Analysis of On/Off servers with Dynamic Voltage Scaling

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A Thesis Submitted to the School of Graduate Studies in the Partial Fulfillment of the Requirements for the Degree Master of Science

McMaster University

Master of Science (2017)

Hamilton, Ontario (Department of Computing and Software)

TITLE: Analysis of On/Off servers with Dynamic Voltage Scaling

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NUMBER OF PAGES: x, 93

Abstract

With rapid adoption of cloud solutions across industries, energy consumed by server farms continues to rise. There are numerous approaches to reduce energy consumption in data centres, and one of the approaches is to use energy-aware policies, which focus on how servers should be operated in order to achieve energy saving and meet service level agreements (SLA). In this thesis, we focus on studying a single server model with dynamic voltage scaling (DVS), presenting a framework with explicit solutions to solve for performance metrics and energy consumption. Our framework is convenient and intuitive, one can easily identify expected response time and expected energy consumption for a given policy. In addition, we also provide insights on how the value of the faster service rate and the choice of when to use speed scaling impact energy consumption and performance metrics.

I would like to dedicate this work to my mom, WanLan Li, and my two beloved sisters, PeiYi Mo and PeiHong Mo. Without their constant support and encouragement, my time as a part time master student could have been unimaginably difficult. They selflessly took care of all the miscellaneous errands for me, so that I could stay focus on both of my academic and industrial careers, which has made this work possible.

Acknowledgements

This work has only made possible through the help of my supervisor Dr. Douglas G. Down and my colleague Ph.D student Vincent Maccio. Dr Down has always given me time and freedom to dive deep into my research topic, which allowed me to learn a lot from advanced topics of stochastic modelling and queueing theory. Vincent, on the other hand, always pays attention to details, which has prevented me from making lots of mistakes while conducting this work.

This research was funded partially by the Natural Sciences and Engineering Research Council of Canada.

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Declaration of Authorship

I, Guang Mo , declare that this thesis titled, “Analysis of On/Off servers with Dynamic Voltage Scaling” and the work presented in it are my own.

Chapter 1

Introduction

1.1 Introduction

The rapid adoption of cloud solutions has led to significant energy consumption by datacentres across the globe. Fortunately, as much as 40 percent of a typical datacentre's energy consumption can be saved [21] when proper energy management practices are used. The use of guidelines on how to operate servers in datacentres is one of the common practices that datacentre operators use to lower energy consumption without compromising service level agreements. This potential energy saving has motivated researchers to look for optimal policies under various settings. Some focus on how to operate multiple servers in an efficient way, while others focus on studying single-server models. Our research interest lies in studying a single-server model where dynamic voltage scaling is employed. The understanding of the behaviour of a single-server system is a useful step in understanding the behaviour of more complex, multi-server systems. A common practice to achieve energy saving for single-server model is to turn off a server when there are no jobs waiting for service, and turn on a server when there are jobs waiting for service. This provides energy saving but also results in performance degradation. Another common practice to achieve energy saving for single-server models is to use dynamic voltage scaling. A dynamic voltage scaling enabled server operates at different speeds depending on the input voltage. As a result, operating policies may vary operating speed to meet energy consumption and performance requirements. We are interested in combining both practices, and to study and understand the resulting system, which is presented in Chapter 4. There we discuss the system and associated metrics for the model. In Chapter 5, we conduct our analysis for the model we presented in Chapter 4, and determine the steady-state distribution. This allows us to provide a set of explicit solutions for system performance metrics and energy consumption. By solving the system in terms of the control parameters, our solution is not limited to providing an optimal policy for one specific performance metric, and our solution is more adaptable to different system settings, such as allowing the operator to turn the server on/off, only use one processing speed, etc. Our framework also allows users to validate whether the system meets an SLA, and allows one to search for the optimal policy for their preferred performance metrics. After the model has been analyzed, Chapter 6 provides a series of numerical experiments based on the obtained closed form expressions, and we provide a number of observations focusing on the effects of dynamic voltage scaling.

Chapter 2

Preliminary Knowledge

This chapter provides necessary background knowledge for understanding this thesis, such as fundamental tools and concepts from stochastic modeling and queueing theory. This chapter could be skipped if readers are already familiar with Continuous Time Markov Chains and queueing theory.

2.1 Stochastic Processes

A stochastic process (also referred to as a random process) is a mathematical object that is used to describe a system's behavior over time. Mathematically speaking, a stochastic process is a set of random variables $\{X_t | t \in T\}$. The index set T is usually viewed as points in time, which corresponds to the system changing over time. In addition, there are two types of stochastic process when the index set is interpreted as time. If the index set T is finite or countable, the stochastic process is considered to be in discrete time, an example would be X_i may be number of customers in a system at the ith minute (discrete). On the other hand, if the index set in T represents intervals of the real line, then the stochastic process is considered to be in continuous time, an example would be

X_i to represent number of customers in a system for all times on the real line between given times t_0 and t_f.

2.1.1 Markov Processes

There are numerous types of stochastic processes. A Markov process is one kind of stochastic process, satisfying the Markov property. The Markov property means that the future values of the process depend on only the current value, in other words they are conditionally independent of the previous values. This property is also referred as "memoryless", because the next value only depends on the current value - what happened previously is not relevant in determining the next value. Mathematically, the Markov property is presented as:

$$P[X_{t_n} \leq x_n | X_{t_0} = x_0, X_{t_1} = x_1, X_{t_2} = x_2, ..., X_{t_{n-1}} = x_{n-1}]$$
$$= P[X_{t_n} \leq x_n | X_{t_{n-1}} = x_{n-1}]$$

where the values of t_i are in order with t_0 being the smallest, and t_n being the largest. The memoryless property typically allows a simplified analytical procedure for such models, as one only needs to understand the relationship between two consecutive states without requiring the entire history of the system. This property is fundamental for analyzing many queueing systems.

2.1.2 Continuous-Time Markov Chains

A continuous-time Markov chain (CTMC) is a type of Markov process, which takes on values in continuous time with a state space consisting of a countable number of states. In other words, it is represented with random variables $\{X_t | t \in T\}$ which take on discrete values from a countable state space S, and the set T is some continuous time interval.

A CTMC is a powerful analysis tool because one can easily write balance equations for the steady-state distribution of the system with the given CTMC. A CTMC transition rate diagram is shown in Figure 2.1, where each node represents one of the elements from the state space S (a system state), and the arrows pointing in and out from each node represent the "transition rates" between elements (system states).

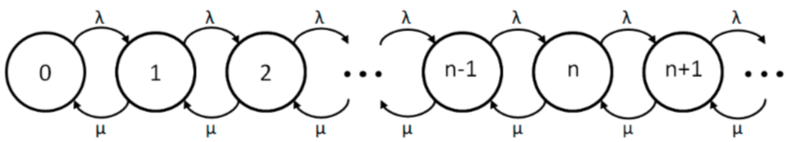


FIGURE 2.1: CTMC of an M/M/1 queue

2.2 Queueing Theory

Queueing theory is the mathematical study of waiting lines or queues. A queueing model is constructed so that one can predict statistics of queue lengths and waiting times based on different characteristics of the system. Queueing models generally describe three key aspects. One aspect focuses on the job arrivals, describing how jobs arrive to the system; one example would be that jobs arrive to the system following a particular random process. The next is the queueing aspect, such as how long the queue length should be. The last aspect concerns how the system processes jobs, which is the counterpart of the arrival aspect, an example being the processing times following a particular distribution. The stochastic nature of the arrival and processing times makes the analysis of queueing systems complicated. This study is further complicated when these systems interconnect with each other. To provide sufficient knowledge to understand this thesis, this chapter presents a simple queueing model along with methods that are typically used to analyse it.

FIGURE 2.2: M/M/1 system [19]

2.2.1 Analyzing an M/M/1 queue

One of the representative queueing models is the M/M/1 queue, shown in Figure 2.2, where M/M/1 is the Kendall's notation that describes the system. The M/M/1 notation describes a system with arrivals following a Poisson process, processing times being exponentially distributed, and one server. The queue is assumed to be infinite in length and processed in first in first out (FIFO) order. The memoryless property of the exponential distribution allows it to be modeled as a CTMC, with the states denoting the number of jobs in the system, which includes jobs that are waiting in the queue and the job that is being processed. The resulting CTMC is shown in Figure 2.1, the number in system increases or decreases by at most one for any given transition (one job more or less in the system), thus the system can also be described as a birth-death process.

The next important part of analyzing an M/M/1 queue is to capture the system behavior in steady-state. The steady-state distribution captures the probability of being in each state n, as time goes to infinity. Given the steady-state distribution, one would be able to compute such quantities as the expected number of jobs in the system. The steady-state distribution is often represented with the quantities π_n, where n represents the state.

In order to determine the steady-state distribution, the relationships among all the states must be understood and representable. To solve a queueing system, balance equations must be derived, which describe the relationship among system states. For a CTMC, it is known that the rate into each state must be equal to the rate of leaving that state, or else the system will not reach steady-state. Based on this observation, let λ, μ be the arrival rate and the processing rate for the system, and $\rho = \frac{\lambda}{\mu}$, the balance equation for states $n > 0$ can be described as $\lambda\pi_{n-1} + \mu\pi_{n+1} = (\lambda + \mu)\pi_n$, which represents that the rate into state n is equal to the rate out of state n. Similarly, the balance equation for the system state 0 is $\lambda\pi_0 = \mu\pi_1$. This gives $\pi_1 = \rho\pi_{0,0}$, and this leads to $\pi_n = \rho^n\pi_0$ by recursion. As a result, all of the steady-state probabilities can be expressed in terms of π_0. As the system must be in on one of its system states at all times, the sum of all of the steady-state probabilities must equal 1. Mathematically, this is expressed as,

$$\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow \pi_0 \sum_{i=0}^{\infty} \rho^i \Rightarrow \pi_0 = 1 - \rho$$

Once π_0 is determined, the remaining probabilities can be expressed as,

$$\pi_n = (1 - \rho)\rho^n$$

The expected number of jobs in the system can be calculated by weighting each π_n by n and summing over all of the states. The expected number of jobs in the system is a key performance indicator. The other important performance indicator is the expected response time, which is the expected time that a job takes from entering to leaving the system. Fortunately, there is an easy way to calculate the expected response time if the expected number of jobs is given. One of the key takeaways from queueing theory is Little's Law, which states the relationship between the expected number of jobs and the expected response time, and this law holds independent of the interarrival time/processing time distribution and queue scheduling policy. Little's Law states:

$$\mathbb{E}[R] = \frac{\mathbb{E}[N]}{\lambda}$$

Thus, the expected response time can be directly calculated via Little's Law given the expected number of jobs. The purpose of this thesis is to quantify the trade off between energy and performance, as a result, the expected energy consumption is another important performance indicator. Energy is consumed at different rates when a server operates in different modes. There are only two modes for the server in this M/M/1 queueing model, the server is either idle (with probability π_0) or operating (with probability $1 - \pi_0$). Assuming that energy cost rates are E_{idle} and E_{busy} for a server in idle and operating modes respectively, the expected energy consumption can be obtained by weighting the steady-state probabilities with the corresponding energy cost as,

$$\mathbb{E}[E] = E_{idle}\pi_0 + E_{busy}\sum_{n=1}^{\infty}\pi_n$$

Since $\pi_0 = 1 - \rho$, this gives the sum of the remaining steady-state probabilities as $1 - \pi_0$, so the expected energy cost for the M/M/1 queueing model is,

$$\mathbb{E}[E] = E_{idle}(1 - \rho) + E_{busy}\rho$$

This completes the example of analyzing an M/M/1 queue - the analytical processes and tools used in this example will be used throughout this thesis. Although this chapter provides background in CTMCs and queueing theory, this only covers the knowledge and examples that are used for this thesis. If readers are interested in looking for deeper and broader knowledge, an independent study in these fields is recommended, see the standard references [14],[15],[17].

Chapter 3

Literature Review

This chapter provides an overview of the rising energy consumption problem in datacentres across the globe, as well as to cover some of the popular methods for energy control in datacentres. Lastly, this chapter examines work that has been done following similar approaches as this thesis.

3.1 Saving Energy in Datacentres

The rapid adoption of cloud computing leads to significant energy consumption by datacentres, and lowering datacentre utility cost is a high priority for every datacentre operational manager. From [2], 50%, 25%, 12%, and 10% of the energy consumed in datacentres is by IT equipment, cooling, air movement, and power distribution, respectively. Many researchers attempt to achieve energy saving by addressing these key energy consumption areas.

One approach to reduce energy consumption is by thermal cooling design and control. The work in [13], [16], [1] are examples of this approach. Datacentre airflow is optimized to avoid hot spots in [13], which are created due to some servers being highly utilized while other servers are under utilized. The work in [13] and [16] also saves energy from cooling by turning up thermostats and with the use of liquid cooling. An algorithm is introduced in [22] that takes both service level agreement (SLA) and heat circulation problems into consideration for scheduling jobs among servers to avoid hotspots. This effectively reduces energy consumption due to cooling.

Another effective energy saving approach is to use virtual machine (VM) consolidation and migration. VM migration allows most applications to be run on a smaller number of servers by increasing the utilization of those servers, allowing servers with low utilization to become idle, and they can then be turned off to realize energy saving and require less cooling. Examples of this approach are [3],[12], they propose similar solutions by implementing a centralized controller, which distributes jobs to local VMs and controls VMs migration on hosting servers. In doing so, the centralized controller can provide efficient VM provisioning over hosting servers, as a result, some servers can be turned off for energy saving.

In the remainder of this chapter, we provide a deeper discussion of the work that focuses on server operating policies for energy saving, which is the topic of this thesis.

3.2 Related Work

There are numerous works on operating policies for single-server models, each exploiting different ways to achieve energy savings while not downgrading performance significantly. A common practice is to turn off a server when there is no job waiting for service, and turn on a server when there are jobs waiting for service. This provides energy saving but also results in performance degradation. There have been a number of other works taking a queueing-theoretic approach to the single-server problem without DVS, see [9],[10],[7],[6],[18], and [7] and [18] are most relevant to the work done in this thesis. The work done by [7] is intended to identify optimality on how to operate a single-server under the energy response time product (ERP) metric, by modelling and analyzing the system as a continuous time Markov chain (CTMC). Using properties of the ERP metric, they identify that the optimal policy is either to always keep the server on, or turn off the server as soon as there are no jobs waiting to be served. Although optimality is identified in [7], the result is limited to the performance metric ERP, which tends to simplify optimality considerations. Work done by [18] follows a similar approach as [7] but with an additional turn on threshold, which intends to provide insights on when the server should be turned on. In addition to solving the underlying model and providing insights on the effect of the turn on threshold, [18] also provides a framework to analyze the system under other cost models, not being limited to ERP. This thesis is highly influenced by [7] and [18], and it can be viewed as a combination of the dynamic voltage scaling aspect into with on/off single-server analysis.

Using dynamic voltage scaling for energy control in datacentres is not new. A dynamic voltage scaling enabled server operates at different speeds depending on the input voltage. As a result, operating policies may vary operating speed to meet energy consumption and performance requirements. The policies in [5],[20],[4] are examples, they focus on multi-server models (except for [20] which is a single server setting) with similar real world datacentre settings, focusing on the overall impact of their proposed policies. They study the relationship between the incoming workload and the adjusted processing speed, where the performance of policies is highly dependent on whether the incoming workload can be predicted accurately. This approach does not give much details as to how a single DVS server should be operated. Others have done work for a single server model with DVS via queueing-theoretic approaches. Wierman et al. [22] focus on a queueing model for a DVS server with processor sharing scheduling. They analyze various scenarios investigating the impact of the operating frequency of the server in relationship to the arrival rates. The work done by [11] is perhaps the most impactful work on varying the service rate for a single queue model, their work focuses on holding costs and operating costs when using different service rates, and they determine solutions for how to optimally control the service rate to achieve minimum incurred cost. This work however could not provide a full picture on how DVS should be operated when it is possible for a server to be turned off, since they assumed the server is always available to serve jobs.

In this chapter, we covered some of the energy saving techniques that are used in datacentres, and we discussed the relevant work for this thesis. We will discuss our approach and how it addresses the problem in Chapter 4.

Chapter 4

Problem Formulation

For a single dynamic voltage scaling server system, which can be operated at different speeds, one can use the system to meet different objectives, such as improving performance by operating at a high speed or lowering energy consumption by operating at a low speed. To analyse the system, the following metrics are of interest: the expected number of jobs in the system, the expected response time of a job in the system, and the expected energy consumption of the system. Typically, a cost function for the system is constructed from these metrics with the objective of minimizing the cost. Deriving a policy that minimizes the constructed cost function is an ultimate goal. However, cost functions are not necessarily universal, and physical system settings could impose hard constraints on the optimal policy. On the contrary, determining the metrics in terms of control parameters provide the tools to understand the system, as well as provide insights on the system behaviors. As a result, the primary objective of this work is to provide a framework to analyse the on/off single server with dynamic voltage scaling and determining the metrics of interest. In addition, an analysis of the system under a particular cost function is conducted to provide insights on how the server should be operated.

4.1 The Model

We present a simplified version of a DVS server, which only contains two different operating speeds, one is the nominal (lower) speed and the other is the scaled speed. This leads to five distinct states for the server. The OFF state represents the server is off; the SETUP state represents the server is being turned on; the BUSY state means the server is operating at nominal speed; the SCALED state represents the server is operating at scaled speed; and lastly, the IDLE state represents the server is idle. We define additional parameters to indicate state transitions for the server. We define a turn on server threshold k_1 to indicate that the server will be transferred from OFF to SETUP as soon as there are k_1 jobs in the system, and the turn on speed scaling threshold k_2 is defined to signal the transition to SCALED. Furthermore, we consider that there are time penalties when turning on the server, and we model this setup time as exponentially distributed with a rate of γ, this transfers the server from SETUP to either BUSY or SCALED. Lastly, we also define the turnoff delay rate α, this parameter captures how long the server waits when there are no jobs, before transferring the server from IDLE to OFF. The system is made of an infinitely long FIFO queue and a DVS server. Jobs arrive to the system following a Poisson process with rate λ. Jobs can only be processed in either the BUSY or SCALED states, the processing time is exponentially distributed with rates μ and $c\mu$ for BUSY and SCALED, respectively, where c is the scaled speed factor. We can model the system as a Continuous Time Markov Chain (CTMC) with the arrival rate, processing rates, and the previously defined parameters. Figure 4.1 and Figure 4.2 capture the possible CTMCs of the system, where the state is represented in the form (n_1,n_2), where n_1 is either 0 or 1 for the server being off and on respectively, and n_2 represents the number of jobs in the system. Figure 4.1 represents a standard way of operating the DVS server, where the nominal speed is used before using the scaled speed, and this allows us to explore the effects on the system when DVS is employed.

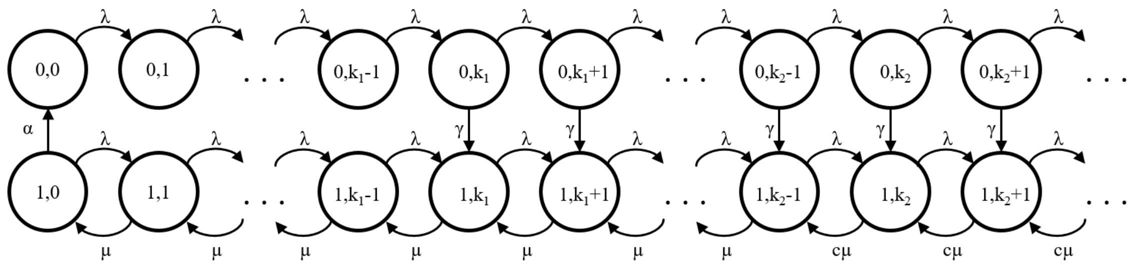


FIGURE 4.1: DVS processor with $k_1 \leq k_2$ CTMC

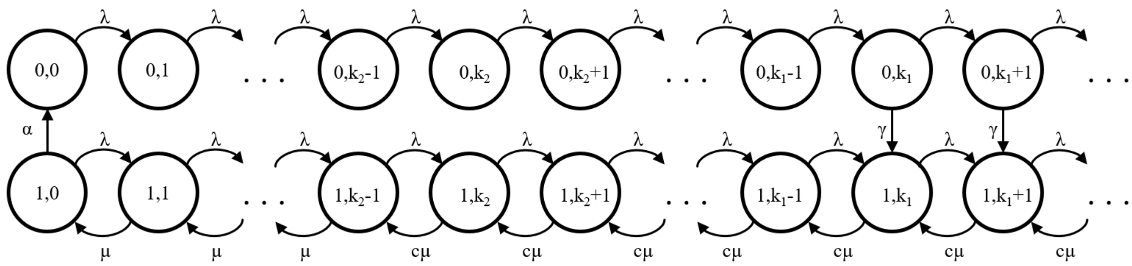


FIGURE 4.2: DVS processor with $k_1 > k_2$ CTMC

Parameter	Description
λ	arrival rate
γ	rate of turning on the server
α	rate that the server waits in IDLE before turning off
c	scaling factor of scaled speed
k_1	number of jobs in the system before turning on the server
k_2	number of jobs in the system before increasing to the scaled processing rate
μ	nominal job processing rate

TABLE 4.1: Table of System Parameters

Figure 4.2 on the other hand represents another possible use case for the DVS server, where the scaled speed is used as soon as the server is on, and the nominal speed is only used when the system is moving towards IDLE. This use case is less interesting in comparison to the use case in Figure 4.1, since it is similar to using a faster server, and this use case does not allow us to study the effect of delaying using DVS. Although both models are valid use cases for the DVS server, our main focus is on the model in Figure 4.1, which allows us to study the effects of both the magnitude of the scaled speed and when to use DVS. For reference, we provide a list of parameters in Table 4.1.

Chapter 5

Analysis

The objective of this analysis is to obtain closed form expressions for system metrics, namely the expected number of jobs, the expected response time, and the expected energy consumption. Once this objective is met, we will make use of these closed form expressions to study the effects of DVS under various system configurations. In order to provide the complete use scenario of the DVS server under our framework, we need to obtain closed form expressions of system metrics for both models defined in Chapter 4. We break down our analysis into two sections, with Section 5.1 for the case $k_1 \leq k_2$ and Section 5.2 for the case $k_2 < k_1$. The approaches we take for solving both cases are very similar, we first determine closed form expressions of the first state in the OFF region $\pi_{0,0}$, then we use a weighted average to obtain closed form expressions for the expected number of jobs and the expected energy consumption. The expected response time is then obtained via Little's Law after we have a closed form expression for the expected number of jobs. As mentioned in Chapter 4, we put more focus on studying and validating Case 1, $k_1 \leq k_2$.

The CTMC model of this case is presented in Figure 4.1. The balance equations for different regions are similar, so we group the balance equations together with a common expression for each region, as follows:

$$\begin{cases} \lambda\pi_{0,n} = \lambda\pi_{0,n-1} & \text{if } 0 \leq n < k_1, OFF\ region \\ (\lambda+\gamma)\pi_{0,n} = \lambda\pi_{0,n-1} & \text{if } k_1 \leq n\ SETUP\ region \\ \lambda\pi_{0,n} = \alpha\pi_{1,n} & \text{if } n = 0\ IDLE \\ \mu\pi_{1,n} = \lambda\pi_{1,n-1} + \lambda\pi_{0,n-1} & \text{if } 0 < n \leq k_1\ BUSY\ to\ IDLE\ region \\ \mu\pi_{1,n} = \lambda\pi_{1,n-1} + \lambda\pi_{0,n-1} & \text{if } k_1 < n < k_2\ BUSY\ to\ SCALED\ region \\ (\lambda+c\mu)\pi_{1,n} = \lambda\pi_{1,n-1} + \gamma\pi_{0,n} + c\mu\pi_{1,n+1} & \text{if } k_2 \leq n\ SCALED\ region \end{cases}$$

To obtain closed form expressions for each probability, we work on expressing each probability in terms of $\pi_{0,0}$.

In the OFF region $0 \leq n < k_1$:

$$\lambda\pi_{0,n} = \lambda\pi_{0,n-1}$$
$$\pi_{0,n} = \pi_{0,0} \tag{5.1}$$

In the SETUP region $k_1 \leq n$:

$$\begin{aligned} (\lambda+\gamma)\pi_{0,n} &= \lambda\pi_{0,n-1} \\ \pi_{0,n} &= \frac{\lambda}{\lambda+\gamma}\pi_{0,n-1} \end{aligned}$$

Note that the SETUP region starts from k_1, and π_{0,k_1-1} belongs to the OFF region, so we have $\pi_{0,k_1-1} = \pi_{0,0}$. With this information, we have:

$$\pi_{0,n} = (\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)}\pi_{0,0} \tag{5.2}$$

In the IDLE region $n = 0$:

$$\begin{aligned} \lambda\pi_{0,0} &= \alpha\pi_{1,0} \\ \pi_{1,0} &= \frac{\lambda}{\alpha}\pi_{0,0} \end{aligned} \tag{5.3}$$

In the BUSY towards IDLE region $0 < n \leq k_1$:

$$\mu\pi_{1,n} = \lambda\pi_{1,n-1} + \lambda\pi_{0,n-1}$$

Let $\rho = \frac{\lambda}{\mu}$

$$\begin{aligned} \mu\pi_{1,1} =& \lambda\pi_{1,0} + \lambda\pi_{0,0} \\ \Rightarrow \pi_{1,1} =& \rho\pi_{1,0} + \rho\pi_{0,0} \\ \mu\pi_{1,2} =& \lambda\pi_{1,1} + \lambda\pi_{0,1} \\ \Rightarrow \pi_{1,2} =& \rho^2\pi_{1,0} + \rho^2\pi_{0,0} + \rho\pi_{0,0} \\ \mu\pi_{1,n} =& \lambda\pi_{1,n-1} + \lambda\pi_{0,0} \\ \Rightarrow \pi_{1,n} =& \rho^n\pi_{1,0} + \pi_{0,0}\sum_{i=1}^{n}\rho^i \\ \pi_{1,n} =& \pi_{0,0}\left(\frac{\lambda}{\alpha}\rho^n + \rho\frac{1-\rho^n}{1-\rho}\right) \end{aligned} \tag{5.4}$$

In the BUSY towards SCALED region $k_1 < n < k_2$:

$$\mu\pi_{1,n} = \lambda\pi_{1,n-1} + \lambda\pi_{0,n-1}$$

At the first state of this region ($n = k_1 + 1$), we have

$$\pi_{1,k_1+1} = \rho\pi_{1,k_1} + \rho\pi_{0,k_1}$$

From (5.2) and (5.4), we have:

$$\begin{aligned}
\pi_{0,k_1} &= \frac{\lambda}{\lambda+\gamma}\pi_{0,0} \\
\pi_{1,k_1} &= \left(\frac{\lambda}{\alpha}\rho^{k_1} + \rho\frac{1-\rho^{k_1}}{1-\rho}\right)\pi_{0,0} \\
\Rightarrow \pi_{1,k_1+1} &= \frac{\lambda}{\lambda+\gamma}\rho\pi_{0,0} + \left(\frac{\lambda}{\alpha}\rho^{k_1+1} + \rho^2\frac{1-\rho^{k_1}}{1-\rho}\right)\pi_{0,0} \\
\pi_{1,k_1+2} &= \rho\pi_{1,k_1+1} + \rho\pi_{0,k_1+1} \\
\Rightarrow \pi_{1,k_1+2} &= \left(\frac{\lambda}{\alpha}\rho^{k_1+2} + \rho^3\frac{1-\rho^{k_1}}{1-\rho}\right)\pi_{0,0} + \frac{\lambda}{\lambda+\gamma}\rho^2\pi_{0,0} + \left(\frac{\lambda}{\lambda+\gamma}\right)^2\rho\pi_{0,0} \\
\pi_{1,k_1+3} &= \rho\pi_{1,k_1+2} + \rho\pi_{0,k_1+2} \\
\Rightarrow \pi_{1,k_1+3} &= \left(\frac{\lambda}{\alpha}\rho^{k_1+3} + \rho^4\frac{1-\rho^{k_1}}{1-\rho}\right)\pi_{0,0} + \frac{\lambda}{\lambda+\gamma}\rho^3\pi_{0,0} + \left(\frac{\lambda}{\lambda+\gamma}\right)^2\rho^2\pi_{0,0} + \left(\frac{\lambda}{\lambda+\gamma}\right)^3\rho\pi_{0,0}
\end{aligned}$$

and this leads to the following expression:

$$\pi_{1,n} = \left(\frac{\lambda}{\alpha}\rho^n + \rho^{n-(k_1-1)}\frac{1-\rho^{k_1}}{1-\rho}\right)\pi_{0,0} + \sum_{i=1}^{n-k_1}\left(\frac{\lambda}{\lambda+\gamma}\right)^i\rho^{n-k_1-(i-1)}\pi_{0,0} \tag{5.5}$$

In the SCALED region $k_2 \le n$:

$$(\lambda + c\mu)\pi_{1,n} = \lambda\pi_{1,n-1} + \gamma\pi_{0,n} + c\mu\pi_{1,n+1} \tag{5.6}$$

The solution to this balance equation can be described as [8]

$$\pi_{1,n} = Ax^{n-(k_2-1)} + B\left(\frac{\lambda}{\lambda+\gamma}\right)^{n-(k_2-1)}$$

where x satisfies:

$$(\lambda + c\mu)x = \lambda + c\mu x^2$$

which yields $x = 1$ or $x = \frac{\lambda}{c\mu}$.
Substitute this relationship along with (5.2) back into (5.6)

$$\begin{aligned}(\lambda + c\mu)\left(Ax^{n-(k_2-1)} + B\left(\frac{\lambda}{\lambda+\gamma}\right)^{n-(k_2-1)}\right) &= \lambda\left(Ax^{n-k_2} + B\left(\frac{\lambda}{\lambda+\gamma}\right)^{n-k_2}\right) + \gamma\left(\left(\frac{\lambda}{\lambda+\gamma}\right)^{n-(k_1-1)}\pi_{0,0}\right) \\ &+ c\mu\left(Ax^{n-k_2+2} + B\left(\frac{\lambda}{\lambda+\gamma}\right)^{n-k_2+2}\right)\end{aligned} \tag{5.7}$$

We can now group terms on both the right hand side and left hand side into those that contain x and those that do not. If we let $x = 1$, the terms that contain x on both right and life hand sides cancel, so the terms that do not contain x on both right and left hand sides must be equal.

As a result, the constant terms in (5.7) are:

$$
\begin{aligned}
(\lambda+c\mu)B\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-(k_2-1)} =& \lambda B\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-k_2} + \gamma\Big(\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-(k_1-1)}\pi_{0,0}\Big) + c\mu B\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-k_2+2} \\
(\lambda+c\mu)B\Big(\frac{\lambda}{\lambda+\gamma}\Big) =& \lambda B + \gamma\Big(\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{k_2-(k_1-1)}\pi_{0,0}\Big) + c\mu B\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{2} \\
\lambda(\lambda+\gamma)^{k_2-k_1}(\lambda+c\mu)B =& \lambda(\lambda+\gamma)^{k_2-k_1+1}B + \gamma\lambda^{k_2-k_1+1}\pi_{0,0} + c\mu B\lambda^2(\lambda+\gamma)^{k_2-k_1-1} \\
\gamma\lambda^{k_2-k_1+1}\pi_{0,0} =& \Big(\lambda(\lambda+\gamma)^{k_2-k_1}(\lambda+c\mu) - \lambda(\lambda+\gamma)^{k_2-k_1+1} - c\mu\lambda^2(\lambda+\gamma)^{k_2-k_1-1}\Big)B \\
\gamma\lambda^{k_2-k_1+1}\pi_{0,0} =& \lambda(\lambda+\gamma)^{k_2-k_1-1}\Big((\lambda+\gamma)(\lambda+c\mu) - (\lambda+\gamma)^2 - c\mu\lambda\Big)B \\
\gamma\lambda^{k_2-k_1}\pi_{0,0} =& (\lambda+\gamma)^{k_2-k_1-1}\Big((\lambda+\gamma)(\lambda+c\mu) - (\lambda+\gamma)^2 - c\mu\lambda\Big)B \\
\gamma\lambda^{k_2-k_1}\pi_{0,0} =& (\lambda+\gamma)^{k_2-k_1-1}(c\mu\gamma - \lambda\gamma - \gamma^2)B \\
B =& \Big(\frac{\lambda^{k_2-k_1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)}\Big)\pi_{0,0}
\end{aligned}
$$

With B solved, we can substitute the result back into (5.7) to solve for A:

$$
\begin{aligned}
(\lambda+c\mu)\Big(Ax^{n-(k_2-1)} + B\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-(k_2-1)}\Big) = \lambda\pi_{1,n-1} + \gamma\Big(\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-(k_1-1)}\pi_{0,0}\Big) \\
+c\mu\Big(Ax^{n-k_2+2} + B\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-k_2+2}\Big)
\end{aligned}
$$

Let $n = k_2$ so that π_{1,k_2-1} falls into the BUSY to SCALED region, which can be expressed from (5.5) as:

$$
\pi_{1,k_2-1} = \Big(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho^{k_2-k_1}\frac{1-\rho^{k_1}}{1-\rho}\Big)\pi_{0,0} + \sum_{i=1}^{k_2-k_1-1}\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{i}\rho^{k_2-k_1-i}\pi_{0,0}
$$

Set $x = 1$ as one of the solutions, then we have:

$$\lambda A = \lambda\Big(\Big(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho^{k_2-k_1}\frac{1-\rho^{k_1}}{1-\rho}\Big)\pi_{0,0} + \sum_{i=1}^{k_2-k_1-1}\Big(\frac{\lambda}{\lambda+\gamma}\Big)^i \rho^{k_2-k_1-i}\pi_{0,0}\Big) + \gamma\Big((\frac{\lambda}{\lambda+\gamma})^{k_2-k_1+1}\pi_{0,0}\Big) \\ -\lambda B\Big(\frac{\lambda^2+\lambda\gamma+c\mu\gamma}{(\lambda+\gamma)^2}\Big)$$

$$\lambda A = \lambda\Big(\Big(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho^{k_2-k_1}\frac{1-\rho^{k_1}}{1-\rho}\Big)\pi_{0,0} + \sum_{i=1}^{k_2-k_1-1}\Big(\frac{\lambda}{\lambda+\gamma}\Big)^i \rho^{k_2-k_1-i}\pi_{0,0}\Big) + \gamma\Big((\frac{\lambda}{\lambda+\gamma})^{k_2-k_1+1}\pi_{0,0}\Big) \\ -\lambda\Big(\frac{\lambda^{k_2-k_1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)}\Big)\pi_{0,0}\Big(\frac{\lambda^2+\lambda\gamma+c\mu\gamma}{(\lambda+\gamma)^2}\Big)$$

$$A = \Big(\Big(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho^{k_2-k_1}\frac{1-\rho^{k_1}}{1-\rho}\Big)\pi_{0,0} + \sum_{i=1}^{k_2-k_1-1}\Big(\frac{\lambda}{\lambda+\gamma}\Big)^i \rho^{k_2-k_1-i}\pi_{0,0}\Big) + \gamma\frac{\lambda^{k_2-k_1}}{(\lambda+\gamma)^{k_2-k_1+1}}\pi_{0,0} \\ -\Big(\frac{\lambda^{k_2-k_1}(\lambda^2+\lambda\gamma+c\mu\gamma)}{(\lambda+\gamma)^{k_2-k_1+1}(c\mu-\lambda-\gamma)}\Big)\pi_{0,0}$$

$$\begin{aligned} A &= \Big(\Big(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho^{k_2-k_1}\frac{1-\rho^{k_1}}{1-\rho}\Big) + \sum_{i=1}^{k_2-k_1-1}\Big(\frac{\lambda}{\lambda+\gamma}\Big)^i \rho^{k_2-k_1-i}\Big)\pi_{0,0} \\ &\quad + \frac{\lambda^{k_2-k_1}}{(\lambda+\gamma)^{k_2-k_1+1}}\Big(\gamma - \frac{\lambda^2+\lambda\gamma+c\mu\gamma}{c\mu-\lambda-\gamma}\Big)\pi_{0,0} \\ &= \Big(\Big(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho^{k_2-k_1}\frac{1-\rho^{k_1}}{1-\rho}\Big) + \sum_{i=1}^{k_2-k_1-1}\Big(\frac{\lambda}{\lambda+\gamma}\Big)^i \rho^{k_2-k_1-i}\Big)\pi_{0,0} \\ &\quad + \frac{\lambda^{k_2-k_1}}{(\lambda+\gamma)^{k_2-k_1+1}}\Big(\frac{-(\lambda+\gamma)^2}{c\mu-\lambda-\gamma}\Big)\pi_{0,0} \\ &= \Big(\Big(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho^{k_2-k_1}\frac{1-\rho^{k_1}}{1-\rho}\Big) + \sum_{i=1}^{k_2-k_1-1}\Big(\frac{\lambda}{\lambda+\gamma}\Big)^i \rho^{k_2-k_1-i}\Big)\pi_{0,0} - \frac{\lambda^{k_2-k_1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)}\pi_{0,0} \\ &= \Big(\Big(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho^{k_2-k_1}\frac{1-\rho^{k_1}}{1-\rho}\Big) + \sum_{i=1}^{k_2-k_1-1}\Big(\frac{\lambda}{\lambda+\gamma}\Big)^i \rho^{k_2-k_1-i} - \frac{\lambda^{k_2-k_1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)}\Big)\pi_{0,0} \end{aligned}$$

With these expressions for A and B, we can now obtain the expression for the probabilities in this region:

$$\pi_{1,n} = A(\frac{\lambda}{c\mu})^{n-(k_2-1)} + B(\frac{\lambda}{\lambda+\gamma})^{n-(k_2-1)}$$

$$\begin{aligned}\pi_{1,n} = &\left(\left(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho^{k_2-k_1}\frac{1-\rho^{k_1}}{1-\rho}\right) + \sum_{i=1}^{k_2-k_1-1}\left(\frac{\lambda}{\lambda+\gamma}\right)^i \rho^{k_2-k_1-i}\right.\\ &\left. - \frac{\lambda^{k_2-k_1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)}\right)\pi_{0,0}(\frac{\lambda}{c\mu})^{n-(k_2-1)}\\ &+ \left(\frac{\lambda^{k_2-k_1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)}\right)\pi_{0,0}(\frac{\lambda}{\lambda+\gamma})^{n-(k_2-1)}\end{aligned}$$

$$\begin{aligned}\Rightarrow \pi_{1,n} = &\left(\frac{\lambda^{n+1}}{\alpha c^{n-(k_2-1)}\mu^n} + \frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\frac{1-\rho^{k_1}}{1-\rho} + \sum_{i=1}^{k_2-k_1-1}\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}(\lambda+\gamma)^i\mu^{n-k_1+1-i}}\right.\\ &\left. - \frac{\lambda^{n-k_1+1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)(c\mu)^{n-(k_2-1)}} + \frac{\lambda^{n-k_1+1}}{(\lambda+\gamma)^{n-k_1}(c\mu-\lambda-\gamma)}\right)\pi_{0,0}\end{aligned} \quad (5.8)$$

We now have expressed all probabilities in terms of $\pi_{0,0}$, we can use the fact that the sum of all of the probabilities must be equal to 1, which leads to an expression for $\pi_{0,0}$. Since the equations developed in (5.3) and (5.4) are in fact the same when $n = 0$, we can combine the IDLE region into the BUSY towards IDLE region, and have the following equation to solve for $\pi_{0,0}$

$$1 = \sum_{n=0}^{k_1-1}\pi_{0,n} + \sum_{n=k_1}^{\infty}\pi_{0,n} + \sum_{n=0}^{k_1}\pi_{1,n} + \sum_{n=k_1+1}^{k_2-1}\pi_{1,n} + \sum_{n=k_2}^{\infty}\pi_{1,n} \quad (5.9)$$

We now subsitute the developed expressions from (5.1), (5.2), (5.4), (5.5), (5.8) into corresponding regions in (5.9).

$$
\begin{aligned}
1 =\ & \pi_{0,0}\sum_{n=0}^{k_1-1} 1 + \pi_{0,0}\sum_{n=k_1}^{\infty}\left(\frac{\lambda}{\lambda+\gamma}\right)^{n-(k_1-1)} + \pi_{0,0}\sum_{n=0}^{k_1}\left(\frac{\lambda}{\alpha}\rho^n + \rho\frac{1-\rho^n}{1-\rho}\right)\\
&+ \pi_{0,0}\sum_{n=k_1+1}^{k_2-1}\left(\left(\frac{\lambda}{\alpha}\rho^n + \rho^{n-(k_1-1)}\frac{1-\rho^{k_1}}{1-\rho}\right) + \sum_{i=1}^{n-k_1}\left(\frac{\lambda}{\lambda+\gamma}\right)^i \rho^{n-k_1-(i-1)}\right)\\
&+ \pi_{0,0}\sum_{n=k_2}^{\infty}\Bigg(\frac{\lambda^{n+1}}{\alpha c^{n-(k_2-1)}\mu^n} + \frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\frac{1-\rho^{k_1}}{1-\rho} + \sum_{i=1}^{k_2-k_1-1}\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}(\lambda+\gamma)^i\mu^{n-k_1+1-i}}\\
&- \frac{\lambda^{n-k_1+1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)(c\mu)^{n-(k_2-1)}} + \frac{\lambda^{n-k_1+1}}{(\lambda+\gamma)^{n-k_1}(c\mu-\lambda-\gamma)}\Bigg)
\end{aligned}
$$

Let $\frac{\lambda}{\mu} = \rho$, $\frac{\lambda}{\lambda+\gamma} = \xi$, we can simplify the equation as:

$$
\begin{aligned}
1 =\ & \pi_{0,0}\sum_{n=0}^{k_1-1} 1 + \pi_{0,0}\sum_{n=k_1}^{\infty}\xi^{n-(k_1-1)} + \pi_{0,0}\sum_{n=0}^{k_1}\left(\frac{\lambda}{\alpha}\rho^n + \rho\frac{1-\rho^n}{1-\rho}\right)\\
&+ \pi_{0,0}\sum_{n=k_1+1}^{k_2-1}\left(\left(\frac{\lambda}{\alpha}\rho^n + \rho^{n-(k_1-1)}\frac{1-\rho^{k_1}}{1-\rho}\right) + \sum_{i=1}^{n-k_1}\left(\xi\right)^i \rho^{n-k_1-(i-1)}\right)\\
&+ \pi_{0,0}\sum_{n=k_2}^{\infty}\Bigg(\frac{\lambda^{n+1}}{\alpha c^{n-(k_2-1)}\mu^n} + \frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\frac{1-\rho^{k_1}}{1-\rho} + \sum_{i=1}^{k_2-k_1-1}\frac{\rho^{n-k_1+1-i}}{c^{n-(k_2-1)}}\xi^i\\
&- \frac{\lambda+\gamma}{(c\mu-(\lambda+\gamma))c^{n-(k_2-1)}}\rho^{n-k_2+1}\xi^{k_2-k_1} + \frac{\lambda}{c\mu-(\lambda+\gamma)}\xi^{n-k_1}
\end{aligned}
$$

$$
\begin{aligned}
1 =\ & \pi_{0,0}\Bigg(k_1 + \sum_{n=0}^{\infty}\xi^n - 1 + \frac{\lambda}{\alpha}\sum_{n=0}^{k_1}\rho^n - \left(\frac{\lambda}{\mu-\lambda}\right)\sum_{n=0}^{k_1}\rho^n + \left(\frac{\lambda}{\mu-\lambda}\right)(k_1+1)\\
&+ \frac{\lambda}{\alpha}\sum_{n=k_1+1}^{k_2-1}\rho^n + \left(\frac{\lambda}{\mu-\lambda}\right)(1-\rho^{k_1})\sum_{n=k_1+1}^{k_2-1}\rho^{n-k_1}\\
&+ \sum_{n=k_1+1}^{k_2-1}\left(\rho^{n-k_1+1}\sum_{i=1}^{n-k_1}\left(\frac{\xi}{\rho}\right)^i\right) + \frac{\lambda c^{k_2-1}}{\alpha}\sum_{n=k_1}^{\infty}\left(\frac{\rho}{c}\right)^n + (c^{k_2-1})(\rho^{-k_1+1})\left(\frac{1-\rho^{k_1}}{1-\rho}\right)\sum_{n=k_1}^{\infty}\left(\frac{\rho}{c}\right)^n\\
&+ \sum_{n=k_2}^{\infty}\left(\frac{\rho^{n-k_1+1}}{c^{n-k_2+1}}\sum_{i=1}^{k_2-1-k_1}\left(\frac{\xi}{\rho}\right)^i\right) - \frac{(\lambda+\gamma)\xi^{k_2-k_1}}{c\mu-(\lambda+\gamma)}\sum_{n=k_2}^{\infty}\left(\frac{\rho}{c}\right)^{n-k_2+1} + \frac{\lambda}{c\mu-(\lambda+\gamma)}\sum_{n=k_2}^{\infty}\xi^{n-k_1}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
1 = \pi_{0,0}\Big(& k_1 + \frac{1}{1-\xi} - 1 + \Big(\frac{\lambda}{\alpha} - (\frac{\lambda}{\mu-\lambda})\Big)\frac{1-\rho^{k_1+1}}{1-\rho} + (\frac{\lambda}{\mu-\lambda})(k_1+1) + \frac{\lambda}{\alpha}\rho^{k_1}(\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}) \\
&+ (\frac{\lambda}{\mu-\lambda})(1-\rho^{k_1})(\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}) + \sum_{n=k_1+1}^{k_2-1}\Big(\rho^{n-k_1+1}\sum_{i=1}^{n-k_1}(\frac{\xi}{\rho})^i\Big) + \frac{\lambda c^{k_2-1}}{\alpha}\sum_{n=k_2}^{\infty}(\frac{\rho}{c})^n \\
&+ (c^{k_2-1})(\rho^{-k_1+1})(\frac{1-\rho^{k_1}}{1-\rho})\sum_{n=k_2}^{\infty}(\frac{\rho}{c})^n + \sum_{n=k_2}^{\infty}\Big(\frac{\rho^{n-k_1+1}}{c^{n-k_2+1}}\sum_{i=1}^{k_2-1-k_1}(\frac{\xi}{\rho})^i\Big) \\
&- \frac{(\lambda+\gamma)\xi^{k_2-k_1}}{c\mu-(\lambda+\gamma)}\sum_{n=k_2}^{\infty}(\frac{\rho}{c})^{n-k_2+1} + \frac{\lambda}{c\mu-(\lambda+\gamma)}\sum_{n=k_2}^{\infty}\xi^{n-k_1}\Big)\Big)
\end{aligned}
\tag{5.10}
$$

This expression has become quite complicated, we will simplify each of the summation terms in (5.10). We first simplify the first sum in the right hand side of (5.10):

$$
\begin{aligned}
\sum_{n=k_1+1}^{k_2-1}\Big(\rho^{n-k_1+1}\sum_{i=1}^{n-k_1}(\frac{\xi}{\rho})^i\Big) &= \sum_{n=k_1+1}^{k_2-1}\Big(\rho^{n-k_1+1}\Big(\frac{\xi}{\rho-\xi} - \frac{\xi(\frac{\xi}{\rho})^{n-k_1}}{\rho-\xi}\Big)\Big) \\
&= \sum_{n=k_1+1}^{k_2-1}\Big(\rho^{n-k_1+1}(\frac{\xi}{\rho-\xi})\Big) - \sum_{n=k_1+1}^{k_2-1}\Big(\frac{\rho\xi^{n-k_1+1}}{\rho-\xi}\Big) \\
&= (\frac{\xi}{\rho-\xi})\rho\sum_{n=k_1+1}^{k_2-1}\rho^{n-k_1} - (\frac{\rho}{\rho-\xi})\xi\sum_{n=k_1+1}^{k_2-1}\xi^{n-k_1} \\
&= (\frac{\rho\xi}{\rho-\xi})\Big(\sum_{n=k_1+1}^{k_2-1}\rho^{n-k_1} - \sum_{n=k_1+1}^{k_2-1}\xi^{n-k_1}\Big) \\
&= (\frac{\rho\xi}{\rho-\xi})\Big(\sum_{n=1}^{k_2-k_1-1}\rho^{n} - \sum_{n=1}^{k_2-k_1-1}\xi^{n}\Big) \\
&= (\frac{\rho\xi}{\rho-\xi})\Big(\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho} - \frac{\xi(1-\xi^{k_2-k_1-1})}{1-\xi}\Big)
\end{aligned}
\tag{5.11}
$$

We continue to simplify the second sum in the RHS of (5.10)

$$\begin{aligned}
&\frac{\lambda c^{k_2-1}}{\alpha}\sum_{n=k_2}^{\infty}(\frac{\rho}{c})^n \\
&= \frac{\lambda c^{k_2-1}}{\alpha}(\frac{\rho}{c})^{k_2}\sum_{n=0}^{\infty}(\frac{\rho}{c})^n \\
&= \frac{\lambda}{c\alpha}\rho^{k_2}(\frac{c}{c-\rho}) \\
&= \frac{\lambda}{\alpha(c-\rho)}\rho^{k_2} \qquad (5.12)
\end{aligned}$$

Next, the third sum in the RHS of (5.10)

$$\begin{aligned}
&c^{k_2-1}\rho^{-k_1+1}(\frac{1-\rho^{k_1}}{1-\rho})\sum_{n=k_2}^{\infty}(\frac{\rho}{c})^n \\
&= c^{k_2-1}\rho^{-k_1+1}(\frac{1-\rho^{k_1}}{1-\rho})(\frac{\rho}{c})^{k_2}\sum_{n=0}^{\infty}(\frac{\rho}{c})^n \\
&= c^{k_2-1}\rho^{-k_1+1}(\frac{1-\rho^{k_1}}{1-\rho})(\frac{\rho}{c})^{k_2}(\frac{c}{c-\rho}) \\
&= \rho^{k_2-k_1+1}(\frac{1-\rho^{k_1}}{1-\rho})(\frac{1}{c-\rho}) \qquad (5.13)
\end{aligned}$$

Next, the fourth sum in the RHS of (5.10)

$$
\begin{aligned}
&\sum_{n=k_2}^{\infty} \frac{\rho^{n-k_1+1}}{c^{n-k_2+1}} \sum_{i=1}^{k_2-k_1-1} (\frac{\xi}{\rho})^i \\
&= \rho^{k_2-k_1} \sum_{n=k_2}^{\infty} \frac{\rho^{n-k_2+1}}{c^{n-k_2+1}} \Big(\frac{\xi}{\rho-\xi} - \frac{\rho\big(\frac{\xi}{\rho}\big)^{k_2-k_1}}{\rho-\xi} \Big) \\
&= \rho^{k_2-k_1} \Big(\sum_{n=k_2}^{\infty} \frac{\rho^{n-k_2+1}}{c^{n-k_2+1}} (\frac{\xi}{\rho-\xi}) - \sum_{n=k_2}^{\infty} \frac{\rho^{n-k_2+1}}{c^{n-k_2+1}} \Big(\frac{\rho\big(\frac{\xi}{\rho}\big)^{k_2-k_1}}{\rho-\xi} \Big) \Big) \\
&= \rho^{k_2-k_1} \Big((\frac{\xi}{\rho-\xi}) \sum_{n=1}^{\infty} (\frac{\rho}{c})^n - \Big(\frac{\rho\big(\frac{\xi}{\rho}\big)^{k_2-k_1}}{\rho-\xi} \Big) \sum_{n=1}^{\infty} (\frac{\rho}{c})^n \Big) \\
&= \Big((\frac{\xi}{\rho-\xi}) - \Big(\frac{\rho\big(\frac{\xi}{\rho}\big)^{k_2-k_1}}{\rho-\xi} \Big) \Big) \frac{\rho^{k_2-k_1+1}}{c-\rho} \\
&= \frac{\rho^{k_2-k_1+1}\xi - \rho^2 \xi^{k_2-k_1}}{(c-\rho)(\rho-\xi)}
\end{aligned} \tag{5.14}
$$

Next, the fifth sum in the RHS of (5.10)

$$
\begin{aligned}
&\frac{(\lambda+\gamma)\xi^{k_2-k_1}}{c\mu-(\lambda+\gamma)} \sum_{n=k_2}^{\infty} (\frac{\rho}{c})^{n-k_2+1} \\
&= \frac{(\lambda+\gamma)\xi^{k_2-k_1}}{c\mu-(\lambda+\gamma)} (\frac{\rho}{c-\rho}) \\
&= \frac{(\lambda+\gamma)\xi^{k_2-k_1}\rho}{(c\mu-(\lambda+\gamma))(c-\rho)}
\end{aligned} \tag{5.15}
$$

Finally, we simplify the last sum in the RHS of (5.10)

$$\begin{aligned}
&\frac{\lambda}{c\mu-(\lambda+\gamma)}\sum_{n=k_2}^{\infty}\xi^{n-k_1}\\
&=\frac{\lambda}{c\mu-(\lambda+\gamma)}\xi^{k_2-k_1}\sum_{n=k_2}^{\infty}\xi^{n-k_2}\\
&=\frac{\lambda}{c\mu-(\lambda+\gamma)}\xi^{k_2-k_1}\sum_{n=0}^{\infty}\xi^{n}\\
&=\frac{\lambda}{c\mu-(\lambda+\gamma)}\xi^{k_2-k_1}(\frac{1}{1-\xi})\\
&=\frac{\lambda\xi^{k_2-k_1}}{(c\mu-(\lambda+\gamma))(1-\xi)}
\end{aligned}\tag{5.16}$$

After we have simplified all of the sums in (5.10), we substute the expressions from (5.11),(5.12),(5.13),(5.14),(5.15),(5.16) back in (5.10), we then have:

$$\begin{aligned}
1=\pi_{0,0}\Big(&k_1+\frac{1}{1-\xi}-1+\Big(\frac{\lambda}{\alpha}-\frac{\lambda}{\mu-\lambda}\Big)\frac{1-\rho^{k_1+1}}{1-\rho}+\Big(\frac{\lambda}{\mu-\lambda}\Big)(k_1+1)+\frac{\lambda}{\alpha}\rho^{k_1}\Big(\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}\Big)\\
&+(\frac{\lambda}{\mu-\lambda})(1-\rho^{k_1})\Big(\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}\Big)+\frac{\rho\xi}{\rho-\xi}\Big(\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}-\frac{\xi(1-\xi^{k_2-k_1-1})}{1-\xi}\Big)\\
&+\frac{\lambda}{\alpha(c-\rho)}\rho^{k_2}+\rho^{k_2-k_1+1}\frac{1-\rho^{k_1}}{1-\rho}\frac{1}{c-\rho}+\frac{\rho^{k_2-k_1+1}\xi-\rho^2\xi^{k_2-k_1}}{(c-\rho)(\rho-\xi)}\\
&-\frac{(\lambda+\gamma)\xi^{k_2-k_1}\rho}{(c\mu-(\lambda+\gamma))(c-\rho)}+\frac{\lambda\xi^{k_2-k_1}}{(c\mu-(\lambda+\gamma))(1-\xi)}\Big)
\end{aligned}\tag{5.17}$$

We continue the simplification by simplifying each term from left to right on the RHS of (5.17), starting off with:

$$\begin{aligned}
&\frac{1}{1-\xi}-1\\
&=\frac{1}{1-\frac{\lambda}{\lambda+\gamma}}-1\\
&=\frac{\lambda}{\gamma}
\end{aligned}\tag{5.18}$$

Then we continue with the next term on the RHS of (5.17):

$$\begin{aligned}&\Big(\frac{\lambda}{\alpha}-\frac{\lambda}{\mu-\lambda}\Big)\frac{1-\rho^{k_1+1}}{1-\rho}\\&=\Big(\frac{\lambda(\mu-\lambda)-\alpha\lambda}{\alpha(\mu-\lambda)}\Big)\frac{(1-\rho^{k_1+1})\mu}{\mu-\lambda}\\&=\frac{\lambda(\mu-\lambda-\alpha)(1-\rho^{k_1+1})\mu}{\alpha(\mu-\lambda)^2}\end{aligned}\tag{5.19}$$

We can sum up (5.18) and (5.19),

$$\begin{aligned}&\frac{\lambda}{\gamma}+\frac{\lambda(\mu-\lambda-\alpha)(1-\rho^{k_1+1})\mu}{\alpha(\mu-\lambda)^2}\\&=\frac{\lambda\alpha(\mu-\lambda)^2+\lambda\gamma\mu(\mu-\lambda-\alpha)-\lambda^2\gamma(\mu-\lambda-\alpha)\rho^{k_1}}{\alpha\gamma(\mu-\lambda)^2}\end{aligned}\tag{5.20}$$

Next, we continue to simplify the next term on the RHS of (5.17),

$$\begin{aligned}&(\frac{\lambda}{\mu-\lambda})(k_1+1)\\&=\frac{\lambda(k_1+1)}{\mu-\lambda}\end{aligned}\tag{5.21}$$

We sum up (5.20) and (5.21) to represent the simplification for the first few terms on the RHS of (5.17),

$$\begin{aligned}&\frac{\lambda\alpha(\mu-\lambda)^2+\lambda\gamma\mu(\mu-\lambda-\alpha)-\lambda^2\gamma(\mu-\lambda-\alpha)\rho^{k_1}}{\alpha\gamma(\mu-\lambda)^2}+\frac{\lambda(k_1+1)}{\mu-\lambda}\\&=\frac{\lambda\alpha(\mu-\lambda)^2+\lambda\gamma\mu(\mu-\lambda-\alpha)-\lambda^2\gamma(\mu-\lambda-\alpha)\rho^{k_1}+\lambda\alpha\gamma(k_1+1)(\mu-\lambda)}{\alpha\gamma(\mu-\lambda)^2}\\&=\frac{(\mu-\lambda+\gamma+\gamma k_1\lambda\alpha(\mu-\lambda)+\lambda\gamma\mu(\mu-\lambda-\alpha)-\lambda^2\gamma(\mu-\lambda-\alpha)\rho^{k_1}}{\alpha\gamma(\mu-\lambda)^2}\end{aligned}\tag{5.22}$$

We continue to simplify the next term on the RHS of (5.17),

$$\begin{aligned}&\frac{\lambda}{\alpha}\rho^{k_1}\left(\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}\right)\\ &=\frac{\lambda}{\alpha}\rho^{k_1}\left(\frac{\lambda(1-\rho^{k_2-k_1-1})}{\mu-\lambda}\right)\\ &=\frac{\lambda^2\rho^{k_1}-\lambda^2\rho^{k_2-1}}{\alpha(\mu-\lambda)}\end{aligned} \tag{5.23}$$

Now, we sum up (5.22) and (5.23),

$$\begin{aligned}&\frac{(\mu-\lambda+\gamma+\gamma k_1\lambda\alpha(\mu-\lambda)+\lambda\gamma\mu(\mu-\lambda-\alpha)-\lambda^2\gamma(\mu-\lambda-\alpha)\rho^{k_1}}{\alpha\gamma(\mu-\lambda)^2}+\frac{\lambda^2\rho^{k_1}-\lambda^2\rho^{k_2-1}}{\alpha(\mu-\lambda)}\\ &=\frac{(\mu-\lambda+\gamma+\gamma k_1)\lambda\alpha(\mu-\lambda)+\lambda\gamma\mu(\mu-\lambda-\alpha)+\alpha\lambda^2\gamma\rho^{k_1}-(\mu-\lambda)\lambda\mu\gamma\rho^{k_2}}{\alpha\gamma(\mu-\lambda)^2}\end{aligned} \tag{5.24}$$

We continue the simplification with the next term on the RHS of (5.17),

$$\begin{aligned}&\left(\frac{\lambda}{\mu-\lambda}\right)(1-\rho^{k_1})\left(\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}\right)\\ &=\frac{\lambda^2(1-\rho^{k_2-k_1-1})(1-\rho^{k_1})}{(\mu-\lambda)^2}\\ &=\frac{\lambda^2-\lambda^2\rho^{k_2-k_1-1}-\lambda^2\rho^{k_1}+\lambda\mu\rho^{k_2}}{(\mu-\lambda)^2}\end{aligned} \tag{5.25}$$

Now we sum up (5.24) and (5.25) to represent the simplification for the first five terms (excluding k_1) on the RHS of (5.17).

$$\begin{aligned}&\frac{(\mu-\lambda+\gamma+\gamma k_1)\lambda\alpha(\mu-\lambda)+\lambda\gamma\mu(\mu-\lambda-\alpha)+\alpha\lambda^2\gamma\rho^{k_1}-(\mu-\lambda)\lambda\mu\gamma\rho^{k_2}}{\alpha\gamma(\mu-\lambda)^2}+\frac{\lambda^2-\lambda^2\rho^{k_2-k_1-1}-\lambda^2\rho^{k_1}+\lambda\mu\rho^{k_2}}{(\mu-\lambda)^2}\\ &=\frac{(\lambda+\alpha-\mu)\lambda\mu\gamma\rho^{k_2}-\alpha\gamma\lambda^2\rho^{k_2-k_1-1}+(\mu-\lambda+\gamma+\gamma k_1)(\lambda\alpha\mu-\alpha\lambda^2)+\lambda\gamma\mu^2-\gamma\mu\lambda^2-\lambda\gamma\mu\alpha+\alpha\gamma\lambda^2}{\alpha\gamma(\mu-\lambda)^2}\\ &=\frac{(\lambda+\alpha-\mu)\lambda\mu\gamma\rho^{k_2}-\alpha\gamma\lambda^2\rho^{k_2-k_1-1}+(\alpha\lambda-2\alpha\mu-\gamma k_1\alpha-\gamma\mu)\lambda^2+\gamma k_1\lambda\alpha\mu+(\gamma+\alpha)\lambda\mu^2}{\alpha\gamma(\mu-\lambda)^2}\end{aligned} \tag{5.26}$$

We then continue to simplify the next term on the RHS of (5.17),

$$\begin{aligned} &-\frac{(\lambda+\gamma)\xi^{k_2-k_1}\rho}{(c\mu-(\lambda+\gamma))(c-\rho)} \\ &=-\frac{(\lambda+\gamma)\xi^{k_2-k_1}\lambda}{(c\mu-(\lambda+\gamma))(c\mu-\lambda)} \end{aligned} \tag{5.27}$$

as well as to simplify the term after that on the RHS of (5.17),

$$\begin{aligned} &\frac{\lambda\xi^{k_2-k_1}}{(c\mu-(\lambda+\gamma))(1-\xi)} \\ &=\frac{(\lambda+\gamma)\lambda\xi^{k_2-k_1}}{(c\mu-(\lambda+\gamma))\gamma} \end{aligned} \tag{5.28}$$

Summing (5.27) and (5.28) yields,

$$\begin{aligned} \Rightarrow\quad &\frac{(\lambda+\gamma)\lambda\xi^{k_2-k_1}(c\mu-\lambda)-\gamma(\lambda+\gamma)\xi^{k_2-k_1}\lambda}{(c\mu-(\lambda+\gamma))\gamma(c\mu-\lambda)} \\ =\ &\frac{\lambda(\lambda+\gamma)\xi^{k_2-k_1}}{\gamma(c\mu-\lambda)} \end{aligned} \tag{5.29}$$

We contine the simplification with the next term on the RHS of (5.17),

$$\begin{aligned} &\frac{\lambda}{\alpha(c-\rho)}\rho^{k_2} \\ &=\frac{\lambda\mu\rho^{k_2}}{\alpha(c\mu-\lambda)} \end{aligned} \tag{5.30}$$

Summing (5.29) and (5.30) yields,

$$\frac{\lambda\gamma\mu\rho^{k_2}+\alpha\lambda(\lambda+\gamma)\xi^{k_2-k_1}}{\alpha\gamma(c\mu-\lambda)} \tag{5.31}$$

Contine with the next term on the RHS of (5.17),

$$\begin{aligned} &\frac{\rho_{k_2-k_1+1}\xi - \rho^2\xi^{k_2-k_1}}{(c-\rho)(\rho-\xi)} \\ &= \frac{\mu^2\rho^{k_2-k_1+1} - \lambda(\lambda+\gamma)\xi^{k_2-k_1}}{(c\mu-\lambda)(\lambda+\gamma-\mu)} \end{aligned} \tag{5.32}$$

Next, we merge (5.31) and (5.32),

$$\begin{aligned} &\frac{\lambda\gamma\mu\rho^{k_2}(\lambda+\gamma-\mu) + \alpha\lambda(\lambda+\gamma)\xi^{k_2-k_1}(\lambda+\gamma-\mu) + \alpha\gamma\mu^2\rho^{k_2-k_1+1} - \alpha\gamma\lambda(\lambda+\gamma)\xi^{k_2-k_1}}{\alpha\gamma(c\mu-\lambda)(\lambda+\gamma-\mu)} \\ &= \frac{\lambda\gamma\mu\rho^{k_2}(\lambda+\gamma-\mu) + \alpha\gamma\mu^2\rho^{k_2-k_1+1} + \alpha(\lambda-\mu)\lambda(\lambda+\gamma)\xi^{k_2-k_1}}{\alpha\gamma(c\mu-\lambda)(\lambda+\gamma-\mu)} \end{aligned} \tag{5.33}$$

Continue the simplification with the next term on the RHS of (5.17),

$$\begin{aligned} &(\rho^{k_2-k_1+1}\Big(\frac{1-\rho^{k_1}}{1-\rho}\Big)\Big(\frac{1}{c-\rho}\Big) \\ &= \frac{(1-\rho^{k_1})\mu^2\rho^{k_2-k_1+1}}{(\mu-\lambda)(c\mu-\lambda)} \end{aligned} \tag{5.34}$$

Now, we sum (5.33) and (5.34), yielding,

$$\begin{aligned} &\frac{\alpha\gamma\mu^2\rho^{k_2-k_1+1}(\lambda+\gamma-\mu) - \alpha\gamma\mu^2\rho^{k_2+1}(\lambda+\gamma-\mu) + \lambda\gamma\mu\rho^{k_2}(\lambda+\gamma-\mu)(\mu-\lambda)}{\alpha\gamma(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)} \\ &+ \frac{\alpha\gamma\mu^2\rho^{k_2-k_1+1}(\mu-\lambda) + \alpha(\lambda-\mu)(\mu-\lambda)\lambda(\lambda+\gamma)\xi^{k_2-k_1}}{\alpha\gamma(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)} \\ &= \frac{\alpha\gamma^2\mu^2\rho^{k_2-k_1+1} + (\mu-\lambda-\alpha)\lambda\gamma\mu\rho^{k_2}(\lambda+\gamma-\mu) - \alpha(\mu-\lambda)^2\lambda(\lambda+\gamma)\xi^{k_2-k_1}}{\alpha\gamma(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)} \end{aligned} \tag{5.35}$$

We continue to simplify the last term on the RHS of (5.17),

$$
\begin{aligned}
&\frac{\rho\xi}{\rho-\xi}\Big(\frac{\rho(1-\rho^{k_{2}k_1-1})}{1-\rho}-\frac{\xi(1-\xi^{k_2-k_1-1})}{1-\xi}\Big)\\
&=\Big(\frac{\lambda^2}{\lambda(\lambda+\gamma-\mu)}\Big)\Big(\frac{\lambda(1-\rho^{k_2-k_1-1})}{\mu-\lambda}-\frac{\lambda(1-\xi^{k_2-k_1-1})}{\gamma}\Big)\\
&=\frac{\lambda^2\gamma(1-\rho^{k_2-k_1-1})-\lambda^2(1-\xi^{k_2-k_1-1})(\mu-\lambda)}{(\lambda+\gamma-\mu)(\mu-\lambda)\gamma}
\end{aligned}
\tag{5.36}
$$

Now we sum (5.35) and (5.36), yielding,

$$
\begin{aligned}
&\frac{\alpha\lambda^2\gamma(1-\rho^{k_2-k_1-1})(c\mu-\lambda)-\alpha\lambda^2(1-\xi^{k_2-k_1-1})(\mu-\lambda)(c\mu-\lambda)}{\alpha\gamma(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)}\\
&+\frac{\alpha\gamma^2\mu^2\rho^{k_2-k_1+1}+(\mu-\lambda-\alpha)\lambda\gamma\mu\rho^{k_2}(\lambda+\gamma-\mu)-\alpha(\mu-\lambda)^2\lambda(\lambda+\gamma)\xi^{k_2-k_1}}{\alpha\gamma(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)}\\
&=\frac{(\lambda+\gamma-c\mu)\alpha\lambda\mu\gamma\rho^{k_2-k_1}+\alpha\lambda^2\mu(c-1)(\mu-\lambda)\xi^{k_2-k_1-1}}{\alpha\gamma(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)}\\
&+\frac{(\mu-\lambda-\alpha)(\lambda+\gamma-\mu)\lambda\gamma\mu\rho^{k_2}+(\lambda+\gamma-\mu)(c\mu-\lambda)\alpha\lambda^2}{\alpha\gamma(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)}
\end{aligned}
\tag{5.37}
$$

Now we sum (5.26) and (5.37), which represents all the terms in the parentheses on the RHS of (5.17) excluding k_1, which yields,

$$
\begin{aligned}
\Rightarrow\quad &\frac{(\lambda+\gamma-c\mu)\alpha\lambda\mu\gamma\rho^{k_2-k_1}(\mu-\lambda)+(\mu-\lambda)(\mu-\lambda-\alpha)(\lambda+\gamma-\mu)\lambda\gamma\mu\rho^{k_2}}{\alpha\gamma(\lambda+\gamma-\mu)(\mu-\lambda)^2(c\mu-\lambda)}\\
&+\frac{\alpha\lambda^2\mu(c-1)(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\mu-\lambda)(\lambda+\gamma-\mu)(c\mu-\lambda)\alpha\lambda^2}{\alpha\gamma(\lambda+\gamma-\mu)(\mu-\lambda)^2(c\mu-\lambda)}\\
&+\frac{(\lambda+\gamma-\mu)(c\mu-\lambda)(\lambda+\alpha-\mu)\lambda\mu\gamma\rho^{k_2}-(\lambda+\gamma-\mu)(c\mu-\lambda)\alpha\gamma\lambda^2\rho^{k_2-k_1-1}}{\alpha\gamma(\lambda+\gamma-\mu)(\mu-\lambda)^2(c\mu-\lambda)}\\
&+\frac{(\lambda+\gamma-\mu)(c\mu-\lambda)(\alpha\lambda-2\alpha\mu-\gamma k_1\alpha-\gamma\mu)\lambda^2+(\lambda+\gamma-\mu)(c\mu-\lambda)\gamma k_1\lambda\alpha\mu+(\lambda+\gamma-\mu)(c\mu-\lambda)(\gamma+\alpha)\lambda\mu^2}{\alpha\gamma(\lambda+\gamma-\mu)(\mu-\lambda)^2(c\mu-\lambda)}
\end{aligned}
$$

Simplify the numerator collecting coefficients of $\rho^{k_2-k_1}$, $\xi^{k_2-k_1-1}$, and ρ^{k_2}, we have:

$$
\begin{aligned}
&\frac{(c-1)(-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+\alpha\lambda^2\mu(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2}}{\alpha\gamma(\lambda+\gamma-\mu)(\mu-\lambda)^2(c\mu-\lambda)}\\
&+\frac{(\gamma k_1\alpha+\mu(\alpha+\gamma))(\mu-\lambda)\lambda(\lambda+\gamma-\mu)(c\mu-\lambda)}{\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)}
\end{aligned}
\tag{5.38}
$$

Now we need to sum (5.38) with k_1 since it has been left over from (5.17) when we started the simplification on each term, yielding,

$$\frac{(c-1)\lambda\mu(-\alpha\mu\gamma^2\rho^{k_2-k_1}+\alpha\lambda(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\mu\gamma\rho^{k_2}}{\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)} + \frac{\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)((\alpha+\gamma)\lambda+\alpha\gamma k_1)}{\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)} \tag{5.39}$$

With (5.17) and (5.39), we obtain the final expression for $\pi_{0,0}$ as:

$$\pi_{0,0} = \frac{\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)}{(c-1)(\alpha\lambda^2\mu(\mu-\lambda)^2\xi^{k_2-k_1-1}-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2})+\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)((\alpha+\gamma)\lambda+\alpha\gamma k_1)} \tag{5.40}$$

The complexity of the above expression makes it difficult to provide meaningful interpretation. However, it does capture the effect of the DVS processor by separating two terms in the denominator. We can see that one term in the denominator heavily relies on the value of c. As a result, we can verify whether our calculation is done correctly by comparing with previous work done in [18]. From [18], for a single-server system with only a turn on server threshold k_1, the closed formed expression for $\pi_{0,0}$ is:

$$\pi_{0,0} = (1-\rho)\frac{\alpha\gamma}{k_1\alpha\gamma+\alpha\lambda+\lambda\gamma}$$

Our model can be easily converted into a single processing speed model by either setting $c = 1$ or $k_2 = \infty$, which means the scaled processing speed is the same as the nominal speed or we never turn on speed scaling. If we substitute $c = 1$ into our expression, we then have:

$$\begin{aligned}\pi_{0,0} &= \frac{\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)}{\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)((\alpha+\gamma)\lambda+\alpha\gamma k_1)} \\ &= \frac{\alpha\gamma(\mu-\lambda)}{\mu((\alpha+\gamma)\lambda+\alpha\gamma k_1)} \\ &= (1-\rho)\frac{\alpha\gamma}{k_1\alpha\gamma+\alpha\lambda+\lambda\gamma}\end{aligned}$$

This substitution gives us the same expression as developed previously in [18], and it gives us confidence for the correctness of our calculation. We then use this expression for $\pi_{0,0}$ to determine the expected number of jobs in the system.

We have expressed the steady-state probabilities for each region in terms of $\pi_{0,0}$. We can make use of this information to derive the number of expected jobs $\mathbb{E}[N]$ in the system by using a weighted sum of the corresponding probabilities.

$$\mathbb{E}[N] = \sum_{n=0}^{k_1-1} n\pi_{0,n} + \sum_{n=k_1}^{\infty} n\pi_{0,n} + \sum_{n=0}^{k_1} n\pi_{1,n} + \sum_{n=k_1+1}^{k_2-1} n\pi_{1,n} + \sum_{n=k_2}^{\infty} n\pi_{1,n} \tag{5.41}$$

Substitute the corresponding steady-state probabilities (5.1), (5.2), (5.4), (5.5), (5.8)

$$\begin{aligned}\mathbb{E}[N] = &\sum_{n=0}^{k_1-1} n\pi_{0,0} + \sum_{n=k_1}^{\infty} n\left(\frac{\lambda}{\lambda+\gamma}\right)^{n-(k_1-1)}\pi_{0,0} + \sum_{n=0}^{k_1} n\pi_{0,0}\left(\frac{\lambda}{\alpha}\rho^n + \rho\frac{1-\rho^n}{1-\rho}\right) \\ &+ \sum_{n=k_1+1}^{k_2-1} n\left(\frac{\lambda}{\alpha}\rho^n + \rho^{n-(k_1-1)}\frac{1-\rho^{k_1}}{1-\rho} + \sum_{i=1}^{n-k_1}\left(\frac{\lambda}{\lambda+\gamma}\right)^i\rho^{n-k_1-(i-1)}\right)\pi_{0,0} \\ &+ \sum_{n=k_2}^{\infty} n\left(\frac{\lambda^{n+1}}{\alpha c^{n-(k_2-1)}\mu^n} + \frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\frac{1-\rho^{k_1}}{1-\rho} + \sum_{i=1}^{k_2-k_1-1}\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}(\lambda+\gamma)^i\mu^{n-k_1-i+1}}\right. \\ &\left. - \frac{\lambda^{n-k_1+1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)(c\mu)^{n-(k_2-1)}} + \frac{\lambda^{n-k_1+1}}{(\lambda+\gamma)^{n-k_1}(c\mu-\lambda-\gamma)}\right)\pi_{0,0}\end{aligned}$$

$$
\begin{aligned}
\frac{\mathbb{E}[N]}{\pi_{0,0}} = & \sum_{n=0}^{k_1-1} n + \sum_{n=k_1}^{\infty} n\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-(k_1-1)} + \frac{\lambda}{\alpha}\sum_{n=0}^{k_1} n(\rho^n) \\
& + \frac{\lambda}{\alpha}\sum_{n=k_1+1}^{k_2-1} n(\rho^n) + \frac{\lambda}{\mu-\lambda}\sum_{n=0}^{k_1} n - \frac{\lambda}{\mu-\lambda}\sum_{n=0}^{k_1} n\rho^n + \frac{1-\rho^{k_1}}{1-\rho}\sum_{n=k_1+1}^{k_2-1} n(\rho^{n-(k_1-1)}) \\
& + \sum_{n=k_1+1}^{k_2-1} n\Big(\rho^{n-k_1+1}\sum_{i=1}^{n-k_1}(\frac{\mu}{\lambda+\gamma})^i\Big) + \sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n+1}}{\alpha c^{n-(k_2-1)}\mu^n}\Big) \\
& + \sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\frac{1-\rho^{k_1}}{1-\rho}\Big) + \sum_{n=k_2}^{\infty} n\Big(\sum_{i=1}^{k_2-1-k_1}\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}(\lambda+\gamma)^i\mu^{n-k_1+1-i}}\Big) \\
& - \sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)(c\mu)^{n-(k_2-1)}}\Big) + \sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{(\lambda+\gamma)^{n-k_1}(c\mu-\lambda-\gamma)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathbb{E}[N]}{\pi_{0,0}} = & \sum_{n=0}^{k_1-1} n + \sum_{n=k_1}^{\infty} n\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-(k_1-1)} + \frac{\lambda}{\alpha}\sum_{n=0}^{k_1} n(\rho^n) + \frac{\lambda}{\alpha}\sum_{n=k_1+1}^{k_2-1} n(\rho^n) + \frac{\lambda}{\mu-\lambda}\sum_{n=0}^{k_1} n \\
& - \frac{\lambda}{\mu-\lambda}\sum_{n=0}^{k_1} n\rho^n + \frac{1-\rho^{k_1}}{1-\rho}\sum_{n=k_1+1}^{k_2-1} n(\rho^{n-(k_1-1)}) + \sum_{n=k_1+1}^{k_2-1} n\Big(\rho^{n-k_1+1}\sum_{i=1}^{n-k_1}(\frac{\mu}{\lambda+\gamma})^i\Big) \\
& + \frac{\lambda c^{(k_2-1)}}{\alpha}\sum_{n=k_2}^{\infty} n\frac{\lambda^n}{c^n\mu^n} + \frac{1-\rho^{k_1}}{1-\rho}\frac{c^{k_2}\mu^{k_1}}{\lambda^{k_1}}\sum_{n=k_2}^{\infty} n(\frac{\lambda^{n+1}}{c^{n+1}\mu^{n+1}}) \\
& + \sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\sum_{i=1}^{k_2-1-k_1}\frac{\mu^i}{(\lambda+\gamma)^i}\Big) \\
& - \frac{(c\mu)^{k_2}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1}}\sum_{n=k_2}^{\infty} n(\frac{\lambda^{n+1}}{(c\mu)^{n+1}}) + \frac{(\lambda+\gamma)^{k_1}}{(c\mu-\lambda-\gamma)\lambda^{k_1}}\sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n+1}}{(\lambda+\gamma)^n}\Big)
\end{aligned}
$$

$$\frac{\mathbb{E}[N]}{\pi_{0,0}} = \sum_{n=0}^{k_1-1} n + \sum_{n=k_1}^{\infty} n\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-(k_1-1)} + \frac{\lambda}{\alpha}\sum_{n=0}^{k_1} n(\rho^n) + \frac{\lambda}{\mu-\lambda}\Big(\sum_{n=0}^{k_1} n - \sum_{n=1}^{k_1} n\rho^n\Big)$$
$$+ \frac{1-\rho^{k_1}}{1-\rho}\sum_{n=k_1+1}^{k_2-1} n\rho^{n-(k_1-1)} + \sum_{n=k_1+1}^{k_2-1} n\Big(\rho^{n-k_1+1}\sum_{i=1}^{n-k_1}(\frac{\mu}{\lambda+\gamma})^i\Big)$$
$$+ \Big(\frac{\mu c^{k_2}}{\alpha} + \frac{1-\rho^{k_1}}{1-\rho}\frac{c^{k_2}\mu^{k_1}}{\lambda^{k_1}} - \frac{(c\mu)^{k_2}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1}}\Big)\sum_{n=k_2}^{\infty} n(\frac{\lambda^{n+1}}{(c\mu)^{n+1}})$$
$$+ \sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\sum_{i=1}^{k_2-1-k_1}\frac{\mu^i}{(\lambda+\gamma)^i}\Big) + \frac{(\lambda+\gamma)^{k_1}}{(c\mu-\lambda-\gamma)\lambda^{k_1}}\sum_{n=k_2}^{\infty} n(\frac{\lambda^{n+1}}{(\lambda+\gamma)^n}) \tag{5.42}$$

We simplify the terms in the big bracket of the fourth sum of (5.42) in reverse order as follows:

$$\frac{\mu c^{k_2}}{\alpha} + \frac{1-\rho^{k_1}}{1-\rho}\frac{c^{k_2}\mu^{k_1}}{\lambda^{k_1}} - \frac{(c\mu)^{k_2}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1}}$$
$$= \frac{\mu c^{k_2}}{\alpha} + \frac{\mu(1-\rho^{k_1})}{\mu-\lambda}\frac{c^{k_2}\mu^{k_1}}{\lambda^{k_1}} - \frac{(c\mu)^{k_2}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1}}$$
$$= \mu c^{k_2}\Big(\frac{1}{\alpha} + \frac{(1-\rho^{k_1})}{\mu-\lambda}\frac{\mu^{k_1}}{\lambda^{k_1}} - \frac{\mu^{k_2-1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1}}\Big)$$
$$= \mu c^{k_2}\frac{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\big((\mu\lambda^{k_1}-\lambda^{k_1+1}) + (\alpha\mu^{k_1}-\alpha\lambda^{k_1})\big) - \mu^{k_2-1}(\mu-\lambda)\alpha}{\alpha(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1}(\mu-\lambda)}$$

So we have:

$$\frac{\mathbb{E}[N]}{\pi_{0,0}} = \sum_{n=0}^{k_1-1} n + \sum_{n=k_1}^{\infty} n\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-(k_1-1)} + \frac{\lambda}{\alpha}\sum_{n=0}^{k_1} n(\rho^n) + \frac{\lambda}{\mu-\lambda}\Big(\sum_{n=0}^{k_1} n - \sum_{n=1}^{k_1} n\rho^n\Big)$$
$$+ \frac{1-\rho^{k_1}}{1-\rho}\sum_{n=k_1+1}^{k_2-1} n\rho^{n-(k_1-1)} + \sum_{n=k_1+1}^{k_2-1} n\Big(\rho^{n-k_1+1}\sum_{i=1}^{n-k_1}(\frac{\mu}{\lambda+\gamma})^i\Big)$$
$$+ \Big(\mu c^{k_2}\frac{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\big((\mu\lambda^{k_1}-\lambda^{k_1+1}) + (\alpha\mu^{k_1}-\alpha\lambda^{k_1})\big) - \mu^{k_2-1}(\mu-\lambda)\alpha}{\alpha(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1}(\mu-\lambda)}\Big)\sum_{n=k_2}^{\infty} n(\frac{\lambda^{n+1}}{(c\mu)^{n+1}})$$
$$+ \sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\sum_{i=1}^{k_2-1-k_1}\frac{\mu^i}{(\lambda+\gamma)^i}\Big) + \frac{(\lambda+\gamma)^{k_1}}{(c\mu-\lambda-\gamma)\lambda^{k_1}}\sum_{n=k_2}^{\infty} n(\frac{\lambda^{n+1}}{(\lambda+\gamma)^n}) \tag{5.43}$$

We now proceed to simplify each of the sums in the RHS of (5.43).

The first sum on the RHS of (5.43):

$$\sum_{n=0}^{k_1-1} n = \frac{k_1(k_1-1)}{2} \tag{5.44}$$

The second sum on the RHS of (5.43) can be simplified as:

$$\begin{aligned}
&\sum_{n=k_1}^{\infty} n\Big(\frac{\lambda}{\lambda+\gamma}\Big)^{n-(k_1-1)} \\
&= \sum_{n=1}^{\infty} (n+(k_1-1))(\frac{\lambda}{\lambda+\gamma})^n \\
&= \sum_{n=1}^{\infty} n(\frac{\lambda}{\lambda+\gamma})^n + (k_1-1)\sum_{n=1}^{\infty}(\frac{\lambda}{\lambda+\gamma})^n \\
&= \frac{\lambda}{\lambda+\gamma}\sum_{n=0}^{\infty} n(\frac{\lambda}{\lambda+\gamma})^{n-1} + (k_1-1)\Big(\sum_{n=0}^{\infty}(\frac{\lambda}{\lambda+\gamma})^n - 1\Big) \\
&= \frac{\lambda}{\lambda+\gamma}\frac{d}{d(\frac{\lambda}{\lambda+\gamma})}\sum_{n=0}^{\infty}(\frac{\lambda}{\lambda+\gamma})^n + \frac{k_1-1}{1-\frac{\lambda}{\lambda+\gamma}} - (k_1-1) \\
&= \frac{\lambda}{\lambda+\gamma}\frac{1}{(1-\frac{\lambda}{\lambda+\gamma})^2} + \frac{(\lambda+\gamma)(k_1-1)-\gamma(k_1-1)}{\gamma} \\
&= \frac{\lambda(\lambda+\gamma)}{\gamma^2} + \frac{\lambda(k_1-1)}{\gamma} \\
&= \frac{\lambda(\lambda+k_1\gamma)}{\gamma^2}
\end{aligned} \tag{5.45}$$

Next, the third sum in the RHS of (5.43),

$$\begin{aligned}
&\frac{\lambda}{\alpha}\sum_{n=0}^{k_2-1} n\rho^n \\
&= \frac{\lambda}{\alpha}\rho\sum_{n=0}^{k_2-1} n\rho^{n-1} \\
&= \frac{\lambda}{\alpha}\rho\frac{d}{d(\rho)}\sum_{n=0}^{k_2-1} \rho^n \\
&= \frac{\lambda}{\alpha}\rho\frac{1-\rho^{k_2}-k_2\rho^{k_2-1}+k_2\rho^{k_2}}{(1-\rho)^2}
\end{aligned} \tag{5.46}$$

Next, the fourth sum in the RHS of (5.43),

$$\begin{aligned}
&\frac{\lambda}{\mu-\lambda}\Big(\sum_{n=0}^{k_1} n - \sum_{n=0}^{k_1} n\rho^n\Big) \\
&= \frac{\lambda}{\mu-\lambda}\Big(\frac{k_1(k_1+1)}{2} - \rho\frac{d}{d(\rho)}\sum_{n=0}^{k_1}\rho^n\Big) \\
&= \frac{\lambda}{\mu-\lambda}\Big(\frac{k_1(k_1+1)}{2} - \rho\frac{1-\rho^{k_1+1}-(k_1+1)\rho^{k_1}+(k_1+1)\rho^{k_1+1}}{(1-\rho)^2}\Big) \\
&= \frac{\lambda}{\mu-\lambda}\Big(\frac{k_1(k_1+1)(\mu-\lambda)^2}{2(\mu-\lambda)^2} - \frac{\rho\mu^2-\mu^2\rho^{k_1+2}-\mu^2(k_1+1)\rho^{k_1+1}+\mu^2(k_1+1)\rho^{k_1+2}}{(\mu-\lambda)^2}\Big) \\
&= \lambda\Big(\frac{k_1(k_1+1)(\mu-\lambda)^2-2\rho\mu^2+2\mu^2\rho^{k_1+2}+2\mu^2(k_1+1)\rho^{k_1+1}-2\mu^2(k_1+1)\rho^{k_1+2}}{2(\mu-\lambda)^3}\Big)
\end{aligned} \tag{5.47}$$

Next, the fifth sum on the RHS of (5.43)

$$
\begin{aligned}
&\frac{1-\rho^{k_1}}{1-\rho}\sum_{n=k_1+1}^{k_2-1} n\rho^{n-(k_1-1)} \\
&= \frac{\lambda(1-\rho^{k_1})}{\mu-\lambda}\sum_{n=k_1+1}^{k_2-1} n\rho^{n-k_1} \\
&= \frac{\lambda(1-\rho^{k_1})}{\mu-\lambda}\sum_{n=1}^{k_2-k_1-1} (n+k_1)\rho^{n} \\
&= \frac{\lambda(1-\rho^{k_1})}{\mu-\lambda}\Big(\sum_{n=1}^{k_2-k_1-1} n\rho^{n} + k_1\sum_{n=1}^{k_2-k_1-1}\rho^{n}\Big) \\
&= \frac{\lambda(1-\rho^{k_1})}{\mu-\lambda}\Big(\rho\frac{d}{d(\rho)}\sum_{n=1}^{k_2-k_1-1}\rho^{n} + k_1\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}\Big) \\
&= \frac{\lambda(1-\rho^{k_1})}{\mu-\lambda}\Big(\rho\frac{d}{d(\rho)}(\frac{\rho-\rho^{k_2-k_1}}{1-\rho}) + k_1\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}\Big) \\
&= \frac{\lambda(1-\rho^{k_1})}{\mu-\lambda}\Big(\rho\frac{(1-\rho)(1-(k_2-k_1)\rho^{k_2-k_1-1})+(\rho-\rho^{k_2-k_1})}{(1-\rho)^2} + k_1\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}\Big) \\
&= \lambda^2(1-\rho^{k_1})\Big(\frac{(\mu-\lambda)(1-(k_2-k_1)\rho^{k_2-k_1-1})+\lambda(1-\rho^{k_2-k_1-1})+k_1(\mu-\lambda)(1-\rho^{k_2-k_1-1})}{(\mu-\lambda)^3}\Big) \\
&= \lambda^2(1-\rho^{k_1})\Big(\frac{\mu+k_1(\mu-\lambda)-(k_2(\mu-\lambda)+\lambda)\rho^{k_2-k_1-1}}{(\mu-\lambda)^3}\Big)
\end{aligned}
\tag{5.48}
$$

Next, the sixth sum on the RHS of (5.43),

$$
\begin{aligned}
&\sum_{n=k_1+1}^{k_2-1} n\Big(\rho^{n-k_1+1}\sum_{i=1}^{n-k_1}(\frac{\mu}{\lambda+\gamma})^i\Big)\\
&=\sum_{n=k_1+1}^{k_2-1} n\Big(\rho^{n-k_1+1}\frac{\mu(1-(\frac{\mu}{\lambda+\gamma})^{n-k_1})}{\lambda+\gamma-\mu}\Big)\\
&=\frac{\lambda}{\lambda+\gamma-\mu}\sum_{n=k_1+1}^{k_2-1}\Big(n\rho^{n-k_1+1}-n\rho^{n-k_1+1}(\frac{\mu}{\lambda+\gamma})^{n-k_1}\Big)\\
&=\frac{\lambda}{\lambda+\gamma-\mu}\Big(\sum_{n=k_1+1}^{k_2-1}(n\rho^{n-k_1+1})-\rho\sum_{n=k_1+1}^{k_2-1}\Big(n(\frac{\lambda}{\lambda+\gamma})^{n-k_1}\Big)\Big)\\
&=\frac{\lambda}{\lambda+\gamma-\mu}\Big(\rho\sum_{n=1}^{k_2-k_1-1}(n+k_1)\rho^n-\rho\sum_{n=1}^{k_2-k_1-1}\Big((n+k_1)(\frac{\lambda}{\lambda+\gamma})^n\Big)\Big)\\
&=\frac{\lambda}{\lambda+\gamma-\mu}\Big(\rho\sum_{n=1}^{k_2-k_1-1}n\rho^{n-1}+k_1\sum_{n=1}^{k_2-k_1-1}\rho^n-\frac{\lambda}{\lambda+\gamma}\sum_{n=1}^{k_2-k_1-1}n(\frac{\lambda}{\lambda+\gamma})^{n-1}-k_1\sum_{n=1}^{k_2-k_1-1}(\frac{\lambda}{\lambda+\gamma})^n\Big)\\
&=\frac{\lambda}{\lambda+\gamma-\mu}\Big(\rho\frac{d}{d(\rho)}\sum_{n=1}^{k_2-k_1-1}\rho^n+k_1\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}-\frac{\lambda}{\lambda+\gamma}\frac{d}{d(\frac{\lambda}{\lambda+\gamma})}\sum_{n=1}^{k_2-k_1-1}n(\frac{\lambda}{\lambda+\gamma})^n\\
&-k_1\frac{\lambda(1-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1})}{\gamma}\Big)\\
&=\frac{\lambda}{\lambda+\gamma-\mu}\Big(\rho\frac{d}{d(\rho)}(\frac{\rho-\rho^{k_2-k_1}}{1-\rho})+k_1\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}-\frac{\lambda}{\lambda+\gamma}\frac{d}{d(\frac{\lambda}{\lambda+\gamma})}\Big(\frac{\frac{\lambda}{\lambda+\gamma}(1-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1})}{1-\frac{\lambda}{\lambda+\gamma}}\Big)\\
&-k_1\frac{\lambda(1-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1})}{\gamma}\Big)\\
&=\frac{\lambda}{\lambda+\gamma-\mu}\Big(\rho\frac{(1-\rho)(1-(k_2-k_1)\rho^{k_2-k_1-1})+\rho-\rho^{k_2-k_1}}{(1-\rho)^2}+k_1\frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho}\\
&-\frac{\lambda}{\lambda+\gamma}\frac{(1-\frac{\lambda}{\lambda+\gamma})(1-(k_2-k_1)(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1})+\frac{\lambda}{\lambda+\gamma}-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1}}{\Big(1-\frac{\lambda}{\lambda+\gamma}\Big)^2}-k_1\frac{\lambda(1-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1})}{\gamma}\Big)\\
&=\frac{\lambda^2}{\lambda+\gamma-\mu}\Big(\frac{\mu+k_1(\mu-\lambda)-(k_2(\mu-\lambda)+\lambda)\rho^{k_2-k_1-1}}{(\mu-\lambda)^2}-\frac{\gamma+k_1\gamma+\lambda-(k_2\gamma+\lambda)(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1}}{\gamma^2}\Big)
\end{aligned}
\tag{5.49}
$$

Next, the seventh sum on the RHS of (5.43),

$$\Big(\mu c^{k_2}\frac{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\big((\mu\lambda^{k_1}-\lambda^{k_1+1})+(\alpha\mu^{k_1}-\alpha\lambda^{k_1})\big)-\mu^{k_2-1}(\mu-\lambda)\alpha}{\alpha(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1}(\mu-\lambda)}\Big)\sum_{n=k_2}^{\infty} n\frac{\lambda^{n+1}}{(c\mu)^{n+1}} \tag{5.50}$$

We first focus on simplifying $\sum_{n=k_2}^{\infty} n\frac{\lambda^{n+1}}{(c\mu)^{n+1}}$:

$$\begin{aligned}
&\sum_{n=k_2}^{\infty} n\frac{\lambda^{n+1}}{(c\mu)^{n+1}}\\
&=(\frac{\lambda}{c\mu})^{k_2}\sum_{n=1}^{\infty}(n+k_2-1)(\frac{\lambda}{c\mu})^{n}\\
&=(\frac{\lambda}{c\mu})^{k_2}\Big(\sum_{n=1}^{\infty}n(\frac{\lambda}{c\mu})^{n}+(k_2-1)\sum_{n=1}^{\infty}(\frac{\lambda}{c\mu})^{n}\Big)\\
&=(\frac{\lambda}{c\mu})^{k_2}\Big(\frac{\lambda}{c\mu}\sum_{n=1}^{\infty}n(\frac{\lambda}{c\mu})^{n-1}+(k_2-1)\frac{\frac{\lambda}{c\mu}}{1-\frac{\lambda}{c\mu}}\Big)\\
&=(\frac{\lambda}{c\mu})^{k_2}\Big(\frac{\lambda}{c\mu}\frac{d}{d(\frac{\lambda}{c\mu})}\Big(\frac{\frac{\lambda}{c\mu}}{1-\frac{\lambda}{c\mu}}\Big)+(k_2-1)\frac{\lambda}{c\mu-\lambda}\Big)\\
&=(\frac{\lambda}{c\mu})^{k_2}\Big(\frac{\lambda}{c\mu}\frac{(c\mu)^2}{(c\mu-\lambda)^2}+(k_2-1)\frac{\lambda}{c\mu-\lambda}\Big)\\
&=(\frac{\lambda}{c\mu})^{k_2}\Big(\frac{\lambda c\mu+(k_2-1)(c\mu-\lambda)\lambda}{(c\mu-\lambda)^2}\Big)\\
&=(\frac{\lambda}{c\mu})^{k_2}\lambda\Big(\frac{c\mu+k_2c\mu-k_2\lambda-c\mu+\lambda}{(c\mu-\lambda)^2}\Big)\\
&=\lambda(\frac{\lambda}{c\mu})^{k_2}\Big(\frac{k_2c\mu-k_2\lambda+\lambda}{(c\mu-\lambda)^2}\Big)
\end{aligned}$$

So (5.50)becomes:

$$\begin{aligned}
&\Big(\mu c^{k_2}\frac{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\big((\mu\lambda^{k_1}-\lambda^{k_1+1})+(\alpha\mu^{k_1}-\alpha\lambda^{k_1})\big)-\mu^{k_2-1}(\mu-\lambda)\alpha}{\alpha(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1}(\mu-\lambda)}\Big)\lambda(\frac{\lambda}{c\mu})^{k_2}\Big(\frac{k_2c\mu-k_2\lambda+\lambda}{(c\mu-\lambda)^2}\Big)\\
&=\Big(\mu\frac{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\big((\mu\lambda^{k_1}-\lambda^{k_1+1})+(\alpha\mu^{k_1}-\alpha\lambda^{k_1})\big)-\mu^{k_2-1}(\mu-\lambda)\alpha}{\alpha(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1-1}(\mu-\lambda)}\Big)(\frac{\lambda}{\mu})^{k_2}\Big(\frac{k_2c\mu-k_2\lambda+\lambda}{(c\mu-\lambda)^2}\Big)
\end{aligned} \tag{5.51}$$

and we continue with the eighth sum on the RHS of (5.43),

$$
\begin{aligned}
&\sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\sum_{i=1}^{k_2-1-k_1}\frac{\mu^i}{(\lambda+\gamma)^i}\Big)\\
&=\sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\frac{(\frac{\mu}{\lambda+\gamma})(1-(\frac{\mu}{\lambda+\gamma})^{k_2-1-k_1})}{1-\frac{\mu}{\lambda+\gamma}}\Big)\\
&=\sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\frac{1-(\frac{\mu}{\lambda+\gamma})^{k_2-1-k_1}}{\lambda+\gamma-\mu}\Big)\\
&=\frac{1}{\lambda+\gamma-\mu}\sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\Big(1-(\frac{\mu}{\lambda+\gamma})^{k_2-1-k_1}\Big)\Big)\\
&=\frac{1}{\lambda+\gamma-\mu}\Big(\sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\Big)-\sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}(\frac{\mu}{\lambda+\gamma})^{k_2-1-k_1}\Big)\Big)\\
&=\frac{1}{\lambda+\gamma-\mu}\Big(\sum_{n=k_2}^{\infty} n\Big(\frac{\lambda^{n-k_1+1}}{c^{n-(k_2-1)}\mu^{n-k_1+1}}\Big)-(\frac{\mu}{\lambda+\gamma})^{k_2-1-k_1}\frac{\lambda^{k_2-k_1+1}}{c\mu^{k_2-k_1}}\sum_{n=0}^{\infty}(n+k_2)\frac{\lambda^n}{(c\mu)^n}\Big)\\
&=\frac{1}{\lambda+\gamma-\mu}\Big(\frac{\lambda^{k_2-k_1+1}}{c\mu^{k_2-k_1}}\sum_{n=0}^{\infty}(n+k_2)(\frac{\lambda}{c\mu})^n-(\frac{1}{\lambda+\gamma})^{k_2-k_1-1}\frac{\lambda^{k_2-k_1+1}}{c\mu}\sum_{n=0}^{\infty}(n+k_2)(\frac{\lambda}{c\mu})^n\Big)\\
&=\frac{\lambda^{k_2-k_1+1}}{c(\lambda+\gamma-\mu)}\Big(\Big(\frac{1}{\mu^{k_2-k_1}}-(\frac{1}{\lambda+\gamma})^{k_2-k_1-1}\frac{1}{\mu}\Big)\sum_{n=0}^{\infty}(n+k_2)(\frac{\lambda}{c\mu})^n\Big)\\
&=\frac{\lambda^{k_2-k_1+1}}{c(\lambda+\gamma-\mu)}\Big(\Big(\frac{(\lambda+\gamma)^{k_2-k_1-1}-\mu^{k_2-k_1-1}}{\mu^{k_2-k_1}(\lambda+\gamma)^{k_2-k_1-1}}\Big)\Big(\frac{\lambda}{c\mu}\sum_{n=0}^{\infty}n(\frac{\lambda}{c\mu})^{n-1}+k_2\frac{c\mu}{c\mu-\lambda}\Big)\Big)\\
&=\frac{\lambda^{k_2-k_1+1}((\lambda+\gamma)^{k_2-k_1-1}-\mu^{k_2-k_1-1})}{c(\lambda+\gamma-\mu)\mu^{k_2-k_1}(\lambda+\gamma)^{k_2-k_1-1}}\Big(\frac{\lambda}{c\mu}\frac{(c\mu)^2}{(c\mu-\lambda)^2}+k_2\frac{c\mu}{c\mu-\lambda}\Big)\\
&=\frac{\lambda^{k_2-k_1+1}((\lambda+\gamma)^{k_2-k_1-1}-\mu^{k_2-k_1-1})}{(\lambda+\gamma-\mu)\mu^{k_2-k_1-1}(\lambda+\gamma)^{k_2-k_1-1}}\Big(\lambda\frac{1}{(c\mu-\lambda)^2}+k_2\frac{1}{c\mu-\lambda}\Big)\\
&=\frac{\lambda^{k_2-k_1+1}((\lambda+\gamma)^{k_2-k_1-1}-\mu^{k_2-k_1-1})}{(\lambda+\gamma-\mu)\mu^{k_2-k_1-1}(\lambda+\gamma)^{k_2-k_1-1}}\Big(\frac{\lambda+(c\mu k_2-\lambda k_2)}{(c\mu-\lambda)^2}\Big)
\end{aligned}
\tag{5.52}
$$

Finally, we simplify the last sum on the RHS of (5.43),

$$
\begin{aligned}
&\frac{(\lambda+\gamma)^{k_1}}{(c\mu-\lambda-\gamma)\lambda^{k_1}}\sum_{n=k_2}^{\infty} n\frac{\lambda^{n+1}}{(\lambda+\gamma)^n}\\
&=\frac{(\lambda+\gamma)^{k_1}\lambda}{(c\mu-\lambda-\gamma)\lambda^{k_1}}\sum_{n=0}^{\infty}(n+k_2)\frac{\lambda^{n+k_2}}{(\lambda+\gamma)^{n+k_2}}\\
&=\frac{(\lambda+\gamma)^{k_1}\lambda}{(c\mu-\lambda-\gamma)\lambda^{k_1}}(\frac{\lambda}{\lambda+\gamma})^{k_2}\sum_{n=0}^{\infty}(n+k_2)(\frac{\lambda}{\lambda+\gamma})^n\\
&=\frac{(\lambda+\gamma)^{k_1}\lambda}{(c\mu-\lambda-\gamma)\lambda^{k_1}}(\frac{\lambda}{\lambda+\gamma})^{k_2}\Big(\sum_{n=0}^{\infty}n(\frac{\lambda}{\lambda+\gamma})^n+k_2\sum_{n=0}^{\infty}(\frac{\lambda}{\lambda+\gamma})^n\Big)\\
&=\frac{(\lambda+\gamma)^{k_1}\lambda}{(c\mu-\lambda-\gamma)\lambda^{k_1}}(\frac{\lambda}{\lambda+\gamma})^{k_2}\Big((\frac{\lambda}{\lambda+\gamma})\sum_{n=0}^{\infty}n(\frac{\lambda}{\lambda+\gamma})^{n-1}+k_2\frac{1}{1-\frac{\lambda}{\lambda+\gamma}}\Big)\\
&=\frac{(\lambda+\gamma)^{k_1}\lambda}{(c\mu-\lambda-\gamma)\lambda^{k_1}}(\frac{\lambda}{\lambda+\gamma})^{k_2}\Big((\frac{\lambda}{\lambda+\gamma})\frac{d}{d(\frac{\lambda}{\lambda+\gamma})}\Big(\frac{1}{1-\frac{\lambda}{\lambda+\gamma}}\Big)+k_2\frac{\lambda+\gamma}{\gamma}\Big)\\
&=\frac{(\lambda+\gamma)^{k_1}\lambda}{(c\mu-\lambda-\gamma)\lambda^{k_1}}(\frac{\lambda}{\lambda+\gamma})^{k_2}\Big(\frac{\lambda}{\lambda+\gamma}\frac{(\lambda+\gamma)^2}{\gamma^2}+k_2\frac{\lambda+\gamma}{\gamma}\Big)\\
&=\frac{(\lambda+\gamma)^{k_1}\lambda}{(c\mu-\lambda-\gamma)\lambda^{k_1}}(\frac{\lambda}{\lambda+\gamma})^{k_2}\frac{\lambda+\gamma}{\gamma}(\frac{\lambda}{\gamma}+k_2)\\
&=\frac{\lambda^{k_2-k_1+1}}{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\gamma}(\frac{\lambda}{\gamma}+k_2)
\end{aligned}
\tag{5.53}
$$

We continue the simplification by summing up the simplified sums from (5.44) to (5.53).

Summing (5.44) and (5.45):

$$
\begin{aligned}
&\frac{k_1(k_1-1)}{2}+\frac{\lambda(\lambda+k_1\gamma)}{\gamma^2}\\
&=\frac{k_1(k_1-1)\gamma^2+2\lambda(\lambda+k_1\gamma)}{2\gamma^2}\\
&=\frac{({k_1}^2\gamma^2-k_1\gamma^2)+(2\lambda^2+2k_1\gamma\lambda)}{2\gamma^2}\\
&=\frac{{k_1}^2\gamma^2+2k_1\gamma\lambda+\lambda^2-k_1\gamma^2+\lambda^2}{2\gamma^2}\\
&=\frac{(k_1\gamma+\lambda)^2-k_1\gamma^2+\lambda^2}{2\gamma^2}
\end{aligned}
\tag{5.54}
$$

and (5.54) and (5.46) :

$$\begin{aligned} &\frac{(k_1\gamma+\lambda)^2-k_1\gamma^2+\lambda^2}{2\gamma^2}+\frac{\lambda}{\alpha}\rho\frac{1-\rho^{k2}-k_2\rho^{k2-1}+k_2\rho^{k2}}{(1-\rho)^2} \\ &=\frac{(k_1\gamma+\lambda)^2-k_1\gamma^2+\lambda^2}{2\gamma^2}+\frac{\lambda^2\mu^{k2}-\lambda^{k2+2}(1+k_2\frac{\mu}{\lambda}-k_2)}{\alpha\mu^{k2-1}(\mu-\lambda)^2} \\ &=\frac{(k_1\gamma+\lambda)^2\alpha\mu^{k2-1}(\mu-\lambda)^2-k_1\gamma^2\alpha\mu^{k2-1}(\mu-\lambda)^2+\lambda^2\alpha\mu^{k2-1}(\mu-\lambda)^2+2\gamma^2\lambda^2\mu^{k2}-2\gamma^2\lambda k_2+2(1+k_2\frac{\mu}{\lambda}-k_2)}{2\gamma^2\alpha\mu^{k2-1}(\mu-\lambda)^2} \end{aligned} \tag{5.55}$$

Next, we sum (5.55) and (5.47). The common denominator between (5.55) and (5.47) is $2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3$, we first simplify (5.47) by multiplying the numerator by $\gamma^2\alpha\mu^{k_2-1}$ yielding

$$\begin{aligned} &=\lambda\Big(\frac{k_1(k_1+1)(\mu-\lambda)^2-2\rho\mu^2+2\mu^2\rho^{k_1+2}+2\mu^2(k_1+1)\rho^{k_1+1}-2\mu^2(k_1+1)\rho^{k_1+2}}{2(\mu-\lambda)^3}\Big) \\ &=\frac{k_1(k_1+1)(\mu-\lambda)^2\lambda-2\mu\lambda^2+2\mu^{-k_1}\lambda^{k_1+3}+2\mu^{1-k_1}(k_1+1)\lambda^{k_1+2}-2\mu^{-k_1}(k_1+1)\lambda^{k_1+3}}{2(\mu-\lambda)^3} \end{aligned}$$

So the numerator of (5.47) becomes:

$$\begin{aligned} &\Big(k_1(k_1+1)(\mu-\lambda)^2\lambda-2\mu\lambda^2+2\mu^{-k_1}\lambda^{k_1+3}+2\mu^{1-k_1}(k_1+1)\lambda^{k_1+2}-2\mu^{-k_1}(k_1+1)\lambda^{k_1+3}\Big)\gamma^2\alpha\mu^{k_2-1} \\ &=k_1(k_1+1)(\mu-\lambda)^2\lambda\gamma^2\alpha\mu^{k_2-1}-2\alpha\gamma^2\mu^{k_2}\lambda^2+2\alpha\gamma^2\mu^{k_2-k_1-1}\lambda^{k_1+3}+2\alpha\gamma^2\mu^{k_2-k_1}(k_1+1)\lambda^{k_1+2} \\ &\quad-2\alpha\gamma^2\mu^{k_2-k_1-1}(k_1+1)\lambda^{k_1+3} \end{aligned} \tag{5.56}$$

Similarly, to have the same common denominator of $2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3$, the numerator of (5.55) becomes:

$$\begin{aligned} &(k_1\gamma+\lambda)^2\alpha\mu^{k_2-1}(\mu-\lambda)^3-k_1\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3+\lambda^2\alpha\mu^{k_2-1}(\mu-\lambda)^3+2\gamma^2\lambda^2\mu^{k_2}(\mu-\lambda) \\ &\qquad-2\gamma^2\lambda^{k_2+2}(1+k_2\frac{\mu}{\lambda}-k_2)(\mu-\lambda) \end{aligned} \tag{5.57}$$

There are similar terms in both of the numerators of (5.57) and (5.56) when we sum the two terms, so we continue the simplifications by merging similar terms one by one:

We first merge terms that containing μ^{k_2-1}:

$$
\begin{aligned}
& k_1(k_1+1)(\mu-\lambda)^2\lambda\gamma^2\alpha\mu^{k_2-1} + (k_1\gamma+\lambda)^2\alpha\mu^{k_2-1}(\mu-\lambda)^3 - k_1\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3 \\
& + \lambda^2\alpha\mu^{k_2-1}(\mu-\lambda)^3 \\
&= \alpha\mu^{k_2-1}(\mu-\lambda)^2\Big({k_1}^2\lambda\gamma^2 + k_1\lambda\gamma^2 + \mu(k_1\gamma)^2 + 2k_1\gamma\lambda\mu + \mu\lambda^2 - \mu k_1\gamma^2 + \mu\lambda^2 - \lambda(k_1\gamma)^2 \\
& - 2k_1\gamma\lambda^2 + \lambda k_1\gamma^2 - 2\lambda^3\Big) \\
&= \alpha\mu^{k_2-1}(\mu-\lambda)^2\Big(2k_1\lambda\gamma^2 - 2k_1\gamma\lambda^2 - 2\lambda^3 + \mu(k_1\gamma)^2 + 2k_1\gamma\lambda\mu - \mu k_1\gamma^2 + 2\mu\lambda^2\Big) \\
&= \alpha\mu^{k_2-1}(\mu-\lambda)^2\Big(2\lambda(k_1\gamma^2 - k_1\gamma\lambda - \lambda^2) + \mu(k_1\gamma)^2 + 2k_1\gamma\lambda\mu - \mu k_1\gamma^2 + 2\mu\lambda^2\Big) \\
&= \alpha\mu^{k_2-1}(\mu-\lambda)^2\Big(({k_1}^2+k_1)\lambda\gamma^2 + (\mu-\lambda)\big((k_1\gamma+\lambda)^2 - k_1\gamma^2 + \lambda^2\big)\Big)
\end{aligned}
\tag{5.58}
$$

and then we merge terms containing μ^{k_2}:

$$
\begin{aligned}
& -2\alpha\gamma^2\mu^{k_2}\lambda^2 + 2\gamma^2\lambda^2\mu^{k_2}(\mu-\lambda) \\
&= 2\gamma^2\lambda^2\mu^{k_2}(\mu-\lambda-\alpha)
\end{aligned}
\tag{5.59}
$$

Next, we merge the terms containing $\mu^{k_2-k_1-1}$:

$$
\begin{aligned}
& 2\alpha\gamma^2\mu^{k_2-k_1-1}\lambda^{k_1+3} - 2\alpha\gamma^2\mu^{k_2-k_1-1}(k_1+1)\lambda^{k_1+3} \\
&= 2\alpha\gamma^2\lambda^{k_1+3}\mu^{k_2-k_1-1}(1-(k_1+1)) \\
&= -2k_1\alpha\gamma^2\lambda^{k_1+3}\mu^{k_2-k_1-1}
\end{aligned}
\tag{5.60}
$$

and we also merge terms contain $\mu^{k_2-k_1}$:

$$2\alpha\gamma^2\mu^{k_2-k_1}(k_1+1)\lambda^{k_1+2} \tag{5.61}$$

Combining the terms containing $\mu^{k_2-k_1}$ (5.61) and $\mu^{k_2-k_1-1}$ (5.60)

$$\begin{aligned} & 2\alpha\gamma^2\mu^{k_2-k_1}(k_1+1)\lambda^{k_1+2} - 2k_1\alpha\gamma^2\lambda^{k_1+3}\mu^{k_2-k_1-1} \\ &= 2\alpha\gamma^2\lambda^{k_1+2}\mu^{k_2-k_1-1}(\mu(k_1+1)-k_1\lambda) \\ &= 2\alpha\gamma^2\lambda^{k_1+2}\mu^{k_2-k_1-1}(k_1(\mu-\lambda)+\mu) \end{aligned} \tag{5.62}$$

Combining terms that contain μ^{k_2} (5.59) and μ^{k_2-1} (5.58)

$$\begin{aligned} & \alpha\mu^{k_2-1}(\mu-\lambda)^2(2\lambda(k_1\gamma^2-k_1\gamma\lambda-\lambda^2)+\mu(k_1\gamma)^2+2k_1\gamma\lambda\mu-\mu k_1\gamma^2+2\mu\lambda^2)+2\gamma^2\lambda^2\mu^{k_2}(\mu-\lambda-\alpha) \\ &= \mu^{k_2-1}\Big(\alpha(\mu-\lambda)^2(2\lambda(k_1\gamma^2-k_1\gamma\lambda-\lambda^2)+\mu(k_1\gamma)^2+2k_1\gamma\lambda\mu-\mu k_1\gamma^2+2\mu\lambda^2) \\ &+ 2\gamma^2\lambda^2\mu(\mu-\lambda-\alpha)\Big) \end{aligned} \tag{5.63}$$

With the simplifications we have done in (5.62) and (5.63), the sum of (5.55) and (5.47) becomes:

$$\begin{aligned} & \frac{2\alpha\gamma^2\lambda^{k_1+2}\mu^{k_2-k_1-1}(\mu k_1-k_1\lambda+\mu)}{2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3} \\ &+ \frac{\mu^{k_2-1}\Big(\alpha(\mu-\lambda)^2\big(2\lambda(k_1\gamma^2-k_1\gamma\lambda-\lambda^2)+\mu(k_1\gamma)^2+2k_1\gamma\lambda\mu-\mu k_1\gamma^2+2\mu\lambda^2\big)+2\gamma^2\lambda^2\mu(\mu-\lambda-\alpha)\Big)}{2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3} \\ &- \frac{2\gamma^2\lambda^{k_2+2}(1+k_2\frac{\mu}{\lambda}-k_2)(\mu-\lambda)}{2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3} \end{aligned} \tag{5.64}$$

We continue to merge the summation terms, and sum up (5.64) and (5.48). We examine (5.48) alone for the common denominator $2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3$, in order to get to this form, we need to multiply (5.48) by $2\gamma^2\alpha$, yielding

$$\lambda^2(1-\rho^{k_1})\Big(\frac{\mu+k_1(\mu-\lambda)-(k_2(\mu-\lambda)+\lambda)\rho^{k_2-k_1-1}}{(\mu-\lambda)^3}\Big)$$
$$=(\lambda^2\mu^{k_1}-\lambda^{k_1+2})\Big(\frac{\mu^{k_2-k_1}+k_1(\mu-\lambda)\mu^{k_2-k_1-1}-(k_2(\mu-\lambda)+\lambda)\lambda^{k_2-k_1-1}}{\mu^{k_2-1}(\mu-\lambda)^3}\Big)$$

Multiply both numerator and denominator by $2\gamma^2\alpha$:

$$2\gamma^2\alpha(\lambda^2\mu^{k_1}-\lambda^{k_1+2})\Big(\frac{\mu^{k_2-k_1}+k_1(\mu-\lambda)\mu^{k_2-k_1-1}-(k_2(\mu-\lambda)+\lambda)\lambda^{k_2-k_1-1}}{2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3}\Big)$$

We now focus on simplying the numerator.

$$\begin{aligned}&(2\alpha\gamma^2\lambda^2\mu^{k_1}-2\alpha\gamma^2\lambda^{k_1+2})\Big(\mu^{k_2-k_1}+k_1(\mu-\lambda)\mu^{k_2-k_1-1}-\big(k_2(\mu-\lambda)+\lambda\big)\lambda^{k_2-k_1-1}\Big)\\&=2\alpha\gamma^2\lambda^2\mu^{k_2}+2\alpha\gamma^2\lambda^2k_1(\mu-\lambda)\mu^{k_2-1}-2\alpha\gamma^2\mu^{k_1}\big(k_2(\mu-\lambda)+\lambda\big)\lambda^{k_2-k_1+1}\\&\quad-2\alpha\gamma^2\lambda^{k_1+2}\mu^{k_2-k_1}-2\alpha\gamma^2k_1(\mu-\lambda)\lambda^{k_1+2}\mu^{k_2-k_1-1}+2\alpha\gamma^2\big(k_2(\mu-\lambda)+\lambda\big)\lambda^{k_2+1}\end{aligned} \tag{5.65}$$

There are no additional changes for (5.64), and we have the numerator of (5.64) as:

$$\begin{aligned}&2\alpha\gamma^2\lambda^{k_1+2}\mu^{k_2-k_1-1}(\mu k_1-k_1\lambda+\mu)\\&+\mu^{k_2-1}\Big(\alpha(\mu-\lambda)^2(2\lambda(k_1\gamma^2-k_1\gamma\lambda-\lambda^2)+\mu(k_1\gamma)^2+2k_1\gamma\lambda\mu-\mu k_1\gamma^2+2\mu\lambda^2)\\&+2\gamma^2\lambda^2\mu(\mu-\lambda-\alpha)\Big)-2\gamma^2\lambda^{k_2+2}(1+k_2\frac{\mu}{\lambda}-k_2)(\mu-\lambda)\end{aligned} \tag{5.66}$$

Similarly, we merge similar terms from both (5.66) and (5.65) as we sum (5.66) and (5.65). We first merge terms containing μ^{k_2}:

$$\begin{aligned}
&2\alpha\gamma^2\lambda^2 + \alpha(\mu-\lambda)^2\Big((k_1\gamma)^2 + 2k_1\gamma\lambda - k_1\gamma^2 + 2\lambda^2\Big) + 2\gamma^2\lambda^2(\mu-\lambda-\alpha) \\
&= 2\alpha\gamma^2\lambda^2 + 2\gamma^2\lambda^2(\mu-\lambda-\alpha) + \alpha(\mu-\lambda)^2((k_1\gamma)^2 + 2k_1\gamma\lambda - k_1\gamma^2 + 2\lambda^2) \\
&= 2\gamma^2\lambda^2(\alpha + (\mu-\lambda-\alpha)) + \alpha(\mu-\lambda)^2((k_1\gamma)^2 + 2k_1\gamma\lambda - k_1\gamma^2 + 2\lambda^2) \\
&= 2\gamma^2\lambda^2(\mu-\lambda) + \alpha(\mu-\lambda)^2((k_1\gamma)^2 + 2k_1\gamma\lambda - k_1\gamma^2 + 2\lambda^2) \qquad (5.67)
\end{aligned}$$

and we merge the terms that contain μ^{k_2-1}:

$$\begin{aligned}
&2\alpha\gamma^2\lambda^2k_1(\mu-\lambda) + 2\lambda\alpha(\mu-\lambda)^2(k_1\gamma^2 - k_1\gamma\lambda - \lambda^2) \\
&= 2\alpha\lambda(\mu-\lambda)(\gamma^2\lambda k_1 + (\mu-\lambda)(k_1\gamma^2 - k_1\gamma\lambda - \lambda^2)) \\
&= 2\alpha\lambda(\mu-\lambda)(\gamma^2\lambda k_1 + \mu k_1\gamma^2 - \mu k_1\gamma\lambda - \mu\lambda^2 - \lambda k_1\gamma^2 + k_1\gamma\lambda^2 + \lambda^3) \\
&= 2\alpha\lambda(\mu-\lambda)(\mu k_1\gamma^2 - \mu k_1\gamma\lambda - \mu\lambda^2 + k_1\gamma\lambda^2 + \lambda^3) \qquad (5.68)
\end{aligned}$$

Combining (5.67) and (5.68):

$$
\begin{aligned}
&2\mu\gamma^2\lambda^2(\mu-\lambda)+\alpha\mu(\mu-\lambda)^2((k_1\gamma)^2+2k_1\gamma\lambda-k_1\gamma^2+2\lambda^2)\\
&\quad+2\alpha\lambda(\mu-\lambda)(\mu k_1\gamma^2-\mu k_1\gamma\lambda-\mu\lambda^2+k_1\gamma\lambda^2+\lambda^3)\\
&=2\lambda(\mu-\lambda)(\mu\gamma^2\lambda+\alpha\mu k_1\gamma^2-\alpha\mu k_1\gamma\lambda-\alpha\mu\lambda^2+\alpha k_1\gamma\lambda^2+\alpha\lambda^3)\\
&\quad+\alpha\mu(\mu-\lambda)^2((k_1\gamma)^2+2k_1\gamma\lambda-k_1\gamma^2+2\lambda^2)\\
&=(\mu-\lambda)(2\mu\gamma^2\lambda^2+2\alpha\mu k_1\lambda\gamma^2-2\alpha\mu k_1\gamma\lambda^2-2\alpha\mu\lambda^3+2\alpha k_1\gamma\lambda^3+2\alpha\lambda^4+\alpha\mu^2(k_1\gamma)^2+\alpha\mu^2 2k_1\gamma\lambda\\
&\quad-\alpha\mu^2k_1\gamma^2+2\alpha\mu^2\lambda^2-\alpha\mu\lambda(k_1\gamma)^2-2\alpha\mu k_1\gamma\lambda^2+\alpha\mu\lambda k_1\gamma^2-2\alpha\mu\lambda^3)\\
&=(\mu-\lambda)\Big(2\mu\gamma^2\lambda^2+\alpha\mu k_1\gamma^2((\mu-\lambda)(k_1-1)+2\lambda)-2\alpha\mu\lambda^2(2k_1\gamma-\mu)-2\alpha\lambda^3(2\mu-k_1\gamma)\\
&\quad+2\alpha\lambda^4+2\alpha\mu^2k_1\gamma\lambda\Big)
\end{aligned}
\tag{5.69}
$$

and we have terms that contain μ^{k_1}:

$$-2\alpha\gamma^2\mu^{k_1}(k_2(\mu-\lambda)+\lambda)\lambda^{k_2-k_1+1} \tag{5.70}$$

and terms that contain $\mu^{k_2-k_1}$:

$$-2\alpha\gamma^2\lambda^{k_1+2} \tag{5.71}$$

and we can merge the terms that contain $\mu^{k_2-k_1-1}$:

$$
\begin{aligned}
&-2\alpha\gamma^2k_1(\mu-\lambda)\lambda^{k_1+2}+2\alpha\gamma^2\lambda^{k_1+2}(\mu k_1-k_1\lambda+\mu)\\
&=2\alpha\gamma^2\lambda^{k_1+2}((\mu k_1-k_1\lambda+\mu)-\mu k_1+k_1\lambda)\\
&=2\alpha\mu\gamma^2\lambda^{k_1+2}
\end{aligned}
\tag{5.72}
$$

By combining (5.68) and (5.72), we get:

$$-2\alpha\mu\gamma^2\lambda^{k_1+2} + 2\alpha\mu\gamma^2\lambda^{k_1+2} = 0$$

Also we can simplify the terms that contain λ^{k_2+1} and λ^{k_2+2}:

$$\begin{aligned}
&2\alpha\gamma^2(k_2(\mu-\lambda)+\lambda)\lambda^{k_2+1} - 2\gamma^2\lambda^{k_2+2}(1+k_2\frac{\mu}{\lambda}-k_2)(\mu-\lambda) \\
&= 2\gamma^2\lambda^{k_2+1}(k_2\mu(\alpha+2\lambda-\mu)-\lambda\mu-\lambda(k_2-1)(\lambda+\alpha))
\end{aligned} \tag{5.73}$$

We can now substitute the results from (5.67) to (5.73) in the sum of (5.64) and (5.48), and we have:

$$\begin{aligned}
&\frac{(\mu-\lambda)\mu^{k_2-1}\left(2\mu\gamma^2\lambda^2+\alpha\mu k_1\gamma^2((\mu-\lambda)(k_1-1)+2\lambda)-2\alpha\mu\lambda^2(2k_1\gamma-\mu)\right)}{2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3} - \frac{2\alpha\lambda^3(2\mu-k_1\gamma)-2\alpha\lambda^4-2\alpha\mu^2k_1\gamma\lambda}{2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3} \\
&-\frac{2\alpha\gamma^2\mu^{k_1}\left(k_2(\mu-\lambda)+\lambda\right)\lambda^{k_2-k_1+1}-2\gamma^2\lambda^{k_2+1}\left(k_2\mu(\alpha+2\lambda-\mu)-\lambda\mu-\lambda(k_2-1)(\lambda+\alpha)\right)}{2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3}
\end{aligned} \tag{5.74}$$

The equation (5.74) represents the simplification for the first to fifth sums on the RHS of (5.43), we continue the simplification for (5.43) by merging the seventh sum (5.51) and the ninth sum (5.53), and we have:

$$\begin{aligned}
&\frac{\alpha(\frac{\lambda}{\gamma}+k_2)(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)^2-(k_2c\mu-k_2\lambda+\lambda)\mu^{k_2-1}(\mu-\lambda)\alpha\gamma}{\alpha(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\gamma\lambda^{k_1-k_2-1}(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)} \\
&+\frac{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)(k_2c\mu-k_2\lambda+\lambda)\gamma((\mu\lambda^{k_1}-\lambda^{k_1+1})+\alpha(\mu^{k_1}-\lambda^{k_1}))}{\alpha(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\gamma\lambda^{k_1-k_2-1}(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)} \\
&=\frac{\alpha(\mu-\lambda)\mu^{k_2-1}((\frac{\lambda}{\gamma}+k_2)(c\mu-\lambda)^2-k_2(c\mu-\lambda)\gamma-\gamma\lambda)}{\alpha(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\gamma\lambda^{k_1-k_2-1}(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)} \\
&+\frac{(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)(k_2c\mu-k_2\lambda+\lambda)\gamma((\mu\lambda^{k_1}-\lambda^{k_1+1})+\alpha(\mu^{k_1}-\lambda^{k_1}))}{\alpha(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\gamma\lambda^{k_1-k_2-1}(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)}
\end{aligned} \tag{5.75}$$

We now also merge the eighth sum term (5.52) of (5.43) with (5.75).

We first force (5.52) to have a common denominator with (5.75), so (5.52) becomes:

$$\frac{\alpha\gamma\mu^{k_1}(c\mu-\lambda-\gamma)(\mu-\lambda)\big((\lambda+\gamma)^{k_2-k_1-1}-\mu^{k_2-k_1-1}\big)\big(\lambda+k_2(c\mu-\lambda)\big)}{\alpha\gamma(\lambda+\gamma-\mu)\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1-k_2-1}(\mu-\lambda)(c\mu-\lambda)^2} \tag{5.76}$$

And now we also need to make (5.75) to have the same common denominator, which only requires multiplication by $(\lambda+\gamma-\mu)$. For better readability, we only show the numerator:

$$\begin{aligned}&\alpha(\mu-\lambda)\mu^{k_2-1}(\lambda+\gamma-\mu)((\frac{\lambda}{\gamma}+k_2)(c\mu-\lambda)^2-k_2(c\mu-\lambda)\gamma-\gamma\lambda)\\&+(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)(k_2c\mu-k_2\lambda+\lambda)\gamma((\mu\lambda^{k_1}-\lambda^{k_1+1})+\alpha(\mu^{k_1}-\lambda^{k_1}))\end{aligned} \tag{5.77}$$

We now simplify the numerator of the sum for the seventh, eighth, and ninth sums of (5.43).

$$\begin{aligned}&\alpha(\mu-\lambda)\mu^{k_2-1}(\lambda+\gamma-\mu)((\frac{\lambda}{\gamma}+k_2)(c\mu-\lambda)^2-k_2(c\mu-\lambda)\gamma-\gamma\lambda)\\&+(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)(k_2c\mu-k_2\lambda+\lambda)\gamma((\mu\lambda^{k_1}-\lambda^{k_1+1})+\alpha(\mu^{k_1}-\lambda^{k_1}))\\&+\alpha\gamma\mu^{k_1}(c\mu-\lambda-\gamma)(\mu-\lambda)\big((\lambda+\gamma)^{k_2-k_1-1}-\mu^{k_2-k_1-1}\big)\big(\lambda+k_2(c\mu-\lambda)\big)\end{aligned}$$

By following a similar simplification approach as we did previously, merging terms that are similar, ((5.69) is an example of this approach), we have the sum of (5.75) and

(5.52) as:

$$\frac{\alpha(\mu-\lambda)\mu^{k_2-1}(\lambda+\gamma-\mu)((\frac{\lambda}{\gamma}+k_2)(c\mu-\lambda)^2)-(c-1)\alpha\gamma(\mu-\lambda)\mu^{k_2}(\lambda+k_2(c\mu-\lambda))}{\alpha\gamma(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1-k_2-1}(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)^2}$$
$$+\frac{\gamma(c\mu-\lambda-\gamma)(\lambda+\gamma)^{k_2-k_1-1}(k_2(c\mu-\lambda)+\lambda)\Big(\alpha\gamma(\mu^{k_1}-\lambda^{k_1})-((\mu-\lambda)^2-(\alpha+\gamma)(\mu-\lambda))\lambda^{k_1}\Big)}{\alpha\gamma(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\lambda^{k_1-k_2-1}(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)^2} \tag{5.78}$$

gmotest Now we want to merge the sixth sum on the RHS of (5.43) into either (5.74) (the sum of the first sum to the fifth sum) or (5.78) (the sum of the seventh sum to the ninth sum). The sixth sum (5.49) is more similar to (5.74) than (5.78), so we merge the sixth sum with (5.74).

We first simplify and arrange the sixth sum (5.49) such that it shows more similar patterns with (5.74)

$$\frac{\lambda^2}{\lambda+\gamma-\mu}\left(\frac{\mu+k_1(\mu-\lambda)-\Big(k_2(\mu-\lambda)+\lambda\Big)\rho^{k_2-k_1-1}}{(\mu-\lambda)^2}-\frac{\gamma+k_1\gamma+\lambda-(k_2\gamma+\lambda)(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1}}{\gamma^2}\right)$$
$$=\frac{\lambda^2}{\lambda+\gamma-\mu}\left(\frac{\gamma^2\mu^{k_2}(\lambda+\gamma)^{k_2-k_1-1}+k_1(\mu-\lambda)\gamma^2\mu^{k_2-1}(\lambda+\gamma^{k_2-k_1-1})-\gamma^2\mu^{k_1}\Big(k_2(\mu-\lambda)+\lambda\Big)\lambda^{k_2-k_1-1}(\lambda+\gamma)^{k_2-k_1-1}}{\gamma^2(\mu-\lambda)^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}}\right.$$
$$-\frac{\gamma\mu^{k_2-1}(\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1}+k_1\gamma\mu^{k_2-1}(\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1}+\lambda\mu^{k_2-1}(\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1}}{\gamma^2(\mu-\lambda)^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}}$$
$$\left.+\frac{\mu^{k_2-1}(\mu-\lambda)^2(k_2\gamma+\lambda)\lambda^{k_2-k_1-1}}{\gamma^2(\mu-\lambda)^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}}\right)$$

We now simplify the numerator of the term within the large parentheses:

$$
\begin{aligned}
&\gamma^2\mu^{k_2}(\lambda+\gamma)^{k_2-k_1-1} + k_1(\mu-\lambda)\gamma^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1} \\
&-\gamma^2\mu^{k_1}(k_2(\mu-\lambda)+\lambda)\lambda^{k_2-k_1-1}(\lambda+\gamma)^{k_2-k_1-1} - (\gamma\mu^{k_2-1}(\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1} \\
&+k_1\gamma\mu^{k_2-1}(\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1} + \lambda\mu^{k_2-1}(\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1} \\
&-\mu^{k_2-1}(\mu-\lambda)^2(k_2\gamma+\lambda)\lambda^{k_2-k_1-1}) \\
=&\gamma^2\mu^{k_2}(\lambda+\gamma)^{k_2-k_1-1} + k_1(\mu-\lambda)\gamma^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1} - \gamma\mu^{k_2-1}(\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1} \\
&-k_1\gamma\mu^{k_2-1}(\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1} - \lambda\mu^{k_2-1}(\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1} \\
&-\gamma^2\mu^{k_1}(k_2(\mu-\lambda)+\lambda)\lambda^{k_2-k_1-1}(\lambda+\gamma)^{k_2-k_1-1} + \mu^{k_2-1}(\mu-\lambda)^2(k_2\gamma+\lambda)\lambda^{k_2-k_1-1} \\
=&\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}(\gamma^2\mu + k_1(\mu-\lambda)\gamma^2 - \lambda(\mu-\lambda)^2 - k_1\gamma(\mu-\lambda)^2 - \lambda(\mu-\lambda)^2) \\
&-\lambda^{k_2-k_1-1}\Big(\gamma^2\mu^{k_1}(k_2(\mu-\lambda)+\lambda)(\lambda+\gamma)^{k_2-k_1-1} - \mu^{k_2-1}(\mu-\lambda)^2(k_2\gamma+\lambda)\Big) \\
=&\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}\Big(\gamma^2(\mu+k_1(\mu-\lambda)) - (\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\Big) \\
&-\lambda^{k_2-k_1-1}\Big(\gamma^2\mu^{k_1}(k_2(\mu-\lambda)+\lambda)(\lambda+\gamma)^{k_2-k_1-1} - \mu^{k_2-1}(\mu-\lambda)^2(k_2\gamma+\lambda)\Big)
\end{aligned}
$$

So (5.49) becomes:

$$
\begin{aligned}
&\frac{\lambda^2}{\lambda+\gamma-\mu}\Bigg(\frac{\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}\Big(\gamma^2(\mu+k_1(\mu-\lambda)) - (\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\Big)}{\gamma^2(\mu-\lambda)^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}} \\
&-\frac{\lambda^{k_2-k_1-1}\Big(\gamma^2\mu^{k_1}(k_2(\mu-\lambda)+\lambda)(\lambda+\gamma)^{k_2-k_1-1} - \mu^{k_2-1}(\mu-\lambda)^2(k_2\gamma+\lambda)\Big)}{\gamma^2(\mu-\lambda)^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}}\Bigg)
\end{aligned} \tag{5.79}
$$

We then make (5.74) have the same common denominator as (5.79):

$$
\begin{aligned}
&\frac{(\mu-\lambda)\mu^{k_2-1}\Big(2\mu\gamma^2\lambda^2+\alpha\mu k_1\gamma^2((\mu-\lambda)(k_1-1)+2\lambda)-2\alpha\mu\lambda^2(2k_1\gamma-\mu)-2\alpha\lambda^3(2\mu-k_1\gamma)+2\alpha\lambda^4+2\alpha\mu^2k_1\gamma\lambda\Big)}{2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3}\\
&-\frac{2\alpha\gamma^2\mu^{k_1}\Big(k_2(\mu-\lambda)+\lambda\Big)\lambda^{k_2-k_1+1}-2\gamma^2\lambda^{k_2+1}\Big(k_2\mu(\alpha+2\lambda-\mu)-\lambda\mu-\lambda(k_2-1)(\lambda+\alpha)\Big)}{2\gamma^2\alpha\mu^{k_2-1}(\mu-\lambda)^3}\\
=&\frac{(\mu-\lambda)(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_2-1}\Big(2\mu\gamma^2\lambda^2+\alpha\mu k_1\gamma^2\Big((\mu-\lambda)(k_1-1)+2\lambda\Big)-2\alpha\mu\lambda^2(2k_1\gamma-\mu)\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&+\frac{(\mu-\lambda)(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_2-1}\Big(-2\alpha\lambda^3(2\mu-k_1\lambda)+2\alpha\lambda^4+2\alpha\mu^2k_1\gamma\lambda\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&+\frac{2\gamma^2(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}\lambda^{k_2+1}\Big(k_2\mu(\alpha+2\lambda-\mu)-\lambda\mu-\lambda(k_2-1)(\lambda+\alpha)\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&-\frac{2\alpha\gamma^2(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_1}\Big(k_2(\mu-\lambda)+\lambda\Big)\lambda^{k_2-k_1+1}}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}
\end{aligned}
\tag{5.80}
$$

We also need to make (5.79) have the same denominator as (5.80):

$$
\begin{aligned}
&\frac{\lambda^2}{\lambda+\gamma-\mu}\Bigg(\frac{\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}\Big(\gamma^2(\mu+k_1(\mu-\lambda))-(\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\Big)}{\gamma^2(\mu-\lambda)^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}}\\
&-\frac{\lambda^{k_2-k_1-1}\Big(\gamma^2\mu^{k_1}(k_2(\mu-\lambda)+\lambda)(\lambda+\gamma)^{k_2-k_1-1}-\mu^{k_2-1}(\mu-\lambda)^2(k_2\gamma+\lambda)\Big)}{\gamma^2(\mu-\lambda)^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}}\Bigg)\\
=&\frac{\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}\lambda^2\Big(\gamma^2(\mu+k_1(\mu-\lambda))-(\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\Big)}{(\lambda+\gamma-\mu)\gamma^2(\mu-\lambda)^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}}\\
&-\frac{\lambda^{k_2-k_1-1}\Big(\gamma^2\mu^{k_1}(k_2(\mu-\lambda)+\lambda)(\lambda+\gamma)^{k_2-k_1-1}-\mu^{k_2-1}(\mu-\lambda)^2(k_2\gamma+\lambda)\Big)}{(\lambda+\gamma-\mu)\gamma^2(\mu-\lambda)^2\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}}\\
=&\frac{2\alpha(\mu-\lambda)\mu^{k_2-1}(\lambda+\gamma)^{k_2-k_1-1}\lambda^2\Big(\gamma^2(\mu+k_1(\mu-\lambda))-(\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&-\frac{2\alpha(\mu-\lambda)\lambda^{k_2-k_1-1}\Big(\gamma^2\mu^{k_1}(k_2(\mu-\lambda)+\lambda)(\lambda+\gamma)^{k_2-k_1-1}-\mu^{k_2-1}(\mu-\lambda)^2(k_2\gamma+\lambda)\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}
\end{aligned}
\tag{5.81}
$$

We follow a similar approach of separating and merging similar terms, obtaining the sum for the first to sixth sums on the RHS of (5.43) as:

$$
\begin{aligned}
&\frac{(\mu-\lambda)(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_2-1}\Big((\lambda+\gamma-\mu)\Big(\alpha\mu k_1\gamma^2(\mu-\lambda)(k_1-1)+2\mu\gamma^2\lambda^2+2\lambda\alpha\mu k_1\gamma^2+2\alpha\lambda(\mu-\lambda)^2(\gamma k_1+\lambda)\Big)\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&+\frac{(\mu-\lambda)(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_2-1}\Big(2\alpha\lambda^2\Big(\gamma^2(\mu+k_1(\mu-\lambda))-(\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\Big)\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&+\frac{2\gamma^2(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}\lambda^{k_2+1}\Big(k_2\mu(\alpha+2\lambda-\mu)-\lambda\mu-\lambda(k_2-1)(\lambda+\alpha)\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&-\frac{2\alpha\gamma^3(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_1}\lambda^{k_2-k_1+1}\Big(k_2(\mu-\lambda)+\lambda\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}+\frac{2\alpha\lambda^{k_2-k_1+1}\mu^{k_2-1}(\mu-\lambda)^3(k_2\gamma+\lambda)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}
\end{aligned}
\tag{5.82}
$$

We have the sum of the first to sixth sums on the RHS of (5.43) in (5.82), as well we have the sum of the seventh to ninth sums on the RHS of (5.43). Now, we need to sum these two to complete the simplification.

We first make the sum of the first to sixth sums (5.82) have the appropriate common denominator by multiplying by $(c\mu-\lambda)^2(c\mu-\lambda-\gamma)$:

$$
\begin{aligned}
&\frac{(c\mu-\lambda)^2(c\mu-\lambda-\gamma)(\mu-\lambda)(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_2-1}\Big((\lambda+\gamma-\mu)\Big(\alpha\mu k_1\gamma^2(\mu-\lambda)(k_1-1)+2\mu\gamma^2\lambda^2+2\lambda\alpha\mu k_1\gamma^2+2\alpha\lambda(\mu-\lambda)^2(\gamma k_1+\lambda)\Big)\Big)}{2(c\mu-\lambda)^2(c\mu-\lambda-\gamma)\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&+\frac{(c\mu-\lambda)^2(c\mu-\lambda-\gamma)(\mu-\lambda)(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_2-1}\Big(2\alpha\lambda^2\Big(\gamma^2(\mu+k_1(\mu-\lambda))-(\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\Big)\Big)}{2(c\mu-\lambda)^2(c\mu-\lambda-\gamma)\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&+\frac{2(c\mu-\lambda)^2(c\mu-\lambda-\gamma)\gamma^2(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}\lambda^{k_2+1}\Big(k_2\mu(\alpha+2\lambda-\mu)-\lambda\mu-\lambda(k_2-1)(\lambda+\alpha)\Big)}{2(c\mu-\lambda)^2(c\mu-\lambda-\gamma)\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&-\frac{2(c\mu-\lambda)^2(c\mu-\lambda-\gamma)\alpha\gamma^3(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_1}\lambda^{k_2-k_1+1}\Big(k_2(\mu-\lambda)+\lambda\Big)}{2\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}\\
&+\frac{2\alpha\lambda^{k_2-k_1+1}\mu^{k_2-1}(\mu-\lambda)^3(k_2\gamma+\lambda)}{2(c\mu-\lambda)^2(c\mu-\lambda-\gamma)\gamma^2\alpha(\lambda+\gamma-\mu)\mu^{k_2-1}(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}
\end{aligned}
\tag{5.83}
$$

Similarly, we also need to make the sum of the seventh to ninth sums (5.84) have the same common denominator by multiplying by $2\gamma(\mu-\lambda)^2$, and (5.84) becomes:

$$
\begin{aligned}
&\frac{2\gamma(\mu-\lambda)^3\lambda^{k_2-k_1+1}\alpha\mu^{k_2-1}(\lambda+\gamma-\mu)\Big((\frac{\lambda}{\gamma}+k_2)(c\mu-\lambda)^2\Big)-(c-1)\alpha\gamma(\mu-\lambda)\mu^{k_2}\Big(\lambda+k_2(c\mu-\lambda)\Big)}{2\gamma^2(\mu-\lambda)^3\alpha(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\mu^{k_2-1}(c\mu-\lambda)^2}\\
&+\frac{2\gamma^2(\mu-\lambda)^2(c\mu-\lambda-\gamma)(\lambda+\gamma)^{k_2-k_1-1}\lambda^{k_2-k_1+1}\Big(k_2(c\mu-\lambda)+\lambda\Big)\Big(\alpha\gamma(\mu^{k_1}-\lambda^{k_1})-\Big((\mu-\lambda)^2-(\alpha+\gamma)(\mu-\lambda)\Big)\lambda^{k_1}\Big)}{2\gamma^2(\mu-\lambda)^3\alpha(\lambda+\gamma-\mu)(\lambda+\gamma)^{k_2-k_1-1}(c\mu-\lambda-\gamma)\mu^{k_2-1}(c\mu-\lambda)^2}
\end{aligned}
\tag{5.84}
$$

Once again, we separate the terms and simplify, similar to what we did in (5.69), we obtain the sum of the first to last sums of (5.43) as:

$$
\begin{aligned}
\frac{\mathbb{E}[N]}{\pi_{0,0}} &= \frac{(\mu-\lambda)(c\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_2-1}(c\mu-\lambda-\gamma)(\lambda+\gamma-\mu)\left(\alpha\mu k_1\gamma^2(\mu-\lambda)(k_1-1)+2\mu\gamma^2\lambda^2\right)}{2\gamma^2\alpha(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)\mu^{k_2-1}(c\mu-\lambda)^2(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}} \\
&+ \frac{(\mu-\lambda)(c\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_2-1}(c\mu-\lambda-\gamma)(\lambda+\gamma-\mu)\left(2\lambda\alpha\mu k_1\gamma^2+2\alpha\lambda(\mu-\lambda)^2(\gamma k_1+\lambda)\right)}{2\gamma^2\alpha(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)\mu^{k_2-1}(c\mu-\lambda)^2(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}} \\
&+ \frac{(\mu-\lambda)(c\mu-\lambda)^2(\lambda+\gamma)^{k_2-k_1-1}\mu^{k_2-1}(c\mu-\lambda-\gamma)2\alpha\lambda^2\left(\gamma^2\left(\mu+k_1(\mu-\lambda)\right)-(\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\right)}{2\gamma^2\alpha(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)\mu^{k_2-1}(c\mu-\lambda)^2(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}} \\
&+ \frac{2\alpha\mu^{k_1}\gamma^3(c\mu-\lambda-\gamma)(\lambda+\gamma)^{k_2-k_1-1}\lambda^{k_2-k_1+1}(\mu-c\mu)\left(k_2(c\mu-\lambda)(\mu-\lambda)+\lambda\left((c\mu-\lambda)+(\mu-\lambda)\right)\right)}{2\gamma^2\alpha(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)\mu^{k_2-1}(c\mu-\lambda)^2(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}} \\
&+ \frac{2\alpha(c-1)\lambda^{k_2-k_1+1}\mu^{k_2}(\mu-\lambda)^3\left((k_2\gamma+\lambda)(c\mu-\lambda)^2-\gamma^2\left(\lambda+k_2(c\mu-\lambda)\right)\right)}{2\gamma^2\alpha(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)\mu^{k_2-1}(c\mu-\lambda)^2(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}} \\
&+ \frac{2\gamma^2(c\mu-\lambda-\gamma)(\lambda+\gamma)^{k_2-k_1-1}\lambda^{k_2+1}(\alpha-\mu+\lambda)(\mu-\lambda-\gamma)(\mu-c\mu)\left(k_2(c\mu-\lambda)(\mu-\lambda)+\lambda\left((c\mu-\lambda)+(\mu-\lambda)\right)\right)}{2\gamma^2\alpha(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)\mu^{k_2-1}(c\mu-\lambda)^2(\mu-\lambda)^3(\lambda+\gamma)^{k_2-k_1-1}}
\end{aligned}
\tag{5.85}
$$

We have obtained the expression for $\pi_{0,0}$ in (5.40). Along with the expression we obtained in (5.85), we have the expression for $\mathbb{E}[N]$ as:

$$
\begin{aligned}
\mathbb{E}[N] &= \frac{(\mu-\lambda)(c\mu-\lambda)^2(c\mu-\lambda-\gamma)\left((\lambda+\gamma-\mu)\left(\alpha\mu k_1\gamma^2(\mu-\lambda)(k_1-1)+2\mu\gamma^2\lambda^2+2\lambda\alpha\mu k_1\gamma^2+2\alpha\lambda(\mu-\lambda)^2(\gamma k_1+\lambda)\right)+2\alpha\lambda^2\left(\gamma^2(\mu+k_1(\mu-\lambda))-(\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\right)\right)}{2\mu\gamma(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)(c\mu-\lambda)^2(\mu-\lambda)^2\left((\alpha+\gamma)\lambda+\alpha\gamma k_1\right)+2\lambda\mu\gamma(c-1)(c\mu-\lambda-\gamma)(c\mu-\lambda)(\mu-\lambda)\left(\alpha\lambda(\mu-\lambda)^2\xi^{k_2-k_1-1}+\rho^{k_2}\left((\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\mu\gamma-\alpha\mu\gamma^2\rho^{-k_1}\right)\right)} \\
&+ \frac{2\lambda\mu(c-1)\left(\alpha\lambda\xi^{k_2-k_1-1}(\mu-\lambda)^3\left((k_2\gamma+\lambda)(c\mu-\lambda)^2-\gamma^2(\lambda+k_2(c\mu-\lambda))\right)+\gamma^2\mu(c\mu-\lambda-\gamma)\rho^{k_2}\left(k_2(c\mu-\lambda)(\mu-\lambda)+\lambda((c\mu-\lambda)+(\mu-\lambda))\right)\left((\mu-\alpha-\lambda)(\mu-\lambda-\gamma)-\alpha\gamma\rho^{-k_1}\right)\right)}{2\mu\gamma(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)(c\mu-\lambda)^2(\mu-\lambda)^2\left((\alpha+\gamma)\lambda+\alpha\gamma k_1\right)+2\lambda\mu\gamma(c-1)(c\mu-\lambda-\gamma)(c\mu-\lambda)(\mu-\lambda)\left(\alpha\lambda(\mu-\lambda)^2\xi^{k_2-k_1-1}+\rho^{k_2}\left((\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\mu\gamma-\alpha\mu\gamma^2\rho^{-k_1}\right)\right)}
\end{aligned}
\tag{5.86}
$$

We have obtained the closed form expression for $\mathbb{E}[N]$ in (5.86). To determine the expected response time, we can apply Little's Law to obtain the expression for $\mathbb{E}[R]$. Little's Law states:

$$\mathbb{E}[R] = \frac{\mathbb{E}[N]}{\lambda}$$

By applying Little's Law to (5.86), we have the expected response time as:

$$
\mathbb{E}[R] = \frac{(\mu-\lambda)(c\mu-\lambda)^2(c\mu-\lambda-\gamma)\left((\lambda+\gamma-\mu)\left(\alpha\mu k_1\gamma^2\lambda^{-1}(\mu-\lambda)(k_1-1)+2\mu\gamma^2\lambda+2\alpha\mu k_1\gamma^2+2\alpha(\mu-\lambda)^2(\gamma k_1+\lambda)\right)+2\alpha\lambda\left(\gamma^2(\mu+k_1(\mu-\lambda))-(\mu-\lambda)^2(\gamma+k_1\gamma+\lambda)\right)\right)}{2\mu\gamma(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)(c\mu-\lambda)^2(\mu-\lambda)^2\left((\alpha+\gamma)\lambda+\alpha\gamma k_1\right)+2\lambda\mu\gamma(c-1)(c\mu-\lambda-\gamma)(c\mu-\lambda)(\mu-\lambda)\left(\alpha\lambda(\mu-\lambda)^2\xi^{k_2-k_1-1}+\rho^{k_2}\left((\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\mu\gamma-\alpha\mu\gamma^2\rho^{-k_1}\right)\right)}
$$

$$+\frac{2\mu(c-1)\left(\alpha\lambda\xi^{k_2-k_1-1}(\mu-\lambda)^3\left((k_2\gamma+\lambda)(c\mu-\lambda)^2-\gamma^2(\lambda+k_2(c\mu-\lambda))\right)+\gamma^2\mu(c\mu-\lambda-\gamma)\rho^{k_2}\left(k_2(c\mu-\lambda)(\mu-\lambda)+\lambda((c\mu-\lambda)+(\mu-\lambda))\right)\left((\mu-\alpha-\lambda)(\mu-\lambda-\gamma)-\alpha\gamma\rho^{-k_1}\right)\right)}{2\mu\gamma(\lambda+\gamma-\mu)(c\mu-\lambda-\gamma)(c\mu-\lambda)^2(\mu-\lambda)^2\left((\alpha+\gamma)\lambda+\alpha\gamma k_1\right)+2\lambda\mu\gamma(c-1)(c\mu-\lambda-\gamma)(c\mu-\lambda)(\mu-\lambda)\left(\alpha\lambda(\mu-\lambda)^2\xi^{k_2-k_1-1}+\rho^{k_2}\left((\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\mu\gamma-\alpha\mu\gamma^2\rho^{-k_1}\right)\right)} \tag{5.87}$$

We also need the expected energy consumption of the system. In order to determine a closed form expression for $\mathbb{E}[E]$, we follow a similar approach that we used for identifying a closed form expression for $\mathbb{E}[N]$. However, energy costs for different system states vary, for example, it is expected to require more energy in the SETUP state versus when the system is in the IDLE state. To be more adaptable to energy cost variations, we express the expected energy consumption as the weighted sum of energy costs from all energy states as follows:

$$\mathbb{E}[E] = E_{SETUP}\pi_{SETUP} + E_{IDLE}\pi_{IDLE} + E_{NORMAL}\pi_{NORMAL} + E_{SCALED}\pi_{SCALED} \tag{5.88}$$

where π_{state} represents the total probability for a system state, for example, π_{SETUP} represents the total probability that the system is in SETUP. As a result, we need to determine expressions for the total probabilities for each system state.

We first solve for π_{SETUP}. The probability of being in the SETUP state is given by (5.2), which leads to:

$$\begin{aligned}\pi_{SETUP} &= \sum_{n=k_1}^{\infty} \pi_{0,n} \\ &= \sum_{n=k_1}^{\infty} (\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)} \pi_{0,0} \\ &= \pi_{0,0} \sum_{n=1}^{\infty} (\frac{\lambda}{\lambda+\gamma})^{n} \\ &= \pi_{0,0} \frac{\frac{\lambda}{\lambda+\gamma}}{1-\frac{\lambda}{\lambda+\gamma}} \\ &= \pi_{0,0} \frac{\lambda}{\gamma} \end{aligned} \tag{5.89}$$

And next we solve for π_{IDLE}, and from (5.3), we have:

$$\begin{aligned}\pi_{IDLE} &= \pi_{1,0} \\ &= \frac{\lambda}{\alpha} \pi_{0,0} \end{aligned} \tag{5.90}$$

We continue to solve for π_{NORMAL}, which contains two regions, one from BUSY to IDLE, and the other one from BUSY to SCALED. With the expressions obtained for both regions (5.4) and (5.5), we have:

$$
\begin{aligned}
&\pi_{NORMAL} = \sum_{n=1}^{k_2-1} \pi_{1,n} \\
&= \sum_{n=1}^{k_1} \pi_{1,n} + \sum_{k_1+1}^{k_2-1} \pi_{1,n} \\
&= \pi_{0,0} \sum_{n=1}^{k_1} \left(\frac{\lambda}{\alpha}\rho^n + \rho\frac{1-\rho^n}{1-\rho}\right) + \pi_{0,0} \sum_{k_1+1}^{k_2-1} \Bigg(\Big(\frac{\lambda}{\alpha}\rho^n + \rho^{n-(k_1-1)} \frac{1-\rho^{k_1}}{1-\rho}\Big) + \sum_{i=1}^{n-k_1} \big(\frac{\lambda}{\lambda+\gamma}\big)^i \rho^{n-k_1-(i-1)} \Bigg) \\
&= \pi_{0,0} \Big(\sum_{n=1}^{k_1} \frac{\lambda}{\alpha}\rho^n + \sum_{n=1}^{k_1} \rho\frac{1-\rho^n}{1-\rho} \Big) + \pi_{0,0} \Big(\sum_{k_1+1}^{k_2-1} \frac{\lambda}{\alpha}\rho^n + \sum_{k_1+1}^{k_2-1} \big(\rho^{n-(k_1-1)} \frac{1-\rho^{k_1}}{1-\rho}\big) \\
&+ \sum_{k_1+1}^{k_2-1} \sum_{i=1}^{n-k_1} \big(\frac{\lambda}{\lambda+\gamma}\big)^i \rho^{n-k_1-(i-1)} \Big) \\
&= \pi_{0,0} \Big(\frac{\lambda}{\alpha} \sum_{n=1}^{k_2-1} \rho^n + \frac{\rho}{1-\rho} \sum_{n=1}^{k_1} (1-\rho^n) \Big) + \pi_{0,0} \Big(\rho \frac{1-\rho^{k_1}}{1-\rho} \sum_{k_1+1}^{k_2-1} \rho^{n-k_1} + \sum_{k_1+1}^{k_2-1} \Big(\rho^{n-k_1+1} \sum_{i=1}^{n-k_1} \big(\frac{\lambda}{\lambda+\gamma}\big)^i \rho^{-i} \Big) \Big) \\
&= \pi_{0,0} \Big(\frac{\lambda}{\alpha} \sum_{n=1}^{k_2-1} \rho^n + \frac{\rho}{1-\rho} \Big(\sum_{n=1}^{k_1} (1) - \sum_{n=1}^{k_1} \rho^n \Big) \Big) + \pi_{0,0} \Big(\rho \frac{1-\rho^{k_1}}{1-\rho} \sum_{n=1}^{k_2-k_1-1} \rho^n \\
&+ \sum_{k_1+1}^{k_2-1} \Big(\rho^{n-k_1+1} \sum_{i=1}^{n-k_1} \big(\frac{\mu}{\lambda+\gamma}\big)^i \Big) \Big) \\
&= \pi_{0,0} \Big(\frac{\lambda}{\alpha} \frac{\rho(1-\rho^{k_2-1})}{1-\rho} + \frac{\rho}{1-\rho} \big(k_1 - \frac{\rho(1-\rho^{k_1})}{1-\rho}\big) \Big) \\
&\quad + \pi_{0,0} \Big(\rho \frac{1-\rho^{k_1}}{1-\rho} \frac{\rho(1-\rho^{k_2-k_1-1})}{1-\rho} + \sum_{k_1+1}^{k_2-1} \Big(\rho^{n-k_1+1} \frac{\frac{\mu}{\lambda+\gamma}\big(1-(\frac{\mu}{\lambda+\gamma})^{n-k_1}\big)}{1-\frac{\mu}{\lambda+\gamma}} \Big) \Big) \\
&= \pi_{0,0} \frac{\rho}{1-\rho} \Big(\frac{\lambda}{\alpha}(1-\rho^{k_2-1}) + k_1 - \frac{\rho(1-\rho^{k_1})}{1-\rho} \Big) \\
&\quad + \pi_{0,0} \Big(\frac{\rho^2(1-\rho^{k_1})(1-\rho^{k_2-k_1-1})}{(1-\rho)^2} + \frac{\frac{\mu}{\lambda+\gamma}}{1-\frac{\mu}{\lambda+\gamma}} \Big(\sum_{k_1+1}^{k_2-1} \rho^{n-k_1+1} - \sum_{k_1+1}^{k_2-1} \big(\rho^{n-k_1+1} \big(\frac{\mu}{\lambda+\gamma}\big)^{n-k_1} \big) \Big) \Big) \\
&= \pi_{0,0} \frac{\rho}{1-\rho} \Big(\frac{\lambda}{\alpha}(1-\rho^{k_2-1}) + k_1 - \frac{\rho(1-\rho^{k_1})}{1-\rho} \Big) \\
&\quad + \pi_{0,0} \Big(\frac{\rho^2(1-\rho^{k_1})(1-\rho^{k_2-k_1-1})}{(1-\rho)^2} + \frac{\mu}{\lambda+\gamma-\mu} \Big(\rho \sum_{n=1}^{k_2-k_1-1} \rho^n - \rho \sum_{n=1}^{k_2-k_1-1} \big(\frac{\lambda}{\lambda+\gamma}\big)^n \Big) \Big) \\
&= \pi_{0,0} \frac{\lambda}{\mu-\lambda} \Big(\frac{\lambda}{\alpha}(1-\rho^{k_2-1}) + k_1 - \frac{\rho(1-\rho^{k_1})}{1-\rho} \Big) + \pi_{0,0} \Big(\frac{\lambda^2(1-\rho^{k_1})(1-\rho^{k_2-k_1-1})}{(\mu-\lambda)^2} \\
&\quad + \frac{\mu}{\lambda+\gamma-\mu} \Big(\frac{\lambda(1-\rho^{k_2-k_1-1})}{\mu-\lambda} - \frac{\lambda\big(1-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1}\big)}{\gamma} \Big) \Big) \\
&= \pi_{0,0} \Big(\frac{\lambda^2(1-\rho^{k_2-1})(\mu-\lambda) + k_1\alpha\lambda(\mu-\lambda) - \alpha\lambda^2(1-\rho^{k_1})}{\alpha(\mu-\lambda)^2} \Big) \\
&\quad + \pi_{0,0} \Big(\frac{\lambda^2(1-\rho^{k_1})(1-\rho^{k_2-k_1-1})}{(\mu-\lambda)^2} + \frac{\gamma\lambda^2(1-\rho^{k_2-k_1-1}) - \lambda^2(\mu-\lambda)\big(1-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1}\big)}{\gamma(\mu-\lambda)(\lambda+\gamma-\mu)} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \pi_{0,0}\Big(\frac{\lambda^2(1-\rho^{k_2-1})(\mu-\lambda)+k_1\alpha\lambda(\mu-\lambda)-\alpha\lambda^2(1-\rho^{k_1})}{\alpha(\mu-\lambda)^2}\Big)\\
&\quad+\pi_{0,0}\left(\frac{\lambda^2\gamma(\lambda+\gamma-\mu)(1-\rho^{k_1})(1-\rho^{k_2-k_1-1})+\gamma(\mu-\lambda)\lambda^2(1-\rho^{k_2-k_1-1})-\lambda^2(\mu-\lambda)^2\Big(1-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1}\Big)}{\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)}\right)\\
&= \pi_{0,0}\left(\frac{\lambda^2\gamma(1-\rho^{k_2-1})(\mu-\lambda)(\lambda+\gamma-\mu)+k_1\alpha\lambda\gamma(\mu-\lambda)(\lambda+\gamma-\mu)-\alpha\lambda^2\gamma\rho^{k_2-k_1-1}(1-\rho^{k_1})(\lambda+\gamma-\mu)}{\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)}\right.\\
&\quad\left.+\frac{\alpha\gamma(\mu-\lambda)\lambda^2(1-\rho^{k_2-k_1-1})-\alpha\lambda^2(\mu-\lambda)^2\Big(1-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1}\Big)}{\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)}\right)
\end{aligned}
\tag{5.91}
$$

Then we continue to solve for π_{OFF}, from (5.1), we have:

$$
\begin{aligned}
\pi_{OFF} &= \sum_{n=0}^{k_1-1}\pi_{0,n}\\
&= \sum_{n=0}^{k_1-1}\pi_{0,0}\\
&= k_1\pi_{0,0}
\end{aligned}
\tag{5.92}
$$

Lastly, to calculate π_{SCALED}, we can use the fact that the sum of the steady-state probabilities must be equal to 1, so we have:

$$
\pi_{SCALED} = 1-\pi_{OFF}-\pi_{IDLE}-\pi_{SETUP}-\pi_{NORMAL} \tag{5.93}
$$

We first substitute the expression for $\pi_{0,0}$ into each of the expressions (5.89), (5.90), (5.91), (5.92). So we have:

$$
\pi_{IDLE} = \frac{\lambda\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)}{(c-1)\Big(-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+\alpha\lambda^2\mu(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2}\Big)+\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)\big((\alpha+\gamma)\lambda+\alpha\gamma k_1\big)}
\tag{5.94}
$$

$$
\pi_{SETUP} = \frac{\alpha\lambda(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)}{(c-1)\Big(-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+\alpha\lambda^2\mu(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2}\Big)+\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)\big((\alpha+\gamma)\lambda+\alpha\gamma k_1\big)}
\tag{5.95}
$$

$$\pi_{NORMAL} = \frac{(c\mu-\lambda)\left(\alpha\lambda^2(\mu-\lambda)^2\xi^{k_2-k_1-1}-\alpha\mu\lambda\gamma^2\rho^{k_2-k_1}-\mu\lambda\gamma(\lambda+\gamma-\mu)(\mu-\lambda-\alpha)\rho^{k_2}\right)+\lambda(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)(\lambda\gamma+k_1\alpha\gamma+\alpha\lambda)}{(c-1)\left(-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+\alpha\lambda^2\mu(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2}\right)+\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)\left((\alpha+\gamma)\lambda+\alpha\gamma k_1\right)} \tag{5.96}$$

$$\pi_{OFF} = \frac{k_1\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)}{(c-1)\left(-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+\alpha\lambda^2\mu(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2}\right)+\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)\left((\alpha+\gamma)\lambda+\alpha\gamma k_1\right)} \tag{5.97}$$

In order to find π_{SCALED}, we just need to use (5.93). Since all of the denominators of (5.94)-(5.97) are the same, the numerator of 1 is just same as the common denominator. From (5.93), we first simplify the numerator first, with substitutions from (5.97), (5.96), (5.95), (5.94), we have the numerator of π_{SCALED} as:

$$(c-1)\Big(-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+\alpha\lambda^2\mu(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2}\Big)$$
$$+\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)\big((\alpha+\gamma)\lambda+\alpha\gamma k_1\big)-k_1\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)$$
$$-\lambda\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)-\alpha\lambda(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)$$
$$-\Bigg(\lambda^2\gamma(1-\rho^{k_2-1})(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)+k_1\alpha\lambda\gamma(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)$$
$$-\alpha\lambda^2\gamma\rho^{k_2-k_1-1}(1-\rho^{k_1})(c\mu-\lambda)(\lambda+\gamma-\mu)$$
$$+\alpha\gamma(\mu-\lambda)(c\mu-\lambda)\lambda^2(1-\rho^{k_2-k_1-1})-\alpha\lambda^2(\mu-\lambda)^2(c\mu-\lambda)\big(1-(\frac{\lambda}{\lambda+\gamma})^{k_2-k_1-1}\big)\Bigg)$$

$$=(c-1)\big(-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+\alpha\lambda^2\mu(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2}\big)$$
$$-\lambda^2\gamma(1-\rho^{k_2-1})(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)+\alpha\lambda^2\gamma\rho^{k_2-k_1-1}(1-\rho^{k_1})(c\mu-\lambda)(\lambda+\gamma-\mu)$$
$$-\alpha\gamma(\mu-\lambda)(c\mu-\lambda)\lambda^2(1-\rho^{k_2-k_1-1})+\alpha\lambda^2(\mu-\lambda)^2(c\mu-\lambda)(1-\xi^{k_2-k_1-1})$$
$$+\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)\big((\alpha+\gamma)\lambda+\alpha\gamma k_1\big)-k_1\alpha\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)$$
$$-\lambda\gamma(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)-\alpha\lambda(\mu-\lambda)^2(\lambda+\gamma-\mu)(c\mu-\lambda)-k_1\alpha\lambda\gamma(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)$$

$$=(c-1)\big((-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2}\big)+\lambda^2(c\mu-\lambda)(\mu-\lambda)(\mu-\lambda-\gamma)(\alpha+\gamma)$$
$$+\lambda^2\gamma(c\mu-\lambda)(\mu-\lambda-\alpha)(\lambda+\gamma-\mu)\rho^{k_2-1}+\alpha\lambda^2\gamma^2(c\mu-\lambda)\rho^{k_2-k_1-1}$$
$$+\alpha\lambda^2(\mu-\lambda)^2\xi^{k_2-k_1-1}\big((c-1)\mu-(c\mu-\lambda)\big)$$
$$+(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)(\alpha\lambda\mu+\gamma\lambda\mu+\alpha\gamma k_1\mu-k_1\alpha\gamma\mu+k_1\alpha\gamma\lambda-\lambda\gamma\mu+\lambda\gamma\lambda-\alpha\lambda\mu+\alpha\lambda\lambda-k_1\alpha\lambda\gamma)$$

$$=\lambda^2(c\mu-\lambda)(\mu-\lambda)(\mu-\lambda-\gamma)(\alpha+\gamma)+\lambda^2(\mu-\lambda)(c\mu-\lambda)(\lambda+\gamma-\mu)(\gamma+\alpha)$$
$$+\lambda\mu\gamma(\mu-\lambda-\alpha)(\lambda+\gamma-\mu)\rho^{k_2}\big((c\mu-\lambda)-(c-1)\mu\big)+\alpha\lambda\mu\gamma^2\rho^{k_2-k_1}\big((c\mu-\lambda)-(c-1)\mu\big)$$
$$+\alpha\lambda^2(\mu-\lambda)^2\xi^{k_2-k_1-1}\big((c-1)\mu-(c\mu-\lambda)\big)$$

$$=\lambda\mu\gamma(\mu-\lambda-\alpha)(\lambda+\gamma-\mu)\rho^{k_2}(\mu-\lambda)+\alpha\lambda\mu\gamma^2\rho^{k_2-k_1}(\mu-\lambda)-\alpha\lambda^2(\mu-\lambda)^2\xi^{k_2-k_1-1}(\mu-\lambda)$$

$$=\lambda\mu\gamma(\mu-\lambda-\alpha)(\lambda+\gamma-\mu)\rho^{k_2}(\mu-\lambda)+\alpha\lambda\mu\gamma^2\rho^{k_2-k_1}(\mu-\lambda)-\alpha\lambda^2(\mu-\lambda)^3\xi^{k_2-k_1-1}$$

As a result, we have the expression for π_{SCALED} as:

$$\pi_{SCALED} = \frac{\lambda\mu\gamma(\mu-\lambda-\alpha)(\lambda+\gamma-\mu)\rho^{k_2}(\mu-\lambda)+\alpha\lambda\mu\gamma^2\rho^{k_2-k_1}(\mu-\lambda)-\alpha\lambda^2(\mu-\lambda)^3\xi^{k_2-k_1-1}}{(c-1)\Big(-\alpha\lambda(\mu\gamma)^2\rho^{k_2-k_1}+\alpha\lambda^2\mu(\mu-\lambda)^2\xi^{k_2-k_1-1}+(\lambda+\gamma-\mu)(\lambda+\alpha-\mu)\lambda\mu^2\gamma\rho^{k_2}\Big)+\mu(c\mu-\lambda)(\lambda+\gamma-\mu)(\mu-\lambda)\big((\alpha+\gamma)\lambda+\alpha\gamma k_1\big)} \tag{5.98}$$

We have determined closed form expressions for all of the required probabilities. Once the energy costs have been specified, one can easily obtain the total expected energy cost. We will show an example in the experiment results in Chapter 6.

With the closed form expression for $\mathbb{E}[E]$, we have determined all of the required closed form expressions for the first case of our study, where the threshold to turn on speed scaling is larger than or equal to the turn on server threshold, namely $k_1 \leq k_2$.

The CTMC model of this case is presented in Figure 4.2. The analysis process is very similar to what we have done for the first case in Chapter 5.1, and as mentioned earlier, our research interest is in the first case where $k_1 \leq k_2$, we only show the necessary analysis for this case.

Similar to Case 1, we first obtain the system balance equations as follows:

$$\begin{cases} \lambda\pi_{0,n} = \lambda\pi_{0,n-1} & \text{if } 0 \leq n < k_1 \ \textit{OFF region} \\ (\lambda+\gamma)\pi_{0,n} = \lambda\pi_{0,n-1} & \text{if } k_1 \leq n \ \textit{SETUP region} \\ \lambda\pi_{0,n} = \alpha\pi_{1,n} & \text{if } n = 0 \ \textit{IDLE} \\ \mu\pi_{1,n} = \lambda\pi_{1,n-1} + \lambda\pi_{0,n-1} & \text{if } 0 < n < k_2 \ \textit{BUSY to IDLE region} \\ c\mu\pi_{1,n} = \lambda\pi_{1,n-1} + \lambda\pi_{0,n-1} & \text{if } k_2 \leq n < k_1 \ \textit{SCALED to BUSY region} \\ (\lambda+c\mu)\pi_{1,n} = \lambda\pi_{1,n-1} + \gamma\pi_{0,n} + c\mu\pi_{1,n+1} & \text{if } k_1 \leq n \ \textit{SCALED region} \end{cases}$$

We now work on expressing the steady-state probabilities in terms of $\pi_{0,0}$. The expressions for the OFF, SETUP, and BUSY towards IDLE regions are identical with those in Case 1 since these portions of the two CTMCs in 4.1 and 4.2 are identical. So we can reuse the expressions developed in (5.1), (5.2),and (5.4). We have identical expressions with Case 1 for the OFF and SETUP regions, with the same expressions as (5.1) and (5.2). We then focus on finding the expression for the SCALED towards BUSY region $k_2 \le n < k_1$:

$$c\mu\pi_{1,n} = \lambda\pi_{1,n-1} + \lambda\pi_{0,n-1}$$

Let $Z = \frac{\lambda}{c\mu}$

$$\begin{aligned}
\pi_{1,k_2} &= Z\pi_{1,k_2-1} + Z\pi_{0,k_2-1} \\
\pi_{1,k_2+1} &= Z\pi_{1,k_2} + Z\pi_{0,k_2} \\
\pi_{1,k_2+1} &= Z^2\pi_{1,k_2-1} + Z^2\pi_{0,k_2-1} + Z\pi_{0,k_2}
\end{aligned}$$

and from (5.1) we have $\pi_{0,n} = \pi_{0,0}$ since the range is within the OFF region. So we have:

$$\begin{aligned}
\pi_{1,n} &= Z^{n-k_2+1}\pi_{1,k_2-1} + \pi_{0,0}\sum_{i=1}^{n-k_2+1} Z \\
&= Z^{n-k_2+1}\pi_{1,k_2-1} + \pi_{0,0}\Big(Z\frac{1-Z^{n-k_2+1}}{1-Z}\Big) \qquad (5.99)
\end{aligned}$$

π_{1,k_2-1} is in the BUSY towards IDLE region, from (5.4), we have:

$$\pi_{1,k_2-1} = \pi_{0,0}\Big(\frac{\lambda}{\alpha}\rho^{k_2-1} + \rho\frac{1-\rho^{k_2-1}}{1-\rho}\Big) \qquad (5.100)$$

Substitute (5.100) back into (5.99)

$$\begin{aligned}\pi_{1,n} &= Z^{n-k_2+1}\pi_{0,0}\Big(\frac{\lambda}{\alpha}\rho^{k_2-1}+\rho\frac{1-\rho^{k_2-1}}{1-\rho}\Big)+\pi_{0,0}(Z\frac{1-Z^{n-k_2+1}}{1-Z}) \\ &= \Big(Z^{n-k_2}(\frac{\lambda}{\alpha}\rho^{k_2-1}+\rho\frac{1-\rho^{k_2-1}}{1-\rho})+\frac{1-Z^{n-k_2+1}}{1-Z}\Big)Z\pi_{0,0}\end{aligned} \tag{5.101}$$

Since $Z = \frac{\lambda}{c\mu}$ and $\rho = \frac{\lambda}{\mu}$, we can substitute these expressions back to (5.101) for further simplification. We first perform simplification on the last term inside the large parentheses in (5.101):

$$\begin{aligned}&\frac{1-Z^{n-k_2+1}}{1-Z} \\ &= \frac{\frac{(c\mu)^{n-k_2+1}-\lambda^{n-k_2+1}}{(c\mu)^{n-k_2+1}}}{\frac{c\mu-\lambda}{c\mu}} \\ &= \frac{(c\mu)^{n-k_2+1}-\lambda^{n-k_2+1}}{(c\mu)^{n-k_2}(c\mu-\lambda)}\end{aligned} \tag{5.102}$$

Then we perform simplification on the first term inside the large parentheses of (5.101), and simplifying terms inside the small parentheses first.

$$\begin{aligned}&\frac{\lambda}{\alpha}\rho^{k_2-1}+\rho\frac{1-\rho^{k_2-1}}{1-\rho} \\ &= \frac{\lambda}{\alpha}\frac{\lambda^{k_2-1}}{\mu^{k_2-1}}+\frac{\lambda}{\mu}\frac{\frac{\mu^{k_2-1}-\lambda^{k_2-1}}{\mu^{k_2-1}}}{\frac{\mu-\lambda}{\mu}} \\ &= \frac{\lambda}{\alpha}\frac{\lambda^{k_2-1}}{\mu^{k_2-1}}+\lambda\frac{\mu^{k_2-1}-\lambda^{k_2-1}}{\mu^{k_2-1}(\mu-\lambda)} \\ &= \Big(\frac{\lambda^{k_2-1}}{\alpha}+\frac{\mu^{k_2-1}-\lambda^{k_2-1}}{\mu-\lambda}\Big)\frac{\lambda}{\mu^{k_2-1}}\end{aligned} \tag{5.103}$$

We now substitute (5.103) into the first term in the large parentheses of (5.101), yielding:

$$\begin{aligned} &Z^{n-k_2}(\frac{\lambda}{\alpha}\rho^{k_2-1}+\rho\frac{1-\rho^{k_2-1}}{1-\rho}) \\ &=\frac{\lambda^{n-k_2+1}}{c^{n-k_2}\mu^{n-1}}\Big(\frac{\lambda^{k_2-1}}{\alpha}+\frac{\mu^{k_2-1}-\lambda^{k_2-1}}{\mu-\lambda}\Big) \end{aligned} \tag{5.104}$$

We can now simplify (5.101) by substituting (5.102) and (5.104):

$$\begin{aligned} &Z^{n-k_2}(\frac{\lambda}{\alpha}\rho^{k_2-1}+\rho\frac{1-\rho^{k_2-1}}{1-\rho})+\frac{1-Z^{n-k_2+1}}{1-Z} \\ &=\frac{\lambda^{n-k_2+1}}{c^{n-k_2}\mu^{n-1}}\Big(\frac{\lambda^{k_2-1}}{\alpha}+\frac{\mu^{k_2-1}-\lambda^{k_2-1}}{\mu-\lambda}\Big)+\frac{(c\mu)^{n-k_2+1}-\lambda^{n-k_2+1}}{(c\mu)^{n-k_2}(c\mu-\lambda)} \\ &=\Big(\frac{\lambda^{n-k_2+1}}{\mu^{k_2-1}}\Big(\frac{\lambda^{k_2-1}}{\alpha}+\frac{\mu^{k_2-1}-\lambda^{k_2-1}}{\mu-\lambda}\Big)+\frac{(c\mu)^{n-k_2+1}-\lambda^{n-k_2+1}}{(c\mu-\lambda)}\Big)\frac{1}{(c\mu)^{n-k_2}} \end{aligned} \tag{5.105}$$

We need to simplify (5.105) further by simplifying the first term inside the large parentheses as follows:

$$\begin{aligned} &\frac{\lambda^{n-k_2+1}}{\mu^{k_2-1}}\Big(\frac{(\mu\lambda^{k_2-1}-\lambda^{k_2})+\alpha\mu^{k_2-1}-\alpha\lambda^{k_2-1}}{\alpha(\mu-\lambda)}\Big) \\ &=\frac{\lambda^{n-k_2+1}}{\mu^{k_2-1}}\Big(\frac{(\mu\lambda^{k_2-1}-\lambda^{k_2}-\alpha\lambda^{k_2-1})+\alpha\mu^{k_2-1}}{\alpha(\mu-\lambda)}\Big) \\ &=\frac{\lambda^{n-k_2+1}}{\mu^{k_2-1}}\Big(\frac{\lambda^{k_2-1}(\mu-\lambda)+\alpha\mu^{k_2-1}-\alpha\lambda^{k_2-1}}{\alpha(\mu-\lambda)}\Big) \\ &=\frac{(\mu-\lambda-\alpha)\lambda^{n}+\alpha\mu^{k_2-1}\lambda^{n-k_2+1}}{\alpha(\mu-\lambda)\mu^{k_2-1}} \end{aligned}$$

The terms with common denominator within the large parentheses in (5.105) become:

$$\frac{(\mu-\lambda-\alpha)\lambda^{n}(c\mu-\lambda)+\alpha\mu^{k_2-1}\lambda^{n-k_2+1}(c\mu-\lambda)}{\alpha(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)}+\frac{\alpha(\mu-\lambda)\mu^{k_2-1}(c\mu)^{n-k_2+1}-\alpha(\mu-\lambda)\mu^{k_2-1}\lambda^{n-k_2+1}}{\alpha(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)} \tag{5.106}$$

We now simplify the numerator of (5.106)

$$(\mu-\lambda-\alpha)\lambda^n(c\mu-\lambda)+\alpha\mu^{k_2-1}\lambda^{n-k_2+1}(c\mu-\lambda)+\alpha(\mu-\lambda)\mu^{k_2-1}(c\mu)^{n-k_2+1}-\alpha(\mu-\lambda)\mu^{k_2-1}\lambda^{n-k_2+1}$$
$$=(\mu-\lambda-\alpha)\lambda^n(c\mu-\lambda)+\alpha\mu^{k_2-1}\lambda^{n-k_2+1}(c\mu-\lambda)-\alpha(\mu-\lambda)\mu^{k_2-1}\lambda^{n-k_2+1}+\alpha(\mu-\lambda)\mu^{k_2-1}(c\mu)^{n-k_2+1}$$
$$=(\mu-\lambda-\alpha)\lambda^n(c\mu-\lambda)+(c\mu-\lambda-\mu+\lambda)\alpha\mu^{k_2-1}\lambda^{n-k_2+1}+\alpha(\mu-\lambda)\mu^{k_2-1}(c\mu)^{n-k_2+1}$$
$$=(\mu-\lambda-\alpha)\lambda^n(c\mu-\lambda)+(c-1)\alpha\mu^{k_2}\lambda^{n-k_2+1}+\alpha(\mu-\lambda)\mu^nc^{n-k_2+1}$$

So the expression in (5.106) becomes:

$$\frac{(\mu-\lambda-\alpha)\lambda^n(c\mu-\lambda)+(c-1)\alpha\mu^{k_2}\lambda^{n-k_2+1}+\alpha(\mu-\lambda)\mu^nc^{n-k_2+1}}{\alpha(\mu-\lambda)\mu^{k_2-1}(c\mu-\lambda)} \tag{5.107}$$

This completes the simplification for the terms within the large parentheses in (5.101), and now we substitute the result from (5.107), along with $Z=\frac{\lambda}{c\mu}$ into (5.101), which yields the expression for this region as:

$$\pi_{1,n}=\left(\frac{(\mu-\lambda-\alpha)\lambda^{n+1}(c\mu-\lambda)+(c-1)\alpha\mu^{k_2}\lambda^{n-k_2+2}+\alpha(\mu-\lambda)\lambda\mu^nc^{n-k_2+1}}{c^{n-k_2+1}\alpha(\mu-\lambda)\mu^n(c\mu-\lambda)}\right)\pi_{0,0} \tag{5.108}$$

We then move to solve for the expression in the SCALED region, where $k_1\leq n$. We have the balance equation for this region as:

$$(\lambda+c\mu)\pi_{1,n}=\lambda\pi_{1,n-1}+\gamma\pi_{0,n}+c\mu\pi_{1,n+1} \tag{5.109}$$

Similar to Case 1, this balance equation once again can be described as:

$$\pi_{1,n}=Ax^{n-(k_1-1)}+B(\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)} \tag{5.110}$$

where x satisfies:

$$(\lambda+c\mu)x=\lambda+c\mu x^2,$$

which yields $x = 1$ or $x = \frac{\lambda}{c\mu}$. Substitute this relationship back into the balance equation (5.109):

$$(\lambda + c\mu)\Big(Ax^{n-(k_1-1)} + B(\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)}\Big) = \lambda\Big(Ax^{n-k_1} + B(\frac{\lambda}{\lambda+\gamma})^{n-k_1}\Big) + \gamma(\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)}\pi_{0,0} + c\mu\Big(Ax^{n-k_1+2} + B(\frac{\lambda}{\lambda+\gamma})^{n-k_1+2}\Big)$$

We apply the same technique as in Case 1 by separating the constant terms and x terms, then we can solve for B:

$$\begin{aligned}
(\lambda + c\mu)B(\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)} &= \lambda B(\frac{\lambda}{\lambda+\gamma})^{n-k_1} + \gamma(\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)}\pi_{0,0} + c\mu B(\frac{\lambda}{\lambda+\gamma})^{n-k_1+2} \\
(\lambda + c\mu)(\frac{\lambda}{\lambda+\gamma})B &= \lambda B + \gamma(\frac{\lambda}{\lambda+\gamma})\pi_{0,0} + c\mu B(\frac{\lambda}{\lambda+\gamma})^2 \\
(\lambda + c\mu)B &= (\lambda+\gamma)B + \gamma\pi_{0,0} + c\mu B\frac{\lambda}{\lambda+\gamma} \\
(\lambda + c\mu)B - (\lambda+\gamma)B - c\mu B\frac{\lambda}{\lambda+\gamma} &= \gamma\pi_{0,0} \\
c\mu B - \gamma B - c\mu B\frac{\lambda}{\lambda+\gamma} &= \gamma\pi_{0,0} \\
c\mu(\lambda+\gamma)B - \gamma(\lambda+\gamma)B - c\mu\lambda B &= \gamma(\lambda+\gamma)\pi_{0,0} \\
(\lambda c\mu B + \gamma c\mu B) - (\lambda\gamma B + \gamma\gamma B) - c\mu\lambda B &= \gamma(\lambda+\gamma)\pi_{0,0} \\
(c\mu B) - (\lambda B + \gamma B) &= (\lambda+\gamma)\pi_{0,0} \\
(c\mu - \lambda - \gamma)B &= (\lambda+\gamma)\pi_{0,0} \\
B &= \frac{\lambda+\gamma}{c\mu - \lambda - \gamma}\pi_{0,0} \qquad (5.111)
\end{aligned}$$

With B solved, we can solve for A with the balance equation written as:

$$(\lambda + c\mu)\Big(Ax^{n-(k_1-1)} + B(\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)}\Big) = \lambda\pi_{1,n-1} + \gamma(\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)}\pi_{0,0} + c\mu\Big(Ax^{n-k_1+2} + B(\frac{\lambda}{\lambda+\gamma})^{n-k_1+2}\Big) \qquad (5.112)$$

When $n = k_1, \pi_{1,n-1} = \pi_{1,k_1-1}$, which belongs to the SCALED to NORMAL region:

$$\pi_{1,k_1-1} = \pi_{0,0} \frac{(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1} + (\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) + \alpha(\mu-\lambda)\lambda\mu^{k_1-1}c^{k_1-k_2}}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)} \tag{5.113}$$

Set $x = 1$ as one of the solutions for (5.112) and substitute (5.113) into (5.112), we then have:

$$\begin{aligned}
(\lambda + c\mu)\big(A + B\frac{\lambda}{\lambda+\gamma}\big) &= \lambda\pi_{0,0}\Big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1} + (\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) + \alpha(\mu-\lambda)\lambda\mu^{k_1-1}c^{k_1-k_2}}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)}\Big) \\
&\quad + \gamma(\frac{\lambda}{\lambda+\gamma}\pi_{0,0}) + c\mu\big(A + B(\frac{\lambda}{\lambda+\gamma})^2\big) \\
\lambda A + (\lambda + c\mu)B\frac{\lambda}{\lambda+\gamma} &= \lambda\pi_{0,0}\Big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1} + (\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) + \alpha(\mu-\lambda)\lambda\mu^{k_1-1}c^{k_1-k_2}}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)}\Big) \\
&\quad + \gamma(\frac{\lambda}{\lambda+\gamma}\pi_{0,0}) + Bc\mu(\frac{\lambda}{\lambda+\gamma})^2 \\
A + (\lambda + c\mu)B\frac{1}{\lambda+\gamma} &= \pi_{0,0}\Big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1} + (\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) + \alpha(\mu-\lambda)\lambda\mu^{k_1-1}c^{k_1-k_2}}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)}\Big) \\
&\quad + \gamma(\frac{1}{\lambda+\gamma}\pi_{0,0}) + Bc\mu\frac{\lambda}{(\lambda+\gamma)^2}
\end{aligned}$$

$$\begin{aligned}
A = \pi_{0,0}\Big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1} + (\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) + \alpha(\mu-\lambda)\lambda\mu^{k_1-1}c^{k_1-k_2}}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)}\Big) + \gamma(\frac{1}{\lambda+\gamma}\pi_{0,0}) \\
+B\Big(\frac{c\mu\lambda - (\lambda+\gamma)(\lambda+c\mu)}{(\lambda+\gamma)^2}\Big)
\end{aligned} \tag{5.114}$$

Sub $B = \frac{\lambda+\gamma}{c\mu-\lambda-\gamma}\pi_{0,0}$ from (5.111) and simplify the B term of (5.114):

$$\begin{aligned}
B\Big(\frac{c\mu\lambda - (\lambda+\gamma)(\lambda+c\mu)}{(\lambda+\gamma)^2}\Big) &= \frac{\lambda+\gamma}{c\mu-\lambda-\gamma}\Big(\frac{c\mu\lambda - (\lambda+\gamma)(\lambda+c\mu)}{(\lambda+\gamma)^2}\Big)\pi_{0,0} \\
&= \frac{1}{c\mu-\lambda-\gamma}\Big(\frac{c\mu\lambda - (\lambda+\gamma)(\lambda+c\mu)}{\lambda+\gamma}\Big)\pi_{0,0} \\
&= -\frac{\lambda(\lambda+\gamma) + \gamma c\mu}{(c\mu-\lambda-\gamma)(\lambda+\gamma)}\pi_{0,0}
\end{aligned}$$

Substitute the simplified B term above into (5.114), so A becomes:

$$\begin{aligned} A &= \pi_{0,0}\Big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1} + (\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) + \alpha(\mu-\lambda)\lambda\mu^{k_1-1}c^{k_1-k_2}}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)}\Big) + \gamma(\frac{1}{\lambda+\gamma}\pi_{0,0}) \\ &- \frac{\lambda(\lambda+\gamma)+\gamma c\mu}{(c\mu-\lambda-\gamma)(\lambda+\gamma)}\pi_{0,0} \\ A &= \pi_{0,0}\Big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1} + (\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) + \alpha(\mu-\lambda)\lambda\mu^{k_1-1}c^{k_1-k_2}}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)}\Big) - \frac{\lambda+\gamma}{c\mu-\lambda-\gamma}\pi_{0,0} \end{aligned} \tag{5.115}$$

Now we merge the $\pi_{0,0}$ terms of (5.115) using a common denominator, and simplify:

$$\begin{aligned} &\frac{(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1}(c\mu-\lambda-\gamma) + (c\mu-\lambda-\gamma)(\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) + \alpha(c\mu-\lambda-\gamma)(\mu-\lambda)\lambda\mu^{k_1-1}c^{k_1-k_2}}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)(c\mu-\lambda-\gamma)} \\ &- \frac{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)(\lambda+\gamma)}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)(c\mu-\lambda-\gamma)} \end{aligned} \tag{5.116}$$

Simplify the numerator of (5.116):

$$\begin{aligned} &(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1}(c\mu-\lambda-\gamma) + (c\mu-\lambda-\gamma)(\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) + \alpha(c\mu-\lambda-\gamma)(\mu-\lambda)\lambda\mu^{k_1-1}c^{k_1-k_2} \\ &- \alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)(\lambda+\gamma) \\ =& (c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1}(c\mu-\lambda-\gamma) + (c\mu-\lambda-\gamma)(\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) \\ &+ \alpha(\mu-\lambda)c^{k_1-k_2}\mu^{k_1-1}\big((c\mu-\lambda)\lambda - \gamma\lambda - (c\mu-\lambda)\lambda - \gamma(c\mu-\lambda)\big) \\ =& (c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1}(c\mu-\lambda-\gamma) + (c\mu-\lambda-\gamma)(\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) - \gamma\alpha(\mu-\lambda)c^{k_1-k_2+1}\mu^{k_1} \end{aligned} \tag{5.117}$$

With the results from (5.117), we have A as:

$$A = \pi_{0,0}\Big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{k_1-k_2+1}(c\mu-\lambda-\gamma) + (c\mu-\lambda-\gamma)(\mu-\lambda-\alpha)\lambda^{k_1}(c\mu-\lambda) - \gamma\alpha(\mu-\lambda)c^{k_1-k_2+1}\mu^{k_1}}{\alpha c^{k_1-k_2}(\mu-\lambda)\mu^{k_1-1}(c\mu-\lambda)(c\mu-\lambda-\gamma)}\Big)$$

With A and B solved, let $x = \frac{\lambda}{c\mu}$. Subsitute these back into (5.110), we now have the expression:

$$\pi_{1,n} = \pi_{0,0}\Big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{n-k_2+2}(c\mu-\lambda-\gamma)+(c\mu-\lambda-\gamma)(\mu-\lambda-\alpha)\lambda^{n+1}(c\mu-\lambda)-\gamma\alpha(\mu-\lambda)c^{k_1-k_2+1}\lambda^{n-k_1+1}\mu^{k_1}}{\alpha c^{n-k_2+1}(\mu-\lambda)\mu^n(c\mu-\lambda)(c\mu-\lambda-\gamma)}\Big) + \frac{\lambda^{n-k_1+1}}{(c\mu-\lambda-\gamma)(\lambda+\gamma)^{n-k_1}}\pi_{0,0} \tag{5.118}$$

We now have expressed all probabilities in terms of $\pi_{0,0}$. As for Case 1, the sum of the probabilities must be 1, so we can solve for $\pi_{0,0}$ with:

$$1 = \sum_{n=0}^{k_1-1}\pi_{0,n} + \sum_{n=k_1}^{\infty}\pi_{0,n} + \sum_{n=0}^{k_2-1}\pi_{1,n} + \sum_{n=k_2}^{k_1-1}\pi_{1,n} + \sum_{n=k_1}^{\infty}\pi_{1,n} \tag{5.119}$$

We now substitute the developed expressions from (5.1), (5.2),(5.4), (5.108), (5.118) into the corresponding regions in (5.119).

$$1 = \sum_{n=0}^{k_1-1}\pi_{0,0} + \sum_{n=k_1}^{\infty}\Big((\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)}\pi_{0,0}\Big) + \sum_{n=0}^{k_2-1}\pi_{0,0}\big(\frac{\lambda}{\alpha}\rho^n + \rho\frac{1-\rho^n}{1-\rho}\big)$$
$$+ \sum_{n=k_2}^{k_1-1}\big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{n-k_2+2}+(\mu-\lambda-\alpha)\lambda^{n+1}(c\mu-\lambda)+\alpha(\mu-\lambda)\lambda\mu^n c^{n-k_2+1}}{\alpha c^{n-k_2+1}(\mu-\lambda)\mu^n(c\mu-\lambda)}\big)$$
$$+ \sum_{n=k_1}^{\infty}\Big(\pi_{0,0}\big(\frac{(c-1)\alpha\mu^{k_2}\lambda^{n-k_2+2}(c\mu-\lambda-\gamma)+(c\mu-\lambda-\gamma)(\mu-\lambda-\alpha)\lambda^{n+1}(c\mu-\lambda)-\gamma\alpha(\mu-\lambda)c^{k_1-k_2+1}\lambda^{n-k_1+1}\mu^{k_1}}{\alpha c^{n-k_2+1}(\mu-\lambda)\mu^n(c\mu-\lambda)(c\mu-\lambda-\gamma)}\big)$$
$$+ \frac{\lambda^{n-k_1+1}}{(c\mu-\lambda-\gamma)(\lambda+\gamma)^{n-k_1}}\pi_{0,0}\Big) \tag{5.120}$$

We perform simplification on each of the sums appearing on the RHS of (5.120), the first sum is:

$$\sum_{n=0}^{k_1-1}\pi_{0,0} = \pi_{0,0}\sum_{n=0}^{k_1-1}1 = k_1\pi_{0,0} \tag{5.121}$$

We continue to the second sum of the RHS of (5.120):

$$\pi_{0,0}\sum_{n=k_1}^{\infty}(\frac{\lambda}{\lambda+\gamma})^{n-(k_1-1)} = \pi_{0,0}\sum_{n=1}^{\infty}(\frac{\lambda}{\lambda+\gamma})^n$$

since

$$\frac{\lambda}{\lambda+\gamma} < 1$$

$$\Rightarrow \quad \pi_{0,0}\sum_{n=1}^{\infty}(\frac{\lambda}{\lambda+\gamma})^n = \pi_{0,0}\frac{\frac{\lambda}{\lambda+\gamma}}{1-\frac{\lambda}{\lambda+\gamma}} = \frac{\lambda}{\gamma}\pi_{0,0} \tag{5.122}$$

We continue to perform simplification on the third sum on the RHS of (5.120),

$$\begin{aligned}
&\sum_{n=0}^{k_2-1}(\frac{\lambda}{\alpha}\rho^n + \rho\frac{1-\rho^n}{1-\rho}) \\
&= \sum_{n=0}^{k_2-1}(\frac{\lambda}{\alpha}(\frac{\lambda}{\mu})^n + \frac{\lambda}{\mu}\frac{1-(\frac{\lambda}{\mu})^n}{1-\frac{\lambda}{\mu}}) \\
&= \sum_{n=0}^{k_2-1}(\frac{\lambda^{n+1}}{\alpha\mu^n} + \lambda\frac{\mu^n-\lambda^n}{\mu^n(\mu-\lambda)}) \\
&= \sum_{n=0}^{k_2-1}\frac{\lambda^{n+1}(\mu-\lambda)+\alpha\lambda\mu^n-\alpha\lambda^{n+1}}{\alpha\mu^n(\mu-\lambda)} \\
&= \sum_{n=0}^{k_2-1}\left(\frac{\lambda^{n+1}}{\alpha\mu^n} + \frac{\lambda}{\mu-\lambda} - \frac{\lambda^{n+1}}{\mu^n(\mu-\lambda)}\right) \\
&= \sum_{n=0}^{k_2-1}\frac{\lambda^{n+1}}{\alpha\mu^n} + \sum_{n=0}^{k_2-1}\frac{\lambda}{\mu-\lambda} - \sum_{n=0}^{k_2-1}\frac{\lambda^{n+1}}{\mu^n(\mu-\lambda)} \\
&= \frac{\lambda}{\alpha}\sum_{n=0}^{k_2-1}\frac{\lambda^n}{\mu^n} + \frac{\lambda}{\mu-\lambda}\sum_{n=0}^{k_2-1}1 - \frac{\lambda}{\mu-\lambda}\sum_{n=0}^{k_2-1}\frac{\lambda^n}{\mu^n}
\end{aligned} \tag{5.123}$$

and

$$\sum_{n=0}^{k_2-1}\frac{\lambda^n}{\mu^n} = \frac{1-(\frac{\lambda}{\mu})^{k_2}}{1-\frac{\lambda}{\mu}} = \frac{\mu^{k_2}-\lambda^{k_2}}{\mu^{k_2-1}(\mu-\lambda)}$$

Substitute this result back to (5.123), so we have:

$$\Rightarrow \frac{\lambda}{\alpha}\frac{\mu^{k_2}-\lambda^{k_2}}{\mu^{k_2-1}(\mu-\lambda)} + \frac{\lambda}{\mu-\lambda}k_2 - \frac{\lambda}{\mu-\lambda}\frac{\mu^{k_2}-\lambda^{k_2}}{\mu^{k_2-1}(\mu-\lambda)}$$
$$= \frac{k_2\mu^{k_2-1}\alpha\lambda(\mu-\lambda)+(\mu^{k_2}-\lambda^{k_2})(\mu-\lambda-\alpha)\lambda}{\mu^{k_2-1}\alpha(\mu-\lambda)^2} \tag{5.124}$$

We merge the first portion of the fourth sum of (5.120) with the first part inside the large parentheses of the fifth sum of (5.120), and perform simplification,

$$\pi_{0,0}\sum_{n=k_2}^{k_1-1}\frac{(c-1)\alpha\mu^{k_2}\lambda^{n-k_2+2}+(\mu-\lambda-\alpha)\lambda^{n+1}(c\mu-\lambda)}{\alpha c^{n-k_2+1}(\mu-\lambda)\mu^n(c\mu-\lambda)}$$
$$+\pi_{0,0}\sum_{n=k_1}^{\infty}\frac{(c-1)\alpha\mu^{k_2}\lambda^{n-k_2+2}+(\mu-\lambda-\alpha)\lambda^{n+1}(c\mu-\lambda)}{\alpha c^{n-k_2+1}(\mu-\lambda)\mu^n(c\mu-\lambda)}$$
$$=\pi_{0,0}\sum_{n=k_2}^{\infty}\frac{(c-1)\alpha\mu^{k_2}\lambda^{n-k_2+2}+(\mu-\lambda-\alpha)\lambda^{n+1}(c\mu-\lambda)}{\alpha c^{n-k_2+1}(\mu-\lambda)\mu^n(c\mu-\lambda)}$$
$$=\pi_{0,0}\sum_{n=1}^{\infty}\frac{(c-1)\alpha\mu^{k_2}\lambda^{n+2}+(\mu-\lambda-\alpha)\lambda^{n+k_2}(c\mu-\lambda)}{\alpha c^{n}(\mu-\lambda)\mu^{n+k_2-1}(c\mu-\lambda)}$$
$$=\pi_{0,0}\sum_{n=1}^{\infty}\Big(\frac{(c-1)\alpha\lambda^{n+1}}{\alpha c^n(\mu-\lambda)\mu^{n-1}(c\mu-\lambda)}+\frac{(\mu-\lambda-\alpha)\lambda^{n+k_2}(c\mu-\lambda)}{\alpha c^n(\mu-\lambda)\mu^{n+k_2-1}(c\mu-\lambda)}\Big) \tag{5.125}$$

We now need to discuss whether the expression within the sum of (5.125) converges or not for further simplification.

$$\sum_{n=1}^{\infty}\frac{(c-1)\alpha\mu\lambda\lambda^n}{\alpha c^n(\mu-\lambda)\mu^n(c\mu-\lambda)}+\sum_{n=1}^{\infty}\frac{(\mu-\lambda-\alpha)\lambda^{k_2}\lambda^n}{\alpha c^n(\mu-\lambda)\mu^{k_2-1}\mu^n}$$
$$=\frac{(c-1)\alpha\mu\lambda}{\alpha(\mu-\lambda)(c\mu-\lambda)}\sum_{n=1}^{\infty}\frac{\lambda^n}{c^n\mu^n}+\frac{(\mu-\lambda-\alpha)\mu\lambda^{k_2}}{\alpha(\mu-\lambda)\mu^{k_2}}\sum_{n=1}^{\infty}\frac{\lambda^n}{c^n\mu^n}$$

Since we have $\frac{\lambda}{c\mu}<1$ for a stable system, so (5.125) becomes:

$$=\pi_{0,0}\Big(\frac{(c-1)\alpha\mu\lambda}{\alpha(\mu-\lambda)(c\mu-\lambda)}\frac{\lambda}{(c\mu-\lambda)}+\frac{(\mu-\lambda-\alpha)\mu\lambda^{k_2}}{\alpha(\mu-\lambda)\mu^{k_2}}\frac{\lambda}{(c\mu-\lambda)}\Big)$$
$$=\pi_{0,0}\Big(\frac{\lambda\mu}{\alpha(\mu-\lambda)(c\mu-\lambda)}\Big(\frac{(c-1)\alpha\lambda}{c\mu-\lambda}+\frac{(\mu-\lambda-\alpha)\lambda^{k_2}}{\mu^{k_2}}\Big) \tag{5.126}$$

We then focus on simplifying the rest of the fourth and fifth sums of (5.120).

$$\pi_{0,0}\sum_{n=k_2}^{k_1-1}\frac{\alpha(\mu-\lambda)\lambda\mu^n c^{n-k_2+1}}{\alpha c^{n-k_2+1}(\mu-\lambda)\mu^n(c\mu-\lambda)}-\pi_{0,0}\sum_{n=k_1}^{\infty}\frac{\gamma\alpha(\mu-\lambda)c^{k_1-k_2+1}\lambda^{n-k_1+1}\mu^{k_1}}{\alpha c^{n-k_2+1}(\mu-\lambda)\mu^n(c\mu-\lambda)(c\mu-\lambda-\gamma)}$$
$$=\pi_{0,0}\sum_{n=k_2}^{k_1-1}\frac{\lambda}{c\mu-\lambda}-\pi_{0,0}\sum_{n=k_1}^{\infty}\frac{\gamma\lambda^{n-k_1+1}}{c^{n-k_1}\mu^{n-k_1}(c\mu-\lambda)(c\mu-\lambda-\gamma)}$$
$$=\pi_{0,0}\Big(\frac{\lambda}{c\mu-\lambda}\sum_{n=k_2}^{k_1-1}1-\frac{\lambda}{c\mu-\lambda}\frac{\gamma}{c\mu-\lambda-\gamma}\sum_{n=k_1}^{\infty}\frac{\lambda^{n-k_1}}{c^{n-k_1}\mu^{n-k_1}}\Big)$$
$$=\pi_{0,0}\Big(\frac{\lambda}{c\mu-\lambda}\sum_{n=k_2}^{k_1-1}1-\frac{\lambda}{c\mu-\lambda}\frac{\gamma}{c\mu-\lambda-\gamma}\sum_{n=0}^{\infty}\frac{\lambda^{n}}{c^{n}\mu^{n}}\Big)$$
$$=\pi_{0,0}\Big(\frac{\lambda}{c\mu-\lambda}(k_1-k_2)-\frac{\lambda}{c\mu-\lambda}\frac{\gamma}{c\mu-\lambda-\gamma}\frac{c\mu}{c\mu-\lambda}\Big)\tag{5.127}$$

Lastly, we perform simplification on the last term of (5.120),

$$\sum_{n=k_1}^{\infty}\frac{\lambda^{n-k_1+1}}{(c\mu-\lambda-\gamma)(\lambda+\gamma)^{n-k_1}}$$
$$=\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{(c\mu-\lambda-\gamma)(\lambda+\gamma)^{n}}$$
$$=\frac{\lambda}{(c\mu-\lambda-\gamma)}\sum_{n=0}^{\infty}\frac{\lambda^{n}}{(\lambda+\gamma)^{n}}$$
$$=\frac{\lambda(\lambda+\gamma)}{(c\mu-\lambda-\gamma)\gamma}\tag{5.128}$$

We can now continue to solve for $\pi_{0,0}$ with all the simplified forms from (5.121), (5.122), (5.124), (5.126), (5.127), and (5.128). Substitute these results into (5.120), yielding,

$$1=k_1\pi_{0,0}+\frac{\lambda}{\gamma}\pi_{0,0}+\pi_{0,0}\frac{k_2\mu^{k_2-1}\alpha\lambda(\mu-\lambda)+(\mu^{k_2}-\lambda^{k_2})(\mu-\lambda-\alpha)\lambda}{\mu^{k_2-1}\alpha(\mu-\lambda)^2}$$
$$+\pi_{0,0}\frac{\lambda\mu}{\alpha(\mu-\lambda)(c\mu-\lambda)}\Big(\frac{(c-1)\alpha\lambda}{(c\mu-\lambda)}+\frac{(\mu-\lambda-\alpha)\lambda^{k_2}}{\mu^{k_2}}\Big)$$
$$+\pi_{0,0}\Big(\frac{\lambda}{c\mu-\lambda}(k_1-k_2)-\frac{\lambda}{(c\mu-\lambda)}\frac{\gamma}{(c\mu-\lambda-\gamma)}\frac{c\mu}{c\mu-\lambda}\Big)+\pi_{0,0}\frac{\lambda(\lambda+\gamma)}{(c\mu-\lambda-\gamma)\gamma}$$

We then follow similar approaches as we did for Case 1, by arranging similar terms together to achieve further simplification, and we obtain the closed form expression for Case 2 as:

$$\begin{aligned}\pi_{0,0} = \alpha(\mu-\lambda)^2(c\mu-\lambda)^2(c\mu-\lambda-\gamma)\Big(\lambda\mu\Big(\lambda\alpha(c-1)(\mu-\lambda)(c\mu-\lambda-\gamma)\\ + 2(\mu-\lambda-\alpha)(c\mu-\lambda-\gamma)(c\mu-\lambda)^2 + k_1\alpha c(\mu-\lambda)^2\lambda^{-1}(c\mu-\lambda)(c\mu-\lambda-\gamma)\\ + k_2\mu^{-1}\alpha(\mu-\lambda)(c\mu-\lambda)(c\mu-\lambda-\gamma)((c-1)\mu+(c\mu-\lambda))\\ + c\alpha(\mu-\lambda)^2(\gamma^{-1}(c\mu-\lambda)^2(c\mu-\lambda-\gamma)-\gamma)\Big)\Big)^{-1}\end{aligned} \tag{5.129}$$

After the expression for $\pi_{0,0}$ is determined, we can follow exactly the same approaches as we did for Case 1 to determine the expressions for $\mathbb{E}[N]$, $\mathbb{E}[R]$, and $\mathbb{E}[E]$. As a result, we omit the detailed derivation for Case 2, and only provide the final expressions here.

$$\begin{aligned}\frac{\mathbb{E}[N]}{\pi_{0,0}} = \frac{(k_1-1)k_1}{2} + \frac{\lambda(\lambda+k_1\gamma)}{\gamma^2} + \frac{\lambda}{\alpha}\rho\frac{1-\rho^{k_2}-k_2\rho^{k_2-1}+k_2\rho^{k_2}}{(1-\rho)^2} + \frac{\lambda}{\mu-\lambda}\Big(\frac{k_2(k_2-1)}{2}\\ -\rho\frac{1-\rho^{k_2}-k_2\rho^{k_2-1}+k_2\rho^{k_2}}{(1-\rho)^2}\Big) + \frac{1}{\alpha(\mu-\lambda)(c\mu-\lambda)}\Big(\frac{(c-1)\mu\alpha\lambda^2}{c\mu-\lambda}\Big(\frac{\lambda-\lambda(\frac{\lambda}{c\mu})^{k_1}}{c\mu-\lambda} + k_2 - (k_1+k_2)(\frac{\lambda}{c\mu})^{k_1}\Big)\\ + (\mu-\lambda-\alpha)(c\mu-\lambda)\frac{\lambda^{k_2+2}}{\mu^{k_2-1}}\frac{-k_1\frac{c\mu}{\lambda}(\frac{\lambda}{c\mu})^{k_1}+k_1(\frac{\lambda}{c\mu})^{k_1}+1-(\frac{\lambda}{c\mu})^{k_1}}{(c\mu-\lambda)^2} + \alpha(\mu-\lambda)\lambda\Big(\frac{(k_1-1)k_1}{2}+k_2k_1\Big)\Big)\\ + \frac{\lambda}{\alpha(\mu-\lambda)(c\mu-\lambda)(c\mu-\lambda-\gamma)}\Big((c\mu-\lambda-\gamma)c^{k_2-1}\Big((c-1)\alpha\mu^{k_2}\lambda^{-k_2+1}+(\mu-\lambda-\alpha)(c\mu-\lambda)\Big)\\ (\frac{\lambda}{c\mu})^{k_1}\frac{c\mu}{c\mu-\lambda}(\frac{\lambda}{c\mu-\lambda}+k_1) - \gamma\alpha(\mu-\lambda)\frac{c\mu}{(c\mu-\lambda)}(\frac{\lambda}{c\mu-\lambda}+k_1)\Big) + \frac{\lambda}{c\mu-\lambda-\gamma}\frac{\lambda+\gamma}{\gamma}(\frac{\lambda}{\gamma}+k_1)\end{aligned}$$

$$\begin{aligned}\frac{\mathbb{E}[R]}{\pi_{0,0}} = \frac{(k_1-1)k_1}{2\lambda} + \frac{(\lambda+k_1\gamma)}{\gamma^2} + \frac{1}{\alpha}\rho\frac{1-\rho^{k_2}-k_2\rho^{k_2-1}+k_2\rho^{k_2}}{(1-\rho)^2} + \frac{1}{\mu-\lambda}\Big(\frac{k_2(k_2-1)}{2}\\ -\rho\frac{1-\rho^{k_2}-k_2\rho^{k_2-1}+k_2\rho^{k_2}}{(1-\rho)^2}\Big) + \frac{1}{\alpha\lambda(\mu-\lambda)(c\mu-\lambda)}\Big(\frac{(c-1)\mu\alpha\lambda^2}{c\mu-\lambda}\Big(\frac{\lambda-\lambda(\frac{\lambda}{c\mu})^{k_1}}{c\mu-\lambda} + k_2 - (k_1+k_2)(\frac{\lambda}{c\mu})^{k_1}\Big)\\ + (\mu-\lambda-\alpha)(c\mu-\lambda)\frac{\lambda^{k_2+2}}{\mu^{k_2-1}}\frac{-k_1\frac{c\mu}{\lambda}(\frac{\lambda}{c\mu})^{k_1}+k_1(\frac{\lambda}{c\mu})^{k_1}+1-(\frac{\lambda}{c\mu})^{k_1}}{(c\mu-\lambda)^2} + \alpha(\mu-\lambda)\lambda\Big(\frac{(k_1-1)k_1}{2}+k_2k_1\Big)\Big)\\ + \frac{1}{\alpha(\mu-\lambda)(c\mu-\lambda)(c\mu-\lambda-\gamma)}\Big((c\mu-\lambda-\gamma)c^{k_2-1}\Big((c-1)\alpha\mu^{k_2}\lambda^{-k_2+1}+(\mu-\lambda-\alpha)(c\mu-\lambda)\Big)\\ (\frac{\lambda}{c\mu})^{k_1}\frac{c\mu}{c\mu-\lambda}(\frac{\lambda}{c\mu-\lambda}+k_1) - \gamma\alpha(\mu-\lambda)\frac{c\mu}{(c\mu-\lambda)}(\frac{\lambda}{c\mu-\lambda}+k_1)\Big) + \frac{1}{c\mu-\lambda-\gamma}\frac{\lambda+\gamma}{\gamma}(\frac{\lambda}{\gamma}+k_1)\end{aligned}$$

The closed form expression for $\mathbb{E}[E]$ is determined by using the required probabilities below:

$$\pi_{OFF} = k_1\pi_{0,0}$$
$$\pi_{SETUP} = \frac{\lambda}{\gamma}\pi_{0,0}$$
$$\pi_{IDLE} = \frac{\lambda}{\alpha}\pi_{0,0}$$
$$\pi_{NORMAL} = \Big(\Big(\frac{\lambda}{\alpha} - \frac{\rho}{1-\rho}\Big)\frac{1-\rho^{k_2}}{1-\rho} - \frac{\lambda}{\alpha} + k_2\frac{\rho}{1-\rho}\Big)\pi_{0,0}$$
$$\pi_{SCALED} = 1 - \pi_{OFF} - \pi_{SETUP} - \pi_{OFF} - \pi_{SCALED}$$

At this point, we have solved for $\mathbb{E}[N]$ and the probability of each energy state for both $k_1 < k_2$ and $k_1 > k_2$ cases. With these expressions, we could easily find out the expected number of jobs in the system and expected energy consumption once the values of the parameters are known. On the other hand, the expressions developed are highly complex, so it is difficult to make general observations based on the expressions themselves, so we will conduct numerical experiments to achieve this purpose in the next chapter.

Chapter 6

Experiments and Observations

In this chapter, the main goal is to explore the effect of dynamic voltage scaling under various system settings via numerical experiments based on the closed form expressions developed in Chapter 5. Apart from meeting this goal, we also investigate how dynamic voltage scaling should be utilized under the energy response time product (ERP) performance metric, which gives an example of the impact of dynamic voltage scaling for a particular cost function.

In order to conduct the experiments, we need to define energy costs for each of the energy states for our system. There are five energy states in our model, and we select E_{NORMAL} as the nominal energy cost, then we express energy costs for the other energy states with respect to E_{NORMAL}. Typically, a server consumes more energy during setup and consumes less energy when it is idling. We capture these effects by assigning energy cost in appropriate proportion to E_{NORMAL}. Once a server starts to operate using dynamic voltage scaling, the energy consumption highly depends on its operating frequency, and according to [5],

$$Dynamic\ CPU\ power = c_1 f^3$$

where c_1 is a processor-dependent constant coefficient, and f is the CPU operating frequency. This gives an exponential correlation with factor of 3 for the operating speeds. Thus, we model energy cost for scaled state as:

$$E_{SCALED} = c^3 E_{NORMAL}$$

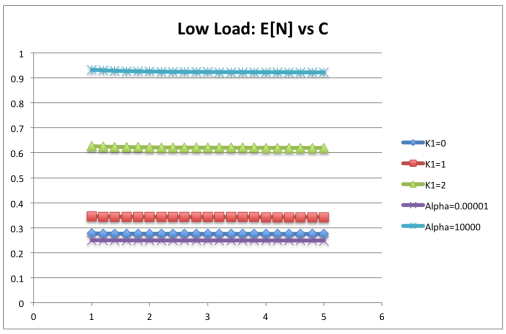
Before conducting the experiments, we must discuss our experimental setup separately. Since the closed form expressions of performance metrics developed in Chapter 5 are composed of numerous parameters, we must fix the values of some parameters in order to see the impact of dynamic voltage scaling. We consider the scaled speed (c) and when to turn on dynamic voltage scaling (k_2) as the main contributors to the impact of dynamic voltage scaling on the system, so we then fix the remaining parameters. We consider various settings. There are four types of system that we consider, Low Load, Medium Load, Heavy Load, and Over Load, which are defined based on the ratio between the arrival rate and the processing rate. Within each type of system, we also select different values for when to turn off the server when no jobs are present (α) and when to turn on the server (k_1). For example, $\alpha = 0.00001$ corresponds to a very slow turn off rate, which means we almost never turn off the server even when the system is idling often; On the other hand $\alpha = 10000$, represents a fast turn off rate, which means we turn off the server almost as soon as the system is in the idle state. We summarize the energy costs for each energy state and the values of fixed parameters in Table 6.1. To illustrate how we use this table for experiments, we describe the system configuration for Low Load as an example. We perform each experiment on five different configurations, which contain different values for either k_1 or α while all the remaining parameters simply follow the values listed in Table 1. For example, for the configuration when $k_1 = 1$, we use $\mu = 1, \gamma = 0.85, k_2 = 5, \alpha = 0.11$ as system parameters to conduct the experiment in Figure 6.1 (a), and we use $\mu = 1, \gamma = 0.85, \alpha = 0.11, c = 2.8$ as system parameters to conduct the experiment in Figure 6.1 (b).

Parameter	Value
E_{OFF}	0
E_{NORMAL}	100
E_{IDLE}	70
E_{SETUP}	140
E_{SCALED}	$c^3 * 100$
Low Load λ	0.2
Med Load λ	0.5
Heavy Load λ	0.8
Over Load λ	1.6
μ	1
γ	0.85
k_1	2 unless specified
k_2	5 unless specified
α	0.11 unless specified
c	2.8 unless specified

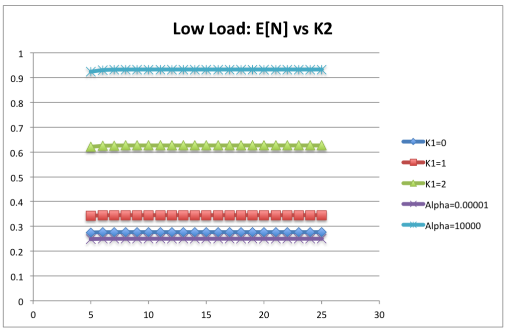
TABLE 6.1: Table of Values for Experiments

6.1 Low Load System

We study the dynamic voltage scaling effects on the expected number of jobs $\mathbb{E}[N]$, the expected energy consumption $\mathbb{E}[E]$, obtaining the results in Figures 6.1 and 6.2. We see both scaled speed c and when to turn on speed scaling k_2 do not affect $\mathbb{E}[N]$ and $\mathbb{E}[E]$ a lot, this is expected since the system is low load, and as the system spends most of

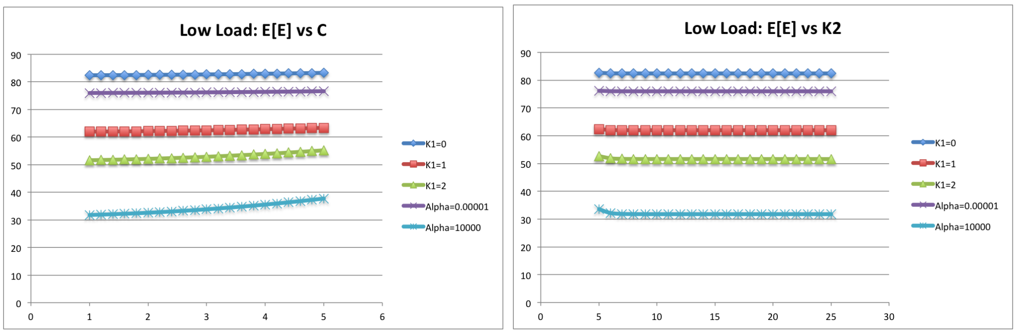


(A) $\mathbb{E}[N]$ vs c



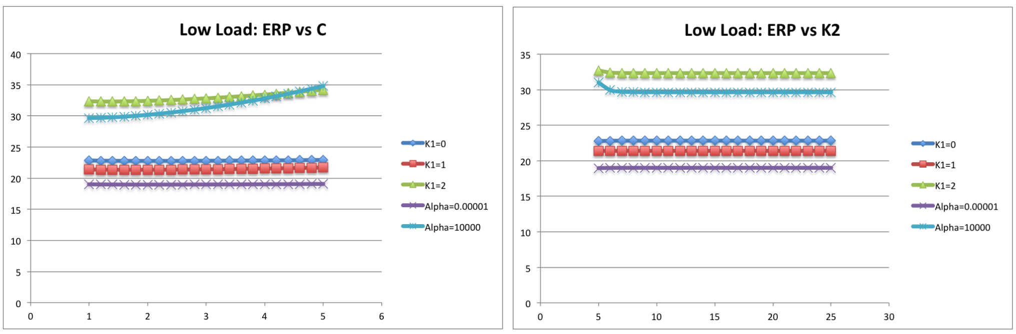
(B) $\mathbb{E}[N]$ vs k_2

FIGURE 6.1: DVS effect on $\mathbb{E}[N]$ in low load system



(A) $\mathbb{E}[E]$ vs c (B) $\mathbb{E}[E]$ vs k_2

FIGURE 6.2: DVS effect on $\mathbb{E}[E]$ in low load system



(A) ERP vs c (B) ERP vs k_2

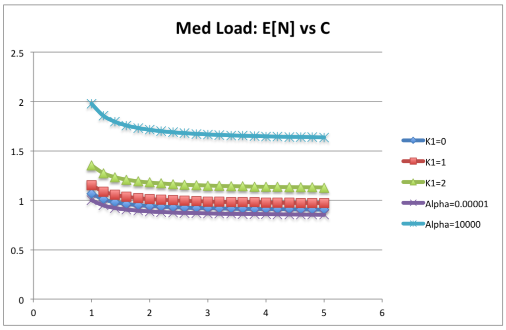
FIGURE 6.3: DVS effect on ERP in low load system

the time operating at the nominal speed, dynamic voltage scaling would be rarely used. There is another interesting observation worth mentioning here, we see both $\mathbb{E}[N]$ and $\mathbb{E}[E]$ are impacted a lot when we change values for k_1 and α. As explained, the low loaded system spends most of the time operating at the nominal rate, as a result, we would expect the system spends most of its time at the left side of the resulting CTMC (Figure 4.1), which is mainly among the OFF, SETUP, BUSY, IDLE states, such that when to turn on server k_1 and how fast we turn off server α impacts the expected number of jobs significantly.

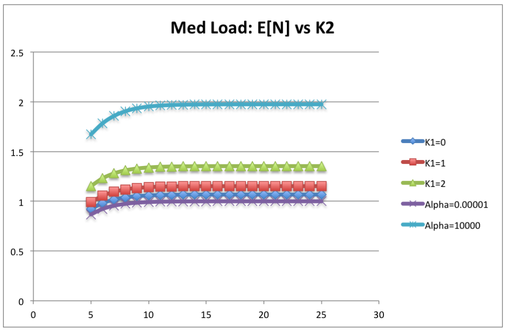
We are also interested in observing how dynamic voltage scaling affects the system under a particular performance metric. To illustrate, we conduct our experiments under the energy response time product (ERP) metric. The effect is shown in Figure 6.3. For a low load system, the DVS effect is not obvious, we see as c increases, the resulting ERP for nearly instant off system and the system with $k_1 = 2$ increases. This is due to the fact that a relatively rapid increase in energy consumption for the two systems, delay in starting the system (a larger value of k_1) and a nearly instant off system (a very large value of α) are more likely to cause a system to use speed scaling to process jobs.

6.2 Medium Load System

We conduct similar experiments for a medium load system. Under medium load, the system spends more time in the middle of the CTMC in Figure 4.1, mainly in the BUSY and SCALED regions, as a result, we expect the effects from the turn on server threshold k_1 and α to reduce. This is reflected in Figure 6.4 and Figure 6.5, as we can see the gaps between curves are reduced compared with Figure 6.1 and Figure 6.2, respectively. In addition, we would expect to see a greater effect of DVS for a medium load system, this is shown in Figure 6.5, as we see $\mathbb{E}[N]$ reduces dramatically with a small increase in c for all system configurations, and $\mathbb{E}[N]$ increases rapidly when we delay the start

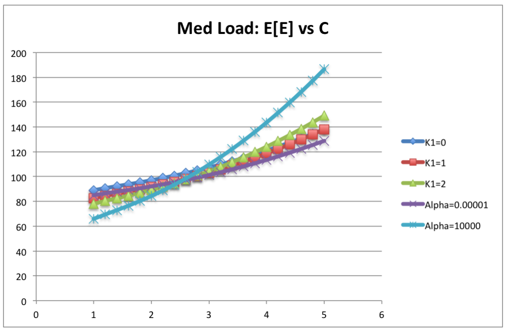


(A) $\mathbb{E}[N]$ vs c

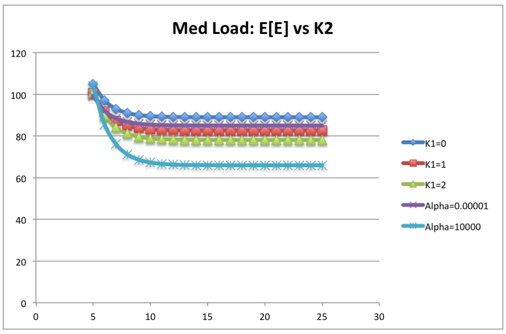


(B) $\mathbb{E}[N]$ vs k_2

FIGURE 6.4: DVS effect on $\mathbb{E}[N]$ in medium load system

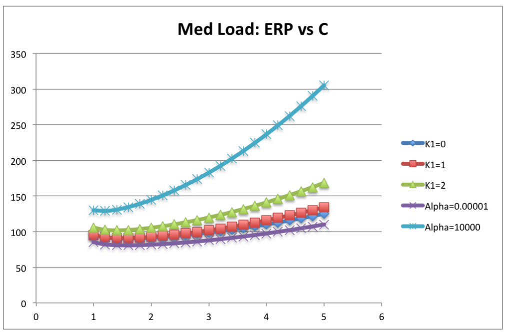


(A) $\mathbb{E}[E]$ vs c

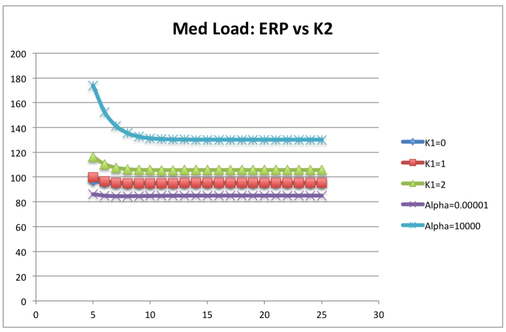


(B) $\mathbb{E}[E]$ vs k_2

FIGURE 6.5: DVS effect on $\mathbb{E}[E]$ in medium load system

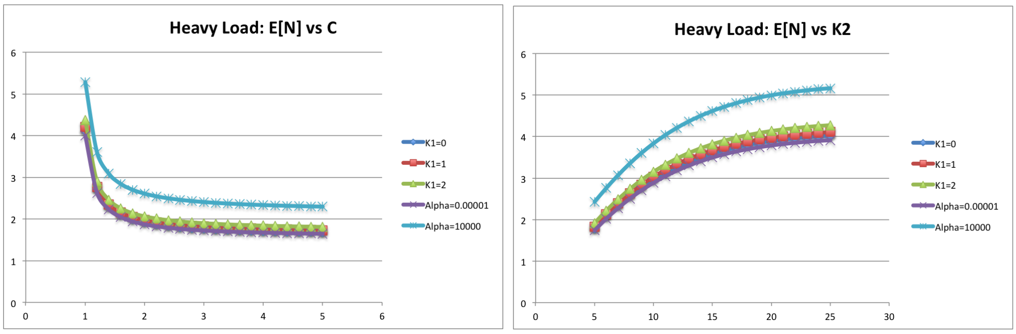


(A) ERP vs c



(B) ERP vs k_2

FIGURE 6.6: DVS effect on ERP in medium load system



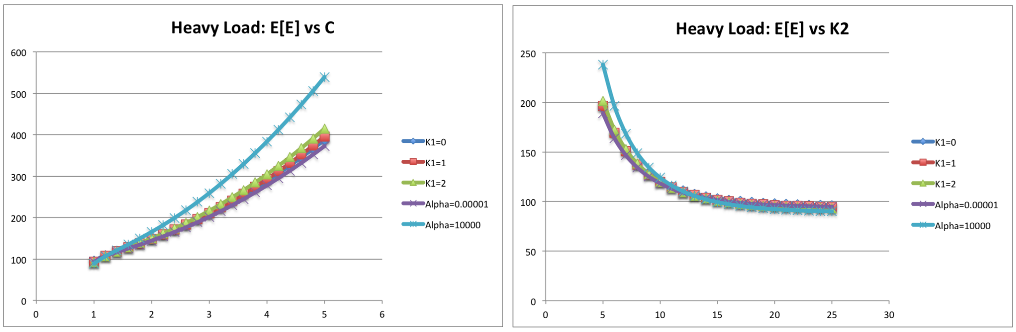
(A) $\mathbb{E}[N]$ vs c (B) $\mathbb{E}[N]$ vs k_2

FIGURE 6.7: DVS effect on $\mathbb{E}[N]$ in heavy load system

of DVS with an increase in k_2. With this observation, we would expect the effect from DVS would be beneficial to performance. From the resulting ERP in Figure 6.6, the ERP drops first as c increases but then it rises again under all system configurations, this suggests there exists an optimal value of c that minimizes ERP. However, the effect of k_2 is not clear. Figure 6.6 (b) suggests DVS should be delayed to start, or in other words, not to use DVS. The actual data points in Figure 6.6 (b) actually show that ERP rises slowly as the value of k_2 increases, but the change is so small that it is difficult to see the benefit of using DVS.

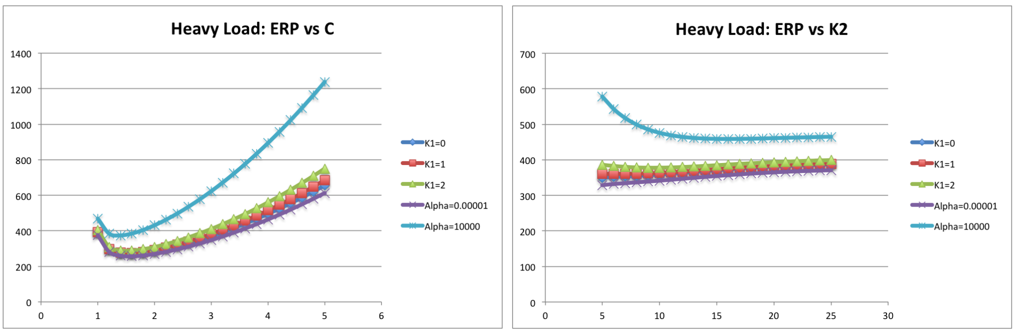
6.3 Heavy Load System

A heavy load system spends more time in the BUSY and SCALED regions towards the right side of the CTMC in Figure 4.1, and we expect the system is impacted more by the DVS effects and less by the effects from k_1 and α. From Figure 6.7 and Figure 6.8, we see the gaps among curves in each figure get smaller, this confirms the system is impacted less by the effects from k_1 and α. There is also a more rapid reduction or increase for $\mathbb{E}[N]$ as we increase c or delay using DVS (increasing k_2), which shows the system benefits from DVS. Examining the resulting ERP in Figure 6.9, it shows the minimum



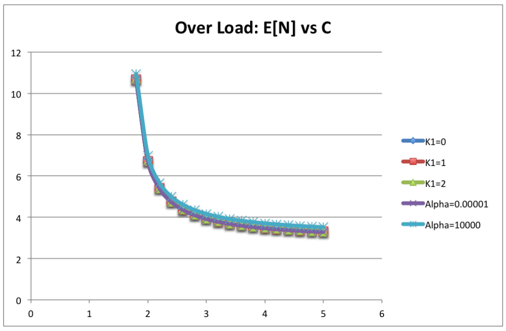
(A) $\mathbb{E}[E]$ vs c (B) $\mathbb{E}[E]$ vs k_2

FIGURE 6.8: DVS effect on $\mathbb{E}[E]$ in heavy load system

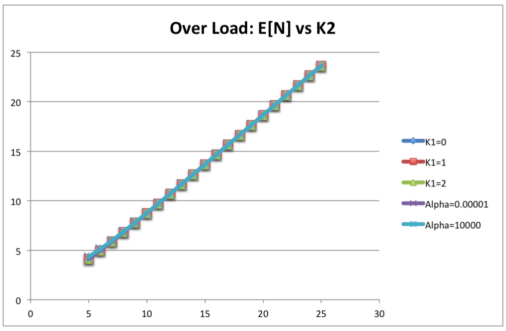


(A) ERP vs c (B) ERP vs k_2

FIGURE 6.9: DVS effect on ERP in heavy load system

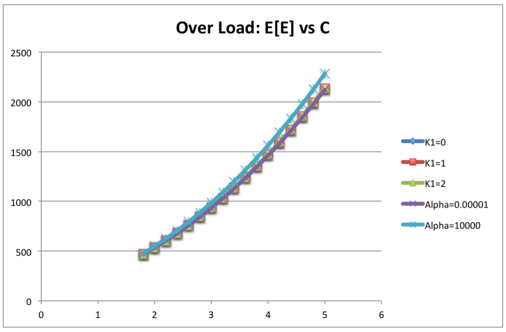


(A) $\mathbb{E}[N]$ vs c

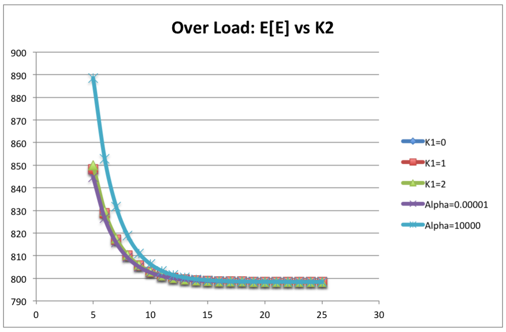


(B) $\mathbb{E}[N]$ vs k_2

FIGURE 6.10: DVS effect to $\mathbb{E}[N]$ in over load system



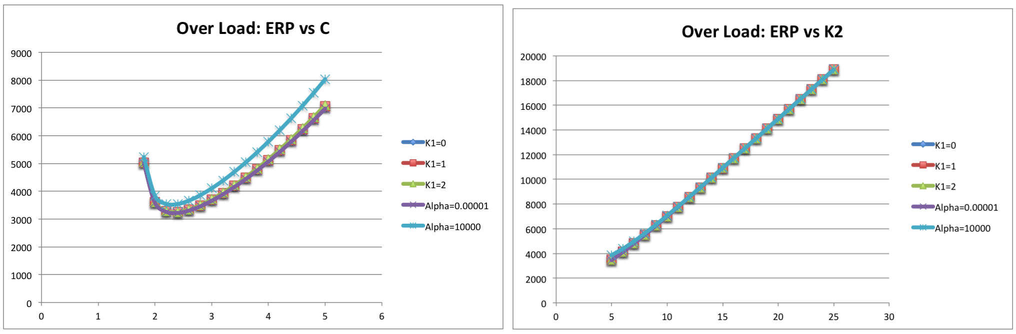
(A) $\mathbb{E}[E]$ vs c



(B) $\mathbb{E}[E]$ vs k_2

FIGURE 6.11: DVS effect to $\mathbb{E}[E]$ in over load system

ERP occurs when we increase the value of c, and it suggests optimality is achieved with a small increase in c. On the other hand, the ERP results with respect to k_2 in Figure 6.9 (b) are once again not clear, as we can not see a clear pattern for how DVS should be operated. We do see that there exists an optimal value of k_2 that minimizes ERP, but it could vary significantly with different system configurations.



(A) ERP vs c (B) ERP vs k_2

FIGURE 6.12: DVS effect to ERP in over load system

6.4 Over Load System

Over load system is a special case, where DVS has to be used in order to have a stable system, since the arrival rate is now larger than the nominal processing speed. Under over load system, the system spends more of its time at the BUSY and SCALED regions (on the right side of the CTMC in Figure 4.1), so we would expect that the over load system has the most benefit from DVS, and we expect the effects from k_1 and α to be minimal. From Figure 6.10 and Figure 6.11, the graphs with different system configurations almost overlap with each other, this confirms the effects to the system from k_1 and α are almost eliminated, and the significant drops and rises from those graphs show that DVS is essential. Even though DVS is required to be used for the over load system, we can still observe insights on how DVS should be used. The resulting ERP in Figure 6.12 shows once again that there exists an optimal value of c that minimizes ERP, and DVS should be used as soon as possible.

6.5 Discussion of Optimality

After making observations on how DVS affects the system under different loads, we want to discuss the optimality of operating DVS under the ERP metric. Despite the fact that the effect from DVS varies under different workloads, optimality consistently happens at system configurations with a very small value for α, which represents the server is nearly never turned off, and this result is consistent with what [7] has claimed, as never turning off the server is one of the optimal policies under the ERP metric. Optimally operating DVS is achieved differently under different system workloads. For low load system, the DVS impact is minimal, and optimality is achieved by not using DVS at all (setting turn on DVS threshold (k_2) to be infinity). For medium, heavy, and overload workloads, optimality is achieved by operating the DVS server at its optimal speed (c), depending on the system configuration. The benefit of using DVS in medium workload is not significant, and a high value of k_2 (even choosing it to be infinity) is recommended to achieve optimality. The benefit of using DVS in heavily loaded and overloaded systems is significant. DVS should be used with an optimal value of k_2 for heavy load system (the value of k_2 depends on the system parameters), and DVS should be used immediately for over load system.

Chapter 7

Conclusion

Energy consumption in datacentres has become a severe problem in our society due to the popularity of cloud computing. We must continue our efforts in researching better energy saving practices. Here we presented our research on the dynamic voltage scaling effect for a single server with on/off energy control. We have established a CTMC model that combines the effects of turning on/off server as well as turning on/off dynamic voltage scaling, and we have solved for explict closed form expressions for performance metrics $\mathbb{E}[N]$, $\mathbb{E}[R]$, and $\mathbb{E}[E]$. With the CTMC model and these expressions, one can easily determine the performance of a given system configuration, or use the given expressions to search for an effective system configuration. Lastly, we also revealed some of the implications for operating a DVS server in our experiments, providing insights for system managers.

We want to extend this work to consider different performance metrics as well as completing Case 2 (Section 5.2). We want to evaluate our work under different performance metrics to have a more complete picture of the DVS effect, in particular, we want to take switching costs into consideration, as it is one of the popular cost considerations in this field. We also want to examine the DVS effect for other performance metrics other than ERP since ERP does tend to simplify optimality analysis. Although we argued Case 2 where $k_2 < k_1$ is not an interesting case to investigate, it is still a possible case of operating DVS. Only by completing Case 2 as well as performing evaluations under different cost functions, and performace metrics, can we have the full picture of the analysis of on/off single servers with dynamic voltage scaling.

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