Optimal Precoder Design for MIMO Communication Systems Equipped with Decision Feedback Receivers

Optimal Precoder Design for MIMO Communication Systems Equipped with Decision Feedback Receivers

By

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To my dearest parents:

Jinjun Liu and Qin Liu

Abstract

We consider the design of the precoders for a multi-input multi-output (MIMO) communication system equipped with a decision feedback equalizer (DFE) receiver. For such design problems, perfect knowledge of the channel state information (CSI) at both the transmitter and the receiver is usually required. However, in the environment of wireless communications, it is often difficult to provide sufficiently timely and accurate feedback of CSI from the receiver to the transmitter for such designs to be practically viable.

In this thesis, we consider the optimum precoder designs for a wireless communication link having M transmitter antennas and N receiver antennas ($M < N$), in which the channels are assumed to be flat fading and may be correlated. We assume that full knowledge of CSI is available at the receiver. At the transmitter, however, only the first- and second-order statistics of the channels are available. Our first goal is to come up with an efficient design of the optimal precoder for such a MIMO system by minimizing the average arithmetic mean-squared error (MSE) of zero-forcing (ZF) decision feedback detection subject to a constraint on the total transmission power. Applying some of the properties of the matrix parameters, this non-convex optimization problem can be transformed into a convex geometrical programming problem which can then be efficiently solved using an interior point method. The performance of the MIMO system equipped with this optimum precoder and a ZF-DFE has also been found to be comparable, and in some cases, superior to that of V-BLAST which necessitates optimally ordered successive interference cancellation

based on the largest post-detection signal-to-noise ratio (SNR). In terms of trade-off between performance and implementation simplicity, the proposed system is certainly an attractive alternative.

In addition, we also utilize these important properties of our system parameters to investigate an "inverse problem" of our first design. That is, we design another precoding matrix by minimizing the total transmission power of the MIMO communication system subject to a constraint on the average MSE. Also, a closed-form solution is derived when the channels are uncorrelated while simulation results for the minimum power precoder designs is given at the end of this thesis.

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Contents

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List of Abbreviations

AWGN	Additive White Gaussian Noise
BER	Bit Error Rate
BLAST	Bell Labs Layered Space-Time
CSI	Channel State Information
DF	Decision Feedback
DFE	Decision Feedback Equalization
FDD	Frequency Division Duplex
IID.	Independent and Identically Distributed
MIMO	Multi-input Multi-output
MIMO-OFDM	MIMO-Orthogonal Frequency Division Multiplexing
MIMO-OFDMA	MIMO-Orthogonal Frequency Division Multiple Access
MLD	Maximum Likelihood Detector
MSE	Mean Square Error
MMSE	Minimum MSE
pdf	Probability Density Function
QAM	Quadrature Amplitude Modulation
SEP	Symbol Error Probability
SER	Symbol Error Rate
SISO	Single-input Single-output

SNR	Signal-to-noise Ratio
STC	Space Time Coding
SVD	Singular Value Decomposition
UMTS	Universal Mobile Telecommunications System
V-BLAST	Vertical BLAST
ZF	Zero-forcing
3GPP	3rd Generation Partnership Project
4G	Fourth-Generation

List of Notations

Lower case: a	Scalars
Lower case bold: $\mathbf{a}$	Column vectors
Upper case bold: $\mathbf{A}$	Matrix
a_{ij}	The (ij)-th entry of $\mathbf{A}$
a_i	The i-th entry of $\mathbf{a}$
$\mathbf{a}_i$	The ith column of matrix $\mathbf{A}$
$\mathbf{A}_{mn}$	A matrix consisting of the first m rows and first n columns of $\mathbf{A}$
$\mathbf{I}$	An identity matrix
$\mathbf{I}_N$	An N by N identity matrix
$\mathbf{A}^\dagger$	Moore-Penrose pseudo inverse of $\mathbf{A}$
$\mathbf{E}[\cdot]$	Expectation
$\mathbf{tr}[\cdot]$	Trace
$(\cdot)^H$	Hermitian transpose of a matrix or a vector
$(\cdot)^T$	The transpose of a matrix or a vector
$\{c_{[m]}\}$	The arrangement of a sequence $\{c_m\}$ in descending order
$\{c_{(m)}\}$	The arrangement of a sequence $\{c_m\}$ in ascending order

Chapter 1

Introduction

1.1 MIMO Communications and Precoding

Wireless technology is one of the most important breakthroughs in modern communications since it enables many applications such as wireless mobile phone, wireless internet access, wireless local area networks, wireless sensor networks and so on. The explosive expansion in wireless communications in recent years has given rise to severe technical challenges which include the demand of transmitting multimedia data at high rates in an environment rich of scattering.

Multi-input multi-output (MIMO) wireless links are important recent developments in wireless communications due to their enormous potential in meeting the challenges caused by fading channels as well as power and bandwidth limitations. MIMO communication systems rely on the use of M transmitter antennas and N receiver antennas which enables the exploitation of the high performance provided by the space diversity available, and the high data rate provided by the capacity obtainable in the MIMO channels [1][2]. One approach which attempts to achieve high data rate is the Vertical Bell Labs Layered Space-Time (V-BLAST) scheme [3] in which data streams are transmitted from each transmitter antenna, and detected at the receiver using nulling and successive interference cancellation. To minimize the probability of error, the order of detection in V-BLAST is based on the post-detection signal-to-noise ratio (SNR) the calculation of which renders the reception procedure computationally demanding.

To ease off the complexity demanded at the receiver, proper design of the precoder at the transmitter has received intensive attention in current digital communication system due to its ability to improve the system performance. An important aspect in the design of the precoder for prescribed receivers is the availability of channel state information (CSI) at the transmission and reception ends. When perfect CSI is available at the transmitter, there exist solutions to various precoder design problems [4], including maximization of information rate [5], maximization of SNR [6], minimization of the mean squared error [6] and minimization of the bit error probability for zero-forcing (ZF) [7] and minimum mean square error (MMSE) equalization [4] [8], as well as the optimum joint design of transmitter-receiver for the ZF-DF and MMSE-DF detectors [9, 10, 11, 12, 13, 14, 15, 16, 17].

It is reasonable to assume that CSI is available at the receiver via training. While having CSI at the transmitter allows for better performance, this may not be possible in practice, due to rapid variations and limited feedback bandwidth such as in the case of frequency division duplex (FDD) systems in which user mobility is high [18]. Nevertheless, it is reasonable to assume that the *channel statistics* are known at the transmitter since these statistics change over much larger time scales than the channel gains. Therefore, designing optimal transmitters based on statistical information of channels is well motivated. When only channel statistical information is available at the transmitter, designs of precoders that minimize an upper bound of the average symbol error probability (SEP) based on a maximum ratio combining receiver has been developed [19] for a MISO communication system. Also, for the correlated VBLAST systems, optimal precoders minimizing the SEP for linear ZF and MMSE receivers have been proposed [18]. These "statistical prefilters", however, are designed only for the relatively simple linear equalization which may cause a significant performance loss. For more sophisticated receivers, when only the statistics of the channels are known at the transmitter, optimal precoder designs are mainly based on a capacity criterion [20, 21, 22, 23, 24]. However, the advantage of such precoders can usually be exploited only if a maximum likelihood detector (MLD) is employed. Indeed, it is well known that the MLD is universally optimal in detection, yielding the minimum detection error probability among all receivers. Unfortunately, it is also computationally demanding which, to a certain extent, restricts its applications in practice. In terms of trade-off between system performance and implementation complexity, the decision feedback (DF) receiver is known to be an attractive alternative detection scheme [25]. Optimal designs of diagonal precoders that minimizes an approximate bit error rate (BER) of the DF detector based on channel information feedback have been obtained in [26] and [27].

1.2 Thesis Contribution

In this thesis, our focus will also be on the use of the DF receiver in a MIMO communication system. Specifically, our goal is to design two optimum precoders for ZF-DF receivers, one of which minimizes the average arithmetic MSE of the ZF-DFE over random channel coefficients subject to a constraint of the total transmission power. The other one is to minimize the total transmission power of the system subject to a constraint on the average MSE. Throughout this thesis, we will assume that perfect CSI is available at the receiver, but only the first- and second-order statistics of the channel is known at the transmitter.

As for modelling the transmission channel, we note that a popular channel model for many precoding matrix design proposed in literature is one in which the channels are independent and identically distributed (IID) zero-mean Gaussian variables. This is an idealized model representing rich uniform scattering and its analysis is relatively straightforward. In practice, however, transmission channels are often correlated.

In this thesis, we examine both models and come up with optimum transmission precoders for both cases.

The main contributions of this thesis include:

- For ZF-DFE, a statistical precoder matrix has been designed based on the minimization of the average MSE while the total transmission power of the MIMO communication system is bounded.
- Theoretical analysis of the important properties of systems matrices is provided. With the aid of these properties, the original non-convex optimization problem has been transformed to a convex one.
- For the uncorrelated channels, a closed-form solution to the precoder design problem is derived. For correlated channel, the problem has been solved by using a numerical convex optimization method and the exact structure of the optimum precoding matrix is given.
- The SER performance of the MIMO system, which shows its superiority to most of existing systems, has been demonstrated by simulation studies.
- As an extension of the optimum precoder design, another precoder design problem that minimizes the total transmission power subject to a constraint on the average MSE has also been considered in this thesis.
- The second design problem has also been transformed to a convex optimization problem by utilizing the same system properties and solved by the interior point method. When the MIMO channels are uncorrelated, a closed-form solution is derived.

1.3 Structure of the Thesis

The thesis is structured as follows:

- In Chapter 2, we first give a brief introduction on the MIMO wireless communication, followed by its properties. After that, the transmission model and assumptions of our design in the whole thesis have been given. Finally, we will discuss various detection schemes for MIMO transmissions.
- In Chapter 3, an optimal precoder that produces a minimal average MSE is designed. After the problem formulation, the convex transformation of the optimization problem is derived. We also derive an analytical solutions for our design problem.
- In Chapter 4, we investigate an "inverse" design problem: minimum power precoder for ZF-DF receivers. We also solve this problem both numerically and analytically according to the channel correlation.
- In Chapter 5, simulations have been studied for different scenarios.
- Conclusion on this thesis and suggestion for future work are discussed in Chapter 6

Chapter 2

Multi-Input Multi-Output Wireless Communications

In this chapter, we will give an overview of MIMO communication system. We first introduce the MIMO architecture and establish the channel model of our designs. After that, a detailed discussion of different MIMO detection schemes, including the maximum likelihood detection, linear receivers, decision feedback receivers and the V-BLAST detection algorithm, is given.

2.1 MIMO Systems

A traditional wireless communication system usually employs a Single-Input Single-Output (SISO) system in which a single transmitter antenna is used for transmission information to a single receiver antenna. However, the most significant disadvantage of SISO transmission is its low transmission data rate. In the past decade, this scenario has been greatly changed with the advent of Multiple-Input Multiple-Output (MIMO) communication systems. Information theoretic results conclude that MIMO systems can offer significant capacity gains and a higher transmission rate than SISO systems [2].

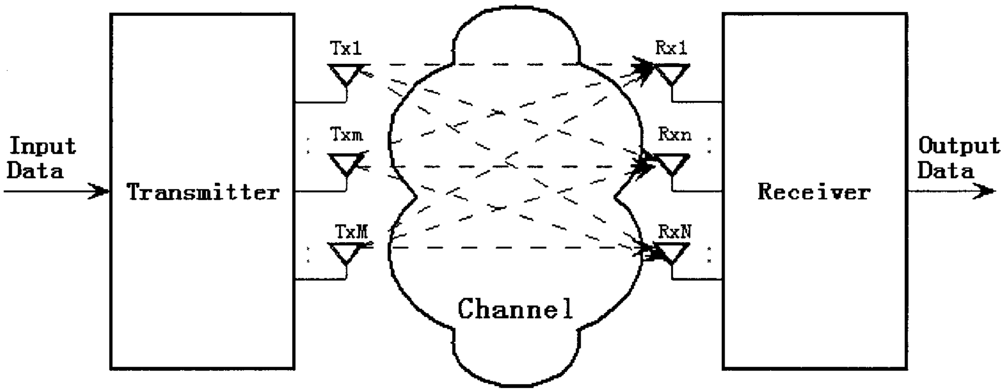

Figure 2.1: A MIMO Communication System

Over the past decade, MIMO has evolved in significance from an academic research to industrial implementation such as the next generation wireless standards, including UMTS (Universal Mobile Telecommunications System) and the IEEE802 standards family. For instance, the IEEE 802.16e standard incorporates MIMO-OFDMA and the IEEE 802.11n standard recommends the MIMO-OFDM. MIMO is also planned to be used in mobile radio telephone standards such as the 3GPP and 3GPP2 standards. All upcoming 4G systems will also employ the MIMO technology. All these applications make MIMO architecture a very attractive scheme for current wireless communications [28].

Fig. 2.1 shows a basic structure of a MIMO wireless system. There are M antennas at the transmitter, and N antennas at the receiver. Each transmitter antenna sends a symbol at each time instance. All the transmitter antennas are working synchronously. Each receiver antenna receives a combination of signals from all M transmitter antennas plus noise. When the channel is modeled as a flat fading channel, which means there is no interference from the previous transmitted symbols, on the receiver side, the received signal vector can be expressed as:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} \tag{2.1}$$

where:

- $\mathbf{x}$ is the transmitted $M \times 1$ vector, with x_i, the ith component of $\mathbf{x}$, being transmitted from transmitter antenna i.
- $\mathbf{H}$ is an $N \times M$ complex channel matrix. The ijth element h_{ij} of this matrix is the channel gain from the jth transmitter antenna to the ith receiver antenna.
- $\mathbf{y}$ is an $N \times 1$ received vector.
- $\mathbf{w}$ is an $N \times 1$ noise vector. The components of $\mathbf{w}$ are zero mean circularly symmetric complex Gaussian variables. If we assume there is no correlation between components of $\mathbf{w}$, the covariance matrix of the noise vector is

$$\mathbf{R}_{ww} = E\{\mathbf{w}\mathbf{w}^H\} = \sigma^2\mathbf{I}_N$$

MIMO communications can increase the system capacity by means of spatial multiplexing and improve transmission quality by taking advantage of spatial diversity. In fact, the MIMO transmission schemes can be mainly categorized as the capacity (or rate) maximization schemes and the diversity maximization schemes [29]. The system capacity or the transmission data rate over MIMO channels can be maximized by means of spatial multiplexing, where independent data streams are transmitted at the same time over multiple transmitter antennas. An example of spatial multiplexing is the Bell Labs BLAST architecture [3]. In this scheme, different symbol streams are simultaneously transmitted from all the transmitter antennas while the receiver antennas receive the superposition of all symbol streams and recover them by using advanced signal processing. On the other hand, the goal of diversity maximization is to reduce the error rate of the received signal. Space time coding (STC) is used for such diversity maximization. In STC systems, the same information symbol stream is transmitted from different transmitter antennas in a proper manner so as to improve the data transmission reliability. Therefore, the advantage of a MIMO channel can be utilized in two main approaches. First, to increase the number of transmitted symbols per unit time. Second, to increase the diversity of the wireless communication system. Although there are some researchers working on exploiting both the diversity gain and multiplexing gain, the tradeoff between the two gains is still a crucial research topic in MIMO communications. In our designs, we consider the MIMO channel that guarantee a higher data rate, the transmission model of which is given by the next section.

2.2 Transmission Model

In this thesis, we consider a precoded MIMO [3] communication system as shown in Fig. 2.2 having M antennas at the transmitter and N antennas at the receiver $(N > M)$ such that the signal transmission can be modelled as

$$\mathbf{y} = \mathbf{HTx} + \mathbf{w}, \tag{2.2}$$

where $\mathbf{y}$ is an $N \times 1$ received signal vector, $\mathbf{H}$ is an $N \times M$ channel matrix, $\mathbf{T}$ is an $M \times M$ precoding matrix, $\mathbf{x}$ is an $M \times 1$ transmitting signal vector and $\mathbf{w}$ is an $N \times 1$ complex noise vector. We make the usual assumptions that the transmitted symbols in $\mathbf{x}$ are uncorrelated with each other and uncorrelated with the circularly-symmetric complex Gaussian channel noise, the covariance of which is $\mathbf{R}_{ww} = \sigma^2\mathbf{I}_N$. The channel matrix $\mathbf{H}$ contains the channels h_{nm} linking the mth transmitter antenna to the nth receiver antenna. Each of the h_{nm} is a zero-mean, circularly-symmetric complex Gaussian distributed random variable with unit variance. Let the nth row of $\mathbf{H}$ be $\mathbf{h}_n^T = [\ h_{n1} \ \cdots \ h_{nM}\ ]$. We assume that

$$\mathrm{E}[\mathbf{h}_\ell \mathbf{h}_n^H] = \begin{cases} \boldsymbol{\Sigma} & \ell = n, \\ \mathbf{0}, & \ell \neq n. \end{cases} \tag{2.3}$$

In other words, we assume that the channels leading the signals to the nth receiver antenna are correlated among themselves, but are uncorrelated with the channel leading the signals to the ℓth receiver antenna if $\ell \neq n$. It is well-known [30] that

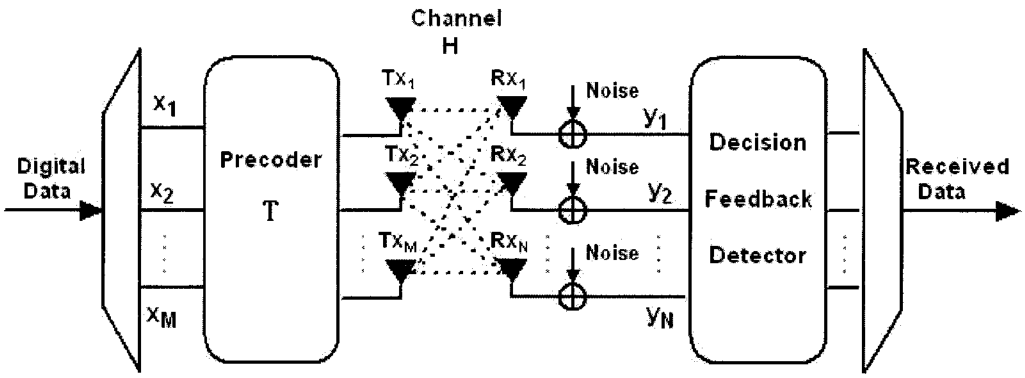

Figure 2.2: Transmission Model

$\mathbf{H}^H\mathbf{H}$ is of Wishart distribution, of N degrees of freedom and with covariance matrix Σ, denoted by $\mathcal{W}_M(N, \Sigma)$. If we consider the precoder matrix $\mathbf{T}$ as a part of the channel such that $\mathbf{H}' = \mathbf{HT}$, then we have

$$\mathrm{E}[\mathbf{h}'_\ell \mathbf{h}'^H_n] = \begin{cases} \mathbf{T}^H \Sigma \mathbf{T} & \ell = n \\ \mathbf{0} & \ell \neq n \end{cases} \tag{2.4}$$

Also, $\mathbf{H}'^H\mathbf{H}' = (\mathbf{HT})^H\mathbf{HT}$ is of Wishart distribution denoted by $\mathcal{W}_M(N, \mathbf{T}^H\Sigma\mathbf{T})$. The quantity $\mathbf{T}^H\Sigma\mathbf{T}$ is of fundamental importance in our application and we will examine its properties in greater details in the ensuing chapter.

2.3 Detection Schemes

The goal of the detector is to obtain an estimate of $\mathbf{x}$ from the given data in $\mathbf{y}$ and $\mathbf{H}$. In this section, we discuss different receivers for MIMO systems.

2.3.1 Maximum Likelihood Detection

The maximum likelihood Detection (MLD) is an optimum receiver for MIMO communication systems. In MLD, the probability of error is minimized such as

$$\min P_e \triangleq P(\mathbf{x} \neq \hat{\mathbf{x}}) \tag{2.5}$$

where $\hat{\mathbf{x}}$ is the estimated symbol vector.

Indeed, minimizing the probability of error is equivalent to maximizing the probability of correctly estimating $\mathbf{x}$ given $\mathbf{y}$ and $\mathbf{H}$, i.e.,

$$\max P(\mathbf{x} = \hat{\mathbf{x}} | \mathbf{y}, \mathbf{H}) \tag{2.6}$$

Note that this probability is equal to:

$$P(\mathbf{x} = \hat{\mathbf{x}} | \mathbf{y}, \mathbf{H}) = \frac{P(\mathbf{x} = \hat{\mathbf{x}})\; p_{\mathbf{y}|\mathbf{x},\mathbf{H}}(\mathbf{y} | \mathbf{x} = \hat{\mathbf{x}}, \mathbf{H})}{p_{\mathbf{y}|\mathbf{H}}(\mathbf{y}|\mathbf{H})} \tag{2.7}$$

where $p_{\mathbf{y}|\mathbf{x},\mathbf{H}}$ and $p_{\mathbf{y}|\mathbf{H}}$ are the conditional probability density functions of $\mathbf{y}$ given $\mathbf{x}, \mathbf{H}$ and $\mathbf{H}$ respectively. Since neither $P(\mathbf{x} = \hat{\mathbf{x}})$ nor $p_{\mathbf{y}|\mathbf{H}}(\mathbf{y}|\mathbf{H})$ depends on $\hat{\mathbf{x}}$, our objective is further equivalent to maximizing $p_{\mathbf{y}|\mathbf{x},\mathbf{H}}(\mathbf{y}|\mathbf{x} = \hat{\mathbf{x}}, \mathbf{H})$.

Thus, the criterion of MLD is given by:

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} p_{\mathbf{y}|\mathbf{x},\mathbf{H}}(\mathbf{y}|\mathbf{x} = \hat{\mathbf{x}}, \mathbf{H}) \tag{2.8}$$

Eq. (2.8) can be further simplified by applying the system equation in Eq. (2.1) to obtain

$$p_{\mathbf{y}|\mathbf{x},\mathbf{H}}(\mathbf{y}|\mathbf{x} = \hat{\mathbf{x}}, \mathbf{H}) = p_{\mathbf{w}}(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}) \tag{2.9}$$

where the probability density function of the white Gaussian noise $\mathbf{w}$ is given by

$$p_{\mathbf{w}}(\mathbf{w}) = \frac{1}{(\pi\sigma^2)^M} \exp\left(-\frac{1}{\sigma^2}\|\mathbf{w}\|\right) \tag{2.10}$$

Therefore, the likelihood of $\hat{\mathbf{x}}$ is obtained by substituting $\mathbf{w}$ with $\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}$ in Eq. (2.10). Since $p_{\mathbf{y}|\mathbf{x},\mathbf{H}}(\mathbf{y}|\mathbf{x} = \hat{\mathbf{x}}, \mathbf{H}) = p_{\mathbf{w}}(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}})$ is maximized by minimizing $\|\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}\|$, the ML estimate of $\mathbf{x}$ is given by

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2 \tag{2.11}$$

The MLD searches through all the vector constellation for the most probable transmitted signal vector, which is a very difficult task. Its computational complexity is prohibitive especially when the number of channels is large. Hence, these receivers are difficult to implement in practical situations.

It is desirable to find other detection schemes which is sub-optimal but has much lower computation complexity.

2.3.2 Linear Equalization

Linear receivers are the class of receivers for which the output symbol estimate $\hat{\mathbf{x}}$ is given by quantizing a linearly transformed version of the received vector $\mathbf{y}$

$$\begin{aligned}\hat{\mathbf{x}} &= \mathcal{Q}\left[\mathbf{x}'\right] \\ &= \mathcal{Q}\left[\mathbf{G}\mathbf{y}\right] \\ &= \mathcal{Q}\left[\mathbf{G}(\mathbf{H}\mathbf{x} + \mathbf{w})\right]\end{aligned}$$

where $\mathbf{G}$ is a matrix that may depend on $\mathbf{H}$ and $\mathcal{Q}\left[\cdot\right]$ is an element-wise quantization operation that maps each element of its argument to the nearest signal point in the constellation (using Euclidian distance). $\mathbf{x}'$ is the equalized signal. Hereby, we introduce both the zero-forcing (ZF) equalization and the minimize mean square error (MMSE) equalization.

- **Zero-foring Equalization**

  Zero-foring Equalization scheme behaves like a linear filter to separate the data streams and thereafter independently detects each stream. It can be applied to the system where the channel matrix $\mathbf{H}$ is invertible. We first multiply $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$ by

$$\mathbf{G} = \mathbf{H}^{\dagger} = (\mathbf{H}^{H}\mathbf{H})^{-1}\mathbf{H} \tag{2.12}$$

where $\mathbf{H}^{\dagger}$ denotes the Moore-Penrose pseudo inverse of $\mathbf{H}$. Thus,

$$\begin{aligned} \mathbf{x}' &= \mathbf{H}^{\dagger}\mathbf{H}\mathbf{x} + \mathbf{H}^{\dagger}\mathbf{w} \\ &= \mathbf{x} + \mathbf{w}' \end{aligned} \tag{2.13}$$

where $\mathbf{w}' = \mathbf{H}^{\dagger}\mathbf{w} = (\mathbf{H}^{H}\mathbf{H})^{-1}\mathbf{H}\mathbf{w}$, and the estimation $\hat{\mathbf{x}}$ is the quantization of $\mathbf{x}'$.

However, the problem of ZF equalization is in the covariance matrix of $\mathbf{w}'$ given by:

$$\begin{aligned} E(\mathbf{w}'\mathbf{w}'^{H}) &= (\mathbf{H}^{H}\mathbf{H})^{-1}\mathbf{H}\mathrm{E}[\mathbf{w}\mathbf{w}^{H}]\mathbf{H}^{H}(\mathbf{H}^{H}\mathbf{H})^{-H} \\ &= \sigma^{2}(\mathbf{H}^{H}\mathbf{H})^{-1}\mathbf{H}\mathbf{H}^{H}(\mathbf{H}\mathbf{H}^{H})^{-1} \\ &= \sigma^{2}(\mathbf{H}^{H}\mathbf{H})^{-1} \end{aligned}$$

When the channel matrix $\mathbf{H}$ is poorly conditioned, the variance of some element in the vector $\mathbf{w}'$ may be too high owing to the inverse of $\mathbf{H}^{H}\mathbf{H}$. This drawback of enhancing the noise power will degrade the error rate performance especially at lower SNRs.

- **MMSE Equalization**

Now, we examine another widely used linear receiver: the MMSE equalization. We want to choose a matrix $\breve{\mathbf{G}}$ that minimizes the mean square error between the equalized signal denoted by $\mathbf{x}'$ and the transmitted signal $\mathbf{x}$. Since the first operation of linear receiver is multiplying Eq. (2.1) by $\breve{\mathbf{G}}$ forming

$$\mathbf{x}' = \breve{\mathbf{G}}\mathbf{y} = \breve{\mathbf{G}}\mathbf{H}\mathbf{x} + \breve{\mathbf{G}}\mathbf{w} \tag{2.14}$$

then the error vector between between $\mathbf{x}'$ and $\mathbf{x}$ can be written as

$$\mathbf{e} = (\breve{\mathbf{G}}\mathbf{H} - \mathbf{I})\mathbf{x} + \breve{\mathbf{G}}\mathbf{w} \tag{2.15}$$

The MSE is given by

$$\epsilon = \mathrm{E}[\mathrm{tr}(\mathbf{e}\mathbf{e}^H)] \tag{2.16}$$

From [31], we know that the minimum value of the MSE is reached when the following equation holds.

$$\mathrm{E}[\mathbf{e}\mathbf{y}^H] = \mathbf{0} \tag{2.17}$$

Substituting Eq. (2.15) into Eq. (2.17), we have

$$\begin{aligned}\mathrm{E}[\mathbf{e}\mathbf{y}^H] &= \mathrm{E}\Big[\big((\breve{\mathbf{G}}\mathbf{H} - \mathbf{I})\mathbf{x} + \breve{\mathbf{G}}\mathbf{w}\big)\big(\mathbf{H}\mathbf{x} + \mathbf{w}\big)^H\Big] \\ &= \breve{\mathbf{G}}\mathbf{H}\mathbf{H}^H - \mathbf{H}^H + \sigma^2\breve{\mathbf{G}} \\ &= \mathbf{0}\end{aligned} \tag{2.18}$$

Hence, the optimal matrix $\breve{\mathbf{G}}$ for MMSE linear receiver is given by

$$\breve{\mathbf{G}} = \mathbf{H}^H\big(\frac{1}{\sigma^2}\mathbf{I} + \mathbf{H}\mathbf{H}^H\big)^{-1} \tag{2.19}$$

The ZF receiver can separate the co-channels' signal at the cost of noise enhancement. The MMSE equalization, on the other hand, has ability to minimize the overall mean square error caused by noise and mutual interference between the co-channel signals, but this is at the cost of separation quality of the signals [32]. In addition, the MMSE receiver needs the noise variance which may not be easily estimated in some situations. But the ZF receiver does not need such information. Therefore, the ZF receiver and the MMSE receiver can be used in different scenarios.

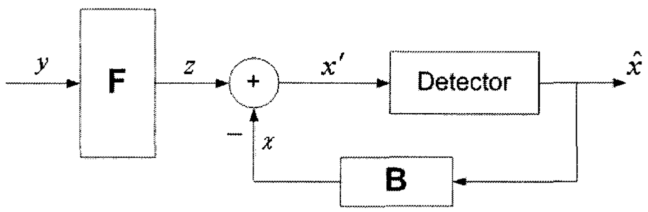

Figure 2.3: A Decision Feedback Receiver

2.3.3 Zero-forcing (ZF) Decision Feedback Receivers

While the ML receiver yields optimum performance but suffers from high complexity and while the linear receiver is simple to implement but suffers from loss of performance, the Decision Feedback (DF) receiver enjoys having a desirable trade-off between good performance and implementation complexity [33]. We therefore focus on the use of DF receiver in this thesis. In particular, we concentrate on the DF receiver which employs the zero-forcing (ZF) strategy, i.e., a ZF-DF receiver.

In the following, a detailed review of decision feedback detection is given as well as its equivalence with the QR-decomposition.

For the detection of block data, the DF receiver (Fig. 2.3) employs a feed-forward filter $\mathbf{F}$ to process the received data vector forming $\mathbf{z} = \mathbf{F}\mathbf{y}$. The estimated symbol vector $\mathbf{x}'$ is given by this processed data vector $\mathbf{z}$ subtracting the output $\boldsymbol{\chi}$ of a feedback filter $\mathbf{B}$. The detected symbol vector $\hat{\mathbf{x}}$ is then the quantized version of the estimated symbol vector. The detection of the mth symbol, x_m, of the transmitted vector $\mathbf{x}$ proceeds sequentially, starting from the last symbol, as follows:

1. Starting from $m = M$, we make $x'_M = z_M$;

2. Let $\chi_m = \sum_{\ell=m+1}^{M} b_{m\ell}\hat{x}_\ell$ be the output of the feedback filter $\mathbf{B}$, with $b_{m\ell}$ being its coefficients. The states of this filter, $\hat{x}_\ell$, are the previously detected symbols in the block and the filter coefficients are different for each element of the block (indexed by m).

3. Let $x'_m = z_m - \chi_m$, $m = M-1, M-2, \ldots, 1$.

Once a given block has been detected, the states of the feedback filter are reset to zero, so that error propagation between blocks is avoided.

Hence, we can write the detection procedure as:

$$\mathbf{x}' = \mathbf{z} - \mathbf{B}\hat{\mathbf{x}} = \mathbf{FHx} + \mathbf{Fw} - \mathbf{B}\hat{\mathbf{x}} \tag{2.20}$$

Since the detection starts from the last symbol $\hat{x}_M$, and the symbol χ_m to be cancelled is a linear combination of the previous detected symbols, the filter $\mathbf{B}$ is a strictly upper triangular matrix such that

$$\mathbf{B} = \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1M} \\ 0 & 0 & b_{23} & \cdots & b_{2M} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{(M-1)M} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where all its elements $b_{m\ell}$, $m = 1, \cdots, M-1$; $\ell = m+1, \cdots, M$ are non-zero.

Assuming that the detected symbols are all correct, i.e., $\hat{\mathbf{x}} = \mathbf{x}$, then

$$\hat{\mathbf{x}} = \mathcal{Q}[\mathbf{x}'] = \mathcal{Q}[(\mathbf{FHT} - \mathbf{B})\mathbf{x} + \mathbf{Fw}] \tag{2.21}$$

Therefore, if we denote the error of detection by $\mathbf{e} = \mathbf{x}' - \mathbf{x}$, then from Eq. (2.20), the error of the DF receiver can be written as:

$$\mathbf{e} = (\mathbf{FH} - \mathbf{I})\mathbf{x} + \mathbf{Fw} - \mathbf{B}\hat{\mathbf{x}} \tag{2.22}$$

Assuming that the previously detected symbols are all correct, i.e., $\hat{\mathbf{x}} = \mathbf{x}$, we have:

$$\mathbf{e} = (\mathbf{FH} - \mathbf{I} - \mathbf{B})\mathbf{x} + \mathbf{Fw}$$

The covariance matrix of this error is:

$$\mathbf{R}_{ee} = (\mathbf{FH} - \mathbf{I} - \mathbf{B})(\mathbf{FH} - \mathbf{I} - \mathbf{B})^H + \mathbf{F}\mathbf{R}_{ww}\mathbf{F}^H \tag{2.23}$$

The arithmetic mean square error (MSE) of the detector input is

$$\begin{aligned} \varepsilon^2 &= \frac{1}{M} \operatorname{tr}\left(\mathrm{E}\left[\mathbf{e}\mathbf{e}\right]^{\mathrm{H}}\right) \\ &= \frac{1}{M} \operatorname{tr}\left(\mathrm{E}\left[(\mathbf{x}' - \mathbf{x})(\mathbf{x}' - \mathbf{x})^H\right]\right) \\ &= \frac{1}{M} \operatorname{tr}\left(\mathbf{R}_{ee}\right) \end{aligned}$$

For zero-forcing criterion (ZF-DFE), in Eq. (2.23), we have

$$\mathbf{FH} - \mathbf{I} - \mathbf{B} = \mathbf{0} \tag{2.24}$$

Hence, we can obtain the ZF feedforward matrix $\mathbf{F}_{\mathrm{ZF}}$ such that

$$\mathbf{F}_{\mathrm{ZF}} = (\mathbf{B} + \mathbf{I})\mathbf{H}^{\dagger} \tag{2.25}$$

where $\mathbf{H}^{\dagger}$ denotes the Moore-Penrose pseudo inverse of $\mathbf{H}$.

And the arithmetic MSE of ZD-DF receiver can be written as:

$$\begin{aligned} \overline{\varepsilon^2} \triangleq \varepsilon_{\mathrm{ZF}}^2 &= \frac{1}{M} \operatorname{tr}\left(\mathbf{R}_{ee}\right) \\ &= \frac{1}{M} \operatorname{tr}\left(\mathbf{F}\mathbf{R}_{ww}\mathbf{F}^H\right) \\ &= \frac{\sigma^2}{M} \operatorname{tr}\left(\mathbf{F}\mathbf{F}^{\mathrm{H}}\right) \end{aligned} \tag{2.26}$$

The mathematical operation of the DF receiver is best described by the QR-decomposition [34] of $\mathbf{HT}$ in Eq. (2.2) such that

$$\mathbf{HT} = \mathbf{QR} \tag{2.27}$$

where $\mathbf{Q}$ is an $N \times M$ orthonormal matrix and $\mathbf{R}$ is an $M \times M$ upper triangular matrix with $r_{ii} > 0$. Then, left-multiplying Eq. (2.2) by $\mathbf{Q}^H$ yields

$$\eta = \mathbf{Q}^H\mathbf{y} = \mathbf{R}\mathbf{x} + \mathbf{w}' \tag{2.28}$$

where $\mathbf{w}' = \mathbf{Q}^H\mathbf{w}$.

Eq. (2.28) can be equivalently written as:

$$\eta_m = r_{mm}x_m + \sum_{k=m+1}^{M} r_{mk}x_k + w'_m, \quad m = 1, \cdots, M \tag{2.29}$$

Now, we employ Eq. (2.28) to estimate the mth transmitted symbol x_m, starting from the last received symbol ($m = M$), such that

$$x'_m = \frac{\left(\eta_m - \sum_{k=m+1}^{M} r_{mk}\hat{x}_k\right) - w'_m}{r_{mm}}, \quad m = M, \cdots, 1 \tag{2.30}$$

The detected symbol $\hat{x}_m$ is the quantized version of x'_m, i.e., $\hat{x}_m = \mathcal{Q}[x'_m]$.

The ZF strategy demands that the term inside the parentheses in the numerator of Eq. (2.30) be zero. Comparing Eq. (2.30) with the operation of the DF receiver in Fig. 2.3, and *assuming no error in the previous detections*, we have

$$\mathbf{F} = \mathrm{diag}(r_{11}^{-1}, r_{22}^{-1}, \cdots, r_{MM}^{-1})\mathbf{Q}^H \tag{2.31a}$$

$$\chi_m = \left(\sum_{k=m+1}^{M} r_{mk}\hat{x}_k\right) \Big/ r_{mm} \tag{2.31b}$$

$$b_{mk} = r_{mk}/r_{mm}, \qquad \text{for } m \neq k \tag{2.31c}$$

where b_{mk} are the non-zero off-diagonal elements of the upper triangular matrix feedback filter $\mathbf{B}$.

Because of the equivalence of ZF-DF receiver and the QR decomposition, under the assumption that the previous symbols have been perfectly detected, the error can be further written as:

$$\mathbf{e} = \mathbf{x} - \mathbf{x}' = \mathrm{diag}(r_{11}^{-1}, r_{22}^{-1}, \cdots, r_{MM}^{-1})\mathbf{Q}^H\mathbf{w} \tag{2.32}$$

Therefore, the arithmetic MSE of the ZF-DF detector, denoted by $\overline{\varepsilon^2}$ as in Eq. (2.26), is further written as

$$\overline{\varepsilon^2} = \frac{\sigma^2}{M} \operatorname{tr}\left(\mathbf{F}\mathbf{F}^{\mathrm{H}}\right) = \frac{\sigma^2}{M} \sum_{m=1}^{M} r_{mm}^{-2} \tag{2.33}$$

where $r_{11}, r_{22}, \cdots, r_{MM}$ are diagonal elements of the $\mathbf{R}$ matrix in the QR-decomposition of $\mathbf{HT}$.

2.3.4 MMSE Decision Feedback Receivers

In the previous section, we have discussed the DF receiver using ZF schemes. In this section, we consider DF detection based on the minimum mean square error criterion.

To determine a condition for the MMSE-DF receiver, the orthogonality principle is employed. Let us first introduce the orthogonality principle. Since the error between $\mathbf{x}'$ and $\mathbf{x}$ is given by Eq. (2.22), which is further equal to $\mathbf{e} = \mathbf{F}\mathbf{y} - (\mathbf{B}+\mathbf{I})\mathbf{x}$. Then, the arithmetic average mean square error (MSE) of the detector input is given by

$$\begin{aligned} \varepsilon^2 &= \frac{1}{M} \operatorname{tr}\left(\mathrm{E}\left[\mathbf{e}\mathbf{e}\right]^{\mathrm{H}}\right) \\ &= \frac{1}{M} \operatorname{tr}\left(\mathrm{E}\big[(\mathbf{F}\mathbf{y} - (\mathbf{B}+\mathbf{I})\mathbf{x})(\mathbf{F}\mathbf{y} - (\mathbf{B}+\mathbf{I})\mathbf{x})^H\big]\right) \\ &= \frac{1}{M}\Big(\operatorname{tr}(\mathbf{F}\ \mathrm{E}[\mathbf{y}\mathbf{y}^H]\ \mathbf{F}^H) - 2\operatorname{tr}((\mathbf{B}+\mathbf{I})\ \mathrm{E}[\mathbf{x}\mathbf{y}^H]\ \mathbf{F}^H) \\ &+ \operatorname{tr}((\mathbf{B}+\mathbf{I})\ \mathrm{E}[\mathbf{x}\mathbf{x}^H]\ (\mathbf{B}+\mathbf{I})^H)\Big) \end{aligned} \tag{2.34}$$

Given the feedback matrix $\mathbf{B}$, the feedforward filter $\mathbf{F}$ is determined by minimizing Eq. (2.34).

Since the first order derivative of Eq. (2.34) with respect to $\mathbf{F}$ is given by

$$\nabla\varepsilon^2 = 2\mathrm{E}\left[\mathbf{e}\mathbf{y}^H\right] \tag{2.35}$$

For the minimum value of the MSE, the first order derivative (or the gradient matrix) $\nabla\varepsilon^2$ must be set to a zero vector, i.e.,

$$\mathrm{E}\left[\mathbf{e}\mathbf{y}^H\right] = \mathbf{0} \tag{2.36}$$

Indeed, the orthogonality principle is the necessary and sufficient condition for the minimized MSE as given by Eq. (2.36). Therefore, the DFE is optimal in the MSE sense when the error $\mathbf{e}$ is orthogonal to $\mathbf{y}$, the input signal of the DFE.

As showed in Fig. 2.3, the received vector of DF receiver is $\mathbf{y} = \mathbf{Hx} + \mathbf{w}$. Hence, the covariance matrix of $\mathbf{y}$ is

$$\begin{aligned} \mathbf{R}_{yy} &= \mathrm{E}\left[\mathbf{y}\mathbf{y}^H\right] \\ &= \mathrm{E}\left[(\mathbf{Hx}+\mathbf{w})(\mathbf{Hx}+\mathbf{w})^H\right] \\ &= \mathbf{H}\mathbf{H}^H + \mathbf{R}_{ww} \end{aligned} \tag{2.37}$$

and the cross correlation matrix of $\mathbf{x}$ and $\mathbf{y}$ is

$$\begin{aligned} \mathbf{R}_{xy} &= \mathrm{E}\left[\mathbf{x}\mathbf{y}^H\right] \\ &= \mathrm{E}\left[\mathbf{x}(\mathbf{Hx}+\mathbf{w})^H\right] \\ &= \mathbf{H}^H \end{aligned} \tag{2.38}$$

and $\mathbf{R}_{xy} = \mathbf{R}_{yx}^H$

Thus,

$$\begin{aligned} \mathrm{E}\left[\mathbf{e}\mathbf{y}^H\right] &= \mathrm{E}\left[(\mathbf{Fy}-(\mathbf{B}+\mathbf{I})\mathbf{x})\mathbf{y}^H\right] \\ &= \mathbf{F}\mathbf{R}_{yy} - (\mathbf{B}+\mathbf{I})\mathbf{R}_{xy} \\ &= \mathbf{0} \end{aligned} \tag{2.39}$$

Therefore, the feedforward matrix of the MMSE-DFE ($\mathbf{F}_{\text{MMSE}}$) is given by:

$$\begin{aligned} \mathbf{F}_{\text{MMSE}} &= (\mathbf{B}+\mathbf{I})\mathbf{R}_{xy}\mathbf{R}_{yy}^{-1} \\ &= (\mathbf{B}+\mathbf{I})\mathbf{H}^H(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1} \end{aligned} \tag{2.40}$$

Substituting Eq. (2.40) into Eq. (2.23), we have

$$\begin{aligned}
\mathbf{R}_{ee} &= ((\mathbf{B}+\mathbf{I})\mathbf{H}^H(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{H}-\mathbf{I}-\mathbf{B})((\mathbf{B}+\mathbf{I})\mathbf{H}^H(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{H}-\mathbf{I}-\mathbf{B})^H \\
&+ (\mathbf{B}+\mathbf{I})\mathbf{H}^H(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{R}_{ww}(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{H}(\mathbf{B}+\mathbf{I})^H \\
&= (\mathbf{B}+\mathbf{I})\big[\mathbf{H}^H(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{H}\mathbf{H}^H(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{H}-2\mathbf{H}^H(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{H} \\
&+ \mathbf{I}+\mathbf{H}^H(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{R}_{ww}(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{H}\big](\mathbf{B}+\mathbf{I})^H \\
&= (\mathbf{B}+\mathbf{I})\big[\mathbf{I}-\mathbf{H}^H(\mathbf{H}\mathbf{H}^H+\mathbf{R}_{ww})^{-1}\mathbf{H}\big](\mathbf{B}+\mathbf{I})^H \qquad (2.41)
\end{aligned}$$

Using the Matrix Inversion Lemma in [35], we can rewrite $\mathbf{R}_{ee}$ of MMSE-DF receiver as:

$$\begin{aligned}
\mathbf{R}_{ee} &= (\mathbf{B}+\mathbf{I})(\mathbf{I}+\mathbf{H}^H\mathbf{R}_{ww}^{-1}\mathbf{H})^{-1}(\mathbf{B}+\mathbf{I})^H \\
&= (\mathbf{B}+\mathbf{I})(\mathbf{I}+\frac{1}{\sigma^2}\mathbf{H}^H\mathbf{H})^{-1}(\mathbf{B}+\mathbf{I})^H \qquad (2.42)
\end{aligned}$$

Now, we perform QR decomposition to the matrix of $(\mathbf{I}+\frac{1}{\sigma^2}\mathbf{H}^H\mathbf{H})^{\frac{1}{2}}$ such that

$$(\mathbf{I}+\frac{1}{\sigma^2}\mathbf{H}^H\mathbf{H})^{\frac{1}{2}} = \bar{\mathbf{Q}}\bar{\mathbf{R}} \qquad (2.43)$$

where $\bar{\mathbf{Q}}$ is also an $N \times M$ orthonormal matrix and $\bar{\mathbf{R}}$ is an $M \times M$ upper triangular matrix with diagonal elements $\bar{r}_{mm} > 0$ for $m = 1, \cdots, M$.

Therefore,

$$\begin{aligned}
\mathbf{I}+\frac{1}{\sigma^2}\mathbf{H}^H\mathbf{H} &= (\bar{\mathbf{Q}}\bar{\mathbf{R}})^H(\bar{\mathbf{Q}}\bar{\mathbf{R}}) \\
&= \bar{\mathbf{R}}^H\bar{\mathbf{R}} \qquad (2.44)
\end{aligned}$$

If we extract the diagonal elements of $\bar{\mathbf{R}}$ matrix as a diagonal matrix such that

$$\begin{aligned}
\bar{\mathbf{R}} &= \text{diag}\,(\bar{r}_{11}, \bar{r}_{22}, \cdots, \bar{r}_{MM}) \begin{bmatrix} 1 & \frac{\bar{r}_{12}}{\bar{r}_{11}} & \cdots & \frac{\bar{r}_{1M}}{\bar{r}_{11}} \\ 0 & 1 & \ddots & \frac{\bar{r}_{2M}}{\bar{r}_{22}} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\
&\triangleq \text{diag}(\bar{r}_{11}, \bar{r}_{22}, \cdots, \bar{r}_{MM})\bar{\mathbf{L}}^H \qquad (2.45)
\end{aligned}$$

with $\bar{\mathbf{L}}^H$ being the upper triangular matrix with all diagonal elements being unity as defined in Eq. (2.45).

Now, substituting Eq. (2.45) into Eq. (2.44), we have

$$\begin{aligned} \mathbf{I} + \frac{1}{\sigma^2}\mathbf{H}^H\mathbf{H} &= \bar{\mathbf{R}}^H\bar{\mathbf{R}} \\ &= \bar{\mathbf{L}} \ \mathrm{diag}(\bar{r}_{11}^2, \bar{r}_{22}^2, \cdots, \bar{r}_{MM}^2) \ \bar{\mathbf{L}}^H \end{aligned} \tag{2.46}$$

Thus, if we let $\mathbf{B} + \mathbf{I} = \bar{\mathbf{L}}^{-1}$, the error covariance matrix of MMSE-DF receiver becomes

$$\begin{aligned} \mathbf{R}_{ee} &= (\mathbf{B} + \mathbf{I})(\mathbf{I} + \frac{1}{\sigma^2}\mathbf{H}^H\mathbf{H})^{-1}(\mathbf{B} + \mathbf{I})^H \\ &= \mathrm{diag}(\bar{r}_{11}^{-2}, \bar{r}_{22}^{-2}, \cdots, \bar{r}_{MM}^{-2}) \end{aligned} \tag{2.47}$$

Hence, the MSE of the MMSE-DF receiver is given by

$$\begin{aligned} \varepsilon_{\text{MMSE}}^2 &= \frac{1}{M} \ \mathrm{tr} \ (\mathbf{R}_{ee}) \\ &= \frac{1}{M} \sum_{m=1}^{M} \bar{r}_{mm}^{-2} \end{aligned} \tag{2.48}$$

where $\bar{r}_{11}, \bar{r}_{22}, \cdots, \bar{r}_{MM}$ are diagonal elements of the $\bar{\mathbf{R}}$ matrix in the QR-decomposition of $(\mathbf{I} + \frac{1}{\sigma^2}\mathbf{H}^H\mathbf{H})^{\frac{1}{2}}$.

2.3.5 V-BLAST Receiver

Successive processing usually leads to the problem of error propagation. If we assume that all the previously detected symbol vectors of the DF receivers are correct, then there is no error propagation, in which case the error rate performance of the DF receiver is dominated by the weakest data stream. To minimize the risk of this problem, the ordered DFE was introduced. This ordered successive interference cancellation scheme is generally called V-BLAST detection algorithm [3].

The optimal ordering in V-BLAST is a method of minimizing the unavoidable error propagation by sorting out the order of the data streams being detected. Basically, from the knowledge of the channel matrix, the V-BLAST scheme decides on the order of detecting the symbols by first selecting the one which associates with the channel vector having the highest SNR. Since the vector with the higher post detection SNR may have lower error probabilities, thus processing these vectors at earlier stages leads to less error.

After threshold detection of the selected symbol, the associated channel vector is eliminated from the original channel matrix forming a diminished matrix. The symbol having the next highest reception SNR is then selected for detection and the procedure of elimination of the corresponding channel vector continues until the sequential detection of all the symbols is complete. This ordering of symbol detection of V-BLAST based on the received symbol SNR yields superior error rate performance compared to that of a similar scheme without the ordering.

Now, we discuss two different V-BLAST (ordered DFE) algorithms [3] [36]: the Zero-forcing V-BLAST algorithm and the MMSE V-BLAST algorithm.

- ZF V-BLAST

Initialization :

$$i \leftarrow 1 \tag{2.49a}$$

$$\mathbf{y}_1 = \mathbf{y} \tag{2.49b}$$

$$\mathbf{G}_1 = \mathbf{H}^{\dagger} \tag{2.49c}$$

$$k_1 = \arg\min_j \|(\mathbf{g}_1)_j^T\|^2 \tag{2.49d}$$

Recursion :

$$\mathbf{w}_{k_i} = (\mathbf{g}_i)_{k_i}^T \tag{2.49e}$$

$$r_{k_i} = \mathbf{w}_{k_i}^T \mathbf{y}_i \tag{2.49f}$$

$$\hat{x}_{k_i} = \mathcal{Q}[r_{k_i}] \tag{2.49g}$$

$$y_{i+1} = y_i - \hat{x}_{k_i} \mathbf{h}_{k_i}^T \tag{2.49h}$$

$$\mathbf{G}_{i+1} = \mathbf{H}_{\bar{k}_i}^{\dagger} \tag{2.49i}$$

$$k_{i+1} = \arg \min_{j \notin \{k_1, \cdots, k_i\}} \|(\mathbf{g}_{i+1})_j^T\|^2 \tag{2.49j}$$

$$i \leftarrow i+1 \tag{2.49k}$$

where $\mathbf{H}_{\bar{k}_i}$ denotes the matrix obtained by setting columns $k_1, \cdots, k_i$ of $\mathbf{H}$ to zero columns. And $\dagger$ denotes the Morre-Penrose pseudo inverse of a matrix. $\mathbf{G}_i$ is the ith iteration of $\mathbf{G}$ and $(\mathbf{g}_i)_j^T$ denotes the jth row of $\mathbf{G}_i$.

In the recursive procedure of ZF V-BLAST, Eq. (2.49d) determines the optimal row of $\mathbf{G}_1 = \mathbf{H}^{\dagger}$ with the strongest SNR. Eq. (2.49e) to Eq. (2.49g) compute the ZF nulling vector, the decision statistic r_{k_i} and the estimated value of $\mathbf{x}$, respectively. And Eq. (2.49h) performs the cacellation of the effect of the detected signal vector from the received signal vector to reduce the detection complexity for the remaining signals. Eq. (2.49i) calculates the new pseudo inverse for the next iteration. This new pseudo inverse is based on the matrix obtained by zeroing columns $k_1, \cdots, k_i$ of $\mathbf{H}$. This is because these columns correspond to components of $\mathbf{x}$, which have already been estimated and canceled [36].

- MMSE V-BLAST

  The V-BLAST combined with the MMSE algorithm is the same as that of ZF V-BLAST but with

$$\mathbf{G}_1 = \left(\mathbf{H}^H\mathbf{H} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{H}^H \tag{2.50}$$

  The MMSE V-BLAST suppresses both the interference and noise components, whereas the ZF V-BLAST only removes the interference components.

We will also compare the performance of our optimally precoded system using the unordered ZF-DF receiver with that of the ZF V-BLAST detection scheme which employs an additional optimal ordering procedure.

Chapter 3

Minimum MSE Precoder for ZF-DF Receivers

3.1 Problem Description

Our purpose in this chapter is to propose an efficient technique for designing a precoder matrix $\mathbf{T}$ that minimizes the average arithmetic MSE of ZF-DF receiver. We assume that the transmitter only knows the first- and second-order statistics of the channel. Referring to Eq. (2.33) for the MSE of the ZF-DF receiver, our problem can be formally stated as:

Problem 3.1. *Design a precoding matrix* $\mathbf{T}$ *such that*

$$\mathbf{T}_{\text{opt}} = \arg \min_{\mathbf{T}} \mathrm{E}_{\mathbf{H}}\Big(\sum_{m=1}^{M} r_{mm}^{-2}\Big) \tag{3.1}$$

subject to the total transmitting power constraint

$$\mathrm{tr}(\mathbf{T}^H\mathbf{T}) \leq p_0$$

where p_0 is the total transmitting power and the notation $\mathrm{E}_{\mathbf{H}}(\cdot)$ denotes the expectation taken over all random channel realizations.

3.2 Analytic Properties of the System Matrices

Since the channel matrix $\mathbf{H}$ and the transmission precoder $\mathbf{T}$ are central to the solution of Problem 3.1, we will examine some properties of these system matrices. First, let us visit the mathematical concept of majorization and Schur convexity [37] [38].

3.2.1 Majorization and Schur Convexity

Majorization theory is a powerful tool that allows us to transform the original problem into a convex problem. In this section, we introduce the basic notions of majorization and state some useful results.

Let $\mathbf{a} = [a_1, a_2, \cdots, a_M]^T$ and $\mathbf{b} = [b_1, b_2, \cdots, b_M]^T$ be two M-dimensional real-valued sequences satisfying $a_{[1]} \geq a_{[2]} \geq \cdots \geq a_{[M]}$ and $b_{[1]} \geq b_{[2]} \geq \cdots \geq b_{[M]}$.

Definition 3.1. *The sequence* $\mathbf{a}$ *is said to be* additively majorized *by sequence* $\mathbf{b}$*, denoted by* $\mathbf{a} \prec_+ \mathbf{b}$*, if*

$$\sum_{m=1}^{K} a_{[m]} \leq \sum_{m=1}^{K} b_{[m]}, \qquad 1 \leq K < M \tag{3.2a}$$

$$\sum_{m=1}^{M} a_{[m]} = \sum_{m=1}^{M} b_{[m]} \tag{3.2b}$$

Parallel to the definition of *additive* majorization is the concept of *multiplicative* majorization.

Definition 3.2. *If* $a_1, a_2, \cdots, a_M$ *and* $b_1, b_2, \cdots, b_M$ *are positive numbers. The sequence* $\mathbf{a}$ *is said to be* multiplicatively majorized *by sequence* $\mathbf{b}$*, denoted by* $\mathbf{a} \prec_\times \mathbf{b}$*, if*

$$\prod_{m=1}^{K} a_{[m]} \leq \prod_{m=1}^{K} b_{[m]}, \qquad 1 \leq K < M \tag{3.3a}$$

$$\prod_{m=1}^{M} a_{[m]} = \prod_{m=1}^{M} b_{[m]} \tag{3.3b}$$

There is also a simple relationship between *additive* majorization and *multiplicative* majorization such that

$$\mathbf{a} \prec_{+} \mathbf{b} \ \text{ if and only if } \ \exp(\mathbf{a}) \prec_{\times} \exp(\mathbf{b}) \tag{3.4}$$

Majorization is a partial ordering among vectors that have the same sum or product. It is a measure of the degree of similarity between the vector elements.

Definition 3.3. *A real-valued function ϕ defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex on $\mathcal{A}$ if*

$$\mathbf{a} \prec_{+} \mathbf{b} \qquad on \quad \mathcal{A} \Rightarrow \phi(\mathbf{a}) \leq \phi(\mathbf{b}) \tag{3.5a}$$

If strict inequality holds such that $\phi(\mathbf{a}) < \phi(\mathbf{b})$ whenever $\mathbf{a} \prec_{+} \mathbf{b}$ but $\mathbf{a}$ is not a permutation of $\mathbf{b}$, then ϕ is said to be strictly Schur-convex on $\mathcal{A}$.

Similarly, ϕ is said to be Schur-concave on $\mathcal{A}$ if

$$\mathbf{a} \prec_{+} \mathbf{b} \qquad on \quad \mathcal{A} \Rightarrow \phi(\mathbf{a}) \geq \phi(\mathbf{b}) \tag{3.5b}$$

and ϕ is strictly Schur-concave on $\mathcal{A}$ if $\phi(\mathbf{a}) > \phi(\mathbf{b})$, for $\mathbf{a}$ not being a permutation of $\mathbf{b}$.

For ϕ being Schur-convex or Schur-concave, equality holds in Eqs. (3.5a) or (3.5b) if $\mathbf{a} = \mathbf{b}$. *For ϕ being* strictly *Schur-convex or Schur-concave, equality in Eqs. (3.5a) or (3.5b) holds* if and only if $\mathbf{a} = \mathbf{b}$.

Obviously, if ϕ is Schur convex on $\mathcal{A}$, then $-\phi$ is Schur concave on $\mathcal{A}$ and vice versa.

3.2.2 QR Decomposition

We first examine the QR decomposition of the matrix product $\mathbf{HT}$ such that

$$\mathbf{H}' \triangleq \mathbf{HT} = \mathbf{QR} \tag{3.6}$$

Then,

$$\mathbf{T}^H\mathbf{H}^H\mathbf{H}\mathbf{T} = (\mathbf{Q}\mathbf{R})^H\mathbf{Q}\mathbf{R} = \mathbf{R}^H\mathbf{R} \tag{3.7}$$

Now, extracting the diagonal elements $\{r_{mm}\}$ for $m = 1, 2, \cdots, M$ as a diagonal matrix such that

$$\begin{aligned} \mathbf{R} &= \text{diag}\left(r_{11}, r_{22}, \cdots, r_{MM}\right) \begin{bmatrix} 1 & \frac{r_{12}}{r_{11}} & \cdots & \frac{r_{1M}}{r_{11}} \\ 0 & 1 & \ddots & \frac{r_{2M}}{r_{22}} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ &\triangleq \text{diag}(r_{11}, r_{22}, \cdots, r_{MM})\mathbf{L}^H \end{aligned} \tag{3.8}$$

with $\mathbf{L}^H$ being the upper triangular matrix with all diagonal elements being unity as defined in Eq. (3.8), we obtain

$$\mathbf{H}'^H\mathbf{H}' = \mathbf{T}^H\mathbf{H}^H\mathbf{H}\mathbf{T} = \mathbf{L}\ \text{diag}(r_{11}^2, r_{22}^2, \cdots, r_{MM}^2)\ \mathbf{L}^H \tag{3.9}$$

from which we can write

$$\det(\mathbf{H}'^H\mathbf{H}') = \prod_{m=1}^{M} r_{mm}^2 \tag{3.10}$$

In order to compute r_{mm}^2, $m = 1, 2, \cdots, M$, we recall that $\mathbf{H}'_{kk}$ denotes the matrix consisting of the first k rows and columns of $\mathbf{H}'$. We note that removing the Mth row and column of $\mathbf{H}'$, the resulting matrix $\mathbf{H}'_{(M-1)(M-1)}$ is equal to product of the matrix $\mathbf{Q}$ with the Mth row removed and the matrix $\mathbf{R}$ with the Mth column removed, i.e., $\mathbf{H}'_{(M-1)(M-1)} = \mathbf{Q}_{(M-1)M}\mathbf{R}_{M(M-1)}$, therefore,

$$\det(\mathbf{H}'^H_{(M-1)(M-1)}\mathbf{H}'_{(M-1)(M-1)}) = \prod_{m=1}^{M-1} r_{mm}^2 \tag{3.11}$$

From Eqs. (3.10) and (3.11) we have

$$\begin{aligned} r_{MM}^2 &= \prod_{m=1}^{M} r_{mm}^2 \Big/ \prod_{m=1}^{M-1} r_{mm}^2 \\ &= \frac{\det\left(\mathbf{H}'^H_{MM}\mathbf{H}'_{MM}\right)}{\det\left(\mathbf{H}'^H_{(M-1)(M-1)}\mathbf{H}'_{(M-1)(M-1)}\right)} \end{aligned} \tag{3.12}$$

Continuing the above process, we obtain a general formula for the evaluation of r_{mm}^2 such that

$$r_{mm}^2 = \frac{\det\left(\mathbf{H'}_{mm}^H \mathbf{H'}_{mm}\right)}{\det\left(\mathbf{H'}_{(m-1)(m-1)}^H \mathbf{H'}_{(m-1)(m-1)}\right)} \tag{3.13}$$

Now, let us define $\tilde{\mathbf{H}} \triangleq \mathbf{\Sigma}^{\frac{1}{2}}\mathbf{T}$, where $\mathbf{\Sigma}$ is the covariance matrix of the Wishart distributed $\mathbf{H}^H\mathbf{H}$. If we apply QR-decomposition to $\tilde{\mathbf{H}}$, then we have

$$\mathbf{T}^H\mathbf{\Sigma}\mathbf{T} = \tilde{\mathbf{L}} \ \mathrm{diag}(\tilde{r}_{11}^2, \tilde{r}_{22}^2, \cdots, \tilde{r}_{MM}^2) \ \tilde{\mathbf{L}}^H \tag{3.14}$$

and following similar steps leading to Eq. (3.13), we obtain

$$\tilde{r}_{mm}^2 = \frac{\det\left(\tilde{\mathbf{H}}_{mm}^H \tilde{\mathbf{H}}_{mm}\right)}{\det\left(\tilde{\mathbf{H}}_{(m-1)(m-1)}^H \tilde{\mathbf{H}}_{(m-1)(m-1)}\right)} \tag{3.15}$$

3.2.3 Cholesky Decomposition

Since $\mathbf{T}^H\mathbf{\Sigma}\mathbf{T}$ is a positive definite matrix, we can apply the Cholesky decomposition [34] such that

$$\mathbf{T}^H\mathbf{\Sigma}\mathbf{T} = \mathbf{L}'\tilde{\mathbf{D}}\mathbf{L}'^H \tag{3.16}$$

where $\mathbf{L}'$ is a lower triangular matrix having unity diagonal elements, and $\tilde{\mathbf{D}} = \mathrm{diag}(\tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_M)$.

Comparing Eq. (3.14) with Eq. (3.16) and using Eq. (3.15), we have

$$\tilde{d}_m = \tilde{r}_{mm}^2 \quad = \quad \frac{\det\left(\tilde{\mathbf{H}}_{mm}^H \tilde{\mathbf{H}}_{mm}\right)}{\det\left(\tilde{\mathbf{H}}_{(m-1)(m-1)}^H \tilde{\mathbf{H}}_{(m-1)(m-1)}\right)} \tag{3.17}$$

for $m = 1, 2, \cdots, M$. The sequence $\{\tilde{d}_m\}$ is called the Cholesky values.

Furthermore, we have the following Theorem.

Theorem 3.1. *[37, 38] Let $\{\lambda_m\}$ and $\{d_m\}$ for $m = 1, 2, \cdots, M$ be the eigenvalue sequence and the Cholesky value sequence of a positive definite $M \times M$ matrix $\mathbf{A}$, respectively. Then, $\{\lambda_m\}_{m=1}^M$ majorizes $\{d_m\}_{m=1}^M$ in the multiplicative sense. Conversely, if $\{\lambda_m\}_{m=1}^M$ majorizes $\{d_m\}_{m=1}^M$ multiplicatively, then, for an arbitrarily given* *desired permutation of* $\{d_m\}_{i=m}^{M}$, $d_{j1}, d_{j2}, \cdots, d_{jM}$, *there exists a matrix* $\mathbf{A}$ *such that* $\{\lambda_m\}_{i=m}^{M}$ *and* $\{d_{jm}\}_{m=1}^{M}$ *are the eigenvalues and the Cholesky values of* $\mathbf{A}$, *respectively.*

At this point, we present three important properties about the Cholesky values of $\mathbf{T}^H\mathbf{\Sigma}\mathbf{T}$.

Property 3.1. *Let* $\{\tilde{\lambda}_m\}$, $m = 1, 2, \cdots, M$ *be the eigenvalues of* $\mathbf{T}^H\mathbf{\Sigma}\mathbf{T}$, *because of Theorem 3.1, we have*

$$\tilde{\mathbf{d}} \prec_{\times} \tilde{\boldsymbol{\lambda}} \tag{3.18}$$

where $\tilde{\boldsymbol{\lambda}} = [\tilde{\lambda}_1, \cdots, \tilde{\lambda}_M]^T$ *and* $\tilde{\mathbf{d}} = [\tilde{d}_1, \cdots, \tilde{d}_M]^T$.

Property 3.2. *[15][38][39] For* $\tilde{\boldsymbol{\lambda}}$ *and* $\tilde{\mathbf{d}}$ *given in Property 3.1, there exist unitary matrices* $\hat{\mathbf{S}}$ *and* $\hat{\mathbf{Q}}$ *such that*

$$\tilde{\mathbf{\Lambda}}^{1/2}\hat{\mathbf{S}} = \hat{\mathbf{Q}}\hat{\mathbf{R}} \tag{3.19}$$

where $\tilde{\mathbf{\Lambda}}^{1/2} = \mathrm{diag}(\sqrt{\tilde{\lambda}_{[1]}}, \cdots, \sqrt{\tilde{\lambda}_{[M]}})$, $\hat{\mathbf{R}}$ *is an upper triangular matrix having* $\hat{r}_{mm} = \sqrt{\tilde{d}_m}$. $\hat{\mathbf{S}}$ *and* $\hat{\mathbf{Q}}$ *can be obtained by the QRS algorithm (Appendix A).*

Property 3.3. *[30] If we let*

$$\begin{aligned} \xi_m &= \frac{\det([\mathbf{T}^H\mathbf{H}^H\mathbf{H}\mathbf{T}]_{mm})}{\det([\mathbf{T}^H\mathbf{H}^H\mathbf{H}\mathbf{T}]_{(m-1)(m-1)})} \cdot \frac{\det([\mathbf{T}^H\mathbf{\Sigma}\mathbf{T}]_{(m-1)(m-1)})}{\det([\mathbf{T}^H\mathbf{\Sigma}\mathbf{T}]_{mm})} \\ &= r_{mm}^2/\tilde{d}_m \end{aligned} \tag{3.20}$$

then, the probability density function (pdf) of ξ_m *is given by*

$$p(\zeta_m) = \frac{1}{\Gamma(N-m+1)}\,\zeta_m^{N-m}e^{-\zeta_m} \qquad \zeta_m > 0 \tag{3.21}$$

for $m = 1, \cdots, M$, *i.e.,* $\xi_1, \xi_2, \cdots, \xi_M$ *are* independent χ^2 distributed *with* $2(N-m+1)$ *degrees of freedom.*

We also have the following theorem [37] and corollaries.

Theorem 3.2. *Let $\{\tilde{\lambda}_m\}$, $\{\tilde{\mu}_m\}$, and $\{\tilde{\nu}_m\}$, where $m = 1, 2, \cdots, M$, be the eigenvalues of the positive definite matrices $\mathbf{T}^H\mathbf{\Sigma}\mathbf{T}$, $\mathbf{T}^H\mathbf{T}$ and $\mathbf{\Sigma}$ respectively, then*

$$\prod_{m=1}^{k} \tilde{\lambda}_{[m]} \leq \prod_{i=1}^{k} \tilde{\mu}_{[m]}\tilde{\nu}_{[m]}, \quad for \quad 1 \leq k < M \tag{3.22a}$$

$$\prod_{m=1}^{M} \tilde{\lambda}_{[m]} = \prod_{m=1}^{M} \tilde{\mu}_{[m]}\tilde{\nu}_{[m]} \tag{3.22b}$$

Proof: See Appendix B.

Corollary 3.1. *Combining Property 3.1 and Theorem 3.2, by transitivity, we have*

$$\prod_{m=1}^{k} \tilde{d}_{[m]} \leq \prod_{i=1}^{k} \tilde{\mu}_{[m]}\tilde{\nu}_{[m]}, \quad for \quad 1 \leq k < M \tag{3.23a}$$

$$\prod_{m=1}^{M} \tilde{d}_{[m]} = \prod_{m=1}^{M} \tilde{\mu}_{[m]}\tilde{\nu}_{[m]} \tag{3.23b}$$

Corollary 3.2. *If the product sequence of $\{\tilde{\mu}_m\tilde{\nu}_m\}_{m=1}^M$ majorizes $\{\tilde{d}_m\}_{m=1}^M$ in the multiplicative sense as in Corollary 3.1, then, for an arbitrarily given desired permutation of $\{\tilde{d}_m\}_{m=1}^M$, $\tilde{d}_{j_1}, \tilde{d}_{j_2}, \cdots, \tilde{d}_{j_M}$, there exists a positive definite matrix $\bar{\mathbf{T}}^H\mathbf{\Sigma}\bar{\mathbf{T}}$ such that $\{\tilde{\mu}_m\tilde{\nu}_m\}_{m=1}^M$ and $\{\tilde{d}_m\}_{m=1}^M$ are the eigenvalues and the Cholesky values of $\bar{\mathbf{T}}^H\mathbf{\Sigma}\bar{\mathbf{T}}$, respectively.*

Proof: See Appendix C.

3.3 Reformulation of the Problem

With the mathematical concepts established in the last section, we are now in a position to reformulate Design Problem 3.1 in Section 3.1 into one that can be more efficiently solved:

From Eq. (3.1) in Problem 3.1, our design objective is to minimize $\mathrm{E}_{\mathbf{H}}\big(\sum_{m=1}^{M} r_{mm}^{-2}\big)$ subject to the power constraint, where r_{mm} is the mth diagonal element of $\mathbf{R}$ in the QR-decomposition of $\mathbf{HT}$. Now, $\mathbf{H}$ is a random matrix, and thus, r_{mm} is a random variable. From Property 3.3, we see that $\xi_m = r_{mm}^2/\tilde{d}_m$ for $m = 1, \cdots, M$ are independent and χ^2 distributed with $2(N-m+1)$ degrees of freedom. Therefore,

$$\begin{aligned}
\mathrm{E}_{\mathbf{H}}\left[\sum_{m=1}^{M} r_{mm}^{-2}\right] &= \sum_{m=1}^{M} \tilde{d}_m^{-1} \int_0^{\infty} \xi_m^{-1} \frac{1}{\Gamma(N-m+1)} \xi_m^{N-m} \mathrm{e}^{-\xi_m} \mathrm{d}\xi_m \\
&= \sum_{m=1}^{M} \frac{\tilde{d}_m^{-1}}{(N-m)!} \int_0^{\infty} \xi_m^{N-m-1} \mathrm{e}^{-\xi_m} \mathrm{d}\xi_m \\
&= \sum_{m=1}^{M} \frac{\tilde{d}_m^{-1}}{(N-m)!} \Big(- \int_0^{\infty} \xi_m^{N-m-1} \mathrm{d}\mathrm{e}^{-\xi_m} \Big) \\
&= \sum_{m=1}^{M} \frac{\tilde{d}_m^{-1}}{(N-m)!} \Big(- \xi_m^{N-m-1} \mathrm{e}^{-\xi_m} \big|_0^{\infty} + (N-M-1) \int_0^{\infty} \xi_m^{N-m-2} \mathrm{e}^{-\xi_m} \mathrm{d}\xi_m \Big) \\
&= \sum_{m=1}^{M} \frac{\tilde{d}_m^{-1}}{(N-m)!} \Big(0 + 0 + \cdots + (N-m-1)! \Big) \\
&= \sum_{m=1}^{M} \frac{\tilde{d}_m^{-1}}{N-m} \qquad (3.24)
\end{aligned}$$

Hence, the original design Problem 3.1 becomes:

Problem 3.2.

$$\mathbf{T}_{\mathrm{opt}} = \operatorname{argmin} \sum_{m=1}^{M} \frac{\tilde{d}_m^{-1}}{N-m} \qquad N > M \qquad (3.25a)$$

$$\text{s.t.} \quad \mathrm{tr}(\mathbf{T}^H \mathbf{T}) \leq p_0 \qquad (3.25b)$$

The formulation of Problem 3.2 simplifies the objective function from its original form. Now, examination of the objective function in Eq. (3.25a) reveals the following two characteristics:

1. The objective function in Eq. (3.25a) is of the form $f = \sum_{m=1}^{M} a_{(m)} \tilde{d}_m^{-1}$ where $\{a_{(m)}\}$ is an increasing sequence.

2. If we arrange the sequence $\{\tilde{d}_m\}$ in ascending order $\{\tilde{d}_{(m)}\}$ such that $\tilde{d}_{(1)} \leq \tilde{d}_{(2)} \leq \cdots \leq \tilde{d}_{(M)}$, then $\{\tilde{d}_{(m)}^{-1}\}$ is a decreasing sequence.

The above observations lead to the following property of the objective function of Eq. (3.25a) essential to the further reformulation of our design problem:

Property 3.4. *Let*

$$f_0 = a_{(1)}\tilde{d}_{(1)}^{-1} + \cdots + a_{(i)}\tilde{d}_{(i)}^{-1} + \cdots + a_{(j)}\tilde{d}_{(j)}^{-1} + \cdots + a_{(M)}\tilde{d}_{(M)}^{-1}$$

If we interchange the position of $\tilde{d}_{(i)}$ with $\tilde{d}_{(j)}$ in f_0 to form

$$f_{ij} = a_{(1)}\tilde{d}_{(1)}^{-1} + \cdots + a_{(i)}\tilde{d}_{(j)}^{-1} + \cdots + a_{(j)}\tilde{d}_{(i)}^{-1} + \cdots + a_{(M)}\tilde{d}_{(M)}^{-1}$$

then,

$$f_0 \leq f_{ij} \quad \forall\, j \geq i \tag{3.26}$$

Proof: Eq. (3.26) can be shown by simply taking the difference between f_0 and f_{ij} such that

$$\begin{aligned} f_0 - f_{ij} &= a_{(i)}\tilde{d}_{(i)}^{-1} - a_{(i)}\tilde{d}_{(j)}^{-1} + a_{(j)}\tilde{d}_{(j)}^{-1} - a_{(j)}\tilde{d}_{(i)}^{-1} \\ &= a_{(i)}(\tilde{d}_{(i)}^{-1} - \tilde{d}_{(j)}^{-1}) - a_{(j)}(\tilde{d}_{(i)}^{-1} - \tilde{d}_{(j)}^{-1}) \\ &= (a_{(i)} - a_{(j)})(\tilde{d}_{(i)}^{-1} - \tilde{d}_{(j)}^{-1}) \leq 0 \end{aligned}$$

This completes the proof of Property 3.4. □

Property 3.4 shows us that for any sum of sequence of the form $f = \sum_{m=1}^{M} a_{(m)}\tilde{d}_m^{-1}$, the minimum value f_0 is reached if $\{\tilde{d}_m\}$ is arranged in an ascending order. Thus, for Problem 3.2, *to achieve the minimum of the objective function, we must keep the sequence $\{\tilde{d}_m\}$ in an ascending order of $\{\tilde{d}_{(m)}\}$* such that $\tilde{d}_{(1)} \leq \tilde{d}_{(2)} \leq \cdots \leq \tilde{d}_{(M)}$.

A Convex Optimization Problem:

We now transform Problem 3.2 into a convex optimization problem. First, we apply the eigen-decomposition to $\mathbf{\Sigma}$ such that

$$\mathbf{\Sigma} = \tilde{\mathbf{U}}\ \mathrm{diag}(\tilde{\nu}_{[1]}, \cdots, \tilde{\nu}_{[M]})\ \tilde{\mathbf{U}}^H$$

Then we let

$$\begin{aligned} \alpha_m &= \ln\ \tilde{d}_{[m]} = \ln\ \tilde{d}_{(M-m+1)} \\ \text{such that} \qquad e^{-\alpha_{M-m+1}} &= \tilde{d}^{-1}_{(m)} && (3.27a) \\ \beta_m &= \ln\ \tilde{\mu}_{[m]} && (3.27b) \\ \gamma_m &= \ln\ \tilde{\nu}_{[m]} && (3.27c) \end{aligned}$$

where $\tilde{\mu}_{[m]}$ and $\tilde{\nu}_{[m]}$ are the mth largest eigenvalues of $\mathbf{T}^H\mathbf{T}$ and $\mathbf{\Sigma}$ respectively. Using Corollary 3.1, Corollary 3.2 and Property 3.4, we can now rewrite the design as the following optimization problem, which is equivalent to Problem 3.2.

Problem 3.3.

$$\begin{aligned} \boldsymbol{\alpha}_{\text{op}} &= \arg\min_{\{\alpha_m\}} \sum_{m=1}^{M} \frac{e^{-\alpha_{M-m+1}}}{N-m} && (3.28a) \\ \text{s.t.} \quad \sum_{m=1}^{k} \alpha_m &\leq \sum_{m=1}^{k} \beta_m + \sum_{m=1}^{k} \gamma_m \quad 1 \leq k < M && (3.28b) \\ \sum_{m=1}^{M} \alpha_m &= \sum_{m=1}^{M} \beta_m + \sum_{m=1}^{M} \gamma_m && (3.28c) \\ \beta_1 \geq \beta_2 \geq &\cdots \geq \beta_M && (3.28d) \\ \sum_{m=1}^{M} e^{\beta_m} &\leq p_0 \qquad \text{(power constraint)} && (3.28e) \end{aligned}$$

Eq. (3.28a) is the objective function written in terms of α_{M-m+1}, Constraints (3.28b) and (3.28c) result from taking logarithm of the majorization property in Corollary 3.1, Constraint (3.28d) follows the order of $\tilde{\boldsymbol{\mu}}$ for majorization, and Constraint (3.28e) is the power constraint written in terms of β_m. The design problem written in the form of Problem 3.3 is called Geometric Programming and is a convex optimization problem that can be efficiently solved using an interior point method [40].

3.4 Optimum Precoders

The design problem described in Problem 3.3 very much depends on the channel matrix $\mathbf{H}$ and the covariance matrix of the channels leading to each receiver antenna $\mathbf{\Sigma}$ which is positive definite. From the formulation of Problem 3.3, we can now obtain the optimum precoder designs for the following two cases:

1. The channels leading to each receiver antenna are known to be uncorrelated, i.e., $\mathbf{\Sigma} = \mathbf{I}$ since each path coefficient h_{nm} is assumed to have unit variance.

2. The channels leading to each receiver antenna are correlated with known covariance matrix $\mathbf{\Sigma}$ but are uncorrelated with those leading to another receiver antenna as depicted in Eq. (2.3).

3.4.1 For Uncorrelated Channels

When the channel gains are independent and identically-distributed complex Gaussian random variables, the objective function Eq. (3.28a) and the power constraint of Eq. (3.28e) remains unchanged. The constraint of Eq. (3.28d) is no longer needed since with $\mathbf{\Sigma} = \mathbf{I}$, $\{\tilde{d}_{[m]}\}$ and $\{\tilde{\mu}_{[m]}\}$ are parameters from the same matrix $\mathbf{T}^H\mathbf{T}$ and the majorization Eqs. (3.29a) and (3.29b) ensures their ordering. The constraints of Eqs. (3.28c) and (3.28b), however, simplify to:

$$\sum_{m=1}^{k} \alpha_m \leq \sum_{m=1}^{k} \beta_m \qquad 1 \leq k < M \tag{3.29a}$$

$$\sum_{m=1}^{M} \alpha_m = \sum_{m=1}^{M} \beta_m \tag{3.29b}$$

Replacing Eqs. (3.28b) and (3.28c) by Eqs. (3.29a) and (3.29b) in Problem 3.3, we can obtain a closed form solution for the optimum MSE precoder for this case as shown in the following:

Theorem 3.3. *For the transmission channels being uncorrelated, the minimum MSE precoder design is a diagonal matrix with the mth diagonal element given by*

$$t_{\text{op}mm} = \left[p_0 \left(\sum_{m=1}^{M} \frac{1}{\sqrt{N-m}} \right)^{-1} \Big/ \left(\sqrt{N-m} \right) \right]^{\frac{1}{2}}, \qquad m = 1, 2, \cdots, M \quad (3.30)$$

These diagonal elements are in an ascending order. The corresponding optimum mean-square error at the decision feedback receiver is given by

$$\overline{\varepsilon^2}_{\text{opt}} = \frac{\sigma^2}{M p_0} \left(\sum_{m=1}^{M} \frac{1}{\sqrt{(N-m)}} \right)^2 \quad (3.31)$$

Proof: The Lagrangian corresponding to the constrained optimization problem is

$$\begin{aligned} L(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{q}) &= \sum_{m=1}^{M} \frac{e^{-\alpha_{M-m+1}}}{N-m} + q_1(\alpha_1 - \beta_1) + q_2(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) + \cdots \\ &+ q_M(\alpha_1 + \alpha_2 + \cdots + \alpha_M - \beta_1 - \beta_2 - \cdots - \beta_M) + q_0 \left(\sum_{m=1}^{M} e^{\beta_m} - p_0 \right) \end{aligned}$$

where $q_1, q_2, \cdots, q_M$ and q_0 are the Lagrange multipliers. The Karush-Kuhn-Tucker (KKT) optimality conditions [40] are given by:

$$\frac{\partial L}{\partial \alpha_1} = q_1 + q_2 + q_3 + \cdots + q_M - \frac{e^{-\alpha_1}}{N-M} = 0 \quad (3.32a)$$

$$\frac{\partial L}{\partial \alpha_2} = q_2 + q_3 + \cdots + q_M - \frac{e^{-\alpha_2}}{N-M+1} = 0 \quad (3.32b)$$

$$\vdots$$

$$\frac{\partial L}{\partial \alpha_M} = q_M - \frac{e^{-\alpha_M}}{N-1} = 0 \quad (3.32c)$$

and

$$\frac{\partial L}{\partial \beta_1} = -q_1 - q_2 - q_3 - \cdots - q_M + q_0 e^{\beta_1} = 0 \quad (3.33a)$$

$$\frac{\partial L}{\partial \beta_2} = -q_2 - q_3 - \cdots - q_M + q_0 e^{\beta_2} = 0 \quad (3.33b)$$

$$\vdots$$

$$\frac{\partial L}{\partial \beta_M} = -q_M + q_0 e^{\beta_M} = 0 \quad (3.33c)$$

Taking the sum of these equations in pairs such that Eq. (3.32a) sums with (3.33a), Eq. (3.32b) sums with (3.33b), etc., yields the following general relationship between e^{α_m} and e^{β_m} such that

$$\frac{e^{-\alpha_m}}{(N-M+m-1)} = q_0\, e^{\beta_m}, \qquad m = 1, 2, \cdots, M \tag{3.34}$$

Summing up all the M equations in Eq. (3.34) and noting the power constraints in Eq. (3.28e), we have:

$$\sum_{m=1}^{M} \frac{e^{-\alpha_{M-m+1}}}{N-m} = q_0 \sum_{m=1}^{M} e^{\beta_m} = q_0 p_0 \tag{3.35}$$

We note that the left side of Eq. (3.35) is the objective function of our reformulated design as shown in Eq. (3.28a). Since the total transmitting power p_0 is fixed, the minimized value of the objective function is obtained when the Lagrangian multiplier q_0 reaches its minimal value, i.e.,

$$\min \sum_{m=1}^{M} \frac{e^{-\alpha_{M-m+1}}}{N-m} = \breve{q}_0 p_0$$

where $\breve{q}_0$ is the minimum value of the Lagrange multiplier q_0.

To obtain $\breve{q}_0$ (and therefore, the minimum value of the MSE), we note that with $\tilde{\nu}_m = 1$ for $m = 1, \cdots, M$, Corollary 3.1 becomes $\tilde{\mathbf{d}} \prec_\times \tilde{\boldsymbol{\mu}}$ which then results in the following majorization relationship,

$$\prod_{m=1}^{k} e^{\alpha_m} \leq \prod_{m=1}^{k} e^{\beta_m}, \qquad 1 \leq k < M \tag{3.36a}$$

$$\prod_{m=1}^{M} e^{\alpha_m} = \prod_{m=1}^{M} e^{\beta_m} \tag{3.36b}$$

Substituting e^{α_m} from Eq. (3.34), $m = 1, 2, \cdots, M$, into Eqs. (3.36) gives

$$\prod_{m=1}^{k} \frac{1}{\sqrt{(N-M+m-1)q_0}} \leq \prod_{m=1}^{k} e^{\beta_m}, \qquad 1 \leq k < M \tag{3.37a}$$

$$\prod_{m=1}^{M} \frac{1}{\sqrt{(N-M+m-1)q_0}} = \prod_{m=1}^{M} e^{\beta_m} \tag{3.37b}$$

Taking logarithm of both sides of Eqs. (3.37), and writing $\theta_m = \ln \frac{1}{\sqrt{(N-M+m-1)q_0}}$, we have $\boldsymbol{\theta} \prec_+ \boldsymbol{\beta}$.

We note that the function $g(\mathbf{x}) = \sum_{m=1}^{M} e^{x_m}$ is strictly convex with respect to $x_1, x_2, \cdots, x_M$, having the Hessian matrix being positive definite (see Appendix D for proof). Therefore $g(\boldsymbol{\beta}) = \sum_{m=1}^{M} e^{\beta_m}$ is Schur-convex with respect to $\beta_1, \beta_2, \cdots, \beta_M$. Consequently, $g(\boldsymbol{\beta}) \geq g(\boldsymbol{\theta})$ which, when both sides are multiplied by the Lagrangian multiplier q_0 ($q_0 \geq 0$), results in

$$q_0 \sum_{m=1}^{M} e^{\beta_m} \geq q_0 \sum_{m=1}^{M} e^{\theta_m} \tag{3.38}$$

i.e.,

$$q_0\, p_0 \geq q_0 \sum_{m=1}^{M} e^{\theta_m} = q_0 \sum_{m=1}^{M} \frac{1}{\sqrt{(N-M+m-1)q_0}} = \sqrt{q_0} \sum_{m=1}^{M} \frac{1}{\sqrt{(N-m)}} \tag{3.39}$$

This yields,

$$q_0 \geq \frac{1}{p_0^2} \left(\sum_{m=1}^{M} \frac{1}{\sqrt{(N-m)}} \right)^2 \tag{3.40}$$

Hence, the minimum value of the objective function is

$$\breve{q}_0\, p_0 = \frac{1}{p_0} \left(\sum_{m=1}^{M} \frac{1}{\sqrt{(N-m)}} \right)^2 \tag{3.41}$$

From Definition 3.3 of Schur-convexity, this minimum value is reached if and only if the inequalities in Eqs. (3.36) become equalities, i.e., $\alpha_m = \beta_m$. In other words, for the optimum precoder product $\mathbf{T}_{\text{op}}^H \mathbf{T}_{\text{op}}$, the Cholesky values $\{\tilde{d}_m\}$ are equal to the eigenvalues $\{\tilde{\mu}_m\}$. This results in

$$\mathbf{T}_{\text{op}}^H \mathbf{T}_{\text{op}} = \tilde{\mathbf{L}}\, \text{diag}(\tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_M)\, \tilde{\mathbf{L}}^H = \mathbf{U}_{\text{op}}^H \text{diag}(\tilde{\mu}_1, \tilde{\mu}_2, \cdots, \tilde{\mu}_M) \mathbf{U}_{\text{op}} \tag{3.42}$$

where $\tilde{\mathbf{L}}$ is a lower triangular matrix with unit diagonal elements such that $\ell_{mm} = 1$, and $\mathbf{U}_{\text{op}}$ is the eigen-vector matrix of $\mathbf{T}_{\text{op}}^H \mathbf{T}_{\text{op}}$. Eq. (3.42) shows that $\tilde{\mathbf{L}}$ is also the eigen-vector matrix of $\mathbf{T}_{\text{op}}^H \mathbf{T}_{\text{op}}$. Hence,

$$
\begin{aligned}
\tilde{\mathbf{L}}^H\tilde{\mathbf{L}} &= \mathbf{I} \\
&= \begin{bmatrix} 1 & \ell_{12} & \cdots & \ell_{1M} \\ 0 & 1 & \ddots & \ell_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \ell_{12} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{1M} & \ell_{2M} & \cdots & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1+\ell_{12}^2+\cdots+\ell_{1M}^2 & \ell_{12}+\ell_{13}\ell_{23}+\cdots+\ell_{1M}\ell_{2M} & \cdots & \ell_{1M} \\ \ell_{12}+\ell_{13}\ell_{23}+\cdots+\ell_{1M}\ell_{2M} & 1+\ell_{23}^2+\cdots+\ell_{2M}^2 & \ddots & \ell_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{1M} & \ell_{2M} & \cdots & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
\end{aligned}
$$

However, this is impossible unless $\ell_{km} = 0$ for $k \neq m$. Since the diagonal elemnts of $\tilde{\mathbf{L}}$ are all equal to unity, thus, $\tilde{\mathbf{L}} = \mathbf{I}$, and thus $\mathbf{T}_{\text{op}}$ is diagonal such that $\mathbf{T}_{\text{op}} = \text{diag}(\sqrt{\tilde{\mu}_{(1)}}, \sqrt{\tilde{\mu}_{(2)}}, \cdots, \sqrt{\tilde{\mu}_{(M)}})$.

The values of the mth diagonal element can be obtained by substituting $\breve{q}_0$, the minimum values of q_0, in Eq. (3.40) into the mth term of Eq. (3.39), arranging them in acending order and taking the square root, Eq. (3.30) then follows. The corresponding optimum arithmetic MSE can be obtained by substituting Eq. (3.41) into Eq. (2.33), and Eq. (3.31) follows. □

The result of Theorem 3.3 is not surprising. Since the transmission channels are uncorrelated, each having unity variance, the optimum precoder, being diagonal, allocates power to each symbols individually according to the order of detection, with the last symbol having the highest power since the last symbol is detected first at the decision-feedback receiver. The allocated power gradually decreases to the lowest value for the first symbol which is received last. The amount of allocated power for each symbol decreases in a way such that the mean-square error is evened out among all the symbols. Indeed, this structure results from the fact that $\xi_m = r_{mm}^2/\tilde{d}_m$ for $m = 1, \cdots, M$ are χ^2 distributed with decreasing values of degrees of freedoms. Consequently, detecting x_{m-1} becomes more reliable than that of x_m for $1 < m \leq M$. Thus, for the ZF-DF receiver with a fixed detection order, to obtain a good overall performance, more power should be allocated to x_m than to x_{m-1} and increasingly so to x_M leading to this diagonal precoder having diagonal elements with an ascending order.

3.4.2 For Correlated Channels

We now turn to the case when the channels leading to the receiver antenna are correlated. The optimization problem of Eqs. (3.28) is known to be convex (Geometric Programming) and can be efficiently solved by interior point methods. The solution of the problem yields numerically the optimum values of $\{\alpha_m\}$ and $\{\beta_m\}$. Therefore, from the optimum values of $\{\alpha_m\}$, using Eq. (3.27a), we can obtain the optimum $\{\tilde{d}_{(m)}\}$, the Cholesky values of $(\mathbf{T}^H\boldsymbol{\Sigma}\mathbf{T})$ and thus, $\{\tilde{r}_{(mm)}^2\}$, since $\{\tilde{d}_{(m)}\} = \{\tilde{r}_{(mm)}^2\}$. Also, from the optimum values of $\{\beta_m\}$, we can obtain the optimum $\{\tilde{\mu}_{[m]}\}$ which are the eigenvalues of $(\mathbf{T}^H\mathbf{T})$. The following are the steps by which we can arrive at an optimum precoder using the optimum values of $\{\tilde{d}_{(m)}\}$ and $\{\tilde{\mu}_{[m]}\}$:

1. Perform an eigen-decomposition on the known channel covariance matrix such that $\boldsymbol{\Sigma} = \tilde{\mathbf{U}}\boldsymbol{\Delta}_\Sigma\tilde{\mathbf{U}}^H$ where $\tilde{\mathbf{U}}$ is the eigenvector matrix of $\boldsymbol{\Sigma}$ and we have written the eigenvalue matrix as $\boldsymbol{\Delta}_\Sigma = \text{diag}(\tilde{\nu}_{[1]}, \cdots, \tilde{\nu}_{[M]})$.

2. With the eigenvector matrix $\tilde{\mathbf{U}}$ and the optimum eigenvalues $\{\tilde{\mu}_{[m]}\}$, a possible form of the optimum precoder is given by

$$\mathbf{T}_{\text{op}} = \tilde{\mathbf{U}}\ \text{diag}(\tilde{\mu}_{[1]}^{1/2}, \cdots, \tilde{\mu}_{[M]}^{1/2})\ \mathbf{S}_{\text{op}} = \tilde{\mathbf{U}}\ \boldsymbol{\Delta}_T^{1/2}\ \mathbf{S}_{\text{op}} \tag{3.43}$$

where $\mathbf{S}_{\text{op}}$ is to be evaluated, and $\boldsymbol{\Delta}_T^{1/2} = \text{diag}(\tilde{\mu}_{[1]}^{1/2}, \cdots, \tilde{\mu}_{[M]}^{1/2})$. This yields

$$\mathbf{T}_{\text{op}}^H\boldsymbol{\Sigma}\mathbf{T}_{\text{op}} = \mathbf{S}_{\text{op}}^H\boldsymbol{\Delta}_T^{1/2}\boldsymbol{\Delta}_\Sigma\boldsymbol{\Delta}_T^{1/2}\mathbf{S}_{\text{op}} = \mathbf{S}_{\text{op}}^H\tilde{\boldsymbol{\Lambda}}\mathbf{S}_{\text{op}} \tag{3.44}$$

where we have written $\tilde{\mathbf{\Lambda}} = \mathbf{\Delta}_T^{1/2}\mathbf{\Delta}_\Sigma\mathbf{\Delta}_T^{1/2}$.

3. To solve for $\mathbf{S}_{\text{op}}$, recall that, using the QRS decomposition algorithm (Appendix A), we can construct unitary matrices $\hat{\mathbf{S}}$ and $\hat{\mathbf{Q}}$ such that $\tilde{\mathbf{\Lambda}}^{1/2}\hat{\mathbf{S}} = \hat{\mathbf{Q}}\hat{\mathbf{R}}$ with the upper triangular matrix $\hat{\mathbf{R}}$ having diagonal elements $\{\hat{r}_{mm}\}$ arranged in any permutation. We apply the QRS decomposition algorithm to $\tilde{\mathbf{\Lambda}}^{1/2}$, then, by Property 3.2, $\hat{r}_{mm} = \sqrt{\tilde{d}_m}$. Keeping $\hat{r}_{mm}$ in *ascending order* (so that $\{\tilde{d}_{(m)}^{-1}\}$ is in descending order to minimize the objective function as shown in Property 3.4), we can obtain $\mathbf{S}_{\text{op}}$. From this, together with the eigenvector matrix $\tilde{\mathbf{U}}$ of $\mathbf{\Sigma}$, and the optimum eigenvalue matrix $\mathbf{\Delta}_T$ of $\mathbf{T}^H\mathbf{T}$, we can obtain the optimum precoder $\mathbf{T}_{\text{op}}$ as shown in Eq. (3.43).

For a system equipped with a ZF-DF detector, when perfect CSI is available at both transmitter and receiver, the optimal precoder is a QRS decomposition of $\mathbf{H}' \triangleq \mathbf{HT}$ having equal diagonal element in the R-factor [11, 13]. However, in the case when CSI is available only at the receiver and channel statistics known at the transmitter, we can see from the above design that the optimal precoder structure is the QRS decomposition of $\tilde{\mathbf{H}} \triangleq \mathbf{\Sigma}^{\frac{1}{2}}\mathbf{T}$ for which the diagonal entries of the R-factor is a non-decreasing sequence.

Chapter 4

Minimum Power Precoder for ZF-DF Receivers

In this Chapter, we investigate an "inverse" problem of Chapter 3, i.e., designing the precoder that minimizes total transmission power subject to a constraint on the average arithmetic MSE of a ZF-DF receiver. We also make the same assumption that perfect CSI is available at the receiver, and only the first- and second-order statistics of the channel is known at the transmitter.

4.1 Problem Description

The minimum power precoder design problem can be stated as:

Problem 4.1. *Design a precoding matrix* $\mathbf{T}$ *such that*

$$\mathbf{T}_{\mathrm{op}} = \arg\min_{\mathbf{T}} \ \mathrm{tr}(\mathbf{T}^H\mathbf{T}) \tag{4.1a}$$

$$\text{s.t.} \quad \frac{\sigma^2}{M}\,\mathrm{E}_{\mathbf{H}}\Big(\sum_{m=1}^{M} r_{mm}^{-2}\Big) \leq \varepsilon \tag{4.1b}$$

where ε is a bound on the MSE of ZF-DF receiver and $\mathrm{E}_{\mathbf{H}}(\cdot)$ denotes the expectation taken over all random channel realizations.

4.2 Reformulation of the Problem

Using the same mathematical concepts in Section 3.2 and also let

$$\begin{aligned} \alpha_m &= \ln \tilde{d}_{[m]} = \ln d_{(M-m+1)} && (4.2a)\\ \Rightarrow \quad \tilde{d}^{-1}_{(m)} &= e^{-\alpha_{M-m+1}} && (4.2b)\\ \beta_m &= \ln \tilde{\mu}_{[m]} && (4.2c)\\ \gamma_m &= \ln \tilde{\nu}_{[m]} && (4.2d) \end{aligned}$$

where $\mu_{[m]}$ and $\nu_{[m]}$ are the mth largest eigenvalues of $\mathbf{T}^H\mathbf{T}$ and $\mathbf{\Sigma}$ respectively, and $\tilde{d}_{[m]}$ is the mth largest Cholesky value of $\mathbf{T}^H\mathbf{\Sigma}\mathbf{T}$.

Then, Design Problem 4.1 can be reformulated to the following convex optimization prblem.

Problem 4.2.

$$\boldsymbol{\beta}_{\text{op}} = \arg\min_{\{\beta_m\}} \quad \sum_{m=1}^{M} e^{\beta_m} \tag{4.3a}$$

$$\text{s.t.} \quad \sum_{m=1}^{k} \alpha_m \le \sum_{m=1}^{k} \beta_m + \sum_{m=1}^{k} \gamma_m \quad 1 \le k < M \tag{4.3b}$$

$$\sum_{m=1}^{M} \alpha_m = \sum_{m=1}^{M} \beta_m + \sum_{m=1}^{M} \gamma_m \tag{4.3c}$$

$$\beta_1 \ge \beta_2 \ge \cdots \ge \beta_M \tag{4.3d}$$

$$\frac{\sigma^2}{M} \sum_{m=1}^{M} \frac{e^{-\alpha_{M-m+1}}}{N-m} \le \varepsilon \quad \text{(MSE constraint)} \tag{4.3e}$$

The problem in Design Problem 4.2 is also a convex Geometric Programming, and can be efficiently solved using an interior point method [40].

4.3 Optimum Precoders

Similar to the minimum MSE precoder design problem, we also have two situations depending on different channel correlation conditions:

1. The channels leading to each receiver antenna are known to be uncorrelated, i.e., $\mathbf{\Sigma} = \mathbf{I}$.

2. The channels leading to each receiver antenna are correlated with known covariance matrix $\mathbf{\Sigma}$ but are uncorrelated with those leading to another receiver antenna as depicted in Eq. (2.3).

Now, we will analyze the two cases respectively.

4.3.1 For Uncorrelated Channels

When the channel gains are independent and identically-distributed complex Gaussian random variables, the design problem for minimum power precoder becomes:

Problem 4.3.

$$\boldsymbol{\beta}_{\text{op}} = \arg\min_{\{\beta_m\}} \sum_{m=1}^{M} e^{\beta_m} \tag{4.4a}$$

$$\text{s.t.} \quad \sum_{m=1}^{k} \alpha_m \leq \sum_{m=1}^{k} \beta_m \quad 1 \leq k < M \tag{4.4b}$$

$$\sum_{m=1}^{M} \alpha_m = \sum_{m=1}^{M} \beta_m \tag{4.4c}$$

$$\sum_{m=1}^{M} \frac{e^{-\alpha_{M-m+1}}}{N-m} = \varepsilon' \quad \text{(MSE constraint)} \tag{4.4d}$$

where $\varepsilon' = \frac{M\varepsilon}{\sigma^2}$.

We can also obtain a closed form solution to Design Problem 4.3. The optimum minimum power precoder and the optimal power are stated in the following Theorem.

Theorem 4.1. *For the transmission channels being uncorrelated, the minimum power precoder design is a diagonal matrix with the mth diagonal element given by*

$$t_{\text{op}mm} = \left[\frac{\sigma^2}{M\varepsilon\sqrt{N-m}}\left(\sum_{m=1}^{M}\frac{1}{\sqrt{N-m}}\right)\right]^{\frac{1}{2}}, \qquad m = 1, 2, \cdots, M \tag{4.5}$$

These diagonal elements are in an ascending order.

In addition, the corresponding optimum power at the decision feedback receiver is given by

$$p_{\text{opt}} = \frac{\sigma^2}{M\varepsilon}\Big(\sum_{m=1}^{M}\frac{1}{\sqrt{N-m}}\Big)^2 \tag{4.6}$$

Proof: The Lagrangian corresponding to the constrained optimization problem is

$$\begin{aligned} L(\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{q}') &= \sum_{m=1}^{M} e^{\beta_m} + q'_1(\alpha_1-\beta_1) + q'_2(\alpha_1+\alpha_2-\beta_1-\beta_2)+\cdots \\ &+ q'_M(\alpha_1+\alpha_2+\cdots+\alpha_M-\beta_1-\beta_2-\cdots-\beta_M) \\ &+ q'_0\left(\sum_{m=1}^{M}\frac{e^{-\alpha_{M-m+1}}}{N-m}-\varepsilon'\right) \end{aligned} \tag{4.7}$$

where $q'_1, q'_2, \cdots, q'_M$ and q'_0 are the Lagrange multipliers. The Karush-Kuhn-Tucker (KKT) optimality conditions [40] are given by:

$$\frac{\partial L}{\partial \alpha_1} = q'_1+q'_2+q'_3+\cdots+q'_M-\frac{q'_0\,e^{-\alpha_1}}{N-M} = 0 \tag{4.8a}$$

$$\frac{\partial L}{\partial \alpha_2} = q'_2+q'_3+\cdots+q'_M-\frac{q'_0\,e^{-\alpha_2}}{N-M+1} = 0 \tag{4.8b}$$

$$\vdots$$

$$\frac{\partial L}{\partial \alpha_M} = q'_M-\frac{q'_0\,e^{-\alpha_M}}{N-1} = 0 \tag{4.8c}$$

and

$$\frac{\partial L}{\partial \beta_1} = -q'_1-q'_2-q'_3-\cdots-q'_M+e^{\beta_1} = 0 \tag{4.9a}$$

$$\frac{\partial L}{\partial \beta_2} = -q'_2-q'_3-\cdots-q'_M+e^{\beta_2} = 0 \tag{4.9b}$$

$$\vdots$$

$$\frac{\partial L}{\partial \beta_M} = -q'_M+e^{\beta_M} = 0 \tag{4.9c}$$

Taking the sum of these equations in pairs such that Eq. (4.8a) sums with (4.9a), Eq. (4.8b) sums with (4.9b), etc., yields the following general relationship between

e^{α_m} and e^{β_m} such that

$$\frac{q_0' \, e^{-\alpha_m}}{N-M+m-1} = e^{\beta_m}, \qquad m = 1, 2, \cdots, M \tag{4.10}$$

Summing up all the M equations in Eq. (4.10) and noting the MSE constraints, we have:

$$\sum_{m=1}^{M} e^{\beta_m} = q_0' \sum_{m=1}^{M} \frac{e^{-\alpha_{M-m+1}}}{N-m} \tag{4.11}$$

$$= q_0' \, \varepsilon' \tag{4.12}$$

The left side of Eq. (4.11) is the objective function of our minimum power precoder design problem. Since the MSE bound ε is fixed, the minimized value of the objective function is obtained when the Lagrangian multiplier q_0' reaches its minimal value, i.e.,

$$\min \sum_{m=1}^{M} e^{\beta_m} = \breve{q}'_0 \, \varepsilon' \tag{4.13}$$

where $\breve{q}'_0$ is the minimum value of the Lagrange multiplier q_0'.

To obtain $\breve{q}'_0$ and the minimum total transmitting power, we again mention the majorization relationship under the condition of uncorrelated channels,

$$\prod_{m=1}^{k} e^{\alpha_m} \leq \prod_{m=1}^{k} e^{\beta_m}, \qquad 1 \leq k < M \tag{4.14a}$$

$$\prod_{m=1}^{M} e^{\alpha_m} = \prod_{m=1}^{M} e^{\beta_m} \tag{4.14b}$$

Substituting e^{α_m} from Eq. (4.10), $m = 1, 2, \cdots, M$, into Eqs. (4.14) gives

$$\prod_{m=1}^{k} \sqrt{\frac{q_0'}{N-M+m-1}} \leq \prod_{m=1}^{k} e^{\beta_m}, \qquad 1 \leq k < M \tag{4.15a}$$

$$\prod_{m=1}^{M} \sqrt{\frac{q_0'}{N-M+m-1}} = \prod_{m=1}^{M} e^{\beta_m} \tag{4.15b}$$

Taking logarithm of both sides of Eqs. (4.15), and writing $\theta'_m = \ln \sqrt{\frac{q_0'}{N-M+m-1}}$, we have $\boldsymbol{\theta}' \prec_+ \boldsymbol{\beta}$. We note that the function $g(\mathbf{x}) = \sum_{m=1}^{M} e^{x_m}$ is strictly convex with respect to $x_1, x_2, \cdots, x_M$, having the Hessian matrix being positive definite. Therefore $g(\boldsymbol{\beta}) = \sum_{m=1}^{M} e^{\beta_m}$ is Schur-convex with respect to $\beta_1, \beta_2, \cdots, \beta_M$, giving

$$g(\boldsymbol{\beta}) \geq g(\boldsymbol{\theta}') \tag{4.16}$$

Substitute Eq. (4.13) into Eq. (4.16) for the objective function, we have:

$$\begin{aligned} q_0' \, \varepsilon' \geq \breve{q}'_0 \, \varepsilon' &= \min \sum_{m=1}^{M} e^{\beta_m} \\ &= \sum_{m=1}^{M} e^{\theta'_m} \\ &= \sqrt{\frac{q_0'}{N-M+m-1}} \\ &= \sum_{m=1}^{M} \sqrt{\frac{q_0'}{N-m}} \end{aligned} \tag{4.17}$$

This yields,

$$q_0' \geq \frac{1}{\varepsilon'^2} \left(\sum_{m=1}^{M} \frac{1}{\sqrt{N-m}} \right)^2 \tag{4.18}$$

Hence, the minimum value of the objective function is

$$\breve{q}'_0 \, \varepsilon' = \frac{1}{\varepsilon'} \left(\sum_{m=1}^{M} \frac{1}{\sqrt{N-m}} \right)^2 = \frac{\sigma^2}{M\varepsilon} \left(\sum_{m=1}^{M} \frac{1}{\sqrt{N-m}} \right)^2 \tag{4.19}$$

Then, Eq. (4.6) follows.

From Definition 3.3 of Schur-convexity and similar analysis of Subsection 3.4.1, the optimum power precoder is also a diagonal matrix having diagonal entries showed in Eq. (4.5)

This completes the proof. □

4.3.2 For Correlated Channels

When the channels leading to the receiver antenna are correlated, Design Problem 4.2 is convex and can then be efficiently solved by numerical method. We first obtain the optimum values of $\{\alpha_m\}$ and $\{\beta_m\}$, and thus the optimum sequences of $\{\tilde{r}^2_{(mm)}\}$ and $\{\tilde{\mu}_{[m]}\}$ can be computed. Employing the same method in Subsection 3.4.2 and the QRS decomposition algorithm, we can also arrive at an optimum precoder for correlated channels as stated in Eq. (3.43).

Chapter 5

Numerical Experiments

In this Chacpter, we first compare the performance of the optimally MSE precoded system employing a ZF-DF receiver developed in Chapter 3 with the performance of other schemes which can also be applied to the scenario in which CSI is not fully available at the transmitter. These include the optimally precoded system using linear receivers [18] and the unprecoded system using ZF-DF receiver. However, our main focus is on the comparison with the celebrated V-BLAST scheme [3].

We also present the simulation results for the minimum power precoder and the relationship between the total transmission power and the MSE of our transmission model.

5.1 Minimum MSE Precoder

In this section, we present simulation results for the minimum MSE precoder designs. We compare the performance of our optimally precoded system with the following systems:

1. The unprecoded system ($\mathbf{T} = \mathbf{I}$) employing the same detection scheme: ZF-DF receiver.

2. Another optimal precoded system designed for ZF linear receiver in [18].

3. The V-BLAST detection scheme (ordered ZF-DF receiver).

4. A combined system equipped with both our optimum precoder and the optimal ordering procedure of V-BLAST

As introduced in Chapter 2, V-BLAST is a MIMO system that employs M ($M \leq N$) transmitter antennas and transmits one symbol per antenna per time slot. Furthermore, V-BLAST utilizes a zero-forcing and symbol cancellation scheme equipped with optimal symbol detection ordering based on the post-detection SNR.

V-BLAST does not use a precoder ($\mathbf{T} = \mathbf{I}$). While its optimal detection order scheme is an important feature and optimizes the detection performance, it increases the complexity of the detection procedure. This is because at each stage, the Moore-Penrose inverse, $\mathbf{H}_i^+$, of the diminished channel matrix at the stage has to be obtained and its row with the minimum norm has to be searched for before the ith symbol is detected. Hence, for simplicity in implementation of the receiver, the ZF-DF detection scheme following the reversed natural order of symbol transmission and successive interference cancellation detection described in Section 2.3.3 is much simpler and preferred. Furthermore, under the condition of high correlation between transmission channels, the sorting out of the post-detection SNR may not be accurate and may lead to the deterioration of the performance of the V-BLAST scheme. It is under these considerations that we compare the performance of the various schemes in the following examples. Also, as a point of interest, we also compare the scheme which combines the optimal precoder for the ZF-DF receiver with an optimally ordered detector. This scheme of course, will have the advantages of both the optimum precoder design as well as the optimum ordering of the V-BLAST detection. However, it also increases the implementation complexity compared to that of the Optimum precoder equipped with the ZF-DF detection scheme following the reversed natural order of symbol transmission.

In the following examples, simulations were carried out for a MIMO system with 6 transmitter antennas and 10 receiver antennas transmitting symbols from a 4-QAM constellation. For each randomly generated channel matrix $\mathbf{H}$, the experiment was carried out for different SNR, and, for each SNR, was repeated 10^4 times with different noise realizations. The average symbol error rates (SER) was then computed for the various values of SNR. To study the effect of antenna correlations, random realizations of correlated channels were generated according to the exponential correlation model [41] such that the mnth element of $\mathbf{\Sigma}$ is given by:

$$\sigma_{mn} = \begin{cases} \rho^{n-m}, & m \leq n \\ \sigma_{nm}^{*} & m > n \end{cases} \qquad m, n = 1, 2, \cdots, M \tag{5.1}$$

where ρ is the correlation coefficient between any two neighboring antennas and "$*$" denotes complex conjugate.

For instance, when the correlation coefficient is set to $\rho = 0.5e^{0.5j}$, the correlation matrix becomes:

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0.5e^{0.5j} & (0.5e^{0.5j})^2 & \cdots & (0.5e^{0.5j})^{M-1} \\ 0.5e^{-0.5j} & 1 & 0.5e^{0.5j} & \cdots & (0.5e^{0.5j})^{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (0.5e^{-0.5j})^{M-1} & (0.5e^{-0.5j})^{M-2} & \cdots & \cdots & 1 \end{bmatrix} \tag{5.2}$$

This correlation model is suitable for our experiments since, in practice, we expect the correlation between neighboring channels to be higher than that between more distant channels. In the following, we examine the performance of the above MIMO systems with various values of ρ such that $|\rho| = 0,\ 0.3,\ 0.5,\ 0.7$, and 0.9.

5.1.1 Example 1 Uncorrelated Channels

We first examine the SER performances of different MIMO systems in uncorrelated channel, i.e., $\rho = 0$. Under this condition, we can use the closed-form solution in Section 3.4.1 to come up with the optimum precoding matrix $\mathbf{T}_{\text{op}}$ for the ZF-DF receiver. Fig. 5.1 shows the SER performance of all the schemes for which the comparison has been carried out. From this figure, it can be observed that the optimal

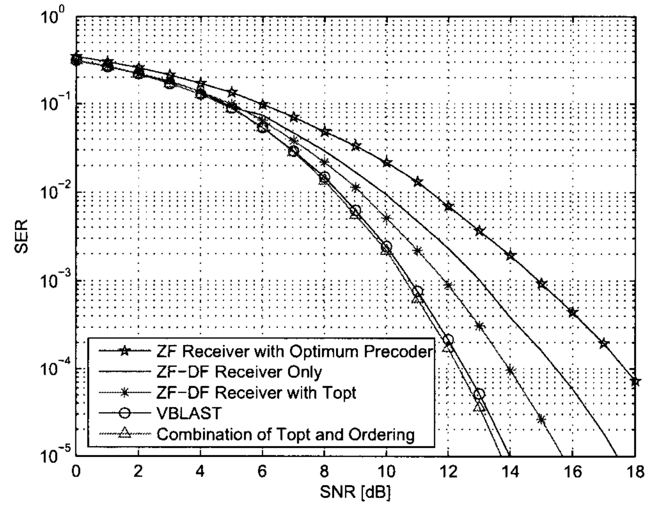

Figure 5.1: Simulation Results when $\rho = 0$

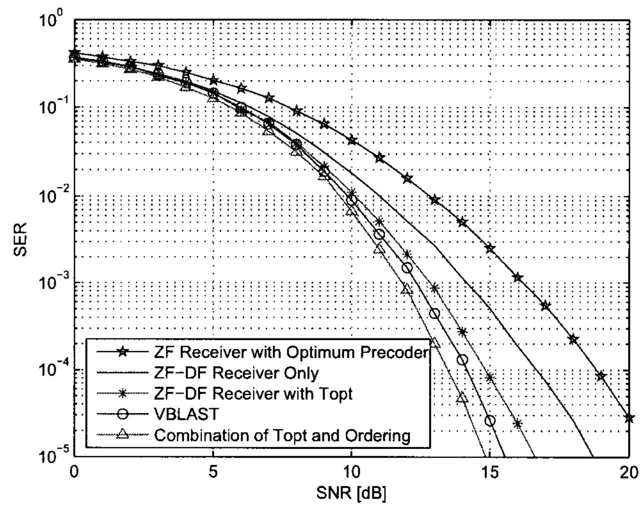

Figure 5.2: Simulation Results when $\rho = 0.3e^{0.5i}$

precoder developed here for the natural order ZF-DF receiver scheme not only has better performance than the unprecoded system using the same ZF-DF receiver, but it also has superior performance compared to that of the optimum precoder in [18] which is developed for the specific linear ZF receiver. When there is no correlation among the channels, the V-BLAST detection algorithm performs considerably better than the precoded system developed here, having almost a relative gain of 2 dB at the SER of 10^{-5}. This, as pointed out earlier, is accomplished at the expense of substantial increase in detection complexity. An additional important observation from the simulation results is that even though the scheme which combines the optimal precoder with the optimal ordering detection shows the best performance, it only offers marginally better error rate than the V-BLAST scheme at higher SNR, and has virtually the same performance as V-BLAST at lower SNR. Thus, we can conclude that under the condition of uncorrelated channels, the new optimal precoder does not offer any performance advantage over the V-BLAST detection scheme, except for lower implementation complexity.

5.1.2 Example 2 Low Channel Correlation

In this example, we consider the scenario in which the random channels are generated basing on a lower level of fading correlation. We let ρ equal to $0.3e^{0.5j}$.

The comparison results are illustrated in Fig. 5.2. We can come to the similar observations as those in Example 1, except that the gap between the combination of optimal precoding and optimal ordering and pure VBLAST detection algorithm becomes larger. To achieve the same SER at 10^{-5}, the traditional VBLAST system needs almost 1dB more power than the optimal combination. It is also noticeable that the curve of our precoded system employing only ZF-DF receiver without the computational demanding ordering technique becomes closer and closer to VBLAST as the correlation level grows from 0 to $0.3e^{0.5i}$.

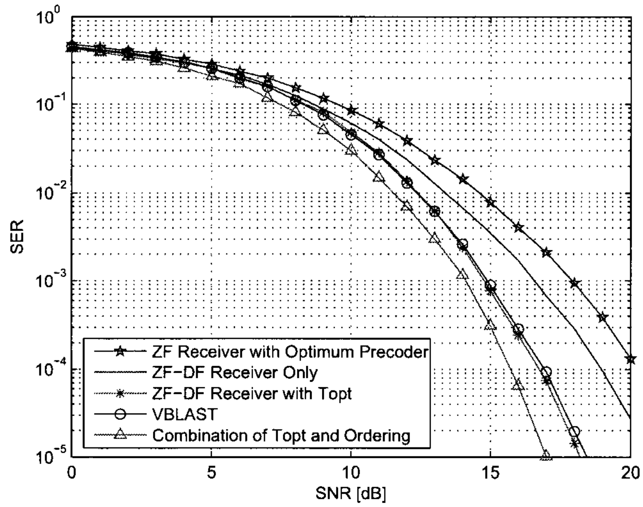

Figure 5.3: Simulation Results when $\rho = 0.5e^{0.5j}$

5.1.3 Example 3 Moderate Channel Correlation

In this example, we investigate the performance of the above MIMO systems under a moderately correlated channel fading environment in which $\rho = 0.5e^{0.5j}$. Fig. 5.3 shows the performance of the various schemes under consideration. It can be observed that there is a considerable deterioration of performance in all schemes in comparison to those in Example 1. However, it is observed that the performance of the two schemes equipped with the optimum precoder design developed in Chapter 3 are both less sensitive to the correlation of the channels. Indeed, the performance of the scheme with optimum precoder and ZF-DF receiver (no re-ordering) matches and surpasses that of the V-BLAST scheme while also having the advantage of substantially lower computational complexity. The combination scheme of optimum precoder and optimally-ordered detection, on the other hand, while suffering from performance deterioration, maintains its performance superiority over the other schemes.

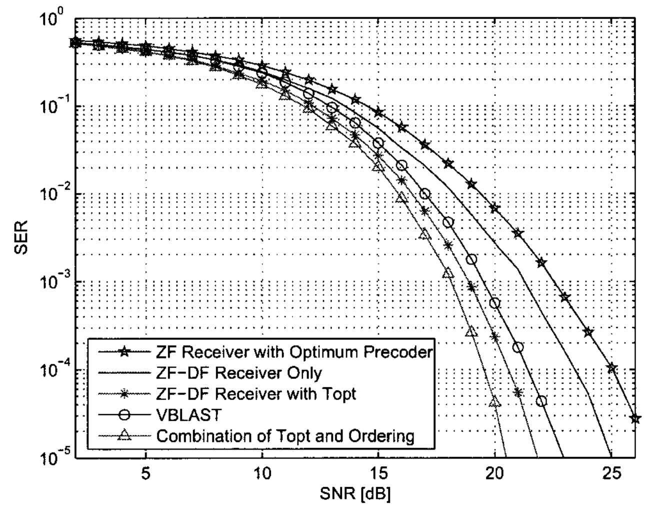

Figure 5.4: Simulation Results when $\rho = 0.7e^{0.5j}$

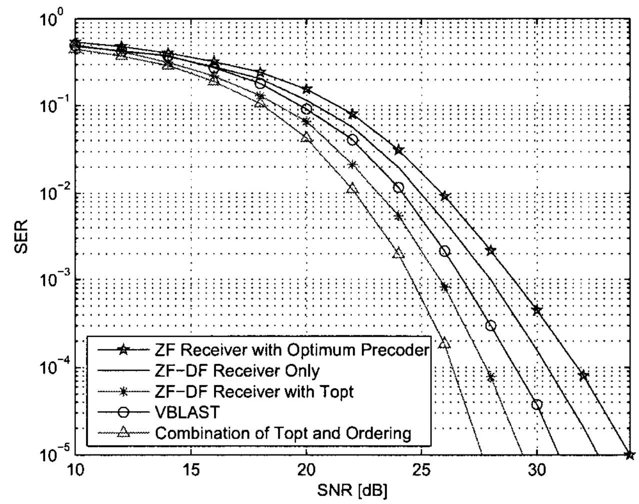

Figure 5.5: Simulation Results when $\rho = 0.9e^{0.5j}$

5.1.4 Example 4 High Channel Correlation

To complete our investigation, in this example, we examine the case of fading channels having higher correlations. Here, the channel correlation coefficients are set to $\rho = 0.7e^{0.5j}$ and $\rho = 0.9e^{0.5j}$ respectively. Fig. 5.4 and 5.5 show the performance of the various schemes under these correlation coefficients respectively. Comparing the performance of the various schemes in the two figures with Figs. 5.2 and 5.3, there is severe deterioration as the correlation coefficient increases. However, as observed in Fig. 5.4 and 5.5, the schemes utilizing the optimum precoder outperform all the other systems, making them the most robust against severe channel correlation.

From both Fig. 5.4 and Fig. 5.5, it is observed that at SER of 10^{-5}, the scheme equipped with the optimum precoder and a ZF-DF receiver has respective SNR gains of approximately 1 dB and 2 dB over V-BLAST for $|\rho| = 0.7$ and $|\rho| = 0.9$ while having the advantage of simplicity in implementation. The gains resulted from the use of the optimum precoder and the optimum ordering of received symbols are even more significant.

5.2 Minimum Power Precoder

In this section, we give simulation results for the other design in this thesis: the minimum power precoder. In particular, we focus on investigating the relationship between the total system transmission power and the MSE of ZF-DFE.

The simulation results in Fig. 5.6 show the total transmission power versus the MSE bound ε for MIMO communication systems with different numbers of transmitter and receiver antennas. It can be observed that the required power decreases with increasing average arithmetic MSE of the ZF-DF receiver. For fixed MSE bound, we can see that the total transmission power increases as the number of transmitter antenna increases. Conversely, the required transmission power decreases as the number of receiver antenna increases.

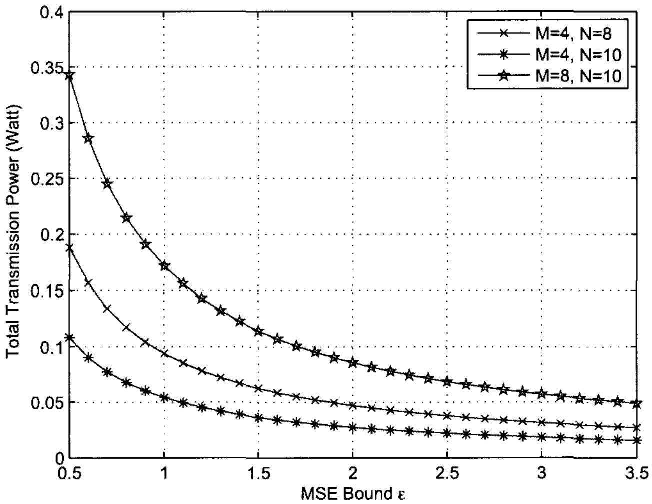

Figure 5.6: Total Transmission Power v.s. MSE Bound ε for Minimum Power Design

We have also shown in Fig. 5.7 the relationship between the SNR, the definition of which is $\text{SNR} = \frac{P}{\sigma^2}$, and the MSE bound ε for the different MIMO scenarios.

From Fig. 5.6, which reveals the relationship between the system transmission power and the MSE, we notice that the transmission power is an injective function (one-to-one function) of the MSE. That is, given a distinct MSE, we can compute a distinct transmission power value. Conversely, given a distinct transmission power, we can then obtain a distinct MSE. This observation can also be proved by our developed closed-form solution for these two precoders and the corresponding optimum objective functions.

First, let us fix a bound on the total transmission power on the first design (Design Problem 3.1) as $\bar{p}$. Because of the closed-form solution in Section 3.4.1, the optimum MSE is

$$\overline{\varepsilon^2}^{\,*} = \frac{\sigma^2}{M\bar{p}} \left(\sum_{m=1}^{M} \frac{1}{\sqrt{(N-m)}} \right)^2 \tag{5.3}$$

Having obtained $\overline{\varepsilon^2}^{\,*}$, we can further compute the closed-form solutions for our second design (Design Problem 4.1). The corresponding optimum power at the decision

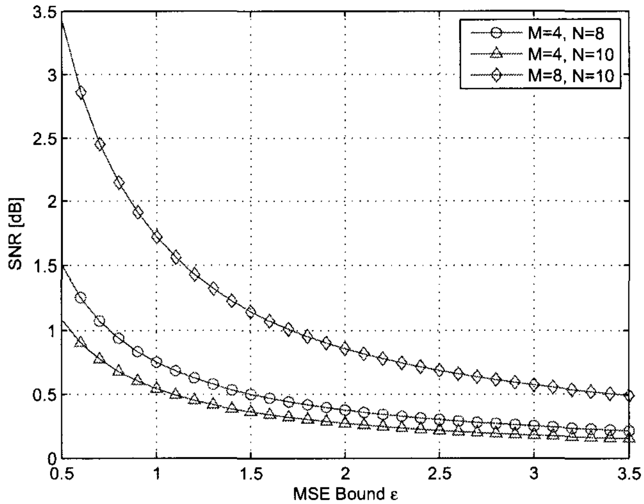

Figure 5.7: SNR v.s. MSE Bound ε for Minimum Power Design

feedback receiver is given by

$$p^* = \frac{\sigma^2}{M\bar{\varepsilon^2} *}\left(\sum_{m=1}^{M}\frac{1}{\sqrt{N-m}}\right)^2 \tag{5.4}$$

Substituting Eq. (5.3) into Eq. (5.4) and canceling all the like-terms on both the denominator and the numerator, we have

$$p^* = \bar{p} \tag{5.5}$$

In other words, if we first solve the minimum MSE problem with a fixed power constraint and obtain the minimum MSE value, then solve the minimum power design problem by letting this MSE value be the MSE constraint, thus, the solution of transmission power will be exactly the same as the previously fixed value of the power constraint of the first design problem. This conclusion further reveals the one-to-one mapping relationship of the MSE and the total transmission power and shows that the two design problems are "inverse problems". This is also true for correlated channels, but for analysis simplicity, we only utilize the closed-form solution under uncorrelated channel conditions to derive this relationship.

Chapter 6

Conclusion and Future Work

6.1 Conclusion

In this thesis, we have presented an efficient technique for the design of two optimal precoders. The first one is to minimize the arithmetic MSE of ZF-DFE subject to a constraint on the total transmission power, while the second one, as an "inverse" design problem, is to minimize the total system transmission power and subject to a constraint on the average MSE. Both of the two optimal precoders are designed for a MIMO system in which full knowledge of CSI is available at the receiver, but only the first- and second-order statistical knowledge is available at the transmitter.

From the analysis of some parameters of the system matrices, and with the aid of the recently developed algorithm of QRS decomposition of a matrix, we have been able to transform these two non-convex optimization problems into convex geometrical programming problems respectively. The interior point method is employed to efficiently solve these two convex optimization problems.

For the MIMO channels, we consider both correlated and uncorrelated situations. For correlated channels, we first use numerical method to obtain the optimal solution of the design problem, and then form the structure of the optimum precoding matrices by using the QRS algorithm. For the case when the transmission channels are uncorrelated, a closed-form solution of the optimum precoders has been obtained by using the Lagrange multipliers method and properties of Schur convexity.

Simulation experiments showed that when the fading channels are independent, the SER performance of the optimum MSE precoder shows marked improvement from that of the unprecoded scheme, while being outperformed by V-BLAST. When channel correlation increases, the performance of scheme equipped with the optimum MSE precoder catches up with and surpasses that of the V-BLAST scheme. For practical applications in which correlated transmission channels are common, the relatively low cost and the superior performance of the optimum MSE precoder with the ZF-DF receiver renders it a particularly attractive alternative to the V-BLAST algorithm.

6.2 Future Work

Based on the studies in this thesis, some interesting research issues arise:

- In this thesis, we proposed two optimal precoders for the ZF-DF receivers. Since MMSE criterion is another strategy for the decision feedback detection. We may apply the properties and theorems developed in this thesis to design precoding matrix for MMSE-DFE. For MMSE criterion, however, the probability density function is no longer as simple as that in Eq. (3.21).

- In the design of the optimum MSE precoder, the criterion of our objective function is the average MSE for ZF-DFE. Our assumption is that the transmitters only have the channel statistics knowledge which leads to an expectation on the objective function. We may therefore extend our design to other criteria, such as minimizing the symbol error rate of the ZF-DF receivers. In fact, the SER expression is very complicated for DFE, especially when we take expectation on it. But using the statistical properties of our system parameters, one may obtain an upper bound or even an exact expression for the result of the expectation. A new objective function will lead to new precoder designs, which may further offers better error rate performance.

Appendix A

QRS Decompositions

QRS decomposition is a collection of algorithms which facilitates the decomposition of an $M \times N$ matrix $\mathbf{A}$ such that $\mathbf{AS} = \mathbf{QR}$, where $\mathbf{Q}$ and $\mathbf{S}$ are unitary matrices and $\mathbf{R}$ has the following structure:

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_{K\times K} & \mathbf{0}_{K\times(N-K)} \\ \mathbf{0}_{(M-K)\times K} & \mathbf{0}_{(M-K)\times(N-K)} \end{pmatrix} \tag{A.1}$$

with $\mathbf{R}_{K\times K}$ being a $K \times K$ upper triangular matrix. Authors in [9, 10, 11, 12, 42] have developed algorithms of decomposition such that the R-factor has its diagonal entries all equal to $\left(\prod_{i=1}^{K} \lambda_i\right)^{1/2K}$. Recently, this equal diagonal QRS decomposition has been extended to a general case in which the diagonal entries of the R-factor are not necessarily identical [39]. For this generalized decomposition, a low-complexity quadratic recursive algorithm which systematically characterizes and constructs all feasible S-factors has been developed [15].

A.1 Generalized QRS Decomposition

We first present the generalized QRS decomposition.

Theorem A.1. *(QRS decomposition) Let the singular value decomposition [43] of* *matrix* $\mathbf{A}$ *be given by*

$$\mathbf{A} = \mathbf{U} \begin{pmatrix} \mathbf{\Lambda}^{1/2} & \mathbf{0}_{K\times(N-K)} \\ \mathbf{0}_{(M-K)\times K} & \mathbf{0}_{(M-K)\times(N-K)} \end{pmatrix} \mathbf{V}^H, \tag{A.2}$$

where $\mathbf{U}$ *and* $\mathbf{V}$ *are unitary matrices, and* $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_K)$ *with* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K > 0$. *Let* d_i *for* $i = 1, 2, \cdots, K$ *be an arbitrarily given positive numbers,* $\mathbf{p} = [j_1, j_2, \cdots, j_K]$ *being an arbitrarily given desired permutation of* $1, 2, \cdots, K$, *and matrix* $\mathbf{D_p} = \mathrm{diag}(d_{j_1}, d_{j_2}, \cdots, d_{j_r})$. *Then, the following three statements are equivalent.*

1. $\{\lambda_i\}_{i=1}^K$ *majorities* $\{d_i\}_{i=1}^K$ *in the product sense.*

2. *For an arbitrarily given desired permutation of* $\{d_i\}_{i=1}^K$, $d_{j_1}, d_{j_2}, \cdots, d_{j_K}$, *there exists a unitary matrix* $\mathbf{S}$ *such that* $\mathbf{AS} = \mathbf{QR}$, *where* $\mathbf{Q}$ *is an* $M \times K$ *column-wise orthonormal matrix and* $\mathbf{R} = [\ \mathbf{R}_{K\times K} \quad \mathbf{0}_{K\times(N-K)}\ ]$ *with* $\mathbf{R}_{K\times K}$ *being a* $K \times K$ *upper triangular matrix and* $[\mathbf{R}_{K\times K}]_i = \sqrt{d_{j_i}}$ *for* $i = 1, \cdots, K$

3. *There exist a unitary matrix* $\mathbf{W}$ *and a vector sequence* $\mathbf{z}_i$ *for* $i = 1, 2, \cdots, K-1$ *such that the canonical information distribution equation (CIDE) generated by* $\mathbf{\Lambda}$ *and* $\mathbf{D_p}$ *holds as follows,*

   (a) *Initialization. The first column vector* $\mathbf{w}_1$ *of* $\mathbf{W}$ *satisfies*

$$\mathbf{w}_1^H \mathbf{\Lambda} \mathbf{w}_1 = d_{j_1} \tag{A.3a}$$

$$\mathbf{w}_1^H \mathbf{w}_1 = 1. \tag{A.3b}$$

   (b) *Recursion. There exists* $\mathbf{z}_i$ *such that* $\mathbf{w}_{i+1} = \mathbf{W}_i^{\perp}\mathbf{z}_i$ *and*

$$\mathbf{z}_i^H \mathbf{C}^{(i)} \mathbf{z}_i = d_{j_{i+1}} \tag{A.3c}$$

$$\mathbf{z}_i^H \mathbf{z}_i = 1, \tag{A.3d}$$

   where $\mathbf{C}^{(i)} = \left((\mathbf{W}_i^{\perp})^H \mathbf{\Lambda}^{-1} \mathbf{W}_i^{\perp}\right)^{-1}$.

In particular, the equal diagonal QRS decomposition always exists; i.e., there exists a unitary matrix $\mathbf{S}$ *such that* $\mathbf{AS} = \mathbf{QR}$*, where* $\mathbf{Q}$ *is an* $M \times K$ *column-wise orthonormal matrix and* $\mathbf{R} = [\ \mathbf{R}_{K\times K} \quad \mathbf{0}_{K\times(N-K)}\ ]$ *with* $\mathbf{R}_{K\times K}$ *being a* $K \times K$ *upper triangular matrix and* $[\mathbf{R}_{K\times K}]_i = (\lambda_1\lambda_2\cdots\lambda_K)^{1/2K}$ *for* $i = 1, \cdots, K$.

Proof: See [15].

The canonical information distribution equations stated in Eq. (A.3) essentially describes a process for successively distributing the total information quantity $D = \det(\mathbf{\Lambda})$ in the whole K-dimensional space over each one dimensional subspace (or sub-channel), which holds a key to characterize all matrices with the prescribed singular values and the R-factor values.

Theorem A.1 not only provides a criterion to judge if a positive sequence is the diagonal elements of the R-factor in the QR decomposition of a given matrix, but also provides a quadratic recursive algorithm to systematically characterize and construct all unitary matrices with a given R-factor values. This recursive algorithm is obtained by successively constructing the vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r$ to satisfy equations (A.3a) - (A.3d).

A.2 Construction of the S-factor

The key to obtaining the QRS decomposition is to find the unitary matrix $\mathbf{S}$. Once we have had $\mathbf{S}$, we can apply the QR decomposition [43] to matrix $\mathbf{AS}$ so as to obtain the generalized QRS decomposition. The following recursive algorithm is to find the S-factor of the QRS decomposition $\mathbf{AS} = \mathbf{QR}$.

Algorithm 1 Construction of the *S*-factor:

1. *SVD*: Perform the SVD (A.2) of $\mathbf{A}$.

2. *Initialization*: Determine the first column of $\widetilde{\mathbf{S}}$ such that

$$\widetilde{\mathbf{s}}_1^H \mathbf{\Lambda} \widetilde{\mathbf{s}}_1 = d_{j_1} \quad \text{(A.4a)}$$

$$\widetilde{\mathbf{s}}_1^H \widetilde{\mathbf{s}}_1 = 1 \quad \text{(A.4b)}$$

3. *Recursion* (reduce the dimension and decouple constraints): Set $\widetilde{\mathbf{s}}_{i+1} = \widetilde{\mathbf{S}}_i^{\perp}\mathbf{z}_{i+1}$,where $\widetilde{\mathbf{S}}_i^{\perp}$ is the orthonormal complement of $\widetilde{\mathbf{S}}_i$ and $\mathbf{z}_{i+1}$ is any vector that satisfies

$$\mathbf{z}_{i+1}^H \mathbf{C}^{(i)} \mathbf{z}_{i+1} = d_{j_i} \quad \text{(A.4c)}$$

$$\mathbf{z}_{i+1}^H \mathbf{z}_{i+1} = 1 \quad \text{(A.4d)}$$

where $\mathbf{C}^{(i)} = \left((\widetilde{\mathbf{S}}_i^{\perp})^H \mathbf{\Lambda}^{-1} \widetilde{\mathbf{S}}_i^{\perp}\right)^{-1}$.

4. *Complete the S-factor*: $\mathbf{S} = [\mathbf{V}_K \widetilde{\mathbf{S}}, \overline{\mathbf{V}}_{1,\cdots,K}]$.

We would like to make the following comments.

- Algorithm 1 tells us that a problem of finding the S-factor for a matrix $\mathbf{A}$ is essentially reduced to a problem of finding the S-factor for its singular value diagonal matrix $\mathbf{\Lambda}^{1/2}$.

- Information decomposition. Actually, the quadratic recursive algorithm is the Schur's decomposition that the determinant of a positive definite matrix is equal to the product of the determinant of an arbitrarily given principal submatrix and the determinant of its Schur complement [43]. Therefore, the algorithm essentially describes a process for successively distributing the total information quantity $D = \det(\mathbf{\Lambda})$ in the whole r-dimensional space over each one dimensional subspace (or R-factor values subchannel).

- Characterization of the S-factor. The quadratic recursive algorithm (A.4a)–(A.4d) characterizes all S-factors such that the resulting matrices $\mathbf{AS}$ possess the prescribed diagonal R-factors. By properly choosing a particular solutions for each recursion one can significantly simplify the complexity of computing a specific S-factor, which we show in the ensuring subsection.

- Characterization of the Q-factor. Notice that if $\mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are the Q-R-S factors of an invertible matrix $\mathbf{A}$, then, $\mathbf{S}, \mathbf{R}^{-1}$ and $\mathbf{Q}$ are the Q-R-S factors of its inverse $\mathbf{A}^{-1}$. This observation implies that there is a companion algorithm for $\mathbf{A}^{-1}$ which has the same structure as Algorithm 1 This inverse algorithm actually characterizes the Q-factor in the QRS decomposition of matrix $\mathbf{A}$, which leads to an efficient algorithm for finding the S and Q factors.

A.3 Specific Closed-form QRS Decompositions

In particular [15], a pair of closed-form QRS decompositions for both a matrix and its inverse can be obtained. This specific closed-form decomposition is stated and proved in the following theorem:

Theorem A.2. *(Closed-form QRS decomposition) Let* $\mathbf{\Lambda}^{1/2} = \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_K})$ *with* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K > 0$. *Also, let* d_i, $i = 1, 2, \cdots, K$, *be an arbitrarily given positive numbers such that* $\{\sqrt{\lambda_i}\}_{i=1}^K$ *majorizes* $\{\sqrt{d_i}\}_{i=1}^K$ *in the product sense. Then, for an arbitrarily given desired permutation of* $\{d_i\}_{i=1}^K$, $d_{j_1}, d_{j_2}, \cdots, d_{j_K}$, *there exists a pair of unitary matrices* $\mathbf{S}_{\boldsymbol{\upsilon\omega}}$ *and* $\mathbf{S}_{\bar{\boldsymbol{\upsilon}}\bar{\boldsymbol{\omega}}}$ *such that* $\mathbf{\Lambda}^{1/2}\mathbf{S}_{\boldsymbol{\upsilon\omega}} = \mathbf{S}_{\bar{\boldsymbol{\upsilon}}\bar{\boldsymbol{\omega}}}\mathbf{R}$. *where* $\mathbf{R}$ *is an upper triangular matrix having diagonal elements* $r_{ii} = \sqrt{d_{j_i}}$ *for* $i = 1, 2, \cdots, K$ *and* $\mathbf{S}_{\boldsymbol{\upsilon\omega}} = \prod_{i=1}^{K-1} \mathbf{S}_{\upsilon_i \ell_i \omega_i}$ *with each* $\mathbf{S}_{\upsilon_i \ell_i \omega_i}$ *be determined by the following algorithm.*

Algorithm 2 (Construction of a pair of closed-form Q- and S-factors):

1. *Construction of a canonical eigen-diagonal matrix sequence* $\{\ell_i, \mathbf{\Lambda}^{(i)}\}_{i=1}^{K-1}$: Initialize $\lambda_i^{(1)} = \lambda_i$ for $i = 1, 2, \cdots, K$. Let ℓ_i be a maximum positive integer such that $\lambda_{\ell_i}^{(i)} \geq d_{j_i} \geq \lambda_{\ell_i+1}^{(i)}$. We define $\lambda_1^{(i+1)} = \lambda_1^{(i)}, \lambda_2^{(i+1)} = \lambda_2^{(i)}, \cdots, \lambda_{\ell_i-1}^{(i+1)} = \lambda_{\ell_i-1}^{(i)}, \lambda_{\ell_i}^{(i+1)} = \frac{\lambda_{\ell_i}^{(i)}\lambda_{\ell_i+1}^{(i)}}{d_{j_i}}, \lambda_{\ell_i+1}^{(i+1)} = \lambda_{\ell_i+2}^{(i)}, \cdots, \lambda_{K-i}^{(i+1)} = \lambda_{K-i+1}^{(i)}$.

2. *Construction of basic rotation sequences*: A basic rotation sequence $\{\upsilon_i, \omega_i\}_{i=1}^{K-1}$ and its inverse basic rotation sequence $\{\bar{\upsilon}_i, \bar{\omega}_i\}_{i=1}^{K}$ are well defined [15], respec-

tively, by

$$\upsilon_i = \sqrt{\frac{d_{j_i} - \lambda^{(i)}_{\ell_i+1}}{\lambda^{(i)}_{\ell_i} - \lambda^{(i)}_{\ell_i+1}}}, \ \omega_i = \sqrt{\frac{\lambda^{(i)}_{\ell_i} - d_{j_i}}{\lambda^{(i)}_{\ell_i} - \lambda^{(i)}_{\ell_i+1}}}, \tag{A.5}$$

and

$$\bar{\upsilon}_i = \sqrt{\frac{\bar{\lambda}^{(i)}_{\ell_i+1} - \bar{d}_{j_i}}{\bar{\lambda}^{(i)}_{\ell_i+1} - \bar{\lambda}^{(i)}_{\ell_i}}}, \ \bar{\omega}_i = \sqrt{\frac{\bar{d}_{j_i} - \bar{\lambda}^{(i)}_{\ell_i}}{\bar{\lambda}^{(i)}_{\ell_i+1} - \bar{\lambda}^{(i)}_{\ell_i}}}, \tag{A.6}$$

where, for notation simplicity and structural symmetry, we denote $\bar{\lambda}^{(i)}_j = (\lambda^{(i)}_j)^{-1}$ and $\bar{d}_{j_i} = (d_{j_i})^{-1}$.

3. *Construction of the Q- and S-factors*: Let $\mathbf{S}_{\boldsymbol{\upsilon}\boldsymbol{\omega}} = \prod_{i=1}^{K-1} \mathbf{S}_{\upsilon_i \ell_i \omega_i}$, where

$$\mathbf{S}_{\upsilon_i\ell_i\omega_i} = \begin{pmatrix} \mathbf{I}_{i-1} & \mathbf{0}_{(i-1)\times 1} & \mathbf{0}_{(i-1)\times(\ell_i-1)} & \mathbf{0}_{(i-1)\times 1} & \mathbf{0}_{(i-1)\times(K-\ell_i-i)} \\ \mathbf{0}_{(\ell-1)\times(i-1)} & \mathbf{0}_{(\ell_i-1)\times 1} & \mathbf{I}_{\ell_i-1} & \mathbf{0}_{(\ell_i-1)\times 1} & \mathbf{0}_{(\ell_i-1)\times(K-\ell_i-i)} \\ \mathbf{0}_{1\times(i-1)} & \upsilon_i & \mathbf{0}_{1\times(\ell_i-1)} & -\omega_i & \mathbf{0}_{1\times(K-\ell_i-i)} \\ \mathbf{0}_{1\times(i-1)} & \omega_i & \mathbf{0}_{1\times(\ell_i-1)} & \upsilon_i & \mathbf{0}_{1\times(K-\ell_i-i)} \\ \mathbf{0}_{(K-\ell_i-i)\times(i-1)} & \mathbf{0}_{(K-\ell_i-i)\times 1} & \mathbf{0}_{(K-\ell_i-i)\times(\ell_i-1)} & \mathbf{0}_{(K-\ell_i-i)\times 1} & \mathbf{I}_{K-\ell_i-i} \end{pmatrix}$$

The same results hold for $\mathbf{S}_{\bar{\boldsymbol{\upsilon}}\bar{\boldsymbol{\omega}}}$ by replacing the basic rotation sequences $\{\upsilon_i, \omega_i\}_{i=1}^{K-1}$ in $\mathbf{S}_{\upsilon\omega}$ by its inverse basic rotation sequences $\{\bar{\upsilon}_i, \bar{\omega}_i\}_{i=1}^{K-1}$.

Proof: Since each $\mathbf{S}_{\upsilon_i\ell_i\omega_i}$ is unitary, so does $\mathbf{S}_{\boldsymbol{\upsilon}\boldsymbol{\omega}}$. Therefore, in the following we only need to prove that $\mathbf{S}^H_{\bar{\boldsymbol{\upsilon}}\bar{\boldsymbol{\omega}}}\boldsymbol{\Lambda}^{1/2}\mathbf{S}_{\boldsymbol{\upsilon}\boldsymbol{\omega}}$ is an upper triangular matrix with a given diagonal entries. To this end, we first prove the following statement using induction on $J : 1 \le J \le K-1$.

$$\left(\prod_{i=1}^{J} \mathbf{S}_{\bar{\upsilon}_i\ell_i\bar{\omega}_i}\right)^H \boldsymbol{\Lambda}^{1/2} \prod_{i=1}^{J} \mathbf{S}_{\upsilon_i\ell_i\omega_i} = \begin{pmatrix} \mathbf{R}_{J\times J} & \mathbf{R}_{J\times(r-J)} \\ \mathbf{0}_{(r-J)\times J} & \sqrt{\boldsymbol{\Lambda}^{(J+1)}} \end{pmatrix}. \tag{A.7}$$

Since $\{\upsilon_i, \omega_i\}_{i=1}^{K}$ and $\{\bar{\upsilon}_i, \bar{\omega}_i\}_{i=1}^{K}$ have structures defined as Eq. (A.5) and Eq. (A.6), then they satisfy

$$\upsilon_i^2 + \omega_i^2 = 1 \tag{A.8a}$$

$$\lambda_{\ell_i}^{(i)}\upsilon_i^2 + \lambda_{\ell_i+1}^{(i)}\omega_i^2 = d_{j_i} \tag{A.8b}$$

$$\bar{\upsilon}_i^2 + \bar{\omega}_i^2 = 1 \tag{A.8c}$$

$$\bar{\lambda}_{\ell_i}^{(i)}\bar{\upsilon}_i^2 + \bar{\lambda}_{\ell_i+1}^{(i)}\bar{\omega}_i^2 = \bar{d}_{j_i} \tag{A.8d}$$

$$\bar{\upsilon}_i = \upsilon_i\sqrt{\frac{\lambda_{\ell_i}^{(i)}}{d_{j_i}}}, \quad \bar{\omega}_i = \omega_i\sqrt{\frac{\lambda_{\ell_i+1}^{(i)}}{d_{j_i}}} \tag{A.8e}$$

Therefore, when $J = 1$, it can be verified by computation that

$$\mathbf{S}_{\bar{\upsilon}_1\ell_1\bar{\omega}_1}\mathbf{\Lambda}^{1/2}\mathbf{S}_{\upsilon_1\ell_1\omega_1} = \begin{pmatrix} \sqrt{d_{j_1}} & \mathbf{R}_{1\times(K-1)} \\ \mathbf{0}_{(K-1)\times 1} & \sqrt{\mathbf{\Lambda}^{(2)}} \end{pmatrix} \tag{A.9}$$

Now we assume that Eq. (A.7) is true for $J = L$. For $J = L+1$, exploiting the induction assumption, we have

$$\begin{aligned}
&\left(\prod_{i=1}^{L+1}\mathbf{S}_{\bar{\upsilon}_i\ell_i\bar{\omega}_i}\right)^H \mathbf{\Lambda}^{1/2}\prod_{i=1}^{L+1}\mathbf{S}_{\upsilon_i\ell_i\omega_i} \\
&= \mathbf{S}_{\bar{\upsilon}_{L+1}\ell_{L+1}\bar{\omega}_{L+1}}\begin{pmatrix} \mathbf{R}_{L\times L} & \mathbf{R}_{L\times(K-L)} \\ \mathbf{0}_{(K-L)\times L} & \sqrt{\mathbf{\Lambda}^{(L+1)}} \end{pmatrix}\mathbf{S}_{\bar{\upsilon}_{L+1}\ell_{L+1}\bar{\omega}_{L+1}} \\
&= \begin{pmatrix} \mathbf{R}_{L\times L} & \mathbf{R}_{L\times 1} & \mathbf{R}_{L\times(K-L-1)} \\ \mathbf{0}_{1\times L} & \sqrt{d_{j_{L+1}}} & \mathbf{R}_{1\times(K-L-1)} \\ \mathbf{0}_{(K-L-1)\times L} & \mathbf{0}_{(K-L-1)\times 1} & \sqrt{\mathbf{\Lambda}^{(L+2)}} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{R}_{(L+1)\times(L+1)} & \mathbf{R}_{(L+1)\times(K-L-1)} \\ \mathbf{0}_{(K-L-1)\times(L+1)} & \sqrt{\mathbf{\Lambda}^{(L+2)}} \end{pmatrix}
\end{aligned}$$

This shows that Eq. (A.7) is also true for $J = L+1$. □

Appendix B

Proof of Theorem 3.2

Let $\lambda_m(\mathbf{A})$ denotes the mth largest eigenvalue of matrix $\mathbf{A}$ for $m = 1, 2, \cdots, M$, where M is the rank of $\mathbf{A}$.

Consider the following two lemmas [37] [43]:

Lemma B.1. *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ are $M \times M$ positive definite Hermitian matrices. Then*

$$\prod_{m=1}^{k} \lambda_m(\mathbf{P}_1\mathbf{P}_2) \leq \prod_{m=1}^{k} \lambda_m(\mathbf{P}_1) \prod_{m=1}^{k} \lambda_m(\mathbf{P}_2), \qquad k = 1, \ldots, M-1, \tag{B.1}$$

$$\prod_{i=1}^{M} \lambda_m(\mathbf{P}_1\mathbf{P}_2) = \prod_{m=1}^{M} \lambda_m(\mathbf{P}_1) \prod_{m=1}^{M} \lambda_m(\mathbf{P}_2). \tag{B.2}$$

Lemma B.2. *Let $\mathbf{C}$ be an $L \times M$ matrix and $\mathbf{D}$ be a $M \times L$ matrix. Then, $\lambda_m(\mathbf{CD}) = \lambda_m(\mathbf{DC})$.*

Proof: By Lemma B.2, we have $\lambda_m(\mathbf{T}^H\mathbf{\Sigma}\mathbf{T}) = \lambda_m(\mathbf{\Sigma}\mathbf{T}\mathbf{T}^H)$ for $i = 1, 2, \cdots, M$. Now, utilizing Lemma B.1 with $\mathbf{P}_1 = \mathbf{\Sigma}$ and $\mathbf{P}_2 = \mathbf{T}\mathbf{T}^H$, we obtain

$$\begin{aligned}\prod_{m=1}^{k} \lambda_m(\mathbf{T}^H\mathbf{\Sigma}\mathbf{T}) = \prod_{m=1}^{k} \lambda_m(\mathbf{\Sigma}\mathbf{T}\mathbf{T}^H) &\leq \prod_{m=1}^{k} \lambda_m(\mathbf{\Sigma})\lambda_m(\mathbf{T}\mathbf{T}^H) \\ &= \prod_{m=1}^{k} \lambda_m(\mathbf{T}^H\mathbf{T})\lambda_m(\mathbf{\Sigma})\end{aligned} \tag{B.3}$$

for $1 \leq k < M$.

And

$$\prod_{m=1}^{M} \lambda_m(\mathbf{T}^H \boldsymbol{\Sigma} \mathbf{T}) = \prod_{m=1}^{M} \lambda_m(\boldsymbol{\Sigma} \mathbf{T} \mathbf{T}^H) = \prod_{m=1}^{M} \lambda_m(\boldsymbol{\Sigma}) \lambda_m(\mathbf{T} \mathbf{T}^H) = \prod_{m=1}^{M} \lambda_m(\mathbf{T}^H \mathbf{T}) \lambda_m(\boldsymbol{\Sigma}) \qquad \text{(B.4)}$$

Since we let $\{\tilde{\lambda}_m\}$, $\{\tilde{\mu}_m\}$, and $\{\tilde{\nu}_m\}$, for $m = 1, 2, \cdots, M$, be the eigenvalues of the positive definite matrices $\mathbf{T}^H \boldsymbol{\Sigma} \mathbf{T}$, $\mathbf{T}^H \mathbf{T}$ and $\boldsymbol{\Sigma}$ respectively, then,

$$\begin{aligned} \tilde{\lambda}_{[m]} &= \lambda_m(\mathbf{T}^H \boldsymbol{\Sigma} \mathbf{T}) \\ \tilde{\mu}_{[m]} &= \lambda_m(\mathbf{T}^H \mathbf{T}) \\ \tilde{\nu}_{[m]} &= \lambda_m(\boldsymbol{\Sigma}) \end{aligned}$$

Thus, equations (B.3) and (B.4) is equivalent to:

$$\prod_{m=1}^{k} \tilde{\lambda}_{[m]} \leq \prod_{i=1}^{k} \tilde{\mu}_{[m]} \tilde{\nu}_{[m]}, \quad \text{for} \quad 1 \leq k < M \qquad \text{(B.5a)}$$

$$\prod_{m=1}^{M} \tilde{\lambda}_{[m]} = \prod_{m=1}^{M} \tilde{\mu}_{[m]} \tilde{\nu}_{[m]} \qquad \text{(B.5b)}$$

This completes the the proof of Theorem 3.2.

Appendix C

Proof of Corollary 3.2

From Theorem 3.1, we know that if Corollary 3.1 holds, there exists a positive definite matrix $\bar{\mathbf{A}}$ such that $\{\tilde{\mu}_m\tilde{\nu}_m\}_{m=1}^M$ and $\{\tilde{d}_m\}_{m=1}^M$ are the eigenvalues and the Cholesky values of $\bar{\mathbf{A}}$, respectively. Now, we will show that matrix $\bar{\mathbf{A}}$ can be further written as a matrix product form such that $\bar{\mathbf{A}} = \bar{\mathbf{T}}^H\mathbf{\Sigma}\bar{\mathbf{T}}$.

Given the sequences of $\{\tilde{\mu}_{[m]}\}$ and $\{\tilde{\nu}_{[m]}\}$ for $m = 1, \cdots, M$, we can form a new sequence $\{\tilde{\mu}_{[m]}\tilde{\nu}_{[m]}\}$ and further construct a diagonal matrix $\text{diag}(\tilde{\mu}_{[1]}\tilde{\nu}_{[1]}, \cdots, \tilde{\mu}_{[M]}\tilde{\nu}_{[M]})$. Also, we let the eigenvalue decomposition of $\mathbf{\Sigma}$ be $\mathbf{\Sigma} = \tilde{\mathbf{U}}\ \text{diag}(\tilde{\nu}_{[1]}, \cdots, \tilde{\nu}_{[M]})\ \tilde{\mathbf{U}}^H$ where $\tilde{\mathbf{U}}$ is the eigenvector matrix of $\mathbf{\Sigma}$ such that $\tilde{\mathbf{U}}^H\tilde{\mathbf{U}} = \mathbf{I}$.

If we let matrix $\text{diag}(\tilde{\mu}_{[1]}\tilde{\nu}_{[1]}, \cdots, \tilde{\mu}_{[M]}\tilde{\nu}_{[M]})$ be the eigenvalue matrix of $\bar{\mathbf{A}}$, then the positive definite matrix $\bar{\mathbf{A}}$ can be decomposed as

$$
\begin{aligned}
\bar{\mathbf{A}} &= \bar{\mathbf{V}}\ \text{diag}(\tilde{\mu}_{[1]}\tilde{\nu}_{[1]}, \cdots, \tilde{\mu}_{[M]}\tilde{\nu}_{[M]})\ \bar{\mathbf{V}}^H \\
&= \bar{\mathbf{V}}\ \text{diag}(\tilde{\mu}_{[1]}^{1/2}, \cdots, \tilde{\mu}_{[M]}^{1/2})\ \tilde{\mathbf{U}}^H\tilde{\mathbf{U}}\ \text{diag}(\tilde{\nu}_{[1]}, \cdots, \tilde{\nu}_{[M]})\ \tilde{\mathbf{U}}^H\tilde{\mathbf{U}}\ \text{diag}(\tilde{\mu}_{[1]}^{1/2}, \cdots, \tilde{\mu}_{[M]}^{1/2})\ \bar{\mathbf{V}}^H \\
&= \bar{\mathbf{T}}^H\mathbf{\Sigma}\bar{\mathbf{T}}
\end{aligned}
$$

where $\bar{\mathbf{T}} = \tilde{\mathbf{U}}\ \text{diag}(\tilde{\mu}_{[1]}^{1/2}, \cdots, \tilde{\mu}_{[M]}^{1/2})\ \bar{\mathbf{V}}^H$ and $\bar{\mathbf{V}}$ is a unitary matrix. □

Appendix D

Proof of the Convexity of $g(\mathbf{x})$

Since $g(\mathbf{x}) = \sum_{m=1}^{M} e^{x_m}$, then the gradient of $g(x)$ with respect to $x_1, x_2, \cdots, x_M$ is given by

$$\nabla g(x) = \begin{bmatrix} e^{x_1} \\ e^{x_2} \\ \vdots \\ e^{x_M} \end{bmatrix} \tag{D.1}$$

then, the Hessian matrix of $g(x)$ is

$$H(\mathbf{x}) = \nabla^2 g(x) = \begin{bmatrix} e^{x_1} & 0 & \cdots & 0 \\ 0 & e^{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{x_M} \end{bmatrix} \tag{D.2}$$

Because all $e^{x_1}, \cdots, e^{x_M}$ are positive, which means the Hessian matrix is positive definite. Therefore, $g(x)$ is strictly convex with respect to $x_1, x_2, \cdots, x_M$.

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