Riesz Sequences and Frames of Exponentials associated with non-full rank lattices.

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Abstract

Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite positive measure (not necessarily bounded). Let $(c_j)_{j=1}^k$ be a given collection of vectors in $\mathbb{R}^d$, and let H be the dual lattice of a full rank lattice $K \subset \mathbb{R}^d$. For $\lambda \in \mathbb{R}^d$, let e_λ denote the exponential

$$e_\lambda(x) := e^{2\pi i\langle \lambda, x\rangle},\ \ x \in \mathbb{R}^d.$$

It is known that, the collection

$$E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\},$$

where $\Lambda = \{(c_j + h) \in \mathbb{R}^d : h \in H, j \in \{1, ..., k\}\}$, forms Riesz basis on $\Omega \subset \mathbb{R}^d$ if the domain Ω is a k-tile domain and if, in addition, it satisfies an extra arithmetic property, called the admissibility condition. The theory of shift invariant spaces generated by the full rank lattice K plays an important role to analyze and solve the above problem.

The main goal of this thesis is to study a variant of the problem above where the dual lattice H is replaced by a non-full rank lattice in $\mathbb{R}^d$. In particular, given an at most countable index set J and a collection of vectors $(c_j)_{j\in J} \subset \mathbb{R}^d$, we examine the existence of Riesz sequences, frames and Riesz bases of the form

$$E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\},$$

where $\Lambda = \{(c_j + h) \in \mathbb{R}^d :\ h \in H,\ j \in J\}$, on $\Omega \subset \mathbb{R}^d$ as above, and H, a non-full rank lattice in $\mathbb{R}^d$. Our results are obtained using an extention of the theory of shift invariant subspaces of $L^2(\mathbb{R}^d)$, where the shifts are now generated by a non-full rank lattice in $\mathbb{R}^d$.

Dedication

To mum & dad and Gifty Obeng, my beloved.

Declaration

I ...., student of the Department of Mathematics & Statistics under the Faculty of Science of the McMaster University, aware of my responsibility of the penal law, declare and certify with my signature that my thesis entitled .... is entirely the result of my own work. I have faithfully and accurately cited all my sources, including books, journals, handouts and unpublished manuscripts, as well as any other media, such as the Internet, letters or significant personal communication.

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"Great is our Lord God, and greatly to be praised. For His mercies over us all, and all His works shall praise Him." I sincerely want to thank God Almighty for His mercies upon me.

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Contents

1 Introduction

The study of the existence of Riesz Basis and frames of exponentials under the action of full rank lattices has some known results. Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite Lebesgue measure. For each $k \in \mathbb{Z}^d$, let e_k denote the exponential

$$e_k(x) := e^{2\pi i\langle k,x\rangle}, \; x \in \mathbb{R}^d.$$

The following theorems (taken from [5] and [6]) give the necessary and sufficient conditions for which the collection $E(\mathbb{Z}^d) := \{e_k : k \in \mathbb{Z}^d\}$ is a Riesz sequence and a frame in $L^2(\Omega)$.

Theorem 1.0.1. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite Lebesgue measure. The collection $E(\mathbb{Z}^d)$ forms a Riesz sequence in $L^2(\Omega)$ if and only if*

$$0 < \sum_{m\in\mathbb{Z}^d} \chi_\Omega(x+m) \leq M, \quad \textit{for a.e } x \in [0,1]^d, \tag{1.1}$$

where the constant $M < \infty$.

Theorem 1.0.2. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite Lebesgue measure. The collection $E(\mathbb{Z}^d)$ forms a frame for $L^2(\Omega)$ if and only if*

$$\sum_{m\in\mathbb{Z}^d} \chi_\Omega(x+m) \leq 1, \quad \textit{for a.e } x \in [0,1]^d. \tag{1.2}$$

The proof of Theorem 1.0.1 and Theorem 1.0.2 is given in [5].

We now turn to a more difficult problem. Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite Lebesgue measure. Given a full rank lattice $H \subset \mathbb{R}^d$ and J, an at most countable index set, we define the set of frequencies $\Lambda \subset \mathbb{R}^d$ by $\Lambda := \{(c_j + h) \in \mathbb{R}^d : \; h \in H, \; j \in J\}$, for $(c_j)_{j\in J}$, some collection of vectors in $\mathbb{R}^d$. Let e_λ denote the exponential

$$e_\lambda(x) := e^{2\pi i\langle \lambda,x\rangle}, \; x \in \mathbb{R}^d.$$

The problem of the existence of such collection of vectors $(c_j)_{j\in J} \subset \mathbb{R}^d$, such that

$$E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\},$$

forms a Riesz basis and a frame in $L^2(\Omega)$ is well analyzed in [1] and [9].

In [9], Kolountzakis establised that, all bounded multi-tile domains support Riesz Basis of exponentials. The latter, in his paper, posed an open problem concerning the case where the domains are unbounded. This open problem was well investigated in [1]. The authors in [1] showed that, in the case where the domains are unbounded, there is the need for an extra condition, which they called "admissibility". The authors in [1] a gave proof of their result via the theory of shift invariant spaces. After a successful proof of their result using both the multitile at level k on the full lattice Λ and the admissibility conditions, they realized by way of an example that, this extra condition, that is, the admissibility condition, is too strong a condition as this example (Example 3.3 in [1]) showed that, an unbounded $k-$tile measurable domain of finite positive measure supports Riesz basis of the form $\{e_{c_j+h} : h \in H, j \in \{1, ..., k\}\}$ without being admissible.

In this thesis, we seek to examine the existence of Riesz sequences and frames of exponentials on measurable sets of finite Lebesgue measure (not necessarily bounded), under the action of **non-full rank lattices**. In particular, we seek to investigate the following problem:

Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite Lebesgue measure and $H \subset \mathbb{R}^d$ be a non-full rank lattice. Let $(c_j)_{j\in J}$ be a given collection of vectors in $\mathbb{R}^d$ associated with the set of frequencies

$$\Lambda = \{(c_j + h) \in \mathbb{R}^d : \ h \in H, \ j \in J\}.$$

We ask the question, under which condition(s) does the collection

$$E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\},$$

form a Riesz sequence and a frame in $L^2(\Omega)$?

In Chapter 2, we consider some basic definitions of the keywords in the thesis, and also, we took a look at the background of the thesis.

The Paley-Wiener spaces associated with the set Ω, is defined by

$$PW_\Omega = \{f \in L^2(\mathbb{R}^d) : \ \hat{f} \in L^2(\Omega)\}.$$

We use subspaces of PW_Ω, which are the family of shift invariant spaces that we are of our interest. Note that, if we let

$$N = span\{e_{c_j+h}\chi_\Omega : h \in H, j \in J\},$$

then the subspace

$$V = span\{f, \hat{f} \in N\}$$

is invariant under translations by the non-full rank lattice H. This motivates our study in Chapter 3 of subspaces of $L^2(\mathbb{R}^d)$ which are invariant under translations by a non-full rank lattice H. We consider there, without loss of generality and for simplicity, non-full rank lattices of the form $\mathbb{Z}^n \times \{0\}^{d-n}$ where $n \in \mathbb{Z}$ with $1 \leq n < d$. Our goal here is to extend some of the results in the theory of shift-invariant subspaces of $L^2(\mathbb{R}^d)$ developed by Ron and Shen in [13] and also by Marcin Bownik in [3]. This theory plays an important role in the study of Riesz sequences and frames and has applications in shift-invariant systems, Weyl-Heisenberg systems, affine (wavelet) systems, and Gramian matrices. An important result which we obtained from the extention of the theory of shift invariant subspaces of $L^2(\mathbb{R}^d)$ is given by Theorem 3.2.10. This theorem reduces the problem of checking whether a system in closed subspaces of $L^2(\mathbb{R}^d)$ is a Riesz sequence or frame to analyzing the fibers in closed subspaces of $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$.

In Chapter 4, we use Theorem 3.2.10 to establish the link between the extended theory of shift invariant spaces and our main problem which was analyzed in $\mathbb{R}^2$. We later considered some examples of our problem, on some bounded subsets of $\mathbb{R}^2$.

In Chapter 5, we use basic changes of variables to extend some of the results that we proved with the lattice $\mathbb{Z}^n \times \{0\}^{d-n}$ to systems generated by general non-full rank lattices in $\mathbb{R}^d$. We conclude by stating a more general result on the above problem.

2 Preliminaries and Background

We shall give definitions of some of the keywords in this thesis. Later in this Chapter, we will also consider the background problem of this thesis.

2.1 Basic Definitions and Notations

Definition 2.1.1 (**Non-full rank Lattice**). *$H \subset \mathbb{R}^d$ is a **non-full rank lattice** if there exists a $d \times d$ invertible matrix $\boldsymbol{Q}$ such that $H = \boldsymbol{Q}[\mathbb{Z}^n \times \{0\}^{d-n}]$.*

Definition 2.1.2 (**Bessel sequence**). *Let $\mathcal{H}$ be a Hilbert space. A countable family of elements $\{f_j\}_{j\in J} \subset \mathcal{H}$ is a Bessel sequence for $\mathcal{H}$ if there exists $B > 0$ such that*

$$\sum_{j\in J} |\langle f_j, g\rangle|^2 \leq B\|g\|^2_{\mathcal{H}}, \quad \forall g \in \mathcal{H}. \tag{2.1}$$

Definition 2.1.3 (**Frame**). *Let $\mathcal{H}$ be a Hilbert space. A countable family of elements $\{f_j\}_{j\in J} \subset \mathcal{H}$ is a frame for $\mathcal{H}$ if there exists constants A, $B > 0$ such that*

$$A\|g\|^2_{\mathcal{H}} \leq \sum_{j\in J} |\langle f_j, g\rangle|^2 \leq B\|g\|^2_{\mathcal{H}}, \quad \forall g \in \mathcal{H}. \tag{2.2}$$

The constants A and B are called frame bounds. In particular, A is called the lower frame bound and B is the upper frame bound. They are not unique.

Definition 2.1.4 (**Riesz Sequence**). *Let $\mathcal{H}$ be a Hilbert space. A countable family of elements $\{f_j\}_{j\in J} \subset \mathcal{H}$ is a Riesz sequence for $\mathcal{H}$ if there exists constants A, $B > 0$ such that*

$$A \sum_{j\in J} |c_j|^2 \leq \left\| \sum_{j\in J} c_j f_j \right\|^2_{\mathcal{H}} \leq B \sum_{j\in J} |c_j|^2, \quad \forall (c_j) \in \ell^2(J). \tag{2.3}$$

The constants A and B are called Riesz bounds. In particular, A is called the lower Riesz bound and B is the upper Riesz bound. They are also not unique.

Definition 2.1.5 (Riesz Basis). *A Riesz basis for $\mathcal{H}$ is the family of the form $\{Ue_j\}_{j=1}^{\infty}$, where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.*

Theorem 2.1.6. *Let $\mathcal{H}$ be a Hilbert space. A countable family of elements $\{f_j\}_{j\in J} \subset \mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_j\}_{j\in J} \subset \mathcal{H}$ is both a frame and a Riesz sequence for $\mathcal{H}$.*

Proof. See Theorem 3.6.6 in [12]. □

Some General Notations:

Throughout this thesis, we will denote $|S|$ as the Lebesgue measure of the measurable set S and denote $\#(S)$ as the number of elements in the set S.
Also denote by $det(\mathbf{Q})$, the determinant of the matrix $\mathbf{Q}$ and denote by J, an at most countable index set.
We denote by $\hat{f}$, the Fourier transform of the integrable function $f : \mathbb{R}^d \to \mathbb{C}$, with

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx,$$

and also, we denote by $\check{f}$, the inverse Fourier transform of f, with

$$\check{f}(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi,$$

where $x \cdot \xi$ is the inner product of x, $\xi \in \mathbb{R}^d$.

2.2 Background Problem

In this section, we state and observe the background problem. The problem is stated with full rank lattices. For the details of the background problem, one may look at [1]. Let us start by considering the definitions below which have been taken from [1].

Definition 2.2.1. $\Lambda \subset \mathbb{R}^d$ *is a **full lattice** if there exist a $d \times d$ invertible matrix M so that $\Lambda := M\mathbb{Z}^d$. A **Fundamental Domain** with respect to the lattice $\Lambda = M\mathbb{Z}^d$ is given by $D = M\mathbb{T}^d$. The **dual lattice of** $\Lambda = M\mathbb{Z}^d$, denoted by H is given by*

$$H = \{h \in \mathbb{R}^d : \langle h, \lambda\rangle \in \mathbb{Z}^d \textit{ for all } \lambda \in \Lambda\}.$$

Definition 2.2.2 (**Multitiling**). *Let k be a positive integer. We say a measurable set $\Omega \subset \mathbb{R}^d$ multi-tiles $\mathbb{R}^d$ at level k on a lattice $\Lambda \subset \mathbb{R}^d$ if for almost every $w \in D$,*

$$\sum_{\lambda \in \Lambda} \chi_\Omega(w + \lambda) = k.$$

Given a measurable set $\Omega \subset \mathbb{R}^d$, and a lattice $\Lambda \subset \mathbb{R}^d$, for every $w \in D$, we denote

$$\Lambda_w(\Omega) = \Lambda_w := \{\lambda \in \Lambda : \ w + \lambda \in \Omega\}.$$

Definition 2.2.3 (**Admissibility**). *Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite measure and $\Lambda \subset \mathbb{R}^d$, be a full lattice. We say that Ω **is admissible for** Λ if there exist a vector $v \in H$ and a number $n \in \mathbb{N}$; such that for almost every $w \in D$, the numbers $\{\langle v, \lambda \rangle : \ \lambda \in \Lambda_w\}$ are distinct elements (mod n). We will emphasize on the dependance on n and v.*

Theorem 2.2.4. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite measure and $\Lambda \subset \mathbb{R}^d$, a full lattice with dual lattice H. If*

(i) Ω multi-tiles $\mathbb{R}^d$ at level k by translation on the lattice $\Lambda \subset \mathbb{R}^d$,

*(ii) Ω **is admissible for** Λ ,*

then, there exist $a_1, ..., a_k \in \mathbb{R}^d$, such that the set $E(H; a_1, ..., a_k)$ is a Riesz basis for $L^2(\Omega)$, where

$$E(H; a_1, ..., a_k) = \{e^{2\pi i (a_j + h).w} : \ h \in H, \ j = 1, ..., k\}.$$

Theorem 2.2.4 is the Main Result obtained in [1] and was proved via shift invariant spaces.

Definition 2.2.5 (**Shift Invariant Space**). *A closed subspace $V \subset L^2(\mathbb{R}^d)$ is a $H-$shift invariant if $f \in V$, then*

$$T_h f \in V, \quad \forall h \in H$$

where $T_h f(x) = f(x - h)$ for every $x \in \mathbb{R}^d$.

Definition 2.2.6 (**Paley-Wiener Spaces**). *The Paley-Wiener Space are family of shift invariant spaces which we are interested in. These spaces are defined by*

$$PW_\Omega = \{f \in L^2(\mathbb{R}^d) : \ \hat{f} \in L^2(\Omega)\}.$$

The following theorem is a result obtained in [1] using the concept of shift invariant spaces. The Main Result in [1] was proved by translating the background problem into part (i) of Theorem 2.2.7. Then, by using the fact that Theorem 2.2.7 (i) and (ii) are equivalent, the proof of Theorem 2.2.4 is given by using Theorem 2.2.7 (ii).

Theorem 2.2.7. *Let $\Omega \subset \mathbb{R}^d$ be a $k-$tile measurable set of finite measure. Given $\phi_1, ..., \phi_k \in PW_\Omega$, we define*

$$T_w = \begin{bmatrix} \hat{\phi}_1(w+\lambda_1) & ..... & \hat{\phi}_k(w+\lambda_1) \\ . & ..... & . \\ . & ..... & . \\ . & ..... & . \\ \hat{\phi}_1(w+\lambda_k) & ..... & \hat{\phi}_k(w+\lambda_k) \end{bmatrix}$$

where $\lambda_j = \lambda_j(w)$ for $j = 1, ..., k$ are the k values of Λ that belongs to Λ_w. Then the subsequent statements are equivalent:

(i) The set $\Phi_H = \{T_h \phi_j : \ h \in H, \ j = 1, ..., k\}$ is a Riesz basis for PW_Ω.

(ii) There exist $A, B > 0$ such that for almost every $w \in D$,

$$A\|x\|^2 \leq \|T_w x\|^2 \leq B\|x\|^2,$$

for every $x \in \mathbb{C}^k$.

Moreover, in this case, the Riesz bounds are given as

$$A = \inf_{w\in D} \|T_w^{-1}\|^2, \ \textit{and} \ B = \sup_{w\in D} \|T_w\|^2.$$

3 Shift Invariant spaces and Non-full rank lattice of $\mathbb{R}^d$

In this chapter, we will start by stating our main problem which was introduced earlier in Chapter 1. In the second section, we will study the extended theory of shift invariant subspaces of $L^2(\mathbb{R}^d)$ under the action of a non-full rank lattice. The last section is the Gramian concept of frames and Riesz sequences. The non-full rank lattice H, is chosen to be $\mathbb{Z}^n \times \{0\}^{d-n}$ without loss of generality and for simplicity.

3.1 Statement of the Main problem

Given a sequence $(a_j)_{j\in J} \subset \mathbb{R}^d$, we consider the associated set of frequencies $\Lambda \subset \mathbb{R}^d$ defined by

$$\Lambda = \{(a_j + k) \in \mathbb{R}^d : \ k \in \mathbb{Z}^n \times \{0\}^{d-n}, \ j \in J\}, \tag{3.1}$$

and the set of exponentials with frequencies in Λ,

$$E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\},$$

where

$$e_\lambda(x) := e^{2\pi i\langle\lambda,x\rangle}, \ \forall x \in \mathbb{R}^d.$$

Main Problem :
Let $E \subset \mathbb{R}^d$ be a measurable set of finite measure. Can we find a sequence $(a_j)_{j\in J} \subset \mathbb{R}^d$ such that $E(\Lambda)$ forms a Riesz sequence or a frame for $L^2(E)$?

Analogous to [1], we analyze the above problem by looking at its equivalent statement in the fibers of $L^2(\mathbb{R}^d)$. In the next section, we will take a look at subspaces of $L^2(\mathbb{R}^d)$ which are invariant under the action of a non-full rank lattice $\mathbb{Z}^n \times \{0\}^{d-n}$.
Both Section 3.2 and 3.3 of this thesis are analogous to Chapter 1 and Chapter 2 of [3]. We give a generalization of the Results obtained in Chapter 1 and Chapter 2 of [3].

3.2 Shift Invariant Spaces

The main goal of this section is achieved in Theorem 3.2.10. The proof of Theorem 3.2.10 is given mainly by using Proposition 3.2.9. Let us start by considering some definitions.

Definition 3.2.1. *A closed subspace $V \subset L^2(\mathbb{R}^d)$ is $\mathbb{Z}^n \times \{0\}^{d-n}-$shift invariant if for every $f \in V$, we have*

$$T_k f \in V, \quad \forall k \in \mathbb{Z}^n \times \{0\}^{d-n},$$

where $T_k f(x) = f(x-k)$ for every $x \in \mathbb{R}^d$.

We will denote by D the set $[0,1]^n$. The Hilbert space $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ is the usual L^2 space associated with the Haar measure $\delta_{\mathbb{Z}^n} \otimes dy$ on $\mathbb{R}^d$. In particular, if $F \in L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$, then

$$\|F\|^2 = \sum_{l \in \mathbb{Z}^n} \int_{\mathbb{R}^{d-n}} |F(l,y)|^2 \, dy.$$

The Hilbert space of square integrable vector functions denoted by

$$\mathbf{H} = L^2(D, L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n}))$$

consists of all vector valued measurable functions $\Phi : D \to L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ whose norm is given by:

$$\|\Phi\|^2 = \int_D \|\Phi(x)\|^2_{L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})} dx \quad < \infty.$$

The fact that the vector valued measurable function $\Phi : D \to L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ is measurable means that $\Phi^{-1}(\mathcal{U})$ is measurable for any $\mathcal{U}$ open subset of $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$. This is equivalent to the property that $\forall \varepsilon > 0$,

$$\{x \in D, \|\Phi(x) - \Phi_0\|^2 < \varepsilon\} \tag{3.2}$$

is a measurable set for fixed $\Phi_0 \in L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$. Φ is weakly measurable if $x \mapsto \langle \Phi(x), F \rangle$ is a measurable scalar function for each $F \in L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$.

Lemma 3.2.2. *The vector valued function $\Phi : D \to L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ is measurable if and only if it is weakly measurable.*

Proof. Firstly, suppose that Φ measurable. Given any continuous function

$$g : L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n}) \to \mathbb{C},$$

we have that $g^{-1}(\mathcal{U})$ is open, where $\mathcal{U}$ is an open subset of $\mathbb{C}$. Therefore,

$$(g \circ \Phi)^{-1}(\mathcal{U}) = \{x \in D, g(\Phi(x)) \in \mathcal{U}\} = \{x \in D, \Phi(x) \in g^{-1}(\mathcal{U})\}$$

is measurable since $g^{-1}(\mathcal{U})$ is open. Hence $g \circ \Phi$ is measurable. In particular if we define $g(\Phi(x)) = \langle \Phi(x), F \rangle$, for any $F \in L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$, then can conclude that $x \mapsto \langle \Phi(x), F \rangle$ is a measurable scalar function.
To show the converse, choose an orthonormal basis $\{e_j\}$ in $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$. Then

$$\|\Phi(x) - \Phi_0\|^2 = \sum_{j \in \mathbb{N}} |\langle \Phi(x) - \Phi_0, e_j \rangle|^2 = \sum_{j \in \mathbb{N}} |\langle \Phi(x), e_j \rangle - \langle \Phi_0, e_j \rangle|^2 . \tag{3.3}$$

Assume Φ is weakly measurable then $x \mapsto \langle \Phi(x), e_j \rangle$ is measurable. Therefore by using the properties of measurable functions and (3.3), we have that $\|\Phi(x) - \Phi_0\|^2$ is a measurable function. Hence, (3.2) is a measurable set. This means that $\Phi : D \to L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ is measurable. $\square$

Proposition 3.2.3. *The mapping $\mathcal{T} : L^2(\mathbb{R}^d) \to \boldsymbol{H}$ defined for $f \in L^2(\mathbb{R}^d)$ by*

$$\mathcal{T} f : D \to L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n}),$$

and

$$\mathcal{T} f(x) = \left\{ \hat{f}(x + l, y) \right\}_{(l,y) \in \mathbb{Z}^n \oplus \mathbb{R}^{d-n}} \quad a.e \quad x \in D,$$

is an isometric isomorphism between $L^2(\mathbb{R}^d)$ and $\boldsymbol{H}$. Furthermore, for any $f \in L^2(\mathbb{R}^d)$, $k \in \mathbb{Z}^n$, we have

$$\mathcal{T} T_k f(x) = e^{-2\pi i x k} \mathcal{T} f(x) \text{ for a.e } x \in D. \tag{3.4}$$

Proof.

$$\begin{aligned}\|\mathcal{T}f\|^2_{\mathbf{H}} &= \int_D \|\mathcal{T}f(x)\|^2_{L^2(\mathbb{Z}^n\oplus\mathbb{R}^{d-n})}dx\\ &= \int_D\int_{\mathbb{Z}^n\oplus\mathbb{R}^{d-n}} |\hat{f}(x+l,y)|^2 dhdx,\end{aligned}$$

where $dh := \delta_{\mathbb{Z}^n}\otimes dy$

Hence,

$$\begin{aligned}\|\mathcal{T}f\|^2_{\mathbf{H}} &= \int_D\sum_{l\in\mathbb{Z}^n}\int_{\mathbb{R}^{d-n}} |\hat{f}(x+l,y)|^2 dydx\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{d-n}} |\hat{f}(x,y)|^2 dydx\\ &= \int_{\mathbb{R}^d} |\hat{f}(x,y)|^2 dxdy, && \text{(by Fubini / Tonelli)}\\ &= \|f\|^2_{L^2(\mathbb{R}^d)}, && \text{(by Plancherel's Identity).}\end{aligned}$$

Also, given any $g \in \mathbf{H}$, we define $f \in L^2(\mathbb{R}^d)$ by its Fourier transform which is given by $\hat{f}(x+l,y) = g(x)|_{(l,y)\in\mathbb{Z}^n\oplus\mathbb{R}^{d-n}}$ for a.e $x \in D$, so that $\mathcal{T}f(x) = g(x)|_{(l,y)}$. Thus, the mapping $\mathcal{T}$ is onto. Clearly, we also have that the mapping $\mathcal{T}$ is one-to-one. Therefore, $\mathcal{T}$ is indeed an isometric isomorphism between $L^2(\mathbb{R}^d)$ and $\mathbf{H}$.

□

Definition 3.2.4. *A range function is a mapping*

$$J : D \to \Big\{ \textit{closed subspaces of } L^2(\mathbb{Z}^n\oplus\mathbb{R}^{d-n}) \Big\}.$$

For $x \in D$, let $P(x) : L^2(\mathbb{Z}^n\oplus\mathbb{R}^{d-n}) \to J(x)$ be the associated orthogonal projections onto $J(x)$. We say that $J(x)$ is measurable if these projections are weakly operator measurable i.e. $x \mapsto \langle P(x)a, b\rangle$ is a measurable scalar function for each $a, b \in L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$.

For a given range function $J(x)$, we define the subspace M_J of $\mathbf{H}$ as

$$M_J = \{\phi \in \mathbf{H} : \ \phi(x) \in J(x) \text{ for } a.e.\ x \in D\}. \tag{3.5}$$

Lemma 3.2.5. *Given any $(\phi_j)_{j\in\mathbb{N}} \subset M_J$ with $\phi_i \to \phi$ in $\boldsymbol{H}$ for $i \to \infty$, there exists a subsequence ϕ_{j_k} such that $\phi_{j_k}(x) \to \phi(x)$ in $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ as $j_k \to \infty$ for a.e $x \in D$.*

Proof. The fact that $\phi_j \to \phi$ in $\mathbf{H}$ means that $\forall \varepsilon > 0$, there exists J such that

$$\int_D \|\Phi_j(x) - \phi(x)\|^2_{L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})} dx < \varepsilon \quad j \geq J.$$

Therefore there exist $j_k \in \mathbb{N}$ such that

$$\begin{aligned}
&\int_D \|\Phi_{j_k}(x) - \phi(x)\|^2_{L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})} dx < 2^{-k}.\\
&\Rightarrow \sum_{k\in\mathbb{N}} \int_D \|\Phi_{j_k}(x) - \phi(x)\|^2_{L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})} dx < \sum_{k\in\mathbb{N}} 2^{-k}.\\
&\Rightarrow \int_D \sum_{k\in\mathbb{N}} \|\Phi_{j_k}(x) - \phi(x)\|^2_{L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})} dx < \infty.\\
&\Rightarrow \sum_{k\in\mathbb{N}} \|\Phi_{j_k}(x) - \phi(x)\|^2_{L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})} < \infty \quad \text{for a.e} \quad x \in D.\\
&\Rightarrow \lim_{j_k\to\infty} \|\Phi_{j_k}(x) - \phi(x)\|^2_{L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})} = 0 \quad \text{for a.e} \quad x \in D.
\end{aligned}$$

□

Remark 3.2.6. *Suppose $J(x)$ is a range function (not necessarily measurable). Then M_J defined in (3.3) is a closed subspace of **H**.*

Proof of Remark 3.2.6. Given any $(\phi_j)_{j\in\mathbb{N}} \subset M_J$ with $\phi_i \to \phi$ in $\mathbf{H}$ for $i \to \infty$, we will have to show that $\phi \in M_J$. Now, $(\phi_i) \subset M_J$ means that $\phi_i(x) \in J(x)$ for a.e $x \in D$. Therefore, by Lemma 3.2.5, for a.e $x \in D$, there exists a subsequence ϕ_{i_j} converging pointwise to ϕ in $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$; i.e $\phi_{i_j}(x) \to \phi(x)$ in $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ as $i_j \to \infty$ for a.e $x \in D$. Since $J(x)$ is closed, $\phi(x) \in J(x)$ for a.e. $x \in D$. Hence $\phi \in M_J$. Therefore M_J is closed.

□

Lemma 3.2.7. *Let $J(x)$ be a measurable range function with associated projection $P(x)$. Let $\mathcal{P}$ be the orthogonal projection of **H** onto M_J. Then for any $\phi \in$ **H**,*

$$(\mathcal{P}\phi)(x) = P(x)(\phi(x)) \text{ for } \text{ a.e } x \in D. \tag{3.6}$$

Proof. Define $\mathcal{P}' : \mathbf{H} \to \mathbf{H}$ by

$$\mathcal{P}'\phi(x) = P(x)(\phi(x)).$$

First all we show that $P(x)(\phi(x))$ is measurable. We do this by showing that $x \mapsto$

$\langle P(x)(\phi(x)), F\rangle$ is measurable for any $F \in L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$. Consider an orthonormal basis $\{e_k\}_{k=1}^{\infty} \in L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$. Then we can write

$$\phi(x) = \sum_{k\in\mathbb{N}} \langle \phi(x), e_k\rangle e_k,$$

so that

$$\begin{aligned}
\langle P(x)(\phi(x)), F\rangle &= \Big\langle P(x)\Big(\sum_{k\in\mathbb{N}} \langle \phi(x), e_k\rangle e_k\Big), F\Big\rangle \\
&= \Big\langle \sum_{k\in\mathbb{N}} P(x)\Big(\langle \phi(x), e_k\rangle e_k\Big), F\Big\rangle \\
&= \sum_{k\in\mathbb{N}} \Big\langle P(x)\Big(\langle \phi(x), e_k\rangle e_k\Big), F\Big\rangle \\
&= \sum_{k\in\mathbb{N}} \langle \phi(x), e_k\rangle \langle P(x)e_k, F\rangle.
\end{aligned}$$

Since both $\phi(x)$ $P(x)$ are measurable, we have that

$$x \mapsto \langle \phi(x), e_k\rangle \quad \text{and} \quad x \mapsto \langle P(x)e_k, F\rangle$$

are measurable. Thus $x \mapsto \langle P(x)(\phi(x)), F\rangle$ is measurable. Since $\|P(x)(\phi(x))\| \leq \|\phi(x)\| < \infty$, for $\phi \in \mathbf{H}$, we have that $P(x)(\phi(x))$ belongs to $\mathbf{H}$. Next, $(\mathcal{P}')^2 = \mathcal{P}'$ as $P(x)$ has this property for a.e $x \in D$. Also, for any $\psi \in \mathbf{H}$ we have that

$$\begin{aligned}
\langle \mathcal{P}'\phi, \psi\rangle &= \int_D \langle \mathcal{P}'\phi(x), \psi(x)\rangle dx \\
&= \int_D \langle P(x)(\phi(x)), \psi(x)\rangle dx \\
&= \int_D \langle \phi(x), P(x)\psi(x)\rangle dx \\
&= \int_D \langle \phi(x), \mathcal{P}'\psi(x)\rangle dx \\
&= \langle \phi, \mathcal{P}'\psi\rangle.
\end{aligned}$$

Hence $\mathcal{P}'$ is also self adjoint. Therefore $\mathcal{P}'$ is an orthogonal projection whose range we shall call M'. If we consider any $g \in M'$, then $g(x) \in J(x)$ for *a.e.* $x \in D$, since $P(x)$ is the projection onto $J(x)$. Therefore, we have that $M' \subset M_J$. To prove the other containment, let us suppose that there exist a nonzero $\psi \in M_J$, which is orthogonal to M'. Then,

$$\begin{aligned} 0 &= \int_D \langle P(x)(\phi(x)), \psi(x) \rangle dx \\ &= \int_D \langle \phi(x), P(x)\psi(x) \rangle dx, \end{aligned}$$

for all $\phi \in \mathbf{H}$, which implies that $P(x)\psi(x) = 0$ for a.e $x \in D$. Therefore, $\psi(x) = P(x)\psi(x) = 0$ for a.e $x \in D$ which is a contradiction. Hence $M' = M_J$. □

Corollary 3.2.8. *Suppose $M_J = M_K$ for some measurable range functions J and K with associated respective projections, P and Q. Then $J(x) = K(x)$ for a.e $x \in D$.*

Proposition 3.2.9. *A closed subspace $V \subset L^2(\mathbb{R}^d)$ is $\mathbb{Z}^n \times \{0\}^{d-n}-$shift invariant if and only if*

$$V = \{f \in L^2(\mathbb{R}^d) : \mathcal{T}f(x) \in J_V(x) \text{ for a.e } x \in D\} \tag{3.7}$$

where J_V is a measurable range function. The correspondence between V and J_V is one-to-one under the convention that the range functions are identified if they are equal almost everywhere. Furthermore, if

$$V = \overline{span}\{T_k f : \ k \in \mathbb{Z}^n \times \{0\}^{d-n}, \ f \in \mathcal{A}\} \tag{3.8}$$

for some countable $\mathcal{A} \subset L^2(\mathbb{R}^2)$, then for almost every $x \in D$,

$$J_V(x) = \overline{span}\{\mathcal{T}f(x) : \ f \in \mathcal{A}\}. \tag{3.9}$$

Therefore by (3.4), $V \subset L^2(\mathbb{R}^d)$ is shift invariant if and only if $M := \mathcal{T}V \subset \mathbf{H}$ is a closed subspace under multiplication by exponentials i.e. $\phi(\cdot) \in M \Rightarrow e^{2\pi i(\cdot)k}\phi(\cdot) \in M$ for all $k \in \mathbb{Z}^n$, where $\cdot$ represent a generic variable.

Proof of proposition 3.2.9. Suppose $V = \overline{span}\{T_k f : \ k \in \mathbb{Z}^n, \ f \in \mathcal{A}\}$ is shift invariant. Let $M = \mathcal{T}V$ and let $J_V(x)$ be given by (3.9). Consider any $\phi \in M$. Then, we can find some sequence (ϕ_i) converging to ϕ such that

$$\mathcal{T}^{-1}\phi_i \in span\{T_k f : f \in A, k \in \mathbb{Z}^n\}.$$

Using (3.4), we see that $\phi_i(x) \in J_V(x)$. By Remark 3.2.6, we conclude that $\phi \in M_J$. Hence $M \subset M_J$.

Take any $\psi \in M_J$ which is orthogonal to M. Then for any $\phi \in \mathcal{T}V$ and $k \in \mathbb{Z}^n$, we have that $e^{-2\pi ixk}\phi(x) \in \mathcal{T}V$. Hence,

$$\begin{aligned} 0 &= \int_D \langle e^{-2\pi ixk}\phi(x), \psi(x)\rangle dx \\ &= \int_D e^{-2\pi ixk}\langle \phi(x), \psi(x)\rangle dx \end{aligned}$$

Therefore, if we let $F \in L^1(D)$ to be defined by $F(x) = \langle \phi(x), \psi(x)\rangle$ then all the Fourier coefficients of F will be zero and thus $F(x) = 0$ for a.e $x \in D$. Hence $\psi(x) \in J_V(x)^\perp$ for a.e $x \in D$. Since $\Psi(x) \in J_V(x)$, we conclude that $\Psi(x) = 0$ for a.e $x \in D$. Thus, there is no non-zero $\psi \in M_J$ which is orthogonal to M, and therefore $M = M_J$.
We finally show that $J_V(x)$ given by (3.9) is measurable. Let $\mathcal{P}$ denote the orthogonal projection of $\mathbf{H}$ onto M, and let $P(x)$ be the associated orthogonal projection onto $J_V(x)$. Take any $\psi \in \mathbf{H}$, then $(I - \mathcal{P})\psi$ is orthogonal to M. We conclude using the above argument that $\psi(x) - \mathcal{P}\psi(x) \in J_V(x)^\perp$ for a.e $x \in D$. Therefore,

$$P(x)(\psi(x)) = P(x)(\mathcal{P}\psi(x)) = \mathcal{P}\psi(x) \text{ for a.e } x \in D \tag{3.10}$$

as $\mathcal{P}\psi(x) \in J_V(x)$ for a.e $x \in D$ with $M = M_J$.
Take any function $a = \psi(x) \in L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$. Since $\mathcal{P}\psi(x)$ is a measurable function by (3.6), we have that $x \longmapsto P(x)a$ is measurable. Thus, J_V is measurable.

Conversely, suppose $J_V(x)$ is a measurable range function, then by Remark 3.2.6,

$$V = \mathcal{T}^{-1}M_J$$

is a closed shift invariant space. By Lemma 3.2.7, it is easy to see that V satisfies (3.7). The one-to-one correspondence between V and $J_V(x)$ is as a result of Corollary 3.2.8.

□

Theorem 3.2.10. *Let $\mathcal{A} \subset L^2(\mathbb{R}^d)$ be countable.*

1. *The family $\{T_k f : \; k \in \mathbb{Z}^n, \; f \in \mathcal{A}\}$ is a frame for V with frame constants A, B if and only if for a.e $x \in D$, $\{\mathcal{T}f(x) : \; f \in \mathcal{A}\} \subset L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ is a frame for $J_V(x)$ with positive constants A,B.*

2. *The family $\{T_k f : \; k \in \mathbb{Z}^n, \; f \in \mathcal{A}\}$ is a Riesz sequence in V with Riesz constants A, B if and only if for a.e $x \in D$, $\{\mathcal{T}f(x) : \; f \in \mathcal{A}\} \subset L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ is a Riesz*

sequence in $J_V(x)$, with Riesz constants A, B.

Lemma 3.2.11. *Let $\mathcal{A} \subset L^2(\mathbb{R}^d)$ be countable and V as in (3.8). Then for all $g \in V$, we have that*

$$\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n} |\langle T_k f, g\rangle|^2 = \sum_{f\in\mathcal{A}} \int_D |\langle \mathcal{T} f(x), \mathcal{T} g(x)\rangle|^2 dx.$$

Proof of Lemma 3.2.11. Let $k \in \mathbb{Z}^n$, $f \in \mathcal{A}$ and $g \in V$. Then, we have that, for a.e $x \in D$,

$$\begin{aligned}
\langle T_k f, g\rangle &= \langle \mathcal{T} T_k f, \mathcal{T} g\rangle \\
&= \int_D \langle \mathcal{T} T_k f(x), \mathcal{T} g(x)\rangle dx \\
&= \int_D \langle e^{-2\pi i x k} \mathcal{T} f(x), \mathcal{T} g(x)\rangle dx \quad \text{by (3.4)} \\
&= \int_D e^{-2\pi i x k} \langle \mathcal{T} f(x), \mathcal{T} g(x)\rangle dx \\
\therefore\ |\langle T_k f, g\rangle|^2 &= |\int_D e^{-2\pi i x k} \langle \mathcal{T} f(x), \mathcal{T} g(x)\rangle dx|^2 \\
&= |\int_D e^{-2\pi i x k} F(x) dx|^2, \quad \text{where } F(x) = \langle \mathcal{T} f(x), \mathcal{T} g(x)\rangle \\
&= |\hat{F}(k)|^2.
\end{aligned}$$

This implies that,

$$\sum_{k\in\mathbb{Z}^n} |\langle T_k f, g\rangle|^2 = \sum_{k\in\mathbb{Z}^n} |\hat{F}(k)|^2 = \int_D |F(x)|^2 dx \quad \text{by (Plancherel's Identity).}$$

Thus,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n} |\langle T_k f, g\rangle|^2 &= \int_D |\langle \mathcal{T} f(x), \mathcal{T} g(x)\rangle|^2 dx \\
\text{and} \quad \sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n} |T_k f, g\rangle|^2 &= \sum_{f\in\mathcal{A}} \int_D |\langle T_k f(x), g(x)\rangle|^2 dx.
\end{aligned}$$

□

Proof of Theorem 3.2.10 (1). Suppose that $\{\mathcal{T} f(x) :\ f \in \mathcal{A}\} \subset L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ is a frame of $J_V(x)$ with frame constants A, B for a.e $x \in D$. Then for a.e $x \in D$, we have that,

$$A\|\mathcal{T}g(x)\|^2_{J_V} \leq \sum_{f\in\mathcal{A}}|\langle\mathcal{T}f(x),\mathcal{T}g(x)\rangle|^2 \leq B\|\mathcal{T}g(x)\|^2_{J_V} \tag{3.11}$$

for any $g \in V$. By integrating (3.11) over D, we obtain

$$A\int_D \|\mathcal{T}g(x)\|^2_{J_V}dx \leq \sum_{f\in\mathcal{A}}\int_D|\langle\mathcal{T}f(x),\mathcal{T}g(x)\rangle|^2dx \leq B\int_D \|\mathcal{T}g(x)\|^2_{J_V}dx$$

By Proposition 3.2.9 and Lemma 3.2.11, we obtain

$$A\|g\|^2_V \leq \sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n}|\langle T_kf,g\rangle|^2 \leq B\|g\|^2_V, \tag{3.12}$$

which means that $\{T_kf : \ k\in\mathbb{Z}^n, \ f\in\mathcal{A}\}$ is a frame for V with frame constants A, B.

Conversely, suppose $\{T_kf : \ k\in\mathbb{Z}^n, f\in\mathcal{A}\}$ is a frame for V with frame constants A, B. This means (3.12) holds. We want to show that $\{\mathcal{T}f(x) : \ f\in\mathcal{A}\} \subset L^2(\mathbb{Z}^n\oplus\mathbb{R}^{d-n})$ is a frame of $J_V(x)$ with frame constants A, B for a.e $x\in D$. Let $\{d_1, d_2, ...\}$ be a dense subset of $L^2(\mathbb{Z}^n\oplus\mathbb{R}^{d-n})$. Then we are to show that,

$$A\|P(x)d_i\|^2 \leq \sum_{f\in\mathcal{A}}|\langle\mathcal{T}f(x),P(x)d_i\rangle|^2 \leq B\|P(x)d_i\|^2 \tag{3.13}$$

for any $i\in\mathbb{N}$, for a.e $x\in D$.

Suppose (3.13) fails, then there exists a measurable set $S\subset D$, with $|S|>0$, such that for some $i_0\in\mathbb{N}$, either

1. $\sum_{f\in\mathcal{A}}|\langle\mathcal{T}f(x),P(x)d_{i_0}\rangle|^2 > B\|P(x)d_{i_0}\|^2$, or

2. $\sum_{f\in\mathcal{A}}|\langle\mathcal{T}f(x),P(x)d_{i_0}\rangle|^2 < A\|P(x)d_{i_0}\|^2$ for $x\in S$.

Let $g \in V$ be given by $\mathcal{T}g(x) = \chi_S(x)P(x)d_{i_0}$. If for example (1) happens, then

$$\begin{aligned}
\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n} |\langle T_k f, g\rangle|^2 &= \sum_{f\in\mathcal{A}} \int_D |\langle \mathcal{T}f(x), \mathcal{T}g(x)\rangle|^2 dx, \quad \text{by Lemma 3.2.9} \\
&= \int_D \sum_{f\in\mathcal{A}} |\langle \mathcal{T}f(x), \mathcal{T}g(x)\rangle|^2 dx \\
&= \int_S \sum_{f\in\mathcal{A}} |\langle \mathcal{T}f(x), P(x)d_{i_0}\rangle|^2 dx \\
&> B\int_D \chi_S(x)\|\mathcal{T}P(x)d_{i_0}\|^2 \\
&= B\int_D \|\mathcal{T}g(x)\|^2 dx \\
&= B\|g\|_V^2, \quad \text{by Proposition 3.2.9.}
\end{aligned}$$

which is a contradiction to (3.12). If (2) happens, we obtain a contradiction by a similar computation. □

Proof of Theorem 3.2.10 (2). Let $\mathcal{C}_{(k,f)\in\mathbb{Z}^n\times\mathcal{A}}$ be a sequence with only finitely many nonzero terms. Then by the Plancherel's Theorem,

$$\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n} |\mathcal{C}_{(k,f)}|^2 = \int_D \sum_{f\in\mathcal{A}} |\mathcal{C}_f(x)|^2 dx, \text{ where } \mathcal{C}_f(x) = \sum_{k\in\mathbb{Z}^n} \mathcal{C}_{(k,f)} e^{-2\pi i k x}, \ x \in D. \quad (3.14)$$

Also see that

$$\begin{aligned}
\left\|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n} \mathcal{C}_{(k,f)} T_k f\right\|^2 &= \left\|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n} \mathcal{C}_{(k,f)} \mathcal{T} T_k f\right\|^2 \\
&= \left\|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n} \mathcal{C}_{(k,f)} e^{-2\pi i x k}\mathcal{T} f\right\|^2 \quad \textit{by } (3.4) \\
&= \left\|\sum_{f\in\mathcal{A}} \mathcal{C}_f(x)\mathcal{T} f\right\|^2 \\
&= \int_D \left\|\sum_{f\in\mathcal{A}} \mathcal{C}_f(x)\mathcal{T} f(x)\right\|^2 dx
\end{aligned}$$

$$i.e \quad \left\|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n}\mathcal{C}_{(k,f)}T_kf\right\|^2 = \int_D \left\|\sum_{f\in\mathcal{A}}\mathcal{C}_f(x)\mathcal{T}f(x)\right\|^2 dx \tag{3.15}$$

Suppose $\{\mathcal{T}f(x) : f \in \mathcal{A}\}$ forms a Riesz sequence for $J_V(x)$ for a.e $x \in D$ with Riesz constants $A, B > 0$. Then for a.e $x \in D$, we have that

$$A\sum_{f\in\mathcal{A}}|\mathcal{C}_f(x)|^2 \leq \|\sum_{f\in\mathcal{A}}\mathcal{C}_f(x)\mathcal{T}f(x)\|^2 \leq B\sum_{f\in\mathcal{A}}|\mathcal{C}_f(x)|^2 \tag{3.16}$$

Integrating (3.16) over D, for a.e $x \in D$, we obtain

$$A\int_D\sum_{f\in\mathcal{A}}|\mathcal{C}_f(x)|^2dx \leq \int_D\|\sum_{f\in\mathcal{A}}\mathcal{C}_f(x)\mathcal{T}f(x)\|^2dx \leq B\int_D\sum_{f\in\mathcal{A}}|\mathcal{C}_f(x)|^2dx \tag{3.17}$$

and using (3.14) and (3.15) gives

$$A\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n}|\mathcal{C}_{(k,f)}|^2 \leq \|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n}\mathcal{C}_{(k,f)}T_kf\|^2 \leq B\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n}|\mathcal{C}_{(k,f)}|^2 \tag{3.18}$$

which means that $\{T_kf : k \in \mathbb{Z}^n, f \in \mathcal{A}\}$ is a Riesz sequence for V with Riesz constants A, B.

Conversely suppose $\{T_kf : k \in \mathbb{Z}^n, f \in \mathcal{A}\}$ forms a Riesz sequence in V with Riesz constants $A, B > 0$. Then by (3.14) and (3.15) , we have that (3.17) holds. Take any family of functions $\{\mathcal{M}_f \in L^\infty(D) : f \in \mathcal{A}\}$ with $\mathcal{M}_f = 0$ except for finitely many f. As a consequences of the Lusin's Theorem, there exists sequence of trigonometric polynomials, call it $\{\mathcal{C}_f^{(i)}\}_{i\in\mathbb{N}}$, such that

$$\begin{cases} \|\mathcal{C}_f^{(i)}\|_\infty \leq \|\mathcal{M}_f\|_\infty & \text{for all } i \in \mathbb{N}, f \in \mathcal{A} \\ \mathcal{C}_f^{(i)}(x) \to \mathcal{M}_f(x), & i \to \infty, \text{ for a.e } x \in D, f \in \mathcal{A}. \end{cases} \tag{3.19}$$

By using (3.19) and the Lebesgue Dominated Convergence Theorem, (3.17) yields the inequalities

$$A\int_D\sum_{f\in\mathcal{A}}|\mathcal{M}_f(x)|^2dx \leq \int_D\|\sum_{f\in\mathcal{A}}\mathcal{M}_f(x)\mathcal{T}f(x)\|^2dx \leq B\int_D\sum_{f\in\mathcal{A}}|\mathcal{M}_f(x)|^2dx \tag{3.20}$$

Let $(d^i)_{i\in\mathbb{N}}$ where $d^i{=}(d^i_f)_{f\in\mathcal{A}}$ be dense in $\ell^2(\mathcal{A})$. We can assume, for each i, $d^i_f = 0$ for finitely many f. The family $\{\mathcal{T}f(x) : f \in \mathcal{A}\}$ forming a Riesz sequence for $J_V(x)$ with

Riesz constants A, B for a.e $x \in D$ means that,

$$A \sum_{f\in\mathcal{A}} |d^i_f|^2 \leq \| \sum_{f\in\mathcal{A}} d^i_f \mathcal{T}f(x)\|^2 \leq B \sum_{f\in\mathcal{A}} |d^i_f|^2 \text{ for all } i \in \mathbb{N} \text{ } for \text{ } a.e \text{ } x \in D. \tag{3.21}$$

Suppose now that (3.21) fails, then there exists a measurable set $S \subset D$, with $|S| > 0$, such that for some $i_0 \in \mathbb{N}$, either

1. $\| \sum_{f\in\mathcal{A}} d^{i_o}_f \mathcal{T}f(x)\|^2 > B \sum_{f\in\mathcal{A}} |d^{i_o}_f|^2$, or
2. $\| \sum_{f\in\mathcal{A}} d^{i_o}_f \mathcal{T}f(x)\|^2 < A \sum_{f\in\mathcal{A}} |d^{i_o}_f|^2$ for $x \in S$.

Consider the family of functions $\mathcal{M}_f = d^{i_o}_f \chi_S$ and assume for example that (1) holds then,

$$\begin{aligned}
\int_D \left\| \sum_{f\in\mathcal{A}} \mathcal{M}_f(x)\mathcal{T}f(x) \right\|^2 dx &= \int_D \left\| \sum_{f\in\mathcal{A}} d^{i_o}\chi_S(x)\mathcal{T}f(x) \right\|^2 dx \\
&> B \int_S \sum_{f\in\mathcal{A}} |d^{i_o}_f|^2 dx \\
&= B \int_D \sum_{f\in\mathcal{A}} |d^{i_o}_f \chi_S(x)|^2 dx \\
&= B \int_D \sum_{f\in\mathcal{A}} |\mathcal{M}_f(x)|^2 dx
\end{aligned}$$

which contradicts (3.20). □

Remark 3.2.12. *Theorem 3.2.10 reduces the problem of checking whether*

$$\{T_k f : \ k \in \mathbb{Z}^n \times \{0\}^{d-n}, f \in \mathcal{A}\}$$

is a frame, or a Riesz sequence in a subspace of $L^2(\mathbb{R}^d)$ *to analyzing the fibers in subspaces of* $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ *parameterized by D, the base space.*

3.3 The Gramian Concept

The goal of this section is achieved in Theorem 3.3.3, which is a corollary of Theorem 3.2.10.

Consider the vectors $\{t_j : j \in J\} \subset L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$, define the operator

$$K : \ell^2(J) \to L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$$

by $K(\mathcal{C}) = \sum_{j\in J} \mathcal{C}_j t_j$ for $\{\mathcal{C}_j\}_{j\in J} \in \ell^2(J)$ and $\mathcal{C}_j = 0$ except for finitely many $j's$.
If K extends to bounded operator, then the adjoint of K, $K^* : L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n}) \to \ell^2(J)$ is given by $K^*(a) = \{\langle a, t_j\rangle\}_{n\in J}$ for $a \in L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$. This is because,

$$\begin{aligned}
\langle K\mathcal{C}, a\rangle &= \langle \sum_{j\in J} \mathcal{C}_j t_j, a\rangle \\
&= \sum_{j\in J} \mathcal{C}_j \langle t_j, a\rangle \\
&= \sum_{j\in J} \mathcal{C}_j \overline{\langle a, t_j\rangle} \\
&= \langle \mathcal{C}, \langle a, t_j\rangle\rangle \\
&= \langle \mathcal{C}, K^* a\rangle
\end{aligned}$$

Definition 3.3.1. *For a.e $x \in D$, consider the vectors $\{t_j : j \in J\} \subset L^2(\mathbb{Z}^j \oplus \mathbb{R}^{d-n})$. The Gramian associated with the collection $\{t_j : j \in J\}$,*

$$G : \ell^2(J) \to \ell^2(J)$$

is defined by $G = K^*K$ and the dual Gramian

$$\tilde{G} : L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n}) \to L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$$

is defined by $\tilde{G} = KK^*$.

By letting $\{e_j\}_{j=1}^{\infty}$ be the canonical orthonormal basis for $\ell^2(J)$, the jk-th entry of the matrix representation of G is

$$\begin{aligned}
G_{jk} &= \langle K^*Ke_k, e_j\rangle \\
&= \langle Ke_k, Ke_j\rangle \\
&= \langle t_k, t_j\rangle \text{ for } j, k \in J.
\end{aligned}$$

Remark 3.3.2. *If G as in the above is a bounded operator on $\ell^2(J)$ then it is self adjoint. Also, if $\tilde{G}$ is a bounded operator on $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ then $\tilde{G}$ is self adjoint. In summary,* *if either $G, \tilde{G}, K$ or K^* is bounded, then $\|G\| = \|\tilde{G}\| = \|K^*\|^2 = \|K\|^2 < \infty$.*

Theorem 3.3.3. *Let $\mathcal{A} = \{f_j : j \in J\} \subset L^2(\mathbb{R}^d)$. For fixed $x \in D$, let $G(x)$ and $\tilde{G}(x)$ denote the Gramian and the dual Gramian of $\{\mathcal{T} f_j(x) : j \in J\} \subset L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$ respectively. Then,*

1. *$\{T_k f_j : k \in \mathbb{Z}^n, j \in J\}$ is a frame with frame constants A,B for its cosed linear span if and only if*
$$A\|a\|^2 \leq \langle \tilde{G}(x)a, a\rangle \leq B\|a\|^2 \tag{3.22}$$
for $a \in span\{\mathcal{T} f_j(x) : j \in J\},\ for\ a.e\ x \in D$.

2. *$\{T_k f_j : k \in \mathbb{Z}^n, j \in J\}$ is a Riesz sequence with Riesz constants A,B if and only if*
$$A\|c\|^2 \leq \langle G(x)c, c\rangle \leq B\|c\|^2 \tag{3.23}$$
for all $c \in \ell^2(J),\ for\ a.e\ x \in D$.

Proof. 1. For fixed $x \in D$, let $t_j(x) = \mathcal{T} f_j(x)$. Then for any $a \in span\{\mathcal{T} f_j(x) : j \in J\}$,

$$\begin{aligned}\langle \tilde{G}(x)a, a\rangle &= \langle K^*a, K^*a\rangle \\ &= \langle \langle a, t_j(x)\rangle, \langle a, t_j(x)\rangle\rangle \\ &= \sum_{j\in J} \langle a, t_j(x)\rangle \overline{\langle a, t_j(x)\rangle} \\ &= \sum_{j\in J} |\langle a, t_j(x)\rangle|^2\end{aligned}$$

Therefore, (3.22) can be written as

$$\|a\|^2 \leq \sum_{j\in J} |\langle a, t_j(x)\rangle|^2 \leq B\|a\|^2 \text{ for } a.e\ x \in D. \tag{3.24}$$

By Theorem (3.2.10), we have that (3.24) is equivalent to $\{T_k f_j : k \in \mathbb{Z}, j \in J\}$ forming a frame with constants A, B.

2. Also, for any $c \in \ell^2(J)$,

$$\begin{aligned}\langle G(x)c, c\rangle &= \langle Kc, Kc\rangle \\ &= \|Kc\|^2 \\ &= \|\sum_{j\in J} c_j t_j(x)\|^2\end{aligned}$$

Therefore, (3.23) can be written as

$$A\|c\|^2 \leq \|\sum_{j\in J} c_j t_j(x)\|^2 \leq B\|c\|^2 \text{ for } a.e\ x \in D. \tag{3.25}$$

By Theorem (3.2.10), we have that (3.25) is equivalent to $\{T_k f_j : k \in \mathbb{Z}, j \in J\}$ forming a Riesz sequence with Riesz constants A, B.

□

4 Main Results.

In this chapter, we will analyze in the special case where $d = 2$, the main problem which was stated in Section 3.1 of Chapter 3. We will state and give proofs of the results on Riesz sequences and frames in Section 4.2 and 4.3 respectively. In the last section of this chapter, we will consider some examples of bounded sets which demonstrate the results obtained on Riesz sequences and frames.

Notations and Settings:
Let $E \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure and $\Lambda \subset \mathbb{R}^2$ be a discrete set of frequencies. We associate with Λ the collection of exponentials

$$E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\}$$

where

$$e_\lambda(x, y) := e^{2\pi i \lambda.(x,y)} \text{ for } \lambda \in \Lambda,\ (x, y) \in E,$$

and $\lambda.(x, y)$ is the inner product of $\lambda \in \Lambda$ and $(x, y) \in E$.

As a particular case of the theory developed in Chapter 3, we define the integer translation on the x-axis by

$$T_k f(x, y) = f(x - k, y), \quad k \in \mathbb{Z}, \quad f \in L^2(\mathbb{R}^2),$$

and

$$\mathcal{T} f(x) = \left\{ \hat{f}(x + l, y) \right\}_{(l,y) \in \mathbb{Z} \oplus \mathbb{R}} \text{ for a.e } x \in [0, 1].$$

For each $x \in [0, 1]$, we define the subset of $\mathbb{Z} \oplus \mathbb{R}$, $\mathbb{F}(E)_x$, by

$$\mathbb{F}(E)_x := \{(l, y) \in \mathbb{Z} \oplus \mathbb{R} :\ (x + l, y) \in E\}.$$

Note that, If we let

$$V_E := \{f \in L^2(\mathbb{R}^2) :\ \hat{f} \in L^2(E)\},$$

V_E is invariant by **any** shift in $\mathbb{R}^2$, in particular, by the shifts T_k, $k \in \mathbb{Z}$, defined above.

Proposition 4.0.1. *Let $E \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure. Then the range function of V_E is characterised as*

$$J_V(x) := \{\mathcal{T}f(x) \in L^2(\mathbb{Z} \oplus \mathbb{R}) : \ supp(\mathcal{T}f(x)) \subseteq \mathbb{F}(E)_x, \ f \in V_E\}.$$

Proof. For each $x \in [0,1]$, we set

$$K_x = \{\mathcal{T}f(x) \in L^2(\mathbb{Z} \oplus \mathbb{R}) : \ supp(\mathcal{T}f(x)) \subseteq \mathbb{F}(E)_x, \ f \in V_E\}.$$

Let $g \in K_x$, and define

$$f(t,y) = \sum_{l \in \mathbb{Z}} \psi_l(t-x) g(l,y) \chi_E(t,y),$$

where

$$\psi_l(x) = \begin{cases} 1, & \text{if } x = l \\ x, & \text{if } l < x \leq l + \frac{1}{2} \\ -x, & \text{if } l - \frac{1}{2} \leq x < l \\ 0, & \text{otherwise} \end{cases}$$

so that,

$$f(x+l,y) = g(l,y)\chi_E(x+l,y).$$

If we define $\tilde{f}$ as f in E, and zero in $\mathbb{R}^2 \setminus E$, then by using the definition of V_E, we have that $\check{\check{f}} \in V_E$. Hence, $\mathcal{T}\check{\check{f}}(x) = f(x+l,y) = g(l,y) \in J_V(x)$. This shows that $K_x \subseteq J_V(x)$.

Conversely suppose that $g \in J_V(x)$. Then $g \in \overline{span}\{\mathcal{T}f(x) : \ f \in V_E\}$. This implies that $supp(g) \subseteq \mathbb{F}(E)_x$, thus $g \in K_x$. Therefore, $J_V(x) \subseteq K_x$.

□

4.1 The connection between Section 3.1 and Section 3.2

In this section, we give a corollary to Theorem 3.2.10 in the special case where $d = 2$ and $n = 1$, using V_E and $J_V(x)$ defined above, as the shift invariant space and range function respectively.

Theorem 4.1.1. *Let $E \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure. Let $(a_j)_{j\in J}$ and $(b_j)_{j\in J}$ be two sequences of real numbers and define*

$$\Lambda = \{(a_j + k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}.$$

Then, the following holds.

1. *The set $E(\Lambda)$ forms a Riesz sequence of $L^2(E)$ with Riesz constants $A, B > 0$ if and only if for a.e $x \in [0,1]$ the set $\{e^{2\pi i a_j l} e^{2\pi i b_j y} \chi_E(x+l, y) : j \in J\} \subset L^2(\mathbb{Z} \oplus \mathbb{R})$ also forms a Riesz sequence in $J_V(x)$ with Riesz constants $A, B > 0$.*

2. *The set $E(\Lambda)$ forms a frame of $L^2(E)$ with frame constants $A, B > 0$ if and only if for a.e $x \in [0,1]$ the set $\{e^{2\pi i a_j l} e^{2\pi i b_j y} \chi_E(x+l, y) : j \in J\} \subset L^2(\mathbb{Z} \oplus \mathbb{R})$ also forms a frame of $J_V(x)$ with frame constants $A, B > 0$.*

Proof. Let $f_j \in V_E$, $j \in J$ be defined by

$$\hat{f}_j = e_{(a_j, b_j)} \chi_E \ , \qquad j \in J, \tag{4.1}$$

and note that

$$E(\Lambda) = \{e_k \hat{f}_j : \ k \in \mathbb{Z}, j \in J\}.$$

By (3.4) in Chapter 3, we have that $e_k \hat{f}_j = \widehat{T_k f_j}$. Hence, the fact that

$$\{e_k \hat{f}_j : \ k \in \mathbb{Z}, j \in J\}$$

forms a Riesz sequence (resp. frame) of $L^2(E)$ is equivalent to the collection

$$\{T_k f_j : \ k \in \mathbb{Z}, j \in J\}$$

being a Riesz sequence (resp. frame) for V_E. By Theorem 3.2.10, the latter is equivalent to

$$\{\mathcal{T} f_j(x) : \ j \in J\}$$

forming a Riesz sequence (resp. frame) for $J_V(x)$ for a.e $x \in [0,1]$. Also,

$$\mathcal{T} f(x) = \{\hat{f}(x+l, y)\}_{(l,y)\in \mathbb{Z}\oplus\mathbb{R}}$$

and by using (4.1), we have that

$$\hat{f}(x+l,y) = e^{2\pi i(a_j x + a_j l)} e^{2\pi i b_j y} \chi_E(x+l,y)$$
$$= e^{2\pi i a_j x} e^{2\pi i a_j l} e^{2\pi i b_j y} \chi_E(x+l,y).$$

Therefore the set $E(\Lambda)$ forms a Riesz sequence (resp. frame) of $L^2(E)$ with Riesz (resp. frame) constants $A, B > 0$ if and only if, for a.e $x \in [0,1]$, the sequence

$$\{e^{2\pi i a_j l} e^{2\pi i b_j y} \chi_E(x+l,y) : j \in J\}$$

also forms a Riesz sequence (resp. frame) of $J_V(x)$ with Riesz (resp. frame) constants $A, B > 0$. □

4.2 Results on Riesz Sequences of Exponentials

Definition 4.2.1. *Consider a measurable set $S \subset \mathbb{R}^d$, and let $w : S \to [0,\infty)$ be a measurable function. The norm on the* ***w-weighted*** L^2 ***space on the set S***, $L^2_w(S)$ *is defined as*

$$\|f\|^2_{L^2_w(S)} = \int_S |f(x)|^2 w(x) dx \ < \infty.$$

Theorem 4.2.2. *Let $E \subset \mathbb{R}^2$ be a measurable set of finite positive measure and for some $(a_{j_o}, b_{j_o}) \in \mathbb{R}^2$ let $\Lambda = \{(a_{j_o}+k, b_{j_o}) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j_o \in J\}$ then the set $E(\Lambda)$ is a Riesz sequence $L^2(E)$ with constants $A, B > 0$ if and only if*

$$A \leq \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}} \chi_E(x+l,y) dy \leq B \quad for\ a.e \ \ x \in [0,1].$$

The theorem below is a special case where $(a_j)_{j \in J} \subset \mathbb{R}$ as given in Theorem 4.1.1, are chosen to be zero.

Theorem 4.2.3. *Let $E \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure. Let $(b_j)_{j \in J}$ be a sequence of real numbers and define*

$$\Lambda = \{(k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}.$$

Then the following statements are equivalent:

(a) $E(\Lambda)$ forms a Riesz sequence in $L^2(E)$ with Riesz constants $A, B > 0$.

(b) For a.e $x \in [0,1]$,

(i)

$$A \leq \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \chi_E(x+l,y)dy \leq B, \text{ and}$$

(ii) the set $\{e^{2\pi i b_j y} : j \in J\}$ is a Riesz sequence in $L^2_{F_x}(E_x)$ with uniform Riesz constants $A, B > 0$, where for each $x \in [0,1]$,

$$E_x := \{y \in \mathbb{R} : \sum_{l\in\mathbb{Z}} \chi_E(x+l,y) \geq 1\},$$

and

$$F_x(y) = \sum_{l\in\mathbb{Z}} \chi_E(x+l,y), \text{ for } y \in E_x.$$

Proof. Let $\{\mathcal{C}_j\}_{j\in J}$ be any sequence with finitely many non-zero terms and observe that,

$$\begin{aligned}\left\|\sum_{j\in J}\mathcal{C}_j e^{2\pi i b_j y}\right\|^2_{L^2_{F_x}(E_x)} &= \int_{E_x}\left|\sum_{j\in J}\mathcal{C}_j e^{2\pi i b_j y}\right|^2 \sum_{l\in\mathbb{Z}}\chi_E(x+l,y)dy \\ &= \sum_{l\in\mathbb{Z}}\int_{E_x}\left|\sum_{j\in J}\mathcal{C}_j e^{2\pi i b_j y}\right|^2 \chi_E(x+l,y)dy \\ &= \sum_{l\in\mathbb{Z}}\int_{E_x}\left|\sum_{j\in J}\mathcal{C}_j e^{2\pi i b_j y}\chi_E(x+l,y)\right|^2 dy \\ &= \left\|\sum_{j\in J}\mathcal{C}_j e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)}\end{aligned}$$

that is,

$$\left\|\sum_{j\in J}\mathcal{C}_j e^{2\pi i b_j y}\right\|^2_{L^2_{F_x}(E_x)} = \left\|\sum_{j\in J}\mathcal{C}_j e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)}. \tag{4.2}$$

Assume (a). Then (b)(i) holds by Theorem 4.2.2. By Theorem 4.1.1, we have that (a) implies

$$\{e^{2\pi i b_j y}\chi_E(x+l,y) : j \in J\}$$

forming a Riesz sequence in $J_V(x)$ with uniform Riesz constants $A, B > 0$, for a.e $x \in [0,1]$. Using (4.2), the latter statement is equivalent to (b)(ii).

Conversely, assume that (b) holds. Then by using (4.2), we have that (b)(ii) is equivalent to the collection

$$\{e^{2\pi i b_j y}\chi_E(x+l,y) : j \in J\}$$

forming a Riesz sequence in $J_V(x)$ with uniform Riesz constants $A, B > 0$, for a.e $x \in [0,1]$. The latter statement is equivalent to (a) by Theorem 4.1.1.

□

Remark 4.2.4. *If $E \subset \mathbb{R}^2$ is bounded, or more generally, if the projection of E onto the $x-axis$ is bounded, then*

$$\sum_{l\in\mathbb{Z}} \chi_E(x+l,y) \leq M \text{ for a.e } y \in \mathbb{R}$$

for some $M \in \mathbb{N}$ and we have

$$\left\|\sum_{j\in J} \mathcal{C}_j e^{2\pi i b_j y}\right\|^2_{L^2(E_x)} \leq \left\|\sum_{j\in J} \mathcal{C}_j e^{2\pi i b_j y}\right\|^2_{L^2_{F_x}(E_x)} \leq M \left\|\sum_{j\in J} \mathcal{C}_j e^{2\pi i b_j y}\right\|^2_{L^2(E_x)}.$$

The above Remark leads to the an immediate corollary of Theorem 4.2.3.

Corollary 4.2.5. *Under the previous assumptions, if E is bounded, or more generally, if the projection of E onto the $x-axis$ is bounded, the set $E(\Lambda)$ forms a Riesz sequence in $L^2(E)$ with Riesz constants $A, B > 0$, if and only if for a.e $x \in [0,1]$, the set $\{e^{2\pi i b_j y} : j \in J\}$ is a Riesz sequence in $L^2(E_x)$, with uniform Riesz constants $A, B > 0$, and $A \leq |E_x| \leq B$, where E_x is as given in Theorem 4.2.3.*

In the theorem below, we give a condition for choosing the sequence $(a_j)_{j\in J} \subset \mathbb{R}$, such that $E(\tilde{\Lambda})$ as defined in Theorem 4.2.6 forms a Riesz sequence in $L^2(E)$ given that $(b_j)_{j\in J} \subset \mathbb{R}$ is chosen so that $E(\Lambda)$ defined in Theorem 4.2.3 forms a Riesz sequence in $L^2(E)$.

Theorem 4.2.6 (**Riesz sequence perturbation result**)**.** *Let $E \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure. Define*

$$D(x) = \{l \in \mathbb{Z}, \int_{\mathbb{R}} \chi_E(x+l,y)dy > 0\}.$$

Let $(b_j)_{j\in J}$ be a collection of real numbers and suppose that the collection $E(\Lambda)$ forms a

Riesz sequence in $L^2(E)$ with Riesz constants $A, B > 0$ where

$$\Lambda = \{(k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}.$$

Let $(a_j)_{j\in J}$ be another collection of real numbers in $[0,1)$ such that

$$\sup_{x\in[0,1]} \sup_{j\in J} \sum_{l\in D(x)} |1 - e^{2\pi i a_j l}|^2 < \frac{A}{2B+2}, \tag{4.3}$$

then the set $E(\tilde{\Lambda})$ forms a Riesz sequence in $L^2(E)$ with Riesz constants $\frac{A}{2B+2}$, and $\frac{AB}{B+1} + 2B$ where $\tilde{\Lambda} = \{(a_j + k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}$.

Proof. By Theorem 4.1.1, $E(\Lambda)$ forms a Riesz sequence in $L^2(E)$ with Riesz constants $A, B > 0$ if and only if $\{e^{2\pi i b_j y}\chi_E(x+l, y) : j \in J\}$ forms a Riesz sequence in $J_V(x)$ with uniform Riesz constants $A, B > 0$. The latter means that,

$$A \sum_{j\in J} |c_j|^2 \leq \left\| \sum_{j\in J} c_j e^{2\pi i b_j y} \chi_E(x+l, y) \right\|^2_{J_v(x)} \leq B \sum_{j\in J} |c_j|^2, \tag{4.4}$$

for any $\{c_j\}_{j\in J} \in \ell^2(J)$.
Observe that, using the inequality, $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, for $a, b \in \mathbb{C}$, we have

$$\begin{aligned}
\left\| \sum_{j\in J} c_j e^{2\pi i a_j l} e^{2\pi i b_j y} \chi_E(x+l, y) \right\|^2_{J_v(x)} &= \sum_{l\in\mathbb{Z}} \int_{E_x} \left| \sum_{j\in J} c_j e^{2\pi i a_j l} e^{2\pi i b_j y} \chi_E(x+l, y) \right|^2 dy \\
&= \sum_{l\in\mathbb{Z}} \int_{E_x} \left| \sum_{j\in J} \left(c_j e^{2\pi i a_j l} e^{2\pi i b_j y} - c_j e^{2\pi i b_j y} + c_j e^{2\pi i b_j y} \right) \chi_E(x+l, y) \right|^2 dy \\
&\leq 2 \sum_{l\in\mathbb{Z}} \int_{E_x} \left| \sum_{j\in J} c_j \left(e^{2\pi i a_j l} - 1 \right) e^{2\pi i b_j y} \chi_E(x+l, y) \right|^2 dy \\
&\quad + 2 \sum_{l\in\mathbb{Z}} \int_{E_x} \left| \sum_{j\in J} c_j e^{2\pi i b_j y} \chi_E(x+l, y) \right|^2 dy \\
&= 2 \sum_{l\in\mathbb{Z}} \int_{E_x} \left| \sum_{j\in J} c_j \left(e^{2\pi i a_j l} - 1 \right) e^{2\pi i b_j y} \chi_E(x+l, y) \right|^2 dy \\
&\quad + 2 \left\| \sum_{j\in J} c_j e^{2\pi i b_j y} \chi_E(x+l, y) \right\|^2_{J_v(x)}
\end{aligned}$$

Further observe that, the term

$$2\sum_{l\in\mathbb{Z}}\int_{E_x}|\sum_{j\in J}c_j(e^{2\pi i a_j l}-1)e^{2\pi i b_j y}|^2\chi_E(x+l,y)dy$$

$$
\begin{aligned}
&= 2\sum_{l\in D(x)}\int_{E_x}|\sum_{j\in J}c_j(e^{2\pi i a_j l}-1)e^{2\pi i b_j y}|^2\chi_E(x+l,y)dy\\
&\leq 2\sum_{l\in D(x)}\int_{E_x}|\sum_{j\in J}c_j(e^{2\pi i a_j l}-1)e^{2\pi i b_j y}|^2\sum_{l'\in\mathbb{Z}}\chi_E(x+l',y)dy\\
&= 2\sum_{l\in D(x)}\sum_{l'\in\mathbb{Z}}\int_{E_x}|\sum_{j\in J}c_j(e^{2\pi i a_j l}-1)e^{2\pi i b_j y}|^2\chi_E(x+l',y)dy\\
&= 2\sum_{l\in D(x)}\sum_{l'\in\mathbb{Z}}\int_{E_x}\left|\sum_{j\in J}c_j(e^{2\pi i a_j l}-1)e^{2\pi i b_j y}\chi_E(x+l',y)\right|^2dy\\
&= 2\sum_{l\in D(x)}\left\|\sum_{j\in J}c_j(e^{2\pi i a_j l}-1)e^{2\pi i b_j y}\chi_E(x+l',y)\right\|^2\\
&\leq 2B\sum_{l\in D(x)}\sum_{j\in J}|c_j|^2|e^{2\pi i a_j l}-1|^2, \quad \text{(by the initial hypothesis)}\\
&= 2B\sum_{j\in J}|c_j|^2\left(\sum_{l\in D(x)}|e^{2\pi i a_j l}-1|^2\right)\\
&\leq 2B\left(\sup_{j\in J}\sum_{l\in D(x)}|1-e^{2\pi i a_j l}|^2\right)\sum_{j\in J}|c_j|^2\\
&< 2B\left(\frac{A}{2B+2}\right)\sum_{j\in J}|c_j|^2 \qquad \text{by (4.3)}\\
&= \frac{BA}{B+1}\sum_{j\in J}|c_j|^2.
\end{aligned}
$$

That is

$$2\sum_{l\in\mathbb{Z}}\int_{E_x}\left|\sum_{j\in J}c_j(e^{2\pi i a_j l}-1)e^{2\pi i b_j y}\chi_E(x+l,y)\right|^2dy\leq\frac{BA}{B+1}\sum_{j\in J}|c_j|^2 \tag{4.5}$$

Therefore,

$$\begin{aligned}&\left\|\sum_{j\in J} c_j e^{2\pi i a_j l} e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)}\\ &\leq \left(\frac{BA}{B+1}\right)\sum_{j\in J}|c_j|^2 + 2\left\|\sum_{j\in J} c_j e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)}.\end{aligned}$$

That is,

$$\left\|\sum_{j\in J} c_j e^{2\pi i a_j l} e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)} \leq \frac{BA}{B+1}\sum_{j\in J}|c_j|^2 + 2B\sum_{j\in J}|c_j|^2,$$

so that

$$\left\|\sum_{j\in J} c_j e^{2\pi i a_j l} e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)} \leq \left(\frac{BA}{B+1} + 2B\right)\sum_{j\in J}|c_j|^2. \tag{4.6}$$

To prove the other inequality, observe that

$$\begin{aligned}A\sum_{j\in J}|c_j|^2 &\leq \left\|\sum_{j\in J} c_j e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)} = \sum_{l\in\mathbb{Z}}\int_{E_x}\left|\sum_{j\in J} c_j e^{2\pi i b_j y}\chi_E(x+l,y)\right|^2 dy\\ &= \sum_{l\in\mathbb{Z}}\int_{E_x}\left|\sum_{j\in J}\left(c_j e^{2\pi i b_j y} - c_j e^{2\pi i a_j l}e^{2\pi i b_j y} + c_j e^{2\pi i a_j l}e^{2\pi i b_j y}\right)\chi_E(x+l,y)\right|^2 dy\\ &\leq 2\sum_{l\in\mathbb{Z}}\int_{E_x}\left|\sum_{j\in J} c_j\left(1-e^{2\pi i a_j l}\right)e^{2\pi i b_j y}\chi_E(x+l,y)\right|^2 dy\\ &\quad +2\sum_{l\in\mathbb{Z}}\int_{E_x}\left|\sum_{j\in J} c_j e^{2\pi i a_j l}e^{2\pi i b_j y}\chi_E(x+l,y)\right|^2 dy\end{aligned}$$

That is,

$$\begin{aligned}A\sum_{j\in J}|c_j|^2 &\leq 2\sum_{l\in\mathbb{Z}}\int_{E_x}\left|\sum_{j\in J} c_j\left(1-e^{2\pi i a_j l}\right)e^{2\pi i b_j y}\chi_E(x+l,y)\right|^2 dy\\ &\quad +2\left\|\sum_{j\in J} c_j e^{2\pi i a_j l}e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)}\end{aligned}$$

and by (4.5), we have that

$$A\sum_{j\in J}|c_j|^2 \le \frac{AB}{B+1}\sum_{j\in J}|c_j|^2 + 2\left\|\sum_{j\in J} c_j e^{2\pi i a_j l} e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)}$$

so that,

$$\left(\frac{A}{2B+2}\right)\sum_{j\in J}|c_j|^2 \le \left\|\sum_{j\in J} c_j e^{2\pi i a_j l} e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)} \tag{4.7}$$

Finally, by combining (4.6) and (4.7), we have that

$$\left(\frac{A}{2B+2}\right)\sum_{j\in J}|c_j|^2 \quad \le \left\|\sum_{j\in J} c_j e^{2\pi i a_j l} e^{2\pi i b_j y}\chi_E(x+l,y)\right\|^2_{J_V(x)} \le \left(\frac{AB}{B+1}+2B\right)\sum_{j\in J}|c_j|^2$$

Thus $\{e^{2\pi i a_j l} e^{2\pi i b_j y}\chi_E(x+l,y) : j\in J\}$ is a Riesz sequence in $J_V(x)$ for a.e $x\in[0,1]$ with uniform Riesz constants $\frac{A}{2B+2}$, and $\frac{AB}{B+1}+2B$, and again by using Theorem 4.1.1, we conclude that the set $E(\tilde{\Lambda})$ forms a Riesz sequence in $L^2(E)$ with Riesz constants $\frac{A}{2B+2}$, and $\frac{AB}{B+1}+2B$. □

Remark 4.2.7. *Note that if $E\subset\mathbb{R}^2$ is bounded, or more generally, if the projection of E onto the $x-axis$ is bounded, the set $D(x)$ above is contained in a finite set $D\subset\mathbb{Z}$. Since the function $x\mapsto \sum_{l\in D}|e^{2\pi i x l}-1|^2$ is continuous and vanishes on $\mathbb{Z}$, there exists a $\delta>0$ such that*

$$\sum_{l\in D}|e^{2\pi i x l}-1|^2 < \frac{A}{2B+2}$$

if $x\in[0,1)$ and each x satisfies $|x|<\delta$ or $|x-1|<\delta$. Thus the condition (4.3) will hold if we choose a_j, $j\in J$, with $|a_j|<\delta$ or $|a_j-1|<\delta$.

4.3 Results on Frames of Exponentials

Theorem 4.3.1. *Let $E\subset\mathbb{R}^2$ be a measurable set of finite Lebesgue measure. Let $(b_j)_{j\in J}$ be a sequence of real numbers and define*

$$\Lambda = \{(k,b_j)\in\mathbb{R}^2 \ : k\in\mathbb{Z}, j\in J\}.$$

Suppose $E(\Lambda)$ is a frame for $L^2(E)$ with frame constants $A, B > 0$. Then, we have the following: For a.e $x \in [0,1]$,

$$\sum_{l\in\mathbb{Z}} \chi_E(x+l,y) \le 1, \quad \textit{for a.e } y \in \mathbb{R}. \tag{4.8}$$

Proof. Assume that $E(\Lambda)$ is a frame for $L^2(E)$ with frame constants A, B, and suppose that (4.8) fails. Then there exists some positive measurable sets $S \subset [0,1]\times\mathbb{R}$, such that,

$$\sum_{l\in\mathbb{Z}} \chi_E(x+l,y) > 1, \quad \text{for all } (x,y) \in S. \tag{4.9}$$

This means that, there exist distinct integers l_1, and l_2 such that

$$\chi_E(x+l_1,y).\chi_E(x+l_2,y) = 1, \text{ for all } (x,y) \in F,$$

where F is some measurable subset of S, otherwise, for any distinct integers l_1, and l_2 ,

$$\chi_E(x+l_1,y).\chi_E(x+l_2,y) = 0, \text{ for a.e } (x,y) \in S.$$

That is,

$$\chi_E(x+l_1,y) = 1, \text{ and } \chi_E(x+l_2,y) = 0,$$

or

$$\chi_E(x+l_1,y) = 0, \text{ and } \chi_E(x+l_2,y) = 1 \text{ for a.e } (x,y) \in S,$$

which is a contradiction to (4.9). Consider the function, $g_0 \in J_V(x)$, defined by

$$g_0(l,y) = \begin{cases} \chi_F(x,y), & \text{if } l = l_1 \\ -\chi_F(x,y), & \text{if } l = l_2 \\ 0, & \text{otherwise.} \end{cases}$$

Then for all $(x, y) \in F$, we have have that,

$$\begin{aligned}\sum_{j\in J}\left|\left\langle g_0, e^{2\pi i b_j y}\chi_E(x+l,y)\right\rangle_{J_V(x)}\right|^2 &= \sum_{j\in J}\left|\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}} g_0(l,y)e^{-2\pi i b_j y}\chi_E(x+l,y)dy\right|^2\\ &= \sum_{j\in J}\left|\int_{\mathbb{R}} e^{-2\pi i b_j y}\Big(\sum_{l\in\mathbb{Z}} g_0(l,y)\chi_E(x+l,y)\Big)dy\right|^2\\ &= 0.\end{aligned}$$

Thus, for a.e $x \in [0, 1]$, $\{e^{2\pi i b_j y}\chi_E(x + l, y) : j \in J\}$ is not complete in $J_V(x)$, hence not a frame for $J_V(x)$. Therefore, by Theorem 4.1.1, the collection $E(\Lambda)$ is not a frame for $L^2(E)$, which contradicts our assumption. Hence, we conclude that, if the collection $E(\Lambda)$ forms a frame of $L^2(E)$ with frame constants $A, B > 0$, then, for a.e $x \in [0, 1]$, (4.8) holds. □

The theorem below is a special case where $(a_j)_{j\in J} \subset \mathbb{R}$ as given in Theorem 4.1.1, are chosen to be zero.

Theorem 4.3.2. *Let $E \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure. Let $(b_j)_{j\in J}$ be a sequence of real numbers and define*

$$\Lambda = \{(k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}.$$

Then the following statements are equivalent:

(a) The set $E(\Lambda)$ forms a frame of $L^2(E)$ with frame constants $A, B > 0$.

(b) (i) For a.e $x \in [0, 1]$, $\sum_{l\in\mathbb{Z}} \chi_E(x + l, y) \leq 1$ for a.e $y \in \mathbb{R}$.

(ii) The collection $\{e^{2\pi i b_j y} : j \in J\}$ forms a frame for $L^2(E_x)$ with uniform frame constants $A, B > 0$ where $E_x := \{y \in \mathbb{R} : \sum_{l\in\mathbb{Z}} \chi_E(x + l, y) = 1\}$.

***Proof*.** Assume (a) holds. Then by Theorem 4.3.1, we have that (b)(i) holds. Also, if (a) holds, by Theorem 4.1.1, the collection

$$\{e^{2\pi i b_j y}\chi_E(x + l, y) : j \in J\}$$

forms a frame for $J_V(x)$ with uniform frame constants $A, B > 0$ for a.e $x \in [0,1]$. The latter means that, for a.e $x \in [0,1]$,

$$A||g||^2 \leq \sum_{j\in J} \left|\langle g, e^{2\pi i b_j y}\chi_E(x+l,y)\rangle\right|^2 \leq B||g||^2, \quad \text{for every } g \in J_V(x). \tag{4.10}$$

Then, for a.e $x \in [0,1]$, we have

$$\begin{aligned}
\sum_{j\in J} \left|\left\langle g, e^{2\pi i b_j y}\chi_E(x+l,y)\right\rangle_{J_V(x)}\right|^2 &= \sum_{j\in J}\left|\sum_{l\in\mathbb{Z}}\int_{E_x} g(l,y)e^{-2\pi i b_j y}\chi_E(x+l,y)dy\right|^2 \\
&= \sum_{j\in J}\left|\int_{E_x}\left(\sum_{l\in\mathbb{Z}} g(l,y)\chi_E(x+l,y)\right)e^{-2\pi i b_j y}dy\right|^2 \\
&= \sum_{j\in J}\left|\int_{E_x} h(y)e^{-2\pi i b_j y}dy\right|^2 \\
&= \sum_{j\in J}\left|\left\langle h, e^{2\pi i b_j y}\right\rangle_{L^2(E_x)}\right|^2,
\end{aligned}$$

where

$$h(y) = \sum_{l\in\mathbb{Z}} g(l,y)\chi_E(x+l,y).$$

That is,

$$\sum_{j\in J}\left|\left\langle h, e^{2\pi i b_j y}\right\rangle_{L^2(E_x)}\right|^2 = \sum_{j\in J}\left|\left\langle g, e^{2\pi i b_j y}\chi_E(x+l,y)\right\rangle_{J_V(x)}\right|^2, \quad \text{for a.e } x \in [0,1]. \tag{4.11}$$

We also see that for a.e $x \in [0,1]$,

$$\begin{aligned}
\|h\|^2_{L^2(E_x)} &= \int_{E_x}\left|\sum_{l\in\mathbb{Z}} g(l,y)\chi_E(x+l,y)\right|^2 dy && (4.12)\\
&= \int_{E_x}\sum_{l\in\mathbb{Z}}|g(l,y)|^2\, dy, \qquad \text{(by using (b)(i))} && (4.13)\\
&= ||g||^2_{J_V(x)}. && (4.14)
\end{aligned}$$

By (4.11), and (4.14), we see that (4.10) can be written as

$$A||h||^2 \leq \sum_{j\in J} |\langle h, e^{2\pi i b_j y}\rangle|^2 \leq B||h||^2, \quad \text{for a.e } x \ \in [0,1], \tag{4.15}$$

for $h \in L^2(E_x)$. Hence (ii)(b) also holds.
Conversely, suppose that (b)(ii) holds, then (4.15) holds. In addition, if (b)(i) holds, the (4.15) is equivalent to (4.10) by (4.14) and (4.11). Now, by using Theorem 4.1.1, we have that (4.10) implies (a).

□

In the theorem below, we give a condition for choosing the sequence $(a_j)_{j\in J} \subset \mathbb{R}$, such that $E(\tilde{\Lambda})$ as defined in Theorem 4.3.3 forms a frame for $L^2(E)$ given that $(b_j)_{j\in J} \subset \mathbb{R}$ is chosen so that $E(\Lambda)$ defined in Theorem 4.3.2 forms a frame for $L^2(E)$.

Theorem 4.3.3 (**Frame perturbation result**). *Let $E \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure. Define*

$$D(x) = \{l \in \mathbb{Z}, \ \int_{\mathbb{R}} \chi_E(x+l, y)dy > 0\}.$$

Let $(b_j)_{j\in J}$ be a collection of real numbers and suppose that the collection $E(\Lambda)$ forms a frame for $L^2(E)$ with frame constants $A, B > 0$ where

$$\Lambda = \{(k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}.$$

Let $(a_j)_{j\in J}$ be another collection of real numbers in $[0,1)$ such that

$$\sup_{x\in[0,1]} \sup_{j\in J} \sum_{l\in D(x)} |1 - e^{-2\pi i a_j l}|^2 < \frac{A}{2B+2}, \tag{4.16}$$

then the set $E(\tilde{\Lambda})$ forms a frame $L^2(E)$ with frame constants $\frac{A}{2B+2}$, and $\frac{AB}{B+1} + 2B$ where $\tilde{\Lambda} = \{(a_j + k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}$.

Proof. By Theorem 4.1.1, $E(\Lambda)$ forms a frame for $L^2(E)$ with frame constants $A, B > 0$ if and only if $\{e^{2\pi i b_j y}\chi_E(x+l, y) : j \in J\}$ forms a frame for $J_V(x)$ with uniform frame constants $A, B > 0$. The latter means that,

$$A||g||^2 \leq \sum_{j\in J} \left| \left\langle g, e^{2\pi i b_j y}\chi_E(x+l,y) \right\rangle_{J_V(x)} \right|^2 \leq B||g||^2, \quad \text{a.e } x \in [0,1] \tag{4.17}$$

for every $g \in J_V(x)$. Observe that, using the inequality, $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, for $a, b \in \mathbb{C}$, we have

$$\begin{aligned}
&\sum_{j\in J} \left| \left\langle g, e^{2\pi i a_j l} e^{2\pi i b_j y}\chi_E(x+l,y) \right\rangle_{J_V(x)} \right|^2 \\
&= \sum_{j\in J} \left| \sum_{l\in\mathbb{Z}} \int_{E_x} g(l,y) e^{-2\pi i a_j l} e^{-2\pi i b_j y} \chi_E(x+l,y)dy \right|^2 \\
&= \sum_{j\in J} \left| \sum_{l\in\mathbb{Z}} \int_{E_x} \left(e^{-2\pi i a_j l} - 1 + 1 \right) g(l,y) e^{-2\pi i b_j y} \chi_E(x+l,y)dy \right|^2 \\
&\leq 2\sum_{j\in J} \left| \sum_{l\in\mathbb{Z}} \int_{E_x} (e^{-2\pi i a_j l} - 1) g(l,y) e^{-2\pi i b_j l} \chi_E(x+l,y)dy \right|^2 \\
&\quad +2\sum_{j\in J} \left| \sum_{l\in\mathbb{Z}} \int_{E_x} g(l,y) e^{-2\pi i b_j l} \chi_E(x+l,y)dy \right|^2.
\end{aligned}$$

Note that, since $E(\Lambda)$ is a frame for $L^2(E)$, for a.e $x \in [0,1]$, we have the inequality

$$\sum_{l\in\mathbb{Z}} \chi_E(x+l,y) \leq 1, \quad \text{for a.e } y \in \mathbb{R}.$$

Therefore, we write the set $E_x := \{y \in \mathbb{R} : \sum_{l\in\mathbb{Z}} \chi_E(x+l,y) = 1\}$ as disjoint union of

$$E_x^l := \{y \in \mathbb{R} : \chi_E(x+l,y) = 1\}.$$

Using this fact together with the Cauchy$-$Schwarz inequality, we obtain

$$2\sum_{j\in J}\left|\sum_{l\in\mathbb{Z}}\int_{E_x}(e^{-2\pi i a_j l}-1)g(l,y)e^{-2\pi i b_j l}\chi_E(x+l,y)dy\right|^2$$

$$=2\sum_{j\in J}\left|\sum_{l\in D(x)}\int_{E_x}(e^{-2\pi i a_j l}-1)g(l,y)e^{-2\pi i b_j l}\chi_E(x+l,y)dy\right|^2$$

$$=2\sum_{j\in J}\left|\sum_{l\in D(x)}(e^{-2\pi i a_j l}-1)\int_{E_x}g(l,y)e^{-2\pi i b_j y}\chi_E(x+l,y)dy\right|^2$$

$$\leq 2\sum_{j\in J}\left(\sum_{l\in D(x)}\left|e^{-2\pi i a_j l}-1\right|^2\right)\left(\sum_{l\in D(x)}\left|\int_{E_x}g(l,y)e^{-2\pi i b_j y}\chi_E(x+l,y)dy\right|^2\right)$$

$$\leq 2\sup_{j\in J}\left(\sum_{l\in D(x)}\left|e^{-2\pi i a_j l}-1\right|^2\right)\sum_{j\in J}\sum_{l\in D(x)}\left|\int_{E_x}g(l,y)e^{-2\pi i b_j y}\chi_E(x+l,y)dy\right|^2$$

$$=2\sup_{j\in J}\left(\sum_{l\in D(x)}\left|e^{-2\pi i a_j l}-1\right|^2\right)\sum_{l\in D(x)}\sum_{j\in J}\left|\sum_{k\in\mathbb{Z}}\int_{E_x^l}g(k,y)\psi_l(k)e^{-2\pi i b_j y}\chi_E(x+k,y)dy\right|^2,$$

where $\psi_l(k) := \delta_l(k)\chi_E(x+k,y)$.

Hence,

$$2\sup_{j\in J}\left(\sum_{l\in D(x)}\left|e^{-2\pi i a_j l}-1\right|^2\right)\sum_{l\in D(x)}\sum_{j\in J}\left|\sum_{k\in\mathbb{Z}}\int_{E_x^l}g(k,y)\psi_l(k)e^{-2\pi i b_j y}\chi_E(x+k,y)dy\right|^2$$

$$=2\sup_{j\in J}\left(\sum_{l\in D(x)}\left|e^{-2\pi i a_j l}-1\right|^2\right)\sum_{l\in D(x)}\sum_{j\in J}\left|\left\langle g\psi_l, e^{2\pi i b_j y}\chi_E(x+k,y)\right\rangle_{J_V(x)}\right|^2$$

$$\leq\left(\tfrac{2AB}{2B+2}\right)\sum_{l\in D(x)}\|g\psi_l\|^2_{J_V(x)}\quad\text{by (4.15) and (4.16)}$$

$$=\left(\tfrac{2AB}{2B+2}\right)\|g\|^2_{J_V(x).}$$

That is,

$$2\sum_{j\in J}\left|\sum_{l\in\mathbb{Z}}\int_{E_x}(e^{-2\pi i a_j l}-1)g(l,y)e^{-2\pi i b_j l}\chi_E(x+l,y)dy\right|^2\leq\left(\frac{2AB}{2B+2}\right)\|g\|^2_{J_V(x).}\quad(4.18)$$

Therefore,

$$\sum_{j\in J}\left|\left\langle g, e^{2\pi i a_j l} e^{2\pi i b_j y}\chi_E(x+l,y)\right\rangle_{J_V(x)}\right|^2 \le \left(\frac{AB}{B+1}+B\right)\|g\|^2_{J_V(x)}. \tag{4.19}$$

To prove the other inequality, again see that

$$\begin{aligned}
A||g||^2 &\le \sum_{j\in J}\left|\left\langle g, e^{2\pi i b_j y}\chi_E(x+l,y)\right\rangle_{J_V(x)}\right|^2\\
&= \sum_{j\in J}\left|\sum_{l\in\mathbb{Z}}\int_{E_x} g(l,y)e^{-2\pi i b_j y}\chi_E(x+l,y)dy\right|^2\\
&= \sum_{j\in J}\left|\sum_{l\in\mathbb{Z}}\int_{E_x}\left(1-e^{-2\pi i a_j l}+e^{-2\pi i a_j l}\right)g(l,y)e^{-2\pi i b_j y}\chi_E(x+l,y)dy\right|^2\\
&\le 2\sum_{j\in J}\left|\sum_{l\in\mathbb{Z}}\int_{E_x}\left(1-e^{-2\pi i a_j l}\right)g(l,y)e^{-2\pi i b_j y}\chi_E(x+l,y)dy\right|^2\\
&\quad +2\sum_{j\in J}\left|\sum_{l\in\mathbb{Z}}\int_{E_x} g(l,y)e^{-2\pi i a_j l}e^{-2\pi i b_j y}\chi_E(x+l,y)dy\right|^2.
\end{aligned}$$

Therefore by (4.18), we conclude that

$$A||g||^2 \le \left(\frac{AB}{B+1}\right)\|g\|^2 \ + 2\sum_{j\in J}\left|\left\langle g, e^{2\pi i a_j l}e^{2\pi i b_j y}\chi_E(x+l,y)\right\rangle_{J_V(x)}\right|^2$$

so that

$$\begin{aligned}
\sum_{j\in J}\left|\left\langle g, e^{2\pi i a_j l}e^{2\pi i b_j y}\chi_E(x+l,y)\right\rangle_{J_V(x)}\right|^2 &\ge \frac{1}{2}\left(A-\frac{AB}{B+1}\right)\sum_{j\in J}\|g\|^2 &(4.20)\\
&= \left(\frac{A}{2B+2}\right)\sum_{j\in J}\|g\|^2 &(4.21)
\end{aligned}$$

Therefore by (4.19) and (4.21),

$$\left(\frac{A}{2B+2}\right)\|g\|^2 \le \sum_{j\in J}\left|\left\langle g, e^{2\pi i a_j l}e^{2\pi i b_j y}\chi_E(x+l,y)\right\rangle_{J_V(x)}\right|^2 \le \left(\frac{AB}{B+1}+2B\right)\|g\|^2. \tag{4.22}$$

Thus $\{e^{2\pi i a_j l} e^{2\pi i b_j y} \chi_E(x+l, y) : j \in J\}$ is a frame for $J_V(x)$ for a.e $x \in [0,1]$ with uniform frame constants $\frac{A}{2B+2}$, and $\frac{AB}{B+1} + 2B$, and again by using Theorem 4.1.1, we conclude that the set $E(\tilde{\Lambda})$ forms a frame for $L^2(E)$ with frame constants $\frac{A}{2B+2}$, and $\frac{AB}{B+1} + 2B$. $\square$

Following Theorem 2.1.6, we combine the results obtained in Riesz sequences and frames as follows:

Theorem 4.3.4 (Riesz Basis). *Let $E \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure. Let $(b_j)_{j\in J}$ be a sequence of real numbers and define*

$$\Lambda = \{(k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}.$$

Then the following statements are equivalent:

(a) The set $E(\Lambda)$ forms a Riesz basis in $L^2(E)$ with Riesz constants $A, B > 0$.

(b) For a.e $x \in [0,1]$,

(i) $\sum_{l\in\mathbb{Z}} \chi_E(x+l, y) \leq 1$ for a.e $y \in \mathbb{R}$,

(ii) $A \leq |E_x| \leq B$,

(iii) the collection $\{e^{2\pi i b_j y} : j \in J\}$ forms a Riesz basis in $L^2(E_x)$ with uniform Riesz constants $A, B > 0$ where $E_x := \{y \in \mathbb{R} : \sum_{l\in\mathbb{Z}} \chi_E(x+l, y) = 1\}$.

Theorem 4.3.5 (Riesz Basis). *Let $E \subset \mathbb{R}^2$ be a measurable set of finite Lebesgue measure. Define*

$$D(x) = \{l \in \mathbb{Z}, \ \int_{\mathbb{R}} \chi_E(x+l, y) dy > 0\}.$$

Let $(b_j)_{j\in J}$ be a collection of real numbers and suppose that the collection $E(\Lambda)$ forms a Riesz basis in $L^2(E)$ with Riesz constants $A, B > 0$ where

$$\Lambda = \{(k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}.$$

Let $(a_j)_{j\in J}$ be another collection of real numbers in $[0,1)$ such that

$$\sup_{x\in[0,1]} \sup_{j\in J} \sum_{l\in D(x)} |1 - e^{-2\pi i a_j l}|^2 < \frac{A}{2B+2}, \tag{4.23}$$

then the set $E(\tilde{\Lambda})$ forms a Riesz basis $L^2(E)$ with Riesz constants $\frac{A}{2B+2}$, and $\frac{AB}{B+1} + 2B$ where $\tilde{\Lambda} = \{(a_j + k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}$.

4.4 Examples with some bounded subsets of $\mathbb{R}^2$

The Beurling Density. For $h > 0$ and $x \in \mathbb{R}^d$, we denote by $Q_h(x)$ the closed cube centered at x with side length h. Let $\Lambda = \{\lambda_j\}_{j\in J} \subset \mathbb{R}^d$ be uniformly discrete, i.e we assume that $|\lambda_j - \lambda_k| \geq \alpha > 0$ for all $\lambda_j \neq \lambda_k$. We denote by

$$D^+(\Lambda) = \lim_{h\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\sup \#(\Lambda \cap Q_h(x))}{h^d}$$

$$D^-(\Lambda) = \lim_{h\to\infty} \inf_{x\in\mathbb{R}^d} \frac{\inf \#(\Lambda \cap Q_h(x))}{h^d}$$

the upper and lower Beurling density of Λ, respectively.

Proposition 4.4.1. *Let $\Lambda = \{(k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}$. Let $D^+((b_j))$ and $D^-((b_j))$ be the upper and lower Beurling density of $(b_j)_{j\in J} \subset \mathbb{R}$ respectively. Then*

1. $D^+((b_j)) = D^+(\Lambda)$
2. $D^-((b_j)) = D^-(\Lambda)$

Proof. Let $Q_h(x, y) := I_h(x) \times I_h(y)$ for some $h > 0$, be a square of side h centered at $(x, y) \in \mathbb{R}^2$ where $I_h(x)$ and $I_h(y)$ are closed intervals both with length h, centered at x and y respectively. Then it is easy to see that

$$\#\{(k, b_j), (k, b_j) \in Q_h(x, y)\} = \#\{k \in \mathbb{Z}, k \in I_h(x)\} \times \#\{b_j, b_j \in I_h(y)\}.$$

Therefore, by using the definition of $D^+(\Lambda)$, we have that

$$\begin{aligned} D^+(\Lambda) &= \lim_{h\to\infty} \sup_{(x,y)\in\mathbb{R}^2} \frac{\sup \#(\{(k, b_j)\} \cap Q_h(x, y))}{h^2} \\ &= \lim_{h\to\infty} \sup_{(x,y)\in\mathbb{R}^2} \frac{\sup \#(\{k\} \cap I_h(x))}{h} \times \lim_{h\to\infty} \sup_{y\in\mathbb{R}} \frac{\sup \#(\{b_j\} \cap I_h(x))}{h} \\ &= D^+(\mathbb{Z}) \times D^+((b_j)) \\ &= D^+((b_j)), \quad \text{since } D^+(\mathbb{Z}) = 1. \end{aligned}$$

A similar proof is used for (2) with

$$D^-((b_j)) := \lim_{h\to\infty} \inf_{y\in\mathbb{R}} \frac{\inf \#(\{b_j\} \cap I_h)}{h}.$$

□

Theorem 4.4.2. *Let $\Lambda = \{\lambda_j : j \in J\} \subset \mathbb{R}$ be uniformly discrete and let $E \subset \mathbb{R}$ be an interval. For the system $E(\Lambda)$ to be a frame for $L^2(E)$, it is necessary that $D^-(\Lambda) \geq |E|$, and it is sufficient that $D^-(\Lambda) > |E|$.*

Theorem 4.4.3. *Let $\Lambda = \{\lambda_j : j \in J\} \subset \mathbb{R}$ be uniformly discrete and let $E \subset \mathbb{R}$ be an interval. For the system $E(\Lambda)$ to be a Riesz sequence in $L^2(E)$, it is necessary that $D^+(\Lambda) \leq |E|$, and it is sufficient that $D^+(\Lambda) < |E|$.*

Proof of 4.4.2 and 3.4.3. See [14]. □

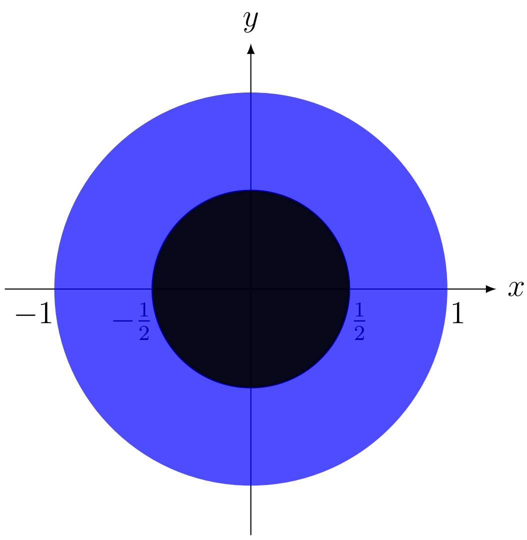
Example 4.4.4 (Disk). *Consider the disk $E \subset \mathbb{R}^2$ of radius $r > 0$ centered at the origin $(0, 0)$; that is*

$$E = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 \leq r^2\}.$$

Let (b_j) be uniformly discrete sequence of real numbers associated with

$$\Lambda = \{(k, b_j) \in \mathbb{R}^2\ : k \in \mathbb{Z}, j \in J\}.$$

We investigate the existence of Riesz sequences and frames of the form $E(\Lambda)$ on the domain E, for fixed $r > 0$.



Frames:

We claim that, the disk defined above will support a frame if $r \in (0, \frac{1}{2}]$. Suppose $r > \frac{1}{2}$, then we see that, the non-overlapping condition (frame necessary condition) in Theorem 4.3.1 fails. Secondly, for each $x \in [-\frac{1}{2}, \frac{1}{2}]$, we find the sets

$$E_x := \{y \in \mathbb{R} : \sum_{l \in \mathbb{Z}} \chi_E(x + l, y) = 1\}.$$

It is seen from the above diagram that,

$$E_x = \Bigg[-\sqrt{r^2 - x^2}\ , \sqrt{r^2 - x^2}\ \Bigg], \text{ for } |x| \leq r,$$

and

$$E_x = \emptyset, \quad \text{when} \ \ r < |x| \leq \frac{1}{2},$$

with Lebesgue measure, $|E_x| = 2\sqrt{r^2 - x^2}$. We now look for the largest of each of the sets E_x for a.e $x \in [-\frac{1}{2}, \frac{1}{2}]$.
Clearly, $\sup_{x \in [-\frac{1}{2}, \frac{1}{2}]} |E_x| = 2r$. Therefore, by Theorem 4.4.2, if $D^-(b_j) > 2r$, the set

$$\{e^{2\pi i b_j y} : j \in J\}$$

will form a frame on E_x defined above, for a.e $x \in [-\frac{1}{2}, \frac{1}{2}]$ with uniform frame bounds A, B. Hence, by Theorem 4.3.2 and Proposition 4.4.1, if

$$r \in \left(0, \frac{1}{2}\right], \quad \text{and} \ \ D^-(\Lambda) > 2r,$$

the collection $E(\Lambda)$ will form frame on the disk $E = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 \leq r\}$ with frame bounds A, B.

Riesz sequences:

For each $x \in [-\frac{1}{2}, \frac{1}{2}]$, we find the sets

$$E_x := \{y \in \mathbb{R} : \sum_{l \in \mathbb{Z}} \chi_E(x + l, y) \geq 1\}.$$

Again, it is seen from that diagram above that, for each $x \in [-\frac{1}{2}, \frac{1}{2}]$,

$$E_x = \Bigg[-\sqrt{r^2 - x^2}\ , \sqrt{r^2 - x^2}\ \Bigg].$$

We also claim that, the disk defined above will support a Riesz sequence with Riesz constants A, B, if $r > \frac{1}{2}$. As before, suppose that $r = \frac{1}{2}$. Then $E_x = \left[-\sqrt{\frac{1}{4} - x^2}\ , \sqrt{\frac{1}{4} - x^2}\ \right]$, with

$$\inf_{x \in [-\frac{1}{2}, \frac{1}{2}]} |E_x| = \inf_{x \in [-\frac{1}{2}, \frac{1}{2}]} 2\sqrt{\frac{1}{4} - x^2} = 0. \tag{4.24}$$

By Theorem 4.2.1, for the disk E to support a Riesz sequence with Riesz constants A, B, it is necessary that

$$A \leq |E_x| \leq B, \quad \text{for a.e } x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

In the case where $r = \frac{1}{2}$, the lower Riesz constant $A = 0$ by (4.24), which is a contradiction to $A > 0$. Hence, for Riesz sequences with Riesz constants A, B, we require that, $r > \frac{1}{2}$ in which case, $|E_x| \neq 0$, for each $x \in [-\frac{1}{2}, \frac{1}{2}]$. We now look for the smallest of each of the sets E_x for a.e $x \in [-\frac{1}{2}, \frac{1}{2}]$.

Clearly, $\inf_{x \in [-\frac{1}{2}, \frac{1}{2}]} |E_x| = 2\sqrt{r^2 - \dfrac{1}{4}}$. Therefore, by Theorem 4.4.3, if $D^+(b_j) < 2\sqrt{r^2 - \frac{1}{4}}$, the set

$$\{e^{2\pi i b_j y} : j \in J\}$$

will form a Riesz sequence on E_x defined above, for a.e $x \in [-\frac{1}{2}, \frac{1}{2}]$ with uniform Riesz bounds A, B. Hence, by Theorem 4.2.3 and Proposition 4.4.1, if

$$r > \frac{1}{2}, \quad \text{and} \quad D^+(\Lambda) < 2\sqrt{r^2 - \frac{1}{4}},$$

the collection $E(\Lambda)$ will form Riesz sequence on the disk $E = \{(x, y) \in \mathbb{R}^2, \ x^2 + y^2 \leq r\}$ with Riesz bounds A, B.

Example 4.4.5 (Rotated square). *Let*

$$Q_{0h} = \left[\frac{-h}{2}, \frac{h}{2}\right] \times \left[\frac{-h}{2}, \frac{h}{2}\right],$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

be a square of length $h > 0$ centered at $(0, 0)$ and a rotation matrix respectively for some $h > 0$ so that

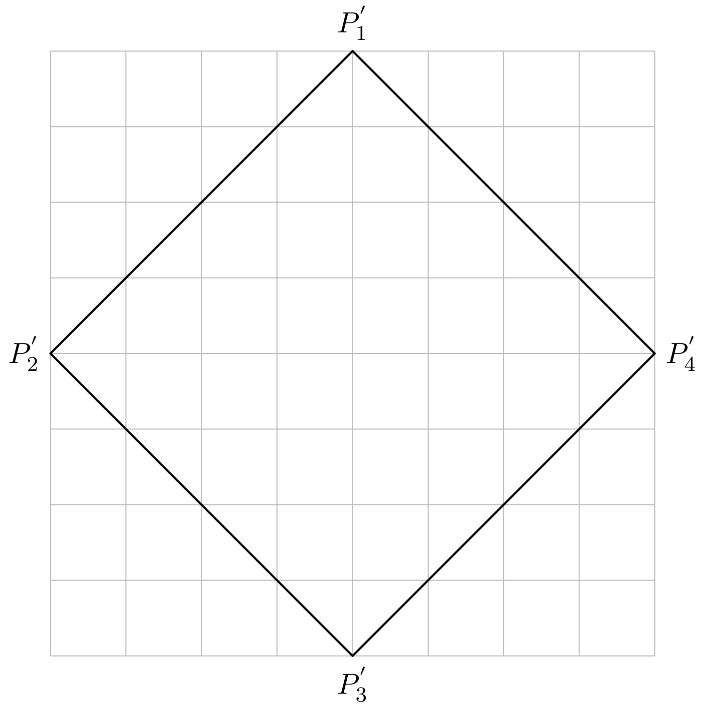
$$Q_h := AQ_{0h}$$

is the rotated square. Let (b_j) be uniformly discrete sequence of real numbers associated with

$$\Lambda = \{(k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}.$$

We seek to investigate the existence of frame or Riesz sequence of the form $E(\Lambda)$ on the *rotated square Q_h for some fixed $h > 0$.*

Let $P_1\left(\frac{h}{2}, \frac{h}{2}\right)$, $P_2\left(-\frac{h}{2}, \frac{h}{2}\right)$, $P_3\left(-\frac{h}{2}, -\frac{h}{2}\right)$, $P_4\left(\frac{h}{2}, -\frac{h}{2}\right)$ be the vertices of Q_h. Then the following are their respective images under A; $P_1'\left(0, \frac{h\sqrt{2}}{2}\right)$, $P_2'\left(-\frac{h\sqrt{2}}{2}, 0\right)$, $P_3'\left(0, -\frac{h\sqrt{2}}{2}\right)$, and $P_4'\left(\frac{h\sqrt{2}}{2}, 0\right)$.



Frame: We claim that $h \leq \frac{1}{\sqrt{2}}$ is a necessary condition for Q_h to support a frame with frame constants A, B. Assume $h > \frac{1}{\sqrt{2}}$, then we see that, the non-overlapping condition (frame necessary condition) in Theorem 4.3.1 fails.
By using the points $P_1'\left(0, \frac{h\sqrt{2}}{2}\right)$ and $P_4'\left(\frac{h\sqrt{2}}{2}, 0\right)$, we obtain the equation of the line

$$y = -x + \frac{h\sqrt{2}}{2},$$

and by using the points $P_3'\left(0, -\frac{h\sqrt{2}}{2}\right)$ and $P_4'\left(\frac{h\sqrt{2}}{2}, 0\right)$, we obtain the equation of the line

$$y = x - \frac{h\sqrt{2}}{2}.$$

Therefore, for each $x \in [-\frac{1}{2}, \frac{1}{2}]$, the sets

$$(Q_h)_x := \{y \in \mathbb{R} : \sum_{l \in \mathbb{Z}} \chi_{Q_h}(x + l, y) = 1\}$$

is given by

$$(Q_h)_x = \left[x - \frac{h\sqrt{2}}{2}\ , -x + \frac{h\sqrt{2}}{2}\ \right], \text{ for } |x| \leq \frac{h}{\sqrt{2}},$$

and

$$(Q_h)_x = \emptyset, \quad \text{when} \quad \frac{h}{\sqrt{2}} < |x| \leq \frac{1}{2},$$

with Lebesgue measure, $|(Q_h)_x| = -2x + h\sqrt{2}$. We now look for the largest of each of the sets $(Q_h)_x$ for a.e $x \in [-\frac{1}{2}, \frac{1}{2}]$.
Clearly, $\sup_{x \in [-\frac{1}{2}, \frac{1}{2}]} |(Q_h)_x| = h\sqrt{2}$. Therefore, by Theorem 4.4.2, if $D^-(b_j) > h\sqrt{2}$, the set

$$\{e^{2\pi i b_j y} : j \in J\}$$

will form a frame on $(Q_h)_x$ defined above, for a.e $x \in [-\frac{1}{2}, \frac{1}{2}]$ with uniform frame bounds A, B. Hence, by Theorem 4.3.2 and Proposition 4.4.1, if

$$h \leq \frac{1}{\sqrt{2}}, \quad \text{and} \quad D^-(\Lambda) > h\sqrt{2},$$

the collection $E(\Lambda)$ will form frame on Q_h with frame bounds A, B.

Riesz sequence. From the diagram above, for each $x \in [-\frac{1}{2}, \frac{1}{2}]$, the sets

$$(Q_h)_x := \{y \in \mathbb{R} : \sum_{l \in \mathbb{Z}} \chi_{Q_h}(x + l, y) = 1\}$$

is given by

$$(Q_h)_x = \left[x - \frac{h\sqrt{2}}{2}\ , -x + \frac{h\sqrt{2}}{2}\ \right].$$

We claim that, for Q_h support a Riesz sequence with Riesz constants A, B, if $h > \frac{1}{\sqrt{2}}$.
As before, suppose that $h = \frac{1}{\sqrt{2}}$. Then $(Q_{\frac{1}{\sqrt{2}}})_x = \left[x - \frac{1}{2}\ , -x + \frac{1}{2}\ \right]$ with

$$\inf_{x \in [-\frac{1}{2}, \frac{1}{2}]} |(Q_{\frac{1}{\sqrt{2}}})_x| = \inf_{x \in [-\frac{1}{2}, \frac{1}{2}]} (-2x + 1) = 0. \tag{4.25}$$

By Theorem 3.2.1, for Q_h to support a Riesz sequence with Riesz constants A, B, it is necessary that

$$A \leq |(Q_h)_x| \leq B, \quad \text{for a.e } x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

In the case where $h = \frac{1}{\sqrt{2}}$, the lower Riesz constant $A = 0$ by (4.25), which is a contradiction to $A > 0$. Hence, for Riesz sequences with Riesz constants A, B, we require that, $r > \frac{1}{2}$ in which case, $|(Q_h)_x| \neq 0$, for each $x \in [-\frac{1}{2}, \frac{1}{2}]$. We now look for the smallest of each of the sets $(Q_h)_x$ for a.e $x \in [-\frac{1}{2}, \frac{1}{2}]$.
Clearly, $\inf_{x \in [-\frac{1}{2}, \frac{1}{2}]} |(Q_h)_x| = h\sqrt{2} - 1$. Therefore, by Theorem 4.4.3, if $D^+(b_j) < h\sqrt{2} - 1$, the set

$$\{e^{2\pi i b_j y} : j \in J\}$$

will form a Riesz sequence on $(Q_h)_x$ defined above, for a.e $x \in [-\frac{1}{2}, \frac{1}{2}]$ with uniform Riesz bounds A, B. Hence, by Theorem 4.2.3 and Proposition 4.4.1, if

$$h > \frac{1}{\sqrt{2}}, \quad \text{and} \quad D^+(\Lambda) < h\sqrt{2} - 1,$$

the collection $E(\Lambda)$ will form Riesz sequence on Q_h with Riesz bounds A, B.

5 Shifts generated by non-full rank lattices in $\mathbb{R}^d$

In this chapter, we extend the results obtained in Chapter 4 under the action of a general non-full rank lattice.

5.1 Correspondence between shifts generated by non-full rank lattices in $\mathbb{R}^d$

Definition 5.1.1. *We define a non-full rank lattice $H \subset \mathbb{R}^d$ by*

$$H = \left\{ \sum_{i=1}^{n} k_i v_i : \; k_i \in \mathbb{Z} \right\}, \text{ where } v_i,\ i = 1, ..., n \text{ are } n \text{ linearly independent vectors in } \mathbb{R}^d.$$

Definition 5.1.2. *A closed subspace $M \subset L^2(\mathbb{R}^d)$ is H-**invariant** if*

$$f \in M \quad \text{then} \quad T_h f \in M \quad \forall h \in H$$

where $T_h f(x) = f(x - h)$.

Given $v_1, ..., v_n$, n linearly independent vectors in $\mathbb{R}^d$, we can choose additional vectors $v_{n+1}, ..., v_d$ such that the collection $\{v_i\}$, $i = 1, ..., d$, forms a basis for $\mathbb{R}^d$.

Proposition 5.1.3. *Given vectors $v_1, ..., v_d$, forming a basis for $\mathbb{R}^d$, the linear mapping $\mathcal{S} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined for $f \in L^2(\mathbb{R}^d)$ by*

$$\mathcal{S} f(x) = f(\boldsymbol{Q} x), \; x \in \mathbb{R}^d$$

*is an **isomorphism**, where the $d \times d$ matrix $\boldsymbol{Q}$ is given by*

$$\boldsymbol{Q} x = \sum_{i=1}^{d} x_i v_i, \; x = (x_1, ..., x_d) \in \mathbb{R}^d.$$

The inverse corresponding to the operator $\mathcal{S}$, $\mathcal{S}^{-1}$ *is given by* $\mathcal{S}^{-1}f(x) = f(\boldsymbol{Q}^{-1}x)$.

Proof. First of all, we see that the $d \times d$ matrix $\mathbf{Q}$ given by $\mathbf{Q}x = \sum_{i=1}^{d} x_i v_i$ is invertible as the vectors $v_i \in \mathbb{R}^d$ are linearly independent, and thus, $det(\mathbf{Q}) \neq 0$. To show that $\mathcal{S}$ is onto, given any $f \in L^2(\mathbb{R}^d)$, we choose $g \in L^2(\mathbb{R}^d)$ defined by $g(x) = f(\mathbf{Q}^{-1}x)$ so that $\mathcal{S}g(x) = f(x)$. Also see that, for any $f \in L^2(\mathbb{R}^d)$ we have that

$$\int_{\mathbb{R}^d} |g(x)|^2 dx = \int_{\mathbb{R}^d} |f(\mathbf{Q}^{-1}x)|^2 dx.$$

Using the change of variables $y = \mathbf{Q}^{-1}x$, we obtain

$$\|g\|^2_{L^2(\mathbb{R}^d)} = det(\mathbf{Q}^{-1}) \int_{\mathbb{R}^d} |f(y)|^2 dy < \infty. \tag{5.1}$$

It is also very easy to see that $Ker(\mathcal{S}) = \{0\}$. Therefore the mapping $\mathcal{S}$ is an *isomorphism* and $\mathcal{S}^{-1}f(x) = f(\mathbf{Q}^{-1}x)$.

□

Lemma 5.1.4. *Let* $\mathcal{S}$ *as in Proposition 5.1.3 and let* $H = \left\{ \sum_{i=1}^{n} k_i v_i : \ k_i \in \mathbb{Z} \right\}$. *Then, for every* $k \in \mathbb{Z}^n \times \{0\}^{d-n}$ *and every* $h \in H$,

$$T_k \mathcal{S} f(x) = \mathcal{S} T_h f(x), \quad f \in L^2(\mathbb{R}^d).$$

Proof. Let $f \in L^2(\mathbb{R}^d)$. Then,

$$\begin{aligned}
T_k \mathcal{S} f(x) &= (\mathcal{S}f)(x-k) && \text{for } \ x \in \mathbb{R}^d, k \in \mathbb{Z}^n \times \{0\}^{d-n} \\
&= f\Big(\mathbf{Q}(x-k)\Big) && \text{for } \ x \in \mathbb{R}^d, k \in \mathbb{Z}^n \times \{0\}^{d-n} \\
&= f(\mathbf{Q}x - \mathbf{Q}k) && \text{for } \ x \in \mathbb{R}^d, k \in \mathbb{Z}^n \times \{0\}^{d-n} \\
&= f(\mathbf{Q}x - h) && \text{for } \ x \in \mathbb{R}^d, h \in H \\
&= T_h f(\mathbf{Q}x) && \text{for } \ x \in \mathbb{R}^d, h \in H \\
&= \mathcal{S} T_h f(x) && \text{for } \ x \in \mathbb{R}^d, h \in H.
\end{aligned}$$

□

Proposition 5.1.5. *A closed subspace* $M \subset L^2(\mathbb{R}^d)$ *is* H***-invariant*** *if and only if the* *set* $\tilde{M}$ *defined by*

$$\tilde{M} = \{\mathcal{S}f,\ f \in M\} \tag{5.2}$$

is $\mathbb{Z}^n \times \{0\}^{d-n}$ *–* ***invariant****.*

Proof. Assume the closed subspace $M \subset L^2(\mathbb{R}^d)$ is H-**invariant**. Then given any $f \in M$, $T_h f \in M$, $\forall h \in H$. By (5.2), we conclude that $\mathcal{S}T_h f \in \tilde{M}$. Therefore we can deduce from Lemma 5.1.4 that $T_k\mathcal{S}f \in \tilde{M}$ for all $k \in \mathbb{Z}^n \times \{0\}^{d-n}$. Thus $\tilde{M}$ is $\mathbb{Z}^n \times \{0\}^{d-n}$ – **invariant**.
Conversely suppose that $\tilde{M}$ is $\mathbb{Z}^n \times \{0\}^{d-n}$ – **invariant**, then we have that $T_k\mathcal{S}f \in \tilde{M}$ for all $k \in \mathbb{Z}^n \times \{0\}^{d-n}$. Therefore by Lemma 5.1.4, we see that $\mathcal{S}T_h f$ also belongs to $\tilde{M}$. By (5.2), we conclude that $T_h f \in M$, $\forall h \in H$. Hence M is H-**invariant**. □

5.2 Riesz sequences, Frames and non-full rank lattices in $\mathbb{R}^d$

The theorem below establishes relationship between Riesz sequences and frames that are generated by the non-full rank lattice $\mathbb{Z}^n \times \{0\}^{d-n}$ and a general non full rank lattice H.

Theorem 5.2.1. *Let* $\mathcal{A} \subset L^2(\mathbb{R}^d)$ *be countable.*

1. *The family* $\{T_h f :\ f \in \mathcal{A},\ h \in H\}$ *is a Riesz sequence for* M *with Riesz constants* $A, B > 0$ *if and only if* $\{T_k\mathcal{S}f :\ f \in \mathcal{A},\ k \in \mathbb{Z}^n \times \{0\}^{d-n}\}$ *is a Riesz sequence of* $\tilde{M}$ *with Riesz constants* $A^* = \frac{A}{det(\boldsymbol{Q})} > 0$ *and* $B^* = \frac{B}{det(\boldsymbol{Q})} > 0$.
2. *The family* $\{T_h f :\ f \in \mathcal{A},\ h \in H\}$ *is a frame for* M *with frame constants* $A, B > 0$ *if and only if* $\{T_k\mathcal{S}f :\ f \in \mathcal{A},\ k \in \mathbb{Z}^n \times \{0\}^{d-n}\}$ *is a frame of* $\tilde{M}$ *with frame constants* $A^* = \frac{A}{det(\boldsymbol{Q})} > 0$ *and* $B^* = \frac{B}{det(\boldsymbol{Q})} > 0$.

Proof*.(1).* Suppose $\{T_h f :\ h \in H,\ f \in \mathcal{A}\}$ is a Riesz sequence of M with Riesz constants $A,\ B > 0$. Then for every sequence $\{\mathcal{C}_{f,h}\}$ indexed by $\mathcal{A} \times H$, with finitely many non-zero terms, we have

$$A \sum_{f\in\mathcal{A}} \sum_{h\in H} |\mathcal{C}_{f,h}|^2 \le \left\| \sum_{f\in\mathcal{A}} \sum_{h\in H} \mathcal{C}_{f,h} T_h f \right\|^2 \le B \sum_{f\in\mathcal{A}} \sum_{h\in H} |\mathcal{C}_{f,h}|^2. \tag{5.3}$$

Since

$$\left\| \sum_{f\in\mathcal{A}} \sum_{h\in H} \mathcal{C}_{f,h} T_h f \right\|^2 = \int_{\mathbb{R}^d} \left| \sum_{f\in\mathcal{A}} \sum_{h\in H} \mathcal{C}_{f,h} T_h f(w) \right|^2 dw, \tag{5.4}$$

letting $w = \mathbf{Q}x$, (5.3) becomes

$$\begin{aligned}
\left\|\sum_{f\in\mathcal{A}}\sum_{h\in H}\mathcal{C}_{f,h}T_h f\right\|^2 &= det(\mathbf{Q})\int_{\mathbb{R}^d}\left|\sum_{f\in\mathcal{A}}\sum_{h\in H}\mathcal{C}_{f,h}T_h f(\mathbf{Q}x)\right|^2 dx \\
&= det(\mathbf{Q})\int_{\mathbb{R}^d}\left|\sum_{f\in\mathcal{A}}\sum_{h\in H}\mathcal{C}_{f,h} f(\mathbf{Q}x-\mathbf{Q}\mathbf{Q}^{-1}h)\right|^2 dx \\
&= det(\mathbf{Q})\int_{\mathbb{R}^d}\left|\sum_{f\in\mathcal{A}}\sum_{h\in H}\mathcal{C}_{f,h} f\Big(\mathbf{Q}(x-\mathbf{Q}^{-1}h)\Big)\right|^2 dx \\
&= det(\mathbf{Q})\int_{\mathbb{R}^d}\left|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}}\mathcal{C}_{f,k} f\Big(\mathbf{Q}(x-k)\Big)\right|^2 dx \\
&= det(\mathbf{Q})\int_{\mathbb{R}^d}\left|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}}\mathcal{C}_{f,k}\mathcal{S}f(x-k)\right|^2 dx \\
&= det(\mathbf{Q})\int_{\mathbb{R}^d}\left|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}}\mathcal{C}_{f,k}T_k\mathcal{S}f(x)\right|^2 dx \\
&= det(\mathbf{Q})\left\|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}}\mathcal{C}_{f,k}T_k\mathcal{S}f\right\|^2.
\end{aligned}$$

That is,

$$\left\|\sum_{f\in\mathcal{A}}\sum_{h\in H}\mathcal{C}_{f,h}T_h f\right\|^2 = det(\mathbf{Q})\left\|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}}\mathcal{C}_{f,k}T_k\mathcal{S}f\right\|^2 \tag{5.5}$$

Therefore by (5.3), we obtain

$$A\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}}|\mathcal{C}_{f,k}|^2 \leq det(\mathbf{Q})\left\|\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}}\mathcal{C}_{f,k}T_k\mathcal{S}f\right\|^2 \leq B\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}}|\mathcal{C}_{f,k}|^2 \tag{5.6}$$

with $A^* = \frac{A}{det(\mathbf{Q})} > 0$ and $B^* = \frac{B}{det(\mathbf{Q})} > 0$. Thus $\{T_k\mathcal{S}f : \ f\in\mathcal{A},\ k\in\mathbb{Z}^n\times\{0\}^{d-n}\}$ is a Riesz sequence of $\tilde{M}$ with Riesz constants A*, B*.

Conversely suppose $\{T_k\mathcal{S}f : \ f\in\mathcal{A},\ k\in\mathbb{Z}^n\times\{0\}^{d-n}\}$ is a Riesz sequence of $\tilde{M}$ with Riesz constants $A^* = \frac{A}{det(\mathbf{Q})} > 0$ and $B^* = \frac{B}{det(\mathbf{Q})} > 0$. Then (5.6) holds. Using (5.5), we see that (5.6) becomes

$$A\sum_{f\in\mathcal{A}}\sum_{h\in H}|\mathcal{C}_{f,h}|^2 \leq \left\|\sum_{f\in\mathcal{A}}\sum_{h\in H}\mathcal{C}_{f,h}T_hf\right\|^2 \leq B\sum_{f\in\mathcal{A}}\sum_{h\in H}|\mathcal{C}_{f,h}|^2. \tag{5.7}$$

Therefore, $\{T_hf : f\in\mathcal{A},\ h\in H\}$ is a Riesz sequence for M with Riesz constants A, B.

(2). Suppose $\{T_hf : f\in\mathcal{A},\ h\in H\}$ is a frame for M with frame constants A, B. Then

$$A\|\ g\|^2 \leq \sum_{f\in\mathcal{A}}\sum_{h\in H}|\langle g, T_hf\rangle|^2 \leq B\|\ g\|^2, \quad \forall g\in M. \tag{5.8}$$

See that

$$\langle g, T_hf\rangle = \int_{\mathbb{R}^d} g(w)T_hf(w)dw \tag{5.9}$$

If we let $w=\mathbf{Q}x$ then (5.9) becomes

$$\begin{aligned}
\langle g, T_hf\rangle &= det(\mathbf{Q})\int_{\mathbb{R}^d} g(\mathbf{Q}x)T_hf(\mathbf{Q}x)dx, \quad h \in H\\
&= det(\mathbf{Q})\int_{\mathbb{R}^d} g(\mathbf{Q}x)f(\mathbf{Q}x-h)dy, \quad h \in H\\
&= det(\mathbf{Q})\int_{\mathbb{R}^d} \mathcal{S}g(x)\mathcal{S}f(x-k)dx, \quad k=\mathbf{Q}^{-1}h\in\mathbb{Z}^n\times\{0\}^{d-n}\\
&= det(\mathbf{Q})\int_{\mathbb{R}^d} \mathcal{S}g(x)T_k\mathcal{S}f(x)dx, \quad k=\mathbf{Q}^{-1}h\in\mathbb{Z}^n\times\{0\}^{d-n}\\
&= det(\mathbf{Q})\langle\mathcal{S}g, T_k\mathcal{S}f\rangle, \quad k=\mathbf{Q}^{-1}h\in\mathbb{Z}^n\times\{0\}^{d-n}.
\end{aligned}$$

Therefore,

$$\sum_{f\in\mathcal{A}}\sum_{h\in H}|\langle g, T_hf\rangle|^2 = |det(\mathbf{Q})|^2\sum_{f\in\mathcal{A}}\sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}}|\langle\mathcal{S}g, T_k\mathcal{S}f\rangle|^2. \tag{5.10}$$

Also, using the change of variables $w=\mathbf{Q}x$, we obtain

$$\|g\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}|g(w)|^2dw = det(\mathbf{Q})\int_{\mathbb{R}^d}|\mathcal{S}g(x)|^2dx = det(\mathbf{Q})\|\ \mathcal{S}g\|^2. \tag{5.11}$$

By (5.8), (5.10) and (5.11), we have that

$$\frac{A}{det(\mathbf{Q})}\| \mathcal{S}g\|^2 \leq \sum_{f\in\mathcal{A}} \sum_{k\in\mathbb{Z}^n\times\{0\}^{d-n}} |\langle \mathcal{S}g, T_k\mathcal{S}f\rangle|^2 \leq \frac{B}{det(\mathbf{Q})}\| \mathcal{S}g\|^2. \tag{5.12}$$

Therefore, $\{T_k\mathcal{S}f : f\in\mathcal{A},\ k\in\mathbb{Z}^n\times\{0\}^{d-n}\}$ is a frame of $\tilde{M}$ with frame constants $A^* = \frac{A}{det(\mathbf{Q})}$ and $B^* = \frac{B}{det(\mathbf{Q})}$.

Conversely, suppose that $\{T_k\mathcal{S}f : f\in\mathcal{A},\ k\in\mathbb{Z}^n\times\{0\}^{d-n}\}$ is a frame of $\tilde{M}$ with frame constants $A^* = \frac{A}{det(\mathbf{Q})}$ and $B^* = \frac{B}{det(\mathbf{Q})}$. Then, (5.12) holds. By (5.10) and (5.11), we have that (5.12) become (5.8). Hence $\{T_hf : f\in\mathcal{A},\ h\in H\}$ is a frame for M with frame constants A, B. □

We now state a consequence of Theorem 5.2.1 in relation to our main problem. We will let H and $\mathbf{Q}$ as in Definition 5.1.1 and Proposition 5.1.3 respectively throughout Corollary 5.2.2 and Theorem 5.2.3.

Corollary 5.2.2. *Let $\Omega \subset \mathbb{R}^d$ be a measurable set of finite Lebesgue measure. Let $(c_j) \subset \mathbb{R}^d$ and $a_j := \boldsymbol{Q}^{-1}c_j$ be given sequences associated with the set of frequencies*

$$\Lambda = \{(c_j + h) \in \mathbb{R}^d :\ h\in H,\ j\in J\}, \tag{5.13}$$

$$\tilde{\Lambda} = \{(a_j + k) \in \mathbb{R}^d :\ k\in \mathbb{Z}^n\times\{0\}^{d-n},\ j\in J\}. \tag{5.14}$$

Then the set $E(\Lambda)$ forming a Riesz sequence (resp. frame) for $L^2(\Omega)$ with Riesz (resp. frame) constants A, B, is equivalent to the set $E(\tilde{\Lambda})$ forming a Riesz sequence (resp. frame) for $L^2(E)$, where $E := \boldsymbol{Q}^T\Omega$, with Riesz (resp. frame) constants $Adet(\boldsymbol{Q})$, $Bdet(\boldsymbol{Q})$.

Proof. Define the shift invariant spaces V_Ω and V_E by

$$V_\Omega = \{f\in L^2(\mathbb{R}^d), \mathrm{supp}\hat{f} \subseteq \Omega\}$$

$$V_E = \{g\in L^2(\mathbb{R}^d), \mathrm{supp}\hat{g} \subseteq E\}.$$

Also, define $f_j \in V_\Omega$ by $\hat{f}_j = e_{c_j}\chi_\Omega$. Then, the fact that collection

$$\begin{aligned} E(\Lambda) &= \{e_{c_j+h},\ h \in H, j \in J\} \\ &= \{e_h e_{c_j},\ h \in H, j \in J\} \\ &= \{e_h \hat{f}_j,\ h \in H, j \in J\} \\ &= \{\widehat{T_h f_j},\ h \in H, j \in J\} \end{aligned}$$

is a Riesz sequence (resp. frame) in $L^2(\Omega)$, is equivalent to the collection

$$\{T_h f_j,\ h \in H, j \in J\} \tag{5.15}$$

forming a Riesz sequence (resp. frame) in V_Ω. Now, for $g_j(x) = f_j(\mathbf{Q}x)$, we have that

$$\hat{g}_j(\xi) = \int_{\mathbb{R}^d} f_j(\mathbf{Q}x)e^{-2\pi i x\cdot\xi}dx = \frac{1}{det(\mathbf{Q})}\int_{\mathbb{R}^d} f_j(w)e^{-2\pi i \mathbf{Q}^{-1}w\cdot\xi}dw = \frac{1}{det(\mathbf{Q})}\hat{f}_j([\mathbf{Q}^{-1}]^T\xi)$$

Thus,

$$\begin{aligned} \hat{g}_j(x) &= \frac{1}{det(\mathbf{Q})}\hat{f}_j([\mathbf{Q}^{-1}]^T x) \\ &= \frac{1}{det(\mathbf{Q})}e_{c_j}([\mathbf{Q}^{-1}]^T x)\chi_\Omega([\mathbf{Q}^{-1}]^T x) \\ &= \frac{1}{det(\mathbf{Q})}e^{2\pi i\langle c_j,\ [\mathbf{Q}^{-1}]^T x\rangle}\chi_{\mathbf{Q}^T\Omega}(x) \\ &= \frac{1}{det(\mathbf{Q})}e^{2\pi i\langle \mathbf{Q}^{-1}c_j,\ x\rangle}\chi_{\mathbf{Q}^T\Omega}(x) \\ &= \frac{1}{det(\mathbf{Q})}e^{2\pi i\langle a_j,\ x\rangle}\chi_E(x) \end{aligned}$$

Hence, the fact that collection

$$\begin{aligned} E(\tilde{\Lambda}) &= \{e_{a_j+k},\ k \in \mathbb{Z}^n \times \{0\}^{d-n},\ j \in J\} \\ &= \{e_k e_{a_j},\ k \in \mathbb{Z}^n \times \{0\}^{d-n}, j \in J\} \\ &= \{det(\mathbf{Q})e_k\hat{g}_j,\ k \in \mathbb{Z}^n \times \{0\}^{d-n}, , j \in J\} \\ &= \{det(\mathbf{Q})\widehat{T_k g_j},\ k \in \mathbb{Z}^n \times \{0\}^{d-n}, j \in J\} \end{aligned}$$

is a Riesz sequence (resp. frame) in $L^2(E)$, is equivalent to the collection

$$\{det(\mathbf{Q})T_k g_j,\ k \in \mathbb{Z}^n \times \{0\}^{d-n},\ j \in J\} \tag{5.16}$$

forming a Riesz sequence (resp. frame) in V_E.
By Theorem 5.2.1, (5.15) is a Riesz sequence (resp. frame) in V_Ω with Riesz (resp. frame) constants A, B if and only if $\{T_k g_j,\ k \in \mathbb{Z}^n \times \{0\}^{d-n},\ j \in J\}$ is a Riesz sequence (resp. frame) in V_E, with Riesz (resp. frame) constants $\frac{A}{det(\mathbf{Q})}, \frac{B}{det(\mathbf{Q})}$. But the collection

$$\{T_k g_j,\ k \in \mathbb{Z}^n \times \{0\}^{d-n},\ j \in J\}$$

is a Riesz sequence (resp. frame) in V_E, with Riesz (resp. frame) constants $\frac{A}{det(\mathbf{Q})}, \frac{B}{det(\mathbf{Q})}$ if and only if (5.16) is a Riesz sequence (resp. frame) in V_E, with Riesz (resp. frame) constants $Adet(\mathbf{Q})$, $Bdet(\mathbf{Q})$.
Therefore, the set $E(\Lambda)$ forming a Riesz sequence (resp. frame) for $L^2(\Omega)$ with Riesz (resp. frame) constants A, B, is equivalent to the set $E(\tilde{\Lambda})$ forming a Riesz sequence (resp. frame) for $L^2(E)$, where $E := \mathbf{Q}^T\Omega$ with Riesz (resp. frame) constants $Adet(\mathbf{Q})$, $Bdet(\mathbf{Q})$. □

6 Conclusion

The problem of existence of Riesz basis and frames of the form $\{e_{c_j+h} : h \in H, j \in J\}$ on a measurable domain $\Omega \subset \mathbb{R}^d$ of finite positive measure, for some collection of vectors $(c_j)_j \subset \mathbb{R}^d$, and H, the dual lattice of a full rank lattice $K \subset \mathbb{R}^d$, is well investigated by the authors in [1] and [9]. In the case where the domain Ω is bounded, Kolountzakis in [9], established that, Ω must be a multitile domain at level $k \in \mathbb{Z}^+$ on the full lattice K, in which case, the index set $J = \{1, ..., k\}$. Later in [8], Kolountzakis posed an open problem concerning the case where the domain Ω is unbounded. He asked whether the unbounded k-tile domain Ω is sufficient enough for the collection $\{e_{c_j+h} : h \in H, j \in \{1, ..., k\}\}$ to form a Riesz basis on this unbounded Ω.

The authors of [1], in an attempt to answer the question by Kolountzakis, examined this problem and discovered that, in the case where Ω is unbounded, there is the need for an extra arithmetic property, which they called the admissibility condition. The authors in [1] a gave proof of their result via the theory of shift invariant spaces. After a successful proof of their result using both the multitile at level k on the full lattice Λ and the admissibility conditions, they realized by way of an example that, this extra condition, that is, the admissibility condition, is too strong a condition as this example (Example 3.3 in [1]) showed that, an unbounded $k-$tile measurable domain of finite positive measure supports Riesz basis of the form $\{e_{c_j+h} : h \in H, j \in \{1, ..., k\}\}$ without being admissible.

In this thesis, we considered the case where $H \subset \mathbb{R}^d$ is a non-full rank lattice, and analyzed the conditions for which the collection $\{e_{c_j+h}, h \in H, j \in J\}$ forms a Riesz sequence or a frame on $\Omega \subset \mathbb{R}^d$ as above, for some collection of vectors $(c_j)_j \subset \mathbb{R}^d$, J, an at most countable index set.

We started by defining a non-full rank lattice $H \subset \mathbb{R}^d$, by $H := \mathbf{Q}G_n$, where $\mathbf{Q}$ is some $d \times d$ invertible matrix, and $G_n = \mathbb{Z}^n \times \{0\}^{d-n}$. Without loss of generality, we chose H as $\mathbb{Z}^n \times \{0\}^{d-n}$ to obtain the following results on Riesz sequences and frames in $\mathbb{R}^2$, via shift invariant spaces under the action of a non-full rank lattice $\mathbb{Z}^n \times \{0\}^{d-n}$.

An important result which we obtained from the extention of the theory of shift invariant subspaces of $L^2(\mathbb{R}^d)$ is given by Theorem 3.2.10. This theorem reduces the problem of checking whether a system in closed subspaces of $L^2(\mathbb{R}^d)$ is a Riesz sequence or frame to analyzing the fibers in closed subspaces of $L^2(\mathbb{Z}^n \oplus \mathbb{R}^{d-n})$. We obtained a corollary of Theorem 3.2.10 but worked in $\mathbb{R}^2$. This corollary establishes the link between the extended theory of shift invariant spaces and our main problem which was analyzed in $\mathbb{R}^2$.

The results on Riesz sequences are seen in Theorem 4.2.2, Theorem 4.2.3 and Theorem 4.2.6. In particular, Theorem 4.2.2 gives a necessary and sufficient condition for which the collection $E(\Lambda)$ to form a Riesz sequence in $L^2(E)$ with constants $A, B > 0$, where $\Lambda = \{(a_{j_o} + k, b_{j_o}) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j_o \in J\}$. Theorem 4.2.2 tells us a necessary and sufficient condition for the collection $E(\Lambda)$, where Λ is as above, to form a Riesz sequence in $L^2(E)$ with constants $A, B > 0$ for a single choice of $j \in J$. Now, in the case where $a_j \neq 0$ for any $j \in J$, it was hard to directly obtain results on Riesz sequences of the form $E(\Lambda)$ in $L^2(E)$, where $\Lambda = \{(a_j + k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}$. This motivated us to initially work with the case where $a_j = 0$ for any $j \in J$, and the results on Riesz sequences in this case was obtained in Theorem 4.2.3. In particular, Theorem 4.2.3 reduces the problem to the real line case which already has some known results on Riesz sequences. We later gave a proof of the case where $a_j \neq 0$ for any $j \in J$, using the knowledge about the choice of $(b_j)_j \subset \mathbb{R}$ (which comes from Theorem 4.2.3) and in addition, choosing $(a_j)_j \subset \mathbb{R}$ such that $|a_j| < \delta$ or $|a_j - 1| < \delta$ for some $\delta > 0$, where $D(x)$ is as given in Theorem 4.2.6.

The results on frames are also seen in Theorem 4.3.1, Theorem 4.3.2 and Theorem 4.3.3. We started by stating a necessary condition for which the collection $E(\Lambda)$ forms a frame for $L^2(E)$ with frame constants $A, B > 0$ where $\Lambda = \{(k, b_j) \in \mathbb{R}^2 \ : k \in \mathbb{Z}, j \in J\}$. In Theorem 4.3.2, we gave a sufficient condition in the real line case, for the collection $E(\Lambda)$, Λ as above to form a frame. Again, we reduced our argument to the real line, as there are some known results on frames with intervals. As before in the case of Riesz sequences, to give a proof on frames in the case where $a_j \neq 0$ for any $j \in J$, we used the knowledge about the choice of $(b_j)_j \subset \mathbb{R}$ (which comes from Theorem 4.3.2) and in addition, choosing $(a_j)_j \subset \mathbb{R}$ such that $|a_j| < \delta$ or $|a_j - 1| < \delta$ for some $\delta > 0$, where $D(x)$ is as given in Theorem 4.3.4.

We combined the hypotheses of both Riesz sequences and frames to give results on Riesz bases. This is seen in Theorem 4.3.4 and Theorem 4.3.5.

As an illustration of our results, we considered the disk in $\mathbb{R}^2$ of radius $r > 0$, and the rotated square at angle $\frac{\pi}{4}$, of side length $h > 0$, as examples. These are both bounded domains, and the sets E_x from Theorem 4.2.3 and Theorem 4.3.2 are intervals. Therefore, via the already known density results on Riesz sequences and frames, we gave conditions for which both the disk and the rotated square will admit Riesz sequences and frames of exponentials of the form $E(\Lambda)$, where Λ is as given in Example 4.4.4 and Example 4.4.5.

We finally established a correspondence between the domain $\Omega \subset \mathbb{R}^d$ and $E \subset \mathbb{R}^d$ and obtained that, the collection $E(\Lambda)$ forming a Riesz sequence (resp. frame) for $L^2(\Omega)$ with Riesz (resp. frame) constants A, B, is equivalent to the collection $E(\tilde{\Lambda})$ forming a Riesz sequence (resp. frame) for $L^2(E)$, with Riesz (resp. frame) constants $Adet(\mathbf{Q})$, $Bdet(\mathbf{Q})$ where the set of frequencies Λ, and $\tilde{\Lambda}$ are given by

$$\Lambda = \{(c_j + h) \in \mathbb{R}^d : \ h \in H, \ j \in J\},$$

$$\tilde{\Lambda} = \{(a_j + k) \in \mathbb{R}^d : \ k \in \mathbb{Z}^n \times \{0\}^{d-n}, \ j \in J\}.$$

with $E := \mathbf{Q}^T\Omega$, $(c_j)_j \subset \mathbb{R}^d$ and $a_j := \mathbf{Q}^{-1}c_j$.

There are still many unsolved questions concerning the existence and the construction of Riesz sequences, frames and Riesz bases of exponentials on a measurable subset of $\mathbb{R}^d$ of finite positve measure. We hope to have shed some light on some particular aspects of this problem in this thesis.

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