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RINGS OF CONDITIONS OF RANK 1 SPHERICAL VARIETIES

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Lay Abstract

We study an algebraic object that describes intersections of certain geometric spaces. An algorithm or formula for computing this object for a given geometric space is not known in general. We provide a technique for computing this algebraic object in a special case.

Abstract

In this thesis, we define and describe the rings of conditions of rank 1 spherical homogeneous spaces G/H. A procedure for computing the ring of conditions of a spherical homogeneous space in general is not known. For the special case of rank 1 spherical homogeneous spaces, we give a proof of the unpublished result of A. Khovanskii that the ring of conditions is isomorphic to the cohomology ring of a certain compactification of G/H. We illustrate this result through the fully worked example of $\mathbb{A}^n \setminus \{0\}$.

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List of Abbreviations and Symbols

$\mathbb{N}$: the set of natural numbers

$\mathbb{Z}$: the set of integers

$\mathbb{Q}$: the set of rational numbers

$\mathbb{R}$: the set of real numbers

$\mathbb{C}$: the set of complex numbers

$\mathbb{C}[x_1, ..., x_n]$: the polynomial ring in n variables over $\mathbb{C}$

$\mathbb{A}^n$: affine n-space

$\mathbb{A}^n \setminus \{0\}$: affine n-space minus the origin

$\mathbb{P}^n$: projective n-space

G/H: a homogeneous space

Chapter 1

Introduction

The ring of conditions has its context within the larger tradition of enumerative geometry and Schubert calculus. For a given geometric space, one can ask how many objects within it satisfy a certain condition or set of conditions. A classic example of such a question was posed by Hermann Schubert, who asked: How many lines in $\mathbb{C}^3$ intersect four given lines? Such a question is representative of more general problems in enumerative geometry, which ask how many points exist in a particular intersection.

In simple cases, the number of geometric objects satisfying certain conditions can be computed using grade school mathematics. More complex problems, however, require sophisticated techniques. Chasles, for example, constructs a formal algebra of conditions on conics [17]. A central idea of his work is that, with this language, one may express certain conditions as linear combinations of others, i.e., as being *generated* by a set of more basic conditions. Then the objects which satisfy the basic conditions also satisfy the generated conditions.

The ring of conditions is an instance of this algebraic approach to intersection theory and enumerative geometry. An important observation is that the ring product is not always well-defined; however, in [[11] Theorem 6.3], De Concini and Procesi prove that the product structure is defined for a spherical homogeneous space. A procedure for actually computing this ring for a given spherical homogeneous space is not known in general. Partial results exist; for example, in [22], Strickland computes the ring of conditions of an adjoint semisimple algebraic group G over the field $\mathbb{C}$ of complex numbers, considered as a $G \times G$-homogeneous space. In this thesis, we consider spherical homogeneous spaces G/H for a connected reductive group G and closed subgroup $H \subseteq G$.

The intended contribution of this thesis is to record an unpublished observation by Askold Khovanskii: namely, that the ring of conditions of certain spherical homogeneous spaces is isomorphic to the cohomology ring of a certain compactification of that space. This leads to a conjecture about rank 1 spherical varieties, where the situation is greatly simplified.

The structure of the thesis is as follows. In Chapter 2, we provide an overview of fundamental objects from algebraic geometry and provide the background necessary for our discussion of spherical geometry. In Chapter 3, we illustrate the definitions in the case of the rank 1 spherical homogeneous space $\mathbb{A}^n \setminus \{0\}$. Chapter 4 introduces the ring of conditions, as well as its connection to cohomology. In Chapter 5, we present an observation of Khovanskii, as mentioned above. We also provide a fully worked out example by computing the ring of conditions of $\mathbb{A}^n \setminus \{0\}$.

Chapter 2

Preliminaries

This chapter provides the background for the remainder of this thesis. In Section 2.1, we introduce relevant foundational objects from algebraic geometry. In Section 2.2, we recall relevant definitions for discussion of homogeneous spaces of algebraic groups. In Section 2.3, we give a brief account of the Luna-Vust theory of spherical embeddings. Some of the combinatorial data associated to spherical homogeneous spaces and their embeddings can be thought of as a generalization of the combinatorial data associated with toric varieties. We will use one of the spherical embeddings of $\mathbb{A}^n \setminus \{0\}$ to compute its ring of conditions in Chapters 4 and 5.

2.1 Basic algebraic geometry

In this section we collect basic notions from algebraic geometry. For simplicity, we will work throughout over the field $\mathbb{C}$ of complex numbers. The majority of the definitions introduced from this section are from Milne's introductory text on algebraic geometry [19].

Definition 2.1.1. [19] Let S be a finite collection of polynomials in $\mathbb{C}[x_1, x_2, ..., x_n]$. We call the set of common zeros in $\mathbb{C}^n$ of the polynomials in S an *algebraic set in* $\mathbb{C}^n$, or simply, an *algebraic set*. When speaking of the algebraic set associated to a particular S, we refer to it as the *vanishing locus* $V(S)$ of S.

An object that connects points in $\mathbb{C}^n$ with elements of $\mathbb{C}[x_1, ..., x_n]$ is the set of polynomials that are zero on a particular subset of points in $\mathbb{C}^n$.

Definition 2.1.2. [19] Let W be a subset of $\mathbb{C}^n$. The *defining ideal* of W, $I(W)$, is the set of polynomials in $\mathbb{C}[x_1, x_2..., x_n]$ that vanish on W.

We next define an important quotient ring whose elements are equivalence classes that can be identified with polynomial functions from an algebraic subset V to $\mathbb{C}$.

Definition 2.1.3. [19] Let V be an algebraic set in $\mathbb{C}^n$ and let $I(V)$ denote its defining ideal. The *coordinate ring* of V is $\mathbb{C}[V] = \mathbb{C}[x_1, x_2, ..., x_n]/I(V)$.

We will consider algebraic subsets as topological spaces with the Zariski topology.

Definition 2.1.4. [19] A topological space (V, τ) is *quasicompact* if every open cover of X has a finite subcover.

Note that we do not require the topological space to be Hausdorff.

Definition 2.1.5. [19] Let V be an algebraic set in $\mathbb{C}^n$ and consider the coordinate ring $\mathbb{C}[V]$ of V. Any function $f \in \mathbb{C}[V]$ defines a function $(p \mapsto f(p))\colon V \to \mathbb{C}$, where $p \in V$. We call such functions $(p \mapsto f(p))$ *regular*.

A continuous map $\phi\colon W \to V$ of algebraic sets is *regular* if each of its components ϕ_i is a regular function.

The next three definitions allow us to state the definition of an algebraic variety very cleanly.

Definition 2.1.6. [19] A pair $(V, \mathcal{O}_V)$ consisting of a topological space V and a sheaf of $\mathbb{C}$-algebras on V is called a $\mathbb{C}$-*ringed space*, or just a *ringed space*.

Definition 2.1.7. [19] An *algebraic prevariety* over $\mathbb{C}$ is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that X is quasicompact and such that every point $x \in X$ has an open neighborhood $U \subseteq X$ for which $(U, \mathcal{O}_{|U})$ is isomorphic to the ringed space of regular functions on an algebraic set over $\mathbb{C}$.

Definition 2.1.8. [19] An algebraic prevariety $(X, \mathcal{O}_X)$ is *separated* if for every pair of regular maps $\phi_1, \phi_2 \colon Z \twoheadrightarrow X$ with Z an affine algebraic variety, the set $\{z \in Z \mid \phi_1(z) = \phi_2(z)\}$ is closed in Z.

Definition 2.1.9. [19] An *algebraic variety* over $\mathbb{C}$ is a separated algebraic prevariety over $\mathbb{C}$.

We will equip all algebraic varieties, or simply *varieties*, with the Zariski topology, unless otherwise stated. One of the most important examples is affine space. Although the n-dimensional affine space has a vector space structure, we will emphasize its definition as an algebraic variety.

Definition 2.1.10. Let $n > 0$ be a positive integer. We view *affine space* of dimension n over $\mathbb{C}$, denoted $\mathbb{A}^n(\mathbb{C})$ or $\mathbb{A}^n$, as the algebraic set $V(\{0\})$.

We will often be interested in varieties that cannot be decomposed into smaller vanishing loci. We formalize this notion as follows.

Definition 2.1.11. [19] An algebraic variety X is *irreducible* iff it is irreducible as a topological space; i.e., it is nonempty and not the union of two proper closed subsets.

We will also be interested in varieties with the following property.

Definition 2.1.12 ([9] Definition 3.4.3.)**.** A variety V is *complete* if for every variety Z, the projection map $\pi_Z \colon V \times Z \to Z$ is a closed map in the Zariski topology.

We will later use rational functions on a variety to help establish a correspondence between geometric and combinatorial objects.

Definition 2.1.13. [24] Let V be an algebraic subset of $\mathbb{C}^n$. We say $\frac{f}{g}$, where $f, g \in \mathbb{C}[V]$, is a *rational function*. We denote the set of rational functions defined on V by $\mathbb{C}(V)$.

We will need a discussion of divisors later on, and so we introduce some necessary terminology here. In addition, normality of an algebraic variety is a hypothesis that we will need to study the spherical embeddings of a spherical homogeneous space.

Definition 2.1.14. [19] A *normal* ring is an integral domain R that is integrally closed in its field of fractions.

A variety is *normal* if we can locally associate to it normal rings. Formally, we have the following.

Definition 2.1.15. [19] A variety V is *normal* if the stalk $\mathcal{O}_{V,v}$ is a normal ring for all $v \in V$.

One of our most important examples is the following.

Definition 2.1.16. [13] Let V be a vector space over $\mathbb{C}$. We denote by $GL(V)$ the group of automorphisms of V, called the *general linear group*. If V is finite-dimensional, then $GL(V) \cong GL_n(\mathbb{C})$, the set of $n \times n$ invertible matrices over $\mathbb{C}$.

Certain prime divisors of normal varieties will be a key piece of the construction of combinatorial objects for spherical embeddings.

Definition 2.1.17. [19] Assume that X is a normal and irreducible algebraic variety. A *prime divisor* of X is an irreducible subvariety of codimension 1.

2.2 Spherical Geometry

In this section, we give basic definitions necessary for our study of the rings of conditions of spherical varieties. An excellent introduction to spherical geometry can be found in notes on the subject by Brion [5].

Definition 2.2.1. [5] An *algebraic group* is an algebraic variety G equipped with the structure of a group, such that the multiplication map

$$\mu\colon G \times G \to G, \quad (g, h) \mapsto gh$$

and the inverse map

$$\iota\colon G \to G, \quad g \mapsto g^{-1}$$

are morphisms of algebraic varieties.

A natural refinement of this idea is to consider algebraic groups that are also affine varieties.

Definition 2.2.2. [5] A *linear algebraic group* is an algebraic group G that is also a Zariski-closed set in $\mathbb{C}^n$ for some natural number n.

By convention, we say that a linear algebraic group G is *connected* iff it is irreducible as an algebraic variety [20].

The hypothesis that a linear algebraic group G is *reductive* will be of great importance for applying results on the rings of conditions of spherical varieties by De Concini and Procesi in Chapters 4 and 5.

Definition 2.2.3. [20] A linear algebraic group G is *reductive* if it does not contain any closed normal unipotent subgroup.

Common examples of reductive linear algebraic groups are $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$. An example of a group which is not reductive is the Borel subgroup of $GL_n(\mathbb{C})$ of upper triangular matrices.

Definition 2.2.4. [20] Let G be a connected linear algebraic group. A *Borel subgroup* of G is a maximal element under inclusion of the set of closed solvable subgroups of G.

Classic examples of Borel subgroups are the subgroups of upper, respectively, lower triangular matrices in $GL_n(\mathbb{C})$.

Definition 2.2.5. [5] A variety X is *homogeneous* if it is equipped with a transitive action of an algebraic group G. A *homogeneous space* is a pair (X, x), where X is a homogeneous variety, and $x \in X$ is a base point.

A very useful classification of all possible homogeneous spaces is stated by Brion as follows.

Fact 2.2.6. *[5] The homogeneous spaces (X, x) with respect to the action of an algebraic group G are exactly the quotient spaces G/H, where $H := G_x$, the stabilizer of x with respect to the same action of G as defines (X, x) as a homogeneous space. The base point of G/H is the coset H.*

We will need some terminology associated to representations of groups, as follows.

Definition 2.2.7. [13] Let V be a finite-dimensional vector space over $\mathbb{C}$, and let G be a group. A *representation* of G is a homomorphism $\rho\colon G \to GL(V)$ from G to the general linear group of V.

We will require the *characters* of group representations to algebraically generate *cones*, which we will see later are the building blocks of fans for spherical embeddings.

Definition 2.2.8. [25] Let G be a group. The *group of characters* of G, denoted $\mathcal{X}$, is the set $\mathrm{Hom}(G, \mathbb{C}^*)$.

2.3 Spherical Geometry and Luna-Vust Theory

In this section, we define spherical homogeneous spaces, spherical embeddings, and give a very condensed account of the Luna-Vust theory of spherical embeddings. The theory of spherical varieties can be thought of as a non-abelian analogue of toric varieties, and the combinatorial data appearing in the classification results of Luna and Vust are reminiscent of the fans which classify toric varieties. We begin with some combinatorial preliminaries.

In order to construct the combinatorial data of fans for spherical varieties, we will require the *B-semi-invariant rational functions* on their corresponding spherical homogeneous space G/H.

We begin with the following fundamental definition.

Definition 2.3.1. [5] Let G be a connected reductive group over $\mathbb{C}$. Fix a Borel subgroup $B \subseteq G$. Let G/H be a homogeneous space for some closed subgroup $H \subseteq G$. We say G/H is *spherical* if the action of B on G/H has an open dense B-orbit.

Example 2.3.2. *Let $G = SL_2(\mathbb{C})$ and consider the closed subgroup H given by*

$$H \coloneqq \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in SL_2\colon a \in \mathbb{C} \right\}.$$

Let B denote the Borel subgroup of upper triangular matrices in $SL_2(\mathbb{C})$. It is not hard to see that $SL_2(\mathbb{C})$ acts transitively on $\mathbb{A}^2 \setminus \{0\}$ and that the stabilizer of $[1 \ \ 0]^T$ is H. Thus the homogeneous space $SL_2(\mathbb{C})/H$ is isomorphic to $\mathbb{A}^2 \setminus \{0\}$. One can check that there is a dense open B-orbit given by

$$U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{A}^2 \colon y \neq 0 \right\},$$

so $SL_2(\mathbb{C})/H$ is an example of a spherical homogeneous space.

Spherical varieties contain spherical homogeneous spaces as a dense open G-orbit.

Definition 2.3.3 ([18] Introduction)**.** A *spherical embedding* is a normal G-variety X with a G-equivariant open embedding $G/H \hookrightarrow X$. We also refer to X as a *spherical variety*.

For what follows, we let $\mathcal{X}$ denote the group of characters of B.

Definition 2.3.4. [18] Let G/H be a spherical homogeneous space, and let $B \subseteq G$ be a Borel subgroup. A rational function on G/H, $f \in \mathbb{C}(G/H)^*$ is *B-semi-invariant* if there is a $\chi \in \mathcal{X}$ such that, for all $g \in B$, $g \cdot f = \chi(g) f$. We denote the set of such functions $\mathbb{C}(G/H)^{(B)}$.

It is not hard to see that if $f, g \in \mathbb{C}(G/H)^{(B)}$ then $f \cdot g \in \mathbb{C}(G/H)^{(B)}$, so $\mathbb{C}(G/H)^{(B)}$ is a group under (pointwise) multiplication.

Special instances of spherical varieties include toric and flag varieties, as we now explain. Let $G = T_{\mathbb{C}} \cong (\mathbb{C}^*)^n$ be a complex torus. In this case G is abelian and the Borel subgroup B is equal to G, i.e. $B = G = (\mathbb{C}^*)^n$. Let X be a toric variety with torus $G = T_{\mathbb{C}}$. Then, by definition, there exists an open dense $T_{\mathbb{C}}$-orbit in X. Since $B = T_{\mathbb{C}}$, this implies X is a spherical variety.

Example 2.3.5. *Let G be a complex reductive linear algebraic group, and let B denote a Borel subgroup. Consider the flag variety G/B. Then the well-known Bruhat decomposition $G/B = \coprod_{w \in W} B\tilde{w}B$ (where W is the Weyl group of G and $\tilde{w} \in N(T)$ denotes a choice of representative of $w \in W = N(T)/T$) implies that there is an open dense B-orbit in G/B given by the so-called "open Bruhat cell" Bw_0B, where w is the longest element in the Weyl group. Hence G/B is a spherical homogeneous space (and also a spherical variety).*

With these preliminaries in place, we summarize below the Luna-Vust classification of spherical embeddings, as presented by Knop in [18]. Throughout, G will be a connected reductive group and $B \subseteq G$ a Borel subgroup. In what follows, G/H denotes a spherical homogeneous space with open dense B-orbit Bx_0.

The aim of the Luna-Vust classification is to determine combinatorial data associated to G/H uniquely identifying the spherical embeddings for a fixed G/H. It is easiest to begin the classification with *simple* spherical embeddings, defined as follows.

Definition 2.3.6 ([18] §2)**.** A spherical embedding $G/H \hookrightarrow X$ is *simple* if X contains exactly one closed G-orbit.

Example 2.3.7. *An example of a simple spherical embedding is the inclusion $\mathbb{A}^2 \setminus \{0\} \hookrightarrow \mathbb{A}^2$. This is because the only closed G-orbit that X contains is $\{0\}$.*

To state the classification result of Luna and Vust, we require some terminology. We begin with the combinatorial objects.

Cones form a general class of combinatorial objects that have independent use in areas such as convex geometry and optimization. In the context of algebraic geometry, we translate information about algebraic varieties into information about their combinatorial counterparts, which is often more convenient to work with.

For the following, let V denote a vector space over $\mathbb{Q}$.

Definition 2.3.8. [18] A *cone* is a subset $\mathcal{C}$ of V which is closed under addition and multiplication by $\mathbb{Q}^+ := \{q \in \mathbb{Q} \,|\, q \geq 0\}$.

For the classification results of Luna and Vust and for our construction of fans for specific spherical embeddings, we will require that cones be *strictly convex*, in the following sense.

Definition 2.3.9. [18] A cone $\mathcal{C}$ is *strictly convex* if it does not contain a nontrivial linear subspace; i.e., $\mathcal{C} \cap (-\mathcal{C}) = 0$.

Definition 2.3.10. [18] The *relative interior* $\mathcal{C}^\circ$ of a cone C, is $\mathcal{C}$ with all proper faces removed.

We will also use some algebraic data, as follows.

Definition 2.3.11. [18] A *valuation* of a normal variety X is a map $\nu\colon \mathbb{C}(X)^* = \mathbb{C}(X) \setminus \{0\} \to \mathbb{Q}$ satisfying
(1) $\nu(f_1 + f_2) \geq \min\{\nu(f_1), \nu(f_2)\}$ whenever $f_1, f_2, (f_1 + f_2) \in \mathbb{C}(X)^*$,
(2) $\nu(f_2 f_2) = \nu(f_1) + \nu(f_2)$ for all $f_1, f_2 \in \mathbb{C}(X)^*$, and
(3) $\nu(\mathbb{C}^*) = 0$.

In particular, we are interested in the *G-invariant* valuations of a G-variety X.

Definition 2.3.12. [18] A valuation ν of a G-variety X is *G-invariant* if $\nu(g \cdot f) = \nu(f)$ for all $g \in G$ and any $f \in \mathbb{C}(X)^*$. We denote the set of G-invariant valuations on X by $\mathcal{V}(X)$.

In the theory of fans associated to toric varieties, one of the natural ingredients is the lattice associated to the torus. In the spherical setting, we use the lattice of B-semi-invariant functions. In fact, we can define a homomorphism from the multiplicative group $\mathbb{C}(G/H)^{(B)}$ to the group $\mathcal{X}$ of characters of B by

$$f \in \mathbb{C}(G/H)^{(B)} \mapsto \chi_f \quad (2.3.12)$$

where χ_f is the (unique) character in $\mathcal{X}$ satisfying $g \cdot f = \chi_f(g) f$, assumed to exist in Definition 2.2.4. It is straightforward to see that (2.3.12.) is a homomorphism of groups.

Definition 2.3.13 ([18] §1 discussion). Fix a spherical homogeneous space G/H. We let $\Lambda = \Lambda(G/H)$ denote the imagine in $\mathcal{X}$ of the homomorphism $\mathbb{C}(G/H)^{(B)} \to \mathcal{X}$ defined by (2.3.12). Since Λ is a subgroup of a lattice, it is a free, finitely generated abelian group.

We have the following simple lemma.

Lemma 2.3.14. *The homomorphism defined by $f \mapsto \chi_f$ induces an isomorphism*

$$\mathbb{C}(G/H)^{(B)} \setminus \{0\}/\mathbb{C}^* \cong \Lambda.$$

Proof. Suppose $f_1, f_2 \in \mathbb{C}(G/H)^{(B)} \setminus \{0\}$ have the same image, i.e. $\chi_{f_1} = \chi_{f_2}$. Then it follows that $\frac{f_1}{f_2} \in \mathbb{C}(G/H)^{(B)} \setminus \{0\}$ has associated character the identity, i.e.

$$g \cdot (\frac{f_1}{f_2}) = \frac{\chi_{f_1}(g)}{\chi_{f_2}(g)}(\frac{f_1}{f_2}) = \frac{f_1}{f_2}$$

In other words, $\frac{f_1}{f_2}$ is B-invariant, i.e. it is constant on B-orbits. Since G/H is spherical, there is an open dense B-orbit in G/H, so in fact $\frac{f_1}{f_2}$ is constant on all of G/H. Hence $f_1 = \lambda f_2$ for some constant $\lambda \neq 0 \in \mathbb{C}$. □

The *rank* of a spherical homogeneous space is of central importance to us in studying the ring of conditions of spherical varieties. In particular, the structure of rank 1 spherical varieties is relatively well understood, as we discuss in Chapter 5.

Definition 2.3.15. [18] The *rank* of a spherical homogeneous space G/H is the rank of $\Lambda = \Lambda(G/H)$. Similarly, if $G/H \hookrightarrow X$ is a spherical embedding, the *rank* of the spherical variety X is the rank of G/H.

Definition 2.3.16 ([18] §1 discussion). Let G/H be a spherical homogeneous space and Λ defined as above. Then we define $\mathcal{Q} = \mathcal{Q}(G/H) \coloneqq \mathrm{Hom}(\Lambda, \mathbb{Q})$.

We now construct a map from the set of G-invariant valuations $\mathcal{V}$ to $\mathcal{Q} = \mathrm{Hom}(\Lambda, \mathbb{Q})$. This will allow us to think of $\mathcal{V}$ as a subset of the vector space $\mathcal{Q}$. More precisely, suppose $\nu \in \mathcal{V}$ is a G-invariant valuation on $\mathbb{C}(G/H)^*$. Restricting to the B-semi-invariants, we immediately obtain a map

$$\nu|_{\mathbb{C}(G/H)^{(B)} \setminus \{0\}} : \mathbb{C}(G/H)^{(B)} \setminus \{0\} \to \mathbb{Q}.$$

By the assumption (2) in Definition 2.3.8., the above is a homomorphism of additive groups, and by assumption (3) in the same definition, it induces an additive homomorphism

$$\varrho_\nu \colon \mathbb{C}(G/H)^{(B)} \setminus \{0\}/\mathbb{C}^* \cong \Lambda \to \mathbb{Q}$$

Thus $\varrho_\nu \in \mathcal{Q}$ as desired. This correspondence defines a map

$$\varrho \colon \mathcal{V} \to \mathcal{Q}, \quad \nu \mapsto \varrho_\nu. \quad (2.3.15)$$

The following fact says that $\varrho\colon \mathcal{V} \to \mathcal{Q}$ as defined above is an embedding.

Fact 2.3.17 ([18] Corollary 1.8.). *Let X be a G-spherical variety with dense open G-orbit $G/H \hookrightarrow X$. Then the map $\mathcal{V}(X) \to \mathcal{Q}(G/H)$ defined by $\nu \mapsto \varrho_\nu$ is injective.*

We next focus attention on certain divisors of G/H. We say a subset Z of G/H is B-stable if $B \cdot Z \subseteq Z$.

Definition 2.3.18. [25] The *palette* of G/H, denoted $\mathcal{D}(G/H)$, is the set of B-stable prime divisors of G/H. An element $D \in \mathcal{D}(G/H)$ is called a *colour*. When the homogeneous space G/H is understood, we denote $\mathcal{D}(G/H)$ by $\mathcal{D}$.

Given a prime divisor $D \in \mathcal{D}(G/H)$, let ν_D denote the standard valuation on $\mathbb{C}(G/H)^*$ associated to D, namely $\nu_D(f) \coloneqq$ the order of vanishing of f along the divisor D.

Now let $Y \subseteq X$ be a G-orbit associated to a spherical embedding $G/H \hookrightarrow X$. Associated to Y we may define several sets, as follows.

$$\begin{aligned}
\mathcal{D}(X) &\coloneqq \{B\text{-stable prime divisors of } X\},\\
\mathcal{D}_Y(X) &\coloneqq \{B\text{-stable prime divisors of } X \text{ which contain } Y\},\\
\mathcal{B}_Y(X) &\coloneqq \{\nu_D \in \mathcal{V}(X) \mid D \in \mathcal{D}_Y(X) \text{ is } G\text{-stable}\}, \text{ and}\\
\mathcal{F}_Y(X) &\coloneqq \{D \cap G/H \mid D \in \mathcal{D}_Y(X) \text{ is not } G\text{-stable}\}.
\end{aligned}$$

When X is a simple embedding and Y its unique closed G-orbit, we sometimes omit the subscript $_Y$. The following is an important first step in the classification of spherical embeddings.

Fact 2.3.19 ([18] Theorem 2.3.). *A simple G/H-embedding X is uniquely determined by the pair $(\mathcal{B}(X), \mathcal{F}(X))$.*

Motivated by the above, now we wish to describe such pairs $(\mathcal{B}(X), \mathcal{F}(X))$ combinatorially, using the notions we have recently introduced. We first introduce a certain cone in $\mathcal{Q}=\mathrm{Hom}(\Lambda, \mathbb{Q})$.

Definition 2.3.20. [18] Let X be a G-spherical variety, and let Y be a closed G-orbit of X. Then we define the cone $\mathcal{C}_Y(X) \subseteq \mathcal{Q}$ as follows:

$$\mathcal{C}_Y(X) = \text{the cone generated by } \{\varrho(\nu_D) \mid D \in \mathcal{F}_Y(X)\} \text{ and } \{\varrho(\nu_D) \mid \nu_D \in \mathcal{B}_Y(X)\}.$$

We denote by $\mathcal{C}^c_Y(X)$ the pair $(\mathcal{C}_Y(X), \mathcal{F}_Y(X))$.

In the case when $G/H \hookrightarrow X$ is a simple embedding and so there is a unique closed G-orbit Y, we sometimes write $\mathcal{C}(X) = \mathcal{C}_Y(X)$ and $\mathcal{C}^c(X) = \mathcal{C}^c_Y(X)$.

These pairs $\mathcal{C}^c_Y(X) = (\mathcal{C}_Y(X), \mathcal{F}_Y(X))$ are instances of the following combinatorial objects.

Definition 2.3.21. [18] A *strictly convex coloured cone, or SCCC*, is a pair $(\mathcal{C}, \mathcal{F})$, where $\mathcal{C} \subseteq \mathcal{Q}$ and $\mathcal{F} \subseteq \mathcal{D}$, satisfying
(1) $\mathcal{C}$ is a strictly convex cone.
(2) $\mathcal{C}$ is generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V}$.
(3) The relative interior of $\mathcal{C}$ intersects $\mathcal{V}$ nontrivially.
(4) The set $\rho(\mathcal{F})$ does not contain 0.

Fact 2.3.22 ([18] Theorem 3.1 proof)**.** *The pair $\mathcal{C}^c_Y(X)$ satisfies all the conditions required to be a SCCC.*

The next results imply that, in the case of simple embeddings, $\mathcal{B}(X)$ can be recovered from $\mathcal{C}(X) := \mathcal{C}_Y(X)$ and $\mathcal{F}(X)$.

Fact 2.3.23 ([18] §2)**.** *Each $\nu_D \in \mathcal{B}(X)$ is uniquely determined by the corresponding ray $\mathbb{Q}^+\nu_D$ in $\mathcal{C}(X)$.*

Fact 2.3.24 ([18] Lemma 2.4.)**.** *The rays $\mathbb{Q}^+\nu_D$ with $\nu_D \in \mathcal{B}(X)$ are exactly the extremal rays of $\mathcal{C}(X)$ which do not contain an element of $\rho(\mathcal{F}(X))$.*

We can now state the classification of simple spherical embeddings in terms of the combinatorial data of strictly convex coloured cones.

Fact 2.3.25 ([18] Lemma 2.4.)**.** *The correspondence $X \mapsto \mathcal{C}^c(X)$ is a bijection between isomorphism classes of simple embeddings and strictly convex colored cones.*

We now wish to state the classification for the general case.

Definition 2.3.26 ([18] §3.). A pair $(\mathcal{C}_0, \mathcal{F}_0)$ is called a *face* of the coloured cone $(\mathcal{C}, \mathcal{F})$ if $\mathcal{C}_0$ is a face of $\mathcal{C}$, $\mathcal{C}_0^\circ \cap \mathcal{V} \neq \emptyset$, and $\mathcal{F}_0 = \mathcal{F} \cap \rho^{-1}(\mathcal{C}_0)$.

The idea is that we can glue SCCC's along faces in such a way that the gluing process preserves the geometric data.

We have the following.

Definition 2.3.27. A [18] *coloured fan* $\mathfrak{F}$ is a (non-empty) collection of coloured cones $(\mathcal{C}, \mathfrak{F})$ such that:

(1) Every face of a cone in $\mathfrak{F}$ is also in $\mathfrak{F}$.

(2) Any element $\nu \in \mathcal{V}$ lies in the interior of at most one cone.

Let G/H be a spherical homogeneous space and $G/H \hookrightarrow X$ a spherical embedding. For such an embedding X, denote $\mathfrak{F}(X) := \{\mathcal{C}^c_Y(X) \,|\, Y \subseteq X \text{ is a closed } G\text{-orbit}\}$.

We are now ready to state the main classification result of Luna and Vust.

Fact 2.3.28 ([18] Theorem 3.3.). *The map $X \mapsto \mathfrak{F}(X)$ induces a bijection between isomorphism classes of spherical embeddings for a fixed G/H and strictly convex coloured fans.*

We conclude this section by summarizing the procedure for computing the fan $\mathfrak{F}(X)$ corresponding to a spherical embedding $G/H \hookrightarrow X$.

1. Enumerate G-orbits of X induced by this choice of embedding. (Each G-orbit corresponds to a cone in $\mathcal{Q}$.)

2. Compute the group $\mathcal{X}$ of characters of B.

3. Compute the set $\mathbb{C}(G/H)^{(B)}$ of B-semi-invariant rational functions on G/H.

4. Compute the image $\Lambda \subseteq \mathcal{X}$ of the homomorphism $\mathbb{C}(G/H)^{(B)} \to \mathcal{X}$ defined by $f \mapsto \chi_f$.

5. Compute $\mathcal{V}(G/H)$, the set of G-invariant valuations of G/H.

6. Compute the image of $\mathcal{V}(G/H)$ in $\mathcal{Q} :=$Hom$(\Lambda, \mathbb{Q})$ under the map $\nu \mapsto \varrho_\nu$.

7. For each G-orbit Y of X, find all B-stable prime divisors $D_1, D_2, ..., D_\ell$ containing Y. This set is $\mathcal{D}_Y(X)$.

8. Compute $\mathcal{C}_Y(X) =$ the cone generated by $\{\varrho(\nu_{D_1}), ..., \varrho(\nu_{D_\ell})\}$ in $\mathcal{Q}$.

9. Among the B-stable prime divisors $D_1, D_2, ..., D_\ell$ containing Y, determine for each D_j whether D_j is also G-stable. Thus, compute $\mathcal{F}_Y(X) = \{D \in \mathcal{D}_Y(X) \,|\, D$ is not G-stable$\}$. Let $\mathcal{C}^c_Y(X) = (\mathcal{C}_Y(X), \mathcal{F}_Y(X))$.

10. Repeat steps 1. - 9. for each closed G-orbit, and glue the cones together to produce the fan corresponding to X.

This procedure always terminates due to the fact that, for a given spherical homogeneous space G/H, any spherical embedding of it contains only finitely many G-orbits [[7] Abstract].

Chapter 3

The Example of $\mathbb{A}^n \setminus \{0\}$

In this chapter, we illustrate the Luna-Vust theory recounted in Chapter 2 for the specific case of the affine space minus the origin, $\mathbb{A}^n \setminus \{0\}$. The simplest case $n = 2$ is described first, and we then generalize to $n \geq 2$.

3.1 The Spherical Homogeneous Space $\mathbb{A}^2 \setminus \{0\}$

As a warm-up, we compute some of the combinatorial data described in Chapter 2 which are associated to the spherical homogeneous space $\mathbb{A}^2 \setminus \{0\}$. Let $G = SL_2(\mathbb{C})$, and let $B \subseteq SL_2$ be the Borel subgroup consisting of upper triangular matrices. Define the subgroup $H \subseteq G$ as

$$H := \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in SL_2 \colon a \in \mathbb{C} \right\}.$$

The action of B on G/H is given by left multiplication. It is not hard to see that $SL_2(\mathbb{C})/H$ can be identified with $\mathbb{A}^2 \setminus \{0\}$, on which $SL_2(\mathbb{C})$ acts transitively, with $\mathrm{Stab}([1\ 0]^T) = H$. There are two B-orbits of $\mathbb{A}^2 \setminus \{0\}$ under this action:

$$U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{A}^2 \colon y \neq 0 \right\}, \quad D = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \colon x \neq 0 \right\}$$

The orbit U is open and dense in $\mathbb{A}^2 \setminus \{0\}$, and thus $\mathbb{A}^2 \setminus \{0\}$ is spherical with respect to this action. Note that D is closed. Since it is equal to the set of points that satisfy the equation $y = 0$, it is a prime divisor. A brief computation shows that D is also B-stable. In fact, it is the only B-stable prime divisor of $\mathbb{A}^2 \setminus \{0\}$. Therefore, when we construct the fans associated to the spherical embeddings of $\mathbb{A}^2 \setminus \{0\}$, each fan will contain at most one colour.

Let $\mathcal{X}$ denote the group of characters of B, which is isomorphic to $\mathbb{Z}$ by the map

$$n \in \mathbb{Z} \mapsto (\chi_n \colon B \to \mathbb{C}^*), \quad \chi_n \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) = a^n.$$

The field of rational functions on $\mathbb{A}^2 \setminus \{0\}$ is $\mathbb{C}(x, y)$. The B-semi-invariant rational functions on $\mathbb{A}^2 \setminus \{0\}$ are those of the form y^n for $n \in \mathbb{Z}$, up to multiplication by scalars. The B-character associated to y^n is χ_n. Thus $\Lambda = \mathcal{X}$ and is generated by y, and $\mathcal{Q} = \operatorname{Hom}(\Lambda, \mathbb{Q})$ is isomorphic to $\mathbb{Q}$, which is spanned by

$$\chi^* \colon \Lambda \to \mathbb{Q}, \ \ \chi_1^*(y) = 1.$$

We now wish to compute the set of G-invariant valuations, $\mathcal{V}$ and its image in $\mathcal{Q}$. Consider the valuation $\nu \colon \mathbb{C}(\mathbb{A}^2 \setminus \{0\})^* \to \mathbb{Q}$ that sends $f \in \mathbb{C}(\mathbb{A}^2 \setminus \{0\})^*$ to its total degree. That is, if we write $f = \frac{f_1(x,y)}{f_2(x,y)}$ for polynomials f_1, f_2, then

$$\nu(f) = mindeg\, f_1 - mindeg\, f_2,$$

where $mindeg\, f$ denotes the minimum among the degrees of monomials which appear in f. From this definition it follows that $\nu(y) = 1$, so $\varrho(\nu)$ is identified with χ_1^*. Another G-invariant valuation ν' can be defined by

$$\nu'(f) = deg\, f_2 - deg\, f_1$$

which satisfies $\nu'(y) = -1$, which means $\varrho(\nu') = -\chi_1^*$.

Hence we may conclude that the image of the map $\mathcal{V} \to \mathcal{Q}$ given by $\nu \mapsto \varrho_\nu$ is all of $\mathcal{Q}$, i.e., the valuation cone is all of $\mathcal{Q} = \operatorname{Hom}(\Lambda, \mathbb{Q}) \cong \mathcal{Q}$.

Informally, the idea is that the cone $\mathcal{R}$ adds to $\mathbb{A}^2 \setminus \{0\}$ "limit points near the origin", and $-\mathcal{R}$ adds to $\mathbb{A}^2 \setminus \{0\}$ "limit points near infinity". This is explained in more detail in the next section. The coloured fans for $\mathbb{A}^2 \setminus \{0\}$ are listed in Table 1 in Section 3.3 after our discussion of the general-n case.

3.2 The Case of $\mathbb{A}^n \setminus \{0\}$

Let $G \coloneqq SL_n(\mathbb{C})$, or simply SL_n, and let $B \subseteq G$ denote the Borel subgroup of upper triangular matrices in SL_n. Treating points in $\mathbb{A}^n$ as column vectors, both SL_n and B act on $\mathbb{A}^n$ and $\mathbb{A}^n \setminus \{0\}$, respectively, by left matrix multiplication.

Claim 3.2.1. *The action of SL_n on $\mathbb{A}^n \setminus \{0\}$ defined by $(g, z) \mapsto g \cdot z$ is transitive.*

Proof. It suffices to show that for each $w = [w_1\, w_2\, \cdots w_n]^T \in \mathbb{A}^n \setminus \{0\}$ there exists $g \in SL_n$ such that $g \cdot e_1 = w$, where e_1 denotes the first standard basis vector in $\mathbb{A}^n$.
Let $w = [w_1\, w_2\, \cdots w_n]^T \in \mathbb{A}^n \setminus \{0\}$. Certainly it is the case that we can choose the first column of g to be w itself. Now the question is whether we can complete the set of columns of g to a basis. This follows from the "Building Up" lemma of linear algebra. □

Consider $H \coloneqq Stab_{SL_n}(e_n)$, where e_n denotes the n^{th} standard basis vector. We have that

$$H = \left\{ \left[\begin{array}{c|c} C & 0 \\ \hline * & 1 \end{array} \right] \mid C \in SL_{n-1}(\mathbb{C}) \right\}$$

where $*$ represents a $1 \times (n-1)$ vector whose entries are arbitrary. Note that H is a closed subgroup of G, since it is a stabilizer subgroup [5]. Since SL_n is a linear algebraic group, by Corollary 2.2.6., $SL_n/H \cong \mathbb{A}^n \setminus \{0\}$ is a homogeneous space.

Claim 3.2.2. *The homogeneous space SL_n/H is spherical with respect to the action of SL_n by left matrix multiplication.*

Proof. We need to show that $SL_n/H \cong \mathbb{A}^n \setminus \{0\}$ has an open dense B-orbit. To see this, it would suffice to show that $\{x_n \neq 0\}$, since $\{x_n \neq 0\} = B \cdot e_n$. For the first inclusion, $B \cdot e_n \subseteq \{x_n \neq 0\}$ since, for all $b \in B$, $b = [b_{ij}]$ has the property $b_{ij} = 0$ if $i > j$. Moreover, $b_{nn} \neq 0$ since B is invertible. Conversely, $\{x_n \neq 0\} \subseteq B \cdot e_n$, since, given any column vector $x = [x_1 \; x_2 \; \cdots \; x_n]^T$ with $x_n \neq 0$, there exists $b \in B$ whose n^{th} column is x and such that $B \cdot e_n = x$.

□

Next, we compute the valuation cone associated to $\mathbb{A}^n \setminus \{0\}$. Toward this, we determine the B-semi-invariant rational functions on G/H. Consider the character group of B,

$$\mathfrak{X} := \mathrm{Hom}(B, \mathbb{C}^\times) = \left\{ \begin{bmatrix} b_{11} & b_{22} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \mapsto b_{11}^{\alpha_1} b_{22}^{\alpha_2} \cdots b_{nn}^{\alpha_n} \;\middle|\; \alpha_i \in \mathbb{Z} \right\} \cong \mathbb{Z}^{n-1}$$

As discussed in Chapter 2, another ingredient of the fans is the set of B-stable prime divisors of SL_n/H. In the case of $SL_n/H = \mathbb{A}^n \setminus \{0\}$, the complement of the open dense B-orbit $B \cdot e_n$ is $D := \{x_n = 0\}$.

We first check that $\{x_n = 0\}$ is a prime divisor of G/H. It suffices to show that its intersection with an open dense subset U of G/H is irreducible of codimension 1. Define $U := \{x_1 \neq 0\} \cong \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^{n-1}$. Then $D \cap U = \{x_1 \neq 0, \, x_n = 0\}$. Consider the coordinate ring $\mathbb{C}[U] = \mathbb{C}[x_0, x_1, ..., x_1^{-1}]$. The defining ideal of $D \cap U$ in U is $I(D \cap U) = \langle x_n \rangle$, which is a prime ideal in $\mathbb{C}[U]$. Thus the closure of D is irreducible in $\mathbb{A}^n \setminus \{0\}$. Therefore D is irreducible of codimension 1 in $\mathbb{A}^n \setminus \{0\}$. Moreover, D is B-stable by the following computation. Let $x = [x_1 \; x_2 \; \cdots \; x_{n-1} \; 0]^T \in D$, and let $b \in B$. Then

$$B \cdot x = \begin{bmatrix} * & * & \cdots & * \\ & * & \cdots & \vdots \\ \mathbf{0} & & \cdots & * \\ & & & * \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} \in \{x_n = 0\}$$

Since $\mathbb{A}^n \setminus \{0\} = (B \cdot e_n) \sqcup D$, the set $\mathcal{D}(\mathbb{A}^n \setminus \{0\})$ of B-stable prime divisors consists of the single element D, i.e. $\mathcal{D}(\mathbb{A}^n \setminus \{0\}) = \{D\}$. In particular, our coloured cones in this case can have at most one colour.

We now wish to enumerate the possible spherical embeddings $G/H \hookrightarrow X$ and construct the corresponding fan for each X. It turns out that these spherical embeddings consist exactly of the following: $\mathbb{A}^n \setminus \{0\}$, $\mathbb{A}^n$, $Bl_0(\mathbb{A}^n)$, $\mathbb{P}^n \setminus \{0\}$, $\mathbb{P}^n$, and $Bl_0(\mathbb{P}^n)$. In Definition 5.2.19, we formally define the blowup construction, of which $Bl_0(\mathbb{A}^n)$ and $Bl_0(\mathbb{P}^n)$ are examples.

To proceed with the enumeration, we determine the B-semi-invariant functions on $\mathbb{A}^n \setminus \{0\}$. We have the following.

Claim 3.2.3. $\mathbb{C}(\mathbb{A}^n \setminus \{0\})^{(B)}$ *=span*$\{z_n^k \mid k \in \mathbb{Z}\}$, *where* z_n *denotes the* n^{th} *coordinate on* $\mathbb{A}^n$.

Proof. Let T denote the maximal torus in B. Any rational function $\frac{f(z_1,\ldots,z_n)}{g(z_1,\ldots,z_n)}$ which is B-semi-invariant must be T-semi-invariant, but the only such are the Laurent monomials $f = z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n}$ where $\beta_i \in \mathbb{Z}$ for all i. Now suppose such a monomial f is B-semi-invariant. Concretely, this means that for all $b = [b_{ij}] \in B$ such that $b_{ij} = 0$ for $i > j$, we have

$$(b_{11}z_1 + b_{12}z_2 + \cdots + b_{1n})^{\beta_1}(b_{22}z_2 + b_{23}z_3 + \cdots + b_{2n})^{\beta_2} \cdots (b_{nn^{-1}}z_n)^{\beta_n}) = \chi_f(b) z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n}.$$

Such an equality can only hold for all $b \in B$ if $\beta_1 = \beta_2 = \cdots = \beta_{n-1} = 0$, i.e. $f = z_n^{\beta_n}$, $\beta_n \in \mathbb{Z}$. Moreoever, in this case we see that $z_n^{\beta_n}$ is indeed B-semi-invariant, with $\chi_f(b) = b_{nn}^{-\beta_n}$. The claim follows. $\square$

The proof of the claim in fact shows that the lattice Λ =image$(\mathbb{C}(\mathbb{A}^n \setminus \{0\})^{(B)}/\mathbb{C}^*) \subseteq \mathcal{X}$ is in fact the $\mathbb{Z}$-span of the character χ_n, where $\chi_n(b) = b_{nn}$. In particular, $\Lambda \cong \mathbb{Z}$ and we conclude that rank$(\mathbb{A}^n \setminus \{0\}) = 1$.

Next, we compute the valuation cone (i.e. the image of $\mathcal{V} = \mathcal{V}(\mathbb{A}^n \setminus \{0\})$ in $\mathcal{Q}$). As in the case of $\mathbb{A}^2 \setminus \{0\}$ in the previous section there are two G-invariant valuations

$$\nu(\frac{f}{g}) = mindeg\, f - mindeg\, g$$

$$\nu'(\frac{f}{g}) = deg\, g - deg\, f$$

and these map to $+\chi_n^*$ and $-\chi_n^*$, respectively. Hence we conclude that the valuation cone is $\mathcal{Q}$, as in the $n = 2$ case. Using the above data for $\mathbb{A}^n \setminus \{0\}$, we now give the classification of the six possible spherical embeddings $\mathbb{A}^n \setminus \{0\} \hookrightarrow X$, and in addition we explicitly describe each of the embeddings corresponding to the possible coloured cones.

We first classify the possible coloured cones. First, since $\mathcal{Q} \cong \mathbb{Q}$ in this case, the only possible 1-dimensional cones are the half-rays $\mathcal{R}$ and $-\mathcal{R}$. Moreover, as noted above, the only possible colour is $D = \{z_n = 0\}$ in $\mathbb{A}^n \setminus \{0\}$, which has associated valuation ν_D with $\varrho_{\nu_D} = \chi_n^*$. Thus the only coloured cone that can possibly have a non-empty set of associated colours is $\mathcal{R}$. These considerations imply that the possible coloured cones for spherical embeddings of $\mathbb{A}^n \setminus \{0\}$ are the six which are pictured in the rightmost column in Table 1.

We claim that these six coloured cones correspond to the spherical varieties $\mathbb{A}^n \setminus \{0\}$, $\mathbb{A}^n$, $Bl_0(\mathbb{A}^n)$, $\mathbb{P}^n \setminus \{0\}$, $\mathbb{P}^n$, and $Bl_0(\mathbb{P}^n)$. To see this, we now compute explicitly the coloured cone associated to each of the above six embeddings following the method outlined in Chapter 2. We start with the spherical embedding $\mathbb{A}^n \setminus \{0\} \hookrightarrow \mathbb{A}^n \setminus \{0\} = X$ itself. The only G-orbit is $\mathbb{A}^n \setminus \{0\}$. Moreover, there are no B-stable divisors containing the whole G-orbit, so $\mathcal{D}_Y(X) = \emptyset$ and hence $\mathcal{B}_Y(X) = \mathcal{F}_Y(X) = \emptyset$, also. So we obtain $(0, \emptyset)$ as the coloured cone for $\mathbb{A}^n \setminus \{0\}$. Informally, this is because we have added nothing to our original space in considering $\mathbb{A}^n \setminus \{0\}$ as a spherical variety over itself.

Next we compute the coloured fan corresponding to the spherical embedding $\mathbb{A}^n \setminus \{0\} \hookrightarrow \mathbb{A}^n = X$. The unique closed G-orbit is $Y := \{(0, 0, ..., 0)\} = \{0\}$. The unique B-stable prime divisor is $\overline{D} = \{z_n = 0\}$, which contains Y. Thus $\mathcal{D}_Y(\mathbb{A}^n) = \{\overline{D}\}$. Since $\overline{D}$ is not G-stable, $\mathcal{B}_Y(X) = \emptyset$ and $\mathcal{F}_Y(X) = \{D\}$. It is not hard to see that ϱ_{ν_D} is χ_n^*. Thus

$\mathcal{C}_Y(X) = \mathcal{R} \subseteq \mathcal{Q}$, where $\mathcal{R} \subseteq \mathcal{Q}$ denotes the cone generated by χ_n^*. Thus the coloured cone associated to $\mathbb{A}^n$ is $(\mathcal{R}, \mathcal{D})$, which corresponds to adding a limit point at the origin to $\mathbb{A}^n \setminus \{0\}$.

Now consider the embedding $\mathbb{A}^n \setminus \{0\} \hookrightarrow \mathbb{P}^n \setminus \{0\} = X$, where we view points in $\mathbb{P}^n$ as having homogeneous coordinates $[z_1 \colon z_2 \colon \cdots \colon z_n \colon w]$ and the embedding is given by $(z_1, ..., z_n) \mapsto [z_1 \colon z_2 \colon \cdots \colon z_n \colon 1]$. Then $Y = \{w = 0\}$ is the unique closed G-orbit. Note also that Y is itself a B-stable prime divisor. Hence $\mathcal{D}_Y(X) = \{Y\}$ and $\mathcal{B}_Y(X) = \{\nu_Y\}$ and $\mathcal{F}_Y(X) = \emptyset$ since Y is G-stable. Thus the cone for $\mathbb{P}^n \setminus \{0\}$ is $(-\mathcal{R}, \emptyset)$.

For the spherical embedding $\mathbb{A}^n \setminus \{0\} \hookrightarrow \mathbb{P}^n = X$, the two closed G-orbits are $\{w = 0\}$ and $\{[0 \colon 0 \colon \cdots \colon 1]\}$. The cones corresponding to $\{w = 0\}$ and $\{[0 \colon 0 \colon \cdots \colon 1]\}$ can be computed by the same reasoning as for the cases of $\mathbb{P}^n \setminus \{0\}$ and $\mathbb{A}^n$, given above. Thus the two coloured cones in the coloured fan of $\mathbb{P}^n$ are $(\mathcal{R}, \mathcal{D})$ and $(-\mathcal{R}, \emptyset)$.

Next consider the spherical embedding $\mathbb{A}^n \setminus \{0\} \hookrightarrow Bl_0(\mathbb{A}^n) = X$ given by identifying $\mathbb{A}^n \setminus \{0\}$ with the complement in $Bl_0(\mathbb{A}^n)$ of the exceptional divisor E at the origin. Here, the unique closed G-orbit is $Y = E$, which is also a B-stable prime divisor. Thus $\mathcal{D}_E(X) = \{E\}$, $\mathcal{B}_E(X) = \{\nu_E\}$, and $\mathcal{F}_Y(X) = \emptyset$ in this case. Since $\varrho(\nu_E) = \chi_1^*$, the cone for $Bl_0(\mathbb{A}^n)$ is $(\mathcal{R}, \emptyset)$.

Lastly, for the spherical embedding $\mathbb{A}^n \setminus \{0\} \hookrightarrow Bl_0(\mathbb{P}^n) = X$, the closed G-orbits are $\{w = 0\}$ and E. Using the same reasoning as above, the coloured cones for $Bl_0(\mathbb{P}^n)$ are $(\mathcal{R}, \emptyset)$ and $(-\mathcal{R}, \emptyset)$.

Table 1. Spherical embeddings for the homogeneous space $\mathbb{A}^n \setminus \{0\}$

Variety	**Closed G-orbits**	**Coloured cones**	**Coloured fan**
$\mathbb{A}^n \setminus \{0\}$	$\mathbb{A}^n \setminus \{0\}$	$(0, \emptyset)$	
$\mathbb{A}^n$	$\{0\}$	$(\mathcal{R}, \mathcal{D})$	
$\mathrm{Bl}_0\mathbb{A}^n$	E	$(\mathcal{R}, \emptyset)$	
$\mathbb{P}^n \setminus \{0\}$	$\{w = 0\}$	$(-\mathcal{R}, \emptyset)$	
$\mathbb{P}^n$	$\{w = 0\}$, $\{0\}$	$(\mathcal{R}, \mathcal{D})$, $(-\mathcal{R}, \emptyset)$	
$\mathrm{Bl}_0\mathbb{P}^n$	$\{w = 0\}$, E	$(\mathcal{R}, \emptyset)$, $(-\mathcal{R}, \emptyset)$	

Chapter 4

The Ring of Conditions

In this chapter, we introduce the ring of conditions of spherical varieties. We review the Kleiman Transversality Theorem and introduce basic geometric notions underlying the definition of the ring of conditions. We also briefly discuss foundational work of De Concini and Procesi [11] related to the ring of conditions for symmetric varieties. (In [10] De Concini remarks that their arguments proceed verbatim for spherical varieties.)

4.1 The Ring of Conditions

In this section we define the ring of conditions. We need some preliminaries. Recall that the dimension of an irreducible algebraic variety X over $\mathbb{C}$ equals $\mathrm{trdeg}_{\mathbb{C}}\mathbb{C}(X)$, the transcendence degree of $\mathbb{C}(X)$ over $\mathbb{C}$ [19].

Definition 4.1.1. [16] Two irreducible algebraic subvarieties Y, Z of an algebraic variety X intersect *properly* if either $Y \cap Z = \emptyset$ or each component of $Y \cap Z$ has dimension $\dim(Y) + \dim(Z) - \dim(X)$.

Definition 4.1.2. [24] An irreducible algebraic variety X is *regular at a point* p if the local ring $\mathcal{O}_{X,p}$ is regular. If X is regular at each point $p \in X$, then we say that X *is regular*.

Definition 4.1.3. [16] Two irreducible algebraic subvarieties Y, Z of an algebraic variety X intersect *transversally* if $Y \cap Z$ is regular and each irreducible component has dimension $\dim(Y) + \dim(Z) - \dim(X)$.

We are interested in whether we can move an algebraic subvariety by translation to make a non-transversal intersection transversal. We need the following.

Definition 4.1.4. Let Y be an algebraic subvariety of a spherical homogeneous space X with respect to the action of an algebraic group G. A *general translate* gY of Y is the translate of Y by an element $g \in G$ on Y, where g is allowed to range over an open dense subset of G.

Let G be a connected algebraic group and X a homogeneous space with respect to G. Let Y, Z be two subvarieties of X. Then there exists a general translate gY of Y such that $gY \cap Z$ is proper, and also transversal if Y, Z are smooth. Formally, we have the following well-known result of Kleiman:

Theorem 4.1.5. *[16] Let G be a connected algebraic group, and let X be a homogeneous space with respect to G. Let Y, Z be algebraic subvarieties of X. Then:*
(1) there exists an open dense subset U of G such that, for each rational point $g \in U$, the translate gY and Z intersect properly, and
(2) in characteristic zero, and for Y, Z smooth, there exists an open dense subset U of G such that, for each rational point $g \in U$, gY and Z intersect transversally.

We now proceed to a definition of the (additive) group of conditions. For what follows, the reader may refer to the discussion in [11], §6.2 for details. We again consider a homogeneous space G/H of dimension n and two subvarieties $Y, Z \subseteq G/H$ of complementary dimension. If we assume G to be a connected group, then it can be seen [[11] §6.2] using the Kleiman transversality theorem that for g in a nonempty open subset of G, the number of points in the intersection $gY \cap Z$ is finite, and, in fact, a constant number, $a \in \mathbb{N}$. This motivates the following definition.

Definition 4.1.6. Let G be a connected algebraic group and X a homogeneous space with respect to G. Let Y, Z be subvarieties of X and assume that $Y, Z \subset X$ are of complementary dimension. We define the *intersection index* $< Y, Z >$ to be $< Y, Z > := \#(gY \cap Z)$ for a general translate gY of Y.

We begin to put a formal algebraic structure on geometric objects with the following definition.

Definition 4.1.7. Let X be a projective variety over $\mathbb{C}$. An *algebraic cycle Z of dimension n*, or an *n-cycle*, is an element of the free abelian group of irreducible subvarieties of X of dimension n, i.e., $Z = \sum_i n_i Z_i$, where $Z_i \subset X$ irreducible and $\dim Z_i = n$, and $n_i \in \mathbb{Z}$.

By linearity, we can extend the definition of intersection index to arbitrary cycles of complementary dimension. Denote by $\mathscr{L}^k(G/H)$ the group of cycles in G/H of dimension k, and denote by $\mathscr{L}^{n-k}(G/H)$ the group of cycles in G/H of dimension $n-k$. Let $< Y, Z >$ be their intersection index, as defined above. Now define

$$\mathscr{B}^k(G/H) \coloneqq \{Y \in \mathscr{L}^k(G/H) \mid < Y, Z >= 0 \text{ for all } Z \in \mathscr{L}^{n-k}(G/H)\}.$$

We then define $C^k(G/H) \coloneqq \mathscr{L}^k(G/H)/\mathscr{B}^k(G/H)$. Finally, we construct the group of conditions as the graded abelian group $C^*(G/H) = \bigoplus_{k=0}^n C^k(G/H)$. If Y is a cycle in G/H, we denote its equivalence class in $C^*(G/H)$ by $\{Y\}$.

Kleiman's Transversality Theorem guarantees that for generic $g \in G$, the intersection $gY \cap Z$ of irreducible subvarieties Y, Z is proper for g in a nonempty open set of G/H. Then, if it were the case that in some (possibly smaller) open subset U the equivalence class $[gY \cap Z]$ is constant for all g in U and depends only on Y and Z, then we could define the product structure on $C^*(X)$ by the formula $\{Y\} \cdot \{Z\} = \{gY \cap Z\}$ for g in this open set. In general, however, this product is not well-defined. For example, consider $G = \mathbb{C}^3$ acting on itself by translation. Then two lines in $\mathbb{C}^3$ are equivalent iff they are parallel. Take the intersection of the surface $xy = z$ and generic translates of the plane $x = 0$. These planes have the parametrized form $x = \lambda$. Then a generic intersection of these two objects is given by $x\lambda = z$, which is a family of lines that are not necessarily parallel—i.e., that are inequivalent.

Remarkably, for spherical homogeneous spaces, the intersection product is always well-defined [11]. Moreover, in [11], De Concini and Procesi relate the ring of conditions of a so-called symmetric variety G/H and the cohomology rings of its equivariant compactifications. In [10], De Concini remarks that their arguments for symmetric varieties proceed verbatim for arbitrary spherical homogeneous spaces. A priori, we might expect these constructions to be related by their definitions in terms of algebraic cycles; for a spherical variety X of a spherical homogeneous space G/H, and for any k-dimensional cycle $Z = \sum c_i Z_i$, one can define the cycle $\overline{Z}$ in X as $\sum c_i \overline{Z_i}$, where $\overline{Z_i}$ is the closure in X of $Z_i \subset G/H$. The cycle $\overline{X_i}$ then defines an element in the cohomology ring $H^*(X)$. De Concini and Procesi make this connection rigorous in their manuscript [11].

Specifically, denote by $\mathcal{S}$ the set of all smooth equivariant compactifications of G/H defined in Section 5.1. This set admits the partial ordering defined such that a compactification X_σ is greater than X_π iff there exists a map $X_\sigma \to X_\pi$ that commutes with the G-action. This map induces a map of cohomology rings $H^*(X_\pi) \to H^*(X_\sigma)$ [11]. Moreover, we have the following.

Fact 4.1.8. *[11] The ring of conditions $C^*(G/H)$ is isomorphic to the direct limit over the set $\mathcal{S}$ of the cohomology rings $H^*(X_\pi)$.*

The above theorem shows that $C^*(G/H)$ can be computed in terms of cohomology rings, but the result is not computationally effective in all cases, since a priori one must consider all possible equivariant compactifications. However, in some situations, it can be seen that it suffices to consider only one equivariant compactification. This is the subject of the next and last chapter of this thesis.

Chapter 5

On the Ring of Conditions of a Rank 1 Spherical Variety

In this last chapter we record in Theorem 5.1.4 an unpublished observation of Khovanskii which simplifies the result of De Concini-Procesi (Theorem 4.2.1) in some special cases. We also formulate a question regarding the rings of conditions of rank 1 spherical varieties which is motivated by Theorem 5.1.4. Finally, we illustrate the result using the example of $\mathbb{A}^n \setminus \{0\}$.

5.1 The Main Result

The analysis of G-equivariant compactifications is central in the work of De Concini and Procesi, as well as for Khovanskii's observation, so we begin with the following definition.

Throughout this chapter, we let G be a connected reductive linear algebraic group over $\mathbb{C}$.

Definition 5.1.1. Let G/H be a spherical homogeneous space with respect to G. A *G-equivariant compactification* of G/H is a G-equivariant embedding $G/H \hookrightarrow X$ where X is a complete G-variety, and the image of the embedding is open and dense in X.

Following [15], we focus on compactifications that satisfy certain intersection conditions with respect to a given cycle. The definition below is modelled after [[15] §4].

Definition 5.1.2. Let G/H be a spherical homogeneous space, and let $Y \subseteq G/H$ be an irreducible subvariety of G/H of dimension k. We say that a G-equivariant compactification $G/H \hookrightarrow X$ is a *good compactification of G/H with respect to Y* if the closure $\overline{Y}$ of Y in X has proper intersection with the boundary $X \setminus (G/H)$.

The above definition concerns the behaviour of a compactification of G/H in connection with a fixed subvariety Y. We wish to analyze compactifications that behave well for (appropriately chosen representatives of) all elements of the cohomology ring. More precisely, we define the following.

Definition 5.1.3. Let G/H be as above. We say that a G-equivariant compactification $G/H \hookrightarrow X$ *satisfies condition (RC)* if
(1) X is smooth and complete,
(2) any element in the cohomology ring $H^*(X)$ can be represented by a cycle $\sum n_i[Y_i]$, $n_i \in \mathbb{Z}$, where each Y_i is an irreducible subvariety of X and $Y_i \cap G/H \neq \emptyset$, and
(3) $G/H \hookrightarrow X$ is a good compactification of G/H with respect to Y for any irreducible subvariety of G/H.

The following is a simple and unpublished observation which we learned from Khovanskii. The statement and the proof are a simplified version of the more general considerations of De Concini-Procesi [[11] Theorem 6.3]. Specifically, Khovanskii's observation gives a sufficient condition for the direct limit appearing in [[11] Theorem 6.3] to be computable using a single compactification as opposed to a directed set thereof. The point of the discussion later on is then to find appropriate compactifications of rank 1 spherical homogeneous spaces to which Theorem 5.1.4. can be applied.

Theorem 5.1.4. *Let G be a connected reductive linear algebraic group, and let G/H be a spherical homogeneous space with respect to G. Suppose $G/H \hookrightarrow X$ is a G-equivariant compactification of G/H satisfying condition (RC). Then the ring of conditions $C^*(G/H)$ of G/H is isomorphic to the cohomology ring $H^*(X)$.*

Proof. We start by defining a map $\phi\colon H^*(X) \to C^*(G/H)$. Let $a \in H^*(X)$ be a homogeneous element. Then by assumption (2) of Definition 5.1.3., we may represent a by a cycle $\sum n_i[Y_i]$, $n_i \in \mathbb{Z}$, such that each Y_i is an irreducible subvariety of X and $Y_i \cap G/H \neq \emptyset$. Since G/H is open and dense in X, this means $Y_i = \overline{Y_i} \cap G/H$ where the closure is taken in X. We would like to define $\phi(a) = \sum n_i(Y_i \cap G/H)$ in the ring of conditions $C^*(G/H)$. First we claim that ϕ is well-defined. To see this, suppose $\sum n_i' Y_i'$ is another cycle representing a, such that each Y_i' is an irreducible subvariety of X and $Y_i' \cap G/H \neq \emptyset$. By a similar argument $Y_i' = \overline{Y_i} \cap G/H$. We must show that

$$\sum n_i\{Y_i \cap G/H\} = \sum n_i'\{Y_i' \cap G/H\} \in C^*(G/H).$$

By definition of the equivalence relation in the group of conditions, it suffices to show that for any cycle D in G/H of complementary codimension we have that

$$\left\langle (D, \sum n_i(Y_i \cap G/H)) \right\rangle = \left\langle (D, \sum n_i'(Y_i' \cap G/H)) \right\rangle. \quad (5.1.5)$$

Without loss of generality we can assume D is an irreducible subvariety of G/H. Consider the closure $\overline{D}$ of D in X, and let $[\overline{D}]$ denote its class in $H^*(X)$. Since X satisfies the condition (RC), $\overline{D}, \overline{Y_i}, \overline{Y_i'}$ for all i intersect $X \setminus (G/H)$ properly. Then by [[11] Proposition 6.1, Corollary 4.6, discussion following], $\langle D, \sum n_i(Y_i \cap G/H)\rangle$ is equal to the evaluation of the cup product $[\overline{D}] \cup a$ against the class of a point in $H^*(X)$, and the same is true of $(D, \sum n_i'(Y_i' \cap G/H))$. Hence ϕ is well-defined.

We next argue that ϕ is injective. Let $a \in H^r(X)$, $a \neq 0$. Then since X is smooth and complete there exists $b \in H^{dim_{\mathbb{R}}(X)-r}(X)$ of complementary degree with $a \cup b \neq 0$. Let $a \cup b = n \cdot p$ where p denotes the class of a point and $n \in \mathbb{Z}$, $n \neq 0$. Then, as in the argument given above for the well-definedness of ϕ, [[11] Corollary 6.1] shows that $\langle \phi(a), \phi(b) \rangle = n \neq 0$. Hence $\phi(a) \neq 0$, so ϕ is injective.

Now we claim ϕ is a ring homomorphism. Since ϕ is additive and by assumptions (2) and (3), it suffices to check that for $a = [\overline{Y}]$, $b = [\overline{Z}]$ in $H^*(X)$ where Y, Z are irreducible subvarieties in G/H (not necessarily of complementary dimension) that $\phi([\overline{Y}] \cup [\overline{Z}])$

is equal to the product of $\phi([\overline{Y}])$ and $\phi([\overline{Z}])$ in $C^*(G/H)$. By definition of ϕ, $\phi([\overline{Y}]) = \{\overline{Y} \cap G/H\} = \{Y\}$ and similarly $\phi([\overline{Z}]) = \{Z\}$. By definition of the product in $C^*(G/H)$, the product of $\{Y\}$ and $\{Z\}$ is $\{Y \cap gZ\}$ for $g \in G$ such that Y and gZ intersect properly in G/H. Since $[g\overline{Z}] = [\overline{Z}]$ for any $g \in G$, to prove ϕ is a ring homomorphism, it therefore suffices to show there exists $g \in G$ such that $\overline{Y} \cap \overline{Z}$ is proper. By assumption (3), both $\overline{Y}$ and $\overline{Z}$ have proper intersection with $X \backslash (G/H)$ and so we may apply [[11] Corollary 6.1] and we obtain the result. Finally, surjectivity of ϕ immediately follows from assumption (2). □

To relate the above theorem to the case of certain spherical varieties, it will be useful to have the following lemma.

Lemma 5.1.5. *Using the same notation as above, let $G/H \hookrightarrow X$ be a G-equivariant compactification of G/H such that*

(a) X is smooth and complete,

(b) $X \setminus (G/H)$ is a (finite) union of G-orbits, each of which is a hypersurface, and

(c) any element in the cohomology ring $H^*(X)$ can be represented by a cycle $\sum n_i[Y_i]$, $n_i \in \mathbb{Z}$, where each Y_i is an irreducible subvariety of X and $Y_i \cap G/H \neq \emptyset$.

Then $G/H \hookrightarrow X$ satisfies condition (RC).

Before proving Lemma 5.1.5, we observe that the following is an immediate consequence of Theorem 5.1.4 and Lemma 5.1.5. This is the result which directly links the above considerations with the rank 1 spherical varieties, as we explain further below.

Corollary 5.1.6. *Let G be a connected reductive linear algebraic group, and let G/H be a spherical homogeneous space with respect to G. Suppose $G/H \hookrightarrow X$ is a G-equivariant compactification satisfying*

(1) X is smooth and complete,

(2) any element in the cohomology ring $H^*(X)$ can be represented by a cycle $\sum n_i[Y_i]$, $n_i \in \mathbb{Z}$, where each Y_i is an irreducible subvariety of X and $Y_i \cap G/H \neq \emptyset$, and

(3) the boundary $X \setminus (G/H) = D_1 \sqcup D_2 \sqcup \cdots \sqcup D_\ell$ is a finite union of G-orbits D_i where each D_i is a hypersurface in X. Then the ring of conditions $C^*(G/H)$ of G/H is isomorphic to $H^*(X; \mathbb{C})$.

We note that it is also possible to prove Corollary 5.1.6 without invoking Theorem 5.1.4. We give a sketch of the argument, which is similar to that of Theorem 5.1.4, here. The point is that the codimension-1 condition on the components of the boundary makes it particularly simple to see that the relevant intersections can be "moved off the boundary". Indeed, suppose $G/H \longrightarrow X$ is a G-equivariant embedding satisfying the hypotheses of Corollary 5.1.6. We wish to define a homomorphism $H^*(X) \to C^*(G/H)$. Proceeding as in the proof of Theorem 5.1.4, in order to prove well-definedness, given a cohomology class $a = \sum n_i[Y_i] = \sum n_i'[Y_i]'$ we wish to show that (5.1.5) holds for an arbitrary irreducible subvariety $D \subseteq G/H$ of complementary dimension. We now claim that to prove the equality it would in fact suffice to prove the following.

Claim 5.1.7. For two irreducible subvarieties D and Y in G/H of complementary dimension, there exists and open dense subset U of G such that $\overline{D} \cap g\overline{Y}$ lies entirely in G/H, i.e. $\overline{D} \cap g\overline{Y} = D \cap gY$ for $g \in U$.

This in turn means that the number of points contained in $D \cap gY$ is precisely the class of a point paired against $[\overline{D}] \cup [\overline{Y}]$ in $H^{top}(X)$, and in particular is independent of the choice of representative of $[\overline{Y}]$. Thus the map $a \mapsto \sum n_i\{Y_i \cap (G/H)\}$ is well-defined. We now proceed to prove the Claim above. Recall

$$X = G/H \sqcup D_1 \sqcup D_2 \cdots \sqcup D_\ell$$

where each D_j is a G-orbit and is a hypersurface in X, i.e. $\text{codim} D_j = 1$. Let j be any integer, $1 \leq j \leq l$. From the proof of Lemma 5.1.5. we know that $\dim(\overline{Y} \cap D_j) < \dim\overline{Y}$ and similarly $\dim(\overline{D} \cap D_j) < \dim(\overline{D})$. Since Y and D are of complementary dimension in X, applying the Kleiman Transversality Theorem to the subvarieties $\overline{D} \cap D_j$ and $\overline{Y} \cap D_j$ in the G-homogeneous space D_j, we conclude that there exists and open dense subset U_j of G such that for all $g \in U_j$, $(\overline{D} \cap D_j) \cap g(\overline{Y} \cap D_j) = \emptyset = \overline{D} \cap g\overline{Y} \cap D_j = \emptyset$. Since there are only finitely many D_j's, and since each U_j is open and dense in G, the intersection

$U = U_1 \cap U_2 \cap \cdots \cap U_l$ is open and dense in G, and for all $g \in U$ we have

$$\overline{D} \cap g\overline{Y} \cap (\bigsqcup_{j=1}^{l} U_j) = \overline{D} \cap g\overline{Y} \cap (X \setminus (G/H)) = \emptyset,$$

as desired. This proves the Claim. We therefore get a well-defined map of vector spaces $H^*(X;\mathbb{C}) \to C^*(G/H)$. The remainder of the argument is essentially the same as that of Theorem 5.1.4. We now proceed to a proof of Lemma 5.1.5.

Proof of Lemma 5.1.5 We wish to prove that if the boundary $Z \coloneqq X \setminus (G/H)$ is a finite union of hypersurfaces, then for any irreducible subvariety Y of G/H, $\overline{Y} \subseteq X$ intersects Z properly. Let $Z = D_1 \sqcup D_2 \sqcup \cdots \sqcup D_\ell$ where each D_j is an irreducible hypersurface. Then we wish to show $\overline{Y} \cap D_j$ is proper, for each j, $1 \leq j \leq l$. Let $d = \dim(Y) = \dim(\overline{Y})$. By assumption, $\dim(D_j) = \dim(X) - 1$. To see that the intersection $\overline{Y} \cap D_j$ is proper, we need to show each irreducible component of $\overline{Y} \cap D_j$ is either $\emptyset$ or $\dim(\overline{Y} \cap D_j) = \dim(Y) - 1$. But this follows from the fact that $\overline{Y} \cap D_j$ (if non-empty) is a proper subvariety of $\overline{Y}$ and that D_j is a hypersurface. □

In some cases of rank 1 spherical homogeneous spaces G/H, we have explicit descriptions of G-equivariant compactifications of G/H which can be explicitly checked to satisfy the conditions (1) - (3) of Corollary 5.1.6. holds. For instance, the example we discuss in §5.4 in the case of $\mathbb{A}^n \setminus \{0\}$, analyzed in detail in §3.3.2. There, it turns out that the embedding $\mathbb{A}^n \setminus \{0\} \hookrightarrow Bl_0(\mathbb{P}^n)$ satisfies all of the conditions (a) - (c), thus allowing us to compute $C^*(\mathbb{A}^n \setminus \{0\})$ concretely.

More generally, in [1] and [6], Ahiezer and Brion classify spherical varieties of rank 1, and each states the following result, as recorded by Timashev in [23].

Theorem 5.1.8 ([23] Proposition 30.4)**.** *The following conditions are equivalent:*
(1) G/H is a spherical homogeneous space of rank 1.
(2) There exists a smooth complete embedding $G/H \hookrightarrow X$ such that $X \setminus (G/H)$ is a union of G-orbits of codimension 1.

The spherical compactifications of Theorem 5.2.1 evidently satisfy the conditions (1) and (3) of Corollary 5.1.6. Thus, the only remaining question is whether or not the condition (2) is also satisfied. In fact, the classification by Akhiezer of the rank 1 spherical varieties gives an explicit list of all such varieties, so in principle it is possible to go through the list and check each example to see whether condition (2) is satisfied. We record the problem here for future work.

Problem. Determine, from Akhiezer's list of all rank 1 spherical varieties, which ones satisfy the condition (2) of Corollary 5.1.6.

A solution to the above problem would then naturally lead to a computation of the ring of conditions of those rank 1 spherical varieties, following Corollary 5.1.6.

5.2 Example: $\mathbb{A}^n \setminus \{0\}$

We now illustrate, using the example of $\mathbb{A}^n \setminus \{0\}$, the technique for computing the ring of conditions of a rank 1 spherical variety, as indicated in Theorem 5.1.4. We begin with a few definitions. Khovanskii's observation implies that we can explicitly compute the ring of conditions of $\mathbb{C}^*(\mathbb{A}^n \setminus \{0\})$ by computing the cohomology ring of the spherical embedding $\mathbb{A}^n \setminus \{0\} \hookrightarrow Bl_0(\mathbb{CP}^n)$, since $Bl_0(\mathbb{CP}^n) = \mathbb{A}^n \setminus \{0\} \sqcup E \sqcup \mathbb{CP}^{n-1}$ is smooth, complete and is a union of $\mathbb{A}^n \setminus \{0\}$ with two prime divisors. The task that remains is to compute the cohomology ring $H^*(Bl_0(\mathbb{CP}^n))$ explicitly. Since $Bl_0(\mathbb{CP}^n)$ is a toric variety, we may use the result of Jurkiewicz-Danilov, for which we need some preparation.

The affine variety $(\mathbb{C}^*)^n$ is a group under component-wise multiplication. A *torus* T is an affine variety isomorphic to $(\mathbb{C}^*)^n$, where T inherits a group structure under the isomorphism.

Definition 5.2.1. A *character* of a torus T is a morphism $\chi\colon T \to \mathbb{C}^*$ that is a group homomorphism. For example, $m = (a_1, ...a_n) \in \mathbb{Z}^n$ gives a character $\chi_m\colon (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by

$$\chi_m(t_1, .., t_n) = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$$

One can show that *all* characters of $(\mathbb{C}^*)^n$ arise this way (see [14] §16). Thus the group of characters of $(\mathbb{C}^*)^n$ is isomorphic to $\mathbb{Z}^n$.

For an arbitrary torus T, its characters form a free abelian group M of rank equal to the dimension of T. By convention, we say that $m \in M$ gives the character $\chi_m \colon T \to \mathbb{C}^*$.

Definition 5.2.2. A *lattice* is a free abelian group of finite rank.

Thus a lattice of rank n is isomorphic to $\mathbb{Z}^n$.

Definition 5.2.3 ([9] Definition 1.1.3.)**.** An (affine) *toric variety* is an irreducible (affine) algebraic variety X, equipped with a complex-torus action, such that the torus $T \coloneqq (\mathbb{C}^*)^n$ has an orbit that is Zariski-open in X.

We now connect cones and affine toric varieties. Fix a pair of dual vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. Our outline omits proofs. We refer the reader to [12] for a detailed discussion.

Definition 5.2.4 ([9] Definition 1.2.1)**.** A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = Cone(S) = \left\{ \sum_{u \in S} \lambda_u u \,|\, \lambda_u \geq 0 \right\} \subseteq N_{\mathbb{R}}$$

where $S \subseteq N_{\mathbb{R}}$ is finite. We say that σ is *generated* by S and that $Cone(\emptyset) = \{0\}$.

A convex polyhedral cone σ is convex in the classical sense, meaning that $x, y \in \sigma$ implies $\lambda x + (1-\lambda)y \in \sigma$ for all $0 \leq \lambda \leq 1$ and is a *cone*, meaning that $x \in \sigma$ implies that $\lambda x \in \sigma$ for all $\lambda \geq 0$. Since we only consider convex cones, the cones satisfying Definition 5.4.4. will be called "polyhedral cones". Basic examples of polyhedral cones include the first quadrant in $\mathbb{R}^2$ and the first octant in $\mathbb{R}^3$.

Definition 5.2.5 ([9] §1.2)**.** The *dimension* $\dim \sigma$ of a polyhedral cone σ is the dimension of the smallest subspace $W = Span(\sigma)$ of $N_{\mathbb{R}}$ containing σ. We call $Span(\sigma)$ the *span* of σ.

Recall that the pairing between $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ is denoted $\langle , \rangle$.

Definition 5.2.6 ([9] Definition 1.2.3)**.** Given a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, its *dual cone* is defined by

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \,|\, \langle m, u \rangle \geq 0\} \,\forall\, u \in \sigma.$$

Duality possesses the following useful and important properties.

Definition 5.2.7 ([9] Proposition 1.2.4.). Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone. Then $\sigma^\vee$ is a polyhedral cone in $M_{\mathbb{R}}$ and $(\sigma^\vee)^\vee = \sigma$.

Given $m \neq 0$ in $M_{\mathbb{R}}$, we get the hyperplane

$$H_m = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \subseteq N_{\mathbb{R}}$$

and the closed half-space

$$H_m^+ = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\} \subseteq N_{\mathbb{R}}.$$

We say that H_m is a *supporting hyperplane* of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ if $\sigma \subseteq H_m^+$, and we say H_m^+ is a *supporting half-space*. Note that H_m is a supporting hyperplane of σ if and only if $m \in \sigma^\vee \setminus \{0\}$.

We can use supporting hyperplanes and half-spaces to define *faces* of a cone.

Definition 5.2.8 ([9] Definition 1.2.5.). A *face of a cone* of the polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$, written $\tau \preceq \sigma$. Using $m = 0$ shows that σ is a face of itself, i.e., $\sigma \preceq \sigma$. Faces $\tau \neq \sigma$ are called *proper faces*, written $\tau \prec \sigma$.

The faces of a polyhedral cone have the following properties.

Fact 5.2.9 ([9] Lemma 1.2.6.). *Let $\sigma = Cone(S)$ be a polyhedral cone. Then:*
(a) Every face of σ is a polyhedral cone.
(b) An intersection of two faces of σ is again a face of σ.
(c) A face of a face of σ is again a face of σ.

Definition 5.2.10 ([9] §1.2). A *facet* of σ is a face τ of codimension 1, i.e., dimτ = dim$\sigma - 1$. An *edge* of σ is a face of dimension 1.

Definition 5.2.11 ([9] §1.2). A polyhedral cone σ is *strongly convex* if the origin is a face of σ.

Let N and M be dual lattices with associated vector spaces $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. For $\mathbb{R}^n$ we usually use the lattice $\mathbb{Z}^n$.

Definition 5.2.12 ([9] Definition 1.2.14)**.** A polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is *rational* if $\sigma = Cone(S)$ for some finite set $S \subseteq N$.

Below are two especially important types of strongly convex rational cones.

Definition 5.2.13 ([9])**.** Definition 1.2.16.] Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.
(a) σ is *smooth* or *regular* if its minimal generators form part of a $\mathbb{Z}$-basis of N,
(b) σ is *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$.

Now we transition to semigroup algebras and affine toric varieties, en route to describing the correspondence between toric varieties and convex geometry.

Definition 5.2.14 ([9] §1.1)**.** A *semigroup* is a set with an associative binary operation and an identity element.

Definition 5.2.15 ([9] §1.1)**.** A semigroup S is *affine* if, further, the following conditions are satisfied:
(1) the binary operation is commutative,
(2) the semigroup is finitely generated, meaning that there is a finite subset $\mathscr{A} \subseteq S$ such that $\mathbb{N}\mathscr{A} = S$, and
(3) the semigroup can be embedded in a lattice M.

Given a rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, the lattice points

$$S_\sigma = \sigma^\vee \cap M \subseteq M$$

form a semigroup. A key fact is that this semigroup is finitely generated.

Fact 5.2.16 ([9] Proposition 1.2.17.)**.** *(Gordan's Lemma). Let S_σ be defined as above. Then $S_\sigma = \sigma^\vee \cap M$ is finitely generated and hence is an affine semigroup.*

Fact 5.2.17 ([9] Theorem 1.2.18.)**.** *Let* $\sigma \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ *be a rational polyhedral cone with semigroup* $S_\sigma = \sigma^\vee \cap M$*. Then*

$$U_\sigma = Spec(\mathbb{C}[S_\sigma]) = Spec(\mathbb{C}[\sigma^\vee \cap M])$$

is an affine toric variety. Further,
$dim U_\sigma = n \Leftrightarrow the\, torus\, of\, U_\sigma\, is\, T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \Leftrightarrow \sigma$ *is strongly convex.*

Henceforth, we will always assume that $\sigma \subseteq N_{\mathbb{R}}$ is strongly convex since we want T_N to be the torus of the affine toric variety U_σ.

Next, we discuss projective toric varieties.

Definition 5.2.18. [3] Let T be a torus. A projective toric variety is a projective variety equipped with a torus action with T as an open orbit.

The canonical first example is $\mathbb{CP}^n$, which is a projective toric variety with torus

$$T_{\mathbb{CP}^n} = \mathbb{CP}^n \setminus V(x_0 \cdots x_n) = \{[a_0 \colon \ldots \colon a_n] \in \mathbb{CP}^n \,|\, a_0 \cdots a_n \neq 0\}$$

$$\cong \{[1 \colon t_1 \colon \ldots \colon t_n] \in \mathbb{CP}^n \,|\, t_1, \ldots, t_n \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^n.$$

The standard action of $T_{\mathbb{CP}^n}$ on itself extends to an action on $\mathbb{CP}^n$, making $\mathbb{CP}^n$ a toric variety.

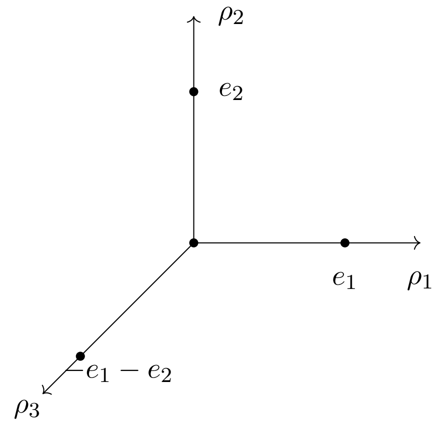
We now wish to describe the projective toric variety which allows us to compute the ring of conditions of $\mathbb{A}^n \setminus \{0\}$. This is the blowup $Bl_0(\mathbb{CP}^n)$ of $\mathbb{CP}^n$ at the origin.

Definition 5.2.19. [3] The *blowup* of $\mathbb{CP}^n$ at a point is the projectivization of the tautological line bundle over $\mathbb{CP}^n$.

We now define the *fan* corresponding to a toric variety.

Definition 5.2.20 ([9] Definition 3.1.2.)**.** A fan $\Sigma \in N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that
(1) every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone,
(2) for all $\sigma \in \Sigma$, each face of σ is also in Σ,

Figure 1. The fan of $\mathbb{CP}^2$



(3) for all $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is also a face of each. Further, if Σ is a fan, then $\Sigma(r)$ is the set of r-dimensional cones of Σ.

Let $N_{\mathbb{R}} = \mathbb{R}^n$, where $N = \mathbb{Z}^n$ has standard basis $e_1, ..., e_n$. Set $e_0 = -e_1 - e_2 - \cdots - e_n$, and let Σ be the fan in $N_{\mathbb{R}}$ comprising the cones generated by all proper subsets of $\{e_0, ..., e_n\}$.

Fact 5.2.21 ([9] Example 3.1.10.)**.** *This fan corresponds to $\mathbb{CP}^n$, considered as a toric variety.*

We will compute the fan corresponding to the blowup of $\mathbb{CP}^n$ at the origin as an application of the following general subdivision procedure.

Definition 5.2.22 ([9] Definition 3.3.13.)**.** Let Σ be a fan in $N_{\mathbb{R}} \cong \mathbb{R}^n$. Let $\sigma = Cone(u_1, ..., u_n)$ be a smooth cone in Σ, so that $u_1, ...u_n$ is a basis for N. Let $u_0 = u_1 + \cdots + u_n$ and let $\Sigma'(\sigma)$ be the set of all cones generated by subsets of $\{u_0, ..., u_n\}$ not containing $\{u_1, ..., u_n\}$. Then

$$\Sigma^*(\sigma) = (\Sigma \setminus \{\sigma\}) \cup \Sigma'(\sigma)$$

is a fan in $N_{\mathbb{R}}$ called the *star subdivision* of Σ along σ.

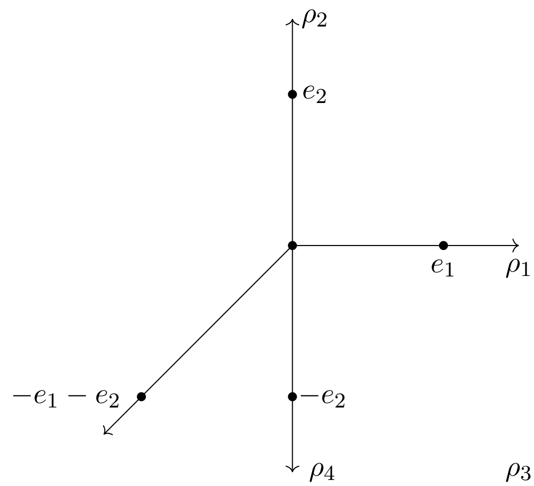
Fact 5.2.23 ([9] Proposition 3.3.15.)**.** *The fan $\Sigma^*(\sigma)$ is a refinement of Σ, and the induced toric morphism*

$$\psi \colon X_{\Sigma^*(\sigma)} \to X_{\Sigma}$$

makes $X_{\Sigma^*(\sigma)}$ the blowup of X_{Σ} at the distinguished point γ_σ corresponding to the cone σ.

We now construct the fan of the blowup of $\mathbb{CP}^n$ at the origin. Let Σ' be the fan obtained from the fan Σ for $\mathbb{CP}^n$ by the following process. Subdivide the cone σ_n by inserting an edge $Cone(-e_n)$. Then the resulting toric variety $X_{\Sigma'}$ is the blowup of $\mathbb{CP}^n$ at the point $V(\sigma_n)$.

Figure 2. The fan of $Bl_0(\mathbb{CP}^2)$



We wish to compute the cohomology of the blowup of $\mathbb{CP}^n$ at the origin. Results of Jurkiewicz and Danilov gives concrete prescription of the cohomology of certain toric varieties, as we now recall.

Definition 5.2.24 ([9] §4.3 discussion)**.** Let Σ be the normal fan of a toric variety X_Σ. Then X_Σ is *simplicial* if every cone $\sigma \in \Sigma$ is simplicial, i.e., the minimal generators of σ are linearly independent over $\mathbb{R}$.

Let X_Σ be a complete and simplicial toric variety and fix a numbering $\rho_1, ..., \rho_r$ for the rays in $\Sigma(1)$. Also let u_i be the minimal generator of ρ_i and introduce a variable x_i for each ρ_i. In the ring $\mathbb{Z}[x_1, ..., x_r]$, let $\mathscr{I}$ be the monomial ideal with square-free generators defined as follows:

$$\mathscr{I} = \langle x_{i_1} \cdots x_{i_s} \,|\, i_j \text{ are distinct and } \rho_{i_1} + \cdots + \rho_{i_s} \text{ is not a cone of } \Sigma \rangle$$

As defined above, $\mathscr{I}$ is called the *Stanley-Reisner* ideal. We also define a second ideal as follows. Let J be the ideal generated by the linear forms

$$\sum_{i=1}^{r} \langle m_i, u_i \rangle x_i \quad (5.3.25)$$

where m ranges over some basis of M.

Example 5.2.25. *For the example of $\mathbb{CP}^2$, fix the basis e_1, e_2 for M. This toric variety has the fan with rays ρ_1, ρ_2, ρ_3 as in Figure 1. These rays have corresponding minimal generators $u_1 = e_1$, $u_2 = e_2$, $u_3 = -e_1 - e_2$. Let x_1, x_2, x_3 be the corresponding variables as above. In this example, it is not hard to see that $\mathscr{I}$ is given by $\langle x_1 x_2 x_3 \rangle$ and the ideal J is equal to $\langle x_1 - x_3, x_2 - x_3 \rangle$.*

Example 5.2.26. *For the example of $Bl_0(\mathbb{CP}^2)$, the fan has four rays $\rho_1, \rho_2, \rho_3, \rho_4$ as in Figure 2 with associated minimal generators $e_1, e_2, -e_1 - e_2, -e_2$. Then the Stanley-Reisner ideal can be seen to be $\mathscr{I} = \langle x_1 x_3, x_2 x_4 \rangle$ and the ideal J can be seen to be $\langle x_1 - x_3, x_2 - x_3 - x_4 \rangle$.*

Since $Bl_0(\mathbb{CP}^n)$ is complete and smooth, we can compute $H^*(Bl_0(\mathbb{CP}^n))$ using the following result of Jurkiewicz-Danilov.

Fact 5.2.27 ([9] Theorem 12.4.4.)**.** *Let X_Σ be a smooth complete toric variety. For the polynomial ring $\mathbb{Z}[x_1, ..., x_r]$ with variables indexed by $\rho_1, ..., \rho_r \in \Sigma(1)$, let $\mathscr{I}$ and J be the ideals in $\mathbb{Z}[x_1, ..., x_r]$ as defined above. Then*

$$H^*(X) \cong \mathbb{Z}[x_1, ..., x_r]/(\mathscr{I} + J).$$

Example 5.2.28. *In the case of $\mathbb{CP}^2$, the computations given in Example 5.3.25 show that $H^*(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, x_3]/\langle x_1 x_2 x_3 = 0, x_1 = x_3, x_2 - x_3 \rangle \cong \mathbb{Z}[x]/\langle x^3 \rangle$.*

Example 5.2.29. *In the case of $Bl_0(\mathbb{CP}^2)$, the computations given in Example 5.3.26 show that $H^*(Bl_0(\mathbb{CP}^2); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, x_3, x_4]/\langle x_1 - x_3, x_2 - x_3 - x_4, x_1 x_3, x_2 x_4 \rangle \cong \mathbb{Z}[x, y]/\langle x^2, y(x+y) \rangle$.*

Example 5.2.30. *In the case of $Bl_0(\mathbb{CP}^2)$, the fan for the blowup has rays $e_0 = -e_1 - e_2 - \cdots - e_n$, $e_1, e_2, ..., e_n, -e_n$. Let the variables corresponding to $e_0, ...e_n, -e_n$ be denoted by $x_0, ..., x_n$, respectively, and denote by y the variable corresponding to the "exceptional divisor", by which we mean the vector $(-e_n)$.*

Then the Stanley-Reisner ideal can be seen to be generated by the following two elements: the degree n monomial $x_0x_1\cdots x_{n-1}$ and the degree-2 monomial x_ny. From the definition of the ideal J, the generators for J are $-x_0+x_1,\ -x_0+x_2,\ ...,-x_0+x_{n-1},-x_0+x_n-y$. This implies that we have relations $x_1=x_0,...,x_{n-1}=x_0$, and $x_n=x_0+y$.

For simplicity, from now on re-label the variable x_0 as just x. Then the Jurkiewicz-Danilov theorem states that the cohomology of the blow-up is $\mathbb{Z}[x,y]/\langle x^n, y(x+y)\rangle$.

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