A THEORETICAL MODEL AND STUDY OF MATHEMATICAL ANXIETY

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Abstract

The study of mathematical anxiety has seen an increased importance in the past few decades in the field of mathematical education. As this topic is of great interest in education research, this thesis investigates the previous contributions made by other researchers via a literature review of mathematical education papers. Furthermore, a literature review of mathematical models of learning is presented. In the hopes of closing the gap between these two streams of research, this thesis conducts a study of mathematical anxiety at the first year university level through a survey and data analysis, and proposes a theoretical model of learning. Throughout the data analysis, the prevalence, effects, and correlates of mathematical anxiety are examined. Using a version of the Mathematical Anxiety Rating Scale refined by Plake & Parker in 1982, factors such as gender, high school performance, and program choices are shown to be correlated to mathematical anxiety, as is consistent with previous literature. On the other hand, the model of learning offers a theoretical perspective in understanding the relationship between knowledge, effort, and anxiety, and how these variables interact during a learning experience. This model suggests that given an individual's aptitude, drive, and susceptibility for anxiety, that they may reach various levels of knowledge, effort, and anxiety throughout an academic term.

Key words. Mathematical anxiety, mathematical model of learning, mathematical education.

Foreword

I wanted to make a small note about how this thesis came to be. Over the years, my experiences as a teaching assistant have allowed me to work with many students. Among these students, there were many that exhibited a fear of mathematics and this fear greatly inhibited their learning. I encountered more and more students with that fear and many of them would say they hated mathematics or that the course they were taking would be the last they'd ever take. It became evident that mathematical anxiety was not just a social construct or stigma associated to students without a "math brain".

This realization inspired me to pursue my passion, both as an academic and an educator, to study this phenomenon. At first, I was not sure how I would go about the study of mathematical anxiety as many researchers have done experiments and longitudinal studies on this topic. I then stumbled upon the comments and criticism by Preece in [20]. In this note, they discuss mathematical modeling of learning and the works of Anderson, [2], and Hicklin, [13]. This intrigued me, and I was now interested in the idea of studying the effects of mathematical anxiety via the modeling of learning.

As such, this thesis is driven by curiosity, passion for mathematics, and the desire to help students overcome this phenomenon of mathematical anxiety.

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Chapter 1

Introduction

The past few decades have seen an increase in the importance of mathematics education. Academic journals such as the *Journal for Research in Mathematics Education* (JRME), the *Journal of Educational Research*, and the *Journal of Educational Psychology* among many others, have multitudes of papers focused particularly on the teaching and learning of mathematics. Over the years, this research has lead to evolution in the field by shaping mathematics curricula and educators teaching techniques. Despite the many advancements, there are still many obstacles that students face when studying mathematics. In particular, the term Mathematical Anxiety (MA) has become prominent in mathematics education. Richardson & Suinn (1972) stated that, "Mathematical anxiety involves feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" [23]. Recently, there has been a greater awareness surrounding the existence and consequences of mathematical anxiety in students, and thus, there have been studies that have explored this phenomenon.

Throughout the years, the study of mathematical modeling of learning has also been a field of interest. In fact, there have been a variety of papers written on the development of mathematical models of learning. Researchers such as Atkinson and Shiffrin (1965), Hicklin et al. (1965),[12], Hicklin (1976),[13], Anderson (1983),[2], and Pritchard (2008),[21], have proposed models to describe the learning process via memory function, mastery learning, content acquisition, and teaching styles. These models have offered insight into how individuals interact with their environment and learn.

Despite mostly being distinct areas of research, there is an obvious connection between these different streams. More precisely, the development of models of learning could be enhanced by including motivational factors such as the effects of mathematical anxiety. As such, the first of three objectives of this thesis will be to present a literature review on the previous research done on mathematical anxiety and mathematical models of learning. The second objective will be to analyze data collected at McMaster University to investigate mathematical anxiety in students enrolled in first year university level mathematics courses. Finally, the third objective will be to propose a theoretical model that could describe the interaction between knowledge, effort, and anxiety. With these three goals, this thesis will hopefully begin to bridge the gap between the qualitative research of mathematical education and the quantitative results of models of learning. A list of important acronyms used throughout this thesis can be found in Appendix A.

1.1 Literature Review

1.1.1 Mathematical Education Research

The advancements that have been made in the field of education since the nineteen-fifties have been significant. They have lead to new and innovative ways to teach mathematics and have improved the ways in which students learn. Here, the main literary contributions, particularly related to the study of anxiety and learning, are presented and discussed. In the early literature, most research dealt with the theme of general anxiety or test anxiety.

Many researchers have found that test anxiety does affect learning. As an example, in [16], Mandler & Sarason conducted a study that consisted of having participants sorted into groups by level of anxiety and randomly into subgroups (success, failure and neutral). Subsequently, participants would undergo six trials of various tests such as the Kohs Block Design and the Digit Symbol Test. Later, the participants were told that they had either done very well, very poorly, or nothing at all regardless of their actual performance. Overall, they found that groups with low anxiety performed better on the trials. Moreover, being told that they had either succeeded or failed improved scores for the low anxiety group, but hindered the performance for the participants with high anxiety [16].

This suggests that individuals are both affected by their levels of anxiety and their perception of their achievements. This result is consistent with the interference model whereby test anxiety reduces performance by inhibiting recollection of prior knowledge [11]. Other researchers have conducted studies on test anxiety that are also consistent with the interference model (see [14],[29]).

Contrary to the interference model, there is also the concept of the deficits model in which poor performance is not caused by test anxiety but rather that the opposite is true [11]. This deficits model, considered in the work of Tobias (1985), [28], suggests that poor performance is a result of insufficient study habits or test taking skills, and that test anxiety develops because of this.

Despite the detrimental effects of anxiety, researchers have also suggested that anxiety could act as a motivating factor and lead to improvement. This was one of the results of the experiment of Mandler & Sarason in [16]. They found that the high anxiety participants were driven to improve over the course of the six trials.

Moreover, arousal theorists, such as Hebb (1955),[10], suggest that there is an optimal level of arousal in terms of performance. In other words, anxiety, among other factors, could be motivating and lead to improvement in an individual's performance.

Though many researchers have been inclined to suggest that mathematical anxiety is a specific subset of test anxiety, research on mathematical anxiety has flourished since the early 1970's. In [26], Suinn found that approximately thirty percent of students had problems with mathematics anxiety. Many studies have suggested that mathematical anxiety is so severe that students may even choose to avoid mathematics altogether. In fact, researchers, such as Richardson & Suinn, 1972 (later Betz, 1978 and Meece, Eccles & Wigfield, 1990) have found that mathematics anxiety has affected course enrollment as well as students career plans.

Due to the prevalence of mathematical anxiety, Frank C. Richardson & Richard M. Suinn developed the Mathematics Anxiety Rating Scale (MARS) in [23]. This scale is a tool to measure a student's level of anxiety when taking on a mathematical task. The MARS is a 98 item questionnaire of math related situations in which students rate their anxiety on a scale from 1 (no anxiety) to 5 (very high anxiety) [23]. The MARS has since been used in many research experiments as a measuring tool to both assess mathematical anxiety and to verify treatments. Not only has the MARS been widely utilized in the literature, it has also been revised, adapted and verified by other researchers. For instance, in [27], published in 1979, Suinn refined the scale for adolescents, and in [7], Brush conducted a study of the validation of the MARS. Brush concluded that the study was successful in correlating with students with negative attitudes towards mathematics and in differentiating between different groups of students [7].

Furthermore, in [19], Plake & Parker created and verified the validity of a more concise version of the MARS. This concise version consists of only 24 items is an efficient index of statistical and mathematical anxiety and is significantly correlated (0.98) with the results of the full MARS [19].

In their paper, they also conduct a factor analysis to analyze the nature of the items in the MARS. This analysis identified two important categories for the items, namely Learning Mathematics Anxiety and Mathematics Evaluation Anxiety. Thus, this solidifies the distinction between anxiety related to learning mathematics and related particularly to test anxiety. This is also consistent with a previous study, [24], by Rounds & Hendel (1980), who found two dominant factors (Mathematics Test Anxiety and Numerical Anxiety) of the original MARS in their factor analysis.

Having sufficient means to measure mathematical anxiety, many researchers have contributed to the literature by investigating the nature of mathematical anxiety and who it affects. Thoughout the literature, it is apparent that mathematical anxiety affects students across all levels of study. In fact, as the meta-analysis by Hembree [11] confirms, mathematical anxiety has been exhibited by students from grade three all the way to the postsecondary level of education. More precisely, in [18], the authors conducted their experiment for young adolescent students in grades seven through nine. Similarly, in 2003, Sherman & Wither wrote their paper on a longitudinal study conducted with students from grades six through ten in schools in Adelaide, Australia. In [5], Betz focused primarily on the prevalence of mathematical anxiety in college students. In [3], Ashcraft and Kirk also focused on undergraduate level students in their investigation of the relationship between working memory, mathematical anxiety and performance.

These studies have found many correlates, predictors, and consequences of mathematical anxiety. In [5], Betz confirms that mathematical anxiety does occur regularly in college students. Further, the author finds that the level of mathematical anxiety does vary within different subgroups of students. As an example, Betz found that the biological sex of an individual made a difference in the way mathematical anxiety had an influence on that person in certain classes, such as psychology and math. This finding is consistent with other studies, such as Fennema and Sherman (in [9], and is confirmed later by Hembree's (1990) meta-analysis, that considered gender and mathematical anxiety. However, it is important to note, that Meece et al. find that mathematical anxiety does not affect the achievement of boys and girls differently, but rather the students report different levels of mathematical anxiety. In [18], the authors also found that a student's expectation of their own achievement had a big effect on their level of anxiety. Thus, student perceptions and self-expectancies can have a direct influence on their levels of mathematical anxiety. Moreover, Ashcraft and Kirk found that mathematical anxiety inhibits working memory, thereby slowing down performance [3]. In essence, throughout the literature, many different findings have arisen on the nature of mathematical anxiety and how it affects individuals of all levels.

In their meta-analysis on the topic of mathematical anxiety, [15], Ma states that arousal theory could lead to the assumption that performance can be depicted as an inverted U shape when considering the effects of anxiety. However, in most papers that investigate the effects of anxiety, authors use methods of statistical analysis and linear correlation coefficients to describe its relationship with performance.

Many studies have shown that there is a significant negative correlation between these two phenomenon. Hembree's meta-analysis of 151 studies confirms this fact and finds negative correlations across all levels of study (-0.3 or more for younger students and -0.27 or more for tertiary students) [11]. Due to this finding, Hembree concludes that mathematical anxiety does impede performance as, "higher achievement consistently accompanies reduction in mathematics anxiety" [11]. What is surprising however, is that according to the author, there is not sufficient evidence to suggest that poor performance leads to mathematics anxiety. In [15], the author also makes note of this apparent negative correlation between achievement and mathematical anxiety.

Similarly, Sherman & Wither's (2003) longitudinal study, [25], found that there was a negative correlation between mathematics anxiety and mathematics achievement. Nonetheless, the authors data suggests that they reject the hypothesis that mathematics anxiety impedes mathematics achievement in subsequent years. This finding certainly questions the results of previous studies.

Despite this fact, it is important to note that Sherman and Wither's experiment gathered information on students mathematical anxiety from one year to the next. Contrarily, other studies were measuring the effect of mathematical anxiety in discrete experiments over short periods of time and this could be the cause of the difference. Thus, this finding does not necessarily contradict the previous results, but rather suggests that mathematical anxiety may have a more immediate effect and may not affect the performance of a student a year later.

These findings, among others, suggest that there is still a need to establish a causal relationship between mathematical anxiety and mathematics achievement. It is obvious from the results of the previous studies that a negative correlation is certainly present. However, the direction of this relationship has not yet been determined. Does mathematical anxiety cause poor performance? Does poor mathematics achievement cause mathematical anxiety? Even more, is there a more complicated relationship at work here? Perhaps the relationship can better be described by more dynamic properties. In fact, this has been suggested in [15], where the author states, "the relationship can change dramatically for students with different social and academic background characteristics". This is one of the theories that motivates the development of the theoretical model in a subsequent chapter.

1.1.2 Models of Learning Research

Just as there have been advances in mathematics education research, there have been many papers from divers fields written on the subject of modeling learning and content acquisition. These papers have allowed researchers to better understand the nature of human learning and how people acquire knowledge. Notable contributions to this research have been made by Atkinson & Shiffrin (1965), Hicklin et al (1965, 1976), Anderson (1983) and Pritchard (2008). Though each of these studies took a slightly different approach, their work is helpful in motivating derivations of new models of learning. These papers will be compared and contrasted to discuss their contributions to the literature, as well as motivate the derivation of a new mathematical model of learning.

In 1965, R. C Atkinson and R. M. Shiffrin wrote, [4], on Mathematical Models for Memory and Learning. In this work, they develop and propose models to explain retention phenomena by assuming that there are distinct differences between short-term and long-term memory systems. This particular assumption is still valid in more recent research such as Anderson (1983).

Atkinson & Shiffrin's models seek to describe a content acquisition situation whereby the individual is presented with a certain number of incoming stimuli (coloured cards, words, etc) for a brief period of time and then are asked to recall the list. By making specific assumptions, the authors set out to predict the probability of a correct response given the length of the list and the test position.

When the individual is presented with the stimulus, it is said to enter the, "sensory buffer". Next, the items enter what the authors call the, "memory buffer". At this stage, items can be recalled instantly unless they did not properly enter the buffer. However, consistent with theory on short-term memory, only a finite number of items may be stored here at once. Thus, as items enter the buffer, others may be pushed out. These items may be lost or forgotten, and could also enter the long-term store (LTS). Furthermore, items that have been lost or forgotten have been removed from the buffer regardless of their state in the long-term store.

The models derived depend mainly on the different assumptions made on the transition of items to the long-term store. The first of three models considers the case where items can be recalled perfectly from the LTS. That is, items that have been stored once and only once in LTS, and that can be retrieved perfectly. The second model assumes that retrieval of information from the LTS is not perfect. In this case, the item that the individual is trying to recall is not in the buffer, and there is a search process to find it in the LTS. As such, the probability of finding the item decreases as the number of items in the LTS increases. Finally, the third model they present is identical to the second in terms of the imperfect retrieval of the items in the LTS. However, if the item cannot be recalled, the search may disrupt the items in the buffer. Thus, though this model is a probabilistic one and depicts only the probability of an item from a list being recalled, it provides a good foundation for assumptions about how an individual's memory works.

In 1965, Hicklin et al., describe the growth and decline between individuals and their environment in regards to their intelligence via a theoretical framework. In particular, there is a state of dynamic equilibrium whereby individuals react with their environment to acquire a level of intellectual development. Moreover, the authors suggest that there is a state of equilibrium between individual and environment due to a small fractional loss of what has been acquired by the individual, which may or may not be reacquired by the environment.

The authors compare the results provided by their solution curves to previous works and find that there is a fair level of agreement to empirical data. Some of the conclusions are that individuals acquire about fifty percent of their knowledge by age four, the maximum occurs between the ages of twenty and twenty-five and that by age fifty-five, the intellectual level has reverted to that of a thirteen year old.

The model derived by Hicklin et al. depicts intelligence over an individual's lifetime given an aptitude constant and a loss rate constant. In fact, this level depends only on the ratio of the acquisition rate and the loss rate constants [12]. With this fact, the concepts of early and late bloomers is realistic, as the age of maximum intellectual level would change accordingly. This theory does not specify any particular rate of decline or age of maximum development, though other theorists have suggested various ages. Hicklin et al. simply suggest that at some point, whether it be at age twenty-five, thirty, or fifty, the individual who remains in a standard environment will eventually reach a maximum after which it will begin to decline.

A special case of their model is noted and is useful in describing the growth of ability, however it does not provide the point at which an individual reaches their maximum status. This model also does not account for the loss of ability in later years. This particular special case also arises in the work of Anderson in [2].

In [13], another paper by Hicklin, a model for mastery learning based on dynamic equilibrium theory is proposed. Mastery learning is a process whereby, “Individuals of different aptitude reacting in an ideal environment to the same total quantity of material will ultimately reach the same mastery status, or gain equal increments of status starting from the same base, in time spans inversely proportional to the aptitude” [13].

Learning in itself is a psychological process, and thus, there have been contributions to models of learning in the field of psychology. Anderson (1983) developed a neuromathematical model to describe the information acquisition process and how it may be applied to science content acquisition. In particular, the author presents a model which describes content acquisition inspired by the workings of the central nervous system’s (CNS) ability to process new information via short and long term memory.

The model accounts for both the internal state of the learner and the characteristics of the information. In order to describe the content acquisition process, there are three rate functions that are established, namely the stability function, the instability function, and the gain function. The stability function describes the fact that there is limited space available in short term memory, whereas the instability function represents the familiarity one develops with a learning task as time goes on and how this makes learning easier. Finally, the gain function describes how the learner would acquire information.

Anderson’s model can result in three distinct learning curves. If the gain function is large enough, despite the modulation factor’s effect, the learning curve will continue to grow. This result depicts a simple learning task that the learner’s abilities can withstand. However, if the gain function coefficient is equal to the stability decay coefficient, the equation of net gain can be written as equation (B.3.11), and the learning curve would approach an asymptote and saturate as is the result of the special case (see equation (B.2.7)) of the model derived by Hicklin et al. in [12]. In the final case, the gain function is not large enough to outweigh the effects of the modulation factor and the curve of the net gain would be bowed. This would represent a difficult learning experience whereby the learner’s capacity would not be able to support the demand of the task.

Throughout their paper, Anderson also gives estimates for the values of constants and parameters by referring to empirical data. Furthermore, the author verifies the accuracy of the predictions of the model by experiment with seventh or eighth grade students (I.Q. of 100-140). They presented tape-recorded information to students with two possible topics, biology of Hydra or human health. After periods of 2-10 minutes, students were asked to recall and write down as many statements from the presentations as possible. In general, the modulation factor's effect suggests that adolescents would benefit from short time periods of information transferral, as there is a peak amount of time that occurs when students are able to perform best. This experiment showed that there was a remarkable correlation between the predicted values of Anderson's neuromathematical model and the empirical data.

Though the model is quite accurate, there are still some limitations that arise. For instance, motivation and fatigue factors have not been accounted for in Anderson's model. Furthermore, the model itself only describes a short interval of time of a few minutes to a half an hour, thus, it does not present the long term behaviour of learning. Despite these limitations, this type of psychological theory can surely be applied to mathematics learning in particular.

Whereas the previous papers discussed presented models which take into account the aptitude of the learner and the characteristics of the information to be acquired, in [21], Pritchard et al. proposes a model inspired by three different teaching methods and an individual's prior knowledge base. This is an interesting approach as it reinforces that diverse teaching methods will lead to different results in learning. Though this particular model is applied to physics students, this model is certainly relevant in the scope of this paper.

In particular, Pritchard et al. propose three different models that vary depending on the style of teaching, More precisely, the authors consider the following learning theories. The first theory is *tabula rasa*, which suggests, "that the mind can be seen as a blank slate which is imprinted with knowledge initially through experience" [21]. The second, *constructivism* theory, considers, "the notion that new knowledge is "constructed" from associations involving knowledge, and thus that increased prior knowledge should positively affect the rate of learning" [21]. The third and final theory that Pritchard et al. make note of is *tutoring*, which is considered as, "one-on-one expert mentoring that is tailored to the particular student" [21].

Using those learning theories, the authors model the student's knowledge after a particular amount of instruction. In order to do this, the authors partition the test domain into what is known and what is unknown. Throughout the paper, a pure memory model, a simple connected model, a connectedness model, and a tutoring model are derived. These models (described in more detail in the appendix B.4), suggest that depending on the motivating philosophy, the learning of the individual changes.

In particular, in the pure memory model, the individual's learning slows as knowledge increases due to the fact that they are trying to memorize information presented to them. Contrarily, the connectedness model suggests that new knowledge is associated to prior knowledge and thus, students learn faster when they have more knowledge. Furthermore, tutoring results in the fastest learning as the tutor can focus on what the student does not know [21]. In general, they find that if the individual has a good base of prior knowledge, the tutoring model is the most effective as the learning process can be more efficiently directed.

Though all of these researchers have taken different approaches appropriate for their fields of study, each of these models can provide insight for future development of models of learning. Most importantly, these models motivate certain assumptions to be made in a subsequent chapter to propose another model of learning including the effects of mathematical anxiety.

Chapter 2

Analysis of Survey

In order to investigate the level of mathematical anxiety present in students, an online survery was administered. This survey was approved by the McMaster Research Ethics Board on June 2nd, 2016 and data was collected in the Summer 2016 and Fall 2016 terms at McMaster University. A copy of the survey that participants answered can be found in Appendix C.

2.1 Hypotheses

Based on the previous research in mathematics education, some hypotheses can be made about what might arise from the survey data. These main hypotheses are listed below.

1. Mathematical anxiety will be higher in female survey participants.
2. The higher the academic achievement in students in high school, the lower the level of MA.
3. The higher the expectation of achievement in students, the higher the MA.
4. The higher the current grades of the students, the lower the level of MA.
5. Students not enrolled in a STEM program will exhibit higher levels of MA.
6. Items on the MARS that deal with examinations or numerical anxiety will foster more anxiety than course preparation items.

2.2 Participants

The online survey was advertised to students enrolled in first year mathematics courses at McMaster University by professors via their course websites and also via email from the Undergraduate Office of the Department of Mathematics and Statistics. Volunteers then participated by following the link to the survey administered via LimeSurvey.

Over the course of both terms, a total of 155 volunteers completed the survey. There were 18 participants who were enrolled in a course during the Summer term and 137 participants during the Fall term. With only 155 volunteer participants, the response rate was low since the total number of students enrolled in first year mathematics courses at McMaster University is usually around 1000 students.

Of these participants, 107 (4 Summer, 103 Fall) were female, 45 (14 Summer, 31 Fall) were male, and 3 (Fall) preferred not disclose their gender. The gender distribution of the students can be seen in Figure 2.1. Note that in this paper, gender is considered to be the way that an individual identifies, and it is not analogous to biological sex. In today's society, gender is considered to be a spectrum as opposed to a binary, and thus, participants were allowed to not respond to this survey item.

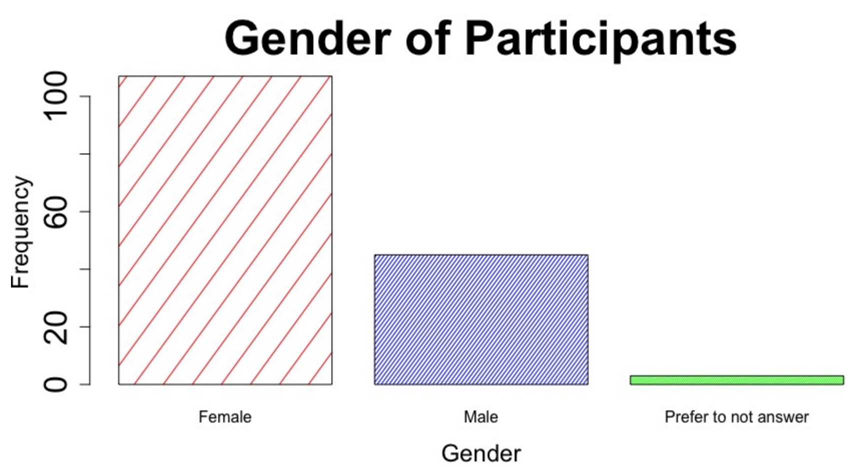


Figure 2.1: Distribution of Participants by Gender

Participants varied between 17-46 years of age, however, most participants were 18 years old ($M_{age} = 18.6$). Figure 2.2 shows the distribution of participants' age.

The participants came from a variety of programs and were enrolled in at

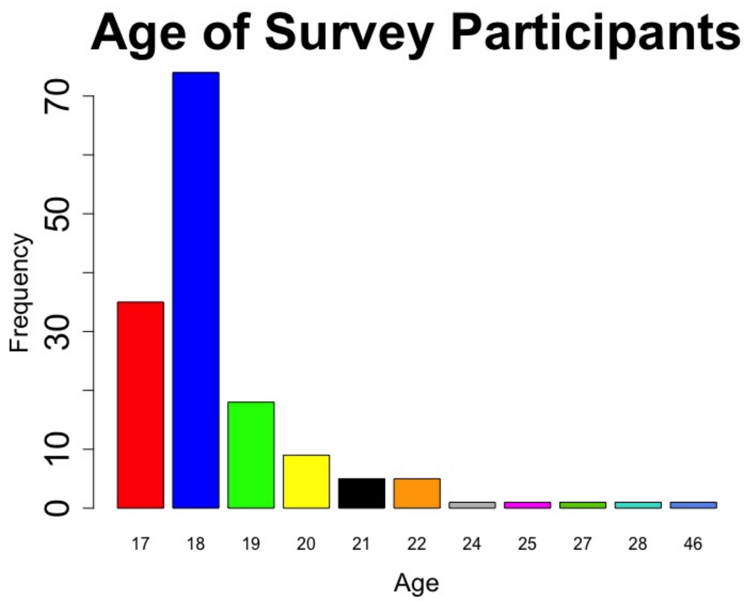


Figure 2.2: Distribution of the Age of Participants in the Summer and Fall 2016 terms

least one of thirteen first year mathematics courses. Table 2.1 and Figure 2.3 show the breakdown of participants according to their program of study and the courses they were enrolled in. Here, it is noted that there is an uneven distribution of students from different programs. For example, there are many students from the Life Sciences and only one from the Computer Sciences program. Furthermore, there is a lack of students enrolled in Math 1K03, a grade twelve equivalent calculus course, whereas many of the participants were enrolled in Math 1LS3, calculus for the Life Sciences.

2.3 Method

The MA survey consisted of four main sections to gather data from the participants. First, participants were asked to provide demographic information such as age, gender, program of study, and course in which they were enrolled. Second, participants self-reported their grades from their previous high school course, their expectation for the current course, as well as any grades that they had received so far throughout the term. The third section consisted of the revised Mathematics Anxiety Rating Scale (MARS), refined by Plake & Parker (1983), where participants were asked to rate their feelings of anxiety

Program	Summer	Fall	Total
Business	1	14	15
Engineering	8	0	8
Life Sciences	0	47	47
Other	3	32	35
Computer Sciences	1	0	1
Kinesiology	0	5	5
Math and Stats	3	37	40
Physics, Chemistry or Biology	2	2	4
Total	18	137	155

Table 2.1: Distribution of Survey Participants by Program and Term

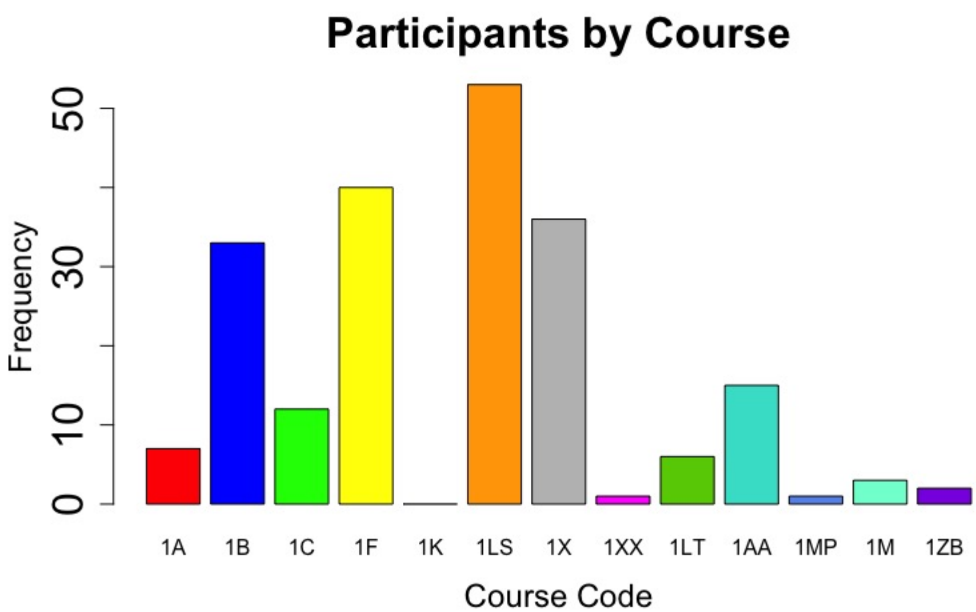


Figure 2.3: Barplot of the Distribution of the Participants by Course

about each statement on a scale from 1 (low anxiety) to 5 (high anxiety). The lowest score possible is 24, whereas the highest is 120. Below are the questions from the revised MARS which may later be referred to as "MARS1-MARS24" for convenience.

Revised MARS Questions

1. Watching a teacher work an algebraic equation on the blackboard.
2. Buying a math textbook.
3. Reading and interpreting graphs or charts.
4. Signing up for a course in Statistics.
5. Listening to another student explain a math formula.
6. Walking into a math class.
7. Looking through the pages in a math text.
8. Starting a new chapter in a math book.
9. Walking on campus and thinking about a math course.
10. Picking up a math textbook to begin working on a homework assignment.
11. Reading the word "Statistics".
12. Working on an abstract math problem, such as: "if x = outstanding bills, and y = total income, calculate how much you have left for recreational expenditures".
13. Reading a formula in chemistry.
14. Listening to a lecture in math.
15. Having to use the tables in the back of a math book.
16. Being told how to interpret probability statements.
17. Being given a homework assignment of many difficult problems which is due the next class meeting.
18. Thinking about an upcoming math test one day before.
19. Solving square root problems.
20. Taking an examination (quiz) in a math class.
21. Getting ready to study for a math test.

22. Being given a “pop” quiz in a math class.

23. Waiting to get a math test returned in which you expected to do well.

24. Taking an examination (final) in a math course.

The fourth and final section of the survey was used to learn about the student awareness when considering the resources available at McMaster University. For instance, participants had to answer whether they had heard about, and if they would attend the Math Help Centre, the Student Wellness Centre, and Student Accessibility Services. The Math Help Centre is available to students on weekdays and provides free Teaching Assistant help to students enrolled in all first year mathematics courses. The Student Wellness Centre provides support to students who may need counselling or help adjusting to university life. Finally, the Student Accessibility Services allow students with disabilities, either physical or learning, to seek assistance with their courses via note taking help, or test taking arrangements etc. These services can be very helpful to students if they are struggling with MA or simply need more support during their studies. As such, we found it important to know whether the students’ awareness correlated in any way with their achievement or level of MA.

2.4 Findings

In this section, we outline the notable results of the analysis of the data collected via the mathematical anxiety survey. In order to perform this analysis, the statistical program R was utilized. The results of the survey will shed light on the nature of MA in students enrolled in first year courses at McMaster University. Furthermore, these findings may help to provide a foundation for the assumptions that are to be made to derive a theoretical model later in this paper. More precisely, we seek to pinpoint any possible trends in the data to identify precursors or indicators of MA in the participants.

2.4.1 Age

As is expected of students enrolled in first year courses in university, most of the students were around 18 years of age. However, due to different circumstances such as avoidance and time constraints, some students may take their mathematics courses later in their post-secondary education. Thus, there were some outliers in the data as there were some more mature students (aged

25-46) present in the variety of the participants. These data points were not removed as they reflect the diversity that is present in students enrolled in first year mathematics courses. Despite this variability in the age of the participants, this did not seem to affect the level of mathematical anxiety of the students. In fact, where there were many participants of the same age, the level of MA varied significantly and thus, there were no notable trends that arose from studying MA by age. Figure 2.4 shows the level of mathematical anxiety by age.

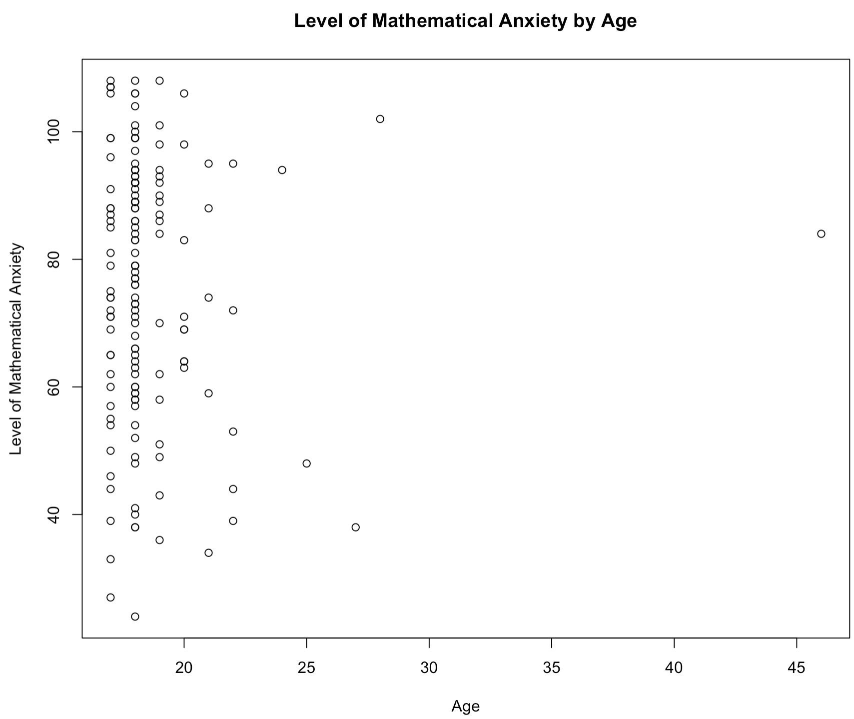


Figure 2.4: Level of Mathematical Anxiety by Age

2.4.2 Gender

Through the analysis of the data, a significant trend in MA was present when considering the gender of the participants. In fact, females exhibited a much higher level of mathematical anxiety than did males. Also, there is an outlier present in the data since one female participant exhibited a much lower level of mathematical anxiety that falls below the first quartile. This can be seen in Figure 2.5.

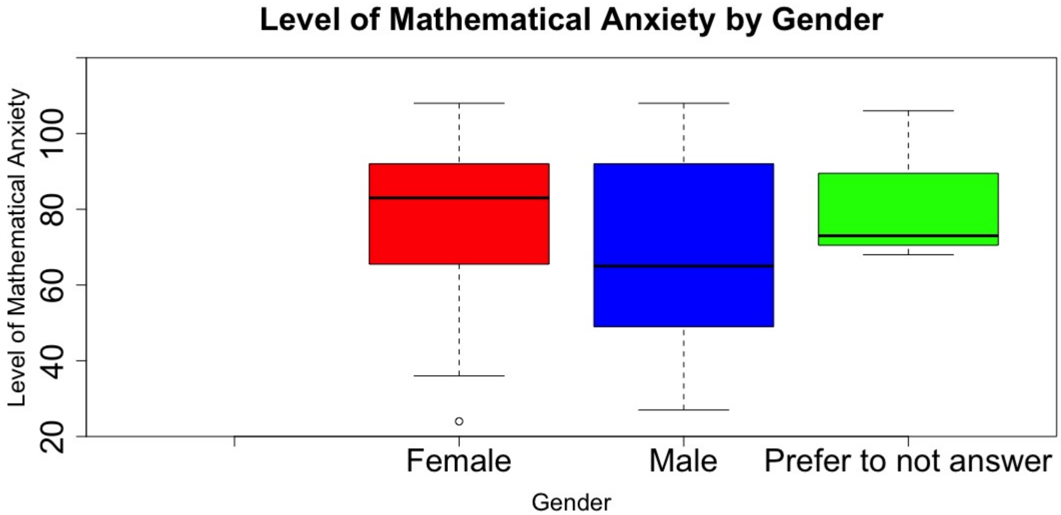


Figure 2.5: Level of Mathematical Anxiety by Gender

However, it is important to note that those who did not disclose their gender had less predictable levels of MA. That being said, the overall result that females exhibit higher levels of MA than do males is consistent with the findings of Hembree's meta-analysis (1990). Not only can this trend be seen in Figure 2.5, but it can also be seen through the average response for each of the MARS questions. This is quite remarkable as there are no questions for which males exhibited higher mathematical anxiety than females. A graph of the average level of MA per question of the MARS by gender appears in Figure 2.6.

This graph is somewhat noisy and so, Figure 2.7 shows a loose fitting trend to the data to illustrate the significant difference in the mathematical anxiety of female and male participants. Since there were only three participants that did not disclose their gender, the green curve for non-binary participants has been omitted from this figure as it was not possible to create a smooth curve for that data.

Despite the apparent trend that arises when analyzing the results of the data in terms of gender, it is important to note that this trend may be a byproduct of the socialization of the participants. More precisely, female participants could be more likely to admit to their feelings of anxiety, whereas male participants may feel pressured to not admit to theirs as is reported by Meece et al. (1990). Furthermore, there has been a very notable stigma surrounding women's abilities in mathematics for many years, and this could very

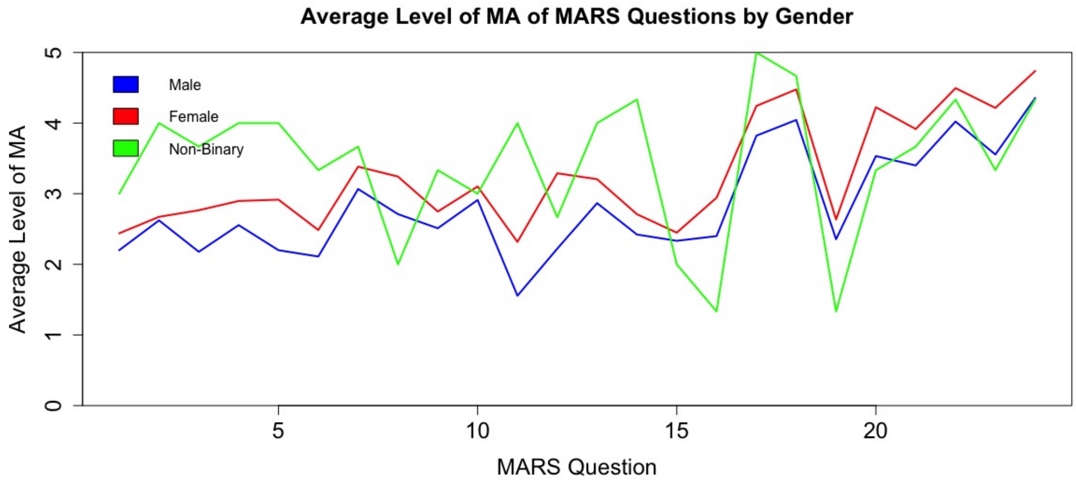


Figure 2.6: Average Level of Mathematical Anxiety for each Question of the MARS by Gender

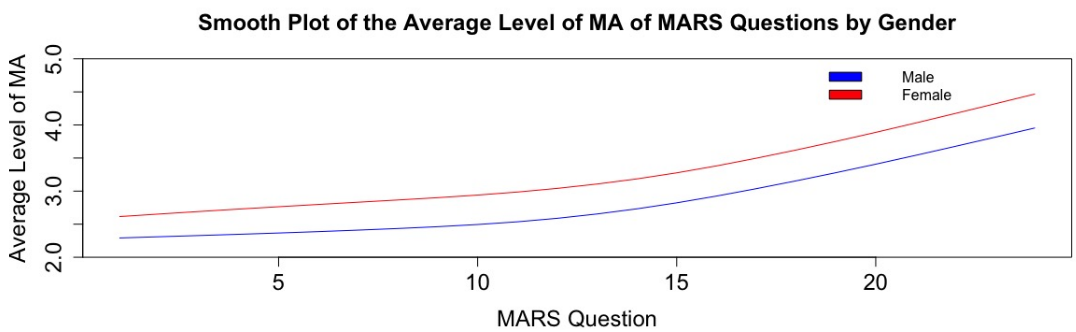


Figure 2.7: Plot of the General Trend of Mathematical Anxiety by Gender on each MARS Question

well contribute to their heightened sense of mathematical anxiety. Moreover, there was no trend for the participants who did not report their gender. This suggests another facet of today's society in that individuals may not identify with a strict binary. Thus, though the results related to gender are consistent with the literature, there may be another explanation for the differences, such as society pressures, personality (see Alcock (2014)) or the individual's use of left or right hemispheres of the brain.

2.4.3 Program

When investigating the level of MA by program of study, there is an apparent difference between programs. However, there could be many reasons for this distinction, such as the students' exposure to mathematics or the familiarity the students should have with the subject within their field of study. For instance, it is possible to assume that students that are enrolled in a program of study where mathematics is necessary (example: engineering) will have a lower MA compared to those who do not need mathematics as much (example: social work). Whatever the reason may be, Figure 2.8 shows the variation in the average MA by program of study. From this figure, it is possible to see that among the participants in the study, the Computer Science program seems to have the highest MA, whereas the Engineering program seems to have the lowest.

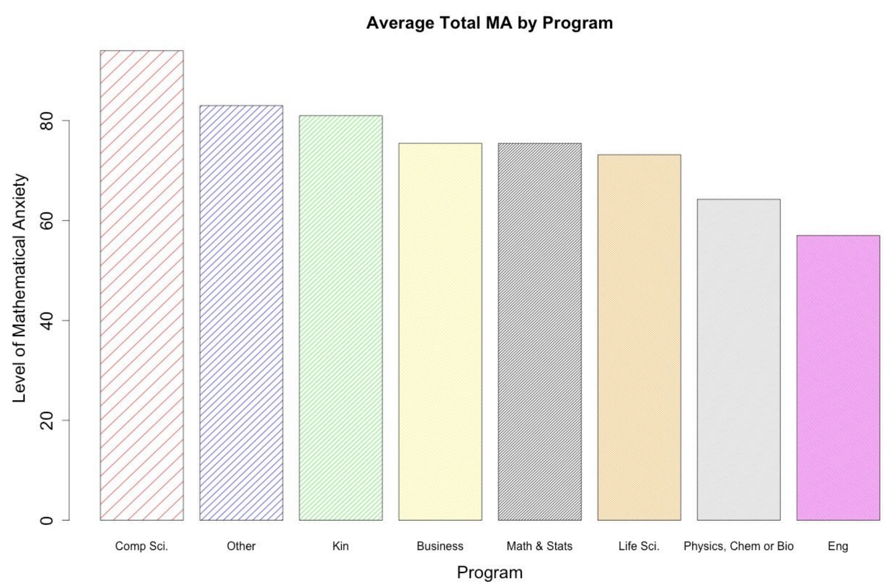


Figure 2.8: Level of Mathematical Anxiety by Program

It is important to note that there was only one participant in the computer science program. Moreover, this particular observation was part of the summer term. Thus, not only is this number biased as it is not an average, but the level of mathematical anxiety could be amplified due to the student needing to take a summer course. If more computer science students had participated in the survey, it could have possibly decreased the average level of mathematical anxiety of this program. Thus, if the assumption is made that the average

MA of the computer science program would be lower with more participants, the "other" category now exhibits the highest level of MA. This would be reasonable based on previous research, and the fact that students that do not need to do a lot of mathematics within their program, could exhibit much higher levels of mathematical anxiety. In particular, the "other" category encompasses students enrolled in social science programs such as psychology, neuroscience and behaviour and other programs where they must take at least one mathematics (or statistics) course.

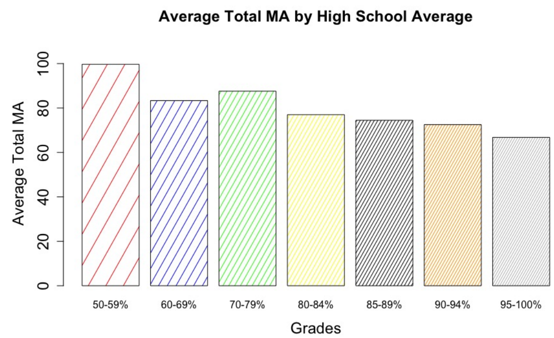
2.4.4 Grades

In terms of the grades section of the dataset, there are three variables that are taken into consideration. Namely, high school average (HSA), expectation for the course (EXP) and the grades so far (GSF) are three variables examined. It is important to investigate the relationship between each of these variables and MA as they may indicate which grades are the best indicators of MA. Identifying which types of grades foster trends in MA could help to distinguish what factors should be taken into account when deriving a mathematical model.

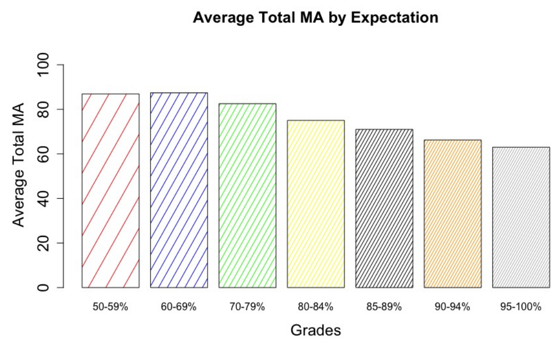
To analyze the level of MA, HSA, EXP and GSF were considered independent variables. Mathematical anxiety was considered the dependent variable, and linear models were fitted. While studying these models, it is in fact possible to notice some trends. Particularly, there is a statistically significant ($p = 0.002186$) negative correlation between HSA and MA (see Figure 2.9a), as well as a statistically significant ($p = 0.0001451$) slight negative trend between EXP and MA (see Figure 2.9b). On the other hand, there is no apparent trend in the GSF case, thus, the plot was omitted. As we would expect current grades to affect MA, we note that this lack of correlation could be attributed to participants not having received many, if any, grades in their courses at the time that they answered the survey.

Furthermore, after observing the significant relationship between HSA and MA, further statistical procedures were utilized to investigate this trend. Precisely, the Cook's distance of each model was computed to reveal any influential points that may have skewed the data. In fact, in the HSA versus MA model, there were a few data points that had relatively high Cook's distances. The Cook's distance is a measure of the influence of data points in a dataset. If this distance is too large, this point is said to have a high influence on the trend of the data.

In Figure 2.10, the Cook's distance of the MA vs HSA model can be seen. Here, there are a few points that are past the red cutoff line. The height of this cutoff was determined by 4 divided by the total number of observations (155)



(a) Level of MA by HSA



(b) Level of MA by EXP

Figure 2.9: Levels of Mathematical Anxiety by High School Average and Expectation

minus the number of items minus one. This value was chosen to be consistent with common statistical procedures for determining a cutoff for the Cook's distance. The observations with a large Cook's distance have the ability to skew the trend in the data. The reason for this is that the influencial points are different than the others in some way. In fact, upon investigation, the influencial points in the high school average versus mathematical anxiety case were of the nature that the participant either had a high level of mathematical anxiety with high achievement in high school, or the opposite. As such, these influencial points were removed to compare the difference in the trend with and without these points. When removing these influencial points, the p-value is improved ($p = 0.0002976$), thus the trend is statistically significant.

Figure 2.11 shows the trend when the most influencial points are removed. In this case, it does create two outliers in the data, however the trend does become more consistent and apparent. Despite these outliers, it seems rea-

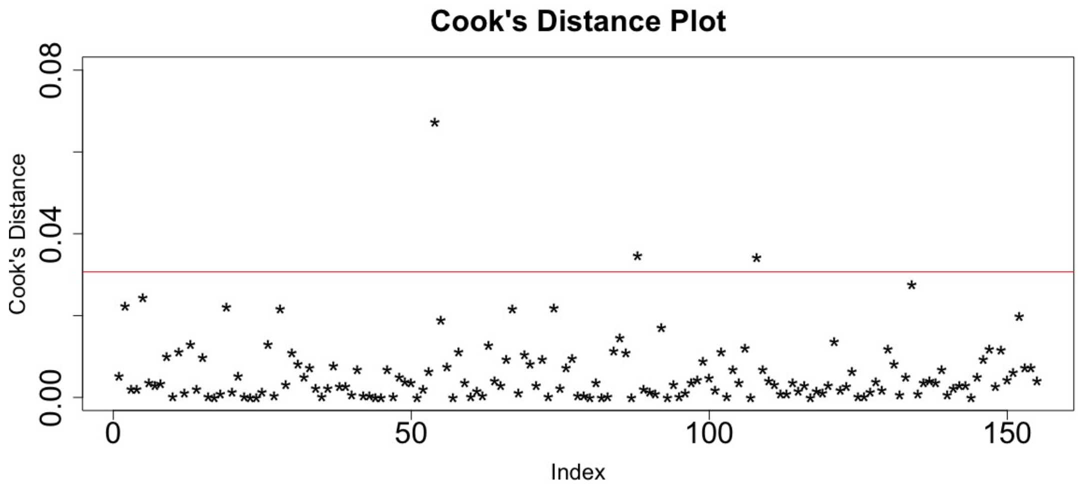


Figure 2.10: Cook's Distance of Model for MA vs HSA

sonable to remove this influencial point to see that the negative correlation between mathematical anxiety and high school average is significant and persists. Similar analysis was conducted with the expectation variable. Since removing the influencial points did not result in a significant difference for the case of expectation, the plot was omitted from this section.

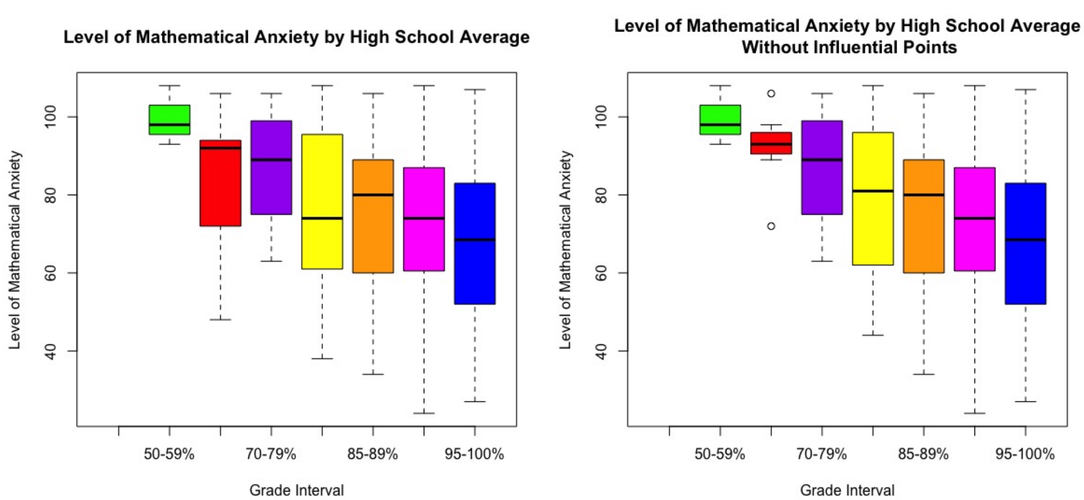


Figure 2.11: Comparison of MA vs HSA Model With and Without the Influencial Point

2.4.5 Factors

While conducting the factor analysis of the dataset, the importance and influence of each of the twenty-four MARS questions was investigated. From the summary of the principal component analysis, eighty percent of the variability in the dataset could be captured by nine principal components. It would thus be possible to reduce the dimensions of the dataset and still account for the variability of the data. When analyzing the loadings provided by the factor analysis, it is clear that the first component is a weighted average of all the MARS questions. The second component, however, is influenced by questions MARS20 and MARS22 (positive effect) versus MARS2 and MARS4 (negative effect). To investigate further, the topics of these questions were noted. In particular, MARS22 and MARS24 refer to situations in which students must take a quiz/exam. On the other hand, MARS2 and MARS4 consider preparations for courses, such as buying a textbook or signing up for a course. This confirms the hypothesis that questions relating to examinations or tests foster more mathematical anxiety compared to questions pertaining to preparation for a course. Furthermore, the third principal component is heavily weighted by MARS12. This question relates to abstract mathematics problems and so, it provides another category. In essence, the MARS questions can be partitioned into text/examination anxiety, mathematical and numerical anxiety, and course preparation. This is consistent with previous literature in the field (see [24]).

2.4.6 Analysis of Resource Section

In this section, a short analysis of the resource section of the survey is presented. This section includes questions that participants answered related to their knowledge and willingness to attend certain services available at McMaster University. One of the reasons for including this section in the survey was for the Research Ethics Board. For participants filling out a survey about mathematical anxiety, it was important to provide them with knowledge of the resources available to them on campus to help with this obstacle. In particular, participants were asked whether they knew about the Math Help Centre, Student Wellness Centre, and Student Accessibility Services, and whether they would attend these available services. Furthermore, it was also included for investigation purposes to see if there is a trend between participants' knowledge of these resources and their own mathematical anxiety levels. This confirms that questions regarding examinations or tests foster more mathematical anxiety, whereas topics dealing with preparation for a course are less likely to lead to mathematical anxiety.

In fact, there is a slight trend in the data. First, participants that did have knowledge of the resources available on campus exhibited lower levels of mathematical anxiety. Though it is not implied here that there is a causal relationship in one way or another, it is noted that this is a very interesting correlation. Furthermore, it suggests that institutions should advertise these resources to students so that they may reach out if need be.

In Figure 2.12, boxplots of the level of mathematical anxiety by knowledge of the Math Help Centre, Student Wellness Centre, and Student Accessibility Services, is presented. Here, it is apparent that the participants that have knowledge of those services exhibit a lower level of mathematical anxiety.

On the other hand, in the examination of participants' willingness to attend those services, there was not a significant trend in the level of mathematical anxiety. This is somewhat unexpected as the previous result would suggest that the participants willingness to attend these services would coincide with their level of mathematical anxiety. However, as this is not the case and the boxplots do not provide much insight, these have been omitted from the thesis.

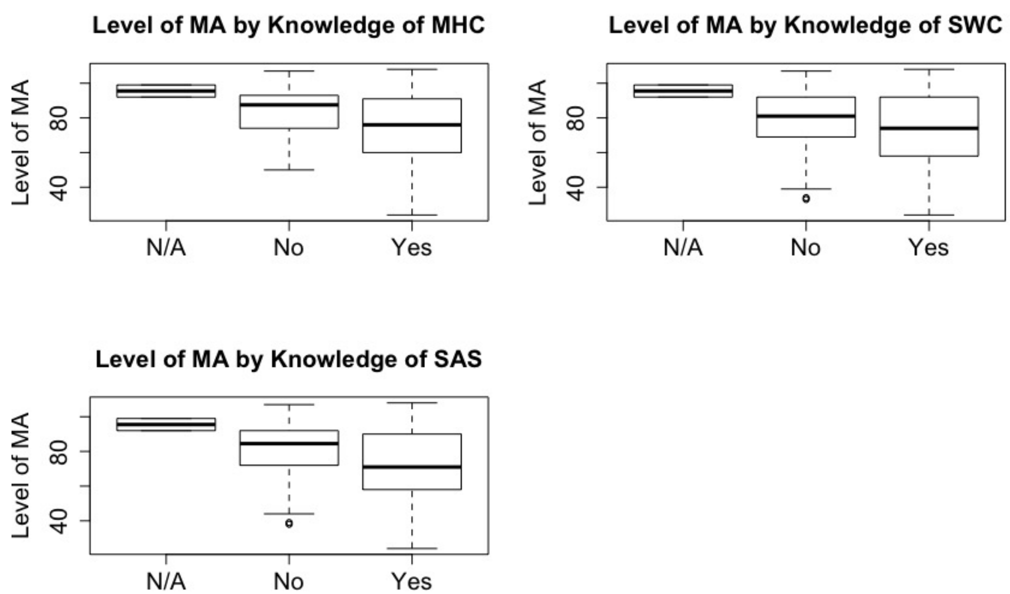


Figure 2.12: Level of Mathematical Anxiety by Knowledge of Services

2.5 Summary and Notes on Hypotheses

Here, the hypotheses that were presented in Section 2.1 are discussed. Furthermore, notes are made on why they may or may not be confirmed by the survey data.

1. It is possible to confirm the first hypothesis that female participants would exhibit higher levels of mathematical anxiety. In this survey, the female participants did report higher levels of mathematical anxiety overall. The average level of anxiety per question was also higher for female participants for every single MARS item. Whether this result was due to female participants being more inclined to admit to their feelings of anxiety, or that females are more affected by the stigma that they are not strong in mathematics, this result is consistent with previous research in the field.

2. There was an apparent and statistically significant trend that arose from analyzing high school average versus mathematical anxiety. As high school average increased, the level of mathematical anxiety decreased in participants. This further confirms the negative correlation between academic achievement and mathematical anxiety reported in previous research.

3. There was a slight and statistically significant negative correlation between student expectation and level of mathematical anxiety. Although it was not as significant as the trend between high school average and mathematical anxiety, this result still suggests that students' perceptions and self-expectations affect their level of mathematical anxiety.

4. Contrary to the hypothesis, the data did not suggest that mathematical anxiety was significantly correlated with the grades the participants had received so far. However, it is noted that participants may not have had any grades at the time that they completed the survey.

5. For mathematical anxiety by program, at first glance, it seems that the hypothesis is incorrect. However, upon further review of the data and knowing that only one student from the computer science program participated in the study, it seems that this hypothesis could still be proven true. In fact, if the computer science program had better representation, it could lower the average mathematical anxiety of that program, and the "other" category would now hold the highest level of mathematical

anxiety. As such, there is room for further investigation to confirm or deny this hypothesis.

6. Based on the factor analysis conducted, it does seem that the loadings suggest that items regarding examination or numerical/abstract problems foster high levels of mathematical anxiety, whereas questions related to preparation for a course foster lower levels of mathematical anxiety.

2.6 Limitations

In terms of limitations, it is important to note that the size of the dataset was troublesome. The lack of response lead to a total of only 155 participants across all first year mathematics courses. Considering the size of the classes, this is not a significant proportion of students and we must take this into consideration when making conclusions about the results.

Due to this lack of response, there may be biases of underrepresentation of certain populations. For instance, in total, there were significantly more female participants than there were male participants (107 vs 45). Unfortunately, this is not reflective of the demographics of the courses.

Furthermore, there were some classes that were not represented in the sample. Notably, Math 1K03 students did not participate in the survey. This could be a significant problem as students in Math 1K03, a course for Advanced Functions and Calculus, could have exhibited high levels of MA which would have provided more insight into the nature of MA.

Moreover, not only were courses misrepresented, but programs were also lacking representation. As mentioned previously, the computer science program is represented by one participant only. Thus, this does not allow for a valid representation of the level of MA in that particular program.

Another limitation of this dataset is that students were not able to accurately report their current grades in the course. It is likely that being able to record the current results of the students on the assessements in the course could provide even more information about how their mathematical anxiety is affecting their performance. Thus, it would have been beneficial to gather this data over a longer period of time, while keeping track of individualized data for each student. In other words, a time series dataset could be of benefit in the future. With time series data, it may be possible to see how MA changes throughout a term, and this might shed more light on the nature of MA. Particularly, if the students were asked to answer the MARS a few times throughout the term in association with their grades, it could lead to a model with better predictive capabilities as opposed to simple correlational analysis. Furthermore, this time series data could certainly help in the derivation and validation of theoretical models of learning like the models to be presented and analyzed in the following chapters of this paper.

Chapter 3

Theoretical Models

In the literature, authors have taken characteristics of the learner, information, and teaching methods into account in the derivation of their models. In the following sections, a variety of mathematical models will be proposed to explain the interaction between the knowledge, the effort, and the anxiety an individual gains and exhibits throughout a term. First, Anderson's model (B.3.10) will be considered and modified to account for motivational factors. Then, inspired by the work of Anderson (1983), another model will be developed. Subsequently, through the use of a system of differential equations, a new theoretical model will be proposed. Having outlined the assumptions of this model, an analysis, and critique of the model will be conducted. Then, the validity and appropriateness of the predictions will be discussed.

3.1 Model I

One of the limitations of Anderson's model is that motivational factors are not taken into account [2]. As was discussed in the mathematics education literature, anxiety can be seen as a motivating factor. Here, it is proposed that parameters for levels of mathematical anxiety and personal drive or effort could be incorporated into Anderson's model.

In order to include a parameter for both mathematical anxiety and effort, it is important to consider the components of the model derived by Anderson in [2]. Firstly, the function $S(t)$, (B.3.2), represents the stability of short-term memory (STM). This function describes the decline in STM capacity. On the other hand, the function $I(t)$, (B.3.4), describes the decline in instability. More precisely, it accounts for the decline in instability as the learner becomes more familiar with a learning task. For convenience, as references will be made to the equations, a more detailed summary of Anderson's model derivation can be found in Appendix B.3.

For this particular model, simplifying assumptions will be made to incorporate new parameters into Anderson's model. These assumptions are listed below.

1. The level of mathematical anxiety (μ) and a learner's drive/effort (ϵ) are considered constants.

2. The level of mathematical anxiety is assumed to affect the stability of short-term memory and thus will be taken into account in the stability funtion (B.3.2).

3. The learner's drive or effort is assumed to affect the instability term in the sense that the more effort made, the greater the decline of instability of a learning task. Thus, it will be taken into account in the instability function (B.3.4).

Due to the above assumptions, it is possible to simply fit two new parameters to Anderson's model. This would result in the following equations ((3.1.1), (3.1.2)) that could be tested with empirical data similar to the data that Anderson used to verify the predictive capabilities of their model.

$$S = S_0 e^{-\frac{\alpha \cdot \beta \cdot \rho \cdot \mu}{\kappa} t}. \tag{3.1.1}$$

$$I = I_0 e^{-\lambda \epsilon t}, \tag{3.1.2}$$

Essentially, by incorporating these new parameters into Anderson's model, mathematical anxiety and effort simply speed up the decline in stability, and the decline in instability of the CNS of the individual. Thus, a larger level of mathematical anxiety would impede the capacity of short term memory, making it harder to acquire knowledge. On the other hand, a higher level of effort would allow the individual to become more familiar with the learning task in a shorter period of time.

This model is limited since it depicts the behaviour of the CNS in the process of content acquisition. In this sense, it only describes a very short period of time span of minutes to approximately a half an hour [2]. Thus, even with the parameters for mathematical anxiety and effort, this model does not appropriately describe behaviour over the period of a school term. Furthermore, the assumption that mathematical anxiety and effort are constants throughout an experiment is too much of a simplification, thus creating a need to propose another model.

3.2 Model II

When studying the work of Anderson (1983), it is possible to draw parallels between the stability and instability functions, and how anxiety and effort would theoretically affect knowledge. Inspired directly by the derivation of Anderson's model, the following model is proposed to describe the effect of effort and anxiety on the acquisition of knowledge during the early stages of a learning experience. Therefore, it is assumed that time is finite and does not account for learning over a lifetime.

Before making note of the necessary assumptions, the notation that will be used for this model is outlined. The three variables of the model considered here are knowledge (K), level of anxiety (A), and level of effort (E). The three functions for these variables will then form the function for the net gain of information, denoted $N(t)$. There are several parameters to describe the characteristics of the learner and the characteristics of the content to be acquired that will be taken into account in this model. A list of the parameters is given below.

- learner's aptitude (α)
- learner's expectation (ϵ)
- learner's personal drive (δ)
- learner's susceptibility for anxiety (σ)
- learner's previous knowledge (ρ)
- characteristic of the content (γ)

For this model, differential equations will be written for each of the variables, and then, they will be solved individually. Some assumptions for the derivation of these differential equations are presented next.

1. Knowledge is considered to be the content that the individual has acquired over time. Thus, knowledge is strictly increasing over time.
2. The net gain ($N(t)$) will represent the effects of effort and anxiety on the individual's acquisition of information.
3. Anxiety is assumed to decrease over time as in this derivation, it is assumed that anxiety decreases as knowledge increases. This said, the anxiety function is analogous to the stability function from [2].

4. It is assumed that effort decreases over time as the individual becomes more comfortable with the course content, and does not need to provide more effort seeing as they have acquired enough knowledge to support the gain of new information. Thus, the effort function is inspired by the instability function from Anderson's model.

3.2.1 Knowledge Function

The rate of increase of knowledge is proportional to the amount of knowledge the individual has K. Further, it is assumed that as the aptitude, α, and drive, δ, of an individual increase, so too would the rate of increase of knowledge K'. However, this increase would be slowed depending on the characteristics of the information, γ. Thus, the rate of change of knowledge would be given by equation (3.2.1).

$$\frac{dK}{dt} = \frac{\alpha\delta}{\gamma}K \tag{3.2.1}$$

Solving the above yields the equation for the knowledge of the individual at time t. In this equation (3.2.2), K_0 denotes the initial amount of knowledge the learner has when going into the learning experience.

$$K = K_0 e^{\frac{\alpha\delta}{\gamma}t} \tag{3.2.2}$$

3.2.2 Effort Function

Next, the rate of change of effort, E', is outlined. In this case, effort is considered to decrease over time, proportional to itself, as the individual gains more knowledge and can make more connections to the new information to be acquired. This is consistent with the constructivism theory presented in [21]. The rate of decrease of effort would be higher with high susceptibility for anxiety, σ, and the characteristics of the information to be acquired, γ. Furthermore, the rate of decrease would be slowed by the individual's drive, δ, and their expectations of themselves, ϵ. Thus, equation (3.2.3) gives the rate of change of effort over time.

$$\frac{dE}{dt} = -\frac{\gamma\sigma}{\delta\epsilon}E \tag{3.2.3}$$

Solving this differential equation leads to the solution curve for effort in equation (3.2.4). Here, E_0 denotes the initial effort level of the individual.

$$E = E_0 e^{-\frac{\gamma\sigma}{\delta\epsilon}t} \tag{3.2.4}$$

3.2.3 Anxiety Function

In this model, the anxiety function mimics the stability function from Anderson's model. As such, anxiety is assumed to decrease over time as the individual is gaining knowledge. Furthermore, the rate of decrease of anxiety would be proportional to the product of itself. This rate of decrease would be higher for individuals with higher aptitude, α, and would be slowed by their susceptibility for anxiety, σ, and the characteristics of the information, γ. Therefore, the rate of change of anxiety is given by equation (3.2.5).

$$\frac{dA}{dt} = -\frac{\alpha}{\sigma\gamma}A \tag{3.2.5}$$

When solving this differential equation with A_0 as the initial level of anxiety, the solution for the level of anxiety over time is given by equation (3.2.6).

$$A = A_0 e^{-\frac{\alpha}{\sigma\gamma}t} \tag{3.2.6}$$

3.2.4 Net Gain Function

As was done in Anderson's paper from 1983, the three functions derived above will be used to form the net gain function. This function is the net gain of information the individual acquires by time t after taking into account the effects of effort and anxiety. The net gain function will be the product of the knowledge function and the difference between the effort and the anxiety functions. Thus, as was the case in [2], there is a modulation effect on knowledge gained. Therefore, we borrow the term modulation factor to describe $E - A$. Equation (3.2.8), decribes the net gain of information over time.

$$N(t) = K_0 e^{\frac{\alpha\delta}{\gamma}t}\left(E_0 e^{-\frac{\gamma\sigma}{\delta\epsilon}t} - A_0 e^{-\frac{\alpha}{\sigma\gamma}t}\right) \tag{3.2.7}$$

For a more concise notation, let $\nu = \frac{\alpha\delta}{\gamma}$, let $\lambda = \frac{\gamma\sigma}{\delta\epsilon}$ and let $\beta = \frac{\alpha}{\sigma\gamma}$. Also, for simplicity, take $E_0 = A_0 = 0$. The net gain function can now be written as equation (3.2.8).

$$N(t) = K_0 e^{\nu t}\left(e^{-\lambda t} - e^{-\beta t}\right) \tag{3.2.8}$$

In order for this model to make practical sense, certain conditions must be met. First, effort must always be greater than anxiety so that the net gain does not become negative. Furthermore, the rate of decay of effort has to be smaller than the rate of decay of anxiety ($\lambda < \beta$).

This model can result in three different curves for the net gain, depending on the nature of the parameters. In the first case, the increase in knowledge is large enough that the effect of the modulation factor is negligeable. The second possibility is that the rate of increase of knowledge is equal to the rate of decrease of effort, and this causes a saturation of the net gain at an asymptote. Finally, if the effect of the modulation factor is too great, then the net gain would reach its maximum and then decrease. These three cases can be seen in Figure 3.1.

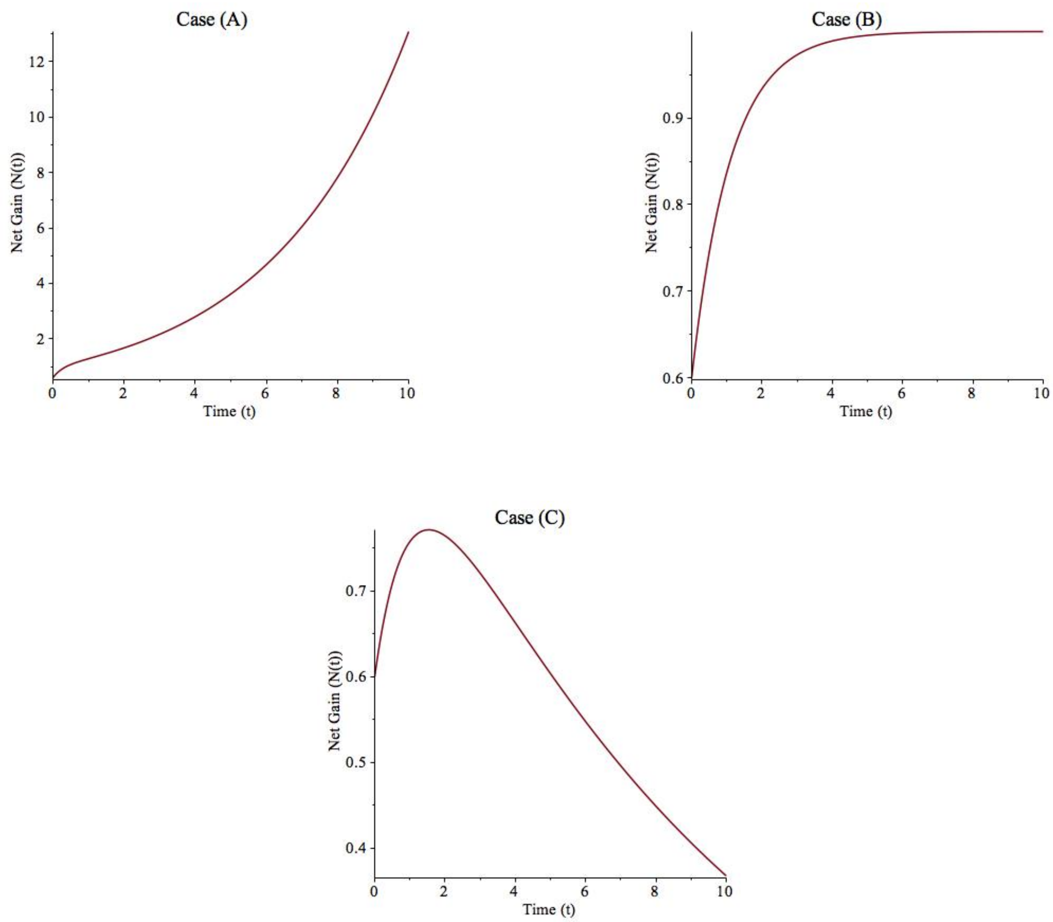


Figure 3.1: Three Possible Shapes of Model II

The biggest shortcoming of this model is that it only describes the learning situation in the early stages. In order to describe a learning situation throughout the course of an academic term, other cases must be considered. In particular, the assumptions that knowledge is always increasing, and that effort and anxiety are always decreasing, are aspects that need to be revised. As is suggested in the mathematical education literature, mathematical anxiety has a dynamic nature and thus, these assumptions do not lead to the best description for the relationship between these variables over an extended period of time.

3.3 Model III

Prior to deriving the next model and proposing appropriate prototype functions, it is important to outline the assumptions that will be made to build the foundation of our theoretical model. The model will seek to explain the short term interactions between the following variables: proportions of knowledge (K), effort (E), and anxiety (A). Throughout the derivation of Model III, the variables represent proportions of knowledge, effort, and anxiety even if it is referred to as a level. The interaction between variables will be described by a model composed of a system of three differential equations for the rates of change of each of the three variables. Each of these variables will be assumed to vary throughout the term. In particular, the variables can increase and decrease over time because they are considered to be proportions. In particular, as the term progresses, there is more content to be learned from every new lecture, thus the proportion of knowledge can decrease. In this sense, knowledge is not strictly increasing. Finally, since the model will describe the interactions between the variables throughout a semester, time is finite. It is noted here that depending on the time scale, not all individual's will be able to reach the equilibrium that they could reach if they had unlimited time. Table 3.1 lists the parameters of the model.

Three main parameters representing the learner's characteristics will be taken into account, namely, the individual's aptitude, their drive and their susceptibility for anxiety. There will be three other parameters, $\tilde{K}$, $\tilde{E}$, and $\tilde{A}$, that represent the critical values of each variable. In theory, these critical values would be dependent on some of the individual's characteristics, the difficulty of the course, and the state of the learning environment. These six parameters will have values between zero and one. Finally, the maximum proportion of knowledge, effort, and anxiety are denoted by $\bar{K}$, $\bar{E}$, and $\bar{A}$ respectively. For all simulations, figures, and bifurcation diagrams these three parameters will be given a value of one. However, for the sake of future mod-

Parameter	Meaning
α	Individual's aptitude
δ	Individual's drive
σ	Individual's susceptibility for anxiety
$\tilde{K}$	Critical level of knowledge with respect to anxiety
$\tilde{E}$	Critical level of effort with respect to knowledge
$\tilde{A}$	Critical level of anxiety with respect to effort
$\bar{K}$	Maximum proportion of knowledge
$\bar{E}$	Maximum proportion of effort
$\bar{A}$	Maximum proportion of anxiety

Table 3.1: Model III Parameters

ifications to the model, they are left as parameters in the definition of the system. Since the critical level of the variables depend on the individual, they are necessarily less than the maximum proportion of knowledge, effort and anxiety. Thus, a necessary condition is that $\bar{K} > \tilde{K}$, $\bar{E} > \tilde{E}$, and $\bar{A} > \tilde{A}$.

The model considered here is,

$$\frac{dK}{dt} = C\left(\alpha - (\tilde{E} - E)K\right)\left(\bar{K} - K\right) \tag{3.3.1}$$
$$\frac{dE}{dt} = C\left(\delta + (\tilde{A} - A)E\right)\left(\bar{E} - E\right) \tag{3.3.2}$$
$$\frac{dA}{dt} = C\left(\sigma + (\tilde{K} - K)A\right)\left(\bar{A} - A\right). \tag{3.3.3}$$

The variables K, E, and A are proportions. The constant rate C can be used to scale time, and has dimensions $\frac{1}{time}$. For simplicity in analyzing the model, let $C = 1$. However, note that if C is chosen accordingly, the time axis could be scaled to represent the duration of an academic term. The model then becomes,

$$\frac{dK}{dt} = \left(\alpha - (\tilde{E} - E)K\right)\left(\bar{K} - K\right) \tag{3.3.4}$$
$$\frac{dE}{dt} = \left(\delta + (\tilde{A} - A)E\right)\left(\bar{E} - E\right) \tag{3.3.5}$$
$$\frac{dA}{dt} = \left(\sigma + (\tilde{K} - K)A\right)\left(\bar{A} - A\right). \tag{3.3.6}$$

The differential equation (3.4.2) describes the rate of change of the proportion of knowledge as it depends on the proportion of effort. In particular, the rate of change of knowledge is proportional to the difference between the current level of knowledge and the maximum proportion, $\bar{K}$. If knowledge is zero, then the rate of change will be proportional to the product of the learner's aptitude α and the maximum amount of knowledge possible for that learner, $\bar{K}$. In the case where $\bar{K}$ is one, this would then mean that the rate of change would be proportional to the individual's aptitude. Thus, if knowledge is zero, it will then increase as its rate of change is positive, and knowledge will not remain zero forever. Furthermore, it will not become negative. If effort exceeds $\tilde{E}$, then the rate of change of knowledge is positive. However, if effort is insufficient, depending on the individual's aptitude, then the rate of change of knowledge could become negative and thus, knowledge could decrease. If the individual's aptitude is zero, regardless of their level of effort, proportion of knowledge will remain zero.

The differential equation (3.4.3), describes the rate of change of the proportion of effort as it depends on the proportion of anxiety. This rate of change is once again proportional to the difference between the current level of effort and the maximum proportion of effort, $\bar{E}$. If effort is zero, it is assumed that the rate of change is proportional to the product of the drive of the individual and the maximum amount of effort, $\bar{E}$. Thus, effort will not remain zero. Once effort is positive, if anxiety becomes too large, the rate of change of effort can become negative, and hence, effort could decrease. Precisely, if anxiety exceeds $\tilde{A}$, then, if the individual's drive δ is low, the individual may not be able to ignore their level of anxiety and the rate of change of effort would become negative. Contrarily, if anxiety is sufficiently low, it will inspire the learner and the rate of change will be positive. Moreover, if the individual's drive is zero, their proportion of effort will remain zero.

The last differential equation of the model is (3.4.4). This differential equation describes the rate of change of the proportion of anxiety as a function of the individual's proportion of knowledge, K. As was the case with the previous differential equations, the rate of change of anxiety will be proportional to how much anxiety the individual has left to gain. That is, it is proportional to the difference between the maximum level of anxiety and the current level of anxiety. If anxiety is zero, then the rate of change of anxiety will be the product of the learner's susceptibility for anxiety, σ, and the maximal level of anxiety, $\bar{A}$. Here, if knowledge grows enough, depending on the individual's susceptibility for anxiety, the rate of change could become negative, and so anxiety could decrease. However, if knowledge is too small, the rate of change of anxiety would be positive and anxiety will increase. Similarly to the previous cases, if susceptibility for anxiety is zero, this individual's level of anxiety will remain zero.

3.4 Analysis of Model III

In this section, an analysis of Model III is presented. Firstly, a verification of the nature of the parameters will be done to validate the system of differential equations. Then, conditions will be imposed on the parameters to ensure the appropriateness of the model and assumption statements will be provided. Subsequently, the equilibria will be listed, along with information about their existence and stability. Finally, the dynamics of the model will be discussed via bifurcation analysis.

A set of gifs were generated by plotting the solutions of Model III while changing one parameter at a time. By changing each parameter, it was possible to see that the basic behaviour of the model makes practical sense. For instance, by changing the individual's aptitude, α, from zero to one, the solutions for the level of knowledge and effort reach their maximum in a shorter period of time, and the solution for the level of anxiety decreases. A similar result ensues for changing the parameter representing the individual's drive, δ. Contrarily, when increasing the individual's susceptibility for anxiety, σ, the level of knowledge and effort decrease, whereas the level of anxiety increases. As such, it is possible to conclude that the parameters in place for the individual's characteristics make practical sense in the way that they affect the solution curves.

As for the critical level parameters $\tilde{K}$, $\tilde{E}$, and $\tilde{A}$, the gifs also support the validation of how these affect the solutions. $\tilde{K}$, representing the critical level of knowledge with respect to anxiety, slows the growth of knowledge and effort, while increasing anxiety as the value of this parameter increases. This makes sense as it represents the level of knowledge for which anxiety could begin to decrease. The parameter $\tilde{E}$ behaves in a similar fashion. Finally, the parameter $\tilde{A}$, that represents the critical level of anxiety with respect to effort, actually increases knowledge and effort as it grows, while decreasing the level of anxiety. Once again, this makes sense as the larger $\tilde{A}$ is, the more anxiety helps to drive the individual's effort.

3.4.1 Assumptions

In order for the model to be well-defined and make theoretical sense, some assumptions must be made. Define the set (3.4.1) as,

$$S = \{(K, E, A) \in \mathbb{R}^3 : 0 \leq K \leq \bar{K}, 0 \leq E \leq \bar{E}, 0 \leq A \leq \bar{A}\}. \tag{3.4.1}$$

The main assumption of the model ensures that the initial values of knowledge, effort and anxiety are between zero and their maximum values,

(1) $(K(0), E(0), A(0)) \in S$.

With assumption (1), it is possible to prove that solutions do not become negative, and that they do not exceed the maximum levels $\bar{K}$, $\bar{E}$, and $\bar{A}$ respectively.

Since the parameters are positive, the proof that solutions do not become negative is straightforward, and is outlined here. Considering the system of differential equations,

$$\frac{dK}{dt} = \left(\alpha - (\tilde{E} - E)K\right)\left(\bar{K} - K\right) \tag{3.4.2}$$
$$\frac{dE}{dt} = \left(\delta + (\tilde{A} - A)E\right)\left(\bar{E} - E\right) \tag{3.4.3}$$
$$\frac{dA}{dt} = \left(\sigma + (\tilde{K} - K)A\right)\left(\bar{A} - A\right), \tag{3.4.4}$$

if $K = 0$, then $\frac{dK}{dt} = (\alpha)(\bar{K})$, and as aptitude is between zero and one, and the maximum proportion of knowledge is positive, the solution curve would increase. The same argument applies for both the rate of change of the proportions of effort and anxiety.

To prove that solutions cannot exceed the maximum value of the variables, assume that $\exists$ a constant solution where $K = \bar{K}$, and the solutions for E and A are in set (3.4.1). If $\exists$ a set of initial conditions $(K_{\bar{t}}, E_{\bar{t}}, A_{\bar{t}})$ for which $K(\bar{T}) = \bar{K}$, then this solution would intersect with the constant solution. If this is the case, this contradicts the uniqueness of solutions. The same arguments persist for the solutions of variables E and A. Thus, it is not possible for a solution, whose initial conditions are in the set (3.4.1) to exceed the maximum values of the variables.

Statements (i)-(iv) are not necessary assumptions, but rather important conditions for the parameters of the model that allow for different dynamics. Their importance and relevance will be explained in more depth in subsequent sections.

(i) $\alpha - \tilde{E}\bar{K} < 0$ so that K' can be negative.

(ii) $\delta + (\tilde{A} - \bar{A})\bar{E} < 0$ so that E' can be negative.

(iii) $\sigma + (\tilde{K} - \bar{K})\bar{A} < 0$ so that A' can be negative.

(iv) $\bar{A} > \hat{A} > \tilde{A}$ and $\bar{E} > \tilde{E} > \hat{E}$ so that K^+ and E^+ exist.

3.4.2 Existence of Equilibria

In table 3.2, the values of the variables that allow for each differential equation to be equal to zero are listed.

$K' = 0$	$E' = 0$	$A' = 0$
$\bar{K}$	$\bar{E}$	$\bar{A}$
$\hat{K} = \frac{\alpha}{\tilde{E}-\bar{E}}$	$\hat{E} = \frac{\delta}{\bar{A}-\tilde{A}}$	$\hat{A} = \frac{\sigma}{\bar{K}-\tilde{K}}$
$K^+ = \frac{\alpha(\bar{A}-\tilde{A})}{\tilde{E}(\bar{A}-\tilde{A})-\delta}$	$E^+ = \frac{\delta(\bar{K}-\tilde{K})}{\sigma-\tilde{A}(\bar{K}-\tilde{K})}$	$A^+ = \frac{\sigma(\tilde{E}-\bar{E})}{\alpha-\tilde{K}(\tilde{E}-\bar{E})}$
K^*	E^*	A^*

Table 3.2: Values of the Variables to Equate the Differential Equations to Zero

However, some of the values above are not feasible. From assumption (1), it is known that $\bar{K} > \tilde{K}$. Thus, the equilibrium $\hat{K} < 0$ and so, as it does not make sense to have a negative value for an individual's proportion of knowledge, $\hat{K}$ does not exist.

K^+, E^+, and A^+ are defined as follows: $K^+ = \frac{\alpha}{\tilde{E}-\hat{E}}$, $E^+ = \frac{\delta}{\hat{A}-\tilde{A}}$, and $A^+ = \frac{\sigma}{\hat{K}-\tilde{K}}$. With this definition, it is possible to see that A^+ does not exist, as it depends on $\hat{K}$.

Furthermore, for K^+ and E^+ to be positive, the following inequalities must be true: $\tilde{E} > \hat{E}$ and $\tilde{A} < \hat{A}$. This confirms the statement (iv) that was noted previously.

Finally, since $\bar{E} > \tilde{E}$, this implies that it is not possible to have a value other than $\bar{K}$ if effort is equal to $\bar{E}$. In Table 3.3, the possible equilibrium points are listed and their conditions for existence are noted.

The equilibria of the system occur when all three differential equations are equal to zero (*i.e.* $K' = 0$, $E' = 0$, and $A' = 0$).

The equilibrium $E_6 = (K^*, E^*, A^*)$ will be referred to as the "interior" equilibrium as it does not include any of the bar or hat values. Summarizing the conditions for existence of this equilibrium is much more complex compared to the other equilibrium points. Thus, a more detailed analysis of the existence conditions will now be presented. In particular, the system below must be solved to find the possible values of K^*, E^*, and A^*.

Equilibrium Point	Existence Condition(s)
$E_1 = (\bar{K}, \bar{E}, \bar{A})$	Always exists.
$E_2 = (\bar{K}, \hat{E}, \bar{A})$	Exists if $\bar{E} > \hat{E}$.
$E_3 = (\bar{K}, \bar{E}, \hat{A})$	Exists if $\bar{A} > \hat{A}$.
$E_4 = (\bar{K}, E^+, \hat{A})$	Exists if $\bar{A} > \hat{A} > \tilde{A}$ and $\frac{\sigma}{\tilde{A}} < \bar{K} - \tilde{K} < \sigma$.
$E_5 = (K^+, \hat{E}, \bar{A})$	Exists if $\bar{E} > \tilde{E} > \hat{E}$ and $\delta < \bar{A} - \tilde{A} < \frac{\delta}{\tilde{E}}$.
$E_6 = (K^*, E^*, A^*)$	Existence conditions to be discussed below.

Table 3.3: Equilibrium points and their existence conditions.

$$\alpha - (\tilde{E} - E^*)K^* = 0 \tag{3.4.5}$$
$$\delta + (\tilde{A} - A^*)E^* = 0 \tag{3.4.6}$$
$$\sigma + (\tilde{K} - K^*)A^* = 0 \tag{3.4.7}$$

When rearranging and solving the system of equations above to solve for K^*, E^*, and A^*, the following expressions are obtained,

$$K^* = \frac{\alpha}{\tilde{E} - E^*}$$
$$E^* = \frac{\delta}{A^* - \tilde{A}}$$
$$A^* = \frac{\sigma}{K^* - \tilde{K}}.$$

From these expressions, a few conditions for the existence of this equilibrium arise. Notably, the following inequalities, $\bar{K} > K^* > \tilde{K}$, $\bar{E} > \tilde{E} > E^*$, and $\bar{A} > A^* > \tilde{A}$, must hold true. In order to solve for any of the values, it is necessary to first find the value of one of the variables. Thus, the equations are rearranged in order to write an expression only in terms of parameters and K^*. The left hand side of the quadratic (3.4.8) will be referred to as $f(K^*)$.

$$(\tilde{A}\tilde{E} + \delta)(K^*)^2 - (\alpha\tilde{A} + \sigma\tilde{E} + \delta\tilde{K} + \tilde{A}\tilde{E}\tilde{K})K^* + \alpha\sigma + \alpha\tilde{A}\tilde{K} = 0. \tag{3.4.8}$$

To find the possible values of K^*, $f(K^*) = 0$ must be solved. The solutions of this quadratic are complicated to analyze due to the number of parameters. However, the Descartes Rule of Signs can be useful here. By the Descartes

Rule of Signs, since $f(K^*)$ has two sign changes, there are either two or zero positive real roots. When considering $f(-K^*)$, there are zero sign changes, and thus, the Descartes Rule of Signs indicates that this quadratic has no negative real roots. As such, this equilibrium could exist when the solutions to (3.4.8) are in the set (3.4.1), and yield values of E^* and A^* with those same properties.

Using Maple [1], it is possible to see that there are parameter values that yield interior equilibrium points. In order to verify this, the function $f(K^*)$ is plotted for given parameter values to see if the x-intercepts are between zero and $\bar{K}$. Subsequently, Maple is used to calculate the corresponding values of E^* and A^* to see if they are in the set (3.4.1). Thus, by investigation, it is possible to find two interior equilibrium points of the form (K^*, E^*, A^*) given certain values of the parameters of the model. However, by studying various sets of parameters, only one interior equilibrium has all three values in the set (3.4.1).

3.4.3 Stability of Equilibria

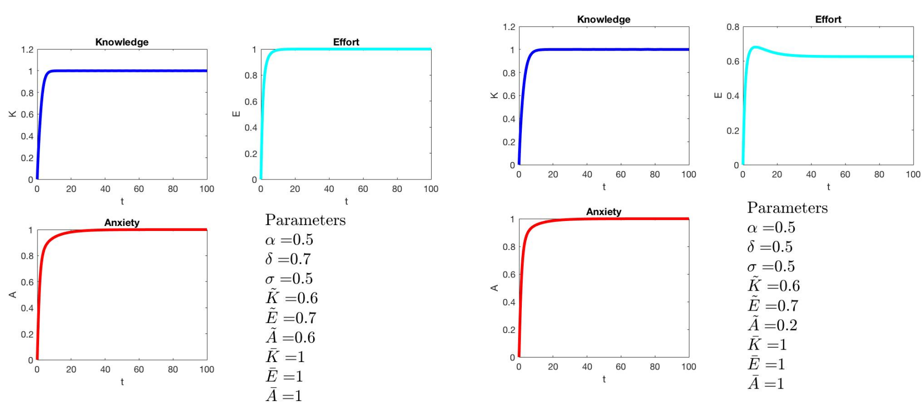
To find the stability of the equilibria, the Jacobian of the system was computed. Next, the eigenvalues of the Jacobian at each of the equilibrium points were found to establish conditions for stability of each respective equilibrium. For an equilibrium to be stable, all of the real parts of the eigenvalues of the Jacobian at that equilibrium must be negative. The conditions for stability of each equlibrium (not including (K^*, E^*, A^*)) are summarized in Table 3.4.

There are a few important notes to make about some of the stability conditions above. For instance, in the case of the first two equilibrium points, if statements (ii) or (iii) hold, these stability conditions cannot be satisfied. In this case, if (ii) or (iii) are true, then equilibrium E_1 and E_2 are both unstable. However, if (ii) and (iii) are not true, then it is possible that equilibrium E_1 or E_2 could be stable, and their stability depends on the first condition in each case. On one hand, if $\bar{K}(\tilde{E} - \bar{E}) - \alpha < 0$ holds, then E_1 is stable. If only $\bar{K}(\tilde{E} - \hat{E}) - \alpha < 0$ is true, then E_2 would be stable. Moreover, given certain parameters, local stability is possible for both of these equilibrium points at once. However, under further scrutiny, it seems that when this is the case, E_2 does not exist as not all variable values are in the set 3.4.1 (*i.e.* $\bar{E} < \hat{E}$). In Figure 3.2a and Figure 3.2b, there are examples of parameter sets, with initial condition $(0, 0, 0)$, for which the solutions converge to E_1 and E_2 respectively.

When examining the stability of E_3, it is possible to see that many sets of parameters yield stability for this equilibrium point. This equilibrium consists of the individual's level of knowledge and effort reaching their maximum po-

Equilibrium	Stability Conditions
$E_1 = (\bar{K}, \bar{E}, \bar{A})$	$\bar{K}(\tilde{E} - \bar{E}) - \alpha < 0$ $\bar{E}(\bar{A} - \tilde{A}) - \delta < 0$ $\bar{A}(\bar{K} - \tilde{K}) - \sigma < 0$
$E_2 = (\bar{K}, \hat{E}, \bar{A})$	$\bar{K}(\tilde{E} - \hat{E}) - \alpha < 0$ $\bar{E}(\bar{A} - \tilde{A}) - \delta < 0$ $\bar{A}(\bar{K} - \tilde{K}) - \sigma < 0$
$E_3 = (\bar{K}, \bar{E}, \hat{A})$	$\bar{K}(\tilde{E} - \bar{E}) - \alpha < 0$ $\bar{E}(\hat{A} - \tilde{A}) - \delta < 0$ $(2\hat{A} - \bar{A})(\bar{K} - \tilde{K}) - \sigma < 0$
$E_4 = (\bar{K}, E^+, \hat{A})$	$\bar{K}(\tilde{E} - E^+) - \alpha < 0$ $(2E^+ - \bar{E})(\hat{A} - \tilde{A}) - \delta < 0$ $(2\hat{A} - \bar{A})(\bar{K} - \tilde{K}) - \sigma < 0$
$E_5 = (K^+, \hat{E}, \bar{A})$	$(2K^+ - \bar{K})(\tilde{E} - \hat{E}) - \alpha < 0$ $(2\hat{E} - \bar{E})(\bar{A} - \tilde{A}) - \delta < 0$ $\bar{A}(K^+ - \tilde{K}) - \sigma < 0$

Table 3.4: Stability conditions for five equilibrium points (E_1-E_5) of Model III



(a) (b)

Figure 3.2: Solutions of Model III

Figure 3.2a shows the solutions converging to E_1, whereas Figure 3.2b shows solutions converging to E_2, both starting from IC $(0, 0, 0)$.

tential, $\bar{K}$ and $\bar{E}$ respectively, whereas their anxiety attains the value $\hat{A}$. In this sense, the stability conditions for E_3 are often satisfied by the sets of parameters chosen. Figure 3.4 shows a set of parameters for which this particular equilibrium is stable.

Furthermore, by investigating different sets of parameters, it is possible to see that both E_4 and E_5 can be locally stable. In this case, the conditions for

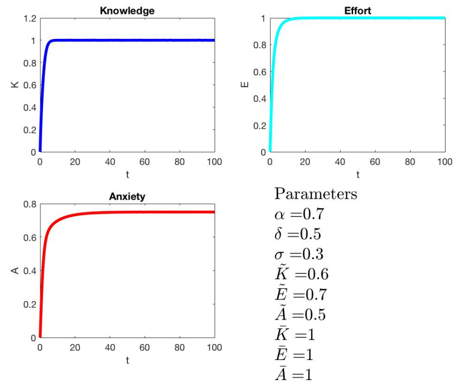
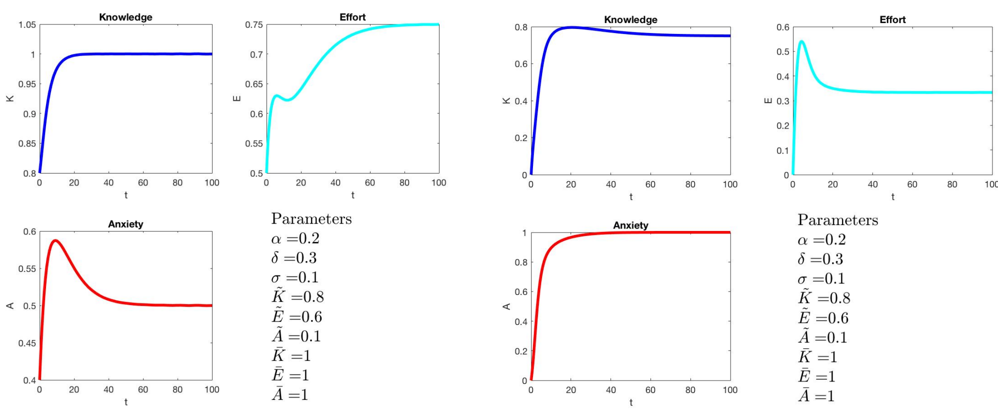


Figure 3.4: Solution of Model III converging to E_3.

stability for both of these equilibria are all satisfied. Whether the individual reaches E_4 or E_5 would depend on the initial conditions given in that particular case. Figures 3.5a and 3.5b show the solutions of the model given a chosen set of parameters and different initial conditions.



(a) (b)

Figure 3.5: Solutions of Model III

Figure 3.5a shows the solutions converging to E_4 from IC $(0.8, 0.5, 0.4)$ whereas Figure 3.5b shows solutions converging to E_5 from IC $(0, 0, 0)$.

As for the stability of the interior equilibrium of the form (K^*, E^*, A^*), it is difficult to determine analytically due to the complicated expressions derived by solving for the values of the variables at this point. The Jacobian of the system can still be used to provide insight. Particularly, when computing the Jacobian at the point (K^*, E^*, A^*), it is possible to find an expression for the characteristic polynomial found by computing $det(J^* - \lambda I)$. The roots of this polynomial yield the eigenvalues of the Jacobian, which would provide information on the stability of this interior equilibrium. The full expression for the characteristic polynomial of the Jacobian at the interior equilibrium, (B.1.1), can be found in the corresponding appendix, but for now, a concise expression, (3.4.9), is given to facilitate discussion.

$$det(J^* - \lambda I) = \lambda^3 - a_1\lambda^2 - a_2\lambda - a_3 \tag{3.4.9}$$

The expressions for the coefficients, a_1, a_2, and a_3, are dependent on the values of the parameters of the system. If it is possible to find the sign of these coefficients, the Routh-Hurwitz criterion could be used to determine if the equilibrium is stable. Unfortunately, due to the nature of the parameters, it was not possible to definitively find the sign of these coefficients. Thus, an analytical answer of whether or not the interior equilibrium is stable is not feasible at this time. However, it is possible to do a numerical analysis to see if this equilibrium is ever stable. Using Maple computation and a MATLAB simulation, it seems that this equilibrium is never stable. Hence, though there is the potential for an interior equilibrium, the model does not allow for it to be stable.

3.4.4 Bifurcation Analysis

In this section, the software XPPAUT, [8], is used to create bifurcation diagrams in order to study the change in stability of the equilibria of the model. Diagrams will be presented to show the bifurcations that occur when changing six of the model parameters, namely, α, δ, σ, $\tilde{K}$, $\tilde{E}$, and $\tilde{A}$. Table 3.5 the significance of different lines and colours are listed to facilitate reading these diagrams. The specific parameter values used to generate each set of bifurcation diagrams will be included in the Appendix B.1.

Type of Line	**Meaning**
Red line	Stable equilibrium
Black line	Unstable equilibrium
Red and black overlapping lines	One stable, one unstable equilibrium
Green dotted line	Stable periodic orbits
Blue dotted line	Unstable periodic orbits

Table 3.5: Meaning of Lines in Bifurcation Diagrams

In the Figures 3.7a, 3.7d, and 3.7g, the bifurcation parameter used was the individual's aptitude, α. These diagrams exemplify the complex nature of the dynamics of Model III. In particular, as α increases, there is a bifurcation from $E_5 = (K^+, \hat{E}, \bar{A})$ to $E_4 = (\bar{K}, E^+, \hat{A})$. Moreover, there is an interval for which bistability between these two equilibrium points occurs. In this region, an individual would be able to achieve different levels depending on their initial conditions. Subsequently, E_4 becomes the only stable equilibrium. Thus, inceasing aptitude allows the individual to reach a higher level of knowledge, a different level of effort, as well as a lower level of anxiety. In these diagrams, it is also possible to see the existence of an interior equilibrium, when aptitude is approximately 0.3, that is unstable. Other interesting dynamics also occur in this series of bifurcation diagrams. Notably, there is the presence of a stable periodic orbit when K is greater than one. Unfortunately, this is currently irrelevant in the scope of Model III as it only considers values between zero and one.

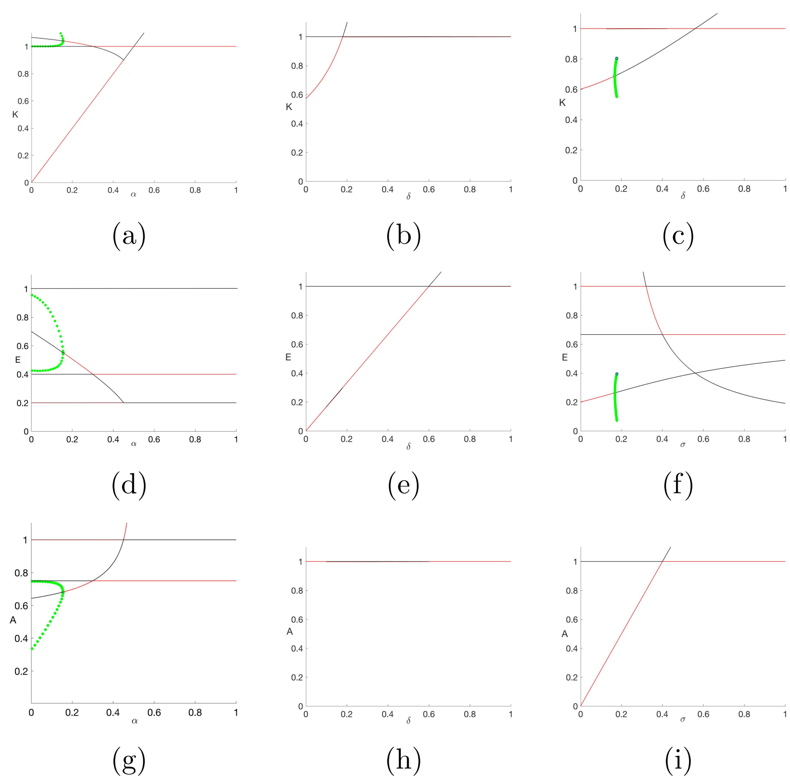


Figure 3.7: Bifurcation diagrams as α, δ, and σ change from zero to one

Next, in the second column including Figures 3.7b, 3.7e, and 3.7h, the bifurcation parameter was the individual's drive, δ. As delta increases from zero to one, there are bifurcations from $E_5 = (K^+, \hat{E}, \bar{A})$ to $E_2 = (K^+, \hat{E}, \bar{A})$, and then to $E_1 = (\bar{K}, \bar{E}, \bar{A})$. This implies that as the individual's drive grows, they may then succeed in increasing their level of knowledge and effort to the maximum. Though anxiety remains at the $\bar{A}$ level, in order to see the full effects of an increase in drive in the bifurcation diagrams, a sufficient value of susceptibility of anxiety must be present. More precisely, if σ is low enough (*e.g.* $\sigma = 0.3$), then the bifurcation that occurs leads to stability of $E_3 = (\bar{K}, \bar{E}, \hat{A})$ instead.

The third column in Figure 3.7 illustrates the bifurcations as the parameter for the susceptibility for anxiety, σ, of the individual increases from zero to one. Once again, the dynamics here seem to be interesting and complex. However, as is the case in the bifurcation diagrams of changing aptitude, these most interesting results occur for values outside of the range of interest. The important behaviour to note from these diagrams is that an increase in susceptibility for anxiety leads to a decrease in the level of effort and an increase in the level of anxiety an individual may reach. In this case, it is equilibrium point $E_3 = (\bar{K}, \bar{E}, \hat{A})$ that is stable, and as σ increases, $E_2 = (K^+, \hat{E}, \bar{A})$ becomes stable.

Another interesting result can be seen when $\tilde{K}$ is used as a bifurcation parameter. This special case is shown in the first column of Figure 3.8. Figures 3.8a, 3.8d, and 3.8g, show the change in knowledge, effort, and anxiety respectively as $\tilde{K}$ increases from zero to one. Here, there is a very wide range of the bifurcation parameter for which bistability of equilibrium points occurs. Thus, depending on the initial conditions of the individual, their outcome could differ. In particular, for these parameter values, the stability shifts from only E_3, to bistability between E_3 and E_5, and finally to only E_5. This demonstrates that as $\tilde{K}$ increases, one's level of knowledge is actually hindered. In fact, as this parameter increases, the level of anxiety rises which in turn, restricts the individual from attaining the maximum level of knowledge. In these diagrams, it is also possible to see that there is in fact an interior equilibrium that is present, but that it is unstable.

In the second column of Figure 3.8, the bifurcation parameter used was $\tilde{E}$. As this parameter increases, there is an interesting bifurcation that appears when considering the level of knowledge. In particular, as it grows, the level of knowledge decreases. Initially, it is equilibrium point E_2 that is stable, whereas as the parameter increases, E_5 becomes the stable equilibrium. Yet another interesting result occurs in the diagrams as there is the presence of an unstable periodic orbit. This orbit, represented by the blue dotted line, only

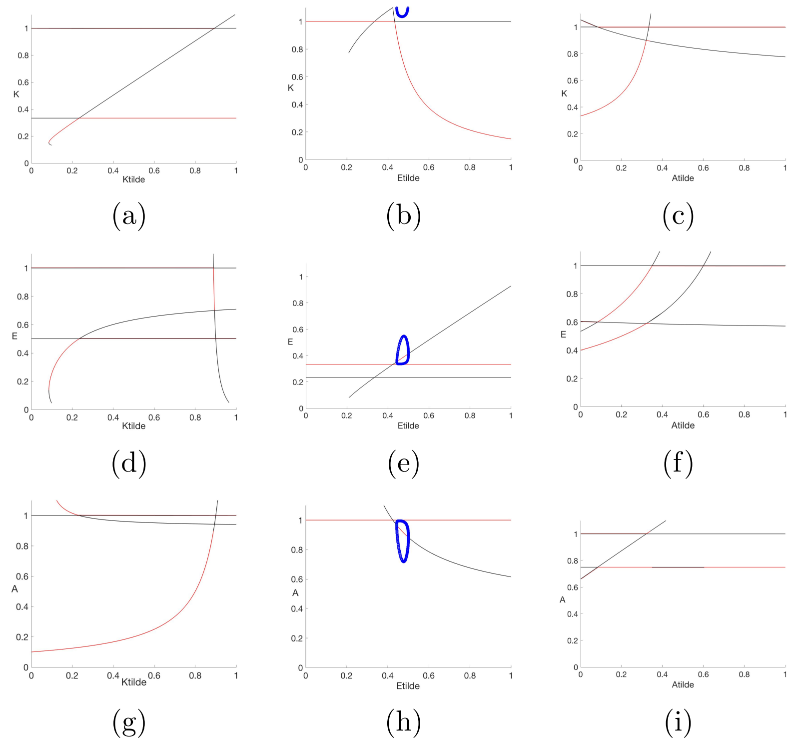


Figure 3.8: Bifurcation diagrams as $\tilde{K}$, $\tilde{E}$, and $\tilde{A}$ changes from zero to one

occurs when the level of knowledge is above one. Unfortunately, due to this fact, this is not a relevant orbit. However, once again, this emphasizes the potential of the model in that the dynamics are very complex.

The last set of bifurcation diagrams, Figures 3.8c,3.8f, and 3.8i, are generated by using $\tilde{A}$ as the bifurcation parameter. In these figures, it demonstrates that there is once again bistability between E_5 and E_4. As the bifurcation parameter increases, E_3 becomes the stable equilibrium point. Thus, as $\tilde{A}$ increases, the level of knowledge and effort increase, whereas the level of anxiety decreases. It is notable that there is also the appearance of an interior equilibrium that is unstable.

After investigating the nature of the bifurcation diagrams given various sets of parameters, it is clear that the dynamics of Model III are complex and offer many different outcomes. The variety of dynamics speak to the potential of the model and suggest that with some refinement even more predictions could be made.

3.4.5 Critique of Model III

A statistician named George Box once stated, "essentially, all models are wrong, but some are useful" [6]. With this in mind, it is noted that Model III does have its shortcomings, which will be discussed in this section. The usefulness of the model is left to the discussion section of this thesis.

The first critique of the model is that the maximal values of the variables ($\bar{K}$, $\bar{E}$, $\bar{A}$) are easily reached. For instance, at times, when the aptitude of an individual is low, they can still reach full knowledge in a short period of time. It would be important to find a more definite range of parameters to better reflect the nature of this phenomenon. Though this may be seen as a reflection of the complex nature of the interaction between knowledge, effort, and anxiety, it could be refined. Despite the fact that the model behaves appropriately when changing one parameter at a time, the dynamics become difficult to predict when several parameters change at once. Furthermore, the parameters were chosen relative to one another and were not based on empirical data. In the future, it would be interesting to try to validate the model using empirical data to find more appropriate ranges for the parameters.

Another critique would be that there are very interesting dynamics that occur outside the valid range of the variables. For instance, there are Hopf bifurcations present when at least one of the variables are greater than one. Thus, there are periodic orbits possible in the dynamics, however they are not in a relevant interval. This said, in the future, it would be wise to refine the model in the hopes that the system would result in these dynamics within an appropriate range of the variables and parameters.

Moreover, the model could take into account even more characteristic parameters such as the individual's expectation, the difficulty of the material, and even their previous performance. For the first version of the model, it was necessary to simplify the parameters being taken into consideration, but more factors can be considered in the future. Furthermore, perhaps the values for the critical levels of knowledge, effort, and anxiety, could depend on other parameters instead of being independent. Thus, there is certainly room for improvement on the level of the parameters of the model.

Finally, Model III is flawed in the fact that it does not predict a stable interior equilibrium. Part of the theoretical motivation for the model is that an individual be able to reach a level of knowledge, effort, and anxiety, that is particular to their characteristics. However, though an interior equilibrium can be found given certain sets of parameters, it fails to be stable. Thus, in this sense, it seems that the model could be improved to allow for a more realistic outcome.

Chapter 4

Discussion

Through the development of Model III, the goal is to provide a theoretical approach to understanding the relationship between the proportions of knowledge, effort, and anxiety. Though these variables are not easily quantifiable, the theory behind the model could one day allow for a better understanding of how individuals react in learning situations. The model itself is reasonable in its behaviour when varying the different parameters of the system. For instance, as aptitude increases, the time it takes the individual to reach full knowledge decreases. On the other hand, when the susceptibility for anxiety becomes too great, the individual can no longer overcome this obstacle, and they are unable to reach the level of full knowledge. In this sense, the initial behaviour of the model is justifiable.

Not only does the model seem reasonable in the behaviour of its solutions, but it also succeeds in the fact that it has several equilibria. Seeing as this model describes a very complex situation, it is appropriate that the model have a variety of outcomes for specific sets of parameters that could represent different individuals. In particular, the bistability that occurs for certain sets of parameters suggests that individuals that have different initial levels of knowledge, effort, and anxiety, could reach a better outcome than other individuals. This could reflect a situation where a student might have to retake a course, and thus, are beginning the term with a better understanding of the course content, a higher level of effort, or even a lower level of anxiety.

With the presence of several feasible equilibria and the amount of parameters, the analysis of the model was quite challenging. Unfortunately, making definite conclusions on the signs of certain analytical expressions such as the eigenvalues of the Jacobian at the equilibrium points was not possible. Thus, the analysis of the model involved a lot of numerical investigations, and examinations of bifurcation diagrams and diagrams of the system's solutions for various sets of parameters.

Overall, the model itself is a step in the right direction to begin thinking about this relationship and how individual's effort and anxiety can affect their level of knowledge. Seeing as motivational factors such as drive and susceptibility for anxiety can greatly affect a person engaged in a learning experience, Model III provides another perspective on the acquisition of knowledge. This perspective may allow researchers to consider the affects of mathematical anxiety on learning in a more quantitative way.

Chapter 5

Future Work

Due to the importance of mathematical education in today's society, it is evident that there is a demand for more advancements in this field of research. The previous literature and the data analysis performed in this paper prove that mathematical anxiety is present in university students, particularly in first year mathematics courses. However, despite these previous findings, there is still a lack of understanding of the nature and causal direction of the correlation between mathematical anxiety and mathematical achievement. Therefore, there is a need to continue to investigate the dynamic nature of mathematical anxiety and its predictors and effects.

This dynamic nature suggests that mathematical anxiety can affect individuals differently, and can have varying consequences. As such, a longitudinal study that examines the nature of mathematical anxiety and its effects on individuals could be beneficial. More precisely, this study could gather time series data to track the level of mathematical anxiety and performance of students over the course of several terms across different programs and mathematics courses. Furthermore, this study should consider other factors such as demographic information, sleep patterns, effort, and self-expectancies that individuals exhibit in a learning situation. All of these factors could impact levels of mathematical anxiety and performance and thus, this could shed light on this complicated relationship.

As the model proposed in this paper is in its early stages, there is also a lot of potential future work that can be done, along with other approaches to modelling that could be utilized. The next important step to improve the model would be to decide on appropriate measures for knowledge, effort, and anxiety, to be able to provide a more precise range of parameters and variable values. This refinement could potentially lead to more interesting dynamics, which would better explain this complex interaction between knowledge, effort, and anxiety.

Particularly, the use of empirical data to validate the dynamics between knowledge, effort, and anxiety would be beneficial to adapt and improve the model. Thus, a well thought out longitudinal study could provide the empirical data needed to verify the appropriateness of the theoretical model from this paper.

Moreover, there are many other methods that could be exploited to refine or expand on the model proposed here. Another avenue could be to investigate the model used in epidemiology that describes the interaction between susceptible, infected, and recovered individuals (*i.e.* the SIR model). More precisely, it would be possible to describe acquisition of knowledge through the effect of the individual's state. In this sense, an individual hoping to acquire knowledge would learn the content if it were successfully stored in long term memory despite the effects of anxiety. An example of this could be to have compartments for (a) content to be acquired, (b) content currently being studied, (c) content acquired, and (d) content returned to the environment. This fourth compartment would account for the information returned to the environment as a result of being unsuccessfully stored due to the effects of mathematical anxiety. Subsequently, depending on the proportion of content acquired, the individual's performance would be described by this proportion and the level of mathematical anxiety that the individual exhibits.

In many cases, learning and anxiety do not have an immediate effect on the individual. For instance, students may put in effort, study course material, and only feel the effects of mathematical anxiety after they receive their grades. This situation could potentially be accounted for in the model by exploring delayed differential equations. Using this method of modelling could more accurately depict the change in levels of knowledge, effort and anxiety by taking into account the time it takes for individuals to undergo the self-regulation process and understand their own comprehension of the course content.

Another potential area of improvement would be to incorporate randomness into the model. In particular, stochasticity could be appropriate to account for the variation in effort that individuals exhibit from week to week throughout the term. This could be beneficial as students do not always provide a constant change in effort from one week to the next due to balancing other classes, work, and social interactions. As such, stochasticity would be an interesting addition to the model to verify the dynamics given the uncertainty of human behaviour.

Chapter 6

Conclusion

Evidently, there is still much work to be done in order to fully understand the complex nature of learning. However, this thesis has provided a different perspective by making a step to bridge the gap between educational research and mathematical models of learning.

Through a brief literature review, some important contributions to research in the field of mathematical education and in mathematical modelling have been recognized. Notable findings, such as establishing the importance of test anxiety and mathematical anxiety in student performance, investigating the factors that affect mathematical anxiety, and developing tools used to measure this particular type of anxiety, have been discussed. On the other hand, various mathematical models of learning were presented in order to provide a foundation for the proposal of a new model of learning.

Moreover, a statistical analysis has been conducted on a dataset collected at McMaster University. This analysis has shown that mathematical anxiety does, in fact, affect the students enrolled in first year courses at the university. Not only does it affect the students at McMaster University, but mathematical anxiety was shown to be correlated with factors such as gender, performance, and even students' program choices, all of which correspond to previous research findings in the field.

Finally, models of learning were developed in order to further drive the theoretical concept of how mathematical anxiety might affect the learning of students. In particular, the model proposed by Anderson in [2], was first used to suggest that mathematical anxiety and drive could be incorporated as parameters into his model. Subsequently, using the theory of Anderson's model, Model II was conceived to reflect the early stages of a learning experience.

Most importantly, Model III was developed to provide a theoretical framework and suggest that the interaction between an individual's level of knowledge, effort, and anxiety, is dynamic and changes throughout an academic term. This model reflects the fact that an individual's aptitude, drive, and susceptibility for anxiety each have an effect on their output in a learning experience. The stability of the equilibria was discussed, and bifurcation analysis was provided to emphasize the dynamics' dependence on the parameters of the model.

There are still many facets of this complex subject that have yet to be discovered, and going forward, it is important to keep investigating the nature of mathematical anxiety and its effects on learning in order to provide even more insight. In fact, many students continue to suffer from this anxiety and cannot appreciate the value and importance of mathematics. Overall, this thesis will hopefully be used as a stepping stone to further drive research in the field of mathematical education, and ultimately lead to new research that will allow more students to discover the beauty of Mathematics.

Appendices

Appendix A

Acronyms

- MA - Mathematical Anxiety
- MARS - Mathematical Anxiety Rating Scale
- CNS - Central Nervous System
- STM - Short Term Memory
- LTS - Long Term Store
- STEM - Science, Technology, Engineering, and Mathematics
- HSA - High School Average
- EXP - Expectation
- GSF - Grades so Far

Appendix B

Details of Models in the Literature

In the following appendix sections, there are extra equations relevant to Model III, and the models derived in the works of Hicklin et al. (1965), Anderson (1983), and Pritchard et al. (2008), are presented. In the case that the reader wishes to refer to the derivations of these models, they have been included for convenience. Thus, these sections do not consist of original work and are simply a reference to previous work conducted by other researchers on the topic of mathematical models of learning. A list of acronyms is also provided.

B.1 Model III Extras

The full expression for the characteristic polynomial of the Jacobian at the interior equilibrium (K^*, E^*, A^*) is given below.

$$\begin{aligned} det(J^* - \lambda I) = \lambda^3 &- \Big[(2K^* - \bar{K})(\tilde{E} - E^*) + (2\tilde{E} - \bar{E})(A^* - \tilde{A}) + (2A^* - \bar{A})(K^* - \tilde{K}) - (\alpha + \delta + \sigma)\Big] \lambda^2 \\ &- ((-(2K^* - \bar{K})(\tilde{E} - E^*) + \alpha - (2E^* - \bar{E})(A^* - \tilde{A}) + \delta)((2A^* - \bar{A})(K^* - \tilde{K}) - \sigma) \\ &- ((2K^* - \bar{K})(\tilde{E} - E^*) - \alpha)((2E^* - \bar{E})(A^* - \tilde{A}) - \delta))\lambda \\ &- ((2K^* - \bar{K})(\tilde{E} - E^*) - \alpha)((2E^* - \bar{E})(A^* - \tilde{A}) - \delta))((2A^* - \bar{A})(K^* - \tilde{K}) - \sigma) \\ &- K^* E^* A^* (A^* - \bar{A})(E^* - \bar{E})(\bar{K} - K^*) \end{aligned} \tag{B.1.1}$$

In Table B.1 the parameters used to generate the bifurcation diagrams in Section 3.4.4 are provided. Note that all the bar parameters ($\bar{K}$,$\bar{E}$, and $\bar{A}$), are equal to one.

Set of Diagrams	Parameters
3.7a, 3.7d, 3.7g	$\alpha \in (0,1)$, $\delta = 0.1$, $\sigma = 0.3$, $\tilde{K} = 0.6$, $\tilde{E} = 0.7$, $\tilde{A} = 0.5$
3.7b, 3.7e, 3.7h	$\alpha = 0.4$, $\delta \in (0,1)$, $\sigma = 0.5$, $\tilde{K} = 0.6$, $\tilde{E} = 0.7$, $\tilde{A} = 0.4$
3.7c, 3.7f, 3.7i	$\alpha = 0.3$, $\delta = 0.4$, $\sigma \in (0,1)$, $\tilde{K} = 0.6$, $\tilde{E} = 0.7$, $\tilde{A} = 0.4$
3.8a, 3.8d, 3.8g	$\alpha = 0.1$, $\delta = 0.1$, $\sigma = 0.1$, $\tilde{K} \in (0,1)$, $\tilde{E} = 0.8$, $\tilde{A} = 0.8$
3.8b, 3.8e, 3.8h	$\alpha = 0.1$, $\delta = 0.2$, $\sigma = 0.5$, $\tilde{K} = 0.6$, $\tilde{E} \in (0,1)$, $\tilde{A} = 0.4$
3.8c, 3.8f, 3.8f	$\alpha = 0.1$, $\delta = 0.4$, $\sigma = 0.5$, $\tilde{K} = 0.6$, $\tilde{E} = 0.7$, $\tilde{A} \in (0,1)$

Table B.1: Parameters for Bifurcation Diagrams

B.2 Hicklin et al. (1965)

In [12] Hicklin et al. present a model to describe content acquisition over an individual's lifetime. This situation is described by a system where (1) represents the environment, (2) the individual and (3) the lost or forgotten category. The richness of the environment is denoted N_{01}. As the individual acquires information, the environment is depleted. What has not yet been acquired at time t is denoted N_1, and N_2 is the status of the individual at time t. The current status of the individual is the difference between what they have acquired and what has been lost.

To write the differential equation for N_1, the rate of decrease in N_1 is assumed to be proportional to the information that has yet to be acquired N_1, thus Hicklin et al. write,

$$\frac{dN_1}{dt} = -k_1 N_1. \tag{B.2.1}$$

Solving this differential equation yields the solution,

$$N_1 = N_{01} e^{-k_1 t}. \tag{B.2.2}$$

Here, this suggests that what is left to be assimilated depends on what is initially present from the environment (N_{01}), and on the aptitude of the individual (k_1).

To derive the expression for the status of the individual, Hicklin et al. make the assumption that during a time interval Δt, the individual gains $N_1 k_1$ units from the environment and loses $N_2 k_2$ units to the lost category. This gives the differential equation,

$$\frac{dN_2}{dt} = k_1 N_1 - k_2 N_2. \tag{B.2.3}$$

Rearranging and using (B.2.2) this differential equation has the solution,

$$N_2 = N_{01}\frac{k_1}{k_1 - k_2}(e^{-k_2 t} - e^{-k_1 t}). \tag{B.2.4}$$

Assuming that the individual is in a standard-unit environment, N_{01} can be set to 1. Thus, the status of the individual at any time t can be described by,

$$N_2 = \frac{k_1}{k_1 - k_2}(e^{-k_2 t} - e^{-k_1 t}). \tag{B.2.5}$$

Hicklin et al. also find the percentage of growth at any time by finding the maximum of N_2, dividing equation (B.2.5) by that value, and multiplying it all by 100. In doing so, it yields the curves given by,

$$Y = \frac{100}{C}(e^{-k_2 t} - e^{-k_1 t}), \tag{B.2.6}$$

where C is a constant that can be determined by the maximum value of N_2 at the time of maximum development.

Moreover, an interesting result of the derivation of the model in the paper by Hicklin et al., is that by assuming that the loss rate is zero ($k_2 = 0$) in equation (B.2.4), the equation becomes,

$$N_2 = N_{01}(1 - e^{-k_1 t}). \tag{B.2.7}$$

B.3 Anderson (1983)

In [2], Anderson proposed a neuromathematical model for content acquisition. This model was directly inspired by the workings of the CNS and the fact that short term memory is limited. Based on this assumption, the author derives three functions to compose the net gain function. These functions are the stability function, the instability function, and the gain function. The derivation of these functions that Anderson's paper presented is outlined below.

B.3.1 Stability Function

In assuming that the stability of short term memory is limited, Anderson proposes that the rate of decline of the stability is proportional to the amount of activity in CNS related to STM, denoted S. Further, it is assumed that as the quality (abstractness β) and quantity (rate of inflow ρ) of information increase, so too would the decrease in stability. However, this decrease would be slowed for learner's of higher aptitude, denoted κ. Thus, the author suggests the following for the rate of decrease of stability,

$$-\frac{ds}{dt} = \frac{\alpha \cdot \beta \cdot \rho}{\kappa} S. \tag{B.3.1}$$

Solving this differential equation yields the solution for the stability function S,

$$S = S_0 e^{-\frac{\alpha \cdot \beta \cdot \rho}{\kappa} t}. \tag{B.3.2}$$

Here, S_0 is the initial value of S at time $t = 0$ and t is the time since the beginning of the learning experience.

B.3.2 Instability Function

Next, the instability function suggests that as the learning task continues, the learner becomes more familiar with the task and the instability decreases. Thus, the rate of change of instability is proportional to the instability I and a rate of decay λ, yielding the following differential equation,

$$\frac{dI}{dt} = -\lambda I. \tag{B.3.3}$$

Solving this differential equation gives,

$$I = I_0 e^{-\lambda t}, \tag{B.3.4}$$

where I_0 is the initial value of the instability factor at time $t = 0$.

B.3.3 Modulation Factor

In order to account for the overall affect of the stability and instability factors, Anderson uses a modulation factor, M, which is defined by the difference between S and I.

$$M = S - I = S_0 e^{-\frac{\alpha \cdot \beta \cdot \rho}{\kappa} t} - I_0 e^{-\lambda t}. \tag{B.3.5}$$

Anderson provides a more concise notation where $\sigma = \frac{\alpha \cdot \beta \cdot \rho}{\kappa}$, which leads to the following expression for M.

$$M = S - I = S_0 e^{-\sigma t} - I_0 e^{-\lambda t}. \tag{B.3.6}$$

B.3.4 Gain Function

Finally, the gain function is derived so that the rate of change of the gain is proportional to N. Furthermore, the rate of change increases for learner's with higher κ, and decreases with more abstract material β and more information ρ. Thus, the differential equation below is established,

$$\frac{dN}{dt} = \frac{\kappa}{\alpha' \cdot \beta \cdot \rho} N. \tag{B.3.7}$$

Solving the above leads to the gain function,

$$N = N_0 e^{\frac{\kappa}{\alpha' \cdot \beta \cdot \rho} t}, \tag{B.3.8}$$

where α' is a constant of proportionality and N_0 is the initial gain at $t = 0$. Once again, this can be written in more concise form if $\gamma = \frac{\kappa}{\alpha' \cdot \beta \cdot \rho}$,

$$N = N_0 e^{\gamma t}, \tag{B.3.9}$$

B.3.5 Composite Function

Using all three functions, Anderson presents the composite equation (B.3.10), where the gain function (B.3.9) is multiplied by the modulation factor (B.3.6),

$$N_t = N_0 e^{\gamma t}(e^{-\sigma t} - e^{-\lambda t}). \tag{B.3.10}$$

Certain conditions must be met to ensure that the net gain cannot be negative. Notably, $S > I$ for all time and the rate of decline of stability must be smaller than the rate of decay of instability ($\sigma < \lambda$).

A special case of the net gain function can be written when $\sigma = \gamma$ as,

$$N_t = N_0[1 - e^{(\sigma - \lambda)t}]. \tag{B.3.11}$$

Note that this special case is analogous to the special case derived in Hicklin et al.'s 1965 paper.

B.4 Pritchard et al. (2008)

In [21], Pritchard et al. propose different models based on the type of teaching methods used. In particular, three theories of learning motivated the development of the models in the authors' paper. The derivation of these models is outlined subsequently.

In Pritchard et al.'s models, the dependent variable is the student's knowledge after a certain amount of instruction. The authors divide the test domain into what is known and what is unknown, denoted $K_T(t)$ and $U_T(t)$ respectively. In this case, t is the amount of instruction the student has undergone. A parameter α is used to denote the probability that a student remembers what was taught. There are four models derived in Prichard et al.'s paper, all of which are inspired by a different learning theory.

B.4.1 Pure Memory Model

Inspired by the tabula rasa learning theory, the pure memory model assumes that a student has a blank slate before starting the learning experience. Thus, the model represents rote memorization since α does not depend on the learner's prior knowledge. There are 4 assumptions for the derivation of this model, that are listed below.

1. The bits of information are uniformly distributed over the test domain.
2. Only the fraction $U_T(t)$ of these bits are unknown and can be learned.
3. α is the probability that the student remembers the bits of information.
4. α does is not dependent on K or U.

These assumptions lead to the following differential equation for $U_T(t)$,

$$\frac{dU_T(t)}{dt} = -\alpha_{memory} U_T(t). \tag{B.4.1}$$

This differential equation has solution,

$$U_T(t) = U_T(0)e^{-\alpha_{memory} t}. \tag{B.4.2}$$

From this, since $U_T(t) = 1 - K_T(t)$, the solution for $K_T(t)$ is given by the following equation,

$$K_T(t) = 1 - (1 - K_{T0})e^{-\alpha_{memory} t}. \tag{B.4.3}$$

B.4.2 Simple Connected Model

The simple connected model uses the constructivism theory as motivation. Thus, its base assumption is that students acquire new information by relating it to knowledge they already have. In this sense, the more prior knowledge one has, the faster the student will learn. The same assumptions as above are used for this model. However, the authors state that, "the learning rate is now proportional to three factors: the probability $U_T(t)$ that the knowledge nugget strikes an unknown region, the probability $K_T(t)$ that the appropriate connecting knowledge is already known, and the probability that the nugget will stick (i.e., the association will be constructed), $\alpha_{connected}$" [21].

These assumptions lead to the following differential equation,

$$\frac{dU_T}{dt} = -\alpha_{connected} U_T(t) K_T(t) \tag{B.4.4}$$

$$= -\alpha_{connected} U_T(t)[1 - U_T(t)]. \tag{B.4.5}$$

This differential equation is solved to yield a logistic function given by the equation,

$$K_T(t) = \frac{1}{1 + (1 - K_{T0}) e^{-\alpha_{connected} t} / K_{T0}} \tag{B.4.6}$$

This model is appropriate to describe learning via peer to peer instruction. On the other hand, a variation of this model is possible when considering that the learning rate is proportional to knowledge that is external to the test domain as opposed to $K_T(t)$. With this slight change, the following differential equation and solutions arise.

$$\frac{dU_T}{dt} = -\alpha_{connexternal} U_T(t) K_{external} \tag{B.4.7}$$

$$K_T(t) = 1 - (1 - K_{T0}) e^{-\alpha_{connexternal} K_{external} t} \tag{B.4.8}$$

In this model, $K_{external}$ does not depend on the instruction t. Thus, it is similar to the pure memory model.

B.4.3 Connectedness Model

This model is a mix of the two previous models. In fact, it is motivated in part by the tabula rasa theory, and in part by the constructivist theory. It mediates between the two pure models by introducing a parameter β for the connectedness of the model. If $\beta = 0$, the model describes the pure memory model, whereas if $\beta = 1$, it is represents the simple connected model. Since a fraction β is connected, and the rest $(1 - \beta)$ is pure memory, the following differential equation is written,

$$\frac{dU_T}{dt} = -U_T(t)[\alpha_{connected}\beta K(t) + \alpha_{memory}(1-\beta)]. \tag{B.4.9}$$

Depending on whether the knowledge connected is internal or external, two solutions occur. These solutions are displayed in equations (B.4.10) and (B.4.12).

$$K_T^{internal}(t) = 1- \tag{B.4.10}$$

$$\frac{(1-K_{T0})[\alpha_{memory}(1-\beta)+\alpha_{connected}\beta]}{(1-K_{T0})\alpha_{connected}\beta + [\alpha_{memory}(1-\beta)+K_{T0}\alpha_{connected}\beta]\exp\{[\alpha_{memory}(1-\beta)+\alpha_{connected}\beta]t\}} \tag{B.4.11}$$

$$K_T^{external}(t) = 1-(1-K_{T0})\exp\{-[\alpha_{memory}(1-\beta)+\alpha_{connected}\beta K_{external}]t\} \tag{B.4.12}$$

B.4.4 Tutoring Model

For this model, the main assumption is that a tutor can provide perfect instruction based on the prior knowledge of the student. Thus, the student need not waste time to relearn what they already know. In this sense, the student can acquire knowledge a the rate k_a, which the authors refer to as the student's maximum assimilation rate. Since the learning is now independent of $K(t)$ and $U(t)$, the learning rate is now uniform,

$$\frac{dU_T}{dt} = -k_a. \tag{B.4.13}$$

Solving this differential equation leads to the following solution,

$$K_T(t) = k_a(t-t_0). \tag{B.4.14}$$

With this solution, a student can learn a finite test domain in a finite time. Note, here the authors restrict $K_T \leq 1$.

Appendix C

Copy of Survey

In the pages that follow, there is a copy of the survey that participants could access online via LimeSurvey. This survey includes demographic information, self-reported grades, the revised MARS items by Plake & Parker, and a section about the resources available at McMaster University.

There are 38 questions in this survey

Preamble and Consent

Preamble Statement

This survey is administered by Savannah Spilotro, Master's Candidate at the Department of Mathematics and Statistics at McMaster University. The purpose of the survey is to provide an overview of the level of mathematical anxiety that students exhibit while taking a first year mathematics course. Information gathered during this survey will be written up as a Master's Thesis. What we learn from this survey will help us understand the effect of mathematical anxiety on learning. To learn more about the survey and the researcher's study, particularly in terms of any associated risks or harms associated with the survey, how confidentiality and anonymity will be handled, withdrawal procedures, how to obtain information about the survey's results, how to find helpful resources should the survey make you uncomfortable or upset etc., please read the accompanying letter of information. This survey should take approximately 20 minutes to complete. People filling out this survey must be enrolled in a first year mathematics course at McMaster University.

This survey is part of a study that has been reviewed and cleared by the McMaster Research Ethics Board (MREB). The MREB protocol number associated with this survey is [insert the MREB protocol number, e.g. MREB 2012 185]. You are free to complete this survey or not. If you have any concerns or questions about your rights as a participant or about the way the study is being conducted, please contact:

McMaster Research Ethics Secretariat
Telephone 1-(905) 525-9140 ext. 23142
c/o Research Office for Administration, Development and Support (ROADS)
E-mail: ethicsoffice@mcmaster.ca

Consent to Participate

Having read the above, I understand that by clicking the "Yes" button below, I agree to take part in this study under the terms and conditions outlined in the accompanied letter of information.*
Please choose **only one** of the following:

- ◯ Yes, I agree to participate
- ◯ No, I do not agree to participate

General Information

Age?
Please write your answer here:__________

[]Gender? *
Please choose only one of the following

- ◯ **Male**
- ◯ **Female**
- ◯ **Prefer to not answer**

Program? *
Please choose **only one** of the following:

- ◯ Life Sciences
- ◯ Kinesiology
- ◯ Business
- ◯ Physics, Chemistry or Biology
- ◯ Math and Stats
- ◯ Engineering
- ◯ Computer Science
- ◯ Other

Course? *
Please choose **only one** of the following:

- ◯ Math 1A03
- ◯ Math 1B03
- ◯ Math 1F03
- ◯ Math 1K03
- ◯ Math 1LS3
- ◯ Math 1X03
- ◯ Other

What was your average in the most recent high school math course you've taken?
Please choose **only one** of the following:

- ◯ 0-49%
- ◯ 50-59%
- ◯ 60-69%
- ◯ 70-79%
- ◯ 80-84%

- ○ 85-89%
- ○ 90-94%
- ○ 95-100%

What grade do you expect in this course?
Please choose **only one** of the following:

- ○ 0-49%
- ○ 50-59%
- ○ 60-69%
- ○ 70-79%
- ○ 80-84%
- ○ 85-89%
- ○ 90-94%
- ○ 95-100%

If you have received marks so far, what is your average?
Please choose **only one** of the following:

- ○ 0-49%
- ○ 50-59%
- ○ 60-69%
- ○ 70-79%
- ○ 80-84%
- ○ 85-89%
- ○ 90-94%
- ○ 95-100%

Revised Mathematics Anxiety Rating Scale

Please rate your feelings of anxiety on a scale of 1 (low anxiety) to 5 (high anxiety) in each of the cases below.

Watching a teacher work an algebraic equation on the blackboard. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Buying a math textbook. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Reading and interpreting graphs or charts. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Signing up for a course in Statistics. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Listening to another student explain a math formula. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Walking into a math class. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Looking through the pages in a math text. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Starting a new chapter in a math book. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Walking on campus and thinking about a math course. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Picking up a math textbook to begin working on a homework assignment. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Reading the word "Statistics." *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Working on an abstract math problem, such as: "if x = outstanding bills, and y = total income, calculate how much you have left for recreational expenditures." *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Reading a formula in chemistry. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Listening to a lecture in math. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Having to use the tables in the back of a math book. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Being told how to interpret probability statements. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Being given a homework assignment of many difficult problems which is due the next class meeting. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Thinking about an upcoming math test one day before. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Solving square root problems. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Taking an examination (quiz) in a math class. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Getting ready to study for a math test. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Being given a "pop" quiz in a math class. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Waiting to get a math test returned in which you expected to do well. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Taking an examination (final) in a math course. *
Please choose **only one** of the following:

○ 1 ○ 2 ○ 3 ○ 4 ○ 5

Helpful Links for Dealing with Mathematical Anxiety

If you're feeling overwhelmed by math anxiety or would like more help and support, there are many places on campus that can be of assistance. Here is a list of resources to help you cope with math anxiety or general concerns about school.

- Math Help Centre (Hamilton Hall 104) : If you need math help for first year math courses, TAs are readily available in the centre to provide free help! See the link for hours of operation.
- Student Wellness Centre : If you're in need of advice, counselling or helpful tips for dealing with stress, the student wellness centre located in the student centre can be of assistance.
- Student Accessibility Services (SAS) : If you have difficulties due to a diagnosed disability or disorder, refer to the SAS webpage for more details on how to receive help.

Did you know about the Math Help Centre? *
Please choose **only one** of the following:

- ○ Yes
- ○ No

Would you seek help from the Math Help Centre? *
Please choose **only one** of the following:

- Yes
- No

Have you heard of the Student Wellness Centre? *
Please choose **only one** of the following:

- Yes
- No

Would you consider going to the Student Wellness Centre for help and advice? *
Please choose **only one** of the following:

- Yes
- No

Were you aware of the Student Accessibility Services? *
Please choose **only one** of the following:

- Yes
- No

If you were in need of assistance, would you make use of the Student Accessibility Services? *
Please choose **only one** of the following:

- Yes
- No

Thank you for taking this survey. Your answers are a valuable part of this research.

Submit your survey.
Thank you for completing this survey.

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