag{2.90}$$

where $\hat{a}$ should be chosen to minimize the mean square error, which is just the variance in this unbiased case. The variance of

$$\bar{X}_h + \hat{a}(\bar{X}_g - E(g(X))) \tag{2.91}$$

can be calculated as follows

$$\mathrm{Var}[\bar{X}_h + \hat{a}(\bar{X}_g - E(g(X)))] = \mathrm{Var}(\bar{X}_h) + \hat{a}^2\mathrm{Var}(\bar{X}_g) + 2\hat{a}\mathrm{Cov}(\bar{X}_h, \bar{X}_g). \tag{2.92}$$

Differentiating it with respect to $\hat{a}$ gives

$$\frac{d\mathrm{Var}[\bar{X}_h + \hat{a}(\bar{X}_g - E(g(X)))]}{d\hat{a}} = 2\hat{a}\mathrm{Var}(\bar{X}_g) + 2\mathrm{Cov}(\bar{X}_h, \bar{X}_g). \tag{2.93}$$

By setting it to zero we obtain that

$$\hat{a} = -\frac{\mathrm{Cov}(\bar{X}_g, \bar{X}_h)}{\mathrm{Var}(\bar{X}_g)}, \tag{2.94}$$

which is the optimal choice of $\hat{a}$. Although the quantities of $\mathrm{Cov}(\bar{X}_g, \bar{X}_h)$ and $Var(\bar{X}_g)$ are not known, we can estimate them using the same simulation sample. Then $\bar{X}_g$ is called control variate. The more highly a control variate is correlated with the Naive Monte Carlo estimator, the larger the possible variance reductions are, conform [16].

2.3.4.2 Application to Asian Option by taking Geometric Average Option Price as a Control Variate

This method uses the price of a related option whose value can be computed analytically. There are many choices that we can use as a control variate, such as European options, Geometric Average options and American option as in [17]. Actually, most Asian options traded are arithmetic average options, that is why we always calculate the Asian option price by using the arithmetic average by default. However, for the price of geometric average option, there is a formula for the price of that using Black-Scholes framework because the geometric average shares a vital property with stock price, which is the lognormal distribution. Besides, geometric Asian call prices often served as a control variate while valuing continuous Asian call option, see [14]. Thus, this research focuses on taking the price of the geometric option as control variates to estimate an arithmetic one. Let an arithmetic mean of stock price be $\bar{S_A}(0,T)$ and let the corresponding geometric mean of stock price be $\bar{S_G}(0,T)$, then

$$\bar{S_A} = \frac{1}{n}\sum_{j=1}^{n} S(t_j), \tag{2.95}$$

$$\bar{S_G} = \exp\left(\frac{1}{n}\sum_{j=1}^{n} \log(S(t_j))\right). \tag{2.96}$$

Therefore, the payoff for a geometric call Asian option at time T will be given by

$$\max\{\bar{S}_G - K, 0\}, \tag{2.97}$$

where K is the strike price.

Black-Scholes Formula for Geometric Call Asian Options

As a control variable, a closed form solution for a geometric Asian option exists[18]. Our next step is to state the explicit formula for the price of the geometric average Asian option. Sample the stock price n times from 0 to T as above with time interval $dt = \frac{T}{n}$. Therefore, let $Z_j \sim N(0,1)$, for $j = 1, \ldots, n$ and the simulated stock price are

$$S(t_1) = S(0) \cdot \exp\left\{\sigma\sqrt{dt}Z_1 + \left(r - \frac{\sigma^2}{2}\right)dt\right\}, \tag{2.98}$$

$$S(t_2) = S(0) \cdot \exp\left\{\sigma\sqrt{dt}(Z_1 + Z_2) + \left(r - \frac{\sigma^2}{2}\right)2dt\right\}, \tag{2.99}$$

$$\cdots$$

$$S(t_k) = S(0) \cdot \exp\left\{\sigma\sqrt{dt}\sum_{j=1}^{k} Z_j + \left(r - \frac{\sigma^2}{2}\right)k \cdot dt\right\}, \tag{2.100}$$

$$\cdots$$

$$S(t_n) = S(T) = S(0) \cdot \exp\left\{\sigma\sqrt{dt}\sum_{j=1}^{n} Z_j + \left(r - \frac{\sigma^2}{2}\right) n \cdot dt\right\}, \tag{2.101}$$

Then the natural log of geometric average stock price from 0 to T, $\bar{S_G}$ will be

$$ln\bar{S_G}(T) = \frac{1}{n}[lnS(t_1) + lnS(t_2) + \cdots + lnS(T)] \tag{2.102}$$

$$= \frac{1}{n}\left[n \cdot \ln S(0) + \sigma \cdot \sqrt{dt}\sum_{i=1}^{n}\sum_{j=1}^{i} Z_j + \left(r - \frac{\sigma^2}{2}\right)\sum_{i=1}^{n} i \cdot dt\right] \tag{2.103}$$

$$= lnS(0) + \left(r - \frac{\sigma^2}{2}\right) \cdot dt\frac{1}{n}\sum_{i=1}^{n} i + \sigma\frac{1}{n}\sqrt{dt}\sum_{i=1}^{n}\sum_{j=1}^{i} Z_j \tag{2.104}$$

$\Longrightarrow$

$$ln\bar{S_G}(T) = lnS(0) + \left(r - \frac{\sigma^2}{2}\right) \cdot \frac{1}{n} \cdot \frac{T}{n} \cdot \sum_{i=1}^{n} i + \sigma\sqrt{\frac{T}{n}}\frac{1}{n} \cdot \sum_{i=1}^{n}\sum_{j=1}^{i} Z_j \tag{2.105}$$

Because $Z_j \sim N(0, 1)$, then

$$E[ln\bar{S_G}(T)] = lnS(0) + \left(r - \frac{\sigma^2}{2}\right)\frac{T}{n^2} \cdot \frac{n(1+n)}{2} + 0 \tag{2.106}$$

$$= lnS(0) + \left(r - \frac{\sigma^2}{2}\right) \cdot \frac{n+1}{2n} \cdot T \tag{2.107}$$

and

$$\mathrm{Var}[ln\bar{S_G}(T)] = \sigma^2 \cdot \frac{T}{n} \cdot \frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} \tag{2.108}$$

$$= \sigma^2 \cdot \frac{(n+1)(2n+1)}{6n^2} \cdot T. \tag{2.109}$$

Since the geometric average price is log-normal distributed, we can apply the usual Black-Scholes formula to get the result. And the usual Black-Scholes formula is given as

Theorem 2.3.3. *For a European call expiring at time T with strike price K, the Black-Scholes-Merton price at time t, if the time-t stock price is x, is*

$$c(t,x) = xN(d1(T-t,x)) - Ke^{-r(T-t)}N(d2(T-t,x)), \tag{2.110}$$

where

$$d1(T-t,x) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right], \tag{2.111}$$

$$d2(T-t,x) = d1(T-t,x) - \sigma\sqrt{T-t}, \tag{2.112}$$

and $N(y)$ is the cumulative standard normal distribution. Conform [15].

In order to apply the Black-Scholes formula directly, let

$$\sigma^{*2} = \sigma^2\frac{(n+1)(2n+1)}{6n^2}. \tag{2.113}$$

Then for finding the r^* we suppose $E[ln\bar{S_G}(T)] = lnS(0) + (r^* - \frac{\sigma^{*2}}{2})T$ so that

$$r^* - \frac{\sigma^{*2}}{2} = \left(r - \frac{\sigma^2}{2}\right)\cdot\frac{n+1}{2n}, \tag{2.114}$$

$\Longrightarrow$

$$r^* = \frac{\sigma^{*2}}{2} + \left(r - \frac{\sigma^2}{2}\right)\cdot\frac{n+1}{2n}, \tag{2.115}$$

Black-Scholes formula for the geometric Asian call option price at $t = 0$ is

$$D1 = \frac{ln\left(\frac{S(0)}{K}\right) + \left(r^* + \frac{1}{2}\sigma^{*2}\right)T}{\sigma^*\sqrt{T}}, \tag{2.116}$$

$$D2 = D1 - \sigma^*\sqrt{T}, \tag{2.117}$$

$$G_{bs} = e^{-r^*T}\left[S(0) \cdot \exp(r^*T) \cdot N(D1) - KN(D2)\right]. \tag{2.118}$$

Choosing $\hat{a}$ to minimize the mean square error

Let P_A be the Naive Monte Carlo estimator of arithmetic Asian option prices and P_G be the Naive Monte Carlo estimator of geometric Asian option prices. If we want to compute

$$P_A + \hat{a}(P_G - G) \tag{2.119}$$

to adjust the results of Naive Monte Carlo, the proper value of $\hat{a}$ has to be pinpointed, where it is better to get less variances for the same sample size.

Define

$$A_{cv} = P_A + \hat{a}(P_G - G_{bs}). \tag{2.120}$$

First, whatever the value of $\hat{a}$ is, A_{cv} is always a unbiased estimator of arithmetic Asian option price at time 0. Because G_{bs} is the expectation of P_G, then

$$E(A_{cv}) = E(P_A) + \hat{a}E(P_G - G_{bs}) = E(P_A) + \hat{a}[E(P_G) - G_{bs}] = E(P_A), \tag{2.121}$$

which means its variance is just the mean square error. In order to find the proper value of $\hat{a}$, let us write the variance of A_{cv}.

$$\text{Var}(A_{cv}) = \text{Var}(P_A) + \hat{a}^2\text{Var}(P_G) + 2\hat{a}\text{Cov}(P_A, P_G). \tag{2.122}$$

Differentiating it to $\hat{a}$ gives

$$\frac{d\text{Var}(A_{cv})}{d\hat{a}} = 2\hat{a}\text{Var}(P_G) + \hat{a}\text{Cov}(P_A, P_G), \tag{2.123}$$

by setting it to zero and we obtain

$$\hat{a} = -\frac{\text{Cov}(P_A, P_G)}{\text{Var}(P_G)}. \tag{2.124}$$

Substitute it in (2.122) and the variance of control variable estimator is

$$\begin{aligned} \text{Var}(A_{cv}) &= \text{Var}(P_A) - \frac{\text{Cov}^2(P_A, P_G)}{\text{Var}(P_G)} \\ &= (1-\rho^2)\text{Var}(P_A), \end{aligned}$$

in which ρ is the correlation coefficient of P_A and P_G. It is obvious that the closer the absolute value of ρ is to 1, the greater the variance can be reduced by this method[19]. Actually, the value of $-\hat{a}$ can be calculated as the slop coefficient of the linear regression line as in [20]. Let

$$P_A = a_0 - \hat{a}(P_G - G_{bs}),$$

so $-\hat{a}$ is just the slope coefficient of the linear regression line of P_A and P_G. The Figure 2.1 presents a scatter plot of P_A and P_G for a sample size $n = 10000$ with $S(0) = 100$, $K = 100$, $\sigma = 0.05$, $r = 0.15$ and $T = 1$.

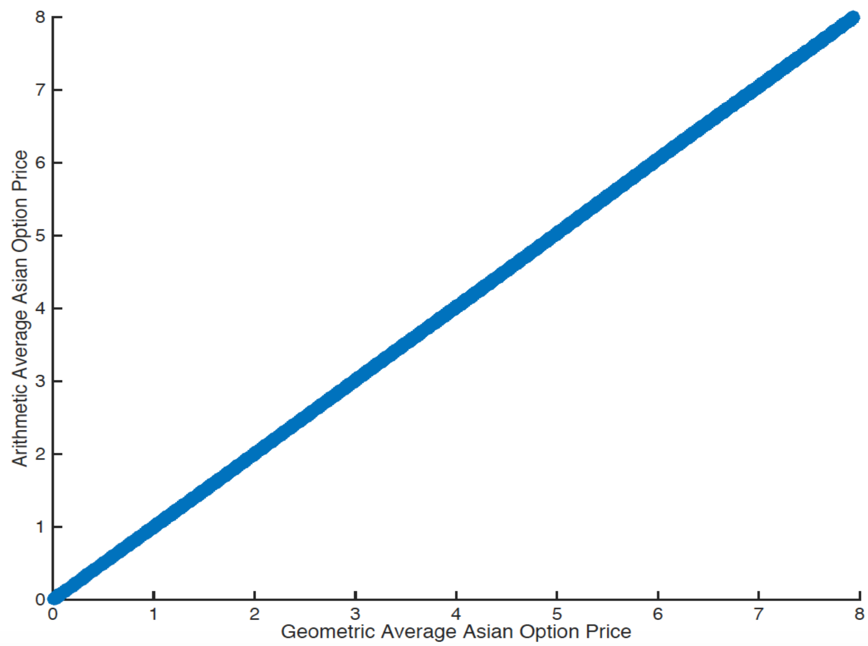

Figure 2.1: Scatter plot of arithmetic average Asian and geometric average Asian option prices

Figures of other parameters can also be done. Because the forms of different parameters are very similar and the coefficients of lines for different parameters always approximate to 1, other figures will not be presented here. Consequently, the correlation of P_A and P_G is seen very close to 1, which means the variance can be highly reduced and it is appropriate to let $\hat{a}$ be -1. Hence, the adjusted estimator can be expressed as

$$A_{cv} = P_A + (G_{bs} - P_G). \tag{2.125}$$

Then it can be implemented in MATLAB so that the corresponding results will be presented in Table 2.4 and followed by the analysis.

2.3.5 Numerical Experiments of Asian Call Option

In this section, taking Večeř method results as benchmarks, comparison of results of different methods for fixed Asian call when $r = 0.15$, $S(0) = 100$ and $T = 1$ will be presented and the performance of all the methods will be evaluated from three aspects in table below, which are absolute error, mean square error and computation time.

when $r = 0.15$, $S(0) = 100$, $T = 1$ and number of simulations is 10000.						
σ	Fixed Strike K	Benchmark (Price at time 0 of Večeř method)	Standard Monte Carlo	Absolute Error	Root Mean Square Error	Computation Time (sec.)
0.05	95	11.094	11.068	0.026	0.0272	6.85
	100	6.795	6.768	0.027	0.0271	7.75
	105	2.745	2.715	0.030	0.0234	8.76
0.10	90	15.399	15.345	0.054	0.0544	8.34
	100	7.028	6.968	0.060	0.0506	7.54
	110	1.414	1.383	0.031	0.0271	6.98
0.20	90	15.642	15.523	0.119	0.1053	9.62
	100	8.409	8.289	0.120	0.0900	8.87
	110	3.556	3.480	0.076	0.0633	6.85
0.30	90	16.513	16.331	0.182	0.1508	7.64
	100	10.210	10.051	0.159	0.1298	7.45
	110	5.731	5.614	0.117	0.1029	6.05

Table 2.2: Comparison of results of Večeř and Naive Monte Carlo method for fixed strike Asian call

when $r = 0.15$, $S(0) = 100$, $T = 1$ and number of simulations is 10000.						
σ	Fixed Strike K	Benchmark (Price at time 0 of Večeř method)	Antithetic Variables Method	Absolute Error	Root Mean Square Error	Computation Time (sec.)
0.05	95	11.094	11.094	0.000	0.0192	11.93
	100	6.795	6.795	0.000	0.0191	13.06
	105	2.745	2.741	0.004	0.0165	9.20
0.10	90	15.399	15.398	0.001	0.0384	9.94
	100	7.028	7.023	0.005	0.0357	10.57
	110	1.414	1.398	0.016	0.0191	8.09
0.20	90	15.642	15.630	0.012	0.0743	10.78
	100	8.409	8.379	0.030	0.0634	11.77
	110	3.556	3.521	0.035	0.0446	11.69
0.30	90	16.513	16.481	0.032	0.1062	11.56
	100	10.210	10.157	0.053	0.0914	9.44
	110	5.731	5.676	0.055	0.0723	10.58

Table 2.3: Comparison of results of Večeř and Antithetic Variates method for fixed strike Asian call

when $r = 0.15$, $S(0) = 100$, $T = 1$ and number of simulations is 10000.						
σ	Fixed Strike K	Benchmark (Price at time 0 of Večeř method)	Control Variables Method	Absolute Error	Root Mean Square Error	Computation Time (sec.)
0.05	95	11.094	11.094	0.000	6.7552×10^{-4}	10.58
	100	6.795	6.795	0.000	6.9862×10^{-4}	6.33
	105	2.745	2.746	0.001	7.6819×10^{-4}	8.03
0.10	90	15.399	15.398	0.001	0.0015	7.82
	100	7.028	7.028	0.000	0.0016	7.13
	110	1.414	1.415	0.001	0.0018	7.42
0.20	90	15.642	15.641	0.001	0.0043	7.68
	100	8.409	8.409	0.000	0.0046	7.66
	110	3.556	3.556	0.000	0.0047	5.99
0.30	90	16.513	16.514	0.001	0.0093	7.59
	100	10.210	10.208	0.002	0.0094	8.52
	110	5.731	5.737	0.006	0.0098	7.37

Table 2.4: Comparison of results of Večeř and Control Variates method for fixed strike Asian call

The results presented in Table 2.2 show the shortcomings of Naive Monte Carlo method compared with the results of Table 2.3 and Table 2.4. As strike price K increases, although the mean square errors decrease, the simulations become less efficient for the reason that the estimators go down much faster than the corresponding mean square errors. Since the convergence of our schemes is where our interest lies, before the computation time is taken into account, the absolute errors and mean square errors should be emphasized first. Obviously, the absolute errors and mean square errors of Naive Monte Carlo in Table 2.2 are much larger than those of Antithetic Variates and Control Variates method. Consequently, Naive Monte Carlo should be considered to be not efficient for Asian options pricing and variance reduction technique, as what has been done in antithetic variates and control variates, is necessary.

Table 2.3 shows figures got from Antithetic Variates method. It is easy to see that all the absolute errors decrease when compared with the figures in Table 2.2. Although the computation time of Antithetic Variates method has the tendency to increase, which is acceptable if we want to get better simulation, mean square errors have been cut down, with each of them in Table 2.3 being almost $\frac{2}{3}$ of those in Table 2.2. So it can be concluded that Antithetic Variates method is more efficient for Asian option than Naive Monte Carlo, however it only achieves a relatively low gain compared to the Control Variate method.

Taking geometric average Asian option as a control variate has also been done with such set of calculations displayed in Table 2.4. Its absolute errors have been accurated to 3 decimals, which means generally lower than them of Antithetic Variates. On the other hand, the computation time data are only a little less than those of Table 2.3 whereas the mean square errors differ dramatically, since the root mean square errors of the estimators with geometric average Asian option as the control variate have been reduced by approximately 90% for every data compared with those of Antithetic Variate method. Besides, Figure 2.2 shows a scatter plot of comparison of Asian option price results from Control Variate method and Antithetic Variate for randomly generating m=10000 samples with $S(0) = 100$, $K = 100$, σ=0.05, $r = 0.15$, $T = 1$ and $dt = 1 \times 10^{-4}$. It can be observed that the prices spread in the Antithetic Variates method is much greater than that of Control Variate estimators, implying variance reduction of Control Variate method.

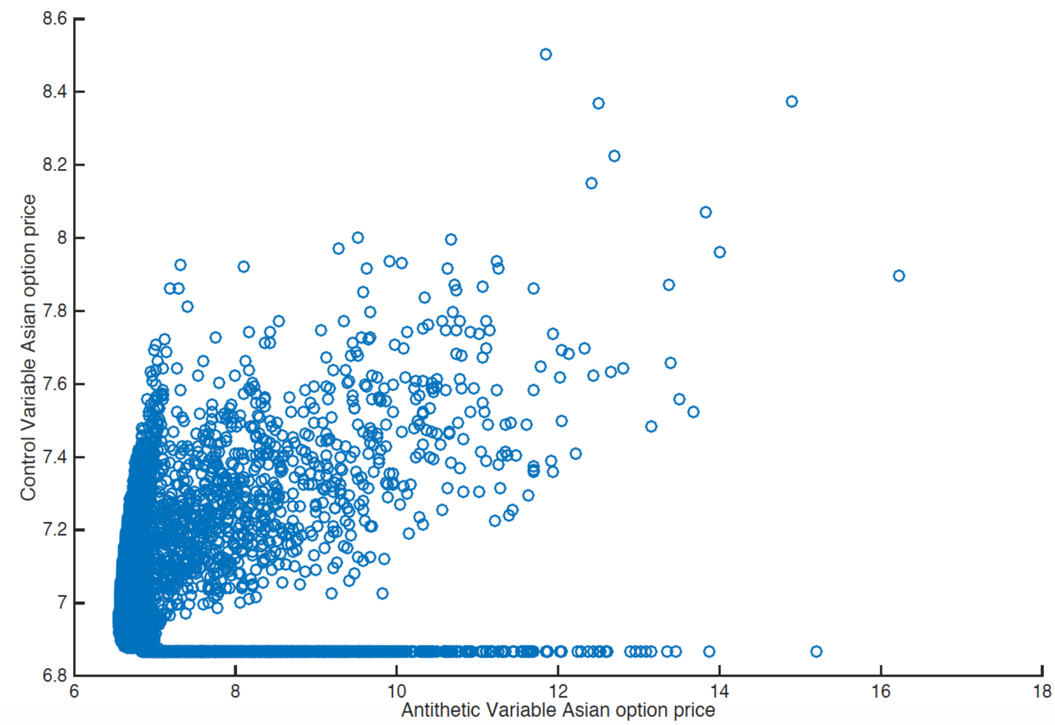

Figure 2.2: Scatter plot of Asian prices with sample size 10000 from Antithetic Variates method V.S. Control Variates method

Thus based on the results obtained from Table 2.2, Table 2.3, and Table 2.4, the Control Variate method is more appropriate and efficient for Asian options pricing.

Chapter 3

Basket Option

3.1 Introduction of Basket Option

Basket Options are popular multivariate derivative securities and their payoff depends on the weighted average of the underlying asset prices. In this study, as most cases do, let us suppose the price of basket options depends on the weighted arithmetic average of the prices of d different assets with correspondingly different weights of the assets, denoted as $\omega_1,\ \omega_2,\ ,\dots,\ \omega_d$. Thus, we can assume that each $\omega_i > 0$ and $\sum_{i=1}^{d} \omega_i = 1$. Let $S_i(t)$ denote the price of the asset i at time t. Then the weighted arithmetic average of the asset prices is given by

$$\sum_{i=1}^{d} \omega_i S_i(T), \tag{3.1}$$

in which $S_i(T)$ is the price of the asset i at maturity T. Consequently, taking Basket Call Option with fixed strike price K as an example, we can write the call payoff as the form

$$V(T) = \left(\sum_{i=1}^{d} \omega_i S_i(T) - K \right)^+, \tag{3.2}$$

where K is a nonnegative constant.

For the prices of the underlying assets, we consider they follow the Geometric Brownian Motion (GBM) model with constant volatilities σ_i, a constant risk free interest rate r and constant continuous dividend yields δ_i, given as

$$dS_i(t) = rS_i(t)dt + \sigma_i S_i(t)dW_i(t) - \delta_i S_i(t)dt, \tag{3.3}$$

and under GBM,

$$S_i(t) = S_i(0) \cdot \exp\left[\left(r - \delta_i - \frac{{\sigma_i}^2}{2}\right)t + \sigma_i W_i(t)\right],\ i = 1, \ldots, d,\ \text{and } 0 \le t \le T. \tag{3.4}$$

Therefore, the price of basket call option at time t prior to the expiration time T is given by the risk-neutral pricing formula as

$$V(t) = \widetilde{\mathbb{E}}[e^{-r(T-t)}V(T)|\mathcal{F}(t)], 0 \le t \le T. \tag{3.5}$$

Because e^{rt} is $\mathcal{F}(t)$ measurable, then

$$V(t) = e^{rt}\widetilde{\mathbb{E}}[e^{-rT}V(T)|\mathcal{F}(t)] \tag{3.6}$$

and

$$e^{-rt} \cdot V(t) = \widetilde{\mathbb{E}}[e^{-rT}V(T)|\mathcal{F}(t)]. \tag{3.7}$$

According to the usual iterated conditioning argument which has been done in 2.7, 2.8 and 2.9, it can be concluded that the discounted price process $\{e^{-rt}V(t)\}$, $t \geq 0$ is a martingale under $\widetilde{\mathbb{P}}$.

3.2 The PDE of Fixed-Strike Basket Call with Constant Volatility and Constant Dividend Yield

To derive the PDE for this basket option call price, suppose $\mathbf{S_t} = (S_1(t), S_2(t), \ldots, S_d(t))^T$. Since $V(T)$ in 3.2 is a function of $S_1(T),\ \ldots,\ S_d(T)$, and 3.5, this implies that there must exist some functions $f(t,\ S_1(t),\ \ldots,\ S_d(t))$ and $g(t,\ S_1(t),\ \ldots,\ S_d(t))$ such that the basket call price $V(t)$ and the discounted basket call price $e^{-rt}V(t)$ are given as

$$V(t) = g(t,\ S_1(t),\ \ldots,\ S_d(t)) \tag{3.8}$$

and

$$e^{-rt}V(t) = f(t,\ S_1(t),\ \ldots,\ S_d(t)) = f(t, \mathbf{S_t}^T) \tag{3.9}$$

$$= e^{-rt}g((t,\ S_1(t),\ \ldots,\ S_d(t)). \tag{3.10}$$

By Itô-Doeblin formula drift-diffusion processes, we have

$$de^{-rt}V(t) = df(t,\ \mathbf{S_t}^T) \tag{3.11}$$
$$= \frac{\partial f}{\partial t}dt + (\nabla_s \mathbf{f})^T d\mathbf{S_t}^T + \frac{1}{2}(d\mathbf{S_t})^T(\mathbf{H_s f})d\mathbf{S_t} \tag{3.12}$$
$$= \left\{\frac{\partial f}{\partial t} + (\nabla_s \mathbf{f})^T \mu_{\mathbf{t}} + \frac{1}{2}Tr[\mathbf{G_t^T}(\mathbf{H_s f})\mathbf{G_t}]\right\}dt$$
$$+ (\nabla_s \mathbf{f})^T \mathbf{G_t} d\mathbf{W}(\mathbf{t}), \tag{3.13}$$

where $\nabla_s \mathbf{f}$ is the gradient of f with respect to (w.r.t.) $S_i(t)$, $\mathbf{H_s f}$ is the Hession matrix of f w.r.t. $S_i(t)$, and Tr is the trace operator. And $\mu_{\mathbf{t}}$, $\mathbf{G_t}$ and $d\mathbf{W}(\mathbf{t})$ are given in the equation 3.15 below, such as

$$d\mathbf{S_t} = \mu_{\mathbf{t}}dt + \mathbf{G_t}d\mathbf{W}(\mathbf{t}). \tag{3.14}$$

Since $dS_i(t) = rS_i(t)dt + \sigma_i S_i(t)dW_i(t) - \delta_i S_i(t)dt$, then we can have

$$\mu_{\mathbf{t}} = [rS_1(t) - \delta_1 S_1(t),\ \dots\ ,\ rS_d(t) - \delta_d S_d(t)]^T, \tag{3.15}$$

$$\mathbf{G_t} = \begin{bmatrix} \sigma_1 S_1(t) & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 S_2(t) & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 S_3(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \sigma_d S_d(t) \end{bmatrix}_{d\times d} \tag{3.16}$$

and

$$d\mathbf{W(t)} = \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ \vdots \\ \vdots \\ dW_d(t) \end{bmatrix}_{d\times 1} . \tag{3.17}$$

By the property of martingale, the dt term in 3.13 has to be 0. As a result, we can have

$$\frac{\partial f}{\partial t} + (\nabla_{\mathbf{s}}\mathbf{f})^T \cdot \mu_{\mathbf{t}} + \frac{1}{2} Tr[\mathbf{G_t^T}(\mathbf{H_s f})\mathbf{G_t}] = 0, \tag{3.18}$$

in which

$$\frac{\partial f}{\partial t} = e^{-rt} \cdot (-r) \cdot g(t,\ S_1(t),\ \ldots,\ S_d(t)) + e^{-rt}\frac{\partial g}{\partial t} \tag{3.19}$$

$$= e^{-rt} \cdot \left[\frac{\partial g}{\partial t} - r \cdot g\right], \tag{3.20}$$

$$\nabla_{\mathbf{s}}\mathbf{f} = \left[\frac{\partial g}{\partial S_1(t)},\ \cdots\ ,\ \frac{\partial g}{\partial S_d(t)}\right]^T \cdot e^{-rt} \tag{3.21}$$

and

$$\mathbf{H_s f} = \begin{bmatrix} \frac{\partial^2 g}{\partial s_1^2} & \frac{\partial^2 g}{\partial s_1 \partial s_2} & \cdots & \cdots & \frac{\partial^2 g}{\partial s_1 \partial s_d} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \frac{\partial^2 g}{\partial s_d^2} \end{bmatrix}_{d\times d} \cdot e^{-rt}. \tag{3.22}$$

Thus the equation 3.18 can be simplified as

$$\frac{\partial g}{\partial t} + \sum_{i=1}^{d} \frac{\partial g}{\partial s_i} \cdot (r - \delta_i) \cdot s_i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i \sigma_j s_i s_j \cdot \frac{\partial^2 g}{\partial s_i \partial s_j} = rg, \tag{3.23}$$

with boundary conditions

$$\begin{cases} g(T, s_1, \ldots, s_d) = (\sum_{i=1}^{d} \omega_i s_i - K)^+, \\ \lim_{(s_1, s_2, \ldots, s_d) \to (0,0,\ldots,0)} g(t, s_1, s_2, \ldots, s_d) = 0. \end{cases} \tag{3.24}$$

Consequently, this is a highly dimensional PDE and it is very hard to get numerical results for it. Based on this fact, Monte Carlo Method of pricing Basket Option will be introduced later.

3.3 Monte Carlo Method for Basket Option with Constant Volatility

3.3.1 Naive Monte Carlo Method for Basket Call Option

According to 3.4 and 3.2, it is clear that the payoff of basket call is

$$V(T) = \left(\sum_{i=1}^{d} \omega_i S_i(T) - K \right)^+, \tag{3.25}$$

where

$$S_i(T) = S_i(0) \cdot \exp\left[\left(r - \delta_i - \frac{{\sigma_i}^2}{2} \right) T + \sigma_i W_i(T) \right], \ i = 1, \ldots, d. \tag{3.26}$$

By the property of GBM, $W_i(T) \sim N(0,T)$ for $i = 1,\ldots,d$ and they are correlated standard Brownian motions with correlations ρ_{ij}. Therefore, the main problem to simulate the stock price will be generating $W_i(T),\ i = 1,\ldots,d$, in which they are correlated with each other. In fact , it is easier for us to generate some independent standard normal variables in practice. So our goal is to find the relationship between $W_i(T),\ i = 1,\ldots,d$ and some independent standard normal variables. To find the relationship of $W_i(T)$ and some independent standard normal variables, for basket option, let us define $\mathbf{R}$ as $d \times d$ correlation matrix with entries ρ_{ij}, such as

$$\mathbf{R} = \begin{bmatrix} & & \\ & \rho_{ij} & \\ & & \end{bmatrix}_{d\times d}, \tag{3.27}$$

and suppose $\mathbf{L}$ is the solution of $\mathbf{L}\mathbf{L}^T = \mathbf{R}$ obtained by Cholesky factorization, in which the entries of the matrix $\mathbf{L}$ are L_{ij}. Because $[W_1(T), W_2(T), \ldots, W_d(T)]^T$ follows a multivariate normal distribution, given as

$$[W_1(T), W_2(T), \ldots, W_d(T)]^T \sim \mathbf{N}(\mathbf{0}, \mathbf{\Sigma}) \tag{3.28}$$

where

$$\mathbf{\Sigma} = \begin{pmatrix} \sqrt{T} & & & & \\ & \sqrt{T} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sqrt{T} \end{pmatrix} \begin{pmatrix} \rho_{11} & \cdots & \cdots & \cdots & \rho_{1d} \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \rho_{d1} & \cdots & \cdots & \cdots & \rho_{dd} \end{pmatrix} \begin{pmatrix} \sqrt{T} & & & & \\ & \sqrt{T} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sqrt{T} \end{pmatrix} = T\mathbf{R} \tag{3.29}$$

From the Linear Transformation Property, if $\xi = [\xi_1, \xi_2, \ldots, \xi_d]^T \sim \mathbf{N}(\mathbf{0}, \mathbf{I})$, in which $\xi_1, \ldots, \xi_d$ are i.i.d. $N(0,1)$, and $X = \mathbf{0} + \mathbf{L}\xi$, then $X \sim \mathbf{N}(\mathbf{0}, \mathbf{LL^T})$ as in [21]. Hence,

$$\mathbf{L}\xi \sim \mathbf{N}(\mathbf{0}, \mathbf{LL^T}). \tag{3.30}$$

Furthermore,

$$\sqrt{T}\mathbf{L}\xi \sim N(\mathbf{0}, T\mathbf{LL^T}) = N(\mathbf{0}, T\mathbf{R}) = N(\mathbf{0}, \mathbf{\Sigma}). \tag{3.31}$$

Obviously, $\sqrt{T}\mathbf{L}\xi$ can be used to replace $[W_1(T), \ldots, W_d(T)]$ in the simulation procedure. Note that the i-th element of the vector $\mathbf{L}\xi$ can be written as $\sum_{j=1}^{i} L_{ij}\xi_j$ as $\mathbf{L}$ is lower triangular since the matrix $\mathbf{R}$ is symmetric. Consequently, the stock price at time T can be written as the form

$$S_i(T) = S_i(0) \exp\left[\left(r - \delta_i - \frac{\sigma_i^2}{2}\right)T + \sigma_i\sqrt{T}\sum_{j=1}^{i} L_{ij}\xi_j\right], \; i = 1, \ldots, d. \tag{3.32}$$

The detailed algorithm of the naive simulation with n different simulation paths is presented as Table 3.1 below.

Algorithm of Naive Monte Carlo Method for Basket Call Option
Conditions: given sample size n, maturity T, number of asset d, weights of assets ω_i, initial asset price $S_i(0)$, strike price K, constant volatility σ_i, divident yields δ_i, correlation matrix $\mathbf{R}$, and risk free interest rate r, for $i = 1, \ldots, d$.
1. Compute the Cholesky factor $\mathbf{L}$ of $\mathbf{R}$, where $\mathbf{LL^T} = \mathbf{R}$.
2. for $m = 1$ to n
3. Generate i.i.d. standard normal variates $\xi_i \sim N(0,1)$, $i = 1, 2, \ldots, d$.
4. Let $S_i(T) = S_i(0) \exp\left[\left(r - \delta_i - \frac{\sigma^2}{2}\right) T + \sigma_i \sqrt{T} \sum_{j=1}^{i} L_{ij}\xi_j\right]$, $i = 1, \ldots, d$
5. Set $Y_m = e^{-rT}\left(\sum_{i=1}^{d} \omega_i S_i(T) - K\right)^+$.
6. end for
7. Compute the sample mean $\bar{Y} = \sum_{m=1}^{n} Y_m/n$ and the standard deviations of Y_i's.
8. Return the estimation $\bar{Y}$ and its 95% confidence interval.

Table 3.1: Algorithm of Naive Monte Carlo Method of Basket Call Option

Concerning the convergence of $\bar{Y}$, as has been proved in section 2.2.2, by using LLN and CLT $\bar{Y}$ can be proved as an unbiased estimation of discounted payoff of basket call $e^{-rT}V(T)$ and it converges to $e^{-rT}V(T)$ when the sample size n goes to infinity. The numerical results are presented in the Table 3.5 and followed the table, the corresponding analysis will be given then.

3.3.2 Classical Control Variate for Basket Call Option

From Naive Monte Carlo of basket call option, the estimator Y_m, for $m = 1, \ldots, n$ can be obtained from every simulation path. As has been stated in section 2.4.1, apart from the original estimator of discounted payoff of basket call, a control variable is necessary to do some correction of the naive result. Suppose Q is the control variate, and $E[Q]$ is its known expectation. Then the every estimator of basket call from every simulation path is given as the from

$$Y_{CV} = Y + a(Q - E[Q]). \tag{3.33}$$

As has been proved in 2.94, the optimal coefficient minimizing the variance is

$$a^* = -\frac{\mathrm{Cov}(Y, Q)}{\mathrm{Var}(Q)}. \tag{3.34}$$

The most important factor to affect the accuracy of the control variate method is the choice of control variate. As has been proved in section 2.4.2, only if the absolute value of correlation coefficient between control variate and the actual variate we want to estimate is close to 1, then the variance of our estimator can be reduced a lot, which means the control variate has to be highly related with the payoff of the basket option. Thus, in classical control variate, the payoff of a geometric average call is used as a control variate, with the payoff function as

$$P_G = (G - K)^+ \tag{3.35}$$

and

$$G = S_1(T)^{\omega_1} \cdot S_2(T)^{\omega_2} \cdots S_d(T)^{\omega_d} = \exp\left(\sum_{i=1}^{d} \omega_i \cdot \log S_i(T)\right). \tag{3.36}$$

There are some reasons to choose P_G as a control variate. The payoffs of the geometric and the arithmetic average options are close to each others because of the close value of $S_i(T)$. Actually, arithmetic and geometric averages yield the same result if all prices $S_1(T), \ldots, S_d(T)$ are the same. So it is a wise choice to use the payoff of the geometric average option P_G as a control variate. Therefore, the simulation estimator for the price (without discount) is

$$Y_{CV} = P_A + a(P_G - \mu_G), \tag{3.37}$$

where P_A is the payoff of the arithmetic average basket call given as

$$P_A = \left(\sum_{i=1}^{d} \omega_i S_i(T) - K \right)^+, \tag{3.38}$$

and

$$\mu_G := E[P_G]. \tag{3.39}$$

Just as what has been done in Naive Monte Carlo method, the estimation of stock prices from different simulation paths can be easily obtained by the form 3.32, so estimators of geometric average payoff can be calculated. It brings our attention to the expectation of P_G. The solution of μ_G has been given in [23], as

$$\mu_G = \exp\left(\mu_{\tilde{s}} + \frac{\sigma_{\tilde{s}}^2}{2} \right) \cdot \Phi(-h + \sigma_{\tilde{s}}) - K \cdot \Phi(-h), \tag{3.40}$$

where $\Phi()$ denotes the cumulative distribution function (cdf) of the standard normal distribution,

$$h = \frac{\log K - \mu_{\tilde{s}}}{\sigma_{\tilde{s}}} \tag{3.41}$$

and

$$\mu_{\tilde{s}} = E[\log G], \tag{3.42}$$

$$\sigma_{\tilde{s}}^2 = \mathrm{Var}(\log G). \tag{3.43}$$

There are also corresponding equations for them, as

$$\mu_{\tilde{s}} = \sum_{i=1}^{d} \omega_i \tilde{\mu}_i, \tag{3.44}$$

and

$$\sigma_{\tilde{s}}^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} \omega_i \omega_j \tilde{\sigma}_i \tilde{\sigma}_j \tilde{\rho}_{ij}, \tag{3.45}$$

where

$$\tilde{\mu}_i = E[logS_i(T)], \tag{3.46}$$

$$\tilde{\sigma}_i^2 = \mathrm{Var}(logS_i(T)) \tag{3.47}$$

and $\tilde{\rho}_{ij}$ is the correlation between $\log S_i(T)$ and $\log S_j(T)$. For basket options,

$$\tilde{\mu}_i = \log S_i(0) + \left(r - \sigma_i - \frac{\sigma_i^2}{2}\right) T, \tag{3.48}$$

$$\tilde{\sigma}_i = \sigma_i \sqrt{T}. \tag{3.49}$$

Since the vector

$$\log S_i(T) = \log S_i(0) + \left(r - \delta_i - \frac{\sigma_i^2}{2}\right) T + \sigma_i W_i(T) \tag{3.50}$$

is a linear transformation of $W(T)$,

$$\tilde{\rho}_{ij} = \rho_{ij}. \tag{3.51}$$

Since according to 3.34 the optimal value a^* can be pinpointed by the formula

$$a^* = -\frac{\mathrm{Cov}(P_A, P_G)}{\mathrm{Var}(P_G)}, \tag{3.52}$$

it is found that for different sample size, different simulation paths and different values of parameters in this case, the value of a^* is always very close to -1 in most cases. Then $a^* = -1$.

The detailed algorithm of the classical control variate method with n different simulation paths is presented as Table 3.2 below as well.

Algorithm of Classical Control Variate for Basket Call Option
Conditions: given sample size n, maturity T, number of asset d, weights of assets ω_i, initial asset price $S_i(0)$, strike price K, constant volatility σ_i, divident yields δ_i, correlation matrix $\mathbf{R}$, risk free interest rate r, for $i = 1, \ldots, d$ and a^*.
1. Compute the Cholesky factor $\mathbf{L}$ of $\mathbf{R}$, where $\mathbf{LL^T} = \mathbf{R}$.
2. Compute μ_G by using the given formula 3.40.
3. for $m = 1$ to n
4. Generate i.i.d. standard normal variates $\xi_i \sim N(0, 1)$, $i = 1, 2, \ldots, d$.
5. Let $S_i(T) = S_i(0) \exp\left[\left(r - \delta_i - \frac{\sigma^2}{2}\right) T + \sigma_i \sqrt{T} \sum_{j=1}^{i} L_{ij}\xi_j\right]$, $i = 1, \ldots, d$
6. Set $P_A = e^{-rT}(\sum_{i=1}^{d} \omega_i S_i(T) - K)^+$ and $P_G = (\exp(\sum_{i=1}^{d} \omega_i \log S_i(T)) - K)^+$.
7. Let $Y_m = e^{-rT}(P_A + a^*(P_G - \mu_G))$
8. end for
9. Compute the sample mean $\bar{Y} = \sum_{m=1}^{n} Y_m/n$ and the standard deviations of Y_i's.
10. Return the estimation $\bar{Y}$ and its 95% confidence interval.

Table 3.2: Algorithm of Classical Control Variate for Basket Call Option

3.3.3 New Control Variate for Basket Call Option

However the variance of our estimation can be reduced by the classical control variate above when compared with the Naive Monte Carlo for basket call option, this technique is not efficient enough, while the computational requirements for obtaining a reasonably accurate estimate become excessive as the number of assets rises, as stated in [22]. Besides, according to the introduction in chapter 2.4.1, the more highly a control variate is correlated with the Naive Monte Carlo estimator, the larger the possible variance can be reduced. So for the arithmetic payoff of basket call, the most highly correlated variate is itself. Therefore, if a partially exact approximation based on the composition of the arithmetic payoff can be used as a control variate, a more accurate simulation method will be found then, as in [23]. By using conditioning, the main part of the arithmetic payoff is decided as a new control variate to see whether a better result can be got, evaluated by the mean square error, length of 95% confidence interval and computation time.

Here an alternative approach is introduced below. In order to find a highly correlated control variate, which can reduce the variance, itself, the arithmetic payoff, is a good choice. However, paths that end up giving 0 payoff should not be sampled. So this approach decomposes the arithmetic payoff of basket call option into two parts by conditioning on the geometric mean price. If let $A = \sum_{i=1}^{d} \omega_i S_i(T)$, then the arithmetic payoff can be expressed as

$$(A-K)^+ = (A-K)^+ \mathbb{1}_{\{G \leq K\}} + (A-K)^+ \mathbb{1}_{\{G>K\}}, \tag{3.53}$$

where G is defined in 3.36 and K is the strike price. Since A represents the arithmetic average and G denotes the geometric average, $A \geq G$ for all possible terminal asset values. Then the second term can be simplified as

$$(A-K)^+ \mathbb{1}_{\{G>K\}} = (A-K) \mathbb{1}_{\{G>K\}}. \tag{3.54}$$

The idea is to use exactly the second term $(A-K)\mathbb{1}_{\{G>K\}}$ as control variate for pricing basket call option. As has been done in last section, arithmetic average stock price and geometric average price is highly correlated, indicating that in most of replications, A and G are generally closer. So the first term in 3.53 can be considered as 0 due to the strong dependence between A and G. The conclusion is the second term $(A-K)\mathbb{1}_{\{G>K\}}$ should be taken as a control variate. Put it in another way, the payoff P_A, defined as 3.38, will be equal to this control variate in most of replications, implying a strong linear dependence. Define

$$Q = (A-K)\mathbb{1}_{\{G>K\}}, \tag{3.55}$$

thus the control variate estimator is simply as

$$Y_{NCV} = P_A + c(Q - E[Q]). \tag{3.56}$$

The simulations of Naive simulation estimator P_A and control variate Q are straightforward to get according to the simulations of stock prices. However the closed form solution of $E[Q]$ has to be found. let $E[Q]$ be denoted by μ_Q. Based on the composition of μ_Q into two parts, μ_Q can be written as

$$\mu_Q = E[(A-K)\cdot \mathbb{1}_{\{G>K\}}] = E[A\mathbb{1}_{\{G>K\}}] - K \cdot P(G>K). \tag{3.57}$$

By using $\mu_{\tilde{s}}$ and $\sigma_{\tilde{s}}$ in 3.44 and 3.45, log geometric average can be standardized as

$$X = \frac{\log G - \mu_{\tilde{s}}}{\sigma_{\tilde{s}}}, \tag{3.58}$$

so $X \sim N(0,1)$. Consequently, the second term in the equation 3.57 can be expressed as

$$K \cdot P(G>K) = K \cdot P(X>h) = K \cdot \Phi(-h), \tag{3.59}$$

where $h = \frac{\log K - \mu_{\tilde{s}}}{\sigma_{\tilde{s}}}$ as defined in 3.41.

As for the first term in equation 3.57, it is evaluated by exploiting the fact that Curran (1994) in [22] gives a closed formula of the conditional expectation of $S_i(T)$, $i = 1, \ldots, d$, as

$$E[S_i(T)|X = x] = \exp\left(\tilde{\mu_i} + a_i x + \frac{\tilde{\sigma_i}^2 - a_i^2}{2}\right), \tag{3.60}$$

where a_i denotes the covariance between X and $logS_i(T)$, and

$$a_i = \text{Cov}(X, \log S_i(T)) = \frac{\tilde{\sigma_i}}{\sigma_{\tilde{s}}} \sum_{j=1}^{d} \omega_j \tilde{\sigma_j} \tilde{\rho_{ij}}, \tag{3.61}$$

where $\tilde{\sigma_i}$, $\sigma_{\tilde{s}}$, $\tilde{\sigma_j}$ and $\tilde{\rho_{ij}}$ are given as above in 3.44, 3.45, 3.48, 3.49, and 3.51. Then

$$\mu_Q = \sum_{i=1}^{d} E[\omega_i S_i(T) \mathbb{1}_{\{G>K\}}], \tag{3.62}$$

$$= \sum_{i=1}^{d} \omega_i \int_{K}^{+\infty} E[S_i(T)|X = x]\Phi'(x)dx. \tag{3.63}$$

By integration, the closed form of μ_Q is

$$\mu_Q = \left(\sum_{i=1}^{d} \omega_i e^{\tilde{\mu_i} + \frac{\tilde{\sigma_i}^2}{2}} \Phi(-h + a_i) - K\Phi(-h)\right). \tag{3.64}$$

Next step will be determining the value of c. According to 3.34 the optimal value c^* can be pinpointed by the formula $c^* = -\frac{\text{Cov}(P_A, W)}{\text{Var}(W)}$. It is found that for different sample size, different simulation paths and different values of parameters in this case, the value of c^* is always very close to -1 in most cases.

The detailed algorithm of the classical control variate method with n different simulation paths is presented as Table 3.3 below.

Algorithm of New Control Variate for Basket Call Option

Conditions: given sample size n, maturity T, number of asset d, weights of assets ω_i, initial asset price $S_i(0)$, strike price K, constant volatility σ_i, divident yields δ_i, correlation matrix $\mathbf{R}$, risk free interest rate r, for $i = 1, \ldots, d$ and c^*.

1. Compute the Cholesky factor $\mathbf{L}$ of $\mathbf{R}$, where $\mathbf{L}\mathbf{L}^{\mathbf{T}} = \mathbf{R}$.
2. Compute μ_W by using the given formula 3.64.
3. for $m = 1$ to n
4. Generate i.i.d. standard normal variates $\xi_i \sim N(0,1)$, $i = 1, 2, \ldots, d$.
5. Let $S_i(T) = S_i(0) \exp\left[\left(r - \delta_i - \frac{\sigma^2}{2}\right) T + \sigma_i \sqrt{T} \sum_{j=1}^{i} L_{ij} \xi_j\right]$, $i = 1, \ldots, d$
6. Set $P_A = \left(\sum_{i=1}^{d} \omega_i S_i(T) - K\right)^+$, $G = \exp\left(\sum_{i=1}^{d} \omega_i \log S_i(T)\right)$ and $A = \sum_{i=1}^{d} \omega_i S_i(T)$.
7. Let $W = (A - K)\mathbb{1}_{\{G>K\}}$
8. Let $Y_m = e^{-rT}(P_A + c^*(W - \mu_W))$
9. end for
10. Compute the sample mean $\bar{Y} = \sum_{m=1}^{n} Y_m / n$ and the standard deviations of Y_i's.
11. Return the estimation $\bar{Y}$ and its 95% confidence interval.

Table 3.3: Algorithm of New Control Variate for Basket Call Option

3.3.4 Numerical Experiments of Basket Call Option

In this section, numerical experiments will be performed through Naive Monte Carlo, the Classical Monte Carlo and the New Monte Carlo For the values of correlations ρ_{ij}, constant volatilities σ_i, dividend yields δ_i, weights of assets ω_i and number of assets d, for $i = 1, \ldots, d$ and $j = 1, \ldots, d$, this case will use the data from popular

G-7 Index-linked guaranteed investment certificates (ILGICs) given in Milevsky and Posner (1998). The basket option is embedding on the products linked to the performance of the G-7 stock market, including Canada, France, Germany, Italy, Japan, U. K. and U. S..

i	Country	Index	Weight	Volatility	Dividend Yield	Correlation ρ_{ij}						
			ω_i	σ_i	δ_i	Canada	France	Germany	Italy	Japan	U. K.	U. S.
1	Canada	TSE 100	0.10	0.1155	0.0169	1.0000	0.3500	0.1000	0.2700	0.0400	0.1700	0.7100
2	France	CAC 40	0.15	0.2068	0.0239	0.3500	1.0000	0.3900	0.2700	0.5000	-0.0800	0.1500
3	Germany	DAX	0.15	0.1453	0.0136	0.1000	0.3900	1.0000	0.5300	0.7000	-0.2300	0.0900
4	Italy	MIB30	0.05	0.1799	0.0192	0.2700	0.2700	0.5300	1.0000	0.4600	-0.2200	0.3200
5	Japan	Nikkei 225	0.20	0.1559	0.0081	0.0400	0.5000	0.7000	0.4600	1.0000	-0.2900	0.1300
6	U. K.	FTSE 100	0.10	0.1462	0.0362	0.1700	-0.0800	-0.2300	-0.2200	-0.2900	1.0000	-0.0300
7	U. S.	S&P 500	0.25	0.1568	0.0166	0.7100	0.1500	0.0900	0.3200	0.1300	-0.0300	1.0000

Table 3.4: Parameters of the G-7 Indices Basket Option

Three methods for pricing basket call options, Naive Monte Carlo, the classical control variate and the new control variate, are implemented when some parameters have been given as above, and other parameters are selected as $S_i(0) = 100$, $i = 1, \ldots, 7$, risk free interest rate $r = 0.063$ and sample size $n = 10000$. The performance of different methods will be evaluated from three aspects, mean square error, length of 95% confidence interval and computation time.

when risk free interest rate $r = 0.063$, $S_i(0) = 100$, $i = 1, \dots, 7$ and sample size $n = 10000$.						
Time T	Fixed Strike K	Naive Monte Carlo	Root Mean Square Error	95% Confidence Interval	Length of the Interval	Computation Time (sec.)
0.5	80	21.6161	0.0668	$[21.4852, 21.7470]$	0.2618	0.0233
	100	3.9034	0.0480	$[3.8093, 3.9975]$	0.1882	0.0179
	120	0.0224	0.0036	$[0.0153, 0.0295]$	0.0142	0.0273
1	80	23.1604	0.0939	$[22.9764, 23.3444]$	0.3680	0.0266
	100	6.2480	0.0722	$[6.1064, 6.3896]$	0.2832	0.0263
	120	0.3481	0.0177	$[0.3135, 0.3828]$	0.0693	0.0201
2	80	26.0681	0.1310	$[25.8113, 26.3248]$	0.5135	0.0228
	100	10.2495	0.1093	$[10.0353, 10.4637]$	0.4284	0.0293
	120	2.0659	0.0540	$[1.9600, 2.1718]$	0.2118	0.0242
3	80	28.7309	0.1585	$[28.4202, 29.0415]$	0.6213	0.0579
	100	13.7795	0.1389	$[13.5073, 14.0517]$	0.5444	0.0157
	120	4.4929	0.0888	$[4.3188, 4.6670]$	0.3482	0.0254

Table 3.5: Results of Naive Monte Carlo Method for Pricing Basket Call Option

when risk free interest rate $r = 0.063$, $S_i(0) = 100$, $i = 1, \dots, 7$ and sample size $n = 10000$.						
Time T	Fixed Strike K	Classical Control Variate	Root Mean Square Error	95% Confidence Interval	Length of the Interval	Computation Time (sec.)
0.5	80	21.6041	0.0031	$[21.5980, 21.6103]$	0.0123	0.0225
	100	3.8830	0.0033	$[3.8765, 3.8895]$	0.0130	0.0213
	120	0.0235	8.5667×10^{-4}	$[0.0218, 0.0251]$	0.0034	0.0248
1	80	23.1453	0.0063	$[23.1330, 23.1576]$	0.0247	0.0215
	100	6.2203	0.0067	$[6.2071, 6.2335]$	0.0264	0.0269
	120	0.3564	0.0039	$[0.3489, 0.3640]$	0.0152	0.0215
2	80	26.0514	0.0127	$[26.0265, 26.0763]$	0.0498	0.0292
	100	10.2164	0.0137	$[10.1896, 10.2431]$	0.0535	0.0202
	120	2.0674	0.0117	$[2.0445, 2.0903]$	0.0458	0.0223
3	80	28.7116	0.0192	$[28.6740, 28.7493]$	0.0753	0.0221
	100	13.7454	0.0206	$[13.7051, 13.7858]$	0.0807	0.0190
	120	4.4760	0.0197	$[4.4374, 4.5147]$	0.0773	0.0206

Table 3.6: Results of Classical Control Variate for Pricing Basket Call Option

when risk free interest rate $r = 0.063$, $S_i(0) = 100$, $i = 1, \ldots, 7$ and sample size $n = 10000$.						
Time T	Fixed Strike K	New Control Variate	Root Mean Square Error	95% Confidence Interval	Length of the Interval	Computation Time (sec.)
0.5	80	21.6022	1.2122×10^{-5}	$[21.6022, 21.6023]$	4.7520×10^{-5}	0.0473
	100	3.8828	6.4116×10^{-4}	$[3.8815, 3.8840]$	0.0025	0.0350
	120	0.0238	3.1308×10^{-4}	$[0.0232, 0.0245]$	0.0012	0.0501
1	80	23.1409	2.0149×10^{-4}	$[23.1405, 23.1413]$	7.8982×10^{-4}	0.0403
	100	6.2196	0.0012	$[6.2173, 6.2219]$	0.0046	0.0244
	120	0.3538	0.0011	$[0.3516, 0.3560]$	0.0044	0.0408
2	80	26.0435	0.0013	$[26.0409, 26.0461]$	0.0052	0.0374
	100	10.2199	0.0032	$[10.2137, 10.2261]$	0.0124	0.0418
	120	2.0542	0.0032	$[2.0480, 2.0604]$	0.0124	0.0504
3	80	28.7007	0.0022	$[28.6965, 28.7050]$	0.0086	0.0257
	100	13.7441	0.0046	$[13.7350, 13.7531]$	0.0181	0.0296
	120	4.4553	0.0056	$[4.4443, 4.4663]$	0.0220	0.0229

Table 3.7: Results of New Control Variate for Pricing Basket Call Option

To compare these three methods, let us focus on Table 3.5 and 3.6 first. Because the mean square errors of every different time and every different strike price in Table 3.6 have been decreased by at least 5 times and a maximum of 20 times, reducing to the two digits after decimal point. For the same situation as in the length of 95% confidence interval, length of the interval in Table 3.5 is much larger than that in Table 3.6. In terms of computation time, there is no obvious difference between these two methods. However, the accuracy of simulation results has been greatly improved by the classical control variate method, which is more important for a simulation method. Therefore, it can be concluded that the performance of classical control variate is better than the naive Monte Carlo.

When it comes to the new control variate, because the classical control variate method did a greater job than naive Monte Carlo, then classical control variate method and new control variate method should be compared next. Given the same conditions, if we take a part of conditional arithmetic payoff as a control variate, the mean square error and length of 95% confidence interval can be reduced drastically. The mean square error and length of 95% confidence interval in Table 3.6 are at least 3 times and 256 times at most as large as those in Table 3.7. Taken computation time into consideration, although the computation time of Table 3.7 is slightly larger than the new control variate, because the new control variate needs to use some variables in the classical one, there is no distinguished increase for the computation time when using new control variate. However, the convergence and accuracy of the simulation are enhanced certainly, implying that the new control variate method is more efficient, compared with the classical one.

In conclusion, based on the results of Table 3.5, 3.6 and 3.7, among these three methods, the new control variate is more useful for pricing basket options from the aspects of convergence and efficiency.

Chapter 4

Conclusion

In this thesis, the novelty is the comprehensive implementation and comparison of different Monte Carlo methods both for Asian options and basket options. In terms of Asian options, a benchmark using Večeř dimensionality reduction approach is calculated first. By comparing the results from naive Monte Carlo, antithetic Monte Carlo and control variate Monte Carlo with the benchmark, we found that for pricing Asian options control variate method has the best performance among these three Monte Carlo methods because of its variance reduction and efficiency. We applied control variate methods to basket options as well. Although there is no benchmark for basket option pricing, the performance of variance reduction can still be expected. After carrying out naive Monte Carlo, the classical control variate and novel control variate approaches, it is obvious that the novel control variate method has the best efficiency and convergence. For further research it would be interesting to study other control variates with better efficiency for pricing options in classical models and new models (e.g. models with price impact).

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