Lattice-based Robust Distributed Coding Scheme for Correlated Sources

LATTICE-BASED ROBUST DISTRIBUTED CODING SCHEME FOR CORRELATED SOURCES

BY

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To my family with all my love.

Abstract

In this thesis we propose two lattice-based robust distributed source coding systems, one for two correlated sources and the other for three correlated sources. We provide a detailed performance analysis under the high resolution assumption. It is shown that, in a certain asymptotic regime, our scheme for two correlated sources achieves the information-theoretic limit of quadratic multiple description coding (MDC) when the lattice dimension goes to infinity, whereas a variant of the random coding scheme by Chen and Berger with Gaussian codes is 0.5 bits away from this limit. Our analysis also shows that, under the same asymptotic regime, when the lattice dimension goes to infinity, the proposed scheme for three correlated sources is very close to the theoretical bound for the symmetric quadratic Gaussian MDC problem with single description and all three descriptions decoders.

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Notation and abbreviations

CEO	Chief Executive Officer
RDSC	Robust Distributed Source Coding
MDC	Multiple Description Coding
MDLVQ	Multiple Description Lattice Vector Quantizer
X	Random variable X
$\mathcal{X}$	Alphabet of X
$\mathbb{E}[\cdot]$	Expectation operator
$\mathcal{RD}$	Rate-distortion region
$H(\cdot)$	Entropy
$h(\cdot)$	Differential entropy
$I(\cdot;\cdot)$	Mutual information
LMMSE	Squared distortion induced by the Linear Minimum Mean Squared error Estimate
σ_X^2	Variance of X
x^n	Row vector
$\mathbb{R}^n$	n-dimensional Euclidean space
$\|\cdot\|$	Norm
$\langle\cdot,\cdot\rangle$	Inner product

$\nu(\cdot)$	Volume of a measurable set
Λ	n-dimensional lattice
$\mathbf{G}$	Generator matrix
$Q_\Lambda(\cdot)$	Nearest-neighbor quantizer
λ	Lattice point
$V_\Lambda(\cdot)$	Voronoi region
$\overline{\mathcal{S}}$	Closure of set $\mathcal{S}$
$x^n \bmod \Lambda$	Modulo-lattice operation
$\mathcal{B}_r$	Open ball of radius r
$\bar{r}_\Lambda$	Covering radius of the lattice Λ
r_Λ	Inscribed radius of the lattice Λ
$G(\cdot)$	Normalized second moment
$N(\Lambda_2 : \Lambda_1)$	Index of Λ_2 with respect to Λ_1
$D(Q, X^n)$	Per sample expected distortion
Λ_c	Central lattice
Λ_s	Side lattice
Λ_{in}	Intermediate lattice
$\Lambda_{s/2}, \Lambda_{s/3}$	Fractional lattice
K	Index of Λ_{in} with respect to Λ_c
M	Index of Λ_s with respect to Λ_{in}
$\mathbb{P}[\cdot]$	Probability
$\mathcal{T}$	Set of coset representatives of Λ_s relative to fractional lattice
$\mathcal{U}$	Set of coset representatives of Λ_s relative to Λ_{in}

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Chapter 1

Introduction

Distributed source coding is a crucial category of source coding problems, which has received significant attention over the past few decades. In distributed source coding, multiple correlated sources are encoded separately and sent to a central decoder for joint decoding. For the case when the central decoder is required to recover both sources losslessly, Slepian and Wolf (Slepian and Wolf, 1973) characterized the achievable rate region. The case when one source is available as side information at the decoder, while the other source may be recovered with some distortion, was solved by Wyner and Ziv (Wyner and Ziv, 1976). A general formulation of the distributed source coding problem in the lossy case was provided by Berger (Berger, 1978) and Tung (Tung, 1978). However, the solution has been found only in certain special cases (Berger and Yeung, 1989; Oohama, 1997; Wagner *et al.*, 2008; Wang *et al.*, 2010; Wang and Chen, 2013, 2014).

A closely related problem is the CEO problem introduced in (Berger *et al.*, 1996), where the correlated sources are noisy observations of a single remote source, whose reconstruction is required at the joint decoder. The rate-distortion region for this problem has been completely characterized in the quadratic Gausian case by Oohama (Oohama, 2005) and Prabhakaran *et al.* (Prabhakaran *et al.*, 2004).

Most of past work assume that the central decoder receives the information sent by all separate encoders. However, in practice this may not be true. For instance, in the case of wireless communications, the quality of the channels may be fluctuating. If the quality of the channel connecting some encoder with the fusion centre becomes very weak, the decoder is no longer able to recover the transmitted information. In such cases a robust system is desired. The robust version of the distributed source coding problem was considered in the CEO setting by Ishwar *et al.* (Ishwar *et al.*, 2005) and Chen and Berger (Chen and Berger, 2008). The design of practical schemes was addressed in (Saxena *et al.*, 2006; Saxena and Rose, 2010; Wu *et al.*, 2016), where iterative algorithms were employed for locally optimal designs. On the other hand, the work of Heegard and Berger (Heegard and Berger, 1985) considers the robust version of the Wyner-Ziv problem and provides a characterization of the rate-distortion region.

The robust distributed source coding (RDSC) problem for the case of two and three correlated sources is considered in this thesis. We propose a structured coding schemes based on lattices and provide a detailed performance analysis under the high resolution assumption. Note that when the sources are identical, the setting being considered coincides with that of the classical multiple description coding (MDC) problem (Ozarow, 1980; Wolf *et al.*, 1980; Gamal and Cover, 1982; Ahlswede, 1985; Zhang and Berger, 1987; Wang *et al.*, 2011b; Wang and Viswanath, 2007, 2009; Chen, 2009; Song *et al.*, 2014). Our analysis of the scheme for two sources indicates that, in a certain asymptotic regime, the performance of the proposed lattice-based scheme approaches the information-theoretic limit of quadratic multiple description coding when the lattice dimension goes to ∞. For comparison we consider a variant of the random coding scheme originally proposed by Chen and Berger (Chen and Berger, 2008) for the robust CEO problem and prove that the sum-rate of the latter system with Gaussian codes is 0.5 bits away from the sum-rate of our proposed approach in the same asymptotic regime. The asymptotic analysis of the scheme for three sources shows that when the lattice dimension approaches infinity, its performance at high resolution is close to the information theoretic limit of the symmetric Gaussian quadratic MDC problem, when only single description decoders and all descriptions decoder are of interest. Specifically, the gap in sum-rate is only 0.207 bits.

Our design is inspired by the prior work on multiple description lattice vector quantizers (MDLVQ) of Vaishampayan *et al.* (Vaishampayan *et al.*, 2001) and Huang and Wu (Huang and Wu, 2006). It is worth pointing out that lattices have been used in prior work in other distributed source coding problems (R. Zamir and Erez, 2002; Servetto, 2007; Krithivasan and Pradhan, 2009; Reani and Merhav, 2015). Most of the aforementioned papers use dithered lattice quantization, except for the work of Servetto (Servetto, 2007), which performs the analysis under the assumption of very high rate and very high correlation.

The thesis is structured as follows. In Chapter 2, we analyze the performance of a random-coding-based RDSC scheme (similar to the one proposed in (Chen and Berger, 2008)) with Gaussian codes and prove that it does not achieve the information-theoretical limit of quadratic MDC in a certain asymptotic regime. We propose a lattice-based robust distributed source coding system for two correlated sources. The asymptotic performance analysis of this lattice-based scheme shows that it is able to achieve the fundamental limit of quadratic MDC in the aforementioned asymptotic regime. We point out that the work in chapter 2 has been submitted for publication. The paper containing the results in chapter 2 is currently under review for possible publication in IEEE Transactions on Information Theory. The Chapter is structured as follows. Section 2.2 intruduce the formulation of the RDSC problem. In Section 2.3 we analyze the performance of a random-coding-based RDSC scheme (similar to the one proposed in (Chen and Berger, 2008)) with Gaussian codes. Section 2.4 presents definitions and notations related to lattices. In Section 2.5 we introduce a lattice-based RDSC scheme. The asymptotic performance analysis of this lattice-based scheme is presented in Section 2.6. Finally, Section 2.7 concludes the Chapter.

In Chapter 3 we present a coding scheme based on lattices for three correlated sources and provide the performance analysis under the high resoluion assumption. Our analysis shows that the performance at high resolution of the proposed scheme is very close to the information theoretic limit of the symmetric Gaussian quadratic MDC problem with single description and all descriptions decoders. Chapter 3 is structured as follows. In Section 3.2 we present the problem formulation. Section 3.4 presents a structured coding scheme based on lattices for RDSC problem for the case of three correlated sources. The asymptotic performance analysis of this lattice-based scheme is carried out Section 3.5 . Finally, Section 3.6 contains the conclusion. Finally, Chapter 4 concludes the thesis.

Chapter 2

Lattice-based Robust Distributed Coding Scheme for Two Correlated Sources

2.1 Introduction

In this chapter we propose a lattice-based robust distributed source coding system for two correlated sources and provide a detailed performance analysis under the high resolution assumption. It is shown that, in a certain asymptotic regime, our scheme achieves the information-theoretic limit of quadratic multiple description coding when the lattice dimension goes to infinity, whereas a variant of the random coding scheme by Chen and Berger with Gaussian codes is 0.5 bits away from this limit.

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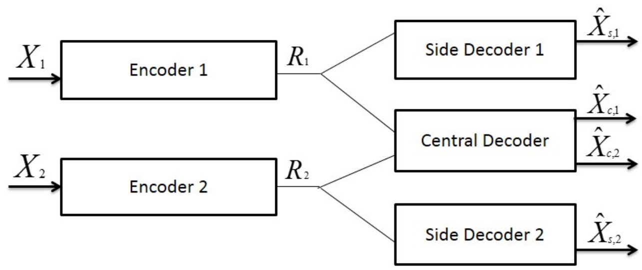

Figure 2.1: Block diagram of robust distributed source coding.

codes and prove that it does not achieve the information-theoretical limit of quadratic MDC in a certain asymptotic regime. Section 2.4 presents definitions and notations related to lattices, while Section 2.5 introduces a lattice-based RDSC scheme. The asymptotic performance analysis of this lattice-based scheme is carried out in Section 2.6, which shows, among other things, that it is able to achieve the fundamental limit of quadratic MDC in the aforementioned asymptotic regime. Finally, Section 2.7 concludes the Chapter.

2.2 Problem Formulation

Consider two sources X_1 and X_2 with joint probability distribution $f_{X_1X_2}$. The two sources generate a jointly i.i.d. random process $(X_{1,k}, X_{2,k})_{k\in\mathbb{N}}$. We will consider an RDSC system as illustrated in Figure 2.1. The system consists of two encoders and three decoders. Encoder i, $i = 1, 2$, has access only to source X_i, while the side decoder i, $i = 1, 2$, receives only the information sent by encoder i and aims at reconstructing source X_i. The central decoder receives the information from both encoders and aims at reconstructing both sources X_1 and X_2.

For each $i = 1, 2$, let $d_i : \mathcal{X}_i \times \hat{\mathcal{X}}_i \to [0, \infty)$ be a distortion measure, where $\mathcal{X}_i$ and $\hat{\mathcal{X}}_i$ are the source alphabet and the reconstruction alphabet for source X_i, respectively. The distortion measures are extended to sequences of length n as follows

$$d_i(x_i^n, \hat{x}_i^n) = \frac{1}{n} \sum_{k=1}^{n} d_i(x_{i,k}, \hat{x}_{i,k}),$$

where $x_i^n = (x_{i,1}, \cdots, x_{i,n})$, $\hat{x}_i^n = (\hat{x}_{i,1}, \cdots, \hat{x}_{i,n})$.

A six-tuple $(R_1, R_2, d_{s,1}, d_{s,2}, d_{c,1}, d_{c,2})$ is said achievable, if for any $\epsilon > 0$ and all sufficiently large n, there exist encoding functions

$$f_i^{(n)} : \mathcal{X}_i^n \to \{1, 2, \cdots, \lfloor 2^{n(R_i+\epsilon)} \rfloor\}, \quad i = 1, 2,$$

and decoding functions

$$g_{s,i}^{(n)} : \{1, 2, \cdots, \lfloor 2^{n(R_i+\epsilon)} \rfloor\} \to \hat{\mathcal{X}}_i^n, \quad i = 1, 2,$$
$$g_{c,i}^{(n)} : \{1, 2, \cdots, \lfloor 2^{n(R_1+\epsilon)} \rfloor\} \times \{1, 2, \cdots, \lfloor 2^{n(R_2+\epsilon)} \rfloor\} \to \hat{\mathcal{X}}_i^n, \quad i = 1, 2,$$

such that

$$\mathbb{E}\left[d_i(X_i^n, \hat{X}_{t,i}^n)\right] \leq d_{t,i} + \epsilon, \quad i = 1, 2, \quad t = s, c,$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator and

$$\hat{X}^n_{t,i} = g^{(n)}_{t,i}(f^{(n)}_i(X^n_i)), \quad i = 1, 2, \quad t = s, c.$$

The RDSC rate-distortion region, denoted by $\mathcal{RD}$, is the set of all such achievable six-tuples.

Furthermore, if Y is a random variable over some discrete alphabet $\mathcal{Y}$, with probability mass function p_Y, and $\mathbb{E}[-\log_2 p_Y]$ is finite, then the entropy of Y is $H(Y) \triangleq \mathbb{E}[-\log_2 p_Y]$. If $X^n \in \mathbb{R}^n$ is a continuous random variable with pdf f_{X^n}, and the quantity $\int_{\mathbb{R}^n} f_{X^n}(x^n) \log_2 f_{X^n}(x^n) dx^n$ is finite, then the differential entropy of X^n is $h(X^n) \triangleq -\int_{\mathbb{R}^n} f_{X^n}(x^n) \log_2 f_{X^n}(x^n) dx^n$.

2.3 A Random-coding-based RDSC Scheme

In this section we adapt a random coding scheme originally proposed by Chen and Berger (Chen and Berger, 2008) for the robust CEO problem to the current setting and analyze the asymptotic performance of this scheme when specialized to the MDC scenario.

Theorem 1 *We have $\mathcal{RD}_{in} \subseteq \mathcal{RD}$, where $\mathcal{RD}_{in}$ denotes the set of rate-distortion tuples $(R_1, R_2, d_{s,1}, d_{s,2}, d_{c,1}, d_{c,2})$ for which there exist auxiliary random variables U_1, U_2, W_1, W_2 (jointly distributed with the generic source variables X_1 and X_2) satisfying the following Markov chain*

$$W_1 \leftrightarrow U_1 \leftrightarrow X_1 \leftrightarrow X_2 \leftrightarrow U_2 \leftrightarrow W_2, \tag{2.1}$$

and deterministic mappings $g_{s,i} : \mathcal{W}_i \to \hat{\mathcal{X}}_i$, $g_{c,i} : \mathcal{U}_1 \times \mathcal{U}_2 \to \hat{\mathcal{X}}_i$, $i = 1, 2$, *such that*

$$R_1 \geq I(X_1; W_1) + I(X_1; U_1|U_2, W_1, W_2),$$
$$R_2 \geq I(X_2; W_2) + I(X_2; U_2|U_1, W_1, W_2),$$
$$R_1 + R_2 \geq I(X_1; W_1) + I(X_2; W_2) + I(X_1, X_2; U_1, U_2|W_1, W_2),$$
$$d_{s,i} \geq \mathbb{E}[d_i(X_i, g_{s,i}(W_i))], \quad i = 1, 2, \tag{2.2}$$
$$d_{c,i} \geq \mathbb{E}[d_i(X_i, g_{c,i}(U_1, U_2))], \quad i = 1, 2. \tag{2.3}$$

The inner bound $\mathcal{RD}_{in}$ in Theorem 1 is achievable by the following random coding scheme. Roughly speaking, encoder i produces (W_i, U_i), where W_i is a (lossy) description of X_i, and U_i is a refinement of W_i, $i = 1, 2$. Moreover, W_i is encoded using the conventional lossy source code while U_i is encoded using the Berger-Tung code (Berger, 1978; Tung, 1978) with (W_1, W_2) as the decoder side information, $i = 1, 2$. Side decoder i can recover W_i and use $g_{s,i}(W_i)$ as an estimate of X_i, $i = 1, 2$. The central decoder can recover (U_1, U_2) (as well as (W_1, W_2)) and use $g_{c,i}(U_1, U_2)$ as an estimate of X_i, $i = 1, 2$. The proof of Theorem 1 is similar to (Chen and Berger, 2008, Theorem 1) and is thus omitted.

In the rest of this paper, we assume $\mathcal{X}_1 = \mathcal{X}_2 = \hat{\mathcal{X}}_1 = \hat{\mathcal{X}}_2 = \mathbb{R}$ and adopt the squared distance as the distortion measure unless specified otherwise. To facilitate the evaluation of the achievable rate-distortion tuples in Theorem 1, we shall focus on so-called Gaussian codes (in the sense of (Zamir, 1999)), which correspond to the following construction. Let

$$U_i = X_i + Z_i, W_i = U_i + Z_i^{'}, \quad i = 1, 2, \tag{2.4}$$

where Z_1, Z_2, Z_1', Z_2' are zero-mean mutually independent Gaussian random variables and are independent of (X_1, X_2). It is clear that U_1, U_2, W_1, W_2 constructed according to (2.4) satisfy the Markov chain condition (2.1). Moreover, we restrict $g_{s,i}$ and $g_{c,i}$, $i = 1, 2$, to be linear MMSE estimators; as such, (2.2) and (2.3) can be rewritten as

$$d_{s,i} \geq \text{LMMSE}(X_i|W_i), \quad i = 1, 2, \tag{2.5}$$

$$d_{c,i} \geq \text{LMMSE}(X_i|U_1, U_2), \quad i = 1, 2, \tag{2.6}$$

where LMMSE denotes the squared distortion induced by the linear MMSE estimate.

Now consider the special case where $X_1 = X_2 = X$, $d_{s,1} = d_{s,2} = d_s$, and $d_{c,1} = d_{c,2} = d_c$. This is exactly the setting of the symmetric MDC problem. We shall assume that the source variable X is of mean zero, variance σ_X^2, and finite differential entropy $h(X)$. It is well-known (see, e.g., (Zamir, 1999; Chen *et al.*, 2006)) that in the asymptotic regime

$$d_s \to 0, \frac{d_c}{d_s} \to 0, \tag{2.7}$$

the minimum sum-rate of symmetric multiple description coding is given by

$$R_{MD}(d_s, d_c) = 2h(X) - \frac{1}{2}\log_2(4(2\pi e)^2 d_s d_c) + o(1). \tag{2.8}$$

We shall show that in the same asymptotic regime the minimum sum-rate of the random-coding-based RDSC scheme in Theorem 1 with Gaussian codes as defined by

(2.4)–(2.6) is given by

$$R_{RC}(d_s, d_c) = 2h(X) - \frac{1}{2}\log_2(2(2\pi e)^2 d_s d_c) + o(1), \tag{2.9}$$

therefore is 0.5 bits away from the fundamental limit.

First note that in the current setting (2.5) and (2.6) can be written equivalently as

$$d_s \geq \left(\frac{1}{\sigma_X^2} + \frac{1}{\sigma_{Z_i}^2 + \sigma_{Z_i'}^2}\right)^{-1}, \quad i = 1, 2, \tag{2.10}$$

$$d_c \geq \left(\frac{1}{\sigma_X^2} + \frac{1}{\sigma_{Z_1}^2} + \frac{1}{\sigma_{Z_2}^2}\right)^{-1}, \tag{2.11}$$

which implies

$$\sigma_{Z_i}^2 + \sigma_{Z_i'}^2 \leq (1 + o(1))d_s, \quad i = 1, 2, \tag{2.12}$$

$$\frac{\sigma_{Z_1}^2 \sigma_{Z_2}^2}{\sigma_{Z_1}^2 + \sigma_{Z_2}^2} \leq (1 + o(1))d_c, \tag{2.13}$$

in the asymptotic regime (2.7). It can be verified that

$$\begin{aligned}
&I(X; W_1) + I(X; W_2) + I(X; U_1, U_2|W_1, W_2) \\
&= I(X; W_1) + I(X; W_2) + I(X; U_1, U_2) - I(X; W_1, W_2) \\
&= h(W_1) - h(Z_1 + Z_1') + h(W_2) - h(Z_2 + Z_2') + h(U_1, U_2) - h(Z_1, Z_2) \\
&\quad - h(W_1, W_2) + h(Z_1 + Z_1', Z_2 + Z_2') \\
&= h(W_1) + h(W_2) + h(U_1, U_2) - h(Z_1, Z_2) - h(W_1, W_2). \qquad (2.14)
\end{aligned}$$

We have

$$\begin{aligned}
h(U_1,U_2)-h(W_1,W_2) &= -I(Z_1',Z_2';X+Z_1+Z_1',X+Z_2+Z_2')\\
&= -I(Z_1',Z_2';Z_1+Z_1'-Z_2-Z_2',X+Z_2+Z_2')\\
&= -I(Z_1',Z_2';Z_1+Z_1'-Z_2-Z_2')\\
&\quad - I(Z_1',Z_2';X+Z_2+Z_2'|Z_1+Z_1'-Z_2-Z_2') \qquad (2.15)
\end{aligned}$$

Substituting (2.15) into (2.14) gives

$$\begin{aligned}
&I(X;W_1)+I(X;W_2)+I(X;U_1,U_2|W_1,W_2)\\
&= h(W_1)+h(W_2)-h(Z_1,Z_2)-I(Z_1',Z_2';Z_1+Z_1'-Z_2-Z_2')\\
&\quad - I(Z_1',Z_2';X+Z_2+Z_2'|Z_1+Z_1'-Z_2-Z_2'). \qquad (2.16)
\end{aligned}$$

Note that

$$\begin{aligned}
&h(Z_1,Z_2)+I(Z_1',Z_2';Z_1+Z_1'-Z_2-Z_2')\\
&= h(Z_1,Z_2)+h(Z_1+Z_1'-Z_2-Z_2')-h(Z_1-Z_2)\\
&= \frac{1}{2}\log_2\left(\frac{(2\pi e)^2\sigma_{Z_1}^2\sigma_{Z_2}^2(\sigma_{Z_1}^2+\sigma_{Z_1'}^2+\sigma_{Z_2}^2+\sigma_{Z_2'}^2)}{\sigma_{Z_1}^2+\sigma_{Z_2}^2}\right)\\
&\le \frac{1}{2}\log_2\big(2(2\pi e)^2 d_s d_c\big)+o(1) \qquad (2.17)
\end{aligned}$$

in the asymptotic regime (2.7), where (2.17) is due to (2.12) and (2.13). Moreover,

$$\begin{aligned}
&I(Z_1', Z_2'; X+Z_2+Z_2'|Z_1+Z_1'-Z_2-Z_2') \\
&= h(X+Z_2+Z_2'|Z_1+Z_1'-Z_2-Z_2') - h(X+Z_2+Z_2'|Z_1+Z_1'-Z_2-Z_2', Z_1', Z_2') \\
&= h(X+Z_2+Z_2'|Z_1+Z_1'-Z_2-Z_2') - h(X+Z_2|Z_1-Z_2) \\
&= h(X+\tilde{Z}_1) - h(X+\tilde{Z}_2), \qquad (2.18)
\end{aligned}$$

where $\tilde{Z}_1 = Z_2+Z_2'-\mathbb{E}[Z_2+Z_2'|Z_1+Z_1'-Z_2-Z_2']$ and $\tilde{Z}_2 = Z_2-\mathbb{E}[Z_2|Z_1-Z_2]$. It can be shown (Linder and Zamir, 1994) that in the asymptotic regime (2.7)

$$\begin{aligned}
h(W_i) &= h(X)+o(1), \quad i=1,2, \\
h(\tilde{Z}_i) &= h(X)+o(1), \quad i=1,2,
\end{aligned}$$

which together with (2.16), (2.17), and (2.18) proves that

$$I(X;W_1)+I(X;W_2)+I(X;U_1,U_2|W_1,W_2) \geq 2h(X) - \frac{1}{2}\log_2\left(2(2\pi e)^2 d_s d_c\right) + o(1).$$

The tightness of this lower bound can be established by choosing $\sigma^2_{Z_i}$, $\sigma^2_{Z_i'}$, $i=1,2$, that satisfy (2.10) and (2.11) with equalities. This completes the proof of (2.9).

There are two possible reasons why the performance of this random-coding-based RDSC scheme with Gaussian codes, when specialized to the symmetric MDC setting, is bounded away from the fundamental limit. Firstly, the restriction to Gaussian codes might be suboptimal. Secondly and more importantly, the random-coding-based RDSC scheme itself might be suboptimal. It is well known (Ozarow, 1980; Gamal and Cover, 1982) that the El Gamal-Cover (EGC) inner bound is tight for the quadratic Gaussian MDC problem. However, the inner bound $\mathcal{RD}_{in}$ in Theorem 1, when specialized to the MDC setting, does not (at least expression-wise) coincide or subsume the EGC inner bound, therefore is unlikely to be tight. For the EGC inner bound, no Markov chain condition is imposed on the relevant auxiliary random variables. On the other hand, it is very difficult (if not impossible) to establish a single-letter inner bound of $\mathcal{RD}$ without a Markov chain condition similar to (2.1). In other words, the conventional random coding argument seems to fall short of providing a RDSC scheme that does not have a performance gap when specialized to the MDC setting. This motivates us to develop an alternative RDSC scheme based on lattices that is able to close the gap in the MDC scenario.

2.4 Lattice-related Definitions and Notations

Before introducing the proposed scheme we need to clarify the lattice-related definitions and notations to be used throughout this work, which is the purpose of this section.

We will denote by x^n row vectors in $\mathbb{R}^n$. For $x^n = (x_1, \cdots, x_n) \in \mathbb{R}$ and $y^n = (y_1, \cdots, y_n) \in \mathbb{R}^n$, let $\langle x^n, y^n \rangle \triangleq \sum_{i=1}^{n} x_i y_i$, and $\|x^n\| \triangleq \sqrt{\langle x^n, x^n \rangle}$. We will use $\mathbf{0}$ for the all-zero n-dimensional vector. For any set $\mathcal{S} \subseteq \mathbb{R}^n$, any $\sigma \in \mathbb{R}$, and any $x^n \in \mathbb{R}^n$, denote

$$\begin{aligned} x^n + \mathcal{S} &\triangleq \{x^n + y^n | y^n \in \mathcal{S}\}, \\ \sigma\mathcal{S} &\triangleq \{\sigma y^n | y^n \in \Lambda\}. \end{aligned}$$

If $\mathcal{S}$ is a measurable set then $\nu(\mathcal{S})$ denotes its volume, i.e.,

$$\nu(\mathcal{S}) \triangleq \int_{\mathcal{S}} dx^n.$$

An n-dimensional lattice Λ is the set of all possible integer linear combinations of the rows of $\mathbf{G}$, for some $n \times n$ non-singular matrix $\mathbf{G}$. In other words, we have

$$\Lambda \triangleq \{\lambda \in \mathbb{R}^n | \lambda = \boldsymbol{i} \cdot \mathbf{G}, \boldsymbol{i} \in \mathbb{Z}^n\}.$$

The nearest-neighbor quantizer associated with the lattice Λ is a function $Q_\Lambda(\cdot)$ which maps each $x^n \in \mathbb{R}^n$ to its nearest lattice point, i.e.,

$$Q_\Lambda(x^n) \triangleq \arg\min_{\lambda \in \Lambda} \|x^n - \lambda\|. \tag{2.19}$$

For every $\lambda \in \Lambda$ the set of all points mapped by Q_Λ to λ is the *Voronoi region* $V_\Lambda(\lambda)$ *of* λ *in* Λ. Note that the ties in (2.19) are broken in a systematic manner such that the following relation holds

$$V_\Lambda(\lambda) = \lambda + V_\Lambda(0), \quad \forall \lambda \in \Lambda.$$

For any set $\mathcal{S} \subseteq \mathbb{R}^n$, let $\overline{\mathcal{S}}$ denote the closure of the set $\mathcal{S}$, i.e., the union of $\mathcal{S}$ with its boundary. Then, the following holds

$$\overline{V_\Lambda(\lambda)} = \{x^n \in \mathbb{R}^n | \|x^n - \lambda\| \leq \|x^n - \lambda'\| \text{ for any } \lambda' \in \Lambda\}.$$

It is worth pointing out that, according to our definition of the Voronoi region, which follows (Zamir, 2014), not all the points on the boundary of $V_\Lambda(\lambda)$ are included in $V_\Lambda(\lambda)$, therefore $\overline{V_\Lambda(\lambda)} \neq V_\Lambda(\lambda)$. We say that two Voronoi regions $V_\Lambda(\lambda_1)$ and $V_\Lambda(\lambda_2)$, where $\lambda_1, \lambda_2 \in \Lambda$, are *adjacent*, if their closures have points in common.

Further, for any $x^n \in \mathbb{R}^n$ define

$$x^n \bmod \Lambda \triangleq x^n - Q_\Lambda(x^n).$$

A *fundamental cell of the lattice* Λ is a bounded set $\mathcal{C}_0$ which, when shifted by the lattice points, generates a partition of $\mathbb{R}^n$ (Zamir, 2014). In other words, the sets $\lambda + \mathcal{C}_0$, for all $\lambda \in \Lambda$, form a partition of $\mathbb{R}^n$. All measurable fundamental cells of a lattice have the same volume (Zamir, 2014). This value is denoted by ν_Λ and we have $\nu_\Lambda = \nu(V_\Lambda(\mathbf{0}))$. Further, for any set $\mathcal{S} \subset \mathbb{R}^n$, denote

$$\bar{r}(\mathcal{S}) \triangleq \sup_{x^n \in \mathcal{S}} \|x^n\|.$$

The open ball of radius r, centered in the origin is denoted by $\mathcal{B}_r$, i.e.,

$$\mathcal{B}_r \triangleq \{x^n \in \mathbb{R}^n | \|x^n\| < r\}.$$

The *covering radius* of the lattice Λ is $\bar{r}_\Lambda \triangleq \bar{r}(V_\Lambda(\mathbf{0}))$. Additionally, we will denote by r_Λ the *inscribed radius* of the lattice Λ, which is defined as the radius of the largest ball centered at the origin and included in $\overline{V_\Lambda(\mathbf{0})}$.

The *normalized second moment* of a measurable set $\mathcal{S} \subseteq \mathbb{R}^n$ is defined as

$$G(\mathcal{S}) \triangleq \frac{\int_{\mathcal{S}} \|x^n\|^2 dx}{n\nu(\mathcal{S})^{\frac{2}{n}+1}}.$$

It is important to notice that the normalized second moment is invariant to scaling. The *normalized second moment of the lattice* Λ, denoted by $G(\Lambda)$, is the normalized second moment of the Voronoi region of $\mathbf{0}$, i.e.,

$$G(\Lambda) \triangleq G(V_\Lambda(\mathbf{0})).$$

A pair of lattices (Λ_1, Λ_2) are said to be *nested* if $\Lambda_2 \subset \Lambda_1$, i.e., if Λ_2 is a sublattice of Λ_1. The lattice Λ_1 is termed the *fine lattice*, while Λ_2 is termed the *coarse lattice.* The index of Λ_2 with respect to Λ_1 is $N(\Lambda_2 : \Lambda_1) \triangleq \frac{\nu_{\Lambda_2}}{\nu_{\Lambda_1}}$. For any $\lambda_1 \in \Lambda_1$, the set $\lambda_1 + \Lambda_2$ is called a *coset of* Λ_2 *relative to* Λ_1. A set $\mathcal{L} \subset \Lambda_1$ is called a *set of coset representatives of* Λ_2 *relative to* Λ_1 if the following two conditions hold

$$\Lambda_1 = \cup_{\lambda_1 \in \mathcal{L}} \left(\lambda_1 + \Lambda_2\right),$$
$$(\lambda_1 + \Lambda_2) \cap (\lambda_1' + \Lambda_2) = \emptyset \text{ for any } \lambda_1 \neq \lambda_1' \in \mathcal{L}.$$

The above conditions imply that any point $\lambda \in \Lambda_1$ can be written in a unique way as $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \in \mathcal{L}$ and $\lambda_2 \in \Lambda_2$. As shown in (Zamir, 2014), if $\mathcal{C}_0$ is a fundamental cell of the coarse lattice Λ_2 then the set $\mathcal{C}_0 \cap \Lambda_1$ is a set of coset representatives of Λ_2 relative to Λ_1.

We use the squared error as a distortion criterion. For any quantizer Q defined on $\mathbb{R}^n$ and any random variable $X^n \in \mathbb{R}^n$ we denote by $D(Q, X^n)$ the per sample expected distortion, i.e.,

$$D(Q, X^n) \triangleq \frac{1}{n}\mathbb{E}\left[\| Q(X^n) - X^n \|^2\right].$$

2.5 Proposed Lattice-based RDSC Scheme

We will assume for the rest of the paper that the marginal probability density functions (pdf) f_{X_1} and f_{X_2} are continuous with finite marginal differential entropies $h(X_1)$ and $h(X_2)$. We assume that X_1 and X_2 have mean zero, variance σ_X^2 and correlation coefficient ρ.

The proposed coding scheme uses four nested lattices in $\mathbb{R}^n$: $\Lambda_s \subset \Lambda_{s/2} \subset \Lambda_{in} \subset \Lambda_c$. The finest lattice, Λ_c, is called the *central* lattice. The central lattice points will be used for the reconstruction at the central decoder. The coarsest lattice, Λ_s, is called *side* lattice since it is used for the reconstruction at the side decoders. The lattices Λ_{in} and $\Lambda_{s/2}$ are auxiliary lattices used in the design. Λ_{in} is called the *intermediate* lattice and it is chosen such that to satisfy a requirement which will be revealed shortly. The lattice $\Lambda_{s/2}$ is called the fractional lattice and it is defined as $\Lambda_{s/2} \triangleq \frac{1}{2}\Lambda_s$. We point out that $\Lambda_s = c\Lambda_{in}$ for some even positive integer c. Therefore, $\Lambda_{s/2}$ defined as above is also a sublattice of Λ_{in}. Let us denote $K \triangleq N(\Lambda_{in} : \Lambda_c)$ and $M \triangleq N(\Lambda_s : \Lambda_{in})$. It follows that $M = c^n$.

In order to simplify the notation we will use in the sequel only the subscript c, in, $s/2$, respectively s, instead of Λ_c, Λ_{in}, $\Lambda_{s/2}$, respectively Λ_s. For instance, we will use $\bar{r}_c$ instead of $\bar{r}_{\Lambda_c}$, for the covering radius of Λ_c.

Let $\Delta_{s/2}$ denote the smallest distance between two points belonging, respectively, to the closures of two non-adjacent Voronoi regions. We assume that the coefficient c is large enough so that the following condition holds

$$\Delta_{s/2} > 3\bar{r}_{in}. \tag{2.20}$$

The above condition is needed for proper operation at the central decoder, as it will be seen in the proof of Proposition 1.

Another important parameter in our construction is $r_0 > 0$. Our scheme is designed such that when the input sequences x_1^n, x_2^n are within the distance r_0 from one another, the central decoder is able to refine the reconstruction of each source using the information received from the other encoder. On the other hand, when the above condition is violated, the reconstruction at the central decoder has essentially the same quality as the reconstruction at the side decoder. For this reason the probability

$$\mathcal{P}(r_0) \triangleq \mathbb{P}[X_2^n - X_1^n \notin \mathcal{B}_{r_0}] \tag{2.21}$$

plays a crucial role in the performance of our scheme. As we will see in the next section the choice of r_0 governs the trade-off between the quality of the reconstruction at the central decoder and the encoder sum-rate.

The choice of the lattice Λ_{in} is related to the value r_0. More specifically, Λ_{in} is chosen as a sublattice of Λ_c satisfying the condition

$$r_0 + 2\bar{r}_c \leq r_{in}.$$

Recall that r_{in} denotes the inscribed radius of the lattice Λ_{in}. The reason for the above requirement is to ensure that the inequalities specified in the following result hold, since they are essential in the operation of the proposed scheme.

Lemma 1 *If* $x_2^n - x_1^n \in \mathcal{B}_{r_0}$, *then*

$$\|Q_c(x_1^n) - Q_c(x_2^n)\| < r_{in},$$
$$\|Q_{in}(Q_c(x_1^n)) - Q_{in}(Q_c(x_2^n))\| < 3\bar{r}_{in}.$$

Let $\lambda_{c,i} \triangleq Q_c(x_i^n)$, and $\lambda_i \triangleq Q_{in}(\lambda_{c,i})$, for $i = 1, 2$. Using the triangle inequality repeatedly, one obtains that

$$\| \lambda_{c,1} - \lambda_{c,2} \| \leq \| \lambda_{c,1} - x_1^n \| + \| x_1^n - x_2^n \| + \| x_2^n - \lambda_{c,2} \| < r_0 + 2\bar{r}_c \leq r_{in}.$$

Additionally,

$$\| \lambda_1 - \lambda_2 \| \leq \| \lambda_1 - \lambda_{c,1} \| + \| \lambda_{c,1} - \lambda_{c,2} \| + \| \lambda_{c,2} - \lambda_2 \| < r_{in} + 2\bar{r}_{in} < 3\bar{r}_{in},$$

which completes the proof.

Further, we will define two labeling functions $\beta_i : \Lambda_{in} \to \Lambda_s$, for $i = 1, 2$. For this, we need to introduce some more notations as follows. Let $\mathcal{T} \triangleq V_s(\mathbf{0}) \cap \Lambda_{s/2}$. Then $\mathcal{T}$ is a set of coset representatives of Λ_s relative to $\Lambda_{s/2}$. Thus, we have $|\mathcal{T}| = N(\Lambda_s : \Lambda_{s/2}) = 2^n$ and

$$\Lambda_{s/2} = \bigcup_{\tau \in \mathcal{T}} (\tau + \Lambda_s).$$

It can be easily seen that the set $\cup_{\tau \in \mathcal{T}} V_{s/2}(\tau)$ is a fundamental cell of Λ_s. Denote $\mathcal{U} \triangleq \cup_{\tau \in \mathcal{T}} V_{s/2}(\tau) \cap \Lambda_{in}$. Then $\mathcal{U}$ is a set of coset representatives of Λ_s relative to Λ_{in}, which implies that $|\mathcal{U}| = N(\Lambda_s : \Lambda_{in}) = M$ and

$$\Lambda_{in} = \bigcup_{\lambda \in \mathcal{U}} (\lambda + \Lambda_s).$$

We will first define β_i for $\lambda \in \mathcal{U}$ as follows

$$\beta_1(\lambda) \triangleq c(\lambda - \tau), \quad \beta_2(\lambda) \triangleq 2\tau - c(\lambda - \tau), \tag{2.22}$$

where $\tau = Q_{s/2}(\lambda)$. Further, the mappings β_1 and β_2 are extended to Λ_{in} using shifting. For this, for arbitrary $\lambda \in \Lambda_{in}$, let $\lambda_{s/2} = Q_{s/2}(\lambda)$, i.e., $\lambda \in V_{s/2}(\lambda_{s/2})$. Then there is a unique pair $(\tau, \lambda_s) \in \mathcal{T} \times \Lambda_s$, such that $\lambda_{s/2} = \lambda_s + \tau$. More specifically, we have $\lambda_s = Q_s(\lambda_{s/2})$ and $\tau = \lambda_{s/2} \bmod \Lambda_s$. Then we define

$$\beta_1(\lambda) \triangleq \beta_1(\lambda - \lambda_s) + \lambda_s = c(\lambda - \lambda_s - \tau) + \lambda_s,$$

$$\beta_2(\lambda) \triangleq \beta_2(\lambda - \lambda_s) + \lambda_s = 2\tau - c(\lambda - \lambda_s - \tau) + \lambda_s.$$

The above definition implies that the mappings β_i satisfy the *shift-invariance property*, i.e., that

$$\beta_i(\lambda + \lambda'_s) = \beta_i(\lambda) + \lambda'_s, \quad \forall \lambda \in \Lambda_{in}, \quad \forall \lambda'_s \in \Lambda_s, \quad i = 1, 2.$$

The shift-invariance property further leads to the following relations, for $i = 1, 2$,

$$\beta_i^{-1}(\lambda_s) = \beta_i^{-1}(\mathbf{0}) + \lambda_s, \forall \lambda_s \in \Lambda_s, \tag{2.23}$$

$$\beta_i^{-1}(\mathbf{0}) = \{\lambda - \beta_i(\lambda) | \lambda \in \mathcal{U}\}. \tag{2.24}$$

Relation (2.23) is obvious. In order to prove (2.24) consider $\lambda' \in \Lambda_{in}$ and let $(\lambda, \lambda_s) \in \mathcal{U} \times \Lambda_s$, be the unique pair such that $\lambda' = \lambda + \lambda_s$. The shift-invariance property implies that $\beta_i(\lambda') = \beta_i(\lambda) + \lambda_s$, which leads to $\lambda_s = \beta_i(\lambda') - \beta_i(\lambda)$. Further, we obtain that

$\lambda' = \lambda + \beta_i(\lambda') - \beta_i(\lambda)$. Consequently, the equality $\beta_i(\lambda') = \mathbf{0}$ is equivalent to $\lambda' = \lambda - \beta_i(\lambda)$, fact which proves the claim.

We point out that the construction of the mappings β_1 and β_2 was inspired by the index assignment $\alpha = (\alpha_1, \alpha_2)$ used in MDLVQ (Vaishampayan *et al.*, 2001; Huang and Wu, 2006), in two ways: 1) by defining the mappings on a set of coset representatives first and then extending them by shifting; 2) by imposing the condition that $\beta_1(\lambda) + \beta_2(\lambda) = 2Q_{s/2}(\lambda)$ for each $\lambda \in \Lambda_{in}$. On the other hand, it is important to note that we cannot simply use the mappings $\alpha_1, \alpha_2 : \Lambda_{in} \to \Lambda_s$ that define the index assignment for MDLVQ[1] in (Vaishampayan *et al.*, 2001; Huang and Wu, 2006) in place of our mappings β_1, β_2, since the requirement at the central decoder in our case is stronger than for MDLVQ. In particular, based on a received pair of side lattice points $\lambda_{s,1}$, $\lambda_{s,2}$, the central decoder of the MDLVQ uniquely identifies a point $\lambda \in \Lambda_{in}$ such that $(\alpha_1(\lambda), \alpha_2(\lambda)) = (\lambda_{s,1}, \lambda_{s,2})$. However, as we will see shortly, the central decoder in our scheme needs to uniquely identify two points $\lambda_1, \lambda_2 \in \Lambda_{in}$ such that $(\beta_1(\lambda_1), \beta_2(\lambda_2)) = (\lambda_{s,1}, \lambda_{s,2})$, using the additional knowledge of $\lambda_1 - \lambda_2$. Using the pair of mappings (α_1, α_2) designed for the MDLVQ in place of (β_1, β_2) does not guarantee that the latter requirement is satisfied.

2.5.1 Proposed Scheme

Before describing the details of the proposed scheme we need the following discussion. Let us denote $\lambda_i = Q_{in}(Q_c(x_i^n))$, for $i = 1, 2$. Our scheme is designed such that side decoder i will be able to recover $\beta_i(\lambda_i)$ always, while the central decoder recovers $\lambda_{c,i} = Q_c(x_i^n)$, for $i = 1, 2$, when the input sequences are sufficiently close, i.e., when

[1]The lattice Λ_{in} takes here the place of the central lattice on which the index assignment is defined for MDLVQ.

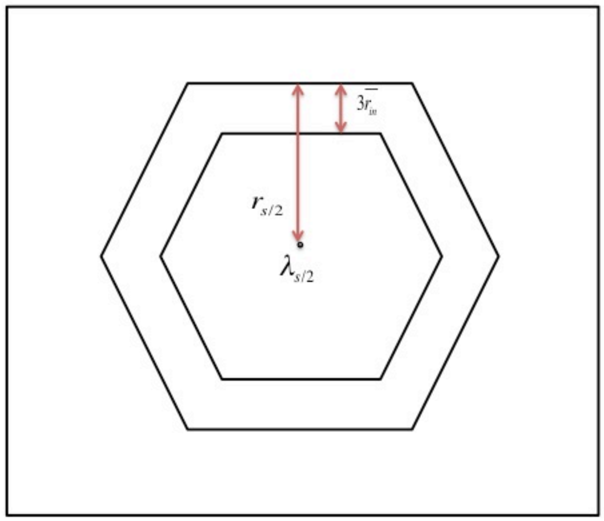

Figure 2.2: The set $\mathcal{C}(\lambda_{s/2})$ is the region between the two hexagons in the Figure.

$x_2^n - x_1^n \in \mathcal{B}_{r_0}$. However, for the central decoder to achieve this goal some additional information needs to be transmitted beside $\beta_1(\lambda_1)$ and $\beta_2(\lambda_2)$. The amount of this additional information is smaller when λ_1 and λ_2 are both in the same Voronoi cell of the lattice $\Lambda_{s/2}$. Encoder i is not able to determine all the time if this is the case or not, since it does not have knowledge of the other source sequence. However, based on Lemma 1, if $\lambda_i \in V_{s/2}(\lambda_{s/2})$ and the distance from λ_i to the boundary of $V_{s/2}(\lambda_{s/2})$ is not smaller than $3\bar{r}_{in}$, then encoder i can infer that the other sequence is also in $V_{s/2}(\lambda_{s/2})$ when $x_2^n - x_1^n \in \mathcal{B}_{r_0}$. Thus, we define the set

$$\mathcal{C} \triangleq \cup_{\lambda_{s/2} \in \Lambda_{s/2}} \mathcal{C}(\lambda_{s/2}), \text{ where } \mathcal{C}(\lambda_{s/2}) \triangleq V_{s/2}(\lambda_{s/2}) \setminus \left(\lambda_{s/2} + \gamma V_{s/2}(\mathbf{0})\right), \quad (2.25)$$

for $\gamma \triangleq 1 - \frac{3\bar{r}_{in}}{r_{s/2}}$, as shown in Figure 2.2. According to Lemma 1, if $\lambda_i \notin \mathcal{C}$, then λ_{3-i} is in the same Voronoi cell of $\Lambda_{s/2}$ as λ_i, when $x_2^n - x_1^n \in \mathcal{B}_{r_0}$. Now we are ready to present the details of the encoder and decoder operation.

Encoder. Encoder i, for $i = 1, 2$, operates as follows. First the input sequence x_i^n is quantized to the closest central lattice point $\lambda_{c,i} \triangleq Q_c(x_i^n)$. Next, the point $\lambda_{c,i}$ is quantized to the closest point in the lattice Λ_{in}, $\lambda_i \triangleq Q_{in}(\lambda_{c,i})$. Let $u_i \triangleq \lambda_{c,i} \bmod \Lambda_{in}$ and $\lambda_{s,i} \triangleq \beta_i(\lambda_i)$. Then encoder i outputs $\lambda_{s,i}$, u_i and b_i, where $b_i = 1$ if $\lambda_i \in \mathcal{C}$ and $b_i = 0$ otherwise. Moreover, if $b_i = 1$ encoder i also transmits $\tau_i \triangleq Q_{s/2}(\lambda_i) \bmod \Lambda_s$. The first component, $\lambda_{s,i}$, will be used at the side decoder i, therefore, it is compressed using entropy coding before transmission. On the other hand, u_1 and u_2 are used only at the central decoder, therefore they will be compressed using Slepian-Wolf coding. Finally, b_i and τ_i will also be used only at the central decoder, thus they may be compressed using Slepian-Wolf coding. However, we will use entropy coding to encode b_i and fixed length codes for τ_i for simplicity of analysis, since, as we will see shortly, the rate overhead is negligible asymptotically.

Decoder. Side decoder i, for $i = 1, 2$, outputs the reconstruction $\hat{x}_{s,i}^n \triangleq \lambda_{s,i}$. The central decoder recovers both values $\lambda_{s,1}$ and $\lambda_{s,2}$, and additionally, u_1, u_2, b_1, b_2. First the decoder checks if the following condition is satisfied

$$\|\lambda_{s,1} - \lambda_{s,2}\| \leq (8 + c)\bar{r}_s + 3\bar{r}_{in}. \tag{2.26}$$

If the condition is violated then the decoder concludes that $x_2^n - x_1^n \notin \mathcal{B}_{r_0}$, and outputs $\lambda_{s,i}$ as the reconstruction for source i, i.e., $\hat{x}_{c,i}^n \triangleq \lambda_{s,i}$, for $i = 1, 2$.

If condition (2.26) is satisfied the decoder assumes that $x_2^n - x_1^n \in \mathcal{B}_{r_0}$, and it proceeds as follows. First the following is computed

$$\tilde{\lambda} \triangleq Q_{in}(u_1 - u_2). \tag{2.27}$$

Next the decoder proceeds based on the values of b_1 and b_2, and of τ_1 and τ_2 (if applicable), according to the following cases.

1) If $b_1 = 0$ or $b_2 = 0$ the decoder evaluates

$$\tilde{\lambda}_{s/2} \triangleq 1/2(\lambda_{s,1} + \lambda_{s,2} + c\tilde{\lambda}), \quad \tilde{\tau} \triangleq \tilde{\lambda}_{s/2} \bmod \Lambda_s, \tag{2.28}$$

$$\tilde{\lambda}_1 \triangleq \tilde{\lambda}_{s/2} + \tfrac{1}{c}(\lambda_{s,1} - \tilde{\lambda}_{s/2} + \tilde{\tau}), \quad \tilde{\lambda}_2 \triangleq \tilde{\lambda}_{s/2} + \tfrac{1}{c}(\tilde{\tau} + \tilde{\lambda}_{s/2} - \lambda_{s,2}), \tag{2.29}$$

and outputs the reconstructions $\hat{x}^n_{c,i} \triangleq \tilde{\lambda}_i + u_i$, for $i = 1, 2$.

2) If $b_1 = b_2 = 1$ and $\tau_1 = \tau_2$ the decoder proceeds as in case 1).

3) If $b_1 = b_2 = 1$ and $\tau_1 \neq \tau_2$ then the decoder computes

$$\tilde{v} \triangleq 1/2(\lambda_{s,1} + \lambda_{s,2} + c\tilde{\lambda} - 2\tau_2 - c(\tau_2 - \tau_1)), \quad \hat{w} \triangleq \tilde{v} \bmod \Lambda_s, \tag{2.30}$$

$$\tilde{w} \triangleq \hat{w} - Q_s(\hat{w} + \tfrac{1}{2}(\tau_2 - \tau_1)), \tag{2.31}$$

$$\tilde{\lambda}_s \triangleq \tilde{v} - (c+1)\tilde{w}, \quad \tilde{\lambda}'_s \triangleq \tilde{\lambda}_s + 2\tilde{w}, \tag{2.32}$$

$$\tilde{\lambda}_1 \triangleq \tilde{\lambda}_s + \tau_1 + \tfrac{1}{c}(\lambda_{s,1} - \tilde{\lambda}_s), \quad \tilde{\lambda}_2 \triangleq \tilde{\lambda}'_s + \tau_2 + \tfrac{1}{c}(2\tau_2 + \tilde{\lambda}'_s - \lambda_{s,2}). \tag{2.33}$$

Finally, the reconstructions are computed as $\hat{x}^n_{c,i} \triangleq \tilde{\lambda}_i + u_i$, for $i = 1, 2$.

Proposition 1 *Let $\lambda_{c,i} \triangleq Q_c(x^n_i)$, $\lambda_i \triangleq Q_{in}(\lambda_{c,i})$, $u_i \triangleq \lambda_{c,i} \bmod \Lambda_{in}$, $\lambda_{s,i} \triangleq \beta_i(\lambda_i)$ and $\tau_i \triangleq Q_{s/2}(\lambda_i) \bmod \Lambda_s$, for $i = 1, 2$. Then when $x^n_2 - x^n_1 \in B_{r_0}$ and the Slepian-Wolf decoding of u_1 and u_2 is successful, we have $\hat{x}^n_{c,i} = \lambda_{c,i}$, for $i = 1, 2$.*

Proof: Assume that $x_2^n - x_1^n \in B_{r_0}$ and that the Slepian-Wolf decoder employed at the central decoder is able to recover u_1 and u_2 correctly. First we need to prove that condition (2.26) is satisfied. To this end we first show that the following relation holds

$$\bar{r}(\beta^{-1}(\mathbf{0})) \leq (4 + c/2)\bar{r}_s. \tag{2.34}$$

Note that relation (2.24) leads to

$$\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq \bar{r}(\mathcal{U}) + \bar{r}(\beta_i(\mathcal{U})). \tag{2.35}$$

Further, since $\mathcal{T} \subset V_s(\mathbf{0})$ and $V_{s/2}(\mathbf{0}) \subset V_s(\mathbf{0})$ we obtain that $\mathcal{U} \subset \cup_{\tau\in\mathcal{T}}(\tau + V_{s/2}(\mathbf{0})) \subset 2V_s(\mathbf{0})$. Thus, $\bar{r}(\mathcal{U}) \leq 2\bar{r}_s$. Moreover, from the definition of β_i given in (2.22), we obtain that $\bar{r}(\beta_i(\mathcal{U})) \leq 2\bar{r}(\mathcal{T}) + c\bar{r}_{s/2} \leq 2\bar{r}_s + c\bar{r}_{s/2}$. The above discussion, together with relation (2.35) and the fact that $\bar{r}_{s/2} = 1/2\bar{r}_s$, implies (2.34).

By applying the triangle inequality and the fact that $\|\lambda - \beta_i(\lambda)\| \leq \bar{r}(\beta^{-1}(\mathbf{0}))$, together with Lemma 1, we obtain

$$\|\lambda_{s,1} - \lambda_{s,2}\| \leq \|\lambda_{s,1} - \lambda_1\| + \|\lambda_1 - \lambda_2\| + \|\lambda_2 - \lambda_{s,2}\| \leq 2\bar{r}(\beta^{-1}(\mathbf{0})) + 3\bar{r}_{in}.$$

By combining the above with (2.34) relation (2.26) follows.

Using Lemma 1 and the fact that $\lambda_{c,i} = \lambda_i + u_i$, $i = 1, 2$, we obtain that

$$r_{in} > \|\lambda_{c,1} - \lambda_{c,2}\| = \|u_1 - u_2 - (\lambda_2 - \lambda_1)\|,$$

which, together with the fact that $\lambda_2 - \lambda_1 \in \Lambda_{in}$, implies that $u_1 - u_2 \in V_{in}(\lambda_2 - \lambda_1)$, i.e., $\lambda_2 - \lambda_1 = Q_{in}(u_1 - u_2)$. This further implies that $\tilde{\lambda}$ computed in (2.27) satisfies the equality

$$\tilde{\lambda} = \lambda_2 - \lambda_1. \tag{2.36}$$

Let $\lambda_s \triangleq Q_s(Q_{s/2}(\lambda_1))$ and $\lambda'_s \triangleq Q_s(Q_{s/2}(\lambda_2))$. Using the fact that $\tau_i \triangleq Q_{s/2}(\lambda_i) \bmod \Lambda_s$, for $i = 1, 2$, it follows that $\lambda_1 \in V_{s/2}(\lambda_s + \tau_1)$ and $\lambda_2 \in V_{s/2}(\lambda'_s + \tau_2)$. Moreover, since $\lambda_{s,i} = \beta_i(\lambda_i)$ for $i = 1, 2$, we obtain that

$$\lambda_{s,1} = c(\lambda_1 - \lambda_s - \tau_1) + \lambda_s \quad \lambda_{s,2} = \lambda'_s + 2\tau_2 - c(\lambda_2 - \lambda'_s - \tau_2). \tag{2.37}$$

Assume now that case 1) holds. According to Lemma 1 we have $\lambda_s + \tau_1 = \lambda'_s + \tau_2$. Since $\tau_1, \tau_2 \in \mathcal{T}$ it follows that $\lambda_s = \lambda'_s$ and $\tau_1 = \tau_2$. Using further equations (2.28), (2.36) and (2.37) we obtain that $\tilde{\lambda}_{s/2} = \lambda_s + \tau_1$. This implies that $\tau_1 = \tilde{\lambda}_{s/2} \bmod \Lambda_s$, i.e., $\tilde{\tau} = \tau_1$. Equations (2.29) imply that $\tilde{\lambda}_i = \lambda_i$ and further that $\hat{x}^n_{c,i} = \lambda_{c,i}$, for $i = 1, 2$.

Assume now that $b_1 = b_2 = 1$. Recall that according to Lemma 1 we have $\|\lambda_1 - \lambda_2\| < 3\bar{r}_{in}$. Condition (2.20) further ensures that $\|\lambda_1 - \lambda_2\| < \Delta_{s/2}$, which implies that $V_{s/2}(\lambda_s + \tau_1)$ and $V_{s/2}(\lambda'_s + \tau_2)$ are either identical or adjacent. Further, if $\tau_1 = \tau_2$ it follows that $\lambda'_s + \tau_2 - (\lambda_s + \tau_1) \in \Lambda_s$. Thus, $V_{s/2}(\lambda_s + \tau_1)$ and $V_{s/2}(\lambda'_s + \tau_2)$ cannot be adjacent. Consequently, the equality $\lambda'_s + \tau_2 = \lambda_s + \tau_1$ holds and the proof proceeds as in case 1).

Assume now that $\tau_1 \neq \tau_2$. Then $\lambda_s + \tau_1 \neq \lambda'_s + \tau_2$. Denote $\Delta\lambda_{s/2} \triangleq \lambda'_s + \tau_2 - (\lambda_s + \tau_1)$. Then $\mathbf{0}$ and $\Delta\lambda_{s/2}$ are adjacent points of the lattice $\Lambda_{s/2}$ (i.e., their Voronoi regions are adjacent). It follows that

$$\Delta\lambda_{s/2} \in \overline{V_s(\mathbf{0})}. \tag{2.38}$$

Let $w \triangleq \frac{1}{2}(\lambda_s' - \lambda_s)$. Using equations (2.30), (2.36) and (2.37) we obtain that

$$\tilde{v} = \lambda_s + \frac{c}{2}(\lambda_s' - \lambda_s) + w. \tag{2.39}$$

Since c is even, it follows that $\frac{c}{2}(\lambda_s' - \lambda_s) \in \Lambda_s$. Thus, $w \bmod \Lambda_s = \tilde{v} \bmod \Lambda_s = \hat{w}$. It follows that $w = \bar{\lambda}_s + \hat{w}$ for some $\bar{\lambda}_s \in \Lambda_s$. Then $\Delta\lambda_{s/2} = 2w + \tau_2 - \tau_1 = 2(\bar{\lambda}_s + \hat{w}) + \tau_2 - \tau_1$. Using further (2.38) leads to $\frac{1}{2}\Delta\lambda_{s/2} = \bar{\lambda}_s + \hat{w} + \frac{1}{2}(\tau_2 - \tau_1) \in \frac{1}{2}\overline{V_s(\mathbf{0})} \subset V_s(\mathbf{0})$, which further implies that $-\bar{\lambda}_s = Q_s(\hat{w} + \frac{1}{2}(\tau_2 - \tau_1))$. It follows that $\tilde{w} = w$, where $\tilde{w}$ is defined in (2.31). Combining with (2.32) and (2.39) we obtain that $\tilde{\lambda}_s = \lambda_s$ and $\tilde{\lambda}_s' = \lambda_s'$. Finally, equations (2.33) imply that $\tilde{\lambda}_i = \lambda_i$ and further that $\hat{x}_{c,i}^n = \lambda_{c,i}$, for $i = 1, 2$.

2.6 Performance Analysis

In this section we will evaluate the performance of the proposed lattice-based scheme, in the high resolution regime. More specifically, we require that the following relations hold simultaneously

$$M\nu_s \to 0, \quad M \to \infty, \quad K \text{ is constant.} \tag{2.40}$$

Recall that $M = \frac{\nu_s}{\nu_{in}} = c^n$ and $K = \frac{\nu_{in}}{\nu_c}$. Note that this asymptotic regime is similar in spirit with that considered in the prior work on MDLVQ (Vaishampayan *et al.*, 2001; Huang and Wu, 2006; Zhang *et al.*, 2011). Clearly, the conditions specified in (2.40) imply that ν_s, ν_{in} and ν_c approach 0. Further, since Λ_{in} is a sublattice of Λ_c

such that $r_0 + 2\bar{r}_c \leq r_{in}$, we also have that

$$r_0 = O(r_c) = O(\nu_c^{\frac{1}{n}})$$

as (2.40) holds. Additionally, the fact that K is constant implies that Λ_c and Λ_{in} are scaled by the same factor θ, i.e., there are some fixed lattices $\Lambda_{c,0}$ and $\Lambda_{in,0}$ such that

$$\Lambda_c = \theta\Lambda_{c,0}, \quad \Lambda_{in} = \theta\Lambda_{in,0}, \quad \Lambda_s = c\theta\Lambda_{in,0}.$$

Then the asymptotic regime specified by (2.40) is equivalently stated in terms of the parameters c and θ as follows

$$\theta \to 0, \quad c \to \infty, \quad c^2\theta \to 0.$$

In order to proceed we need to introduce a few more notations. For $i = 1, 2$, let $d_{s,i}$ denote the distortion of source i at side decoder i and let $d_{c,i}$ denote the distortion of source i at the central decoder. For each $\lambda_s \in \Lambda_s$ and $i = 1, 2$, let $\mathcal{A}_i(\lambda_s) \triangleq \{x_i^n | \hat{x}_{s,i}^n = \lambda_s\}$. Further, for each $\lambda \in \Lambda_{in}$, denote $\mathcal{M}(\lambda) \triangleq \cup_{\lambda_c \in V_{in}(\lambda) \cap \Lambda_c} V_c(\lambda_c)$. Then $\mathcal{A}_i(\lambda_s) = \cup_{\lambda \in \beta_i^{-1}(\lambda_s)} \mathcal{M}(\lambda)$. Clearly, we have $\mathcal{M}(\lambda) = \lambda + \mathcal{M}(\mathbf{0})$ for all $\lambda \in \Lambda$. This fact together with relation (2.23) implies that

$$\mathcal{A}_i(\lambda_s) = \mathcal{A}_i(\mathbf{0}) + \lambda_s, \quad \forall \lambda_s \in \Lambda_s. \tag{2.41}$$

Obviously, we have $d_{s,i} = D(Q_{\mathcal{A}_i}, X_i^n)$, where $Q_{\mathcal{A}_i}$ denotes the quantizer which maps each input sequence $x_i^n \in \mathcal{A}_i(\lambda_s)$ to λ_s, for $\lambda_s \in \Lambda_s$.

Further, let us denote $\Delta_{i,sup} \triangleq \sup_{x_i^n \in \mathbb{R}^n} \|x_i^n - \hat{x}_{c,i}^n\|$, $i = 1, 2$. Additionally, Let

$\mathcal{P}_{e,SW}$ denote the probability that the Slepian-Wolf decoder fails. In view of definition (2.21) of $\mathcal{P}(r_0)$ and of Proposition 1 it follows that, for $i = 1, 2$,

$$D(Q_c, X_i^n) \leq d_{c,i} \leq (\mathcal{P}(r_0) + \mathcal{P}_{e,SW})\Delta_{i,sup}^2 + D(Q_c, X_i^n).$$

The following lemma, proved in Appendix B, gives an upper bound for $\Delta_{i,sup}$.

Lemma 2 *There is some constant κ_0 such that for each $i = 1, 2$, and c sufficiently large, the following holds*

$$\Delta_{i,sup} \leq \kappa_0 \left(M\nu_s\right)^{\frac{1}{n}}.$$

It is known that the probability that the Slepian-Wolf decoder fails can be made arbitrarily small by increasing the block length used for Slepian-Wolf encoding. Since $\Delta_{i,sup}$ is bounded, it follows that the impact on the distortion of the Slepian-Wolf decoder failure can also be made arbitrarily small. Therefore, in the limit as the block length of Slepian-Wolf encoder approaches infinity, the following holds

$$D(Q_c, X_i^n) \leq d_{c,i} \leq \kappa_0^2 \mathcal{P}(r_0) \left(M\nu_s\right)^{\frac{2}{n}} + D(Q_c, X_i^n). \tag{2.42}$$

In order to evaluate the quantity $D(Q_c, X_i^n)$ at high resolution we can directly use Lemma 1 in (Linder and Zeger, 1994), and obtain that

$$D(Q_c, X_i^n) = G_c \nu_c^{\frac{2}{n}} (1 + o(1)) \text{ as } \nu_c \to 0. \tag{2.43}$$

Furthermore, in order to evaluate the rate we need the following notation, for $i = 1, 2$,

$$\mathcal{P}_i \triangleq \mathbb{P}[Q_{in}(Q_c(X_i^n)) \in \mathcal{C}],$$

where $\mathcal{C}$ is defined in (2.25). We will use the following lemma, which is proved in Appendix B.

Lemma 3 *For $i = 1, 2$, we have $\lim_{(2.40)} \mathcal{P}_i = 0$.*

Now we are ready to present the main result of this section.

Theorem 2 *For $i = 1, 2$, the following relations hold in the asymptotic regime specified by (2.40)*

$$d_{s,i} = \frac{1}{4} G_{s/2} (M\nu_s)^{\frac{2}{n}} (1 + o(1)), \tag{2.44}$$

$$G_c \nu_c^{\frac{2}{n}} (1 + o(1)) \leq d_{c,i} \leq \kappa_0^2 \mathcal{P}(r_0) \, (M\nu_s)^{\frac{2}{n}} + G_c \nu_c^{\frac{2}{n}} (1 + o(1)), \tag{2.45}$$

$$R_1 + R_2 = h(X_1) + h(X_2) - \frac{2}{n} \log_2 \frac{\nu_s}{K^{1/2}} + \frac{1}{n} H(U_2|U_1) + o(1). \tag{2.46}$$

Additionally, we have

$$H(U_2|U_1) \leq \log_2 K, \tag{2.47}$$

while if $r_0 \leq r_c$, the following is true

$$H(U_2|U_1) \leq 1 + \left(1 - \left(1 - \frac{r_0}{r_c}\right)^n + \mathcal{P}(r_0)\right) \log_2 K + o(1), \tag{2.48}$$

in the limit of (2.40).

Proof: Relation (2.44) is proved in Appendix A.1. Relation (2.45) follows based on (2.42) and (2.43). Let us prove now equality (2.46). For this notice that the rate used to transmit $\beta_i(\lambda_i)$ is $\frac{1}{n} H(Q_{\mathcal{A}_i}(X_i^n))$.

The rate needed for b_i is $\frac{1}{n} \left(-(1 - \mathcal{P}_i) \log_2 (1 - \mathcal{P}_i) - \mathcal{P}_i \log_2 \mathcal{P}_i\right)$. The rate used for encoding τ_i equals $\frac{1}{n}\mathcal{P}_i \log_2 |\mathcal{T}| = \mathcal{P}_i$. Finally, the rate needed for encoding u_1 and u_2 using Slepian-Wolf coding equals $\frac{1}{n}H(U_1, U_2)$. Summarizing we obtain

$$R_1+R_2 = \frac{1}{n}\sum_{i=1}^{2}[H(Q_{\mathcal{A}_i}(X_i^n))-(1-\mathcal{P}_i)\log_2(1-\mathcal{P}_i)+\mathcal{P}_i(-\log_2\mathcal{P}_i+n)]+\frac{1}{n}H(U_1,U_2). \tag{2.49}$$

Since $\lim_{(2.40)} \bar{r}(\mathcal{A}_i(\mathbf{0})) = 0$, as shown in the proof of relation (2.44), we can apply Lemma 2 from (Linder and Zeger, 1994)[2] and, using the fact that $\nu(\mathcal{A}_i(\mathbf{0})) = \nu_s$, obtain that

$$\lim_{(2.40)} \frac{1}{n}\left(H(Q_{\mathcal{A}_i}(X_i^n)) + \log_2(\nu_s)\right) = h(X_i). \tag{2.50}$$

Equations (2.49), (2.50) and Lemma 3 imply that

$$\lim_{(2.40)} \left(R_1 + R_2 + \frac{2}{n}\log_2(\nu_s) - \frac{1}{n}H(U_1,U_2)\right) = h(X_1) + h(X_2).$$

Finally, relation (2.46) follows using the following equality, which is proved in Appendix A.1,

$$\lim_{(2.40)} H(U_i) = \log_2 K, \text{ for } i = 1, 2. \tag{2.51}$$

Further, inequality (2.47) is based on $H(U_2|U_1) \leq H(U_2) = \log_2 K$, while inequality (2.48) is proved in Appendix A.1.

The following corollary deals with the case when $\mathcal{P}(r_0)$ is sufficiently small to make the central distortion dominated by $G_c\nu_c^{\frac{2}{n}}$.

Corollary 1 *Assume that the random variables X_1 and X_2 have the same pdf, denoted by f_X, and that*

$$\mathcal{P}(r_0) \leq \frac{\epsilon}{M^{\frac{4}{n}}}, \tag{2.52}$$

[2]This result was proved by Csiszar in (Csiszár, 1973).

where $\lim_{(2.40)} \epsilon = 0$. *Then the following relations hold in the limit of (2.40),*

$$d_{c,i} = G_c \nu_c^{\frac{2}{n}} (1 + o(1)), \tag{2.53}$$

$$R_1 + R_2 = 2h(X) + \frac{1}{2} \log_2 \frac{G_c G_{s/2}}{4 d_{s,i} d_{c,i}} + \frac{1}{n} H(U_2|U_1) + o(1). \tag{2.54}$$

If additionally we have $\lim_{(2.40)} \frac{r_0}{r_c} = 0$ *then the following is true*

$$R_1 + R_2 = 2h(X) + \frac{1}{2} \log_2 \frac{G_c G_{s/2}}{4 d_{s,i} d_{c,i}} + o(1). \tag{2.55}$$

Proof: Notice that $(M\nu_s)^{\frac{2}{n}} = (M^2 K \nu_c)^{\frac{2}{n}}$. By plugging (2.52) in (2.45) and using the fact that K and κ_0 are constants, relation (2.53) follows. Further, equalities (2.44) and (2.53) imply that $d_{s,i} d_{c,i} = \frac{1}{4} G_{s/2} G_c (M \nu_s \nu_c)^{\frac{2}{n}} (1 + o(1)) = \frac{1}{4} G_{s/2} G_c \left(\frac{\nu_s^2}{K}\right)^{\frac{2}{n}} (1 + o(1))$. By substituting this in (2.46), relation (2.54) follows.

In order to prove (2.55) we first apply Fano's inequality and obtain that

$$H(U_2|U_1) \leq H_b(\mathbb{P}[U_1 \neq U_2]) + \mathbb{P}[U_1 \neq U_2] \log_2 K, \tag{2.56}$$

where $H_b(\cdot)$ denotes the binary entropy function. Next we assume that $r_0 \leq r_c$ and use the following inequality proved in Appendix A.1 (in the proof of relation (2.48))

$$\mathbb{P}[U_1 \neq U_2] \leq 1 - \left(1 - \frac{r_0}{r_c}\right)^n + \mathcal{P}(r_0) + o(1).$$

The fact that $\lim_{(2.40)} \frac{r_0}{r_c} = 0$, together with $\lim_{(2.40)} \mathcal{P}(r_0) = 0$, further imply that $\lim_{(2.40)} \mathbb{P}[U_1 \neq U_2] = 0$. Combining this with (2.56), leads to $\lim_{(2.40)} H(U_2|U_1) = 0$. By applying this result in (2.54), relation (2.55) follows.

Let us assume now that the marginal pdf's of X_1 and X_2 are equal with the pdf of some random variable X with variance σ_X^2. We are interested in finding a sufficient condition on ρ for which relation (2.52) holds. To this aim we can use Markov's inequality applied to the random variable $\|X_2^n - X_1^n\|^2$, which leads to

$$\mathcal{P}(r_0) = \mathbb{P}[\|X_2^n - X_1^n\|^2 > r_0^2] < \frac{n\sigma_{X_2-X_1}^2}{r_0^2} = \frac{2n(1-\rho)\sigma_X^2}{r_0^2}.$$

By imposing further the condition $\frac{n\sigma_{X_2-X_1}^2}{r_0^2} \leq \frac{\epsilon}{M^{\frac{4}{n}}}$, and using the fact that $r_0 = O(\nu_c^{\frac{1}{n}})$ we obtain that

$$\sigma_{X_2-X_1}^2 = o\left(\frac{\nu_c^{\frac{2}{n}}}{M^{\frac{4}{n}}}\right), \text{ leading to } \rho = 1 - o\left(\frac{\nu_c^{\frac{2}{n}}}{M^{\frac{4}{n}}}\right). \tag{2.57}$$

This implies that r_0 can be chosen such that $r_0 = o(\nu_c^{\frac{1}{n}})$ while (2.52) still holds.

According to Theorem 2 and Corollary 1, we have

$$d_{s,i} = \tfrac{1}{4}G_{s/2}M^{\frac{4}{n}}K^{\frac{2}{n}}\nu_c^{\frac{2}{n}}(1+o(1)),$$
$$d_{c,i} = G_c\nu_c^{\frac{2}{n}}(1+o(1)).$$

Then relation (2.57) is equivalent to $(1-\rho)\frac{d_{s,i}}{d_{c,i}^2} \to 0$, and further note that the limits in (2.40) are equivalent to

$$d_{s,i} \to 0, \quad \frac{d_{c,i}}{d_{s,i}} \to 0, \quad (1-\rho)\frac{d_{s,i}}{d_{c,i}^2} \to 0.$$

Let us make the notations $d_s = \frac{d_{s,1}+d_{s,2}}{2}$ and $d_c = \frac{d_{c,1}+d_{c,2}}{2}$. Then the above limits imply that

$$d_s \to 0, \quad \frac{d_c}{d_s} \to 0, \quad (1-\rho)\frac{d_s}{d_c^2} \to 0. \tag{2.58}$$

Further, for $i = 1, 2$, let us denote by $R_i(n, d_s, d_c, \rho)$[3] the rate at encoder i when using the proposed scheme with the smallest sum rate $R_1 + R_2$, which achieves d_s as the average of side distortions and d_c as the average of central distortions, for lattice dimension n and correlation coefficient ρ. Assume that the lattices used in the construction achieve the smallest second moment for the corresponding dimension, denoted by $G_{opt,n}$. Applying this result in Corollary 1 we further obtain that

$$R_1(n, d_s, d_c, \rho) + R_2(n, d_s, d_c, \rho) = 2h(X) + \frac{1}{2}\log_2 \frac{G_{opt,n}^2}{4d_s d_c} + \epsilon(n, d_s, d_c, \rho), \tag{2.59}$$

where $\lim_{(2.58)} \epsilon(n, d_s, d_c, \rho) = 0$.

Now we will compare the proposed lattice-based RDSC scheme with MDLVQ. As a byproduct we obtain the optimality of the proposed lattice-based coding system when specialized to the MDC scenario in the asymptotic regime (2.7) and with the additional assumption that $n \to \infty$. Thus, consider $X_1 = X_2 = X$ and an MDLVQ as in (Vaishampayan *et al.*, 2001). Further, let $R_{MD}(n, d_s, d_c)$ denote the sum-rate for an MDLVQ with lattice dimension n, achieving side distortion d_s and central distortion d_c. Let us denote by S_n the n-dimensional sphere of radius 1. We point out that the rate-distortion analysis in (Vaishampayan *et al.*, 2001) was also performed for the asymptotic regime (2.7). According to (Vaishampayan *et al.*, 2001) the following

[3]This quantity is defined for those quadruples (n, d_s, d_c, ρ) for which there exists a lattice-based scheme of dimension n, achieving average side distortion d_s and average central distortion d_c, when the correlation coefficient is ρ.

holds

$$R_{MD}(n, d_s, d_c) = 2h(X) + \frac{1}{2}\log_2 \frac{G_{opt,n}G(S_n)}{4d_sd_c} + \zeta(n, d_s, d_c),$$

for some $\zeta(n, d_s, d_c)$ satisfying

$$\lim_{\substack{d_s \to 0 \\ \frac{d_c}{d_s} \to 0}} \zeta(n, d_s, d_c) = 0.$$

Combining the above relations with (2.59) leads further to

$$\lim_{\substack{d_s \to 0 \\ \frac{d_c}{d_s} \to 0}} \left(R_1(n, d_s, d_c, \rho)|_{\rho=1} + R_2(n, d_s, d_c, \rho)|_{\rho=1} - R_{MD}(n, d_s, d_c) \right) = \frac{1}{2}\log_2 \frac{G_{opt,n}}{G(S_n)}.$$

The above equality shows that there is a small rate gap between the proposed scheme and MDLVQ for fixed n. However, this gap disappears as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \lim_{\substack{d_s \to 0 \\ \frac{d_c}{d_s} \to 0}} \left(R_1(n, d_s, d_c, \rho)|_{\rho=1} + R_2(n, d_s, d_c, \rho)|_{\rho=1} - R_{MD}(n, d_s, d_c) \right) = 0. \quad (2.60)$$

Recall the definition of $R_{MD}(d_s, d_c)$ in (2.8). Using the fact that $\lim_{n\to\infty} G_{opt,n} = \frac{1}{2\pi e}$ (Zamir and Feder, 1996), leads to

$$\lim_{n\to\infty} \lim_{\substack{d_s \to 0 \\ \frac{d_c}{d_s} \to 0}} \left(R_{MD}(n, d_s, d_c) - R_{MD}(d_s, d_c) \right) = 0,$$

which together with (2.60) implies

$$\lim_{n\to\infty} \lim_{\substack{d_s \to 0 \\ \frac{d_c}{d_s} \to 0}} \left(R_1(n, d_s, d_c, \rho)|_{\rho=1} + R_2(n, d_s, d_c, \rho)|_{\rho=1} - R_{MD}(d_s, d_c) \right) = 0.$$

Note that another RDSC scheme which achieves the fundamental limits of multiple descriptions is a scheme which uses the encoders and decoders of an MDLVQ. Therefore It is interesting to find out whether there is any advantage in using the proposed RSDC scheme rather than directly applying an MDLVQ.

More specifically, in an MDLVQ-based RDSC system as (Vaishampayan *et al.*, 2001), encoder i maps the input sequence x_i^n to $\lambda_{c,i} = Q_c(x_i^n)$, next applies the index assignment $\alpha = (\alpha_1, \alpha_2) : \Lambda_c \to \Lambda_s \times \Lambda_s$ and outputs the side lattice point $\alpha_i(\lambda_{c,i})$. Side decoder i uses the received side lattice point $\lambda_{s,i}$ as the source reconstruction, while the central decoder looks for the central lattice point λ_c satisfying $(\lambda_{s,1}, \lambda_{s,2}) = (\alpha_1(\lambda_c), \alpha_2(\lambda_c))$, and uses λ_c as the common reconstruction for both sources. The problem with this scheme is when $\lambda_{c,1} \neq \lambda_{c,2}$ since in this case the central distortion is essentially as high as the side distortion. To see this, note first that the mappings α_1, α_2 are constructed such that $\alpha_1(\lambda'_c)+\alpha_2(\lambda'_c) = 2Q_{s/2}(\lambda'_c)$ for each $\lambda'_c \in \Lambda_c$. Assume now that $Q_{s/2}(\lambda_{c,1}) = Q_{s/2}(\lambda_{c,2}) = \tau$ and $\lambda_{c,1} \neq \lambda_{c,2}$. Then $\alpha_1(\lambda_{c,1}) \neq \alpha_1(\lambda_{c,2})$ because otherwise we would also have $\alpha_2(\lambda_{c,1}) = \alpha_2(\lambda_{c,2})$ contradicting the fact that α is injective. Further, we obtain that $\alpha_1(\lambda_{c,1}) + \alpha_2(\lambda_{c,2}) \neq 2\tau$, which implies that the point λ_c chosen by the central decoder is not in the same Voronoi region of the fractional lattice $\Lambda_{s/2}$ as $\lambda_{c,1}$ and $\lambda_{c,2}$. Then, if $\|\lambda_{c,i} - \tau\| < 1/2r_{s/2}$, the error in the reconstruction is at least $1/2r_{s/2}$.

2.7 Conclusion

We have proposed a constructive lattice-based scheme for robust distributed coding of two correlated sources. The analysis shows, among other things, that, in a certain asymptotic regime, our scheme is capable of approaching the information-theoretic limit of quadratic MDC whereas a variant of the random-coding-based RDSC scheme by Chen and Berger with Gaussian codes is strictly sub-optimal. Note that in standard random coding arguments, to facilitate the joint typicality analysis, the block-length is often sent to ∞. However, in the infinite block-length limit, the condition needed to ensure joint typicality in the distributed setting is much more restrictive than its counterpart in the centralized setting; as a consequence, the resulting distributed coding schemes, when specialized to the centralized setting, may fail to achieve the fundamental performance limit. In contrast, for lattice-based schemes, the performance analysis can be carried out under fixed block-length (i.e., fixed lattice dimension), which reveals a smooth transition from the distributed setting to the centralized setting. In this sense, our result echoes the recent finding in (Shirani and Pradhan, 2014) regarding the importance of finite block-length schemes in distributed source coding.

Chapter 3

Lattice-based Robust Distributed Coding Scheme for Three Correlated Sources

3.1 Introduction

In this chapter we consider the robust distributed source coding problem for the case of three correlated sources. We propose a coding scheme based on lattices inspired by prior work on multiple description lattice vector quantizer (MDLVQ) of Vaishampayan et at (Vaishampayan *et al.*, 2001) and Huang and Wu (Huang and Wu, 2006). We provide the analysis under the high resolution assumption. Our analysis shows that the performance at high resolution of the proposed scheme is very close to the information theoretic limit of the symmetric Gaussian quadratic MDC problem with single description and all descriptions decoders when the lattice dimension goes to

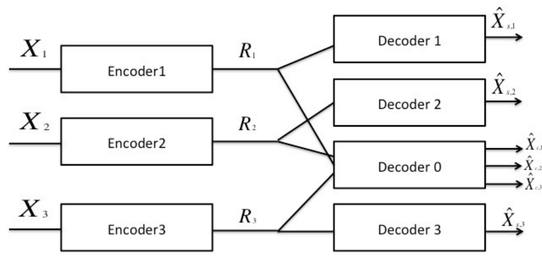

Figure 3.3: Block diagram of robust distributed source coding for three corelated sources.

∞. This chapter is structured as follows. Section 3.2 presents the problem formulation. In Section 3.4 we present a structured coding scheme based on lattices for RDSC problem for the case of three correlated sources. In Section 3.5 we derive the distortion and the rate of the proposed scheme in a certain asymptotic regime, and compare its performance with MDLVQ (Zhang *et al.*, 2011) and with the theoretical limt of the MDC when the lattice dimension goes to ∞. Finally, Section 3.6 contains the conclusion.

3.2 System Model and Problem Statement

Consider a three-component continuous memoryless source (X_1, X_2, X_3) with joint pdf f_{X_1,X_2,X_3}. This memoryless source generates a jointly i.i.d. random process $(X_{1i}, X_{2i}, X_{3i})_{i\in\mathbb{N}}$. The marginal density function of each X_j will be denoted by f_{X_j}, $j = 1, 2, 3$. We will construct a coding scheme for the robust distributed source coding problem illustrated in Figure 3.3. The scheme consists of three encoders and four decoders. Encoder i, $i = 1, 2, 3$ has access only to source X_i, while the side decoder i, $i = 1, 2, 3$, receives only the information sent by encoder i. The central decoder receives the information from all encoders.

For each $i = 1, 2, 3$, let $d_i : \mathcal{X}_i \times \hat{\mathcal{X}}_i \to [0, \infty)$ be a distortion measure, where $\mathcal{X}_i$ and $\hat{\mathcal{X}}_i$ are the source alphabet and the reconstruction alphabet for source X_i, respectively. The distortion measures are extended to sequences of length n as follows

$$d_i(x_i^n, \hat{x}_i^n) = \frac{1}{n} \sum_{k=1}^{n} d_i(x_{i,k}, \hat{x}_{i,k}),$$

where $x_i^n = (x_{i,1}, \cdots, x_{i,n})$, $\hat{x}_i^n = (\hat{x}_{i,1}, \cdots, \hat{x}_{i,n})$.

A nine-tuple $(R_1, R_2, R_3, d_{s,1}, d_{s,2}, d_{s,3}, d_{c,1}, d_{c,2}, d_{c,3})$ is said achievable, if for any $\epsilon > 0$ and all sufficiently large n, there exist encoding functions

$$f_i^{(n)} : \mathcal{X}_i^n \to \{1, 2, \cdots, \lfloor 2^{n(R_i+\epsilon)} \rfloor\}, \quad i = 1, 2, 3,$$

and decoding functions

$$g_{s,i}^{(n)} : \{1, 2, \cdots, \lfloor 2^{n(R_i+\epsilon)} \rfloor\} \to \hat{\mathcal{X}}_i^n, \quad i = 1, 2, 3,$$
$$g_{c,i}^{(n)} : \{1, 2, \cdots, \lfloor 2^{n(R_1+\epsilon)} \rfloor\} \times \{1, 2, \cdots, \lfloor 2^{n(R_2+\epsilon)} \rfloor\} \times \{1, 2, \cdots, \lfloor 2^{n(R_3+\epsilon)} \rfloor\} \to \hat{\mathcal{X}}_i^n, \quad i = 1, 2, 3,$$

such that

$$\mathbb{E}\left[d_i(X_i^n, \hat{X}_{t,i}^n)\right] \le d_{t,i} + \epsilon, \quad i = 1, 2, 3, \quad t = s, c,$$

where

$$\hat{X}^n_{t,i} = g^{(n)}_{t,i}(f^{(n)}_i(X^n_i)), \quad i = 1, 2, 3, \quad t = s, c.$$

The RDSC rate-distortion region, denoted by $\mathcal{RD}$, is the set of all such achievable nine-tuples.

3.3 Lattice-based RDSC Scheme

The proposed coding scheme uses five nested lattices in $\mathbb{R}^n$: $\Lambda_s \subset \Lambda_{s/3} \subset \Lambda_f \subset \Lambda_{in} \subset \Lambda_c$. The finest lattice, Λ_c, is called the *central* lattice. The central lattice points will be used for the reconstruction at the central decoder. The coarsest lattice, Λ_s, is called *side* lattice since it is used for the reconstruction at the side decoders. Λ_{in} is called the *intermediate* lattice . The lattice Λ_f is defined as $\Lambda_f = c_o\Lambda_{in}$, while $\Lambda_s = c_1\Lambda_f$, where $c_1 = 3c_o$ and c_o is a positive integer. It follows that $\Lambda_s = c_oc_1\Lambda_{in}$. The lattice $\Lambda_{s/3}$ is called the fractional lattice and is defined as $\Lambda_{s/3} \triangleq \frac{1}{3}\Lambda_s$. $\Lambda_{s/3}$ defined as above is also a sublattice of Λ_{in}. Let us denote $K \triangleq N(\Lambda_{in} : \Lambda_c)$ and $M \triangleq N(\Lambda_s : \Lambda_{in})$. It follows that $M = c_o^n c_1^n$.

As in the previous chapter in order to simplify the notation we will use in the sequel only the subscript c,in,f,$s/3$, respectively s, instead of Λ_c, Λ_{in}, Λ_f, $\Lambda_{s/3}$, respectively Λ_s. For instance, we will use $\bar{r}_c$ instead of $\bar{r}_{\Lambda_c}$, for the covering radius of Λ_c.

Our construction hinges on the fact that the three sources are highly correlated so that there is some small $r_0 > 0$ such that the probability

$$\mathcal{P}(r_0) \triangleq \mathbb{P}[X^n_1 - X^n_2 \notin \mathcal{B}_{r_0} \text{ or } X^n_2 - X^n_3 \notin \mathcal{B}_{r_0} \text{ or } X^n_1 - X^n_3 \notin \mathcal{B}_{r_0}] \tag{3.61}$$

is sufficiently small.

Denote now

$$r \triangleq r_0 + 2\bar{r}_c. \tag{3.62}$$

Then the lattice Λ_{in} is chosen as a sublattice of Λ_c such that

$$r \leq r_{in}. \tag{3.63}$$

Recall that r_{in} denotes the inscribed radius of the lattice Λ_{in}. The following property is a direct consequence of Lemma 1 from Chapter 2.

Lemma 4 *For $i, j \in \{1, 2, 3\}$, if $x_j^n - x_i^n \in \mathcal{B}_{r_0}$, then*

$$\|Q_c(x_i^n) - Q_c(x_j^n)\| < r,$$
$$\|Q_{in}(Q_c(x_i^n)) - Q_{in}(Q_c(x_j^n))\| < 3\bar{r}_{in}.$$

Further, we will define three labeling functions $\beta_i : \Lambda_{in} \to \Lambda_s$, for $i = 1, 2, 3$. For this, we need to introduce some more notations as follows. Let $\mathcal{T} \triangleq V_s(\mathbf{0}) \cap \Lambda_{s/3}$. Then $\mathcal{T}$ is a set of coset representatives of Λ_s relative to $\Lambda_{s/3}$ and we have $|\mathcal{T}| = N(\Lambda_s : \Lambda_{s/3}) = 3^n$ and

$$\Lambda_{s/3} = \bigcup_{\tau \in \mathcal{T}} (\tau + \Lambda_s).$$

It can be easily seen that the set $\cup_{\tau \in \mathcal{T}} V_{s/3}(\tau)$ is a fundamental cell of Λ_s.

Denote

$$\mathcal{U} \triangleq \{\tau + \tilde{\lambda}_f + \tilde{u}_f | \tau \in \mathcal{T}, \tilde{u}_f \in V_f(0) \cap \Lambda_{in}, \tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f\}. \tag{3.64}$$

Then $\mathcal{U}$ is a set of coset representatives of Λ_s relative to Λ_{in}, which implies that $|\mathcal{U}| = N(\Lambda_s : \Lambda_{in}) = M$ and

$$\Lambda_{in} = \bigcup_{\lambda \in \mathcal{U}} (\lambda + \Lambda_s) .$$

For each $\lambda \in \Lambda_{in}$, let $\lambda_f \triangleq Q_f(\lambda)$, $\tilde{u_f} \triangleq \lambda \bmod \Lambda_f = \lambda - Q_f(\lambda)$, $\tilde{\lambda_f} \triangleq \lambda_f \bmod \Lambda_{s/3} = \lambda_f - Q_{s/3}(\lambda_f)$ and $\lambda_{s/3} \triangleq Q_{s/3}(\lambda_f)$. Moreover, let $\tau \triangleq \lambda_{s/3} \bmod \Lambda_s = \lambda_{s/3} - Q_s(\lambda_{s/3})$, and $\lambda_s \triangleq Q_s(\lambda_{s/3})$. Then

$$\lambda = \tilde{u}_f + \tilde{\lambda}_f + \tau + \lambda_s. \tag{3.65}$$

Also note that $\tilde{u}_f \in V_f(0) \cap \Lambda_{in}$, $\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f$ and $\tau \in V_s(0) \cap \Lambda_{s/3}$. Additionally for $\lambda \in \mathcal{U}$ we have $\lambda_s = 0$. We will first define β_i for $\lambda \in \mathcal{U}$ as follows,

$$\beta_1(\lambda) \triangleq c_1\tilde{\lambda_f}, \quad \beta_2(\lambda) \triangleq c_o c_1 \tilde{u_f}, \quad \beta_3(\lambda) \triangleq 3\tau - c_1\tilde{\lambda_f} - c_o c_1 \tilde{u_f}. \tag{3.66}$$

Further, the mappings β_1, β_2 and β_3 are extended to Λ_{in} using shifting. More specifically,

$$\beta_1(\lambda) \triangleq c_1\tilde{\lambda_f} + \lambda_s,$$

$$\beta_2(\lambda) \triangleq c_o c_1 \tilde{u_f} + \lambda_s,$$

$$\beta_3(\lambda) = 3\tau - c_1\tilde{\lambda_f} - c_o c_1 \tilde{u_f} + \lambda_s.$$

The above definition implies that the mappings β_i satisfy the *shift-invariance property*, i.e., that

$$\beta_i(\lambda + \lambda'_s) = \beta_i(\lambda) + \lambda'_s, \forall \lambda \in \Lambda_{in}, \forall \lambda'_s \in \Lambda_s, i = 1, 2, 3.$$

The shift-invariance property further leads to the following relations, for $i = 1, 2, 3$,

$$\beta_i^{-1}(\lambda'_s) = \beta_i^{-1}(\mathbf{0}) + \lambda'_s, \forall \lambda'_s \in \Lambda_s, \tag{3.67}$$

$$\beta_i^{-1}(\mathbf{0}) = \{\lambda - \beta_i(\lambda) | \lambda \in \mathcal{U}\}. \tag{3.68}$$

Note that the proof of (3.68) is similar to the proof (2.24) in Chapter 2.

3.4 Proposed Scheme

Before describing the details of the proposed scheme we need the following discussion. Let us denote $\lambda_i = Q_{in}(Q_c(x_i^n))$, for $i = 1, 2, 3$. Our scheme is designed such that side decoder i will be able to recover $\beta_i(\lambda_i)$, while the central decoder recovers $\lambda_{c,i} = Q_c(x_i^n)$, for $i = 1, 2, 3$, when the input sequences are sufficiently close, i.e., $x_j^n - x_i^n \in \mathcal{B}_{r_0}$ for all $i, j \in \{1, 2, 3\}$. However, for the central decoder to achieve this goal some additional information needs to be transmitted beside $\beta_1(\lambda_1)$, $\beta_2(\lambda_2)$ and $\beta_3(\lambda_3)$. The amount of this additional information is smaller when λ_1, λ_2 and λ_3 are all in the same Voronoi cell of the lattice Λ_f. Encoder i is not able to determine all the time if this is the case or not since it does not have knowledge of the other source sequence. However, based on Lemma 4, if $\lambda_i \in V_f(\lambda_f)$ and the distance from λ_i to the boundary of $V_f(\lambda_f)$ is not smaller than $3\bar{r}_{in}$, then encoder i can infer that the other sequence are also in $V_f(\lambda_f)$ when $x_j^n - x_i^n \in \mathcal{B}_{r_0}$ for all $i, j \in \{1, 2, 3\}$. Thus, we define the set

$$\mathcal{W} \triangleq \cup_{\lambda_f \in \Lambda_f} \mathcal{W}(\lambda_f), \text{ where } \mathcal{W}(\lambda_f) \triangleq V_f(\lambda_f) \setminus (\lambda_f + \eta V_f(\mathbf{0})), \tag{3.69}$$

for $\eta \triangleq 1 - \frac{3\bar{r}_{in}}{r_f}$ as shown in Figure 3.4.

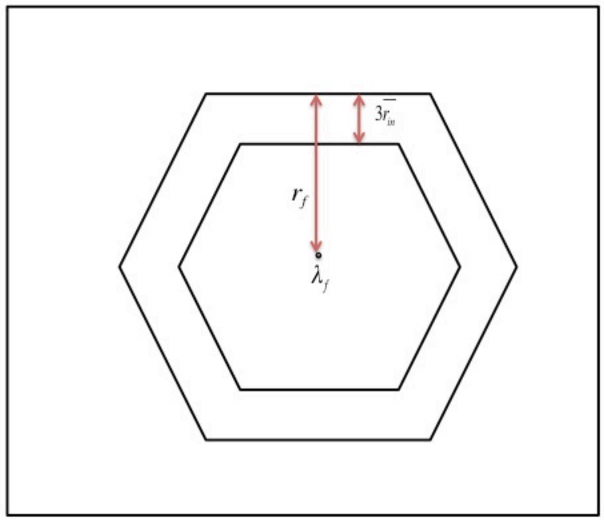

Figure 3.4: The set $\mathcal{W}(\lambda_f)$ is the region between the two hexagons in the figure.

For $\lambda_{s/3} \in \Lambda_{s/3}$ define $\tilde{V}_{s/3}(\lambda_{s/3}) = \cup_{\lambda_f \in V_{s/3}(\lambda_{s/3}) \cap \Lambda_f} V_f(\lambda_f)$. Note that if $Q_f(\lambda_i) \neq Q_f(\lambda_j)$, it is important to determine if $Q_{s/3}(Q_f(\lambda_i))$ and $Q_{s/3}(Q_f(\lambda_j))$ are equal or not. If $\lambda_i \in \tilde{V}_{s/3}(\lambda_{s/3})$ and the distance from λ_i to the boundary of $\tilde{V}_{s/3}(\lambda_{s/3})$ is not smaller than $3\bar{r}_{in}$, then encoder i can infer that for any other j, λ_j is also in $\tilde{V}_{s/3}(\lambda_{s/3})$. Thus, we define the set $\mathcal{S}(\lambda_{s/3})$ as the set of points in $\tilde{V}_{s/3}(\lambda_{s/3})$ such that the distance to the boundary of $\tilde{V}_{s/3}(\lambda_{s/3})$ is smaller than or equal to $3\bar{r}_{in}$. Further, let $\mathcal{S} \triangleq \cup_{\lambda_{s/3} \in \Lambda_{s/3}} \mathcal{S}(\lambda_{s/3})$. Note that $\mathcal{S} \subseteq \mathcal{W}$.

According to Lemma 4, if $\lambda_i \notin \mathcal{W}$, then $\lambda_j, j \in \{1,2,3\} \setminus \{i\}$ is in the same Voronoi cell of Λ_f as λ_i, when $x_i^n - x_j^n \in \mathcal{B}_{r_0}$. Similarly, according to Lemma 4, if $\lambda_i \notin \mathcal{S}$, then for any $\lambda_j, j \in \{1,2,3\} \setminus \{i\}$ then $Q_{s/3}(Q_f(\lambda_i)) = Q_{s/3}(Q_f(\lambda_j))$. Now we are ready to present the details of the encoder and decoder operation.

Encoder. Encoder i, for $i = 1, 2, 3$, operates as follows. First the input sequence x_i^n is quantized to the closest central lattice point $\lambda_{c,i} \triangleq Q_c(x_i^n)$. Next, the point $\lambda_{c,i}$ is quantized to the closest point in the lattice Λ_{in}, $\lambda_i \triangleq Q_{in}(\lambda_{c,i})$. Let $u_i \triangleq \lambda_{c,i} \bmod \Lambda_{in}$ and $\lambda_{s,i} \triangleq \beta_i(\lambda_i)$. Then the encoder outputs $\lambda_{s,i}$, u_i, a_i, where $a_i = 1$ if $\lambda_i \in \mathcal{W}$ and $a_i = 0$ otherwise. Moreover, if $a_i = 1$ the encoder also transmits $\tilde{\lambda}_{fi} \triangleq Q_f(\lambda_i) \bmod \Lambda_{s/3}$ and b_i, where $b_i = 1$ if $\lambda_i \in \mathcal{S}$ and $b_i = 0$ otherwise. Moreover, if $b_i = 1$ the encoder also transmits $\tau_i \triangleq Q_{s/3}(Q_f(\lambda_i)) \bmod \Lambda_s$. The first component, $\lambda_{s,i}$, will be used at the side decoder i, therefore, it is compressed using entropy coding before transmission. On the other hand, u_1, u_2 and u_3 are used only at the central decoder, therefore they will be compressed using Slepian-Wolf coding. Finally, a_i, b_i,$\tilde{\lambda}_{fi}$ and τ_i will also be used only at the central decoder, thus they may be compressed using Slepian-Wolf coding. However, we will use entropy coding to encode a_i, b_i and fixed length codes for τ_i and $\tilde{\lambda}_{fi}$ for simplicity of analysis, since, as we will see shortly, the rate overhead is negligible asymptotically.

Decoder. Side decoder i, for $i = 1, 2, 3$, outputs the reconstruction $\hat{x}_{s,i}^n \triangleq \lambda_{s,i}$. The central decoder recovers both values $\lambda_{s,1}$, $\lambda_{s,2}$ and $\lambda_{s,3}$, and additionally, u_1, u_2, u_3, a_1, a_2, a_3 and b_1, b_2, b_3, if applicable. First the decoder checks if the following condition is satisfied

$$\|\lambda_{s,i} - \lambda_{s,j}\| \leq (10 + 4c_o)\bar{r}_s + 3\bar{r}_{in}, \tag{3.70}$$

for all $i, j \in \{1, 2, 3\}$.

If the condition is violated for at least pair (i, j) then the decoder concludes that $x_j^n - x_i^n \notin \mathcal{B}_{r_0}$, and outputs $\lambda_{s,i}$ as the reconstruction for source i, i.e., $\hat{x}_{c,i}^n \triangleq \lambda_{s,i}$, for

$i = 1, 2, 3$.

If condition (3.70) is satisfied for all pairs (i, j) the decoder assumes that $x_j^n - x_i^n \in \mathcal{B}_{r_0}$, for all pairs (i, j) and it proceeds as follows. First the following are computed

$$\tilde{\lambda}_a \triangleq Q_{in}(u_1 - u_2), \tag{3.71}$$

$$\tilde{\lambda}_b \triangleq Q_{in}(u_1 - u_3), \tag{3.72}$$

$$\tilde{\lambda}_c \triangleq Q_{in}(u_2 - u_3). \tag{3.73}$$

Next the decoder proceeds based on the values of a_1, a_2 and a_3, and of $\tilde{\lambda}_{f1}, \tilde{\lambda}_{f2}, \tilde{\lambda}_{f3}, b_1, b_2, b_3$, τ_1, τ_2 and τ_3 (if applicable), according to the following cases.

1) If $a_1 = 0$ or $a_2 = 0$ or $a_3 = 0$ the decoder evaluates

$$\tilde{\lambda}_{s/3} \triangleq 1/3(\lambda_{s,1} + \lambda_{s,2} + \lambda_{s,3} + 3c_o^2\tilde{\lambda}_c), \quad \tilde{\tau} \triangleq \tilde{\lambda}_{s/3} \bmod \Lambda_s, \tag{3.74}$$

$$\tilde{\lambda}_s \triangleq Q_s(\tilde{\lambda}_{s/3}), \tag{3.75}$$

$$\bar{\tilde{\lambda}}_f \triangleq \tfrac{1}{c_1}(\lambda_{s,1} - \tilde{\lambda}_s), \tag{3.76}$$

$$\bar{\tilde{u}}_{f_2} \triangleq \tfrac{1}{3c_o^2}(\lambda_{s,2} - \tilde{\lambda}_s), \tag{3.77}$$

$$\bar{\tilde{u}}_{f_1} \triangleq \bar{\tilde{u}}_{f_2} - \tilde{\lambda}_a, \tag{3.78}$$

$$\bar{\tilde{u}}_{f_3} \triangleq \bar{\tilde{u}}_{f_2} + \tilde{\lambda}_c, \tag{3.79}$$

$$\tilde{\lambda}_i \triangleq \tilde{\lambda}_s + \tilde{\tau} + \bar{\tilde{\lambda}}_f + \bar{\tilde{u}}_{fi}, \tag{3.80}$$

and outputs the reconstructions $\hat{x}_{c,i}^n \triangleq \tilde{\lambda}_i + u_i$, for $i = 1, 2, 3$.

2) If $a_1 = a_2 = a_3 = 1$ and $\tilde{\lambda}_{f1} = \tilde{\lambda}_{f2} = \tilde{\lambda}_{f3}$ the decoder proceeds as in case 1).

3) If $a_1 = a_2 = a_3 = 1$ and $\tilde{\lambda}_{f1}, \tilde{\lambda}_{f2}, \tilde{\lambda}_{f3}$ are not all three equal and $b_i = 0$ for at

least one $i \in \{1, 2, 3\}$ then the decoder computes

$$\tilde{\lambda}_{s/3} \triangleq \tfrac{1}{3}(\lambda_{s,1} + \lambda_{s,2} + \lambda_{s,3} + 3c_o(1 - c_o)\tilde{\lambda}_{f_3} + 3c_o^2(\tilde{\lambda}_{f_2} + \tilde{\lambda}_c) - 3c_o\tilde{\lambda}_{f_1}, \quad (3.81)$$

$$\tilde{\lambda}_s \triangleq Q_s(\lambda_{s/3}), \quad (3.82)$$

$$\tilde{\tau} \triangleq \tilde{\lambda}_{s/3} \bmod \Lambda_s, \quad (3.83)$$

$$\bar{\tilde{u}}_{f_3} \triangleq \frac{1}{3c_o^2}(\tilde{\lambda}_s + 3\tau - 3c_o\tilde{\lambda}_{f_3} - \lambda_{s,3}), \quad (3.84)$$

$$\bar{\tilde{u}}_{f_1} \triangleq \tilde{\lambda}_{f_3} - \tilde{\lambda}_{f_1} + \tilde{u}_{f_3} - \tilde{\lambda}_b, \quad (3.85)$$

$$\bar{\tilde{u}}_{f_2} \triangleq \tilde{\lambda}_{f_3} - \tilde{\lambda}_{f_2} + \tilde{u}_{f_3} - \tilde{\lambda}_c, \quad (3.86)$$

$$\tilde{\lambda}_i \triangleq \tilde{\lambda}_s + \tilde{\tau} + \tilde{\lambda}_{f_i} + \bar{\tilde{u}}_{f_i}, \quad (3.87)$$

and outputs the reconstructions $\hat{x}^n_{c,i} \triangleq \tilde{\lambda}_i + u_i$, for $i = 1, 2, 3$.

4) If $a_1 = a_2 = a_2 = 1$ and $b_1 = b_2 = b_3 = 1$, then the decoder computes

$$\tilde{\lambda}_{s_1} \triangleq \lambda_{s,1} - 3c_o\tilde{\lambda}_{f_1}, \quad (3.88)$$

$$\nu_2 \triangleq Q_s(\tau_2 - \tau_1), \quad (3.89)$$

$$\tilde{\lambda}_{s_2} \triangleq \tilde{\lambda}_{s1} - \nu_2, \quad (3.90)$$

$$\bar{\tilde{u}}_{f_2} \triangleq \tfrac{1}{3c_o^2}(\lambda_{s,2} - \tilde{\lambda}_{s_2}), \quad (3.91)$$

$$\tilde{\lambda}_2 \triangleq \tilde{\lambda}_{s_2} + \tau_2 + \tilde{\lambda}_{f_2} + \bar{\tilde{u}}_{f_2}, \quad (3.92)$$

$$\tilde{\lambda}_1 \triangleq \tilde{\lambda}_2 - \tilde{\lambda}_a, \quad (3.93)$$

$$\tilde{\lambda}_3 \triangleq \tilde{\lambda}_2 + \tilde{\lambda}_c. \quad (3.94)$$

Finally, the reconstructions are computed as $\hat{x}^n_{c,i} \triangleq \tilde{\lambda}_i + u_i$, for $i = 1, 2, 3$.

Proposition 2 *Let* $\lambda_{c,i} \triangleq Q_c(x_i^n)$, $\lambda_i \triangleq Q_{in}(\lambda_{c,i})$, $u_i \triangleq \lambda_{c,i}$ *mod* Λ_{in}, $\lambda_{s,i} \triangleq \beta_i(\lambda_i)$, $\tilde{u}_{fi} = \lambda_i$ *mod* Λ_f, $\tilde{\lambda}_{fi} = Q_f(\lambda_i)$ *mod* $\Lambda_{s/3}$, $\tau_i \triangleq Q_{s/3}(Q_f(\lambda_i))$ *mod* Λ_s, *and* $\lambda_{s_i} \triangleq Q_s(Q_{s/3}(Q_f(\lambda_i))$ *for* $i = 1, 2, 3$. *Then when* $x_j^n - x_i^n \in B_{r_0}$, *for* $i, j \in \{1, 2, 3\}$, *the Slepian-Wolf decoding of* u_1, u_2 *and* u_3 *is successful, and* c_o *is sufficiently large we have* $\hat{x}_{c,i}^n = \lambda_{c,i}$, *for* $i = 1, 2, 3$.

Proof: Assume that $x_j^n - x_i^n \in B_{r_0}$, for all $i, j \in \{1, 2, 3\}$ and that the Slepian-Wolf decoder employed at the central decoder is able to recover u_1, u_2 and u_3 correctly. First we need to prove that condition (3.70) is satisfied. For this we first show that the following relation holds

$$\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq (5 + 2c_o)\bar{r}_s. \tag{3.95}$$

Note that relation (3.68) leads to

$$\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq \bar{r}(\mathcal{U}) + \bar{r}(\beta_i(\mathcal{U})). \tag{3.96}$$

Further, since $\mathcal{T} \subset V_s(\mathbf{0})$ and $V_{s/3}(\mathbf{0}) \subset V_s(\mathbf{0})$ we obtain that $\mathcal{U} \subset \cup_{\tau \in \mathcal{T}}(\tau + V_{s/3}(\mathbf{0})) \subset 2V_s(\mathbf{0})$. Thus, $\bar{r}(\mathcal{U}) \leq 2\bar{r}_s$. Moreover, from the definition of β_i given by (3.66), we obtain that $\bar{r}(\beta_i(\mathcal{U})) \leq 3\bar{r}(\mathcal{T}) + c_1\bar{r}_{s/3} + c_1c_o\bar{r}_f \leq 3\bar{r}_s + c_1\bar{r}_{s/3} + c_1c_o\bar{r}_f$. The above discussion, together with relation (3.96) and the fact that $\bar{r}_{s/3} = 1/3\bar{r}_s$, $\bar{r}_f = \frac{1}{c_1}\bar{r}_s$, and using the fact that $c_o = \frac{c_1}{3}$ implies that $\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq (5 + 2c_o)\bar{r}_s$ proves (3.95).

By applying the triangle inequality and the fact that $\|\lambda - \beta_i(\lambda)\| \leq \bar{r}(\beta^{-1}(\mathbf{0}))$, together with Lemma 4, we obtain

$$\|\lambda_{s,i} - \lambda_{s,j}\| \leq \|\lambda_{s,i} - \lambda_i\| + \|\lambda_i - \lambda_j\| + \|\lambda_j - \lambda_{s,j}\| \leq 2\bar{r}(\beta^{-1}(\mathbf{0})) + 3\bar{r}_{in}.$$

By combining the above with (3.95) relation (3.70) follows.

Using Lemma 4 and the fact that $\lambda_{c,i} = \lambda_i + u_i$, $i = 1, 2, 3$, we obtain that

$$r_{in} > \|\lambda_{c,i} - \lambda_{c,j}\| = \|u_i - u_j - (\lambda_j - \lambda_i)\|,$$

which, together with the fact that $\lambda_j - \lambda_i \in \Lambda_{in}$, implies that $u_i - u_j \in V_{in}(\lambda_j - \lambda_i)$, i.e., $\lambda_j - \lambda_i = Q_{in}(u_i - u_j)$. This further implies that $\tilde{\lambda}_a$, $\tilde{\lambda}_b$, $\tilde{\lambda}_c$ computed in (3.71), (3.72), (3.73) satisfy the equalities

$$\tilde{\lambda}_a = \lambda_2 - \lambda_1, \tag{3.97}$$

$$\tilde{\lambda}_b = \lambda_3 - \lambda_1, \tag{3.98}$$

$$\tilde{\lambda}_c = \lambda_3 - \lambda_2. \tag{3.99}$$

Recall that

$$\lambda_1 = \lambda_{s1} + \tau_1 + \tilde{u}_{f_1} + \tilde{\lambda}_{f_1}, \tag{3.100}$$

$$\lambda_2 = \lambda_{s2} + \tau_2 + \tilde{u}_{f2} + \tilde{\lambda}_{f2}, \tag{3.101}$$

$$\lambda_3 = \lambda_{s3} + \tau_3 + \tilde{u}_{f3} + \tilde{\lambda}_{f_3}. \tag{3.102}$$

Moreover, since $\lambda_{s,i} = \beta_i(\lambda_i)$, for $i = 1, 2, 3$, we obtain that

$$\lambda_{s,1} = c_1\tilde{\lambda}_{f_1} + \lambda_{s_1} \tag{3.103}$$

$$\lambda_{s,2} = c_1 c_o \tilde{u}_{f_2} + \lambda_{s_2}, \tag{3.104}$$

$$\lambda_{s,3} = \lambda_{s_3} + 3\tau_3 - c_1\tilde{\lambda}_{f_3} - c_1 c_o \tilde{u}_{f3}. \tag{3.105}$$

Assume now that case 1) holds. Based on Lemma 4, it follows that $Q_f(\lambda_1) = Q_f(\lambda_2) = Q_f(\lambda_3)$, which implies that $\tilde{\lambda}_{f_1} = \tilde{\lambda}_{f_2} = \tilde{\lambda}_{f_3}$, $\tau_1 = \tau_2 = \tau_3$ and $\lambda_{s_1} = \lambda_{s_2} = \lambda_{s_3}$. Using further equations (3.74), (3.99) and (3.103), (3.104), (3.105), (3.101) and (3.102) we obtain that $\tilde{\lambda}_{s/3} = \lambda_{s_1} + \tau_1$. This implies that $\tau_1 = \tilde{\lambda}_{s/3} \bmod \Lambda_s$, i.e., $\tilde{\tau} = \tau_1$, and $\tilde{\lambda}_s = \lambda_{s_1}$. Using (3.103) and (3.76) we obtain $\bar{\tilde{\lambda}}_f = \tilde{\lambda}_{f_1}$. Using further equations (3.77), (3.104) and the fact $c_o = \frac{c_1}{3}$ we obtain that $\bar{\tilde{u}}_{f_2} = \tilde{u}_{f_2}$. Using further equations (3.78),(3.97),(3.100) and (3.101) we obtain that $\bar{\tilde{u}}_{f_1} = \tilde{u}_{f_1}$. Moreover, using equations (3.79), (3.99), (3.101) and (3.102) we obtain that $\bar{\tilde{u}}_{f_3} = \tilde{u}_{f_3}$. These imply that $\tilde{\lambda}_i = \lambda_i$ and further that $\hat{x}^n_{c,i} = \lambda_{c,i}$, for $i = 1, 2, 3$.

Assume now that $a_1 = a_2 = a_3 = 1$. Since for c_o sufficiently large the distane between λ_i and λ_j is very small in comparison with the size of a Voronoi cell of Λ_f, it follows that $V_f(\lambda_{s1} + \tilde{\lambda}_{f1} + \tau_1)$, $V_f(\lambda_{s2} + \tilde{\lambda}_{f2} + \tau_2)$ and $V_f(\lambda_{s3} + \tilde{\lambda}_{f3} + \tau_3)$ are either identical or adjacent. Further, if $\tilde{\lambda}_{f1} = \tilde{\lambda}_{f2} = \tilde{\lambda}_{f3}$ it follows that $V_f(\lambda_{s1} + \tau_1 + \tilde{\lambda}_{f1})$, $V_f(\lambda_{s2} + \tau_2 + \tilde{\lambda}_{f2})$ and $V_f(\lambda_{s3} + \tau_3 + \tilde{\lambda}_{f3})$ are identical. Thus, the proof of case 2) proceeds as in case 1).

Assume now that case 3) holds. The fact that $b_i = 0$ and Lemma 4 imply that $\lambda_{s1} + \tau_1 = \lambda_{s2} + \tau_2 = \lambda_{s3} + \tau_3$. Using equations (3.81), (3.103), (3.104), (3.105), (3.99), (3.101) and (3.102) we obtain that $\tilde{\lambda}_{s/3} = \lambda_{s1} + \tau_1$. This implies that $\tau_1 = \tilde{\lambda}_{s/3} \bmod \Lambda_s$, i.e., $\tilde{\tau} = \tau_1$ and that $\tilde{\lambda}_s = \lambda_{s1}$. Using further equations (3.84), (3.105) and using the fact $c_o = \frac{c_1}{3}$ we obtain that $\bar{\tilde{u}}_{f_3} = \tilde{u}_{f_3}$. Using further equations (3.85), (3.98), (3.100) and (3.102) we obtain that $\bar{\tilde{u}}_{f_1} = \tilde{u}_{f_1}$. Moreover, using equations (3.86), (3.99), (3.101) and (3.102) we obtain that $\bar{\tilde{u}}_{f_2} = \tilde{u}_{f_2}$. Equations (3.87) imply that $\tilde{\lambda}_i = \lambda_i$ and further that $\hat{x}^n_{c,i} = \lambda_{c,i}$, for $i = 1, 2, 3$.

Assume now that case 4) holds. Here we will use the following result.

Assertion: If $\lambda_{s/3} \in \Lambda_{s/3}$ and $V_{s/3}(\lambda_{s/3})$ is adjacent to $V_{s/3}(0)$, then $\lambda_{s/3} \in V_s(0)$.

Proof: Let $\lambda_{s/3} \in \Lambda_{s/3}$, then $3\lambda_{s/3} \in \Lambda_s$. Since $V_{s/3}(\lambda_{s/3})$ and $V_{s/3}(0)$ are adjacent it follows that $V_s(3\lambda_{s/3})$ and $V_s(0)$ are adjacent. Then $\frac{3\lambda_{s/3}}{2}$ is on the boundary of $V_s(0)$. The point $\lambda_{s/3}$ is on the interior of the segment connecting 0 and $\frac{3\lambda_{s/3}}{2}$, Therefore, $\lambda_{s/3}$ is in the interior of $V_s(0)$. This conclude the proof of Assertion.

Now consider that case 4) holds. Since for $c_o c_1$ sufficiently large the distance between λ_i and λ_j is very small in comparison with the size of a Voronoi cell of $\Lambda_{s/3}$, it follows that $V_{s/3}(\lambda_{s1}+\tau_1)$, $V_{s/3}(\lambda_{s2}+\tau_2)$ are either identical or adjacent. According to Assertion it follows that $\lambda_{s_2}+\tau_2-(\lambda_{s_1}+\tau_1) \in V_s(0)$, which leads to $\mathbf{0} = Q_s(\lambda_{s_2}-\lambda_{s_1}+\tau_2-\tau_1) = \lambda_{s_2}-\lambda_{s_1}+Q_s(\tau_2-\tau_1)$. Using (3.89) it follows that $\nu_2 = \lambda_{s_1}-\lambda_{s_2}$. Further, (3.88) and (3.103) imply that $\tilde{\lambda}_{s_1} = \lambda_{s_1}$, and further that $\tilde{\lambda}_{s_2} = \lambda_{s_2}$. Combining with (3.91) and (3.104) we obtain that $\bar{\tilde{u}}_{f_2} = \tilde{u}_{f_2}$. Further equation (3.92) imply that $\tilde{\lambda}_2 = \lambda_2$. Then $\tilde{\lambda}_1 = \lambda_1$ and $\tilde{\lambda}_3 = \lambda_3$ and the conclusion follows.

3.5 Performance analysis

In this section we will evaluate the performance of the proposed lattice-based RDSC scheme. We will perform the analysis as ν_c, ν_{in} and ν_s approach 0, while M approaches ∞. As in Chapter 2, we assume that K is constant and we will consider some fixed lattices $\Lambda_{c,0}$ and $\Lambda_{in,0}$ and scale factor θ such that

$$\Lambda_c = \theta\Lambda_{c,0}, \quad \Lambda_{in} = \theta\Lambda_{in,0}, \tag{3.106}$$

$$\Lambda_s = c_1 c_o \theta\Lambda_{in,0}, \quad \Lambda_f = c_o\theta\Lambda_{in,0}. \tag{3.107}$$

We require that the following relations hold

$$\theta \to 0, \quad c_o \to \infty, \quad c_1 \to \infty, \quad c_1 c_o^2 \theta \to 0. \tag{3.108}$$

We will evaluate the distortions and rates corresponding to the proposed scheme, in the limit of (3.108). For $i = 1, 2, 3$, let $d_{s,i}$ denote the distortion of source i at side decoder i and let $d_{c,i}$ denote the distortion of source i at the central decoder.

3.5.1 Central Distortion

In this subsection we will evaluate the central distortion. Denote $\Delta_{i,sup} \triangleq \sup_{x_i^n \in \mathbb{R}^n} \|x_i^n - \hat{x}_{c,i}^n\|$, for $i = 1, 2, 3$ and let $\mathcal{P}_{e,SW}$ denote the probability that the Slepian-Wolf decoder fails. In view of definition (3.61) of $\mathcal{P}(r_0)$ and of Proposition 2 it follows that, for $i = 1, 2, 3$,

$$D(Q_c, X_i^n) \leq d_{c,i} \leq (\mathcal{P}(r_0) + \mathcal{P}_{e,SW})\Delta_{i,sup}^2 + D(Q_c, X_i^n).$$

The following lemma, proved in Appendix D, gives an upper bound for $\Delta_{i,sup}$.

Lemma 5 *There is some constant κ_1 such that for each $i = 1, 2, 3$, and c_o sufficiently large, the following holds*

$$\Delta_{i,sup} \leq \kappa_1 c_o \nu_s^{\frac{1}{n}}.$$

It is known that the probability that the Slepian-Wolf decoder fails can be made arbitrarily small by increasing the block length used for Slepian-Wolf encoding. Since $\Delta_{i,sup}$ is bounded, it follows that the impact on the distortion of the Slepian-Wolf decoder failure can also be made arbitrarily small. Therefore, in the limit as the block length of Slepian-Wolf encoder approaches infinity, the following holds

$$D(Q_c, X_i^n) \leq d_{c,i} \leq \mathcal{P}(r_0)\kappa_1^2 c_o^2 \nu_s^{\frac{2}{n}} + D(Q_c, X_i^n). \tag{3.109}$$

In order to evaluate the quantity $D(Q_c, X_i^n)$ at high resolution we can directly use Lemma 1 in (Linder and Zeger, 1994), and obtain that

$$D(Q_c, X_i^n) = G_c \nu_c^{\frac{2}{n}}(1 + o(1)) \tag{3.110}$$

in the limit of (3.108). The following corollary deals with the case when $\mathcal{P}(r_0)$ is sufficiently small to make the central distortion dominated by $G_c \nu_c^{\frac{2}{n}}$.

Corollary 2 *Assume that*

$$\mathcal{P}(r_0) \leq \frac{\epsilon}{c_o^2 M^{\frac{2}{n}}}, \tag{3.111}$$

where $\lim_{(3.108)} \epsilon = 0$*. Then the following relations hold in the limit of (3.108),*

$$d_{c,i} = G_c \nu_c^{\frac{2}{n}}(1 + o(1)), \tag{3.112}$$

Proof: By plugging (3.111) in (3.109), using the fact that κ_1 is constant and using (3.110) relation (3.112) follows.

3.5.2 Side Distortion

In this subsection, we will evaluate the side distortion of the proposed scheme in the asymptotic regime specifed by (3.108). The expression of the side distortion is given in the following theorem, which is proved in Appendix C.

Theorem 3 *The following relation holds in the asymptotic regime specifed by (3.108).*

$$\frac{d_{s,1}+d_{s,2}+d_{s,3}}{3}=\frac{4}{3}c_o^2 G(\Lambda_s)\nu_s^{\frac{2}{n}}(1+o(1)). \tag{3.113}$$

3.5.3 Rate Computation

Let us evaluate now the rate R_i, in bits per source sample, at encoder $i, i=1,2,3$. Let us first denote

$$\bar{\mathcal{P}}_i \triangleq \mathbb{P}[Q_{in}(Q_c(X_i^n)) \in \mathcal{W}],$$

$$\tilde{\mathcal{P}}_i \triangleq \mathbb{P}[Q_{in}(Q_c(X_i^n)) \in \mathcal{S}],$$

where $\mathcal{W}$ is defined in (3.69). Further, notice that the rate used to transmit $\beta_i(\lambda_i)$ is $\frac{1}{n}H(Q_{\mathcal{A}_i}(X_i^n))$. The rate needed for a_i is $\frac{1}{n}\left(-(1-\bar{\mathcal{P}}_i)\log_2(1-\bar{\mathcal{P}}_i)-\bar{\mathcal{P}}_i\log_2\bar{\mathcal{P}}_i\right)$. The rate used for encoding $\tilde{\lambda_{fi}}$ equals $\frac{1}{n}\bar{\mathcal{P}}_i\log_2|V_{s/3}(0)\cap\Lambda_f|=\bar{\mathcal{P}}_i\log_2(c_o)$. Since b_i is transmited only when $a_i=1$, the rate needed for b_i is $\frac{1}{n}\bar{\mathcal{P}}_i\left(-(1-\tilde{\mathcal{P}}_i)\log_2(1-\tilde{\mathcal{P}}_i)-\tilde{\mathcal{P}}_i\log_2\tilde{\mathcal{P}}_i\right)$. The rate used for encoding τ_i equals $\frac{1}{n}\bar{\mathcal{P}}_i\tilde{\mathcal{P}}_i\log_2|\mathcal{T}|=\bar{\mathcal{P}}_i\tilde{\mathcal{P}}_i\log_2 3$. Finally, the rate needed for encoding u_1,u_2 and u_3 using Slepian-Wolf coding, i.e., $\frac{1}{n}H(U_1,U_2,U_3)$, is equally divided between the three encoders. Summarizing we obtain

$$\begin{aligned} R_i &= \frac{1}{n}\left[H(Q_{\mathcal{A}_i}(X_i^n))+\frac{1}{3}H(U_1,U_2,U_3)-(1-\bar{\mathcal{P}}_i)\log_2(1-\bar{\mathcal{P}}_i)-\bar{\mathcal{P}}_i\log_2\bar{\mathcal{P}}_i\right] \\ &+ \bar{\mathcal{P}}_i\log_2(c_o)+\frac{1}{n}\left[\bar{\mathcal{P}}_i\left(-(1-\tilde{\mathcal{P}}_i)\log_2(1-\tilde{\mathcal{P}}_i)-\tilde{\mathcal{P}}_i\log_2\tilde{\mathcal{P}}_i+\tilde{\mathcal{P}}_i\log_2 3^n\right)\right]\end{aligned}$$

Since $\lim_{(3.108)}\bar{r}(\mathcal{A}_i(\mathbf{0}))=0$, as shown in the proof of Theorem 3.113, we can apply Lemma 2 from (Linder and Zeger, 1994)[1] and, using $\nu(\mathcal{A}_i(\mathbf{0}))=\nu_s=M\nu_{in}$, obtain

[1]This result was proved by Csiszar in (Csiszár, 1973).

that

$$\lim_{(3.108)} \frac{1}{n} \left(H(Q_{\mathcal{A}_i}(X_i^n)) + \log_2 (\nu_s) \right) = h(X_i).$$

Additionally, we will use the following lemma, which is proved in Appendix D.

Lemma 6 *For $i = 1, 2, 3$, the following holds*

$$\lim_{(3.108)} \bar{\mathcal{P}}_i = 0 \textit{ and } \lim_{(3.108)} \bar{\mathcal{P}}_i \log_2 c_o = 0.$$

Based on the above discussion we conclude that

$$\lim_{(3.108)} \left(R_i + \frac{1}{n} \log_2 (\nu_s) - \frac{1}{3n} H(U_1, U_2, U_3) \right) = h(X_i). \tag{3.114}$$

Thus, the following approximation holds in the limit of (3.108)

$$R_i \approx h(X_i) - \frac{1}{n} \log_2 (\nu_s) + \frac{1}{3n} H(U_1, U_2, U_3).$$

3.5.4 Comparison with MDLVQ

In this section we will compare the proposed coding scheme with MDLVQ as in (Zhang *et al.*, 2011). Consider $X_i = X_j$, this implies that $U_i = U_j$ for $i, j = 1, 2, 3$. Note that limits (3.108) are equivalent to $d_s \to 0$, and $\frac{d_c}{d_s} \to 0$, where $d_c = d_{c,1} = d_{c,2} = d_{c,3}$ and $d_s = \frac{d_{s,1}+d_{s,2}+d_{s,3}}{3}$.

Then relation (3.114) becomes

$$\lim_{d_s \to 0, \frac{d_c}{d_s} \to 0} \left(R_i - h(X_i) + \frac{1}{n} \log_2 (\nu_s) - \frac{1}{3n} H(U_i) \right) = 0. \tag{3.115}$$

Notice that as (3.108) holds, U_i approaches a uniform distribution, for $i = 1, 2, 3$.

Then

$$\lim_{(3.108)} H(U_i) = \log_2 K. \tag{3.116}$$

The proof is similar to the proof of (2.51).

Plugging (3.116) in (3.115), we obtain that

$$\lim_{d_s \to 0, \frac{d_c}{d_s} \to 0} \left(R_i - h(X_i) + \frac{1}{n} \log_2 (\nu_s) - \frac{1}{3n} \log_2(K) \right) = 0. \tag{3.117}$$

Consider now $X_i = X_j$ for $i, j \in \{1, 2, 3\}$ and an MDLVQ as in (Zhang *et al.*, 2011) for lattice dimension n. Further, let R_{MD} denote the rate of each description and let $d_{s,MD}$, denote the side distortion. For comparison we will assume that the central lattice used in the MDLVQ is the same lattice Λ_c as in our scheme. Additionally, we also assume that $d_{s,MD} = d_s$ and $d_{c,MD} = d_c$. Recall that S_n denotes for the n-dimensional sphere of radius 1. Then according to (Zhang *et al.*, 2011) when $d_s \to 0, \frac{d_c}{d_s} \to 0$ we have

$$d_{s,MD} = \frac{2}{3^{\frac{3}{2}}} \bar{K}^{\frac{3}{n}} G(S_{2n}) \nu_c^{\frac{2}{n}} (1 + o(1)), \tag{3.118}$$

and

$$\lim_{d_s \to 0, \frac{d_c}{d_s} \to 0} \left(R_{MD} - h(X_1) + \frac{1}{n} \log_2 (\tilde{\nu_s}) \right) = 0, \tag{3.119}$$

where $\tilde{\nu_s}$ is the volume of the Voronoi region of the side lattice used in the MDLVQ, and $\bar{K} = \frac{\tilde{\nu_s}}{\nu_c}$.

Using the fact that $\tilde{\nu_s} = \bar{K}\nu_c$, (3.117) and (3.122) we obtain

$$\lim_{d_s \to 0, \frac{d_c}{d_s} \to 0} (R_{MD} - R_i) = \lim_{d_c \to 0, d_s \to 0, \frac{d_c}{d_s} \to 0} \left(\frac{1}{3n} \log_2 \left(\left(\frac{\nu_s}{\tilde{\nu_s}} \right)^3 \left(\frac{\nu_c}{\nu_{in}} \right) \right) \right). \tag{3.120}$$

Using $d_{s,MD} = d_s$, we obtain that

$$\frac{2}{3^{\frac{3}{2}}} G(S_{2n}) \left(\frac{\tilde{\nu_s}^{\frac{3}{n}}}{\nu_c^{\frac{1}{n}}} \right) = \frac{4}{9} G_s(\Lambda_s) \left(\frac{\nu_s^{\frac{3}{n}}}{\nu_{in}^{\frac{1}{n}}} \right), \tag{3.121}$$

which leads to

$$\frac{\nu_c}{\nu_{in}} \approx \left(\frac{\sqrt{3} G(S_{2n})}{2G(\Lambda_s)} \right)^n \left(\frac{\tilde{\nu_s}}{\nu_s} \right)^3. \tag{3.122}$$

From (3.120) and (3.122) we see that for fixed n, there is a gap between R_i and R_{MD}, namely

$$\lim_{d_s \to 0, \frac{d_c}{d_s} \to 0} (R_{MD} - R_i) = \frac{1}{3} \log_2 \left(\frac{\sqrt{3} G(S_{2n})}{2G(\Lambda_s)} \right).$$

Now we will discuss the situation when $n \to \infty$. It was shown in (Zamir and Feder, 1996) that there is a sequence of lattices Λ_n such that $\lim_{n \to \infty} G(\Lambda_n) = \frac{1}{2\pi e}$. It follows that the gap is very small as $n \to \infty$, i.e.,

$$\lim_{n \to \infty} \lim_{d_s \to 0, \frac{d_c}{d_s} \to 0} (R_{MD} - R_i) = \frac{1}{3} \log_2 \frac{\sqrt{3}}{2} = -0.0692.$$

It follows that the gap in the sum-rate is 0.207 bits . It was shown in (Zhang *et al.*, 2011) that the MDLVQ scheme with $d_s \to 0$, $\frac{d_c}{d_s} \to 0$ and $n \to \infty$ approaches the theoretical bound of the symmetric Gaussian quadratic MDC problem, when only single description and all descriptions decoder are of interest.

3.6 Conclusion

In this chapter we have proposed a lattice-based coding scheme for robust distributed source coding for three correlated sources. We derive the distortion and the rate for the proposed scheme under the high resolution assumption. It is shown that, in a certain asymptotic regime, the performance of our scheme is very close to the theoretical bound of the symmetric Gaussian quadratic MDC problem with single description and all descriptions decoders, with a gap of 0.0692 bits for single rate and 0.207 bits in sum rate.

Chapter 4

Conclusion

We have proposed two constructive lattice-based scheme for robust distributed coding one for two correlated sources and the other for three correlated sources. We have perform the rate and the distortion analysis under high resolution assumption. The analysis of the proposed lattice coding scheme for two correlated sources shows, among other things, that, in a certain asymptotic regime, our scheme is capable of approaching the information-theoretic limit of quadratic MDC whereas a variant of the random-coding-based RDSC scheme by Chen and Berger with Gaussian codes is strictly sub-optimal. Note that in standard random coding arguments, to facilitate the joint typicality analysis, the block-length is often sent to ∞. However, in the infinite block-length limit, the condition needed to ensure joint typicality in the distributed setting is much more restrictive than its counterpart in the centralized setting; as a consequence, the resulting distributed coding schemes, when specialized to the centralized setting, may fail to achieve the fundamental performance limit. In contrast, for lattice-based schemes, the performance analysis can be carried out under fixed block-length (i.e., fixed lattice dimension), which reveals a smooth transition from the distributed setting to the centralized setting. In this sense, our result echoes the recent finding in (Shirani and Pradhan, 2014) regarding the importance of finite block-length schemes in distributed source coding. For the case of three sources the analysis shows, that performance of the proposed scheme under a certain asymptotic regime is very close to the bound of MDC in case of symmetric Gaussian quadratic source when only the single description and all descriptions receivers are of intrest with a gap of 0.0692 for sigle rate and 0.207 in term of the sum- rate.

Appendix A

Appendix

A.1 Proof of Relations (2.44), (2.51) and (2.48)

Proof of Relation (2.44)

First let us fix i. We will split the proof into two parts. In Part 1 we show that if the limit $\lim_{(2.40)} \frac{G(\mathcal{A}_i(\mathbf{0}))}{M^{\frac{2}{n}}}$ exists then we have

$$\lim_{(2.40)} \frac{D(Q_{\mathcal{A}_i}, X_i^n)}{(M\nu_s)^{\frac{2}{n}}} = \lim_{(2.40)} \frac{G(\mathcal{A}_i(\mathbf{0}))}{M^{\frac{2}{n}}}. \tag{A.1}$$

In Part 2 we prove that

$$\lim_{(2.40)} \frac{G(\mathcal{A}_i(\mathbf{0}))}{M^{\frac{2}{n}}} = \frac{1}{4} G(\Lambda_{s/2}). \tag{A.2}$$

Part 1.[1] The proof is based on the idea that in the limit of (2.40) the pdf $f_{X_i^n}$ can be approximated by a uniform density function over each set $\mathcal{A}_i(\lambda_s)$. This density function is $f_{\theta,c} : \mathbb{R}^n \to [0, \infty)$ defined as follows. For each $\lambda_s \in \Lambda_s$ and $x^n \in \mathcal{A}_i(\lambda_s)$,

[1] This proof uses ideas from the proof of (Linder and Zeger, 1994, Lemma 1).

let

$$f_{\theta,c}(x^n) = \frac{\mathbb{P}[X_i^n \in \mathcal{A}_i(\lambda_s)]}{\nu(\mathcal{A}_i(\lambda_s))} = \frac{1}{\nu(\mathcal{A}_i(\lambda_s))} \int_{\mathcal{A}_i(\lambda_s)} f_{X_i^n}(y^n) dy^n.$$

Let $X_{\theta,c}^n$ denote the random variable with pdf $f_{\theta,c}$. Note that

$$\begin{aligned} &\tfrac{1}{(M\nu_s)^{\frac{2}{n}}} |D(Q_{\mathcal{A}_i}, X_{\theta,c}^n) - D(Q_{\mathcal{A}_i}, X_i^n)| \leq \\ &\tfrac{1}{n(M\nu_s)^{\frac{2}{n}}} \textstyle\sum_{\lambda_s \in \Lambda_s} \int_{\mathcal{A}_i(\lambda_s)} \|x^n - \lambda_s\|^2 |f_{\theta,c}(x^n) - f_{X_i^n}(x^n)| dx^n \leq \\ &\tfrac{1}{n(M\nu_s)^{\frac{2}{n}}} \textstyle\sum_{\lambda_s \in \Lambda_s} \bar{r}(\mathcal{A}_i(\mathbf{0}))^2 \int_{\mathcal{A}_i(\lambda_s)} |f_{\theta,c}(x^n) - f_{X_i^n}(x^n)| dx^n = \\ &\tfrac{\bar{r}(\mathcal{A}_i(\mathbf{0}))^2}{n(M\nu_s)^{\frac{2}{n}}} \textstyle\int_{\mathbb{R}^n} |f_{\theta,c}(x^n) - f_{X_i^n}(x^n)| dx^n, \end{aligned} \quad (A.3)$$

where the second inequality is based on the fact that $\mathcal{A}_i(\lambda_s) = \lambda_s + \mathcal{A}_i(\mathbf{0})$, which implies that $\max_{x^n \in \mathcal{A}_i(\lambda_s)} \|x^n - \lambda_s\|^2 = \bar{r}(\mathcal{A}_i(\mathbf{0}))$. Let us analyze now the quantity $\bar{r}(\mathcal{A}_i(\mathbf{0}))$. Recall that $\mathcal{A}_i(\mathbf{0}) = \cup_{\lambda \in \beta_i^{-1}(\mathbf{0})}(\lambda + \mathcal{M}(\mathbf{0}))$, where $\mathcal{M}(\mathbf{0}) \triangleq \cup_{\lambda_c \in V_{in}(\mathbf{0}) \cap \Lambda_c} V_c(\lambda_c)$. Then it follows that

$$\bar{r}(\mathcal{A}_i(\mathbf{0})) \leq \bar{r}(\beta_i^{-1}(\mathbf{0})) + \bar{r}(\mathcal{M}(\mathbf{0})). \quad (A.4)$$

Further,

$$\bar{r}(\mathcal{M}(\mathbf{0})) \leq \bar{r}_{in} + \bar{r}_c \leq 2\bar{r}_{in} = 2\theta \bar{r}_{in,0}. \quad (A.5)$$

Since we are interested in computing the limits in (A.1) as (2.40) holds, we may assume that c is conveniently large. In particular, in the sequel we will assume that $c \geq 8$ so that relation (2.34) leads to the following

$$\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq c^2 \theta \bar{r}_{in,0}. \quad (A.6)$$

Finally, relations (A.4)-(A.6) together with the fact that $M = c^n$ and $\nu_s = c^n\theta^n\nu_{in,0}$, lead to

$$\frac{\bar{r}(\mathcal{A}_i(\mathbf{0}))}{(M\nu_s)^{\frac{1}{n}}} \leq \frac{2\theta\bar{r}_{in,0} + c^2\theta\bar{r}_{in,0}}{c^2\theta\nu_{in,0}^{\frac{1}{n}}} \to \frac{\bar{r}_{in,0}}{2\nu_{in,0}^{\frac{1}{n}}}, \tag{A.7}$$

in the limit of (2.40). The above result also implies that $\bar{r}(\mathcal{A}_i(\mathbf{0})) \to 0$ as (2.40) holds. This enables us to apply Lemma 7, which is stated and proved in Appendix B, and we obtain that $f_{\theta,c}(x^n) \to f_{X_1}^n(x^n)$ for each $x^n \in \mathbb{R}^n$, as (2.40) holds. Using further Scheffe's theorem (Scheffé, 1947), it follows that $\int_{\mathbb{R}^n} |f_{\theta,c}(x^n) - f_{X_1^n}(x^n)|dx^n \to 0$ as (2.40) holds. Combining further with (A.3) and (A.7) we obtain that

$$\lim_{(2.40)} \frac{1}{(M\nu_s)^{\frac{2}{n}}}|D(Q_{\mathcal{A}_i}, X_{\theta,c}^n) - D(Q_{\mathcal{A}_i}, X_i^n)| = 0. \tag{A.8}$$

Using now the fact that $f_{\theta,c}$ is uniform over each quantizer cell $\mathcal{A}_i(\lambda_s)$ we obtain that

$$\begin{gathered}D(Q_{\mathcal{A}_i}, X_{\theta,c}^n) = \frac{1}{n}\sum_{\lambda_s\in\Lambda_s}\int_{\mathcal{A}_i(\lambda_s)}\|x^n-\lambda_s\|^2 f_{\theta,c}(x^n)dx^n = \\ \frac{1}{n}\sum_{\lambda_s\in\Lambda_s}\frac{\mathbb{P}[X_i^n\in\mathcal{A}_i(\lambda_s)]}{\nu(\mathcal{A}_i(\lambda_s))}\int_{\mathcal{A}_i(\lambda_s)}\|x^n-\lambda_s\|^2dx^n \stackrel{(a)}{=} \\ \frac{1}{n\nu(\mathcal{A}_i(\mathbf{0}))}\int_{\mathcal{A}_i(\mathbf{0})}\|x^n\|^2dx^n\sum_{\lambda_s\in\Lambda_s}\mathbb{P}[X_i^n\in\mathcal{A}_i(\lambda_s)] = \\ \frac{1}{n\nu(\mathcal{A}_i(\mathbf{0}))}\int_{\mathcal{A}_i(\mathbf{0})}\|x^n\|^2dx^n\mathbb{P}[X_i^n\in\mathbb{R}^n] = G(\mathcal{A}_i(\mathbf{0}))(\nu(\mathcal{A}_i(\mathbf{0})))^{\frac{2}{n}} \stackrel{(b)}{=} G(\mathcal{A}_i(\mathbf{0}))\nu_s^{\frac{2}{n}},\end{gathered} \tag{A.9}$$

where (a) uses the fact that $\mathcal{A}_i(\lambda_s) = \lambda_s + \mathcal{A}_i(\mathbf{0})$, while (b) is based on the fact that $\nu(\mathcal{A}_i(\mathbf{0})) = \nu_s$ since $\mathcal{A}_i(\mathbf{0})$ is a fundamental cell of the lattice Λ_s. Relations (A.8) and (A.9) prove the claim of Part 1.

Part 2. In order to prove (A.2) we will first evaluate $\int_{\mathcal{A}_i(\mathbf{0})}\|x^n\|^2dx^n$. Using the fact that $\mathcal{A}_i(\mathbf{0}) = \cup_{\lambda \in \beta_i^{-1}(\mathbf{0})}(\lambda + \mathcal{M}(\mathbf{0}))$ and relation (2.24) we obtain that

$$\mathcal{A}_i(\mathbf{0}) = \cup_{\lambda \in \mathcal{U}} \left(\lambda - \beta_i(\lambda) + \mathcal{M}(\mathbf{0})\right). \tag{A.10}$$

Using further Lemma 8, which is stated and proved in Appendix B, we obtain that

$$\int_{\lambda-\beta_i(\lambda)+\mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n = \int_{\mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n + 2\langle \int_{\mathcal{M}(\mathbf{0})} x^n dx^n, \lambda - \beta_i(\lambda)\rangle + \|\lambda-\beta_i(\lambda)\|^2 \nu(\mathcal{M}(\mathbf{0})). \tag{A.11}$$

It is easy to see that $\mathcal{M}(\mathbf{0})$ is a fundamental cell of the lattice Λ_{in}, therefore, $\nu(\mathcal{M}(\mathbf{0})) = \nu_{in}$. Further, relations (A.10) and (A.11) lead to

$$\int_{\mathcal{A}_i(\mathbf{0})} \|x^n\|^2 dx^n = \underbrace{|\mathcal{U}| \int_{\mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n}_{T_1} + \underbrace{2\sum_{\lambda \in \mathcal{U}} \langle \int_{\mathcal{M}(\mathbf{0})} x^n dx^n, \lambda - \beta_i(\lambda)\rangle}_{T_{2,i}} + \underbrace{\nu_{in} \sum_{\lambda \in \mathcal{U}} \|\lambda - \beta_i(\lambda)\|^2}_{T_{3,i}}.$$

Then

$$\frac{G(\mathcal{A}_i(\mathbf{0}))}{M^{\frac{2}{n}}} = \frac{T_1}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} + \frac{T_{2,i}}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} + \frac{T_{3,i}}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}}. \tag{A.12}$$

We will prove first that the first two terms in the right hand side of the above equality, approach 0 in the limit of (2.40). Consider the first term. Note that $\int_{\mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n \leq (\bar{r}(\mathcal{M}(\mathbf{0})))^2 \nu_{in}$. Combining further with (A.5) and with the fact that $|\mathcal{U}| = M$ it follows that

$$\frac{T_1}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} \leq \frac{4M\theta^2 \bar{r}_{in,0}^2 \nu_{in}}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} = \frac{4\bar{r}_{in,0}^2}{nM^{\frac{4}{n}} \nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (2.40) holds.} \tag{A.13}$$

It is easy to see that the closure of a lattice Voronoi cell of the origin is symmetric about the origin. Therefore, if Λ_{in} is a clean sublattice of Λ_c, i.e., there are no points of Λ_c on the boundary of $V_{in}(\mathbf{0})$, then the set $\Lambda_c \cap V_{in}(\mathbf{0})$ is symmetric about the origin. The above considerations further imply that the closure of the set $\mathcal{M}(\mathbf{0})$ is symmetric about the origin, thus $\int_{\mathcal{M}(\mathbf{0})} x^n dx^n = 0$. Then the second term in (A.12) is 0. When Λ_{in} is not a clean sublattice of Λ_c, the aforementioned term still approaches 0 in the limit of (2.40), as we prove next.

$$\begin{aligned}
|T_{2,i}| &= 2\left|\sum_{\lambda\in\mathcal{U}}\int_{\mathcal{M}(\mathbf{0})}\langle x^n, \lambda-\beta_i(\lambda)\rangle dx^n\right| \\
&\leq 2\sum_{\lambda\in\mathcal{U}}\int_{\mathcal{M}(\mathbf{0})}\left|\langle x^n, \lambda-\beta_i(\lambda)\rangle\right| dx^n \\
&\overset{(a)}{\leq} 2\sum_{\lambda\in\mathcal{U}}\int_{\mathcal{M}(\mathbf{0})}\|x^n\|\,\|\lambda-\beta_i(\lambda)\|\,dx^n \\
&= 2\int_{\mathcal{M}(\mathbf{0})}\|x^n\|\,dx^n\sum_{\lambda\in\mathcal{U}}\|\lambda-\beta_i(\lambda)\| \\
&\overset{(b)}{\leq} 2\bar{r}(\mathcal{M}(\mathbf{0}))\nu_{in}M(\max_{\lambda\in\mathcal{U}}\|\lambda\|+\max_{\lambda\in\mathcal{U}}\|\beta_i(\lambda)\|) \\
&\overset{(c)}{\leq} 4\theta\bar{r}_{in,0}\nu_{in}Mc^2\theta\bar{r}_{in,0} \overset{(d)}{=} 4\theta^2\nu_{in}M^{1+\frac{2}{n}}\bar{r}^2_{in,0},
\end{aligned} \tag{A.14}$$

where (a) follows from the Cauchy-Schwarz inequality and (b) is based on the fact that $\int_{\mathcal{M}(\mathbf{0})}\|x^n\|dx^n \leq \bar{r}(\mathcal{M}(\mathbf{0}))\nu_{in}$ and $|\mathcal{U}| = M$. Additionally, (c) follows from (A.5) and the discussion in the paragraph below equation (2.35). Finally, (d) is based on the fact that $c = M^{\frac{1}{n}}$. Further, relation (A.14) implies that

$$\frac{|T_{2,i}|}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} \leq \frac{4\theta^2\nu_{in}M^{1+\frac{2}{n}}\bar{r}^2_{in,0}}{nM^{1+\frac{4}{n}}\nu_{in}\theta^2\nu_{in,0}^{\frac{2}{n}}} = \frac{4\bar{r}^2_{in,0}}{nM^{\frac{2}{n}}\nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (2.40) holds.} \tag{A.15}$$

Let us evaluate now $\frac{T_{3,i}}{\nu_{in}}$. We need to treat separately the cases $i = 1$ and $i = 2$. Recall that $\mathcal{U} = \cup_{\tau\in\mathcal{T}} V_{s/2}(\tau) \cap \Lambda_{in}$. We will denote $\hat{V}_{s/2}(\tau) \triangleq V_{s/2}(\tau) \cap \Lambda_{in}$. Using further (2.22) we obtain that

$$\begin{aligned}
\frac{T_{3,1}}{\nu_{in}} &= \sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \|\lambda - c(\lambda-\tau)\|^2 \\
&= \sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \|(1-c)(\lambda-\tau)+\tau\|^2 \\
&= \sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \big(\|(1-c)(\lambda-\tau)\|^2 + \|\tau\|^2 + 2\langle(1-c)(\lambda-\tau),\tau\rangle\big) \\
&= \sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} (1-c)^2\|\lambda-\tau\|^2 + \sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \|\tau\|^2 \\
&+ \underbrace{2\sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \langle(1-c)(\lambda-\tau),\tau\rangle}_{T_4} \\
&\overset{(a)}{=} \underbrace{(1-c)^2|\mathcal{T}| \sum_{\lambda\in\hat{V}_{s/2}(\mathbf{0})} \|\lambda\|^2}_{T_5} + \underbrace{\frac{M}{|\mathcal{T}|}\sum_{\tau\in\mathcal{T}} \|\tau\|^2}_{T_6} + T_4,
\end{aligned} \tag{A.16}$$

where (a) is based on the fact that $\hat{V}_{s/2}(\tau) = \tau + \hat{V}_{s/2}(\mathbf{0})$ and $|\hat{V}_{s/2}(\mathbf{0})| = \frac{M}{|\mathcal{T}|}$. Relation (A.16) leads to

$$\frac{T_{3,1}}{nM^{\frac{2}{n}}(M\nu_{in})^{1+\frac{2}{n}}} = \frac{T_4}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_5}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_6}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}}. \tag{A.17}$$

We will show first that the first and last term on the right hand side of (A.17) approach 0 in the limit of (2.40). For this we need to introduce the following notation. For any two nested lattices $\Lambda_2 \subset \Lambda_1$ in $\mathbb{R}^n$ denote $\mathcal{C}_{\Lambda_2:\Lambda_1} \triangleq \cup_{\lambda_1\in V_{\Lambda_2}(\mathbf{0})\cap\Lambda_1} V_{\Lambda_1}(\lambda_1)$. Using

Lemma 9, which is stated and proved in Appendix B, we obtain

$$\begin{aligned}\frac{T_6}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} &= \frac{\frac{M}{2^n}n2^n}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}}\left(G(\mathcal{C}_{\Lambda_s:\Lambda_{s/2}})\nu_s^{\frac{2}{n}} - G(\Lambda_{s/2})\nu_{s/2}^{\frac{2}{n}}\right)\\ &= \frac{1}{M^{\frac{2}{n}}}\left(G(\mathcal{C}_{\Lambda_s:\Lambda_{s/2}}) - \frac{1}{4}G(\Lambda_{s/2})\right),\end{aligned}$$

where the last equality is based on $\nu_s = M\nu_{in}$ and $\nu_{s/2} = M\nu_{in}/2^n$. As the parameters c and θ vary, both lattices Λ_s and $\Lambda_{s/2}$ are scaled by the same factor, therefore the set $\mathcal{C}_{\Lambda_s:\Lambda_{s/2}}$ is scaled by that factor. Since the second moment is invariant under scaling it follows that $G(\mathcal{C}_{\Lambda_s:\Lambda_{s/2}}) - \frac{1}{4}G(\Lambda_{s/2})$ remains constant as θ and c vary. Consequently, we have that

$$\lim_{(2.40)} \frac{T_6}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} = 0. \tag{A.18}$$

Consider now the first term on the right hand side of (A.17). The following holds

$$\begin{aligned}|T_4| &\leq 2|c-1|\sum_{\tau\in\mathcal{T}}\sum_{\lambda\in\hat{V}_{s/2}(\tau)}|\langle\lambda-\tau,\tau\rangle|\\ &\overset{(a)}{\leq} 2|c-1|\sum_{\tau\in\mathcal{T}}\sum_{\lambda\in\hat{V}_{s/2}(\tau)}\|\lambda-\tau\|\|\tau\|\\ &\leq 2|c-1|M\max_{\tau\in\mathcal{T}}\|\tau\|\max_{\lambda\in\hat{V}_{s/2}(\mathbf{0})}\|\lambda\|\\ &\leq 2cM\bar{r}_s\bar{r}_{s/2},\end{aligned}$$

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $c = M^{\frac{1}{n}}$, while $\bar{r}_{s/2} = \bar{r}_s/2 = M^{\frac{1}{n}}\bar{r}_{in}/2$, leads to

$$\frac{|T_4|}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} \leq \frac{M^{1+\frac{3}{n}}\bar{r}_{in}^2}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} = \frac{\bar{r}_{in,0}^2}{nM^{\frac{1}{n}}\nu_{in,0}^{\frac{2}{n}}} \rightarrow 0 \text{ as (2.40) holds.} \tag{A.19}$$

In order to evaluate the second term in (A.17) we use again Lemma 9 and obtain that

$$\begin{aligned}\frac{T_5}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} &= \frac{(c-1)^2 nM}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}}\left(G(\mathcal{C}_{\Lambda_{s/2}:\Lambda_{in}})\nu_{s/2}^{\frac{2}{n}} - G(\Lambda_{in})\nu_{in}^{\frac{2}{n}}\right) \\ &= \frac{(M^{\frac{1}{n}}-1)^2}{M^{\frac{4}{n}}}\left(G(\mathcal{C}_{\Lambda_{s/2}:\Lambda_{in}})\frac{M^{\frac{2}{n}}}{4} - G(\Lambda_{in})\right),\end{aligned}$$

where the last equality relies on the fact that $c = M^{\frac{1}{n}}$, while $\nu_{s/2} = M\nu_{in}/2^n$. Further, we obtain that

$$\lim_{(2.40)} \frac{T_5}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}}} = \lim_{(2.40)} \frac{G(\mathcal{C}_{\Lambda_{s/2}:\Lambda_{in}})}{4} = \frac{G(\Lambda_{s/2})}{4}, \tag{A.20}$$

where the last equality follows from Lemma 10, which is stated and proved in Appendix B.

Relations (A.17)-(A.20) imply that

$$\lim_{(2.40)} \frac{T_{3,1}}{nM^{1+\frac{4}{n}}\nu_{in}^{\frac{2}{n}+1}} = \frac{1}{4}G(\Lambda_{s/2}). \tag{A.21}$$

Combining the above with (A.12), (A.13) and (A.15), we obtain that (A.2) holds for

$i = 1$. In order to prove the claim for $i = 2$ we need to evaluate now $\frac{T_{3,2}}{\nu_{in}}$.

$$\begin{aligned}
\frac{T_{3,2}}{\nu_{in}} &= \sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \|\lambda - 2\tau + c(\lambda-\tau)\|^2 \\
&= \sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \|(1+c)(\lambda-\tau) - \tau\|^2 \\
&= \sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \left(\|(1+c)(\lambda-\tau)\|^2 + \|\tau\|^2 + 2\langle(1+c)(\lambda-\tau),\tau\rangle\right) \\
&= \sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \left(\|(1+c)(\lambda-\tau)\|^2 + \|\tau\|^2\right) + 2\sum_{\tau\in\mathcal{T}} \sum_{\lambda\in\hat{V}_{s/2}(\tau)} \langle(1+c)(\lambda-\tau),\tau\rangle.
\end{aligned}$$

Next the conclusion follows using similar arguments as for $i = 1$. This observation concludes the proof.

Proof of Relation (2.51)

In order to prove the claim we will show that U_i approaches a uniform distribution. To prove this let $u \in V_{in}(\mathbf{0}) \cap \Lambda_c$.

The general idea of the proof is that, as the limits of (2.40) are approached, the pdf $f_{X_i^n}$ can be approximated by a pdf which is uniform on each set $\mathcal{M}(\lambda)$. Then the following relations hold, in the limit of (2.40),

$$\begin{aligned}
\mathbb{P}[U_i = u] &= \sum_{\lambda\in\Lambda_{in}} \int_{V_c(\lambda+u)} f_{X_i^n}(x^n)dx^n \overset{(a)}{\approx} \sum_{\lambda\in\Lambda_{in}} f_{X_i^n}(\lambda)\nu_c \\
&= \sum_{\lambda\in\Lambda_{in}} f_{X_i^n}(\lambda)\frac{\nu_{in}}{K} \overset{(b)}{\approx} \frac{1}{K}\sum_{\lambda\in\Lambda_{in}} \int_{\mathcal{M}(\lambda)} f_{X_i^n}(x^n)dx^n = \frac{1}{K}.
\end{aligned}$$

Next we provide a rigorous treatment of relations (a) and (b).

Define a density function $f_{\theta,c} : \mathbb{R}^n \to [0,\infty)$, which is uniform on each set $\mathcal{M}(\lambda)$, as follows

$$f_{\theta,c}(x^n) = \frac{1}{\nu(\mathcal{M}(\lambda))}\int_{\mathcal{M}(\lambda)} f_{X_i^n}(y^n)dy^n,$$

if $x^n \in \mathcal{M}(\lambda)$. Then in view of Lemma 7 (stated and proved in Appendix B), we have that $f_{\theta,c}(x^n) \to f_{X_i^n}(x^n)$ for every $x^n \in \mathbb{R}^n$, as (2.40) holds. Further, we have

$$\begin{aligned}\mathbb{P}[U_i = u] &= \int_{\cup_{\lambda\in\Lambda_{in}} V_c(\lambda+u)} \left(f_{X_i^n}(x^n) - f_{\theta,c}(x^n) + f_{\theta,c}(x^n)\right) dx^n \\ &\leq \int_{\cup_{\lambda\in\Lambda_{in}} V_c(\lambda+u)} |f_{X_i^n}(x^n) - f_{\theta,c}(x^n)|dx^n + \sum_{\lambda\in\Lambda_{in}} \int_{V_c(\lambda+u)} f_{\theta,c}(x^n)dx^n.\end{aligned} \tag{A.22}$$

Note that

$$\int_{\cup_{\lambda\in\Lambda_{in}} V_c(\lambda+u)} |f_{X_i^n}(x^n) - f_{\theta,c}(x^n)|dx^n \leq \int_{\mathbb{R}^n} |f_{X_i^n}(x^n) - f_{\theta,c}(x^n)|dx^n \to 0, \text{ as (2.40) holds,} \tag{A.23}$$

where the last relation is valid in view of Scheffe's theorem Scheffé (1947).

Further, since $f_{\theta,c}$ is constant on each $\mathcal{M}(\lambda)$, we have

$$\begin{aligned}\sum_{\lambda\in\Lambda_{in}} \int_{V_c(\lambda+u)} f_{\theta,c}(x^n)dx^n &= \sum_{\lambda\in\Lambda_{in}} f_{\theta,c}(\lambda)\frac{\nu_{in}}{K} \\ &= \frac{1}{K}\sum_{\lambda\in\Lambda_{in}} \int_{\mathcal{M}(\lambda)} f_{X_i^n}(x^n)dx^n = \frac{1}{K}.\end{aligned} \tag{A.24}$$

Relations (A.22)-(A.24) together with the fact that the size of the alphabet of U_i is K and K is constant, prove the claim. With this observation the proof is complete.

Proof of Relation (2.48)

Using a variant of Fano's inequality we obtain that

$$H(U_2|U_1) \leq 1 + \mathbb{P}[U_1 \neq U_2] \log_2 K, \tag{A.25}$$

where we used the fact that $H(U_2) = \log_2 K$. Let $\lambda_{c,1} = Q_c(x_1^n)$. Notice that if $x_2^n - x_1^n \in \mathcal{B}(r_0)$ and the distance from x_1^n to the boundary of the Voronoi cell $V_c(\lambda_{c,1})$ is larger than or equal to r_0, then it is guaranteed that $x_2^n \in V_c(\lambda_{c,1})$, thus $u_2 = u_1$. Then let us denote $\mathcal{E}(\lambda_c) \triangleq V_c(\lambda_c) \setminus \left(1 - \frac{r_0}{r_c}\right) V_c(\lambda_c)$, for each $\lambda_c \in \Lambda_c$, and $\mathcal{E} \triangleq \cup_{\lambda_c \in \Lambda_c} \mathcal{E}(\lambda_c)$. It follows that

$$\mathbb{P}[U_1 \neq U_2] \leq \mathcal{P}(r_0) + \mathbb{P}[X_1^n \in \mathcal{E}]. \tag{A.26}$$

Further, we obtain

$$\mathbb{P}[X_1^n \in \mathcal{E}] \leq \int_{\mathcal{E}} |f_{X_1^n}(x^n) - f_{\theta,c}(x^n)| dx^n + \int_{\mathcal{E}} f_{\theta,c}(x^n) dx^n, \tag{A.27}$$

where $f_{\theta,c}$ was defined in the proof of relation (2.51). According to that proof the first integral in (A.27) approaches 0 in the limit of (2.40). Since $f_{\theta,c}$ is uniform over each Voronoi region of the central lattice, we have

$$\begin{aligned}\int_{\mathcal{E}} f_{\theta,c}(x^n) dx^n &= \sum_{\lambda_c \in \Lambda_c} \int_{\mathcal{E}(\lambda_c)} f_{\theta,c}(x^n) dx^n = \sum_{\lambda_c \in \Lambda_c} f_{\theta,c}(\lambda_c) \nu(\mathcal{E}(\lambda_c)) \\ &= \left(1 - \left(1 - \frac{r_0}{r_c}\right)^n\right) \sum_{\lambda_c \in \Lambda_c} f_{\theta,c}(\lambda_c) \nu_c = 1 - \left(1 - \frac{r_0}{r_c}\right)^n. \end{aligned} \tag{A.28}$$

Relations (A.27)-(A.28), together with the fact that the first integral in (A.27) approaches 0 in the limit of (2.40), imply that

$$\mathbb{P}[U_1 \neq U_2] \leq 1 - \left(1 - \frac{r_0}{r_c}\right)^n + \mathcal{P}(r_0) + o(1).$$

Finally, by applying the above inequality in (A.25), the conclusion follows.

Appendix B

Appendix

B.1 Proofs of Lemmas in Chapter 2

Proof of Lemma 2

According to equation (A.6) at the beginning of the proof of relation (2.44), for $i = 1, 2$ and $c \geq 8$, we have

$$\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq c\bar{r}_s = c^2\theta\bar{r}_{in,0}. \tag{B.29}$$

Using the fact that $\lambda_{c,i} = \lambda_i + u_i$ and the triangle inequality we obtain that

$$\begin{aligned} \|x_i^n - \hat{x}_{c,i}^n\| &= \|x_i^n - \lambda_{c,i} + u_i + \lambda_i - \hat{x}_{c,i}^n\| \\ &\leq \|x_{c,i}^n - \lambda_{c,i}\| + \|u_i\| + \|\lambda_i - \hat{x}_{c,i}^n\| \\ &\leq \bar{r}_c + \bar{r}_{in} + \|\lambda_i - \hat{x}_{c,i}^n\| \leq 2\bar{r}_s + \|\lambda_i - \hat{x}_{c,i}^n\|. \end{aligned} \tag{B.30}$$

If condition (2.26) is violated then $\hat{x}^n_{c,i} = \lambda_{s,i}$. Thus, we have

$$\|\lambda_i - \hat{x}^n_{c,i}\| = \|\lambda_i - \lambda_{s,i}\| \leq \bar{r}(\beta_i^{-1}(\mathbf{0})) \leq c\bar{r}_s, \tag{B.31}$$

for $c \geq 8$. Relations (B.30) and (B.31) imply that

$$\|x^n_i - \hat{x}^n_{c,i}\| \leq (c+2)\bar{r}_s \leq 2c\bar{r}_s = 2c^2\theta\bar{r}_{in,0}, \tag{B.32}$$

for $c \geq 8$.

Let us assume now that condition (2.26) is satisfied and that Case 3) holds at the decoder, i.e., $b_1 = b_2 = 1$ and $\tau_1 \neq \tau_2$. Thus, $\hat{x}^n_{c,i} = \tilde{\lambda}_i + u_i$, where $\tilde{\lambda}_i$ is given in (2.33). Then

$$\|\lambda_i - \hat{x}^n_{c,i}\| \leq \|\lambda_i - \tilde{\lambda}_i\| + \|u_i\| \leq \|\lambda_i - \tilde{\lambda}_i\| + \bar{r}_{in}. \tag{B.33}$$

Let us consider now $i = 1$. Using (2.33) and the triangle inequality we obtain that

$$\begin{aligned} \|\lambda_1 - \tilde{\lambda}_1\| &\leq \|\lambda_1 - \tilde{\lambda}_s\| + \|\tau_1\| + \frac{1}{c}\|\lambda_{s,1} - \tilde{\lambda}_s\| \\ &\leq \|\lambda_1 - \lambda_{s,1}\| + \|\lambda_{s,1} - \tilde{\lambda}_s\| + \bar{r}_s + \frac{1}{c}\|\lambda_{s,1} - \tilde{\lambda}_s\| \\ &\leq c\bar{r}_s + \bar{r}_s + \left(1 + \frac{1}{c}\right)\|\lambda_{s,1} - \tilde{\lambda}_s\|, \end{aligned} \tag{B.34}$$

where the last inequality is based on $\|\lambda_1 - \lambda_{s,1}\| \leq \bar{r}(\beta_i^{-1}(\mathbf{0})) \leq c\bar{r}_s$. Using now (2.32) in conjunction with the triangle inequality leads to

$$\begin{aligned} \|\lambda_{s,1} - \tilde{\lambda}_s\| &\leq \|\lambda_{s,1} - \tilde{v}\| + (c+1)\|\tilde{w}\| \\ &\leq \|\lambda_{,s1} - \tilde{v}\| + 2(c+1)\bar{r}_s, \end{aligned} \tag{B.35}$$

where the last inequality follows based on (2.31) and on

$$\|\tilde{w}\| \leq \|\hat{w} + \frac{1}{2}(\tau_2 - \tau_1) - Q_s(\hat{w} + \frac{1}{2}(\tau_2 - \tau_1))\| + \|\frac{1}{2}(\tau_2 - \tau_1)\| \leq 2\bar{r}_s.$$

Finally, based on (2.30) we obtain that

$$\begin{aligned} \|\lambda_{s,1} - \tilde{v}\| &= \left\|\frac{1}{2}(\lambda_{s,1} - \lambda_{s,2}) - \frac{1}{2}c\tilde{\lambda} + \left(1 + \frac{c}{2}\right)\tau_2 - \frac{c}{2}\tau_1\right\| \\ &\leq \frac{1}{2}\|(\lambda_{s,1} - \lambda_{s,2})\| + \frac{1}{2}c\|\tilde{\lambda}\| + \left(1 + \frac{c}{2}\right)\|\tau_2\| + \frac{c}{2}\|\tau_1\|. \end{aligned} \tag{B.36}$$

Notice that relation (2.26) implies that

$$\|(\lambda_{s,1} - \lambda_{s,2})\| \leq 2c\bar{r}_s. \tag{B.37}$$

Additionally, from (2.27) we obtain that

$$\|\tilde{\lambda}\| \leq \|u_1 - u_2\| + \|(u_1 - u_2) - Q_{in}(u_1 - u_2)\| \leq 2\bar{r}_{in} + \bar{r}_{in} = 3\bar{r}_{in}. \tag{B.38}$$

Plugging (B.37) and (B.38) in (B.36) leads to

$$\|\lambda_{s,1} - \tilde{v}\| \leq c\bar{r}_s + \frac{3}{2}c\bar{r}_{in} + (1+c)\bar{r}_s \leq (\frac{5}{2} + 2c)\bar{r}_s,$$

The above relations and (B.35) imply that

$$\|\lambda_{s,1} - \tilde{\lambda}_s\| \leq (4c + \frac{9}{2})\bar{r}_s.$$

Combining now the above inequality with (B.30), (B.33) and (B.34) we obtain that

$$\begin{aligned}\|x_1^n - \hat{x}_{c,1}^n\| &\leq 2\bar{r}_s + \bar{r}_{in} + (c+1)\bar{r}_s + \left(1 + \frac{1}{c}\right)(4c + \frac{9}{2})\bar{r}_s \\ &\leq 6c\bar{r}_s = 6c^2\theta\bar{r}_{in,0}, \qquad \text{(B.39)}\end{aligned}$$

for c sufficiently large. The proof for $i = 2$ and for the remaining cases follows along the same lines.

Proof of Lemma 3

Let us fix i. Denote $\tilde{\mathcal{C}}(\lambda_{s/2}) \triangleq \{x_i^n \in \mathbb{R}^n : Q_{in}(Q_c(x_i^n)) \in \mathcal{C}(\lambda_{s/2})\}$ and $\tilde{\mathcal{C}} \triangleq \cup_{\lambda_{s/2}\in\Lambda_{s/2}}\tilde{\mathcal{C}}(\lambda_{s/2})$. A moment of thought reveals that $\tilde{\mathcal{C}}(\lambda_{s/2}) \subset \big(\lambda_{s/2} + \gamma_1 V_{s/2}(\mathbf{0})\big) \setminus \big(\lambda_{s/2} + \gamma_2 V_{s/2}(\mathbf{0})\big)$, where $\gamma_1 = 1 + \frac{\bar{r}_{in}+\bar{r}_c}{r_{s/2}}$ and $\gamma_2 = \gamma - \frac{\bar{r}_{in}+\bar{r}_c}{r_{s/2}}$. The above relation implies that

$$\nu(\tilde{\mathcal{C}}(\lambda_{s/2})) \leq (\gamma_1^n - \gamma_2^n)\nu(V_{s/2}(\lambda_{s/2})). \qquad \text{(B.40)}$$

Let $\tilde{\mathcal{V}}(\lambda_{s/2}) \triangleq \{x_i^n \in \mathbb{R}^n | Q_{in}(Q_c(x_1^n)) \in V_{s/2}(\lambda_{s/2})\}$. Clearly, $\nu(\tilde{\mathcal{V}}(\lambda_{s/2})) = \nu_{s/2}$. The proof of the lemma hinges on the fact that, as (2.40) holds, the pdf of X_i^n can be approximated by a pdf which is uniform over $\tilde{\mathcal{V}}_{s/2}(\lambda_{s/2})$. The general idea of the proof is the following.

$$\mathbb{P}[Q_{in}(Q_c(X_i^n)) \in \mathcal{C}(\lambda_{s/2})] \overset{(a)}{\approx} f_{X_i^n}(\lambda_{s/2})\nu(\tilde{\mathcal{C}}(\lambda_{s/2})) \leq f_{X_i^n}(\lambda_{s/2})\nu(V(\lambda_{s/2}))(\gamma_1^n - \gamma_2^n),$$

where the last inequality follows from (B.40). The above relations lead to

$$\mathbb{P}[Q_{in}(Q_c(X_i^n)) \in \cup_{\lambda_{s/2}\in\Lambda_{s/2}}\mathcal{C}(\lambda_{s/2})] \leq \sum_{\lambda_{s/2}\in\Lambda_{s/2}} f_{X_i^n}(\lambda_{s/2})\nu(V(\lambda_{s/2}))(\gamma_1^n-\gamma_2^n) \overset{(b)}{\approx} \gamma_1^n-\gamma_2^n,$$

where (b) follows from the assumption that the pdf is uniform over $V_{s/2}(\lambda_{s/2})$, thus $\sum_{\lambda_{s/2}\in\Lambda_{s/2}} f_{X_i^n}(\lambda_{s/2})\nu(V(\lambda_{s/2})) = 1$. Finally, it is easy to see that $\gamma_1 \to 1$ and $\gamma_2 \to 1$ as (2.40) holds, thus $\lim_{(3.108)}(\gamma_1^n - \gamma_2^n) = 0$.

Next we provide a detailed proof including a rigorous treatment of relations (a) and (b). Note that the sets $\tilde{\mathcal{V}}(\lambda_{s/2})$ with $\lambda_{s/2} \in \Lambda_{s/2}$, form a partition of $\mathbb{R}^n$. Define a density function $f_{\theta,c} : \mathbb{R}^n \to [0,\infty)$, which is uniform on each set $\tilde{\mathcal{V}}(\lambda_{s/2})$, as follows

$$f_{\theta,c}(x^n) = \frac{1}{\nu(\tilde{\mathcal{V}}(\lambda_{s/2}))}\int_{\tilde{\mathcal{V}}(\lambda_{s/2})} f_{X_i^n}(y^n)dy^n, \tag{B.41}$$

if $x^n \in \tilde{\mathcal{V}}(\lambda_{s/2})$. Then in view of Lemma 7, which is stated and proved after the proof of this lemma, we have that $f_{\theta,c}(x^n) \to f_{X_i^n}(x^n)$ for every $x^n \in \mathbb{R}^n$, as (2.40) holds. Further, we have

$$\begin{aligned}\mathbb{P}[X_i^n \in \tilde{\mathcal{C}}] &= \int_{\tilde{\mathcal{C}}} \left(f_{X_i^n}(x^n) - f_{\theta,c}(x^n) + f_{\theta,c}(x^n)\right) dx^n \\ &\leq \int_{\tilde{\mathcal{C}}} |f_{X_i^n}(x^n) - f_{\theta,c}(x^n)|dx^n + \sum_{\lambda_{s/2}\in\Lambda_{s/2}} \int_{\tilde{\mathcal{C}}(\lambda_{s/2})} f_{\theta,c}(x^n)dx^n.\end{aligned}$$

Note that

$$\int_{\tilde{\mathcal{C}}} |f_{X_i^n}(x^n) - f_{\theta,c}(x^n)|dx^n \leq \int_{\mathbb{R}^n} |f_{X_i^n}(x^n) - f_{\theta,c}(x^n)|dx^n \to 0, \text{ as (2.40) holds,}$$

where the last relation is valid in view of Scheffe's theorem (Scheffé, 1947). Further, since the density $f_{\theta,c}$ is uniform over each $\tilde{\mathcal{V}}(\lambda_{s/2})$ and $\tilde{\mathcal{C}}(\lambda_{s/2}) \subset \tilde{\mathcal{V}}(\lambda_{s/2})$, we obtain that

$$\begin{aligned}
\sum_{\lambda_{s/2}\in\Lambda_{s/2}} \int_{\tilde{\mathcal{C}}(\lambda_{s/2})} f_{\theta,c}(x^n)dx^n &= \sum_{\lambda_{s/2}\in\Lambda_{s/2}} f_{\theta,c}(\lambda_{s/2})\nu(\tilde{\mathcal{C}}(\lambda_{s/2})) \\
&\stackrel{(c)}{\leq} \sum_{\lambda_{s/2}\in\Lambda_{s/2}} f_{\theta,c}(\lambda_{s/2})\nu(V_{s/2}(\lambda_{s/2}))(\gamma_1^n - \gamma_2^n) \\
&= (\gamma_1^n - \gamma_2^n) \sum_{\lambda_{s/2}\in\Lambda_{s/2}} f_{\theta,c}(\lambda_{s/2})\nu(V_{s/2}(\lambda_{s/2})) \\
&\stackrel{(d)}{=} (\gamma_1^n - \gamma_2^n) \sum_{\lambda_{s/2}\in\Lambda_{s/2}} \int_{\tilde{\mathcal{V}}(\lambda_{s/2})} f_{X_1^n}(y^n)dy^n \\
&= (\gamma_1^n - \gamma_2^n) \int_{\mathbb{R}^n} f_{X_i^n}(y^n)dy^n = \gamma_1^n - \gamma_2^n,
\end{aligned}$$

where (c) follows from (B.40) and (d) is based on relation (B.41) and on the fact that $\nu(V_{s/2}(\lambda_{s/2})) = \nu(\tilde{\mathcal{V}}(\lambda_{s/2}))$. This observation concludes the proof.

Lemma 7 *Let Λ be a lattice and $\sigma > 0$ a scale factor. Let $\mathcal{C}_\sigma$ be a measurable fundamental cell of the scaled lattice $\sigma\Lambda$ such that $\lim_{\sigma\to 0} \bar{r}(\mathcal{C}_\sigma) = 0$. Let $f : \mathbb{R}^n \to [0, \infty)$ be a continuous density function. For each σ define the function $f_\sigma : \mathbb{R}^n \to [0, \infty)$ as follows. For each $\lambda_\sigma \in \sigma\Lambda$ and $x^n \in \lambda_\sigma + \mathcal{C}_\sigma$, let*

$$f_\sigma(x^n) \triangleq \frac{1}{\nu(\mathcal{C}_\sigma)} \int_{\lambda_\sigma+\mathcal{C}_\sigma} f(y^n)dy^n. \tag{B.42}$$

Then for every $x^n \in \mathbb{R}^n$ the following holds

$$\lim_{\sigma\to 0} f_\sigma(x^n) = f(x^n). \tag{B.43}$$

Let us fix $x^n \in \mathbb{R}^n$ and let $\lambda_\sigma \in \sigma\Lambda$ such that $x^n \in \lambda_\sigma + \mathcal{C}_\sigma$. Then

$$|f_\sigma(x^n) - f(x^n)| \leq \frac{1}{\nu(\mathcal{C}_\sigma)} \int_{\lambda_\sigma + \mathcal{C}_\sigma} |f(y^n) - f(x^n)| dy^n \leq \max_{y^n \in \overline{\lambda_\sigma + \mathcal{C}_\sigma}} |f(y^n) - f(x^n)| \tag{B.44}$$

it further follows that

$$\lim_{\sigma \to 0} \max_{y^n \in \overline{x^n + \mathcal{B}_{2\bar{r}(\mathcal{C}_\sigma)}}} |f(y^n) - f(x^n)| = 0. \tag{B.45}$$

Relations (B.44) and (B.45) imply that (B.43) holds.

Lemma 8 *For any set $\mathcal{A} \subseteq \mathbb{R}^n$ and any $u \in \mathbb{R}^n$, the following holds,*

$$\int_{u+\mathcal{A}} \|x^n\|^2 dx^n = \int_{\mathcal{A}} \|x^n\|^2 dx^n + 2\langle \int_{\mathcal{A}} x^n dx^n, u\rangle + \|u\|^2 \nu(\mathcal{A}).$$

Applying the change of variable $x^n = u + y^n$ we obtain that

$$\begin{aligned} \int_{u+\mathcal{A}} \|x^n\|^2 dx^n &= \int_{\mathcal{A}} \|y^n + u\|^2 dy^n = \int_{\mathcal{A}} \|y^n\|^2 dy^n + \int_{\mathcal{A}} 2\langle y^n, u\rangle dy^n + \int_{\mathcal{A}} \|u\|^2 dy^n \\ &= \int_{\mathcal{A}} \|y^n\|^2 dy^n + 2\langle \int_{\mathcal{A}} x^n dx^n, u\rangle + \|u\|^2 \nu(\mathcal{A}). \end{aligned}$$

Lemma 9 *Let $\Lambda_2 \subset \Lambda_1$ be two nested lattices in $\mathbb{R}^n$. Let $N_0 \triangleq N(\Lambda_2 : \Lambda_1)$ and $\mathcal{C}_{\Lambda_2:\Lambda_1} \triangleq \cup_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} V_{\Lambda_1}(\lambda_1)$. Then*

$$\sum_{\lambda_1 \in V_{\Lambda_2}(\mathbf{0}) \cap \Lambda_1} \|\lambda_1\|^2 = nN_0 \left(G(\mathcal{C}_{\Lambda_2:\Lambda_1}) \nu_{\Lambda_2}^{\frac{2}{n}} - G(\Lambda_1) \nu_{\Lambda_1}^{\frac{2}{n}} \right).$$

It can be easily seen that $\mathcal{C}_{\Lambda_2:\Lambda_1}$ is a fundamental region of the lattice Λ_2, thus $\nu(\mathcal{C}_{\Lambda_2:\Lambda_1}) = \nu(\Lambda_2) = N_0\nu(\Lambda_1)$. Using further the definition of $G(\mathcal{C}_{\Lambda_2:\Lambda_1})$ one obtains that

$$nN_0G(\mathcal{C}_{\Lambda_2:\Lambda_1})\nu_{\Lambda_2}^{\frac{2}{n}} = \frac{1}{\nu_{\Lambda_1}}\int_{\mathcal{C}_{\Lambda_2:\Lambda_1}} \|x^n\|^2 dx^n.$$

Using the fact that $V_{\Lambda_1}(\lambda_1) = \lambda_1 + V_{\Lambda_1}(\mathbf{0})$, we obtain that

$$\begin{aligned}
\frac{1}{\nu_{\Lambda_1}}\int_{\mathcal{C}_{\Lambda_2:\Lambda_1}} \|x^n\|^2 dx^n &= \frac{1}{\nu_{\Lambda_1}}\sum_{\lambda_1\in V_{\Lambda_2}(\mathbf{0})\cap\Lambda_1}\int_{\lambda_1+V_{\Lambda_1}(\mathbf{0})}\|x^n\|^2 dx^n \\
&\stackrel{(a)}{=} \frac{1}{\nu_{\Lambda_1}}\sum_{\lambda_1\in V_{\Lambda_2}(\mathbf{0})\cap\Lambda_1}\left(\int_{V_{\Lambda_1}(\mathbf{0})}\|x^n\|^2 dx^n + 2\left\langle\int_{V_{\Lambda_1}(\mathbf{0})} x^n dx^n, \lambda_1\right\rangle + \|\lambda_1\|^2\nu_{\Lambda_1}\right) \\
&\stackrel{(b)}{=} \frac{N_0}{\nu_{\Lambda_1}}\int_{V_{\Lambda_1}(\mathbf{0})}\|x^n\|^2 dx^n + \sum_{\lambda_1\in V_{\Lambda_2}(\mathbf{0})\cap\Lambda_1}\|\lambda_1\|^2 \\
&\stackrel{(c)}{=} nN_0G(\Lambda_1)\nu_{\Lambda_1}^{\frac{2}{n}} + \sum_{\lambda_1\in V_{\Lambda_2}(\mathbf{0})\cap\Lambda_1}\|\lambda_1\|^2,
\end{aligned}$$

where (a) is based on Lemma 8. Moreover, (b) uses the fact that $\int_{V_{\Lambda_1}(\mathbf{0})} x^n dx^n = \mathbf{0}$ and $|V_{\Lambda_2}(\mathbf{0})\cap\Lambda_1| = N_0$, while (c) is based on the definition of $G(\Lambda_1)$. Now the claim follows.

Lemma 10 *Consider two nested lattices $\Lambda_{2,0} \subset \Lambda_{1,0}$ and scale coefficients ω_1, ω_2 such that lattices $\Lambda_2 = \omega_2\Lambda_{2,0}$ and $\Lambda_1 = \omega_1\Lambda_{1,0}$ are still nested. Let $N_0 \triangleq N(\Lambda_2 : \Lambda_1)$ and $\mathcal{C}_{\Lambda_2:\Lambda_1} \triangleq \cup_{\lambda_1\in V_{\Lambda_2}(\mathbf{0})\cap\Lambda_1} V_{\Lambda_1}(\lambda_1)$. Then the following holds:*

$$\lim_{\frac{\omega_2}{\omega_1}\to\infty} G(\mathcal{C}_{\Lambda_2:\Lambda_1}) = G(\Lambda_2).$$

Note that since the lattices Λ_2 and Λ_1 are scaled by different scale factors, the value $G(\mathcal{C}_{\Lambda_2:\Lambda_1})$ is not constant. On the other hand, $G(\Lambda_2)$ is constant. Notice further that the set $\mathcal{C}_{\Lambda_2:\Lambda_1}$ is a fundamental region of the lattice Λ_2, thus its volume equals ν_{Λ_2}. Then the following holds

$$G(\mathcal{C}_{\Lambda_2:\Lambda_1}) - G(\Lambda_2) = \frac{1}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} \left(\int_{\mathcal{C}_{\Lambda_2:\Lambda_1}} \|x^n\|^2 dx^n - \int_{V_{\Lambda_2}(\mathbf{0})} \|x^n\|^2 dx^n \right).$$

For simplicity let us denote $\mathcal{A} = \mathcal{C}_{\Lambda_2:\Lambda_1}$, $\mathcal{B} = V_{\Lambda_2}(\mathbf{0})$ and $\Delta\nu = \nu(\mathcal{A}) - \nu(\mathcal{A} \cap \mathcal{B})$. Since $\nu(\mathcal{A}) = \nu(\mathcal{B})$ it follows that $\Delta\nu = \nu(\mathcal{B}) - \nu(\mathcal{A} \cap \mathcal{B})$. Then we obtain that

$$\begin{aligned} |G(\mathcal{A}) - G(\mathcal{B})| &= \frac{1}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} \left| \int_{\mathcal{A}\setminus\mathcal{A}\cap\mathcal{B}} \|x^n\|^2 dx^n - \int_{\mathcal{B}\setminus\mathcal{A}\cap\mathcal{B}} \|x^n\|^2 dx^n \right| \\ &\leq \frac{1}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} \left(\bar{r}(\mathcal{A})^2\Delta\nu + \bar{r}(\mathcal{B})^2\Delta\nu \right) \\ &\leq \frac{\Delta\nu}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} \left((\bar{r}_{\Lambda_2} + \bar{r}_{\Lambda_1})^2 + \bar{r}_{\Lambda_2}^2 \right) \\ &\leq \frac{5\bar{r}_{\Lambda_2}^2\Delta\nu}{n\nu_{\Lambda_2}^{1+\frac{2}{n}}} = \frac{5\omega_2^2\bar{r}_{\Lambda_{2,0}}^2\Delta\nu}{n\omega_2^2\nu_{\Lambda_{2,0}}^{\frac{2}{n}}\nu_{\Lambda_2}} = \frac{5\bar{r}_{\Lambda_{2,0}}^2}{4n\nu_{\Lambda_{2,0}}^{\frac{2}{n}}} \frac{\Delta\nu}{\nu_{\Lambda_2}}. \end{aligned}$$

According to the above relations in order to prove the claim of the lemma it is sufficient to show that $\lim_{\frac{\omega_2}{\omega_1}\to\infty} \frac{\Delta\nu}{\nu_{\Lambda_2}} = 0$, which is equivalent to

$$\lim_{\frac{\omega_2}{\omega_1}\to\infty} \frac{\nu(\mathcal{A}\cap\mathcal{B})}{\nu_{\Lambda_2}} = 1. \tag{B.46}$$

It is easy to see that for any point $x^n \in V_{\Lambda_2}(\mathbf{0})$ which is at a distance larger than $\bar{r}_{\Lambda_1}$ from the boundary of $V_{\Lambda_2}(\mathbf{0})$, we have $Q_{\Lambda_1}(x^n) \in V_{\Lambda_2}(\mathbf{0})$, thus $x^n \in \mathcal{A}$. This observation implies that the interior of the set $\gamma V_{\Lambda_2}(\mathbf{0})$ is included in $\mathcal{A} \cap \mathcal{B}$, where $\gamma = 1 - \frac{\bar{r}_{\Lambda_1}}{r_{\Lambda_2}} = 1 - \frac{\omega_1}{\omega_2}\frac{\bar{r}_{\Lambda_{1,0}}}{r_{\Lambda_{2,0}}}$. Then we have $\gamma^n \leq \frac{\nu(\mathcal{A}\cap\mathcal{B})}{\nu_{\Lambda_2}} \leq 1$, which implies that (B.46) holds. With this the proof is completed.

Appendix C

Appendix

C.1 Proof of Theorem 3

Proof of Theorem 3

For each $\lambda_s \in \Lambda_s$ and $i = 1, 2, 3$, let $\mathcal{A}_i(\lambda_s) \triangleq \{x_i^n | \hat{x}_{s,i}^n = \lambda_s\}$. Further, for each $\lambda \in \Lambda_{in}$, denote $\mathcal{M}(\lambda) \triangleq \cup_{\lambda_c \in V_{in}(\lambda) \cap \Lambda_c} V_c(\lambda_c)$. Then $\mathcal{A}_i(\lambda_s) = \cup_{\lambda \in \beta_i^{-1}(\lambda_s)} \mathcal{M}(\lambda)$. Clearly, we have $\mathcal{M}(\lambda) = \lambda + \mathcal{M}(\mathbf{0})$ for all $\lambda \in \Lambda$. This fact together with relation (3.67) implies that

$$\mathcal{A}_i(\lambda_s) = \mathcal{A}_i(\mathbf{0}) + \lambda_s, \quad \forall \lambda_s \in \Lambda_s. \tag{C.47}$$

Obviously, we have $d_{s,i} = D(Q_{\mathcal{A}_i}, X_i^n)$, where $Q_{\mathcal{A}_i}$ denotes the quantizer which maps each input sequence $x_i^n \in \mathcal{A}_i(\lambda_s)$ to λ_s, for $\lambda_s \in \Lambda_s$.

First let us fix i. We will split the proof into two parts. In Part 1 we show that if $\lim_{(3.108)} \frac{G(\mathcal{A}_i(\mathbf{0}))}{c_o^2}$ exists then we have

$$\lim_{(3.108)} \frac{D(Q_{\mathcal{A}_i}, X_i^n)}{c_o^2(\nu_s)^{\frac{2}{n}}} = \lim_{(3.108)} \frac{G(\mathcal{A}_i(\mathbf{0}))}{c_o^2}. \tag{C.48}$$

In Part 2 we will evaluate that

$$\lim_{(3.108)} \frac{G(\mathcal{A}_i(\mathbf{0}))}{c_o^2}. \tag{C.49}$$

Part 1.[1]

The proof is based on the idea that in the limit of (3.108) the pdf $f_{X_i^n}$ can be approximated by a uniform density function over each set $\mathcal{A}_i(\lambda_s)$. This density function is $f_{\theta,c} : \mathbb{R}^n \to [0, \infty)$ defined as follows. For each $\lambda_s \in \Lambda_s$ and $x^n \in \mathcal{A}_i(\lambda_s)$, let

$$f_{\theta,c}(x^n) = \frac{\mathbb{P}[X_i^n \in \mathcal{A}_i(\lambda_s)]}{\nu(\mathcal{A}_i(\lambda_s))} = \frac{1}{\nu(\mathcal{A}_i(\lambda_s))} \int_{\mathcal{A}_i(\lambda_s)} f_{X_i^n}(y^n) dy^n. \tag{C.50}$$

Let $X_{\theta,c}^n$ denote the random variable with pdf $f_{\theta,c}$. Note that

$$\begin{aligned}
&\tfrac{1}{c_o^2 \nu_s^{\frac{2}{n}}} |D(Q_{\mathcal{A}_i}, X_{\theta,c}^n) - D(Q_{\mathcal{A}_i}, X_i^n)| \leq \\
&\tfrac{1}{nc_o^2 (\nu_s)^{\frac{2}{n}}} \textstyle\sum_{\lambda_s \in \Lambda_s} \int_{\mathcal{A}_i(\lambda_s)} \|x^n - \lambda_s\|^2 |f_{\theta,c}(x^n) - f_{X_i^n}(x^n)| dx^n \leq \\
&\tfrac{1}{nc_o^2 (\nu_s)^{\frac{2}{n}}} \textstyle\sum_{\lambda_s \in \Lambda_s} \bar{r}(\mathcal{A}_i(\mathbf{0}))^2 \int_{\mathcal{A}_i(\lambda_s)} |f_{\theta,c}(x^n) - f_{X_i^n}(x^n)| dx^n = \\
&\tfrac{\bar{r}(\mathcal{A}_i(\mathbf{0}))^2}{nc_o^2 (\nu_s)^{\frac{2}{n}}} \textstyle\int_{\mathbb{R}^n} |f_{\theta,c}(x^n) - f_{X_i^n}(x^n)| dx^n,
\end{aligned} \tag{C.51}$$

where the second inequality is based on the fact that $\mathcal{A}_i(\lambda_s) = \lambda_s + \mathcal{A}_i(\mathbf{0})$, which implies that $\max_{x^n \in \mathcal{A}_i(\lambda_s)} \|x^n - \lambda_s\|^2 = \bar{r}(\mathcal{A}_i(\mathbf{0}))$. Let us analyze now the quantity $\bar{r}(\mathcal{A}_i(\mathbf{0}))$. Recall that $\mathcal{A}_i(\mathbf{0}) = \cup_{\lambda \in \beta_i^{-1}(\mathbf{0})} (\lambda + \mathcal{M}(\mathbf{0}))$, where $\mathcal{M}(\mathbf{0}) \triangleq \cup_{\lambda_c \in V_{in}(\mathbf{0}) \cap \Lambda_c} V_c(\lambda_c)$. Then it follows that

$$\bar{r}(\mathcal{A}_i(\mathbf{0})) \leq \bar{r}(\beta_i^{-1}(\mathbf{0})) + \bar{r}(\mathcal{M}(\mathbf{0})). \tag{C.52}$$

[1]This proof is similar to the proof of part 1 of theorem (2.44) from Chapter 2.

Further,

$$\bar{r}(\mathcal{M}(\mathbf{0})) \leq \bar{r}_{in} + \bar{r}_c \leq 2\bar{r}_{in} = 2\theta\bar{r}_{in,0}. \tag{C.53}$$

Using (3.95) and the fact that $\bar{r}_s = 3c_o^2\theta\bar{r}_{in,0}$, it follows that $\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq (5 + 2c_o)3c_o^2\theta\bar{r}_{in,0}$. Since we are interested in computing the limit in (C.49) as (3.108) holds, we may assume that c_o is conveniently large. Then the following holds

$$\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq 9c_o^3\theta\bar{r}_{in,0}. \tag{C.54}$$

Finally, relations (C.52)-(C.54) together with the fact that $c_1 = 3c_o$ and $\nu_s = c_1^n c_o^n \theta^n \nu_{in,0} = (3c_o^2)^n \theta^n \nu_{in,0}$, lead to

$$\frac{\bar{r}(\mathcal{A}_i(\mathbf{0}))}{c_o(\nu_s)^{\frac{1}{n}}} \leq \frac{2\theta\bar{r}_{in,0} + 3c_1c_o^2\theta\bar{r}_{in,0}}{c_1c_o^2\theta\nu_{in,0}^{\frac{1}{n}}} \to \frac{3\bar{r}_{in,0}}{\nu_{in,0}^{\frac{1}{n}}}, \tag{C.55}$$

in the limit of (3.108). The above result also implies that $\bar{r}(\mathcal{A}_i(\mathbf{0})) \to 0$ as (3.108) holds. This enables us to apply Lemma 7, which is stated and proved in Appendix B, and we obtain that $f_{\theta,c}(x^n) \to f_{X_1}^n(x^n)$ for each $x^n \in \mathbb{R}^n$, as (3.108) holds. Using further Scheffe's theorem (Scheffé, 1947), it follows that $\int_{\mathbb{R}^n} |f_{\theta,c}(x^n) - f_{X_1^n}(x^n)|dx^n \to 0$ as (3.108) holds. Combining further with (C.51) and (C.55) we obtain that

$$\lim_{(3.108)} \frac{1}{c_o^2(\nu_s)^{\frac{2}{n}}} |D(Q_{\mathcal{A}_i}, X_{\theta,c}^n) - D(Q_{\mathcal{A}_i}, X_i^n)| = 0. \tag{C.56}$$

Using now the fact that $f_{\theta,c}$ is uniform over each quantizer cell $\mathcal{A}_i(\lambda_s)$ we obtain that

$$D(Q_{\mathcal{A}_i}, X^n_{\theta,c}) = \tfrac{1}{n}\sum_{\lambda_s\in\Lambda_s}\int_{\mathcal{A}_i(\lambda_s)}\|x^n-\lambda_s\|^2 f_{\theta,c}(x^n)dx^n =$$
$$\tfrac{1}{n}\sum_{\lambda_s\in\Lambda_s}\tfrac{\mathbb{P}[X^n_i\in\mathcal{A}_i(\lambda_s)]}{\nu(\mathcal{A}_i(\lambda_s))}\int_{\mathcal{A}_i(\lambda_s)}\|x^n-\lambda_s\|^2 dx^n \stackrel{(a)}{=}$$
$$\tfrac{1}{n\nu(\mathcal{A}_i(\mathbf{0}))}\int_{\mathcal{A}_i(\mathbf{0})}\|x^n\|^2dx^n\sum_{\lambda_s\in\Lambda_s}\mathbb{P}[X^n_i\in\mathcal{A}_i(\lambda_s)] =$$
$$\tfrac{1}{n\nu(\mathcal{A}_i(\mathbf{0}))}\int_{\mathcal{A}_i(\mathbf{0})}\|x^n\|^2dx^n\mathbb{P}[X^n_i\in\mathbb{R}^n] =$$
$$G(\mathcal{A}_i(\mathbf{0}))(\nu(\mathcal{A}_i(\mathbf{0})))^{\frac{2}{n}} \stackrel{(b)}{=} G(\mathcal{A}_i(\mathbf{0}))\nu_s^{\frac{2}{n}}, \tag{C.57}$$

where (a) uses the fact that $\mathcal{A}_i(\lambda_s) = \lambda_s + \mathcal{A}_i(\mathbf{0})$, and (b) is based on the fact that $\nu(\mathcal{A}_i(\mathbf{0})) = \nu_s$ since $\mathcal{A}_i(\mathbf{0})$ is a fundamental cell of the lattice Λ_s. Relations (C.56) and (C.57) prove the claim of Part 1.

Part 2.

We will first evaluate $\int_{\mathcal{A}_i(\mathbf{0})}\|x^n\|^2dx^n$. Using the fact that $\mathcal{A}_i(\mathbf{0}) = \cup_{\lambda\in\beta_i^{-1}(\mathbf{0})}(\lambda + \mathcal{M}(\mathbf{0}))$ and relation (3.68) we obtain that

$$\mathcal{A}_i(\mathbf{0}) = \cup_{\lambda\in\mathcal{U}}\left(\lambda - \beta_i(\lambda) + \mathcal{M}(\mathbf{0})\right). \tag{C.58}$$

Using further Lemma 8, which is stated and proved in Appendix B, we obtain that

$$\int_{\lambda-\beta_i(\lambda)+\mathcal{M}(\mathbf{0})}\|x^n\|^2dx^n = \int_{\mathcal{M}(\mathbf{0})}\|x^n\|^2dx^n + 2\langle\int_{\mathcal{M}(\mathbf{0})}x^ndx^n, \lambda-\beta_i(\lambda)\rangle + \|\lambda-\beta_i(\lambda)\|^2\nu(\mathcal{M}(\mathbf{0})). \tag{C.59}$$

It is easy to see that $\mathcal{M}(\mathbf{0})$ is a fundamental cell of the lattice Λ_{in}, therefore, $\nu(\mathcal{M}(\mathbf{0})) =$

ν_{in}. Further, relations (C.58) and (C.59) lead to

$$\int_{\mathcal{A}_i(\mathbf{0})} \|x^n\|^2 dx^n = \underbrace{|\mathcal{U}| \int_{\mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n}_{T_1} + \underbrace{2 \sum_{\lambda \in \mathcal{U}} \langle \int_{\mathcal{M}(\mathbf{0})} x^n dx^n, \lambda - \beta_i(\lambda) \rangle}_{T_{2,i}} + \underbrace{\nu_{in} \sum_{\lambda \in \mathcal{U}} \|\lambda - \beta_i(\lambda)\|^2}_{T_{3,i}}.$$

Then

$$\frac{G(\mathcal{A}_i(\mathbf{0}))}{c_o^2} = \frac{T_1}{nc_o^2 (M\nu_{in})^{1+\frac{2}{n}}} + \frac{T_{2,i}}{nc_o^2 (M\nu_{in})^{1+\frac{2}{n}}} + \frac{T_{3,i}}{nc_o^2 (M\nu_{in})^{1+\frac{2}{n}}}. \tag{C.60}$$

We will prove first that the first two terms in the right hand side of the above equality approach 0 in the limit of (3.108). Consider the first term. Note that $\int_{\mathcal{M}(\mathbf{0})} \|x^n\|^2 dx^n \leq (\bar{r}(\mathcal{M}(\mathbf{0})))^2 \nu_{in}$. Combining further with (C.53) and with the fact that $|\mathcal{U}| = M$ it follows that

$$\frac{T_1}{nc_o^2 (M\nu_{in})^{1+\frac{2}{n}}} \leq \frac{4M\theta^2 \bar{r}_{in,0}^2 \nu_{in}}{nc_o^2 (M\nu_{in})^{1+\frac{2}{n}}} = \frac{4\bar{r}_{in,0}^2}{nc_o^2 M^{\frac{2}{n}} \nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (3.108) holds.}$$

It is easy to see that the closure of a lattice Voronoi cell of the origin is symmetric about the origin. Therefore, if Λ_{in} is a clean sublattice of Λ_c, i.e., there are no points of Λ_c on the boundary of $V_{in}(\mathbf{0})$, then the set $\Lambda_c \cap V_{in}(\mathbf{0})$ is symmetric about the origin. The above considerations further imply that the closure of the set $\mathcal{M}(\mathbf{0})$ is symmetric about the origin, thus $\int_{\mathcal{M}(\mathbf{0})} x^n dx^n = 0$. Then the second term in (C.60) is 0. When Λ_{in} is not a clean sublattice of Λ_c, the aforementioned term still approaches

0 in the limit of (3.108), as we prove next.

$$\begin{aligned}
|T_{2,i}| &= 2\left|\sum_{\lambda\in\mathcal{U}}\int_{\mathcal{M}(\mathbf{0})}\langle x^n, \lambda-\beta_i(\lambda)\rangle dx^n\right| \\
&\leq 2\sum_{\lambda\in\mathcal{U}}\int_{\mathcal{M}(\mathbf{0})}|\langle x^n, \lambda-\beta_i(\lambda)\rangle|\, dx^n \\
&\stackrel{(a)}{\leq} 2\sum_{\lambda\in\mathcal{U}}\int_{\mathcal{M}(\mathbf{0})}\|x^n\|\,\|\lambda-\beta_i(\lambda)\|\, dx^n \\
&= 2\int_{\mathcal{M}(\mathbf{0})}\|x^n\|\, dx^n \sum_{\lambda\in\mathcal{U}}\|\lambda-\beta_i(\lambda)\| \\
&\stackrel{(b)}{\leq} 2\bar{r}(\mathcal{M}(\mathbf{0}))\nu_{in} M\bar{r}(\beta_i^{-1}(\mathbf{0})) \\
&\stackrel{(c)}{\leq} 4\theta\bar{r}_{in,0}\nu_{in}M(9c_o^3\theta\bar{r}_{in,0}) \stackrel{(d)}{=} 12c_o\theta^2\nu_{in}M^{1+\frac{1}{n}}\bar{r}_{in,0}^2, \qquad \text{(C.61)}
\end{aligned}$$

where (a) follows from the Cauchy-Schwarz inequality and (b) follows from (3.68) and based on the fact that $\int_{\mathcal{M}(\mathbf{0})}\|x^n\|dx^n \leq \bar{r}(\mathcal{M}(\mathbf{0}))\nu_{in}$ and $|\mathcal{U}| = M$. Additionally, (c) follows from (C.53) and (C.54). Finally, (d) is based on the fact that $3c_o^2 = M^{\frac{1}{n}}$. Further, relation (C.61) implies that

$$\frac{|T_{2,i}|}{nc_o^2(M\nu_{in})^{1+\frac{2}{n}}} \leq \frac{12c_o\nu_{in}M^{1+\frac{1}{n}}\theta^2\bar{r}_{in,0}^2}{nc_o^2M^{1+\frac{2}{n}}\theta^2\nu_{in,0}^{\frac{2}{n}}} = \frac{4\bar{r}_{in,0}^2}{nc_o^3\nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (3.108) holds.} \qquad \text{(C.62)}$$

Let us evaluate now $\frac{T_{3,i}}{\nu_{in}}$. We need to treat separately the cases $i = 1$, $i = 2$ and $i = 3$. Recall that $\mathcal{U} \triangleq \{\tau + \tilde{\lambda}_f + \tilde{u}_f | \tau \in \mathcal{T}, \tilde{u}_f \in V_f(0)\cap\Lambda_{in}, \tilde{\lambda}_f \in V_{s/3}(0)\cap\Lambda_f\}$. Using further (3.66) we obtain that

$$
\begin{aligned}
\frac{T_{3,1}}{\nu_{in}} &= \sum_{\lambda\in\mathcal{U}} \|\lambda - c_1\tilde{\lambda}_f\|^2 \\
&= \sum_{\lambda\in\mathcal{U}} \Big(\|(1-c_1)(\tilde{\lambda_f})\|^2 + \|\tilde{u}_f\|^2 + \|\tau\|^2\Big) \\
&+ \sum_{\lambda\in\mathcal{U}} \Big(2(1-c_1)\langle\tilde{\lambda}_f,\tau\rangle + 2(1-c_1)\langle\tilde{\lambda}_f,\tilde{u}_f\rangle + \langle\tau,\tilde{u}_f\rangle\Big) \\
&\overset{(a)}{=} \underbrace{\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} (1-3c_o)^2\|\tilde{\lambda}_f\|^2}_{T_2} + \underbrace{\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tilde{u}_f\ \|^2}_{T_3} \\
&+ \underbrace{\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tau\ \|^2}_{T_4} + \underbrace{2(1-3c_o)\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \langle\tilde{u}_f,\tilde{\lambda}_f\rangle}_{T_5} \\
&+ \underbrace{2(1-3c_o)\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \langle\tilde{\lambda}_f,\tau\rangle}_{T_6} + \underbrace{2\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \langle\tilde{u}_f,\tau\rangle}_{T_7} \\
&= T_2 + T_3 + T_4 + T_5 + T_6 + T_7, \qquad \text{(C.63)}
\end{aligned}
$$

where (a) is based on the fact that $c_o = \frac{c_1}{3}$.

Relation (C.63) leads to

$$
\begin{aligned}
\frac{T_{3,1}}{nc_o^2(M\nu_{in})^{1+\frac{2}{n}}} &= \frac{T_2}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_3}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_4}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} \\
&+ \frac{T_5}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_6}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_7}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} \qquad \text{(C.64)}
\end{aligned}
$$

Consider now the first term on the right

$$\begin{aligned} T_2 &= (1-3c_o)^2 \sum_{\tau \in \mathcal{T}} \sum_{\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f} \sum_{\tilde{u}_f \in V_f(0) \cap \Lambda_{in}} \| \tilde{\lambda}_f \|^2 \\ &= 3^n c_o^n (1-3c_o)^2 \sum_{\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f} \| \tilde{\lambda}_f \|^2. \end{aligned} \tag{C.65}$$

Using Lemma 9, which is stated and proved in Appendix B, we obtain

$$\begin{aligned} \frac{T_2}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &= \frac{n 3^n c_o^{2n} (1-3c_o)^2}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_{s/3}:\Lambda_f}) \nu_{s/3}^{\frac{2}{n}} - G(\Lambda_f) \nu_f^{\frac{2}{n}} \right) \\ &= \frac{(1-6c_o+9c_o^2)}{9c_o^6} \left(c_o^4 G(\mathcal{C}_{\Lambda_{s/3}:\Lambda_f}) - c_o^2 G(\Lambda_f) \right), \end{aligned}$$

where the last equality relies on the fact that $M = (3c_o^2)^n$, while $\nu_f = c_o^n \nu_{in}$ and $\nu_{s/3} = c_o^{2n} \nu_{in}$. Further, we obtain that

$$\begin{aligned} \lim_{(3.108)} \frac{T_2}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &= \lim_{(3.108)} \frac{(1-6c_o+9c_o^2)}{9c_o^6} \left(c_o^4 G(\mathcal{C}_{\Lambda_{s/3}:\Lambda_f}) - c_o^2 G(\Lambda_f) \right) \\ &\overset{(a)}{=} G(\Lambda_{s/3}) = G(\Lambda_s). \end{aligned} \tag{C.66}$$

where (a) follows from Lemma 10, which is stated and proved in Appendix B and the last equality follows from the fact that the normalized second moment is invariant to scaling.

We will show now that the last five terms on the right hand side of (C.64) approach 0 in the limit of (3.108). Consider now the second term on the right hand side of

(C.64). The following holds

$$\begin{aligned} T_3 &= \sum_{\tau \in \mathcal{T}} \sum_{\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f} \sum_{\tilde{u}_f \in V_f(0) \cap \Lambda_{in}} \|\tilde{u}_f\|^2 \\ &= 3^n c_o^n \sum_{\tilde{u}_f \in V_f(0) \cap \Lambda_{in}} \|\tilde{u}_f\|^2. \end{aligned} \tag{C.67}$$

In order to evaluate (C.67) we use Lemma 9 and obtain that

$$\begin{aligned} \frac{T_3}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &= \frac{n 3^n c_o^{2n}}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_f : \Lambda_{in}}) \nu_f^{\frac{2}{n}} - G(\Lambda_{in}) \nu_{in}^{\frac{2}{n}} \right) \\ &= \frac{1}{M^{\frac{2}{n}} c_o^2} \left(G(\mathcal{C}_{\Lambda_f : \Lambda_{in}}) c_o^2 - G(\Lambda_{in}) \right), \end{aligned}$$

where the last equality relies on the fact that $M = (3c_o^2)^n$, while $\nu_f = c_o^n \nu_{in}$. Further, we obtain that

$$\lim_{(3.108)} \frac{T_3}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} = \lim_{(3.108)} \frac{1}{M^{\frac{2}{n}} c_o^2} \left(G(\mathcal{C}_{\Lambda_f : \Lambda_{in}}) c_o^2 - G(\Lambda_{in}) \right) = 0, \tag{C.68}$$

where the last equality follows from Lemma 10.

We will show that the third term on the right hand side of (C.64) approaches 0 in the limt of (3.108). Note that

$$T_4 = c_o^{2n} \sum_{\tau \in \mathcal{T}} \|\tau\|^2. \tag{C.69}$$

Using Lemma 9, we obtain

$$\begin{aligned}
\frac{T_4}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} &= \frac{Mn}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}}\left(G(\mathcal{C}_{\Lambda_s:\Lambda_{s/3}})\nu_s^{\frac{2}{n}} - G(\Lambda_{s/3})\nu_{s/3}^{\frac{2}{n}}\right) \\
&= \frac{1}{9c_o^2}\left(9G(\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}) - G(\Lambda_{s/3})\right),
\end{aligned}$$

where the last equality is based on $\nu_s = M\nu_{in}$ and $\nu_{s/3} = M\nu_{in}/3^n$. As the parameters c and θ vary, both lattices Λ_s and $\Lambda_{s/3}$ are scaled by the same factor, therefore the set $\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}$ is scaled by that factor. Since the second moment is invariant under scaling it follows that $9G(\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}) - G(\Lambda_{s/3})$ remains constant as the parameters θ and c vary. Consequently, we have that

$$\lim_{(3.108)} \frac{T_4}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} = 0. \tag{C.70}$$

Consider now the fourth term in (C.64).

$$\begin{aligned}
|T_5| &\leq 2|1-3c_o|\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}|\langle\tilde{u}_f,\tilde{\lambda}_f\rangle| \\
&\overset{(a)}{\leq} 2|1-3c_o|\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\|\tilde{u}_f\|\|\tilde{\lambda}_f\| \\
&\leq 2|1-3c_o|M\max_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\|\tilde{u}_f\|\max_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\|\tilde{\lambda}_f\| \\
&\leq 2|1-3c_o|M\bar{r}_f\bar{r}_{s/3},
\end{aligned}$$

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $\bar{r}_f = c_o\theta\bar{r}_{in,o}$ and $\bar{r}_{s/3} = c_o^2\theta\bar{r}_{in,o}$, leads to

$$\begin{aligned}\frac{|T_5|}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} &\leq \frac{2|1-3c_o|c_o^3 M\theta^2\bar{r}_{in,o}^2}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} \\ &= \frac{2|c_o-3c_o^4|\bar{r}_{in,0}^2}{n9c_o^6\nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (3.108) holds.} \qquad \text{(C.71)}\end{aligned}$$

Consider now the fifth term on the right hand side of (C.64). The following holds

$$\begin{aligned}|T_6| &\leq 2|1-3c_o|\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}|\langle\tau,\tilde{\lambda}_f\rangle| \\ &\overset{(a)}{\leq} 2|1-3c_o|\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\|\tau\|\|\tilde{\lambda}_f\| \\ &\leq 2|1-3c_o|M\max_{\tau\in\mathcal{T}}\|\tau\|\max_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\|\tilde{\lambda}_f\| \\ &\leq 2|1-3c_o|M\bar{r}_s\bar{r}_{s/3},\end{aligned}$$

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $\bar{r}_{s/3} = c_o^2\theta\bar{r}_{in,o}$ and $\bar{r}_s = 3c_o^2\theta\bar{r}_{in,o}$, leads to

$$\begin{aligned}\frac{|T_6|}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} &\leq \frac{6|1-3c_o|c_o^4 M\theta^2\bar{r}_{in,o}^2}{nc_o^2 M^{1+\frac{2}{n}}\theta^2\nu_{in,o}^{\frac{2}{n}}} \\ &= \frac{2|c_o^2-3c_o^3|\bar{r}_{in,0}^2}{3nc_o^4\nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (3.108) holds.} \qquad \text{(C.72)}\end{aligned}$$

We will evaluate the last term in (C.64).

$$\begin{aligned}
|T_7| &\leq 2\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}|\langle\tau,\tilde{u}_f\rangle| \\
&\overset{(a)}{\leq} 2\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\|\tau\|\|\tilde{u}_f\| \\
&\leq 2M\max_{\tau\in\mathcal{T}}\|\tau\|\max_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\|\tilde{u}_f\| \\
&\leq 2M\bar{r}_s\bar{r}_f,
\end{aligned}$$

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $\bar{r}_f = c_o\theta\bar{r}_{in,o}$ and $\bar{r}_s = 3c_o^2\theta\bar{r}_{in,o}$, leads to

$$\frac{|T_7|}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} \leq \frac{6c_o^3M\theta^2\bar{r}_{in,o}^2}{nc_o^2M^{1+\frac{2}{n}}\theta^2\nu_{in,o}^{\frac{2}{n}}} = \frac{2c_o\bar{r}_{in,0}^2}{3nc_o^4\nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (3.108) holds.} \tag{C.73}$$

Relations (C.64)-(C.73) imply that

$$\lim_{(3.108)}\frac{T_{3,1}}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}+1}} = G(\Lambda_s). \tag{C.74}$$

Let us evaluate now $\frac{T_{3,2}}{\nu_{in}}$.

$$\begin{aligned}
\frac{T_{3,2}}{\nu_{in}} &= \sum_{\lambda\in\mathcal{U}} \|\lambda - c_o c_1 \tilde{u}_f\|^2 \\
&= \sum_{\lambda\in\mathcal{U}} \Big(\|(1-c_o c_1)\tilde{u}_f\|^2 + \|\tilde{\lambda}_f\|^2 + \|\tau\|^2 \Big) \\
&+ \sum_{\lambda\in\mathcal{U}} \Big(2\langle\tilde{\lambda}_f,\tau\rangle + 2(1-c_o c_1)\langle\tilde{\lambda}_f,\tilde{u}_f\rangle + (1-c_o c_1)\langle\tau,\tilde{u}_f\rangle \Big) \\
&\overset{(a)}{=} \underbrace{\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} (1-3c_o^2)^2\|\tilde{u}_f\|^2}_{T_8} + \underbrace{\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tilde{\lambda}_f\|^2}_{T_9} \\
&+ \underbrace{\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tau\|^2}_{T_{10}} + \underbrace{2(1-3c_o^2)\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \langle\tilde{u}_f,\tilde{\lambda}_f\rangle}_{T_{11}} \\
&+ \underbrace{\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \langle\tilde{\lambda}_f,\tau\rangle}_{T_{12}} + \underbrace{2(1-3c_o^2)\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \langle\tilde{u}_f,\tau\rangle}_{T_{13}} \\
&= T_8 + T_9 + T_{10} + T_{11} + T_{12} + T_{13}, \qquad \text{(C.75)}
\end{aligned}$$

where (a) is based on the fact that $c_o = \frac{c_1}{3}$. Relation (C.75) leads to

$$\begin{aligned}
\frac{T_{3,2}}{nc_o^2(M\nu_{in})^{1+\frac{2}{n}}} &= \frac{T_8}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_9}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_{10}}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} \\
&+ \frac{T_{11}}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_{12}}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_{13}}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}}. \qquad \text{(C.76)}
\end{aligned}$$

Consider now the first term on the right hand side of (C.76)

$$\begin{aligned} T_8 &= (1-3c_o^2)^2 \sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f \in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tilde{u}_f\|^2 \\ &= 3^n c_o^n (1-3c_o^2)^2 \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tilde{u}_f\|^2. \end{aligned} \tag{C.77}$$

In order to evaluate (C.77), we use again Lemma 9, and obtain that

$$\begin{aligned} \frac{T_8}{nc_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &= \frac{n3^n c_o^{2n}(1-3c_o^2)^2}{nc_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_f:\Lambda_{in}}) \nu_f^{\frac{2}{n}} - G(\Lambda_{in}) \nu_{in}^{\frac{2}{n}} \right) \\ &= \frac{(1-6c_o^2+9c_o^4)}{9c_o^6} \left(c_o^2 G(\mathcal{C}_{\Lambda_f:\Lambda_{in}}) - G(\Lambda_{in}) \right), \end{aligned}$$

where the last equality relies on the fact that $M = (3c_o^2)^n$, while $\nu_f = c_o^n \nu_{in}$. Further, we obtain that

$$\begin{aligned} \lim_{(3.108)} \frac{T_8}{nc_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &= \lim_{(3.108)} \frac{(1-6c_o^2+9c_o^4)}{9c_o^6} \left(c_o^2 G(\mathcal{C}_{\Lambda_f:\Lambda_{in}}) - c_o^2 G(\Lambda_{in}) \right) \\ &\overset{(a)}{=} G(\Lambda_f) = G(\Lambda_s), \end{aligned} \tag{C.78}$$

where (a) follows from Lemma 10 and the last equality follows from the fact that the normalized second moment is invariant to scaling. We will show that the last five terms on the right hand side of (C.76) approach 0 in the limit of (3.108). Consider now the second term on the right hand side of (C.76). The following holds

$$\begin{aligned} T_9 &= \sum_{\tau \in \mathcal{T}} \sum_{\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f} \sum_{\tilde{u}_f \in V_f(0) \cap \Lambda_{in}} \|\tilde{\lambda}_f\|^2 \\ &= 3^n c_o^n \sum_{\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f} \|\tilde{\lambda}_f\|^2. \end{aligned} \tag{C.79}$$

In order to evaluate (C.79) we use Lemma 9 and obtain that

$$\begin{aligned} \frac{T_9}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &= \frac{n 3^n c_o^{2n}}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_{s/3}:\Lambda_f}) \nu_{s/3}^{\frac{2}{n}} - G(\Lambda_f) \nu_f^{\frac{2}{n}} \right) \\ &= \frac{1}{9 c_o^6} \left(c_o^4 G(\mathcal{C}_{\Lambda_{s/3}:\Lambda_f}) - c_o^2 G(\Lambda_f) \right), \end{aligned}$$

where the last equality relies on the fact that $M = (3c_o^2)^n$, while $\nu_f = c_o^n \nu_{in}$ and $\nu_{s/3} = c_o^{2n} \nu_{in}$ Further, we obtain that

$$\lim_{(3.108)} \frac{T_9}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} = \lim_{(3.108)} \frac{1}{9 c_o^6} \left(G(\mathcal{C}_{\Lambda_{s/3}:\Lambda_f}) c_o^4 - c_o^2 G(\Lambda_f) \right) = 0, \tag{C.80}$$

where the last equality follows from Lemma 10.

We will show that the third term on the right hand side of (C.76) approaches 0 in the limt of (3.108). Using Lemma 9, we obtain

$$\begin{aligned} \frac{T_{10}}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &= \frac{Mn}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}) \nu_s^{\frac{2}{n}} - G(\Lambda_{s/3}) \nu_{s/3}^{\frac{2}{n}} \right) \\ &= \frac{1}{9 c_o^2} \left(9 G(\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}) - G(\Lambda_{s/3}) \right), \end{aligned}$$

where the last equality is based on $\nu_s = M \nu_{in}$ and $\nu_{s/3} = M \nu_{in}/3^n$. As the parameters

c_o and θ vary, both lattices Λ_s and $\Lambda_{s/3}$ are scaled by the same factor, therefore the set $\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}$ is scaled by that factor. Since the second moment is invariant under scaling it follows that $9G(\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}) - G(\Lambda_{s/3})$ remains constant as the parameters θ and c vary. Consequently, we have that

$$\lim_{(3.108)} \frac{T_{10}}{nc_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} = 0. \tag{C.81}$$

We will consider the fourth term in (C.76).

$$\begin{aligned} |T_{11}| &\leq 2|1-3c_o^2| \sum_{\tau \in \mathcal{T}} \sum_{\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f} \sum_{\tilde{u}_f \in V_f(0) \cap \Lambda_{in}} |\langle \tilde{u}_f, \tilde{\lambda}_f \rangle| \\ &\overset{(a)}{\leq} 2|1-3c_o^2| \sum_{\tau \in \mathcal{T}} \sum_{\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f} \sum_{\tilde{u}_f \in V_f(0) \cap \Lambda_{in}} \|\tilde{u}_f\| \|\tilde{\lambda}_f\| \\ &\leq 2|1-3c_o^2| M \max_{\tilde{u}_f \in V_f(0) \cap \Lambda_{in}} \|\tilde{u}_f\| \max_{\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f} \|\tilde{\lambda}_f\| \\ &\leq 2|1-3c_o^2| M \bar{r}_f \bar{r}_{s/3}, \end{aligned}$$

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $\bar{r}_f = c_o\theta\bar{r}_{in,o}$ and $\bar{r}_{s/3} = c_o^2\theta\bar{r}_{in,o}$. leads to

$$\frac{|T_{11}|}{nc_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} \leq \frac{2|1-3c_o^2| c_o^3 M \theta^2 \bar{r}_{in,o}^2}{nc_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} \tag{C.82}$$

$$= \frac{2|c_o - 3c_o^3| \bar{r}_{in,0}^2}{n 9 c_o^4 \nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (3.108) holds.} \tag{C.83}$$

Consider now the fifth term on the right hand side of (C.76). The following holds

$$\begin{aligned} |T_{12}| &\leq \sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} |\langle\tau,\tilde{\lambda}_f\rangle| \\ &\overset{(a)}{\leq} \sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tau\|\|\tilde{\lambda}_f\| \\ &\leq M\max_{\tau\in\mathcal{T}}\|\tau\| \max_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\|\tilde{\lambda}_f\| \\ &\leq M\bar{r}_s\bar{r}_{s/3}, \end{aligned}$$

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $\bar{r}_{s/3} = c_o^2\theta\bar{r}_{in,o}$ $\bar{r}_s = 3c_o^2\theta\bar{r}_{in,o}$, leads to

$$\frac{|T_{12}|}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} \leq \frac{3c_o^4M\theta^2\bar{r}_{in,o}^2}{nc_o^2M^{1+\frac{2}{n}}\theta^2\nu_{in,o}^{\frac{2}{n}}} \tag{C.84}$$

$$= \frac{\bar{r}_{in,0}^2}{n3c_o^2\nu_{in,0}^{\frac{2}{n}}} \rightarrow 0 \text{ as (3.108) holds.} \tag{C.85}$$

We will evaluate the last term in (C.76).

$$\begin{aligned} |T_{13}| &\leq 2|1-3c_o^2|\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} |\langle\tau,\tilde{u}_f\rangle| \\ &\overset{(a)}{\leq} 2|1-3c_o^2|\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tau\|\|\tilde{u}_f\| \\ &\leq 2|1-3c_o^2|M\max_{\tau\in\mathcal{T}}\|\tau\| \max_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\|\tilde{u}_f\| \\ &\leq 2|1-3c_o^2|M\bar{r}_s\bar{r}_f, \end{aligned}$$

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $\bar{r}_f = c_o\theta\bar{r}_{in,o}$ $\bar{r}_s = 3c_o^2\theta\bar{r}_{in,o}$, leads to

$$\frac{|T_{13}|}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} \leq \frac{6|1-3c_o^2|c_o^3 M\theta^2\bar{r}_{in,o}^2}{nc_o^2 M^{1+\frac{2}{n}}\theta^2\nu_{in,o}^{\frac{2}{n}}}0 \quad \text{(C.86)}$$

$$= \frac{2|1-3c_o^2|c_o\bar{r}_{in,0}^2}{3nc_o^4\nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (3.108) holds.} \quad \text{(C.87)}$$

Relations (C.76)-(C.86) imply that

$$\lim_{(3.108)} \frac{T_{3,2}}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}+1}} = G(\Lambda_s). \quad \text{(C.88)}$$

Let us evaluate now $\frac{T_{3,3}}{\nu_{in}}$.

$$\begin{aligned}
\frac{T_{3,3}}{\nu_{in}} &= \sum_{\lambda\in\mathcal{U}} \|\lambda - 3\tau + c_1\tilde{\lambda}_f + c_1 c_o \tilde{u}_f\|^2 \\
&\overset{(a)}{=} \sum_{\lambda\in\mathcal{U}} \Big(\|(1+c_1)(\tilde{\lambda_f})\|^2 + \|(1+c_oc_1)\tilde{u}_f\|^2 + \|2\tau\|^2 \Big) \\
&+ \sum_{\lambda\in\mathcal{U}} \Big(-4(1+c_1)\langle\tilde{\lambda}_f, \tau)\rangle + 2(1+c_1)(1+c_oc_1)\langle\tilde{\lambda}_f, \tilde{u}_f)\rangle - 4(1+c_oc_1)\langle\tau, \tilde{u}_f)\rangle \Big) \\
&\overset{(b)}{=} \underbrace{\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} (1+3c_o)^2\|\tilde{\lambda}_f\|^2}_{T_{14}} \\
&+ \underbrace{\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} (1+3c_o^2)\|\tilde{u}_f\ \|^2}_{T_{15}} \\
&+ \underbrace{4\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tau\ \|^2}_{T_{16}} \\
&+ \underbrace{2(1+3c_o)(1+3c_o^2)\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \langle\tilde{u}_f, \tilde{\lambda}_f\rangle}_{T_{17}} \\
&- \underbrace{4(1+3c_o)\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \langle\tilde{\lambda}_f, \tau\rangle}_{T_{18}} \\
&- \underbrace{4(1+3c_o^2)\sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \langle\tilde{u}_f, \tau\rangle}_{T_{19}} \\
&= T_{14} + T_{15} + T_{16} + T_{17} + T_{18} + T_{19}, \qquad \text{(C.89)}
\end{aligned}$$

where (a) is using the fact that $\lambda = \tau + \tilde{\lambda}_f + \tilde{u}_f$ and (b) is based on the fact that

$c_o = \frac{c_1}{3}$. Relation (C.89) leads to

$$\begin{aligned} \frac{T_{3,3}}{nc_o^2(M\nu_{in})^{1+\frac{2}{n}}} &= \frac{T_{14}}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_{15}}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_{16}}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} \\ &+ \frac{T_{17}}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_{18}}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} + \frac{T_{19}}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}}. \end{aligned} \tag{C.90}$$

Consider now the first term on the right hand side of (C.90)

$$\begin{aligned} T_{14} &= (1+3c_o)^2 \sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f \in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f \in V_f(0)\cap\Lambda_{in}} \|\tilde{\lambda}_f\|^2 \\ &= 3^n c_o^n (1+3c_o)^2 \sum_{\tilde{\lambda}_f \in V_{s/3}(0)\cap\Lambda_f} \|\tilde{\lambda}_f\|^2. \end{aligned} \tag{C.91}$$

In order to evaluate (C.91), we use Lemma 9, and we obtain

$$\begin{aligned} \frac{T_{14}}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} &= \frac{n3^n c_o^{2n}(1+3c_o)^2}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_{s/3}:\Lambda_f})\nu_{s/3}^{\frac{2}{n}} - G(\Lambda_f)\nu_f^{\frac{2}{n}} \right) \\ &= \frac{(1+6c_o+9c_o^2)}{9c_o^6} \left(c_o^4 G(\mathcal{C}_{\Lambda_{s/3}:\Lambda_f}) - c_o^2 G(\Lambda_f) \right), \end{aligned}$$

where the last equality relies on the fact that $M = (3c_o^2)^n$, while $\nu_f = c_o^n \nu_{in}$ and $\nu_{s/3} = c_o^{2n}\nu_{in}$. Further, we obtain that

$$\begin{aligned} \lim_{(3.108)} \frac{T_{14}}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} &= \lim_{(3.108)} \frac{(1+6c_o+9c_o^2)}{9c_o^6} \left(c_o^4 G(\mathcal{C}_{\Lambda_{s/3}:\Lambda_f}) - c_o^2 G(\Lambda_f) \right) \\ &\overset{(a)}{=} G(\Lambda_{s/3}) = G(\Lambda_s), \end{aligned} \tag{C.92}$$

where (a) follows from Lemma 10 and the last equality based on the fact that the normalized second moment is invariant to scaling.

Consider now the first term on the right hand sie of (C.90)

$$\begin{aligned} T_{15} &= (1+3c_o^2)^2 \sum_{\tau\in\mathcal{T}} \sum_{\tilde{\lambda}_f \in V_{s/3}(0)\cap\Lambda_f} \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tilde{u}_f\ \|^2 \\ &= 3^n c_o^n (1+3c_o^2)^2 \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}} \|\tilde{u}_f\ \|^2. \end{aligned} \tag{C.93}$$

In order to evaluate (C.93), we use again Lemma 9, and obtain that

$$\begin{aligned} \frac{T_{15}}{nc_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &= \frac{n3^n c_o^{2n}(1+3c_o^2)^2}{nc_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} \left(G(\mathcal{C}_{\Lambda_f:\Lambda_{in}})\nu_f^{\frac{2}{n}} - G(\Lambda_{in})\nu_{in}^{\frac{2}{n}} \right) \\ &= \frac{(1+6c_o^2+9c_o^4)}{9c_o^6} \left(c_o^2 G(\mathcal{C}_{\Lambda_f:\Lambda_{in}}) - G(\Lambda_{in}) \right), \end{aligned}$$

where the last equality relies on the fact that $M = (3c_o^2)^n$, while $\nu_f = c_o^n \nu_{in}$. Further, we obtain that

$$\begin{aligned} \lim_{(3.108)} \frac{T_{15}}{nc_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &= \lim_{(3.108)} \frac{(1+6c_o^2+9c_o^4)}{9c_o^6} \left(c_o^2 G(\mathcal{C}_{\Lambda_f:\Lambda_{in}}) - c_o^2 G(\Lambda_{in}) \right) \\ &\overset{(a)}{=} G(\Lambda_f) = G(\Lambda_s). \end{aligned} \tag{C.94}$$

where (a) follows from Lemma 10, which is proved in Appendix B, and the last equality follows from the fact that the normalized second moment is invariant to scaling.

We will show that the last four terms on the right hand side of (C.90) approach 0 in the limit of (3.108).

Consider the third term on the right hand side of (C.90). Using Lemma 9, we obtain that

$$\begin{aligned}\frac{T_{16}}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} &= \frac{4Mn}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}}\left(G(\mathcal{C}_{\Lambda_s:\Lambda_{s/3}})\nu_s^{\frac{2}{n}} - G(\Lambda_{s/3})\nu_{s/3}^{\frac{2}{n}}\right)\\ &= \frac{4}{9c_o^2}\left(9G(\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}) - G(\Lambda_{s/3})\right),\end{aligned}$$

where the last equality is based on $\nu_s = M\nu_{in}$ and $\nu_{s/s} = M\nu_{in}/3^n$. As the parameters c, and θ vary, both lattices Λ_s and $\Lambda_{s/3}$ are scaled by the same factor, therefore the set $\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}$ is scaled by that factor. Since the second moment is invariant under scaling it follows that $9G(\mathcal{C}_{\Lambda_s:\Lambda_{s/3}}) - G(\Lambda_{s/3})$ remains constant as the parameters θ and c vary. Consequently, we have that

$$\lim_{(3.108)} \frac{T_{16}}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} = 0. \tag{C.95}$$

We will evaluate the fourth term in (C.90).

$$\begin{aligned}|T_{17}| &= |2(1+c_o)(1+3c_o^2)\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\langle\tilde{u}_f,\tilde{\lambda}_f\rangle|\\ &= |2(1+c_o)(1+3c_o^2)\sum_{\tau\in\mathcal{T}}\langle\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\tilde{u}_f, \sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\tilde{\lambda}_f\rangle|.\end{aligned}$$

In order to proced we need to intruduce more notation. Let $(V_f(0)\cap\Lambda_{in})_b$ denote the set of points which are in $V_f(0)\cap\Lambda_{in}$ and on the boundary of $V_f(0)$ and let $(V_{s/3}(0)\cap\Lambda_f)_b$ denote the set of points which are in $V_{s/3}(0)\cap\Lambda_f$ and on the boundary of $V_{s/3}(0)$.

Moreover, let $M_c = |(V_f(0) \cap \Lambda_{in})_b|$ and $N_c = |(V_{s/3}(0) \cap \Lambda_f)_b|$. Note that $V_f(0) \cap \Lambda_{in} \setminus (V_f(0) \cap \Lambda_{in})_b$ is symmetric about the origin. Thus $\sum_{\tilde{u}_f \in V_f(0) \cap \Lambda_{in} \setminus (V_f(0) \cap \Lambda_{in})_b} \tilde{u}_f = 0$ likewise $\sum_{\tilde{\lambda}_f \in V_{s/3}(0) \cap \Lambda_f \setminus (V_{s/3}(0) \cap \Lambda_f)_b} \tilde{\lambda}_f = 0$. Then

$$\begin{aligned}
|T_{17}| &= |2(1+3c_o)(1+3c_o^2) \sum_{\tau \in \mathcal{T}} \sum_{\tilde{\lambda}_f \in (V_{s/3}(0)\cap\Lambda_f)_b} \sum_{\tilde{u}_f \in (V_f(0)\cap\Lambda_{in})_b} \langle \tilde{u}_f, \tilde{\lambda}_f \rangle| \\
&\leq 2(1+3c_o)(1+3c_o^2) \sum_{\tau \in \mathcal{T}} \sum_{\tilde{\lambda}_f \in (V_{s/3}(0)\cap\Lambda_f)_b} \sum_{\tilde{u}_f \in (V_f(0)\cap\Lambda_{in})_b} |\langle \tilde{u}_f, \tilde{\lambda}_f \rangle| \\
&\overset{(a)}{\leq} 2(1+3c_o^2)(1+3c_o) \sum_{\tau \in \mathcal{T}} \sum_{\tilde{\lambda}_f \in (V_{s/3}(0)\cap\Lambda_f)_b} \sum_{\tilde{u}_f \in (V_f(0)\cap\Lambda_{in})_b} \|\tilde{u}_f\| \|\tilde{\lambda}_f\| \\
&\leq 2(1+3c_o^2)(1+3c_o) 3^n N_c M_c \max_{\tilde{u}_f \in (V_f(0)\cap\Lambda_{in})_b} \|\tilde{u}_f\| \max_{\tilde{\lambda}_f \in (V_{s/3}(0)\cap\Lambda_f)_b} \|\tilde{\lambda}_f\| \\
&\leq 2(1+3c_o^2)(1+3c_o) 3^n N_c M_c \bar{r}_f \bar{r}_{s/3},
\end{aligned}$$

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $\bar{r}_f = c_o \theta \bar{r}_{in,o}$ and $\bar{r}_{s/3} = c_o^2 \theta \bar{r}_{in,o}$ leads to

$$\begin{aligned}
\frac{|T_{17}|}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} &\leq \frac{2(1+3c_o)(1+3c_o^2) c_o^3 3^n N_c M_c \theta^2 \bar{r}_{in,o}^2}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} \\
&\overset{(a)}{=} \left(\frac{2(1+3c_o)(1+3c_o^2) c_o^3 \bar{r}_{in,0}^2}{n 9 c_o^6 \nu_{in,0}^{\frac{2}{n}}} \right) \left(\frac{N_c \nu_f}{\frac{3^n c_o^n \nu_f}{3^n}} \right) \left(\frac{M_c \nu_{in}}{c_o^n \nu_{in}} \right) \\
&\overset{(b)}{=} \left(\frac{2(1+3c_o^2+3c_o+9c_o^3) c_o^3 \bar{r}_{in,0}^2}{n 9 c_o^6 \nu_{in,0}^{\frac{2}{n}}} \right) \left(\frac{N_c \nu_f}{\nu_{s/3}} \right) \left(\frac{M_c \nu_{in}}{\nu_f} \right) \quad \text{(C.96)}
\end{aligned}$$

where (a) based on the fact that $M = (3c_o^2)^n$, and (b) is based on the fact $\nu_{s/3} = 3^n c_o^n \nu_f / 3^n$ and $\nu_f = c_o^n \nu_{in}$.

Now we will show that $\left(\frac{N_c \nu_f}{\nu_{s/3}} \right) \to 0$ as (3.108) holds. Note that $N_c \nu_f \leq (\phi_1^n - \phi_2^n) \nu_{s/3}$, where $\phi_1 = 1 + \frac{\bar{r}_f}{\bar{r}_{s/3}}$ and $\phi_2 = 1 - \frac{\bar{r}_f}{\bar{r}_{s/3}}$. Then,

$$\begin{aligned} \left(\frac{N_c \nu_f}{\nu_{s/3}}\right) &\leq \left(\frac{(\phi_1^n - \phi_2^n)\nu_{s/3}}{\nu_{s/3}}\right) \\ &= \phi_1^n - \phi_2^n. \end{aligned} \tag{C.97}$$

Now let us show that $\lim_{(3.108)}(\phi_1^n - \phi_2^n) = 0$. Note that

$$(\phi_1^n - \phi_2^n) = (\phi_1 - \phi_2)(\phi_1^{n-1} + \phi_1^{n-2}\phi_2 + \phi_1^{n-3}\phi_2^2 + ... + \phi_2^{n-1}) \leq (\phi_1 - \phi_2)n\phi_1^{n-1}.$$

Since $\phi_1^{n-1} \to 1$ as (3.108) holds, it is sufficient to show that $(\phi_1 - \phi_2) \to 0$ as (3.108) holds. We have

$$(\phi_1 - \phi_2) = \left(1 + \frac{\bar{r}_f}{\bar{r}_{s/3}} - 1 + \frac{\bar{r}_f}{\bar{r}_{s/3}}\right) = \left(\frac{2\theta c_o \bar{r}_{in,o}}{c_o^2 \theta \bar{r}_{in,o}}\right) = \left(\frac{2}{c_o}\right) \to 0 \tag{C.98}$$

as (3.108) holds. Similarly, it can be shown that

$$\lim_{(3.108)} \frac{M_c \nu_{in}}{\nu_f} = 0. \tag{C.99}$$

From relations (C.96)-(C.99), we obtain that

$$\lim_{(3.108)} \frac{|T_{17}|}{n c_o^2 M^{1+\frac{2}{n}} \nu_{in}^{\frac{2}{n}}} = 0. \tag{C.100}$$

Consider now the fifth term on the right hand side of (C.90). The following holds

$$\begin{aligned}|T_{18}| &\leq 4(1+3c_o)\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}|\langle\tau,\tilde{\lambda}_f\rangle|\\ &\overset{(a)}{\leq} 4|1+3c_o|\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\|\tau\|\|\tilde{\lambda}_f\|\\ &\leq 4|1+3c_o|M\max_{\tau\in\mathcal{T}}\|\tau\|\max_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\|\tilde{\lambda}_f\|\\ &\leq 4|1+3c_o|M\bar{r}_{s/3}\bar{r}_s,\end{aligned}$$

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $\bar{r}_{s/3} = c_o^2\theta\bar{r}_{in,o}$ and $\bar{r}_s = 3c_o^2\theta\bar{r}_{in,o}$, leads to

$$\begin{aligned}\frac{|T_{18}|}{nc_o^2M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} &\leq \frac{12|1+3c_o|c_o^4M\theta^2\bar{r}_{in,o}^2}{nc_o^2M^{1+\frac{2}{n}}\theta^2\nu_{in,o}^{\frac{2}{n}}}\\ &= \frac{4|c_o^2+3c_o^3|\bar{r}_{in,0}^2}{3nc_o^4\nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (3.108) holds.}\end{aligned}\qquad\text{(C.101)}$$

We will evaluate the last term in (C.90).

$$\begin{aligned}|T_{19}| &\leq 4(1+3c_o^2)\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_in}|\langle\tau,\tilde{u}_f\rangle|\\ &\overset{(a)}{\leq} 4(1+3c_o^2)\sum_{\tau\in\mathcal{T}}\sum_{\tilde{\lambda}_f\in V_{s/3}(0)\cap\Lambda_f}\sum_{\tilde{u}_f\in V_f(0)\cap\Lambda_in}\|\tau\|\|\tilde{u}_f\|\\ &\leq 4(1+3c_o^2)M\max_{\tau\in\mathcal{T}}\|\tau\|\max_{\tilde{u}_f\in V_f(0)\cap\Lambda_{in}}\|\tilde{u}_f\| \leq 4(1+3c_o^2)M\bar{r}_f\bar{r}_s,\end{aligned}$$

(C.103)

where (a) is based on the Cauchy-Schwartz inequality. Using further the fact that $\bar{r}_f = c_o\theta\bar{r}_{in,o}$, $\bar{r}_s = 3c_o^2\theta\bar{r}_{in,o}$ leads to

$$\begin{aligned} \frac{|T_{19}|}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{\frac{2}{n}}} &\leq \frac{12(1+3c_o^2)c_o^3 M\theta^2\bar{r}_{in,o}^2}{nc_o^2 M^{1+\frac{2}{n}}\theta^2\nu_{in,o}^{\frac{2}{n}}} \\ &= \frac{(1+3c_o^2)c_o\bar{r}_{in,0}^2}{3nc_o^4\nu_{in,0}^{\frac{2}{n}}} \to 0 \text{ as (3.108) holds.} \end{aligned} \tag{C.104}$$

Relations (C.90)-(C.104) imply that

$$\lim_{(3.108)} \frac{T_{3,3}}{nc_o^2 M^{1+\frac{2}{n}}\nu_{in}^{1+\frac{2}{n}}} = 2G(\Lambda_s). \tag{C.105}$$

Using (C.60), (C.61), (C.1), (C.74), (C.105), and (C.74), we obtain that

$$\lim_{(3.108)} \frac{1}{3c_o^2}\sum_{i=1}^{3} G(\mathcal{A}_i(\mathbf{0})) = \frac{4}{3}G(\Lambda_s). \tag{C.106}$$

Corroborating further with (C.48), relation (3.113) follows.

Appendix D

Appendix

D.1 Proofs of Lemmas in Chapter 3

Proof of Lemma 5

According to equation (3.95) for $i = 1, 2, 3$, and $c_o \geq 5$ we have

$$\bar{r}(\beta_i^{-1}(\mathbf{0})) \leq 3c_o\bar{r}_s. \tag{D.107}$$

Using the fact that $\lambda_{c,i} = \lambda_i + u_i$ and the triangle inequality we obtain that

$$\begin{aligned}
\|x_i^n - \hat{x}_{c,i}^n\| &= \|x_i^n - \lambda_{c,i} + u_i + \lambda_i - \hat{x}_{c,i}^n\| \\
&\leq \|x_{c,i}^n - \lambda_{c,i}\| + \|u_i\| + \|\lambda_i - \hat{x}_{c,i}^n\| \\
&\leq \bar{r}_c + \bar{r}_{in} + \|\lambda_i - \hat{x}_{c,i}^n\| \leq \bar{r}_s + \|\lambda_i - \hat{x}_{c,i}^n\|.
\end{aligned} \tag{D.108}$$

If condition (3.70) is violated then $\hat{x}^n_{c,i} = \lambda_{s,i}$. Thus, we have

$$\|\lambda_i - \hat{x}^n_{c,i}\| = \|\lambda_i - \lambda_{s,i}\| \leq \bar{r}(\beta_i^{-1}(\mathbf{0})) \leq 3c_o\bar{r}_s, \tag{D.109}$$

for $c_o \geq 5$ relations (D.108) and (D.109) imply that

$$\|x^n_i - \hat{x}^n_{c,i}\| \leq (3c_o + 1)\bar{r}_s \leq 4c_o\bar{r}_s \tag{D.110}$$

Let us assume now that condition (3.70) is satisfied and that Case 1) holds at the decoder, i.e., $a_1 = 0$ or $a_2 = 0$ or $a_3 = 0$. Then

$$\|\lambda_i - \hat{x}^n_{c,i}\| \leq \|\lambda_i - \tilde{\lambda}_i\| + \|u_i\| \leq \|\lambda_i - \tilde{\lambda}_i\| + \bar{r}_{in}. \tag{D.111}$$

Let us consider now $i = 1$. Using (3.80) and the triangle inequality we obtain that

$$\begin{aligned}\|\lambda_1 - \tilde{\lambda}_1\| &\leq \|\lambda_1 - \tilde{\lambda}_s\| + \|\tilde{\tau}\| + \|\bar{\tilde{\lambda}}_f\| + \|\bar{\tilde{u}}_{f,1}\| \\ &\overset{(a)}{\leq} \|\lambda_1 - \lambda_{s,1}\| + \|\lambda_{s,1} - \tilde{\lambda}_s\| + \bar{r}_s + \frac{1}{3c_o}\|\lambda_{s,1} - \tilde{\lambda}_s\| + \|\frac{1}{3c_o^2}(\lambda_{s,2} - \tilde{\lambda}_s) - \tilde{\lambda}_a\| \\ &\leq 3c_o\bar{r}_s + \bar{r}_s + \left(1 + \frac{1}{3c_o}\right)\|\lambda_{s,1} - \tilde{\lambda}_s\| + \frac{1}{3c_o^2}\|\lambda_{s,2} - \tilde{\lambda}_s\| + \|\tilde{\lambda}_a\|, \end{aligned} \tag{D.112}$$

where (a) is based on using (3.76), (3.77) and (3.78). The last inequality is based on $\|\lambda_1 - \lambda_{s,1}\| \leq \bar{r}(\beta_i^{-1}(\mathbf{0})) \leq 3c_o\bar{r}_s$ and on Lemma 4.
Using the fact that $\tilde{\lambda}_s = \tilde{\lambda}_{s/3} - \tilde{\tau}$ from (3.74), (3.75) and using the triangle inequality

leads to

$$\begin{aligned}\|\lambda_{s,1} - \tilde{\lambda}_s\| &\leq \|\lambda_{s,1} - \tilde{\lambda}_{s/3}\| + \|\tilde{\tau}\| \\ &\leq \|\lambda_{s,1} - \tilde{\lambda}_{s/3}\| + \bar{r}_s. \qquad \text{(D.113)}\end{aligned}$$

Using (3.74) in conjunction with the triangle inequality leads to

$$\begin{aligned}\|\lambda_{s,1} - \tilde{\lambda}_{s/3}\| &\leq \|\lambda_{s,1} - \frac{1}{3}(\lambda_{s,1} + \lambda_{s,2} + \lambda_{s,3} + 3c_o^2\tilde{\lambda}_c)\| \\ &\leq \frac{1}{3}\|\lambda_{s,1} - \lambda_{s,2}\| + \frac{1}{3}\|\lambda_{s,1} - \lambda_{s,3}\| + c_o^2\|\tilde{\lambda}_c\| \\ &\leq \frac{2}{3}\left((10 + 4c_o)\bar{r}_s + 3\bar{r}_{in}\right) + c_o^2\|\tilde{\lambda}_c\|, \qquad \text{(D.114)}\end{aligned}$$

Where the last inequality follows from (3.70). Now we will derive an upper bound for $\|\tilde{\lambda}_a\|$ and $\|\tilde{\lambda}_c\|$. Note that $\tilde{\lambda}_a = Q_{in}(u_1 - u_2) - (u_1 - u_2)\text{mod}Q_{in}$. Then

$$\|\tilde{\lambda}_a\| \leq \|u_1\| + \|u_2\| + \|(u_1 - u_2) \text{ mod } Q_{in}(u_1 - u_2)\| \leq 3\bar{r}_{in}. \qquad \text{(D.115)}$$

Similarly, we obtain

$$\|\tilde{\lambda}_c\| \leq 3\bar{r}_{in}. \qquad \text{(D.116)}$$

Plugging (D.114) and (D.116) in (D.113) and using the fact that $\bar{r}_s = 3c_o^2\bar{r}_{in}$ leads to

$$\|\lambda_{s,1} - \tilde{\lambda}_s\| \leq 2\bar{r}_s + \frac{2}{3}(10 + 4c_o)\bar{r}_s + 2\bar{r}_{in}. \qquad \text{(D.117)}$$

Now we will evaluate the following term

$$\begin{aligned}\|\lambda_{s,2} - \tilde{\lambda}_s\| &\leq \|\lambda_{s,2} - \tilde{\lambda}_{s/3}\| + \|\tilde{\tau}\| \\ &\leq \|\lambda_{s,2} - \tilde{\lambda}_{s/3}\| + \bar{r}_s. \qquad \text{(D.118)}\end{aligned}$$

Using (3.74) in conjunction with the triangle inequality leads to

$$\begin{aligned}\|\lambda_{s,2} - \tilde{\lambda}_{s/3}\| &\leq \|\lambda_{s,2} - \frac{1}{3}(\lambda_{s,1} + \lambda_{s,2} + \lambda_{s,3} + 3c_o^2\tilde{\lambda}_c)\| \\ &\leq \frac{1}{3}\|\lambda_{s,2} - \lambda_{s,1}\| + \frac{1}{3}\|\lambda_{s,2} - \lambda_{s,3}\| + c_o^2\|\tilde{\lambda}_c\| \\ &\leq \frac{2}{3}\left((10 + 4c_o)r_s + 3\bar{r}_{in}\right) + 3c_o^2\bar{r}_{in}, \qquad \text{(D.119)}\end{aligned}$$

where the last equality follows from (3.70) and (D.116). Plugging (D.119) in (D.118) and using the fact that $\bar{r}_s = 3c_o^2\bar{r}_{in}$ leads to

$$\|\lambda_{s,2} - \tilde{\lambda}_s\| \leq 2\bar{r}_s + \frac{2}{3}(10 + 4c_o)\bar{r}_s + 2\bar{r}_{in}. \qquad \text{(D.120)}$$

Plugging (D.117), (D.115) and (D.120) in (D.112) leads to

$$\|\lambda_1 - \tilde{\lambda}_1\| \leq \left(13 + 6c_o + \frac{34}{9c_o} + \frac{26}{9c_o^2}\right)\bar{r}_s + 7\bar{r}_{in}. \qquad \text{(D.121)}$$

Combining now the above inequality with (D.111), (D.108) we obtain that

$$\|x_1^n - \hat{x}_{c,1}^n\| = \left(13 + 6c_o + \frac{34}{9c_o} + \frac{26}{9c_o^2}\right)\bar{r}_s + \bar{r}_s + 8\bar{r}_{in} \leq 7c_o\bar{r}_s, \qquad \text{(D.122)}$$

for c_o sufficiently large. The proof for $i = 2, 3$ and for the remaining cases follows along the same lines.

Proof of Lemma 6

Let us fix i. Denote $\tilde{\mathcal{W}}(\lambda_f) \triangleq \{x_i^n \in \mathbb{R}^n : Q_{in}(Q_c(x_i^n)) \in \mathcal{W}(\lambda_f)\}$ and $\tilde{\mathcal{W}} \triangleq \cup_{\lambda_f \in \Lambda_f} \tilde{\mathcal{W}}(\lambda_f)$. A moment of thought reveals that $\tilde{\mathcal{W}}(\lambda_f) \subset (\lambda_f + \eta_1 V_f(\mathbf{0})) \backslash (\lambda_f + \eta_2 V_f(\mathbf{0}))$, where $\eta_1 = 1 + \frac{\bar{r}_{in}+\bar{r}_c}{r_f}$ and $\eta_2 = \eta - \frac{\bar{r}_{in}+\bar{r}_c}{r_f}$. The above relation implies that

$$\nu(\tilde{\mathcal{W}}(\lambda_f)) \leq (\eta_1^n - \eta_2^n)\nu_f \tag{D.123}$$

Let $\tilde{\mathcal{V}}(\lambda_f) \triangleq \{x_i^n \in \mathbb{R}^n | Q_{in}(Q_c(x_1^n)) \in V_f(\lambda_f)\}$. Clearly, $\nu(\tilde{\mathcal{V}}(\lambda_f)) = \nu_f$. The proof of the lemma hinges on the fact that, as (3.108) holds, the pdf of X_i^n can be approximated by a pdf which is uniform over $\tilde{\mathcal{V}}_f(\lambda_f)$.

Note that the sets $\tilde{\mathcal{V}}(\lambda_f)$ with $\lambda_f \in \Lambda_f$, form a partition of $\mathbb{R}^n$. Define a density function $f_{\theta,c} : \mathbb{R}^n \to [0, \infty)$, which is uniform on each set $\tilde{\mathcal{V}}(\lambda_f)$, as follows

$$f_{\theta,c}(x^n) = \frac{1}{\nu(\tilde{\mathcal{V}}(\lambda_f))} \int_{\tilde{\mathcal{V}}(\lambda_f)} f_{X_i^n}(y^n) dy^n, \tag{D.124}$$

if $x^n \in \tilde{\mathcal{V}}(\lambda_f)$. Then according to Lemma 7, we have that $f_{\theta,c}(x^n) \to f_{X_i^n}(x^n)$ for every $x^n \in \mathbb{R}^n$, as (3.108) holds. Further, we have

$$\begin{aligned}
\mathbb{P}[X_i^n \in \tilde{\mathcal{W}}] &= \int_{\tilde{\mathcal{W}}} \left(f_{X_i^n}(x^n) - f_{\theta,c}(x^n) + f_{\theta,c}(x^n) \right) dx^n \\
&\leq \int_{\tilde{\mathcal{W}}} |f_{X_i^n}(x^n) - f_{\theta,c}(x^n)| dx^n + \sum_{\lambda_f \in \Lambda_f} \int_{\tilde{\mathcal{W}}(\lambda_f)} f_{\theta,c}(x^n) dx^n.
\end{aligned}$$

Note that

$$\int_{\tilde{\mathcal{W}}} |f_{X_i^n}(x^n) - f_{\theta,c}(x^n)| dx^n \leq \int_{\mathbb{R}^n} |f_{X_i^n}(x^n) - f_{\theta,c}(x^n)| dx^n \to 0, \text{ as (3.108) holds,}$$

where the last relation is valid in view of Scheffe's theorem Scheffé (1947). Further, since the density $f_{\theta,c}$ is uniform over each $\tilde{\mathcal{V}}(\lambda_f)$ and $\tilde{\mathcal{W}}(\lambda_f) \subset \tilde{\mathcal{V}}(\lambda_f)$, we obtain that

$$\begin{aligned}
\sum_{\lambda_f \in \Lambda_f} \int_{\tilde{\mathcal{W}}(\lambda_f)} f_{\theta,c}(x^n) dx^n &= \sum_{\lambda_f \in \Lambda_f} f_{\theta,c}(\lambda_f) \nu(\tilde{\mathcal{W}}(\lambda_f)) \\
&\overset{(c)}{\leq} \sum_{\lambda_f \in \Lambda_f} f_{\theta,c}(\lambda_f) \nu_f(\eta_1^n - \eta_2^n) \\
&= (\eta_1^n - \eta_2^n) \sum_{\lambda_f \in \Lambda_f} f_{\theta,c}(\lambda_f) \nu_f \\
&\overset{(d)}{=} (\eta_1^n - \eta_2^n) \sum_{\lambda_f \in \Lambda_f} \int_{\tilde{\mathcal{V}}(\lambda_f)} f_{X_1^n}(y^n) dy^n \\
&= (\eta_1^n - \eta_2^n) \int_{\mathbb{R}^n} f_{X_i^n}(y^n) dy^n = \eta_1^n - \eta_2^n,
\end{aligned}$$

where (c) follows from (D.123) and (d) is based on relation (D.124) and on the fact that $\nu_f = \nu(\tilde{\mathcal{V}}(\lambda_f))$. Finally, it is easy to see that $\eta_1 \to 1$ and $\eta_2 \to 1$ as (3.108) holds, thus $\lim_{(3.108)}(\eta_1^n - \eta_2^n) = 0$. Further, we need to show that $\lim_{(3.108)}(\eta_1^n - \eta_2^n) \log_2 c_o = 0$. Note that

$$(\eta_1^n - \eta_2^n) = (\eta_1 - \eta_2)(\eta_1^{n-1} + \eta_1^{n-2}\eta_2 + \eta_1^{n-3}\eta_2^2 + \ldots\ldots + \eta_2^{n-1}) \leq (\eta_1 - \eta_2) n \eta_1^{n-1}$$

Since $\eta_1^{n-1} \to 1$ as (3.108) holds, it is sufficient to show that$(\eta_1 - \eta_2) \log_2 c_o \to 0$ as (3.108) holds. For this notice that

$$(\eta_1 - \eta_2) \log_2 c_o = \left(\frac{5\bar{r}_{in} + 2\bar{r}_c}{\bar{r}_f} \right) \log_2 c_o = \left(\frac{5\theta\bar{r}_{in,o} + 2\theta\bar{r}_{c,o}}{c_o \theta \bar{r}_{in,o}} \right) \log_2 c_o \to 0 \text{ as (3.108) holds.}$$

This observation concludes the proof.

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