GAME THEORETIC OPTIMIZATION OF PRODUCTION PLANNING

GAME THEORETIC APPROACHES TO PETROLEUM REFINERY PRODUCTION PLANNING – A JUSTIFICATION FOR THE ENTERPRISE LEVEL OPTIMIZATION OF PRODUCTION PLANNING

by

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A Thesis
Submitted to the School of Graduate Studies
In Partial Fulfillment of the Requirements for the Degree
Doctor of Philosophy

McMaster University

DOCTOR OF PHILOSOPHY (2017)	McMaster University
(Chemical Engineering)	Hamilton, Ontario

TITLE: Game theoretic approaches to petroleum refinery production planning – a justification for the enterprise level optimization of production planning

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NUMBER OF PAGES: xiv,158

LAY ABSTRACT

This thesis presents a mathematical framework in which refinery production planning problems are solved to optimal solutions in competing scenarios. Concepts from game theory are used to formulate these competitive problems into mathematical programs under single objective functions which coordinate the interests of the competing refiners. Several different cases are considered presenting refinery planning problems as static and dynamic programs in which decisions are time independent or dependent, respectively. A theoretical development is also presented in the concept of the mixed integer game, a game theoretic problem containing both continuous and discrete valued variables and which must satisfy both continuous and discrete definitions of Nash equilibrium. This latter development is used to examine refinery problems in which individual refiners have access to numerous unit upgrades which can potentially improve performance. The results are used to justify a game theoretic approach to enterprise optimization.

ABSTRACT

This thesis presents frameworks for the optimal strategic production planning of petroleum refineries operating in competition in multiple markets. The game theoretic concept of the Cournot oligopoly is used as the basic competitive model, and the Nash equilibrium as the solution concept for the formulated problems, which are reformulated into potential games. Nonlinear programming potential game frameworks are developed for static and dynamic production planning problems, as well for mixed integer nonlinear expansion planning problems in which refiners have access to potential upgrades increasing their competitiveness. This latter model represents a novel problem in game theory as it contains both integer and continuous variables and thus must satisfy both discrete and continuous mathematical definitions of the Nash equilibrium. The concept of the mixed-integer game is introduced to explore this problem and the theoretical properties of the new class of games, for which conditions are identified defining when a class of two-player games will possess Nash equilibria in pure strategies, and conjectures offered regarding the properties of larger problems and the class as a whole. In all examples, petroleum refinery problems are solved to optimality (equilibrium) to illustrate the competitive utility of the mathematical frameworks. The primary benefit of such frameworks is the incorporation of the influence of market supply and demand on refinery profits, resulting in rational driving forces in the underlying production planning problems. These results are used to justify the development of frameworks for enterprise optimization as a means of decision making in competitive industries.

ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr. Vladimir Mahalec, whose knowledge and experience inspired this thesis, and without whose guidance I would not have been able to accomplish all that I have. Thank you.

I would like to thank my committee members Drs. Thomas Adams II and Elkafi Hassini, both of whom have provided invaluable input to my work over the last four years.

Special thanks are due to the professors of the Department of Chemical Engineering, and in the wider McMaster community, who have contributed to ten years of university education. I shall put it all to good use.

Perhaps most of all I would like to thank my mother, who has been encouraging my education for as long as I can remember, and without whose influence I would not have come so far. I would also like to thank my grandmother, who did not get to see me complete this thesis, but who would have been very proud.

Finally, I would like to thank my colleagues, past and present, in the McMaster Advanced Control Consortium. It was certainly fun.

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Declaration of academic achievement

I, Philip A. Tominac, declare this thesis to be my own work. I am the sole author of this document with exception to those chapters included as works published, submitted, or accepted for publication in research journals in which case authorship, credit, and copyright are duly noted with respect to each such included item.

As this thesis contains materials published, submitted, or accepted for publication in journals, all steps have been taken to ensure that the necessary copyright limitations and rights have been respected.

My supervisor, Dr. Vladimir Mahalec, and supervisory committee members Drs. Thomas Adams II and Elkafi Hassini have provided guidance to me throughout my Ph.D.; I completed all the research work included within this thesis.

Chapter 1

Introduction

Motivation

The work presented in this thesis is defined by the common goal of investigating supply chain production planning problems common to the field of chemical engineering in competitive contexts. A competitive supply chain production planning problem is interpreted to be one in which two or more parties are involved, and where there exists some interaction between the parties resulting in a dependency of each individual participant's outcome on the decisions made not only by itself but also by all other parties. In particular, this thesis focuses upon the case in which this interaction is antagonistic as opposed to cooperative, i.e., where each participant possesses its own objective unique from those of its opposition, and fulfillment of that objective conflicts with the fulfillment of opposing parties' objectives. Concepts from game theory are employed to model and quantify competitive behaviour in supply chain planning problems. In particular, the Cournot oligopoly model of competitive market behaviour is used as the basic competitive framework for competing suppliers, and the concept of the Nash equilibrium to define the solutions to such problems. These two conceptual entities are used throughout this thesis and form the basis upon which qualitative interpretations are drawn from results. Petroleum refining is used consistently in this thesis as an example supply chain planning problem; the phrasing refinery production planning is used with regard to these examples. The results presented in this thesis form a groundwork for the use of game theoretic planning as a means of achieving optimization across an entire enterprise, incorporating both large scale strategic objectives, decisions concerning production and associated operational constraints, the presence of competitors, and market distribution channels. By combining all of these decision levels within a single planning framework in which optimization is conducted at the enterprise level, suboptimal decisions resulting from optimization at multiple discrete planning levels can be avoided. The mechanism and primary benefit of the game theoretic enterprise planning framework is the unification of all objectives under a single economic objective function.

In this chapter the background material necessary for the thesis is introduced, providing a unified conceptual framework for various elements of theory. Later chapters consist of work published in or submitted for publication in peer-reviewed journals, and the theory discussed in those chapters may contain discrepancies in notation where the same concept emerges in multiple chapters. In part, the purpose of a reintroduction of the background material here unifies the notation for comparative purposes, and allows for some elaboration which is not feasible in articles intended for publication, even while keeping the background discussion brief.

Background and literature review

This section contains a general overview of the topics addressed in subsequent thesis chapters and is intended to unify certain nomenclature. There is thus some overlap between the review in this section and those in each of the following chapters. It is attempted not to reproduce in detail that which is addressed in other chapters, but to provide a unified overview of the theoretical background drawn upon throughout this work. With this objective in mind, overviews of supply chain production planning and game theoretic concepts are included here. The specifics of refinery operations and modelling which form the basis for the example scenarios in this thesis are presented in Chapter 2, with little need for reproduction in this section. Detailed refinery models and parameter data are attached as appendices.

Supply chain production planning

Production planning is a relevant problem within multiple disciplines, with varying interpretations. Stadtler reviews and outlines the general structure of supply chain management literature, in which production planning is a foundational element to overall competitiveness[1]. This thesis is primarily concerned with the interpretation of production planning in the field of process systems engineering, sometimes termed optimal production planning or simply optimal planning. Production planning is one of the primary elements of supply chain management and optimization; it is concerned with the determination of material and product inventories and production activities required to ensure that a process is capable of meeting demands placed upon it by future operations, and provides objectives to process scheduling and control initiatives which take into account process operational characteristics[2].

The body of literature in optimal planning is large. For the interested reader, Shapiro[3], Shah[4], Papageorgiou[5], and Sahebi, Nickel, and Ashayeri[6] provide reviews of relevant problems in and the state of the art of optimal planning problems, the latter with a particular focus on crude oil supply chains. The common theme among these reviews is the interpretation and modelling of supply chain planning problems as mathematical programs, and relationship between developments in the field of computational optimization and supply chain planning. Grossmann[7] and Trespalacios and Grossmann[8] provide reviews of MINLP methods used in process systems engineering. Floudas and Lin review MILP algorithms with a focus on process scheduling[9]. This thesis is concerned primarily with optimal planning problem structures employed in refinery planning. Joly[10] defines the role of production planning in the context of the Brazilian refining sector, in particular defining the relevant planning problems addressed in terms of strategic planning for future expansions, annual (long term) planning of production and maintenance, monthly (mid term) planning of operational targets, and short term planning of operations. Neiro and Pinto[11] provide a framework for the modelling and optimization of a wider petroleum supply chain including multiple refineries and the associated supply and distribution channels.

Game theory

The Nash equilibrium

Competitive game theoretic problems inherently quantify the conflicting interests of multiple parties. The Nash equilibrium provides a solution concept for these problems. A game theoretic problem is denoted G, with a set of competing, rational players R. Rationality in the game theoretic sense is understood to mean that players are assumed to behave in a prescribed manner; i.e., following some mathematical description. Each player r possesses a set of available options in the game, referred to as strategies. The set of strategies available to player r is Ξ_r, and an individual strategy in that set is denoted ξ_r. The outcome of a game is quantified in terms of payoffs to individual players resulting from their chosen strategies, as well as the strategies chosen by their competitors. The payoff to player r is defined as $J_r(\xi_r, \xi_{-r})$ where ξ_{-r} is used to indicate all other players except r. We assume without loss of generality that players in the game G seek to maximize their payoffs. A player's Nash equilibrium strategy is ξ_r^* and is defined according to Eq. (1).

$$J_r(\xi_r^*, \xi_{-r}^*) \geq J_r(\xi_r, \xi_{-r}^*) \quad \forall \xi_r \in \Xi_r, r \in R \tag{1}$$

The interpretation of this definition is that player r has no better payoff than that which can be achieved than by playing its Nash equilibrium strategy when every other player is also playing its Nash equilibrium strategy. This does not preclude the existence of a better payoff resulting to a player from a combination of strategies in which two or more players avoid the Nash strategy, but it relates how strategies are chosen by individual players such that certain combinations of strategies are not rational and not realizable.

Nash proved the existence of at least one equilibrium solution in games with finitely valued strategy sets Ξ_r; this solution may exist as a probability-weighted combination of strategies referred to as a mixed strategy Nash equilibrium (MSNE)[12]. A special case arises in which each player in such a game selects a single strategy with probability one, and is referred to as a pure strategy Nash equilibrium (PSNE). In games in which player strategy sets are infinitely valued (i.e., are continuous variables) there will exist at least one Nash equilibrium, analogously in pure strategies. The existence of a Nash equilibrium in infinitely-valued games is due independently to Debreu[13], Glicksberg[14], and Fan[15].

Static games

The defining characteristic of a static game is that all strategies in the game are executed at once, and without any player able to observe other players' strategic decisions prior to making their own[16]. Typical static games involve a single decision from multiple players, but this is not the limit of the form. Consider a game defined over multiple time instants; the static realization of this game is that in which all strategic decisions in all time instants are made prior to the realization of the game. In the context of petroleum refining this could be thought of as a scenario in which two refiners are forced to commit to a month-long production plan before either finds out what the other decided, and in which case each is unable to alter that plan once it is realized. The concept of equilibrium in static games is exactly as defined in Eq. (1).

Dynamic games

The concept of time becomes relevant in dynamic games; players become capable of making multiple decisions throughout the game, and have knowledge of their opposition's historic strategies as they become available. In this thesis, discussion of dynamic games is limited to Cournot oligopoly models – a subject for which there exists a large body of literature notwithstanding – for cohesiveness and consistency. Most of the theoretical elements discussed with respect to such games will be general in nature, but discussed with respect to the dynamic Cournot oligopoly model introduced by Simaan and Takayama[17]. There are two important features of this dynamic game which determine the properties of the game and methods by which it is solved. These are whether the game is formulated in continuous or discretized time, and whether the time horizon is finite or infinite in length.

Important developments in dynamic game theory have occurred through study of continuous time infinite horizon games, which are in some senses the most general form. Such games are problems of optimal control, where players' strategies are feedback control functions of past strategies[17]. Fershtman and Kamien made additional developments to the study of dynamic Cournot oligopolies in the continuous time finite horizon case, and demonstrated the turnpike properties of Nash equilibrium strategies: namely that the strategic profile initially approaches the infinite horizon equilibrium as it evolves through time, but deviates as the end of the time horizon approaches[18,19]. The reasoning behind this difference is a function of the time horizon itself: strategies become viable in the endgame which are not acceptable in the early game (typically because they would result in unacceptable payouts in future periods) and which are generally not acceptable in the infinite horizon game.

The variation considered in this thesis is the discrete time finite horizon dynamic game, which has several desirable properties. The discrete time finite horizon dynamic Cournot oligopoly problem is a potential game with a readily derived potential function; see Chapter 3 for additional discussion and a derivation of such a function. As a potential game, the dynamic Cournot oligopoly can be solved directly using numerical optimization rather than integration, which is the approach used to solve optimal control problems[17]. Since the model can be posed as an optimization problem, the incorporation of process constraints for the modelling of realistic systems is trivial. Furthermore, problems of significant size can be solved using existing numerical optimization tools.

An important difference between static and dynamic games is the interpretation of the Nash equilibrium. In the discrete time finite horizon game G with R players and N discrete time periods, player strategy sets are denoted Ξ_{nr} to indicate the additional time dimension. Player payoff functions become $J_r(\xi_{rn}, \xi_{-rn})$, and are functions of a strategy profile which evolves over time as a function of previous time instances. The Nash equilibrium definition is shown in Eq. (2).

$$J_r(\xi_{rn}^*, \xi_{-rn}^*) \geq J_r(\xi_{rn}, \xi_{-rn}^*) \quad \forall \xi_{rn} \in \Xi_{rn}, r \in R, n \in N \tag{2}$$

Equilibria in finite games

The payoffs in a game with finitely valued strategies constitute a payoff matrix. To distinguish finite payoffs from continuously valued payoff functions in infinite games, the notation $j_r(\xi_r, \xi_{-r})$ is introduced and is used to indicate scalar payoff values in finitely valued games. Pure strategy Nash equilibria in finitely valued games are identified as instances in which all players simultaneously possess a payoff maximum in the matrix dimension corresponding to their strategy vector. Instances in which this is not the case reflect finite games possessing only equilibrium in mixed strategies. Finding all equilibria in a finite game is NP hard[20].

A PSNE in a finite game is illustrated in Eq. (3) where the players R and C participate in a prisoners' dilemma[21]. The equilibrium is identified by placing accent marks $\dot{\xi}$ on the strategy vector minima (in this particular case) for each player; i.e., R's column minima and C's row minima. Individual payoffs in the matrix are presented as vectors $[j_R(\xi_R, \xi_C), j_C(\xi_R, \xi_C)]$. The PSNE occurs where the accent marks indicate both players possess a mutual minimum payoff subject to the opposing player's decision.

$$\begin{array}{cc} & C \\ R & \begin{bmatrix} 1,1 & 10,\dot{0} \\ \dot{0},10 & \dot{3},\dot{3} \end{bmatrix} \end{array} \tag{3}$$

The canonical example of a finite game possessing only a MSNE is the classic rock-paper-scissors[21]. The payoff matrix for this game is shown in Eq. (4), and illustrates that no PSNE can be identified based on matrix dimension maxima; there is always an alternative strategy one player can select which improves their own payoff while simultaneously reducing that of the opponent. As an aside, the equilibrium in mixed strategies is easily deduced in this game: each strategy should be selected with probability 1/3.

$$\begin{array}{cc} & C \\ R & \begin{bmatrix} 0,0 & 0,\dot{1} & \dot{1},0 \\ \dot{1},0 & 0,0 & 0,\dot{1} \\ 0,\dot{1} & \dot{1},0 & 0,0 \end{bmatrix} \end{array} \tag{4}$$

Equilibria in infinite games

In finite games payoff functions are typically functions of continuous strategy variables. Each participant in a game seeks to maximize its payoff with respect to those variables over which it has control. The Nash equilibria of a continuous game are defined as the solutions to the set of payoff function derivatives defined in Eq. (5).

$$\frac{\partial J_r(\xi_r, \xi_{-r})}{\partial \xi_r} = 0 \quad \forall \xi_r \in \Xi_r, r \in R \tag{5}$$

This set of equations is also referred to as the set of best response functions, as each player's payoff is optimized with respect to every opponent's similarly payoff-maximizing strategy[22]. Nash equilibrium in continuous games is thus also defined as a maximization problem as in Eq. (6).

$$\xi_r^* = argmax\big(J_r(\xi_r, \xi_{-r})\big) \quad \forall \xi_r, \xi_{-r} \in \Xi_r, r \in R \tag{6}$$

Generalized Nash equilibrium

In certain cases players' strategy spaces will not be independent of each other. Such strategy sets are generally referred to as having coupling constraints and are written as $\Xi_r(\xi_{-r})$. The Nash equilibrium concept in such cases is modified to account for these coupling constraints, and is referred to as a generalized Nash equilibrium[23,24]. Generalized Nash equilibria are in general not unique. Rosen developed a process known as normalization based on the weighting of dual variables to define a unique Nash equilibrium in such games[23,25,26]. In this thesis the concept of the generalized Nash equilibrium is important as an interpretation; in later chapters problems will be solved in which coupling constraints may or may not be active in a given equilibrium solution, changing the interpretation of the type of equilibrium obtained.

Potential games

The definitions of Nash equilibrium presented so far have many definitions arising as optimization arguments. The class of potential games are those in which the set of players behaves independently in such a way as to maximize a single objective function whose solution is a Nash equilibrium to the game[27,28,29]. The objective to a potential games is referred to as the potential function Z and has the property in Eq. (7) that the potential function derivative with respect to a player's strategy variable is exactly equal to the derivative of that player's payoff function derivative.

$$\frac{\partial Z(\xi_r, \xi_{-r})}{\partial \xi_r} = \frac{\partial J_r(\xi_r, \xi_{-r})}{\partial \xi_r} \quad \forall \xi_r \in \Xi_r, r \in R \tag{7}$$

The concept of the potential game also extends to finite games in which the potential to a game payoff matrix of dimension $|\Xi_1| \times \ldots \times |\Xi_r| \times \ldots \times |\Xi_R| \times |R|$ containing the payoff values $j_r(\xi_r, \xi_{-r})$ is a matrix of dimension $|\Xi_1| \times \ldots \times |\Xi_r| \times \ldots \times |\Xi_R|$ containing the potential values $z(\xi_r, \xi_{-r})$ which satisfy the relationship in Eq. (8)[29].

$$j_r(\xi_r, \xi_{-r}) - j_r(\xi_r', \xi_{-r}) = z(\xi_r, \xi_{-r}) - z(\xi_r', \xi_{-r}) \quad \forall \xi_r, \xi_r' \in \Xi_r, r \in R \tag{8}$$

This is the definition of an exact potential game in which the change in payoff resulting from a change in strategy by a player is exactly equal to the change in value in the potential corresponding to the same strategies, and holds true for all players. Alternative definitions of finite game potential exist; these are the weighted potential, in which the potential definition in Eq. (8) is satisfied subject to the weighting of the potential difference as $w_r\big(z(\xi_r, \xi_{-r}) - z(\xi'_r, \xi_{-r})\big)$, and the ordinal potential in which the signs of the two differences need to be the same, but the magnitudes do not. More in-depth discussion of the theoretical background of finite potential games is presented in Chapter 4.

The Cournot oligopoly

This model has been the subject of a much study in economic literature, and models a scenario in which multiple producers of a uniform product supply a market; the price of that good varies inversely with the total market supply. Thus producers must consider that their profits are a function not only of their own production volume, but also those of their opponents. In this thesis the canonical form of the model is avoided in favour of the version developed by Tominac and Mahalec[30] in which some apparent unit inconsistencies are resolved and – more importantly – market demand is used to modify producer behaviour. The market price π in this model is defined as a function of three parameters: A the value above nominal price of the first unit of product to enter the market, the nominal price of a product B corresponding to a nominal market supply level of D. Supplier production volumes are indicated by the nonnegative variables q_r. The relationship is expressed in Eq. (9). A general cost function $C_r(q_r)$ is considered for each player such that profits are as in Eq. (10).

$$\pi = A + B - \frac{A}{D}\sum_{r} q_r \tag{9}$$

$$J_r(q_r, q_{-r}) = \left(A + B - \frac{A}{D}\sum_{r'} q_{r'}\right) q_r - C_r(q_r) \quad \forall r \in R \tag{10}$$

This modified Cournot model is used throughout this thesis as a static game, a dynamic game, and a game of mixed continuous and integer variables and is discussed in those capacities in each respective chapter. It possesses an exact potential function of the form in Eq. (11).

$$Z(q_r, q_{-r}) = \sum_r \left((A+B)q_r - \frac{A}{D} q_r^2 - C_r(q_r) \right) - \frac{A}{D} \sum_r \sum_{\substack{r' \\ r'<r}} (q_r q_{r'}) \qquad \textbf{(11)}$$

Contributions

This thesis contains arguments for the use of game theoretic analysis in strategic refinery production planning. The primary argument upon which this thesis is predicated is that most production planning literature implicitly assumes a monopolistic market structure in its formulation. This structure yields mathematical results that are not realizable in their implementation, primarily due to the presence of competitors in real systems that make certain optimal production planning strategies unviable. This thesis presents static and dynamic frameworks for competitive refinery production planning across multiple markets and products, and identifies a new class of games in the mixed integer game in order to solve refinery expansion problems. These arguments are assembled within the broader context of this thesis as an argument for game theoretic optimization at the enterprise level.

Static and dynamic game theoretic planning frameworks

These frameworks are the first contribution that this thesis makes. Potential game structures are used to capture the presence of competitors in multi-market production planning problems with the objective of generating rational strategic planning results. The models used in these frameworks include producer economics using a detailed refining model, as well as the market economics of supply and demand in what are modelled as domestic and global markets. By modifying Cournot price variations to include nominal market demand levels refinery output and profit levels are linked to market supply volumes. This framework could allow refiners to optimize production with respect to changing demands and prices in markets, and to avoid costly planning errors. In addition, these frameworks provide a link between enterprise level decision making procedures as well as plant operating decisions and constraints. The dynamic game theoretic production planning framework is interesting in this respect as it also provides an indication of future economic repercussions to decisions made in the present; a feature absent in most production planning approaches. An example considered throughout this thesis is a scenario representative of Western Canada in which a small refinery competes with much larger opponents. The question in this scenario is whether the opposition should simply force the small refiner into closure. Based on the economic impacts it is found that the answer is highly dependent on market conditions, and the results tend to mirror the structure of the refining assets in Western Canada, suggesting that similar market forces may be at play, and can be accounted for in strategic planning approaches.

Mixed integer games and expansion planning

Expansion planning is a strategic planning problem in which future refinery upgrades are included as discrete variables such that the optimal future plant configuration can be determined along with the strategic operating conditions of that plant. Such problems are typically MINLP models and to date were not solvable by game theoretic approaches. This thesis provides a basis for the theory of mixed integer games, starting with the algebraic conditions under which two player, two strategy games are guaranteed to possess a Nash equilibrium in pure strategies. This contribution has implications in relevant engineering problems, and also as an academic line of inquiry into the properties and structure of this new class of game theoretic problems. This thesis offers as much in the way of developments as have been possible in the identification of the existence and behaviour of these problems, but does not offer mathematical proof, which remains as a line of inquiry.

Thesis overview

Chapter 2 of this thesis consists of the submitted text of a paper published in AIChE Journal. This paper details the static game elements of the potential game theoretic framework for strategic production planning. It outlines in detail prior works in the field of engineering supply chain planning that have used elements of game theory, then proceeds to outline how a potential game approach allows relevant production planning problems to be formulated and solved in a competitive context. This paper and its associated supplementary material (included as an appendix with this thesis) provide a detailed refinery production planning model, and an account of all data sources from which data were obtained with the goal of modelling competitive behaviour in the petroleum refining industry in a Canadian context. Necessary elements of game theory are introduced in this paper including a detailed derivation of the potential function used to link the refinery planning objectives to a market supply objective. Results in this paper demonstrate the benefits of using a game theoretic approach to production planning in contrast with typical single refiner fixed price approaches; primarily these are a more conservative prediction of profits, and a production regime which is robust to changes in opposing refiners' strategies. A scenario is also examined in which a small, high-cost refinery competes with larger, lower cost opponents, and the conditions under which the opponents seek to force the high-cost refiner to shut down. Game theoretic analysis of this scenario yields economic curves as functions of domestic and global market demand indicating the point at which the threat of elimination is realized, and are contrasted with similar curves obtained when a fixed price approach is used. In general, the fixed price approach does not capture the game theoretic results, which vindicate the continued existence of small refineries in Western Canada under certain market conditions. The citation for this work is presented below.

Tominac P, Mahalec V. A game theoretic framework for petroleum refinery strategic production planning. *AICHE J*, 2017; 63(7): 2751-2763.

Chapter 3 of this thesis consists of the text submitted for publication to AIChE Journal and which is in review at the time of writing. This work extends the static potential game framework of Chapter 2 to a dynamic potential game framework allowing refiners to observe the past behaviour of their opponents and to respond to it over the planning horizon. This manuscript reviews applications of dynamic games in engineering problems, presents the required background in dynamic Nash equilibria, and based on the static competitive refining model of Chapter 2, derives and verifies the properties of a dynamic potential function for the modified Cournot oligopoly model. This manuscript focuses primarily on the problem of the high-cost refiner in dynamic setting. In this work, the high-cost refiner may attempt to avert the threat of elimination by upgrading its facilities and becoming competitive with the low-cost refiners. Upgrading takes time, and changes the properties of the market in which the refiners operate; thus the threat of elimination also changes, potentially occurring in an earlier time period. The high-cost refiner must thus be able to complete its upgrades prior to being shut down, else it is unable to escape the threat. An interesting result emerges from this work in that the threat of elimination may not be legitimate; the low-cost refiners may decide not to eliminate the high-cost refiner at all, or at a time in the future which is never realized due to the rolling horizon nature of the implemented model. In such cases the high-cost refiner is safe if it does not initiate upgrade procedures, but the threat may become legitimate if it elects to do so. The submitted title of this manuscript is as below:

> Tominac P, Mahalec V. A dynamic game theoretic framework for process plant competitive upgrade and production planning.

Chapter 4 of this thesis is another manuscript submitted for publication, this one in the European Journal of Operational Research. The focus of this manuscript is on the properties of finite games with the objective of solving games possessing both discrete and continuous variables as mixed integer programs. At time of writing, there is no theory regarding games of this type, and the manuscript refers to them simply as mixed integer games, or mixed integer potential games with reference to the type of problem that is investigated. This work is heavily based on theoretical developments made by Monderer and Shapley[18] in their formalization of finite and infinite potential games. This paper examines those developments and extends the work to the case of two-player, two-strategy mixed integer Cournot oligopoly games. The conditions under which such a game is guaranteed to possess a Nash equilibrium in pure strategies are derived, and are established as conditions upon which it can be determined whether a given game of the indicated structure can be solved as an MINLP with the resulting solution being a Nash equilibrium. An example is presented where these conditions are applied, and the resulting game enumerated to validate the result. A second example consists of a refinery expansion game in higher dimensions, and thus the conditions derived in the paper cannot be applied; however, the game presented is enumerated to demonstrate that it also possesses a PSNE which is correctly obtained from the solution of the MINLP formulation, and it is noted that it no case was a game found which lacked a PSNE. Conjectures are then offered regarding the properties of mixed integer games. This work defines a new class of game theoretic problems and opens up a new line of inquiry for further research. The working title of the submitted manuscript is as below.

> Tominac P, Mahalec V. Conjectures regarding the existence and properties of mixed integer potential games.

Chapter 5 concludes the thesis and offers remarks on the work enclosed, the overall theme, and potential avenues for future research. The included appendices are the texts submitted as supplementary material to the publication in Chapter 2, and the submitted manuscript in Chapter 4.

Author's statement of contribution

I am the author of this thesis and the first author of all works submitted or accepted for publication included in this thesis.

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Chapter 2

A game theoretic framework for petroleum refinery strategic production planning

The content of this chapter is a revision of the manuscript text accepted for publication under the following citation:

Tominac P, Mahalec V. A game theoretic framework for petroleum refinery strategic production planning. *AICHE J*, 2017; 63(7): 2751-2763.

A game theoretic framework for petroleum refinery strategic production planning

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Topical Heading – Process Systems Engineering

Keywords – refinery planning; strategic planning; game theory; potential game; Nash equilibrium

Abstract

A game theoretic framework for strategic refinery production planning is presented in which strategic planning problems are formulated as non-cooperative potential games whose solutions represent Nash equilibria. The potential game model takes the form of a nonconvex nonlinear program (NLP) and we examine an additional scenario extending this to a nonconvex mixed integer nonlinear program (MINLP). Tactical planning decisions are linked to strategic decision processes through a potential game structure derived from a Cournot oligopoly-type game in which multiple crude oil refineries supply several markets. Two scenarios are presented which illustrate the utility of the game theoretic framework in the analysis of production planning problems in competitive scenarios. Solutions to these problems are interpreted as mutual best responses yielding maximum profit in the competitive planning game. The resulting production planning decisions are rational in a game theoretic sense and are robust to deviations in competitor strategies.

Introduction

Strategic production planning plays a vital role in modern organizations as a tool for strategic and tactical decision making at an organization-wide level[1]. In a comprehensive review of refinery supply chain planning models Sahebi, Nickel, and Ashayeri identify crude oil supply chain planning optimization as an imperative source of competitive advantage in the refining business[2]. Few papers exist in which refinery production planning has been examined in a competitive context where the presence of separate refiners competing for limited market share is taken into account at the strategic or tactical planning levels. Game theory provides the tools to investigate competitive interactions and has seen wide use in process systems engineering in areas where the interactions between competing entities are of fundamental interest. Of note is the area of electricity market modelling in deregulated power markets, where the ability of interested power suppliers to "game" established auction and distribution systems is well known. Bajpai and Singh review game theoretic methodologies used in modelling strategic decision making processes in electrical markets[3]. Also of note is the area of distributed model predictive control (MPC) in which the control actions of separate but interacting controllers are managed using game theoretic principles. Scattolini reviews game theoretic and other distributed MPC architectures[4].

Game theoretic principles have seen use in engineering supply chain planning literature to solve cooperative and competitive problems. Gjerdrum, Shah, and Papageorgiou have implemented Nash bargaining objective functions to determine fair profit allocation among members of multi-enterprise supply chains[5,6]. Pierru used Aumann-Shapley cost sharing to allocate carbon dioxide emissions to various products in an oil refinery[7]. Bard, Plumer, and Sourie used a bilevel formulation to investigate interactions between governments and biofuels producers as a Stackelberg game where the government leads by enacting policy[8]. Bai, Ouyang, and Pang have used a bilevel formulation to solve a competitive biofuel refinery location and planning problem as a Stackelberg game wherein the biofuel refiner takes the role of the leader and farmers follow by adjusting their land use[9]. Yue and You used KKT conditions to reduce the bilevel program describing a Stackelberg game into a single nonconvex MINLP whose global optimum is a Stackelberg equilibrium[10]. Zamarripa *et al* have developed a framework for solving cooperative and competitive supply chain problems through enumeration of the payoff matrix in multi-objective scenarios, yielding Nash equilibria in almost all cases[11,12,13].

With the exception to the works of Zamarripa *et al*, the applications of game theoretic principles in engineering supply chain literature do not yield Nash equilibrium planning results, and rely instead on other game theoretic constructs. In particular, the use of a Stackelberg game allows the planning decisions of a leader to be optimized such that the followers are constrained to Nash equilibrium strategies. The Stackelberg framework is not appropriate if no single competitor can be identified as a leader or does not have the capacity to implement a strategy before competitors can react[14,15]. The framework proposed by Zamarripa *et al* yields Nash equilibria in most cases, but does not under certain conditions, as they observed in their work[13]. Since their method is based on enumeration of a finite strategy matrix, and the framework examines only pure strategy solutions (as opposed to mixed strategies) a Nash equilibrium is not guaranteed to exist in all cases[16]. There is thus a gap in engineering supply chain literature where supply chain planning problems in competitive scenarios cannot be effectively solved to Nash equilibrium strategies. We address this problem with a game theoretic framework for strategic and tactical production planning which generates production plans representative of Nash equilibria between competing producers and we illustrate the properties of this framework using a set of competing oil refiners. Our framework treats production planning problems as continuous games (also referred to as infinite games) which guarantees that at least one Nash equilibrium will exist[17,18,19]. Problems are formulated as potential games, and Nash equilibrium solutions are identified as the global maxima of a potential function objective[20]. This potential game framework circumvents many of the problems which arise in the application of game theoretic models to production planning as the planning and game theoretic aspects of the problem are defined by a single objective function which can be solved using conventional NLP and MINLP solvers. The contributions and novel elements of this work are:

- A framework under which strategic production planning problems can be solved in a game theoretic context using a potential game formulation yielding solutions forming Nash equilibria;
- A modification to the Cournot oligopoly model which uses a defined demand level as a modifier of price behaviour;
- Two case studies which illustrate the utility of the game theoretic framework in relevant planning scenarios which exemplify its potential applications to strategic and tactical production planning.

Background

Nash equilibrium

The concept of the Nash equilibrium as a solution to a noncooperative game has been studied extensively and has different interpretations in various types of game theoretic problems[16,21,22]. We present elements of Nash equilibrium theory pertinent to the development of our potential game framework. Denoting the game as G and the strategy sets of each of N players as S_n with strategies $s_n \in S_n$ then a Nash equilibrium of G is defined as a set of strategies $G\{s_1^*, \dots, s_N^*\}$ where s_n^* represents player n's equilibrium strategy. Each player has an objective function $J_n\{s_n, s_{-n}\}$; a Nash equilibrium strategy has the property in Eq. (1).

$$J_n\{s_n, s_{-n}^*\} \leq J_n\{s_n^*, s_{-n}^*\} \quad \forall n \in N, s_n \in S_n \tag{1}$$

A non-strict inequality in this definition allows multiple equilibria to exist with the same value, referred to in such cases as weak Nash equilibria. Where an equilibrium satisfies the definition to strict inequality, the resulting Nash equilibrium is termed strict[23]. Nash equilibrium strategies are interpreted as a set of mutual best responses among all players; deviation from equilibrium will not yield an increase in objective value. The Nash equilibrium may also be interpreted as a maximizer of the set of player objectives in Eq. (2).

$$s_n^* = argmax\{J_n(s_n, s_{-n}^*)\} \quad \forall n \in N, s_n \in S_n \tag{2}$$

Each player's objective is maximized with regard to the best responses of all other players, which are usually not the global maximizers of $J_n(s_n, s_{-n})$ with respect to both strategy sets S_n and S_{-n}. Where the players' objectives are continuous and differentiable functions of strategy variables $s_n \in S_n$ the Nash equilibrium is defined by solving the set of equations in Eq. (3) [22].

$$\frac{\partial J_n(s_n, s_{-n})}{\partial s_n} = 0 \quad \forall s_n \in S_n, n \in N \tag{3}$$

Multiple Nash equilibria may exist in a continuous game. Calculation of all Nash equilibria which exist in a game is an NP-hard problem, although heuristics exist which allow additional equilibria to be characterized[24,25].

Games can be defined such that participants' strategy spaces are not independent. Such games are referred to as generalized Nash equilibrium problems (GNEP) [26,27,28]. In a GNEP player strategies are defined in terms of a strategy set $s_n \in S_n(s_{-n})$ which is dependent on competing players' chosen strategies. Constraints on player strategies make analytical solutions more difficult to obtain[28]. The solution to a GNEP is referred to as a generalized Nash equilibrium, and shares many of the same properties of a Nash equilibrium, with the definition in Eq. (4).

$$s_n^* = argmax\{J_n(s_n, s_{-n}^*)\} \quad \forall n \in N, s_n \in S_n(s_{-n}) \tag{4}$$

The generalized Nash equilibrium is defined by the KKT conditions corresponding to players' problems, and multiple generalized Nash equilibria may be defined this way. Normalization is a process through which a single equilibrium is defined as an appropriate solution and is accomplished by imposing a set of relative weightings on the dual variables which, for convex games, guarantees that a unique normalized Nash equilibrium exists for each unique set of weightings[29,30].

Potential games and the potential function

For a subclass of games called potential games, the system of equations defining Nash equilibria can be used to formulate a potential function whose maxima correspond to the Nash equilibria of the game. Early work demonstrating existence of the potential function was formalized by Bergstrom and Varian in 1985[31], and Slade in 1989[32] and 1994[33]. The class of potential games and the associated nomenclature were characterized by Monderer and Shapley in 1996[20]. Potential games can be solved using optimization tools, and the equilibria defined may be strict, weak, or of the generalized type[34,35].

A potential function is derived from the objective functions $J_n(s_n, s_{-n})$. All objective functions must be of the form in Eq. (5).

$$J_n(s_n, s_{-n}) = \Psi(s_n, s_{-n}) + \Omega_n(s_n) + \Theta_n(s_{-n}) \quad \forall n \in N \tag{5}$$

In this form each player's objective consists of three parts: Ψ is a term common to all players and a function of all players' strategy variables; Ω_n is a term unique to each player and is a function exclusively of that player's strategy variables; and Θ_n is a term unique to each player which contains only the variables associated with the other players. The potential function is formulated as in Eq. (6).

$$Z(s_n, s_{-n}) = \Psi(s_n, s_{-n}) + \sum_n \Omega_n(s_n) \tag{6}$$

This yields the same definition of the Nash equilibrium as defined in Eq. (3): the derivative with respect to any individual player's strategy variable yields the derivative of that player's objective function, as in Eq. (7).

$$\frac{\partial Z(s_n, s_{-n})}{\partial s_n} = \frac{\partial}{\partial s_n}\big(\Psi(s_n, s_{-n}) + \Omega_n(s_n)\big) = \frac{\partial J_n(s_n, s_{-n})}{\partial s_n} \tag{7}$$

The maxima of the potential function are also solutions to the set of partial differential equations obtained by equating each player's derivative to zero, and are therefore Nash equilibria by definition. These concepts extend to constrained games and the generalized Nash equilibrium; i.e., the maxima of the potential function subject to coupling constraints are generalized Nash equilibria[29].

Problem statement

We examine strategic refinery production planning in a game theoretic framework to investigate the effects of competition on strategic planning decisions. In this framework individual refineries are owned and operated by single, competing refiners such that each refinery in a given market is considered to be an individual competitor in a game theoretic sense. Each refiner produces the same set of petroleum products as the others and has access to the same crude oil stocks. Refineries are identical in configuration, but vary in capacity.

Refiners are faced with a production planning problem in which multiple target markets exist and each market is characterized by its own nominal demand levels, corresponding nominal prices, and status as either a domestic or a global market. Domestic markets consist of the geographical area in which refiners are physically located and corresponding points of sale. Refiners are collectively obligated to satisfy product supply constraints in their domestic market, and refiners outside that market cannot export product there for sale due to a lack of shipping infrastructure; e.g., we consider pipelines that carry product from domestic markets to global markets, but not between domestic markets. In this way, shipping between domestic markets by other means is prohibitively costly such that domestic market refiners have an oligopoly[36].

Each domestic market's maximum and minimum demand for each product are known, and will be satisfied by the refiners operating in that market. Domestic markets will absorb product levels between their upper and lower demand limits at the corresponding price and clear the market. Since this assumption of demand obligation can render the problem infeasible (i.e., if combined production capacity is too low) we assume that refiners are able to import finished product at a fixed price from other assets owned by the same entity, located elsewhere, and which are thus not market purchases. This import option provides the slack necessary to maintain feasibility. The products imported in this way cannot be exported to global markets.

Global markets are considered to be points of sale in geographical regions not occupied by refineries. Since global markets do not contain refiners, they are reliant upon imports from refiners in domestic markets. Global markets are connected to domestic markets by pipelines, and any refiner with access to a pipeline may export product to a global market without limit, as no shipping constraints are placed on global markets; we assume that the pipelines have sufficient capacity in this regard.

The refinery market is formulated with Cournot oligopoly pricing. Product prices in each market are variable functions of the collective market supply of that product; refiners do not control prices, but do influence them with their production decisions. Pricing is based on the concept of inverse demand; prices decrease in response to a market supply in excess of demand, and increase when supply falls short of demand. This pricing structure assumes that prices adjust to a point where all supplied product is sold and the market clears. Each refiner has the objective of maximizing its profit independently of the others. A refiner's individual problem is thus to:

- Determine the amounts of each product which should be sold in its domestic market in order to satisfy domestic supply constraints in concert with its competitors, and whether any product should be imported, in order to maximize its own profit (a strategic decision).
- Determine the amounts of each product which should be sold to global markets accounting for all competitors with market access in order to maximize its profit (a strategic decision).
- Determine how much of each crude oil stock to purchase and how to process the purchased stocks into the desired products in the most cost efficient manner (production planning decisions).

Each refiners' decision variables are:

- Crude stock purchase volumes.
- Blend volumes and unit operating modes.
- Product volumes and shipping destinations, including imports.

This game theoretic production planning problem is formulated as a potential game taking the form of an NLP. Domestic and global maxima of the potential function objective are defined as Nash equilibrium strategies in terms of refiner production decisions, and may be generalized Nash equilibria[29,33]. Due to the domestic market supply constraint forcing refiners to satisfy production within specified limits, the solutions obtained from this model may be characterized as generalized Nash equilibria when the constraint is enforced. Figure 1 illustrates an arrangement of refiners, consumers, and markets with three domestic markets and two global markets; one refiner has a domestic market monopoly because no other refiners have access to its point of sale, the rest compete in domestic oligopolies, and all compete in either one or two global market oligopolies.

Two scenarios are presented under this framework in which multiple refiners compete under different conditions. Each scenario explores different aspects of production planning problems in a competitive context. The first scenario illustrates the properties of game theoretic planning solutions to competitive problems; the second defines a complex scenario in which a competitor may be eliminated from a market, and illustrates how game theoretic planning can be implemented to make rational solutions to such problems. The scenarios are described in the following sections.

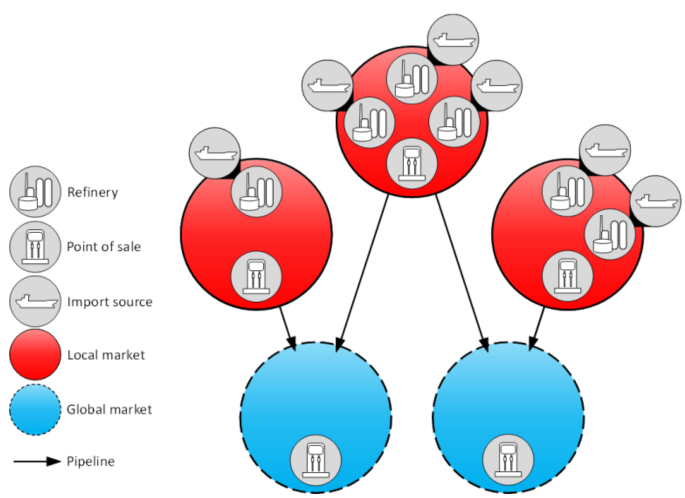

Figure 1. Sample arrangement of domestic and global markets.

Scenario 1 (S1) - Competition for market share

This scenario examines refiners competing in the petroleum market, serves to illustrate the properties of game theoretic planning solutions, and forms a point of comparison with other production planning approaches. In this scenario unit capacity constraints in the production planning model limit the ability of any individual refiner to process more than a certain total throughput regardless of market driving forces. It will be shown by removing these unit capacity constraints and allowing refiners to produce potentially infinite volumes that there exist Nash equilibria as the global maxima of the unconstrained potential function and that both types of solution share similar properties. We present this as variant S1-G. The production planning problem is also solved under a fixed price profit maximization objective for comparison using the sum of refiner profits as the objective function. With fixed prices the refiners are not in competition; this variant is intended to present the classical approach to planning (by which we mean profit maximization under the assumption of fixed prices) and in particular, to illustrate the motivating forces which define an optimal solution in the context of the defined market problem for the purpose of comparison with game theoretic results. This variant is defined as S1-F. Our results show that the classical approach results in non-optimal solutions to the defined problem. We discuss comparisons between the classic planning results with the game theoretic results.

Scenario 2 (S2) - Elimination of inefficient competitors

This scenario examines refiners in competition where the market structure may change. A subset of refineries are considered to be more efficient and competitive than the remainder and are labeled low-cost refineries, denoted by the subset $LCN \subseteq N$. The remaining refineries are, typically by virtue of their size or age, rendered less competitive than the low-cost refineries and are termed high-cost refineries, denoted as part of the subset $HCN \subseteq N$. In addition, high-cost refineries are characterized as being isolated from other assets. All refiners are in competition regardless of low or high-cost status, and the low cost refiners need to decide whether to shut down high-cost refineries and obtain additional market share for themselves, or to continue operating in the existing market structure. Examples of such industry structures exist in western Canada where several small refiners compete with large capacity refiners, as described by the Government of Canada National Energy Board in a listing of refining assets and the corresponding distribution pipeline network in Western Canada[37]. We draw attention to the trans-mountain and plateau pipelines which link five independent refineries in a distribution channel, three of which have quoted capacities of at least 100 Mb/d, one with a quoted capacity of 55 Mb/d, and the smallest with a capacity of 12 Mb/d and which is isolated from other pipelines in the network and other assets owned by the same parent company. Examination of the existence of this type of scenario has been cited as an industrially relevant problem at the enterprise planning level in a private communication with V. Mahalec in 2015. Similar problems examining refinery competitiveness as a function of scale have been identified by industry experts[38].

Cournot limit theorem states that all else being equal a market with fewer competitors maintains higher prices[39]. Based on this theorem, any option to reduce the number of competitors is a positive decision for the remaining refiners. We differentiate our scenario from this theorem by assuming that a high-cost refinery requires higher market prices than its low-cost competitors to remain profitable. In our model, as long as high-cost refineries remain active in a domestic market, all refiners gain the benefit of the higher domestic market prices; if the high-cost refineries shut down, prices drop to reflect the competitive margins of the low-cost refineries. We do not examine cases of differentiated products, thus high-cost refiners influence markets only through the differences in prices that they effect. Low-cost refiners have the option either to drive a high-cost refiner out of business by aggressively supplying the domestic market, or allowing the high-cost refiner to continue to operate; this decision occurs by consensus among the low-cost refiners in order to avoid cartel game mechanics[40]. The consensus decision is modelled as a binary variable such that this scenario is modeled as a nonconvex MINLP. The question in this game is under what market conditions a high-cost refiner is allowed to remain in operation. It will be shown that an inclusion region can be characterized based on market demand levels and the price increase associated with the high-cost refiner. We also defined variants S2-G and S2-F analogous to those defined for S1. We compare the inclusion regions defined in S2 and S2-F to illustrate how the decision to shut down a competitor varies under analysis by game theoretic and classical approaches.

Models and Formulation

Production planning model

The refinery production planning model consists of the set of equations which describe how crude oil is transformed into intermediates and products. We use a simplified linear yield-based model similar to that used by Castillo Castillo and Mahalec[41]. A schematic of the refinery is shown in Figure 2 illustrating the pathways that crude oil, production intermediates, and products take through the process units. Each refinery consists of a crude distillation unit (CDU) two hydrotreaters (HT1 and HT2) a hydrocracker (HC) a fluid catalytic cracker (FCC) and a catalytic reformer (CR). Blending of intermediates into products occurs in a gas blender (GB) and a diesel blender (DB). The eight intermediates of interest are straight run light naphtha (srln) hydrocracker light naphtha (hcln) catalytic cracker light naphtha (fccln) heavy naphtha (fcchn) light cycle oil (fcclco) straight run distillate (srds) hydrocracker distillate (hcds) and reformate (rft). The six products are regular, mid-grade, and premium gasoline (reg, mid, and pre) and diesel grades 1, 2, and 4 (de1, de2, and de4). We examine a planning horizon of one year divided into two six-month periods.

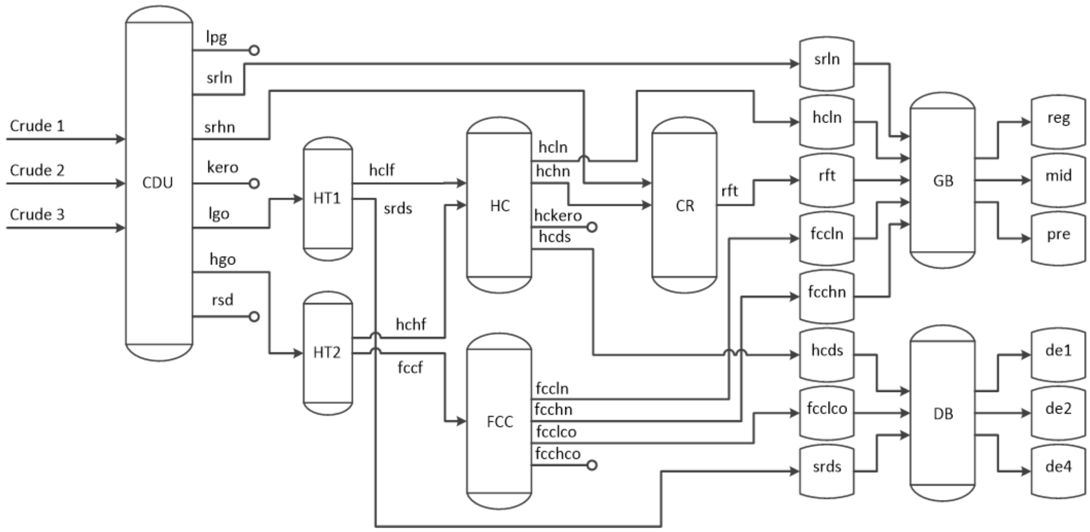

Figure 2. Refinery model schematic.

The plant processes crude oil following the layout in Figure 2. For each crude oil which enters the CDU there is a yield corresponding to the type of crude processed and operating mode (either max diesel mode or max naphtha mode) which dictates the amounts of outputs produced. The refinery model considers primarily those streams involved in the production of gasoline and diesel products. The streams denoting the CDU output of light product gasses (lpg) kerosene (kero) and residuals (rsd) are assumed to be sold at fixed price in order to keep the model relatively small. This assumption impacts neither the qualitative pattern of the results nor the conclusions. Similarly, the HC kerosene stream (hckero) and FCC heavy cycle oil (fcchco) are calculated but not included in profit calculations. All other streams in Figure 2 indicate movements of material through the refinery linking crude oil to gasoline and diesel products.

Refinery efficiency cost reflects the cost a refiner faces due to operating away from its efficient operating throughput. It is meant to represent complex unit and process operating costs incurred from nonstandard plant operation. The efficiency cost curve is modelled as a quadratic function with vertex coordinates $\left(EC_H(n), EC_K(n), EC_P(n)\right)$ for each refinery n, where $EC_H(n)$ indicates the design throughput with the efficient cost $EC_K(n)$. $EC_P(n)$ represents parabolic focal length and determines how efficiency costs increase with deviation from $EC_H(n)$. A quadratic efficiency function is calculated with the parameters in Eqs. (8), (9), and (10).

$$EC_A(n) = \frac{1}{4\big(EC_P(n)\big)} \tag{8}$$

$$EC_B(n) = -\frac{EC_H(n)}{2\big(EC_P(n)\big)} \tag{9}$$

$$EC_C(n) = \frac{\big(EC_H(n)\big)^2}{4\big(EC_P(n)\big)} + EC_K(n) \tag{10}$$

The total efficiency cost experienced by a refiner is the quadratic efficiency cost multiplied by the total output from the refinery cumulatively over all products and planning periods, and is cubic overall. Total efficiency cost is defined by Eq. (11), where the variable $Prc(t,p,n)$ indicates the amount of a product produced in a given time period by a refiner.

$$\begin{aligned} TEC(n) = {} & EC_A(n)\left[\sum_t \sum_p Prc(t,p,n)\right]^3 \\ & + EC_B(n)\left[\sum_t \sum_p Prc(t,p,n)\right]^2 \\ & + EC_C(n)\sum_t \sum_p Prc(t,p,n)\,, \forall n \in N \end{aligned} \tag{11}$$

The use of quadratic functions to represent refinery costs follows the arguments of economies of scale, which we implement in a simple form as a cost which increases with deviation from an ideal operating capacity[42]. In real refinery unit operations increased throughput can decrease efficiency because equipment may operate at non-optimal conditions. Olefin recovery efficiency downstream of FCC units, for example, can decrease at higher throughputs[43]. The production planning model equations are included as supplementary material. The refinery production planning model consists of Eqs. (B1) to (B41). Model equation variants specific to a scenario are outlined in the following subsections.

A demand-based Cournot oligopoly

A brief summary of the Cournot oligopoly is presented in the supplementary material available online. We present here a modified Cournot oligopoly designed for use with this work which assumes that if the total market production level of a product is equal to a nominal market demand level, denoted $D(p,w)$, then the market price of that product will take a value $B(p,w)$. As in the classic Cournot model, price varies linearly with total market supply, where individual producer amounts are denoted $Tpd(p,n,w)$ with a product index p. The marginal value of the first unit of a product to enter the market is defined as $A(p,w)+B(p,w)$, thus the price of a product p varies according to Eq. (12).

$$Pr(p,w) = A(p,w) + B(p,w) - \frac{A(p,w)}{D(p,w)}\sum_{n} Tpd(p,n,w) \quad \forall p \in P, w \in W \tag{12}$$

The competitor profit function is defined in Eq. (13) using the definition of market price in Eq. (12) and cost total $TotCost(n)$. This demand-based Cournot oligopoly problem is a potential game and forms the basis of the game theoretic refinery planning framework.

$$J(n) = \sum_{p}\sum_{w}\big(\mathrm{Pr}(p,w)\, Tpd(p,n,w)\big) - TotCost(n) \quad \forall n \in N \tag{13}$$

Potential function formulation

The potential function corresponding to the set of objectives defined in Eq. (13) consists of the common part Ψ and the unique parts $\Omega(n)$ of the objectives $J(n)$, and is defined in Eq. (14) with definitions for Ψ and $\Omega(n)$ in Eqs. (15) and (16).

$$\max Z = \Psi + \sum_{n} \Omega(n) \tag{14}$$

$$\Psi = \sum_{p,w}\left[-\frac{A(p,w)}{D(p,w)}\sum_{n}\sum_{\substack{n' \\ n'<n}} Tpd(p,n,w)Tpd(p,n',w)\right] \tag{15}$$

$$\Omega(n) = \sum_{p,w}\left[\left(A(p,w) + B(p,w) - \frac{A(p,w)}{D(p,w)}Tpd(p,n,w)\right)Tpd(p,n,w)\right] - TotCost(n) \quad \forall n \in N \tag{16}$$

This form of the potential function serves as the model objective and its maxima are Nash equilibria of strict, weak, or generalized types depending on the included constraints. The total product leaving a refinery $Tpd(p,n,w)$ is defined in Eq. (17) as the sum of the product the refinery produces and the amounts which it imports. These variables link the refinery planning model to the potential function.

$$Tpd(p,n,w) = \sum_{t} Dlv(t,p,n',w) + Imp(p,n,w) \tag{17}$$

Fixed-price analysis

Current practices generally use fixed prices in refinery planning models, generally assuming that prices will become known prior to the planning period[2]; we compare the outcomes of such analyses with game theoretic results. Our scenarios are examined under a fixed-price profit maximization framework using the objective in Eq. (18). This objective is the total profit of all refiners. Revenues are calculated based on fixed market prices $F(p,w)$ and are linear calculations; the only nonlinearity in this variant is the efficiency cost calculation.

$$\max Z = \sum_{n}\left(\sum_{p}\big(F(p,w)Tpd(p,n,w)\big) - TotCost(n)\right) \tag{18}$$

The domestic market supply constraints defined by Eqs. (B40) and (B41) enforce refiner coordination. With exception to these constraints, refiners are independent of one another in terms of their decision making; their profits are not interdependent under this objective.

Model alterations for Scenario 2

The game theoretic model is modified such that high-cost refiners are linked to a binary variable that is incorporated into the high-cost refiner model equations in order to allow all flow rates, inventories, and outputs to be set to zero, effectively shutting down those refiners. The continued participation of high-cost refiners is dependent on a binary variable $Y_{HCN}(w)$. High-cost refiners are also limited to a decreased production level $HCNset$ using Eq. (19), and are prevented from making import purchases in this scenario. These changes define the characteristics of the high-cost refiner, along with its parameter values.

$$\sum_{t}\sum_{p} Prc(t,p,n) = Y_{HCN}(w)HCNset \quad \forall\,(n,w) \in WHCN(n,w) \tag{19}$$

The participation binary is also used to relax variable bound constraints in Eqs. (B3), (B4), (B10), (B12), (B13), (B19), (B20), (B27), and (B29)-(B32); each use the binary variable to reduce a constraint value to zero if the value of $Y_{HCN}(w)$ is zero in order to deactivate the high-cost refiner model. The potential function term $\Omega(n)$ is altered to include the price increase $AHC(p,w)$ corresponding to the presence or absence of the high-cost refiner defined in Eq. (20).

$$\begin{aligned}\Omega(n) = \sum_{p} \sum_{\substack{w \\ w \in WN(n,w)}} \Bigg[\Bigg(& A(p,w) + AHC(p,w) \sum_{\substack{w' \\ w' \in WLN}} Y_{HCN}(w') \\ & + B(p,w) - \frac{A(p,w)}{D(p,w)} Tpd(p,n,w) \Bigg) Tpd(p,n,w) \Bigg] \\ & - TotCost(n) \quad \forall n \in N\end{aligned} \tag{20}$$

This version of $\Omega(n)$ contains a bilinear term of the form $Y_{HCN}(w)Tpd(p,n,w)$ which has an exact linearization obtained by introducing two variables and the constraints in Eqs. (21), (22), and (23). This linearization technique reduces the number of model nonlinearities and is described in more detail by You and Grossmann[44]. The presented formulation allows multiple high-cost refiners to exist in a single domestic market, and dictates their activity on an all-or-none basis. The upper bound $\overline{Tpd}(p,n,w)$ represents the total combined processing capacity of a refiner plus its product import limit.

$$TP(p,n,w,w') + TP1(p,n,w,w') = Tpd(p,n,w) \qquad \forall p \in P, n \in N, (w,w') \in W \tag{21}$$

$$TP(p,n,w,w') \leq Y_{HCN}(w')\overline{Tpd}(p,n,w) \quad \forall p \in P, n \in N, (w,w') \in W \tag{22}$$

$$TP1(p,n,w) \leq \big(1 - Y_{HCN}(w')\big)\overline{Tpd}(p,n,w) \quad \forall p \in P, n \in N, (w,w') \in W \tag{23}$$

With this linearization Eq. (20) can be rewritten as in Eq. (24), which is the form of the equation implemented in the elimination scenario and is denoted $\Omega_{\mathrm{K}}(n)$ in order to differentiate it from the version used in other scenarios.

$$\Omega_{\mathrm{K}}(n) = \sum_{p} \sum_{\substack{w \\ w \in WN(n,w)}} \Bigg[\big(A(p,w) + B(p,w)\big)Tpd(p,n,w) + \left(AHC(p,w) \sum_{\substack{w' \\ w' \in WLN}} TP(p,n,w,w') \right) - \frac{A(p,w)}{D(p,w)}\big(Tpd(p,n,w)\big)^2 \Bigg] - TotCost(n) \quad \forall n \in N \tag{24}$$

The elimination scenario is intended to determine whether low-cost refiners in a domestic market are better off with or without high-cost refiners. High-cost refiners' production decisions cannot be part of the potential function in this scenario. Exclusion of the high-cost refiners assures that the comparison made regarding the elimination decision includes only the interests of the low-cost refiners and their preferences regarding the market structure. Since the high-cost refiners are not included in the objective function, their decision variables must be fixed to game theoretically rational values. A two-stage solution process is used to solve this problem. In the first stage all refiners are active under a full potential function generating the optimal game theoretic production decisions for all refiners. This first stage amounts to solving a Scenario 1 problem where the high-cost refiner is unable to import product and effects a price increase in its domestic market. In the second stage of the solution process the potential function is generated with $\Omega_{\mathrm{K}}(n)$ for the low-cost refiners which captures their profits and the price increase effected by the high-cost refiner. The variable Ψ is generated over all refiners and captures the decrease in market prices caused by their collective production decisions, including the fixed first stage values assigned to the high-cost refiner. The second stage thus represents the interests of only the low-cost refiners. Any product profile could be assigned to the high-cost refiner in order to solve the Scenario 2 problem; generating the high-cost refiner's profile using the first stage ensures that the decision is rational with respect to game theoretic analysis. In this scenario we assume that high-cost refiners behave as they would in S1 if active; we do not examine cases where high-cost refiners could modify their behaviour in order to remain active, i.e., limiting their production to levels that would encourage low-cost refiners to allow them to remain operational. The potential function used in the second stage is formulated as in Eq. (25).

$$\max Z = \Psi + \sum_{n \in LCN} \Omega_{\mathrm{K}}(n) \tag{25}$$

The fixed price approach to the elimination problem is formulated using the objective function in Eq. (26).

$$\max Z = \sum_{\substack{n \\ n \in LCN}} \left(\sum_{p} \left(\left(F(p,w) + AHC(p,w) \sum_{\substack{w' \\ w' \in WLN}} Y_{HCN}(w') \right) Tpd(p,n,w) \right) - TotCost(n) \right) \tag{26}$$

Results and Discussion

Scenario basis and data

The scenarios presented in this work are based as much as possible on the Canadian national fuel market using data from 2014. Each scenario is based on the same example involving three refineries acting in a domestic market $LM1$ with access to an global market $EM1$. Product demand is scaled down to an appropriate level corresponding to the total combined production capacity existing among the three refineries. All numerical and structural problem data are included as supplementary material.

Market demands $D(p, w)$ for the six products in the scenarios are calculated based on historical Canadian national consumption using data from Statistics Canada for gasoline and diesel products. The published net consumer sales of gasoline and diesel provide the baseline for demand, but the reported net gasoline and diesel sales are not listed by grade[45]. The fraction of demand associated with each product grade is calculated using the Canadian gasoline and diesel totals assuming that the sales by grade can be approximated using consumer sales data for the relevant gasoline and diesel product grades in the USA in 2014 made available by the EIA[46,47]. These values are scaled to 18% and 21% of the real total in order to create domestic and global market demand totals with values scaling to the same order of magnitude as the combined production capacity of the three refineries.

The product pricing structure is based on weekly national average price data from 2014 published by Natural Resources Canada. Data are available for regular, mid-grade, and premium gasoline[48,49,50]. Data is also available for diesel fuel, but due to changes in the sale of diesel fuels the majority of diesel fuel sold for commercial purposes consists of a single grade[51,52]. We take the average price of each corresponding fuel in 2014 as $B(p, w)$ in both markets. In order to account for the different grades of diesel, the average diesel price obtained from Natural Resources Canada data is assigned to fuel oil type 2 and the price for fuel oil types 1 and 4 are calculated as one standard deviation above and below that price based on the available data. The marginal value of the first unit of each product on the market is calculated using a value of $A(p,w)$ equal to three standard deviations of price. Thus the price of the first unit of product corresponds roughly to the highest observed price in the data, and the price resulting from a supply of twice $D(p,w)$ to the lowest, fitting the product demand space to the available price data. The values of $AHC(p,w)$ are taken to be 5% of the value of one standard deviation of the calculated prices, e.g., 5% of ⅓ of $A(p,w)$ in the domestic market and zero elsewhere; $CI(p,w)$ is 120% of $B(p,w)$, and $F(p,w)$ is calculated as the sum of $A(p,w)$ and $B(p,w)$ meaning that the fixed price in the corresponding examples is the price of the first unit sold in the equivalent game theoretic scenario.

The primary operating cost burden to the refiner is the purchase price of crude oil. In this work crude oil prices are denoted by parameter $Cost(i)$ and are considered to be constant. Three crude oil stocks are available on the market: a light sweet, a medium, and a heavy sour variety. The prices for these crude oils are chosen from representative average monthly prices of benchmark crude oils reported by Natural Resources Canada, using Canadian Light Chicago, Canadian Light Sweet, Canadian Heavy Chicago, and Canadian Heavy Hardisty prices to generate three representative crude stock prices at 610.20, 577.30, and 535.04 dollars per cubic meter[53]. Yield values in the production planning model are calculated based on assays for three crude oil stocks produced by ExxonMobil: Hibernia, a light blend, Terra Nova, a medium crude, and Cold Lake, a heavy sour crude[54,55,56]. Yields are computed for max naphtha and max diesel CDU operating modes using data from Fu, Sanchez, and Mahalec[57].

The two scenarios presented in this work are referred to using the notation S1 and S2 for convenience. Associated with each of these scenarios are two additional variations. The first is a version of the game theoretic scenario excluding upper capacity limits on refinery units such that refiners are capable of processing unlimited amounts of material. This variant is intended to illustrate the relationship between unit capacity constraints and Nash equilibria, and is denoted by adding –G to the scenario name. The second variant solves the scenario problem under fixed prices for contrast with the game theoretic planning results. The fixed price variants are referred to by appending each model name with –F. Equation listings for each scenario are given in Table 1. All set and parameter data is included as supplementary material.

Table 1. Scenario nomenclature and corresponding equations.

Scenario	Equation list
S1	(11), (14) to (17), and (B1) to (B41)
S1-G	(11), (14) to (17), (B1) to (B41), excluding: (B4), (B13), (B20), and (B30)
S1-F	(11), (18), and (B1) to (B41)
S2	(11), (17), (19), (21) to (25), and (B1) to (B41)
S2-G	(11), (17), (19), (21) to (25), and (B1) to (B41), excluding: (B4), (B13), (B20), and (B30)
S2-F	(11), (19), (26), and (B1) to (B41)

Scenario 1 results

Results are characterized entirely by refiner production decisions interpreted through the potential function as profits. Market prices and individual refiner profits are implicitly defined in the potential function and do not appear directly in the model. The production volumes resulting in S1 are presented in Figure 3 and correspond to the variables $Tpd(p, n, w)$. Domestic and import production volumes satisfy domestic market demand while production excesses of gasoline products are exported in order to take advantage of higher global market prices. Refiners have similar optimal production plans which scale according to refinery capacity. In the case that all three refiners are identical in size, their production volumes will be symmetric. Import volumes scale inversely with refinery size; smaller refiners import more than larger ones.

Refiners do not have the capacity to satisfy domestic market diesel demand to the minimum constraint level and must import to do so. Each refiner produces some portfolio of diesel products up to its capacity, and then imports the balance volumes of each of these products in such a way that all refiners supply the same volume of each diesel product in order to satisfy domestic market demands. Larger refiners are able to produce more diesel in-house and import less. The cumulative amount produced by the refiners satisfies precisely the lower supply limit. Similarly, refiners produce identical amounts of diesel 1 without imports; the cumulative volume satisfies the minimum supply. This pattern is not observed for gasoline products, in which case refiners produce amounts varying with their capacity for both domestic and global markets. This result suggests that game theoretic planning drives refiners to compete on marginally more profitable products and to allocate production equally for less profitable products. We note that the observed results are strongly dependent on refinery production capacities; e.g., refineries configured for diesel would have a reversed pattern of production volumes.

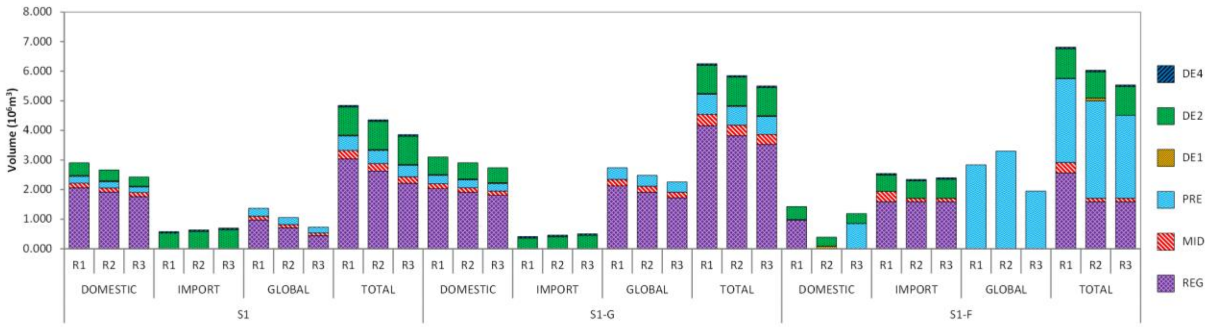

Figure 3. S1 production volume breakdown and totals by refiner and scenario variant.

The results of S1-G are presented in Figure 3 for comparison with S1. The total amount of diesel product is unchanged from S1 (minimum domestic supply is satisfied) but the amounts of gasoline product produced are increased. No product is supplied in the domestic market at an amount large enough to reach the upper constraint level; the solution in the domestic market is defined by Nash equilibrium prices, not domestic supply constraints. Import amounts in S1-G are unchanged from S1. This outcome suggests that with unlimited production capacity it is not profitable to produce any more diesel products than those required in the domestic market, and that the most cost effective means of obtaining those products is to produce a fraction and import the balance, where the decision is driven by prices, not capacity constraints.

Refiners export more gasoline product to the global market in S1-G than S1. Despite unconstrained capacities the refiners halt production at a price point representing a Nash equilibrium, contrasting with the different equilibrium obtained in S1 defined by capacity constraints. Refiners continue to plan production according to their sizes due to efficiency costs.

The production volumes obtained from the solution of S1-F are shown in Figure 3. Total domestic market supply satisfies the upper and lower demand constraints, but production volumes associated with each refiner do not follow a competitively optimal pattern. In S1-F prices are fixed and the objective is total refiner profit. The allocation of production is that which maximizes total market profit regardless of individual profits. This result is unobtainable barring a monopoly; an individual refiner will not yield profits because a competitor has a better marginal gain. This fixed price approach does not generate competitively optimal behaviour. Refiners only export premium gasoline as there is no consideration of market demand levels. We present this result to illustrate the driving forces at play in fixed priced models.

The results of S1, S1-G, and S1-F also serve to illustrate the role of flexibility in the refinery model. Although capacity to produce diesel is limited, the refinery models implemented in this work are flexible in the production of gasoline products. The refiners are capable of much larger throughput than in S1, as seen in S1-G and S1-F where total product is higher. They are also capable of changing product ratios, as seen by comparing S1 and S1-F. This flexibility allows refiners to reach a Nash equilibrium defined primarily by throughput constraints rather than individual product limitations.

The prices and profits associated with these three cases and for all scenarios are collected in Table 2 and Table 3. In S1-G refiners produce more than in S1; refiners R1 and R2 lose profits in S1-G while R3 gains. This illustrates the rationality of Nash equilibria: refiners will not be worse off in terms of their own profits if any others behave differently; they can make gains if competitors deviate from equilibrium strategies. In this case R1 lost profits in S1-G relative to S1, but could stand to gain if R2 or R3 made a non-equilibrium plan. The profits reported in S1-F by comparison are much higher, and are unrealistic in a game-theoretic sense.

Table 2. Profit values by scenario (10^6 CAD).

	R1	R2	R3
S1	97.82	52.81	4.24
S1-G	71.61	50.73	30.18
S1-F	1102.98	971.85	899.27
S2	47.96	0.72	121.11

S2-G	132.81	95.60	0.00
S2-F	1112.83	1014.83	646.16

The prices in scenarios S1 and S1-G illustrate the Cournot property: as refiners supply more of a product, its price drops. In S1-F the prices reported are the highest possible due to the assumption of fixed prices (under FIXED heading). The equivalent game theoretic prices (under EQUIV. heading) corresponding to the market supplies in S1-F are correspondingly lower and, in the case of the price of premium gasoline in the global market, substantially lower than those observed in the game theoretic version where consideration of market demand and price limits the total volume of premium gasoline supplied to a rational level.

Table 3. Scenario prices (CAD/m^3). S1-F and S2-F show fixed price values and the equivalent game theoretic Cournot prices corresponding to the production levels in those scenarios.

MARKET	PRODUCT	S1	S1-G	S1-F		S2	S2-G	S2-F	
				FIXED	EQUIV.			FIXED	EQUIV.
LOCAL	REG	912.18	912.18	1130.49	909.72	916.22	912.18	1134.53	909.72
	MID	977.00	958.01	1183.36	911.64	980.82	977.00	1187.18	911.64
	PRE	1028.12	971.91	1238.94	959.67	1032.02	1021.10	1242.84	959.67
	DE1	1089.39	1089.39	1238.58	1042.34	1092.15	1089.39	1241.34	1042.34
	DE2	1034.13	1034.13	1183.33	1036.14	1036.90	1034.13	1186.09	1036.14
	DE4	978.88	978.88	1128.07	980.89	981.64	978.88	1130.83	980.89
GLOBAL	REG	1003.27	924.58	1049.64	1049.64	1008.15	958.00	1049.64	1043.54
	MID	1017.59	938.91	1106.93	1106.93	1022.47	977.10	1106.93	1086.84
	PRE	1031.07	952.39	1160.86	-367.68	1035.96	995.07	1160.86	70.25
	DE1	1183.33	1183.33	1183.33	1183.33	1183.33	1183.33	1183.33	1183.33
	DE2	1128.07	1128.07	1128.07	1128.07	1128.07	1128.07	1128.07	1128.07
	DE4	1072.81	1072.81	1072.81	1072.81	1072.81	1072.81	1072.81	1072.81

Scenario 2 results

The purpose of this scenario is to examine conditions under which strategic planning can incorporate large scale decisions affecting the structure of the market. Results are presented for S2 in Figure 4 in which the high-cost refiner R3 continues to participate in the market, and are similar to the production plans in S1, but the high-cost refiner cannot import in this scenario and has limited production. The high-cost refiner distributes its market share in the same way as the low-cost refiners; it produces less than in S1, since its share is constrained.

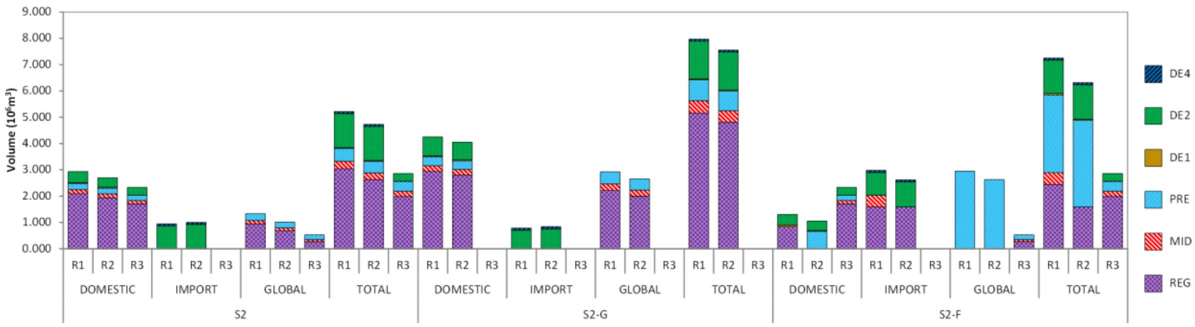

Figure 4. S2 production volume breakdown and totals by refiner and scenario variant.

An unconstrained version of this problem is presented as S2-G in which low-cost refiners have unconstrained capacity while the high-cost refiner (if active) is limited to producing the profile determined from the first stage calculation. In S2-G the high-cost refiner is shut down, as can be seen in Figure 4. In this scenario the low-cost refiners benefit more from the increased domestic market share obtained by shutting down the high-cost refiner than by having higher domestic market prices. The optimal decision in S2-G differs from that in S2; with unconstrained capacity the low-cost refiners are better off without the high-cost refiner, whereas in S2 the high-cost refiner’s contributions in the domestic market allow the low-cost refiners greater access to global markets. There are thus multiple factors, both refinery-specific and market-based, influencing the participation of the high-cost refiner.

A fixed price case is presented as S2-F. As in S2 the high-cost player remains active, but the production decisions made by the low-cost refiners are non-optimal, following the same patterns as S1-F, even though the high-cost refiner’s allocation is rational from a game theoretic standpoint.

In order to investigate the influence of domestic and global market demands on the high-cost refiner’s participation in scenario S2 and S2-F these two cases are solved over a grid of demand values and for six different values of the high-cost refiner price increase

$AHC(p,w)$. Demand scaling factors for all products are taken at 19 even intervals ranging from 6% to 25.3% of the demand values $D(p,w)$ included as supplementary data. The test values for the price increase $AHC(p,w)$ are taken at 10% intervals ranging from 5% up to 55% of the standard deviation of product price calculated from Statistics Canada data as described earlier in this work. These values are reported in the supplementary material available online.

Objective values are calculated for the six values of $AHC(p,w)$ and are used to define contours characterizing the inclusion region boundary. These boundary lines are visualized in Figure 5 as solid lines labeled with the associated price increase percentage. As the high-cost refiner's presence brings larger price increases, the minimum domestic and global demands at which it will be shut down decrease. This result is intuitive; as the benefit accrued by low-cost refiners increases, they become more tolerant of the high-cost refiner in smaller markets.

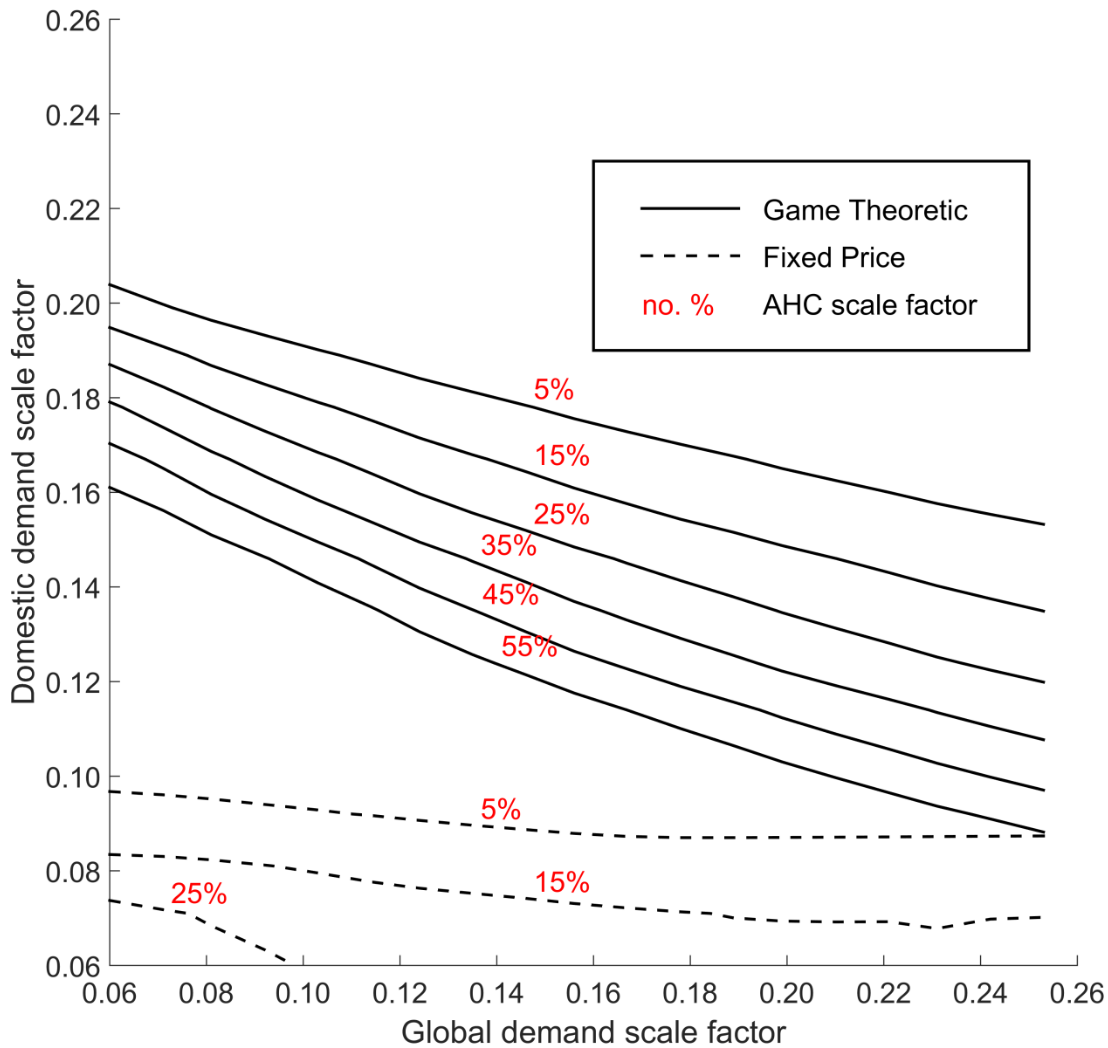

Figure 5. Inclusion region boundary characterization for S2; boundaries define paired domestic and global demand levels (as scaled values) below which low-cost refiners shut down the high-cost refiner with the indicated price increase.

Results are shown for the inclusion region associated with S2-F as dashed lines. Boundary lines are shown for values of $AHC(p, w)$ equal to 5%, 15%, and 25% of $A(p, w)$ for domestic market prices; no such boundary lines are found for values of 35% or greater, in which case the high cost refiner is allowed to remain active for all demand levels tested. Under fixed price analysis, the low-cost refiners will mistakenly allow the high-cost refiner to operate when its impacts will reduce their profits, instead of increasing them as they would predict.

Existence of multiple equilibria

Multiple Nash equilibria may exist in continuous games; our potential game formulation is nonconvex and we identify equilibria of the generalized type as globally optimal solutions. The existence of equilibria with equal objective value is of interest to determine whether normalization is required. To ascertain whether equal-valued equilibria exist in this problem, we constrain the potential function to the obtained optimal value Z^* and modify the objective function to drive refiners' individual decision variables to different values corresponding to the same optimum. Changes in $Dlv(t,p,n,w)$ or $Imp(p,n,w)$ yielding Z^* constitute equal valued alternative equilibria. We were unable to find alternative Nash equilibria using this approach, suggesting that there is a single globally optimal Nash equilibrium under the implemented formulation and data set. Since the equilibrium is unique, we are not concerned with normalization to characterize a best equilibrium solution[30].

Model solution statistics

All results are generated on a Dell Optiplex 9010 computer with Intel Core-i7-3770 CPU and a 3.40 GHz processor running the Windows 10 64-bit operating system. Models are solved using GAMS 24.7.1 with ANTIGONE 1.1[58] warm starting with CONOPT 3.17A or DICOPT. Solution data for each scenario are given in Table 4 including preprocessing results.

Solution results consist of the objective function value and optimality gap data reported by the solver, as well as the model statistics generated by GAMS. The model status indicates the optimality of the solution achieved; scenarios with a model status of 1 are solved to global optimality; a model status of 2 indicates local optimality. All scenarios presented are solved to global optimality with a relative gap of 1×10^{-9}. The processed model sizes generated by ANTIGONE are also reported; these indicate the types of equations, variables, and nonlinear terms detected by the solver and provide additional model information. The relative gap and CPU time reported by ANTIGONE are also included. The CPU times are essentially the same as those reported by GAMS, but exclude model generation time.

Table 4. Solution data.

	S1	S1-G	S1-F	S2	S2-G	S2-F
Model statistics (GAMS)						
Single equations	1833	1713	1753	2121	2009	1825
Single variables	1245	1245	1165	1376	1376	1140
Nonlinear entities	111	111	3	55	55	20
Solve summary (ANTIGONE)						
Objective value	871.7855	1182.4504	2974.0996	354.0593	864.1746	2127.6622
Model status	1	1	1	1	1	1
Resource usage	0.364	1.534	0.151	0.583	0.885	0.541
After pre-processing						
Variables	531	531	513	498	498	438
Continuous	531	531	513	497	497	437
Binary				1	1	1
Equations	1153	1153	1063	1172	1162	1004
Linear	1077	1077	1060	1132	1122	1000
Convex nonlinear						
Nonconvex nonlinear	76	76	3	40	40	4
Nonlinear terms	348	348	6	194	194	80
Bilinear/quadratic	345	345	3	192	192	78
Sigmoidal	3	3	3	2	2	2
Solve statistics						
Relative gap	1.00E-09	1.00E-09	1.00E-09	1.00E-09	1.00E-09	1.00E-09
Total time (CPU s)	0.35	1.43	0.15	0.56	0.81	0.44

Conclusions

We have presented a game theoretic strategic production planning framework based on a modified Cournot oligopoly formulated as a potential game which we use to solve strategic refinery production planning problems to Nash equilibrium solutions. Two scenarios have been presented illustrating competitive behaviour in production planning problems. The first scenario illustrates competitive behaviour in the game theoretic sense and contrasts those results with equivalent fixed price planning results. The second scenario extends the framework to include a decision to shut down a competitor by claiming its market share. This scenario illustrates how competitive behaviour manifests in problems involving market restructuring and that the inclusion of planning decisions influences the outcome. In both scenarios, the results of production planning in a game theoretic framework were contrasted with those obtained by solving the same problem under the assumption of fixed prices. Making production planning decisions in a competitive context is a non-obvious problem, particularly so when market restructuring decisions are involved, and using fixed price methods does not yield competitively optimal solutions in either case. The importance of competitive planning arises from the reality that most industries operate under competition. The proposed potential game framework demonstrates that refinery production planning benefits from game theoretic analysis.

Notation

Sets

BL	(bl) set of blenders
I	(i) set of process streams
IC	(i) set of crude oils streams entering refinery
M	(m) set of unit operating modes
N	(n) set of refineries
P	(p) set of products
Q	(q) set of quality properties
W	(w) set of markets
T	(t) set of time periods in which planning takes place
TK	(tk) set of tanks for intermediates
U	(u) set of process units

BL_{BLEND}	(i) all streams entering a blender
BL_{IN}	(bl, i) streams entering blender bl
BL_{OUT}	(bl, p) streams leaving a blender
$BL_{OUT,VOL}$	(bl, p, q) product p with volume-based properties q leaves blender bl
$BL_{OUT,WT}$	(bl, p, q) product p with weight-based properties q leaves blender bl
$BL_{OUT,NL}$	(bl, p, q) product p with nonlinear properties q leaves blender bl
$BLIP$	(bl, i, p) product of BL_{IN} and BL_{OUT}
DWN	(p', p) product p' may be mixed with product p for delivery to market
LCN	(n) refineries classified as low cost
HCN	(n) refineries classified as high cost
$NPrest$	(n, p) refineries with production limits on product p
P_g	(p) gasoline products
P_d	(p) diesel products
Q_g	(q) gasoline properties
Q_d	(q) diesel properties
Q_{VOL}	(q) volume-based quality properties
Q_{WT}	(q) weight-based quality properties
W_E	(w) global markets
W_L	(w) domestic markets
WLN	(n, w) refinery n is located in domestic market w
WLE	(w, w') refiners in domestic market w can sell to global market w'
WN	(n, w) refiner n can sell to market w
$WLCN$	(n, w) low cost refiner n is located in market w
$WHCN$	(n, w) high cost refiner n is located in market w
TK_{IN}	(tk, i) streams entering intermediate tank tk
TK_{OUT}	(tk, i) streams leaving intermediate tank tk
U_{IN}	(u, i) streams i entering unit u
U_{OUT}	(u, i) streams i leaving unit u

U_C	(u) subset of U for certain constraints
UM	(u, m) units u which can operate in a mode m
UM_C	(u, m) subset of UM for certain constraints
$UM_{OUT,C}$	(i, u, m) product of U_{OUT} and UM_C

Parameters

$A(p, w)$	Marginal value of first unit of product p in market w
$AHC(p, w)$	Additional marginal value associated with product p if high cost refineries are active in market w
$B(p, w)$	Marginal value of product p when market w supply is $D(p, w)$
$CI(p, w)$	Import cost of product p to domestic market w from elsewhere
$D(p, w)$	Expected market demand for product p in market w
$\underline{D}(p, w)$	Minimum demand for product p in domestic market w
$\overline{D}(p, w)$	Maximum demand for product p in domestic market w
$F(p, w)$	Fixed sale price for product p in market w
$Pr(p, w)$	Price of product p in market w
$HCNset$	High-cost refinery production level, as a fraction of market demand
$EC_H(n)$	Efficiency cost parameter H for n
$EC_K(n)$	Efficiency cost parameter K for n
$EC_P(n)$	Efficiency cost parameter P for n
$EC_A(n)$	Efficiency cost parameter A for n
$EC_B(n)$	Efficiency cost parameter B for n
$EC_C(n)$	Efficiency cost parameter C for n
Cap	Percentage rated capacity
$Cost(i)$	Cost of crude oil stream $i \in IC$
$MaxProd(u)$	Maximum production rate on unit u
$MinProd(u)$	Minimum production rate on unit u

$\overline{V}(tk)$	Maximum holding in intermediate tank tk
$\underline{V}(tk)$	Minimum holding in intermediate tank tk
$V_{ini}(tk)$	Initial holding in intermediate tank tk
$\overline{VP}(p)$	Maximum holding in product tank p
$\underline{VP}(p)$	Minimum holding in product tank p
$VP_{ini}(p)$	Initial holding in product tank p
$BlendMax(bl)$	Maximum blending rate for blender bl
$BlendMin(bl)$	Minimum blending rate for blender bl
$BLcost(bl)$	Cost of operating blender bl
$\tau(t)$	Duration of time period t
$OpCost(u,m)$	Operating cost of unit u in mode m
$ProdRest(n,p)$	Restriction on refinery n production level of product p
$qq(i,q)$	Quality property q of stream i
$\overline{Q}(q,p)$	Maximum quality specification of property q for product p
$\underline{Q}(q,p)$	Minimum quality specification of property q for product p
$\overline{R}(i,p)$	Maximum specification of stream i for product p
$\underline{R}(i,p)$	Minimum specification of stream i for product p
TC	Time scaling cost factor
$\overline{Tpd}(p,n,w)$	Total product variable upper bound
$Y(i,m,u)$	Yield of stream i from unit u operating in mode m
$X(i,m,i')$	CDU yield of stream i from feed of crude oil $i' \in IC$ operating in mode m

Continuous variables

Z	Objective function value
Ψ	Potential function term
Ω_n	Potential function term
$TEC(n)$	Total efficiency cost for n

$FA(t,u,n)$	Inlet feed to unit u in period t for refinery n
$FV(t,i,n)$	Volumetric flow of stream i in period t for refinery n
$FVM(m,t,u,n)$	Inlet feed to unit u in period t in mode m for refinery n
$FVM_{IN}(m,t,i,n)$	Inlet feed of stream i in period t in mode m
$FVM_{OUT}(i,m,t,u,n)$	Volumetric flow of stream i leaving unit u in mode m in period t
$VP(t,p,n)$	Product tank inventory of p in period t
$V(tk,t,n)$	Intermediate tank inventory tk in period t
$VB(i,p,t,n)$	Volume of intermediate i used to produce product p in period t
$VBlend(t,p,n)$	Volume of product p blended in period t
$VBlendT(t,bl,n)$	Total volume blended by blender bl in period t
$Prc(t,p,n)$	Volume of product p produced in period t by refiner n
$Dlv(t,p,n,w)$	Volume of product p for delivery to market w produced in period t
$Imp(p,n,w)$	Volume of product p imported by refiner n in domestic market w
$CrudeOilCost(n)$	Total cost of all crude oil purchased by refinery n
$UnitOpCost(n)$	Total unit operating cost in refinery n
$BlendOpCost(n)$	Total blending cost in refinery n
$ImpCost(n)$	Cost of imports for refiner n
$TimeCost(n)$	Cost of production timing for refinery n
$TotCost(n)$	Total cost for refiner n excluding upgrades
$Tpd(p,n,w)$	Total product p leaving refinery n for sale to market w
$TP(p,n,w)$	Linearization variable for $TotProd(p,n,w)$
$TP1(p,n,w)$	Linearization variable for $TotProd(p,n,w)$

Binary variables

$Y_{HCN}(w)$	Decision variable dictating whether high-cost refiners remains in a market w

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Acknowledgements

This work has been funded by the Natural Sciences and Engineering Research Council of Canada, the Government of Ontario, and the McMaster Advanced Control Consortium whom we thank for their support. We would also like to extend our gratitude to our reviewers whose commentary is appreciated.

Chapter 3

A dynamic game theoretic framework for process plant competitive upgrade and production planning

The content of this chapter is a revision of the manuscript text submitted for publication under the following title:

Tominac P, Mahalec V. A dynamic game theoretic framework for process plant competitive upgrade and production planning.

A dynamic game theoretic framework for process plant competitive upgrade and production planning

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Topical Heading – Process Systems Engineering

Keywords – production planning; strategic planning; dynamic game; potential game; Nash equilibrium

Abstract

A dynamic potential game theoretic production planning framework is presented in which production plants are treated as individual competing entities and competition occurs dynamically over a discrete finite time horizon. A modified Cournot oligopoly with sticky prices provides the basis for dynamic game theoretic competition in a multi-market nonlinear and nonconvex production planning model wherein market price adapts to a value that clears cumulative market supply. The framework is used to investigate a petrochemical refining scenario in which a single inefficient refiner faces elimination by its competitors; we demonstrate that there exist conditions under which the threatened refiner may upgrade itself in order to become competitive and escape the threat, or alternatively in which the threat of elimination will never be carried out and the refiner is effectively safe in the given market configuration. Globally optimal dynamic Nash equilibrium production trajectories are presented for each case.

Introduction

We develop a dynamic game theoretic framework for competitive production planning and re-examine the static game theoretic competitor elimination problem introduced by Tominac and Mahalec[1]. In this problem a plant can be forced to shut down through the actions of its competitors and its market share be claimed by them at the expense of decreased market prices. Extension of optimal game theoretic production planning to the dynamic domain facilitates the analysis of decision-making processes in time-varying competitive markets. In particular, we investigate the elimination problem in a petrochemical refining setting from the perspective of the threatened refiner and the decisions it faces as it attempts to prolong its survival. Game theory provides a means of analyzing the complex interactions between competing and cooperating entities. A number of authors have used game theoretic principals to examine cooperative and competitive interactions in chemical engineering supply chain literature. Gjerdrum, Shah, and Papageorgiou examine fair profit allocation among supply chain participants using transfer prices and Nash bargaining equilibria[2,3]. Bai, Ouyang, and Pang incorporate agricultural decisions into the biorefinery supply chain using a Stackelberg framework to analyze business interactions between biofuels and food crop producers[4]. Yue and You propose a mixed integer nonlinear programming (MINLP) Stackelberg framework for the optimal design of multi-echelon supply chain systems[5]. Zamarripa *et al* solve cooperative and competitive supply chain planning problems using a framework in which Nash equilibria are identified through the assembly of a payoff matrix[7,8,15]. Game theory has seen use in engineering design problems as well: Bard, Plummer, and Sourie use a Stackelberg game framework to determine optimal biofuel tax credit policies[9]. Pierru uses Aumann-Shapley cost sharing to allocate refinery carbon dioxide emissions over finished products[10]. Chew *et al* present a game theoretic model for water integration in industrial parks[11]. Recent reviews of supply chain optimization literature by Papageorgiou[12] and Sahebi, Nickel, and Ashayeri[13] identify few papers taking advantage of game theoretic principles, and of the research identified in those reviews and here, none make use of dynamic game theoretic models. The model used by Tominac and Mahalec[1], for example, is a multiperiod planning problem formulated under a static Cournot game objective, precluding the ability of competitors to react to temporally changing market conditions.

Dynamic game theoretic problems are problems of optimal control with solutions that are Nash equilibria, and have seen extensive research and use in the area of process control[14]. Due to the breadth of the field of dynamic (or differential) game theory, we restrict ourselves to a review of theory regarding Cournot oligopoly models, which emit exact potential functions. Dynamic Cournot oligopoly models owe their present day form to developments made by Simaan and Takayama, whose pricing model is based on an adjustment function relating price at some instant to all historical prices[15]. Fershtman and Kamien presented a variation on this model in which the rate of price adjustment is allowed to take an arbitrary value; their model is referred to as the Cournot oligopoly with sticky prices[16]. In their subsequent paper, equilibrium price trajectories are analyzed on finite time horizons, and are shown to exhibit asymptotic steady state (or turnpike) properties in which the equilibrium trajectory approaches the infinite horizon steady state price trajectory from the outset of the time horizon, but deviates as the game terminates[17]. Fershtman and Kamien examine primarily closed loop equilibrium trajectories representative of subgame perfect Nash equilibria; Cellini and Lambertini examine open loop equilibrium steady state properties[18], and steady state properties of both open and closed loop strategies in oligopoly games with product differentiation[19]. Wiszniewska-Matyszkiel, Bodnar, and Mirota extend the analysis of the dynamic Cournot oligopoly to off-steady states[20]. These dynamic oligopoly models assume that competitors' individual actions result in significant impact on the system. Wiszniewska-Matyszkiel presents dynamic game models, in both discrete[21] and continuous[22] time domains, involving competitor continua of sizes large enough to render an individual's influence negligible. Such frameworks are important in the analysis of renewable resource usage. Zazo *et al* analyze a number of constrained dynamic game models as potential games[23].

Taking advantage of the potential game emitted by the dynamic Cournot oligopoly structure, we propose a dynamic closed-loop game theoretic process plant production planning framework based on a discrete time potential game formulation. To the best of our knowledge, such a framework has not been proposed within the body of literature of engineering supply chain planning. The closed loop dynamic game formulation allows competitors to respond to each other temporally, reacting to competing players' decisions at prior time points[16]. This dynamic framework synergizes with multiperiod strategic and tactical production planning models; planning decisions at each period are also interpreted as competitive best responses to decisions from prior periods. Competitor decision variables in this framework are infinitely valued, assuring from a theoretical standpoint that at least one Nash equilibrium strategy exists[24,25,26]. A Nash equilibrium trajectory to a dynamic potential game is defined as the maximum of the dynamic potential function; thus the proposed framework guarantees an equilibrium solution exists and is obtainable using numerical optimization[27]. The properties of this dynamic potential game framework are illustrated using a set of competing petrochemical refiners with a focus on the interpretation of the competitor elimination scenario. The novel elements presented in this work are:

- A dynamic potential game theoretic framework for production planning problems yielding Nash equilibrium trajectories;
- The derivation of the dynamic potential function corresponding to the modified Cournot oligopoly model of Tominac and Mahalec[1] and the demonstration of the validity of this potential function;
- A case study in which a refiner threatened with closure predicts competitor hostility throughout phases of vulnerable growth and expansion in order to determine whether the threat of closure is legitimate and if so, whether closure can be prevented through upgrading of its facilities;
- If facilities are to be upgraded, the proposed framework will indicate the time available in which those upgrades must be completed before a closure would occur.

Background

Dynamic Nash equilibrium

The dynamic Nash equilibrium is defined in the same way as the static Nash equilibrium: for a set of players R playing a competitive game G over N discrete periods of time and having available strategies $\xi_{nr} \in \Xi_{nr}$ (where ξ and Ξ are used to denote unique strategies and strategy sets, respectively) with objective functions $J_r(\xi_{rn}, \xi_{-rn})$ of both each player's own and opposing players' strategies (denoted $-r$), then the set of Nash equilibrium strategy trajectories (denoted as player-indexed vectors $\bar{\xi}_r = [\xi_{r1}, \dots, \xi_{rN}]$) to the game, $G\{\bar{\xi}_1^*, \dots, \bar{\xi}_R^*\}$ have the property in Eq. (1)[17].

$$J_r(\bar{\xi}_r, \bar{\xi}_{-r}^*) \leq J_r(\bar{\xi}_r^*, \bar{\xi}_{-r}^*) \quad \forall r \in R, \bar{\xi}_r \in \bar{\Xi}_r \tag{1}$$

There exist several important distinctions between dynamic and static Nash equilibrium, and we restrict ourselves to a discussion of these as they relate to the Cournot oligopoly with sticky prices. The dynamic closed-loop Nash equilibrium trajectory steady state value in games of infinite duration is not equivalent to the corresponding static game Nash equilibrium value[16]. Closed-loop dynamic games of finite duration exhibit a turnpike property prior to termination, but will exhibit an intermediate steady state value at some point following the beginning of a sufficiently long time horizon and before its end; this intermediate steady state is not equivalent to the corresponding static game Nash equilibrium[17]. As the number of firms increases to infinity, the steady state values of these games approach the static Nash equilibrium values, representing a dynamic perfectly competitive environment[18]. Games of finite duration terminate at the static game Nash equilibrium; players become perfectly competitive when there is no future time in which to compete[18,28]. While the equilibrium steady state value is unique, the strategy trajectories which yield this value are generally not unique[17].

Problem statement

We examine a dynamic version of the game theoretic refinery production planning problem presented by Tominac and Mahalec[1], in which multiple refineries are considered in a framework in which each refinery is a private firm seeking to maximize its profits. Refiners are rational in the game theoretic sense and are competitive. All refiners are capable of producing identical products and have access to the same set of crude feedstocks, but refineries vary in size and capacity. Geographic regions of sale are identified as markets in which each product has a nominal demand level and a price determined by a modified linear Cournot model[1]. Markets containing refineries are identified as domestic markets and receive all products from those refiners, who are in turn obligated to supply domestic markets with product levels within some predefined socially acceptable window, and thus also a predefined price window. The potential for domestic market supply infeasibility (in which refiners cannot produce the minimum domestic market supply of all products) is avoided by allowing domestic market refiners to import finished product at a fixed cost from other refiner-owned assets. Markets lacking any local refiner presence are defined as global markets, and are connected to domestic markets by pipelines. All refiners in connected domestic markets may export to global markets without any restrictions on total market supply, or thus price. In all cases, markets are assumed to clear at the Cournot price corresponding to total market supply in a given period of time; i.e., the market supply is sold entirely without residual demand. The dynamic Cournot oligopoly pricing model is such that price in a given period of time is a function of the initial product price and all prior supply as well as the product supplied within the current period. Under these conditions refiners are faced with the following problem:

- To determine the product amounts which must be provided and the delivery time of those products at the domestic market level in order to satisfy supply obligations together with other competing domestic market refiners, including whether and when any product imports are required (a strategic decision).

- To determine the timing and amounts of each product for export to available global markets in which a refiner is competing with both local domestic refiners and refiners in other domestic markets (a strategic decision).
- To determine the required amounts of available crude stocks and purchase timing, and the blend plan to be used to produce products in the means yielding the greatest profit (production planning decisions).

We focus our attention on the scenario in which a high-cost refiner may be eliminated from a domestic market and examine this problem in the dynamic game theoretic production planning framework. This scenario is based on the notion that market prices are higher while the high-cost refiner is active, allowing all refiners to benefit. The remaining low-cost refiners may unanimously elect to shut down the high-cost refiner, at which point they gain the high-cost refiner's market share. The problem is interpreted as a trade-off between higher prices and a larger market from the low-cost refiners' perspective; Tominac and Mahalec[1] discuss the ramifications of this interpretation of the problem in depth, which is based on observations of refining assets and pipeline networks in western Canada[29]. From a modelling perspective, this scenario represents a case in which price parameters are larger when the high-cost refiner is active; since Cournot prices are a result of production volumes, it is possible to observe lower market prices in the presence of the high-cost refiner than without.

We consider instead the high-cost refiner's perspective on the problem, which is inherently one of survival. In a dynamic game theoretic framework, the high-cost refiner may be shut down in any period. From the perspective of the low-cost refiners, this will occur when and if it maximizes their own profits to do so. While the high cost refiner would rather avoid being shut down, its available strategies are limited in order to investigate a scenario in which it either transitions into a low-cost refiner or is shut down. The high-cost refiner is not allowed to lower its production levels in order to prevent closure; its production levels are fixed to the values which result in the equivalent dynamic scenario in which no elimination threat exists. We consider the transition of a high-cost refiner to a low-cost refiner through an implicit upgrade argument: as the high-cost refiner implements upgrades, the higher prices associated with its presence will diminish. If the high-cost refiner can implement upgrades which reduce its associated higher prices to zero (i.e., market prices become the unmodified Counrot prices, or equivalently the high-cost refiner induces a zero-level increase in Cournot price) then it is considered to have become competitive with the low-cost refiners (it has in effect become a low-cost refiner) and cannot be eliminated by them any longer. If the high-cost refiner is unable to complete upgrades in this manner, then it is shut down. The assumptions made regarding the high-cost refiner's problem are as follows:

- A high-cost refiner may become a low-cost refiner through upgrading.
- High-cost refiners are unable to import product from other assets; at some time after transitioning to low-cost refiner status and not threatened with elimination, a high-cost refiner will establish an import source.
- Low-cost refiners cannot be eliminated from markets.
- The elimination decision is made unanimously by the set of low-cost refiners; this is in part to avoid gaming effects on that decision[30].
- A high-cost refiner is vulnerable until it becomes a low-cost refiner; i.e., it is vulnerable until its upgrades are complete.
- The process of upgrading is represented entirely by changes to the high-cost refiner's price increase parameter; an upgrade increasing the high-cost refiner's efficiency lowers the value of this parameter, and the high-cost refiner has completed transition to a low-cost refiner if the value of this parameter becomes zero
- Multiple upgrade stages may be examined by sequential solution of the dynamic model using changing values of the price increase and setting the initial price to the current period price in which an upgrade is completed.
- If the high-cost refiner is not eliminated within the finite horizon, it is never eliminated. The static game Nash equilibrium includes the high-cost refiner.

Under these assumptions, the high-cost refiner's problem is to determine if an upgrade plan is available which results in its continued participation in the market as either a high-cost or low-cost refiner. This is a strategic decision on the high-cost refiner's part, and is a function of the competing low-cost refiners' profits, not its own. We examine a scenario with three refiners, two of which are low-cost and one high-cost, all operating out of a single domestic market which has export access to one global market. All refiners in this scenario produce two products: regular gasoline and type 2 consumer diesel. This scenario is laid out in Figure 1.

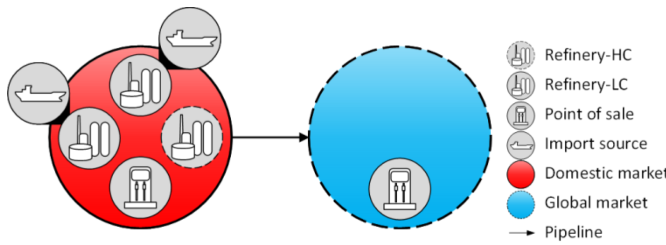

Figure 1. Market scenario layout

Models and Formulation

Deriving a dynamic potential function

The basis of the discrete dynamic Cournot oligopoly model is that the price of a homogenous product in a period of time is equal to a weighted combination of its price in the previous period of time and the corresponding static Cournot oligopoly price in the current period. Price is represented as π_{npw}, static Cournot price is $\tilde{\pi}_{npw}$, and the weighting factor is a scalar s with value between zero and one inclusive as in Eq. (2).

$$\pi_{npw} = (1 - s)\tilde{\pi}_{npw} + s\pi_{(n-1)pw} \tag{2}$$

The static price term represents a linear Cournot pricing model. We use the demand-based model of Tominac and Mahalec[1] in Eq. (3), where A_{pw} and B_{pw} represent the initial price of a product p in market w and the price corresponding to a supply level of D_{pw}, respectively, and where q_{nrpw} is the amount of product p supplied by refiner r to market w in period n.

$$\tilde{\pi}_{npw} = A_{pw} + B_{pw} - \frac{A_{pw}}{D_{pw}} \sum_{r} q_{nrpw} \tag{3}$$

Price π_{npw} is shown in its general recursive expression in Eq. (4), and with $\tilde{\pi}_{npw}$ expanded in Eq. (5). The dependence of price on the initial price π_{0pw} is apparent in the recursive expression; it is assumed to be $\pi_{0pw} = A_{pw} + B_{pw}$.

$$\pi_{npw} = \sum_{n'=1}^{n} \left[s^{n-n'}(1-s)\tilde{\pi}_{n'pw} \right] + s^n \pi_{0pw} \tag{4}$$

$$\pi_{npw} = \sum_{n'=1}^{n} \left[s^{n-n'}(1-s)\left(A_{pw} + B_{pw} - \frac{A_{pw}}{D_{pw}} \sum_{r} q_{n'rpw} \right) \right] + s^n \pi_{0pw} \tag{5}$$

We assume a general refiner cost function of the form $C_{nr}(q_{nrpw})$ such that refiner profit J_r may be defined as in Eq. (6), and expressed as in Eq. (7). The parameter e^{-in} is a discounting factor decreasing the weight of future profits using the parameter i to determine the amount.

$$J_r = \sum_{n} e^{-in} \left(\sum_{p} \sum_{w} \left[\pi_{npw} q_{nrpw} \right] - C_{nr}(q_{nrpw}) \right) \tag{6}$$

$$J_r = \sum_{n} e^{-in} \left(\sum_{p} \sum_{w} \left[\left(\sum_{n'=1}^{n} \left[s^{n-n'}(1-s)\tilde{\pi}_{n'pw} \right] + s^{n} \pi_{0pw} \right) q_{nrpw} \right] - C_{nr}(q_{nrpw}) \right) \tag{7}$$

The potential function is defined according to the methods defined by Monderer and Shapley[27] and Slade[31,32] and is defined by Eq. (8). We separate the usual Ω into $\Omega_r^S + \Omega_r^C$ to isolate costs from the main part of the term. The definitions of Ψ, Ω_r^S, and Ω_r^C are provided in Eqs. (9), (10), and (11).

$$Z = \Psi + \sum_{r} (\Omega_r^S + \Omega_r^C) \tag{8}$$

$$\Psi = \sum_{p}\sum_{w}\sum_{n}\sum_{n'=1}^{n}\left[e^{-in}s^{n-n'}(1 - s)\left(-\frac{A_{pw}}{D_{pw}}\right)\left(\sum_{r}\sum_{\substack{r' \\ r'\neq r}} q_{nrpw}q_{n'r'pw}\right)\right] \tag{9}$$

$$\Omega_r^S = \sum_{p}\sum_{w}\sum_{n}\left(\sum_{n'=1}^{n}\left[e^{-in}s^{n-n'}(1-s)\left(\left(A_{pw}+B_{pw}\right)q_{nrpw} - \frac{A_{pw}}{D_{pw}}q_{nrpw}q_{n'rpw}\right)\right] + \mathrm{s}^{n}\pi_{0pw}q_{nrpw}\right) \tag{10}$$

$$\Omega_r^C = -\sum_{n} e^{-in}C_{nr}\left(q_{nrpw}\right) \tag{11}$$

Verification of the dynamic potential function

The dynamic potential function defined by Eqs. (8), (9), (10), and (11) is verified as a potential function for the dynamic game by demonstrating that it simultaneously maximizes the profits of all competing refiners $r \in R$[31,32]. The condition that must be satisfied by the potential function is defined in Eq. (12). The derivative of profit J_r is defined in Eq. (13).

$$\frac{\partial \mathrm{Z}}{\partial q_{nrpw}} = \frac{\partial J_r}{\partial q_{nrpw}} \quad \forall n \in N, r \in R, p \in P, w \in W \tag{12}$$

$$\frac{\partial J_r}{\partial q_{nrpw}} = e^{-in}\left\{\sum_{n'=1}^{n}\left[s^{n-n'}(1-s)\left(A_{pw}+B_{pw}-\frac{A_{pw}}{D_{pw}}\sum_{r'}q_{n'r'pw}\right)\right] + (1-s)\left(-\frac{A_{pw}}{D_{pw}}q_{nrpw}\right) + \mathrm{s}^{n}\pi_{0pw} - \frac{\partial C_{nr}\left(q_{nrpw}\right)}{\partial q_{nrpw}}\right\} \tag{13}$$

For the dynamic potential function derivatives for each component are determined to be as in Eqs. (14), (15), and (16).

$$\frac{\partial \Psi}{\partial q_{nrpw}} = e^{-in} \sum_{n'=1}^{n} \left[s^{n-n'}(1-s) \left(-\frac{A_{pw}}{D_{pw}} \sum_{\substack{r' \\ r' \neq r}} q_{n'r'pw} \right) \right] \tag{14}$$

$$\begin{aligned} \frac{\partial \Omega_r^S}{\partial q_{nrpw}} = e^{-in} \Bigg(& \sum_{n'=1}^{n} \left[s^{n-n'}(1-s) \left(A_{pw} + B_{pw} - \frac{A_{pw}}{D_{pw}} q_{n'rpw} \right) \right] \\ & + (1-s)\left(-\frac{A_{pw}}{D_{pw}} q_{nrpw} \right) + \mathrm{s^n} \pi_{0pw} \Bigg) \end{aligned} \tag{15}$$

$$\frac{\partial \Omega_r^S}{\partial q_{nrpw}} = -\frac{\partial C_{nr}(q_{nrpw})}{\partial q_{nrpw}} \tag{16}$$

Addition of the dynamic potential function derivatives yields the profit function derivative in Eq. (13), and satisfies the condition in Eq. (12); therefore the discretized dynamic potential function is correct.

A potential function for the elimination game

The competitor elimination scenario uses the definition of $\tilde{\pi}_{npw}$ in Eq. (17) which incorporates the parameter A_{pw}^{HC} and binary variable y_{nw} in order to increase market prices when the high-cost refiner is active. The high cost refiner is defined to be active when the value of y_{nw} is one, and inactive when it is zero. The variable y_{nw} is constrained such that it is fixed to zero following the first period in which it becomes zero; thus the high-cost refiner is never allowed back into operation once it has been eliminated. Since the high-cost refiner is assumed to be active at the zeroth period, initial price is taken as π_{0pw}=$A_{pw} + B_{pw} + A_{pw}^{HC}$.

$$\tilde{\pi}_{npw} = A_{pw} + B_{pw} + A_{pw}^{HC} \sum_{\substack{w' \\ w' \in RW_D}} (y_{nw'}) - \frac{A_{pw}}{D_{pw}} \sum_{r} q_{nrpw} \tag{17}$$

The potential function for the competitor elimination game taking this change into account is defined by Eqs. (18), (19), (20), and (21). The demonstration that this potential function is also correct follows the same logic as the general case.

$$Z = \Psi + \sum_{\substack{r \\ r \in R_L}} (\Omega_r^S + \Omega_r^C) \tag{18}$$

$$\Psi = \sum_p \sum_w \sum_n \sum_{n'=1}^{n} \left[e^{-in} s^{n-n'} (1 - s) \left(-\frac{A_{pw}}{D_{pw}} \right) \left(\sum_r \sum_{\substack{r' \\ r' \neq r}} q_{nrpw} q_{n'r'pw} \right) \right] \tag{19}$$

$$\Omega_r^S = \sum_p \sum_w \sum_n \left(\sum_{n'=1}^{n} \left[e^{-in} s^{n-n'} (1 - s) \left(\left(A_{pw} + B_{pw} + A_{pw}^{HC} \sum_{\substack{w' \\ w' \in RW_D}} y_{nw'} \right) q_{nrpw} - \frac{A_{pw}}{D_{pw}} q_{nrpw} q_{n'rpw} \right) \right] + s^n \pi_{0pw} q_{nrpw} \right) \tag{20}$$

$$\Omega_r^C = -\sum_n e^{-in} C_{nr}(q_{nrpw}) \tag{21}$$

Introduction of the binary variable y_{nw} yields a bilinear term of the form $y_{nw} q_{nrpw}$ for which there exists an exact linearization[33]. We replace the binary term identified in Eq. (22) with the equivalent linearization, and add the constraints in Eqs. (23), (24), and (25) to the model to preserve the original functionality. The parameter $\bar{q}_{nrpw}$ is used to represent the maximum value of q_{nrpw}.

$$\sum_{\substack{w' \\ w' \in RW_D}} y_{nw'} q_{nrpw} \equiv \sum_{\substack{w' \\ w' \in RW_D}} \gamma_{nrpww'} \tag{22}$$

$$\gamma_{nrpww'} + \gamma^{*}_{nrpww'} = q_{nrpw} \quad \forall n \in N, r \in R, p \in P, (w, w') \in W \tag{23}$$

$$\gamma_{nrpww'} \leq y_{nw'} \bar{q}_{nrpw} \quad \forall n \in N, r \in R, p \in P, (w, w') \in W \tag{24}$$

$$\gamma^{*}_{nrpww'} \leq (1 - y_{nw'}) \bar{q}_{nrpw} \quad \forall n \in N, r \in R, p \in P, (w, w') \in W \tag{25}$$

The potential function in the elimination scenario takes into account the high-cost refiner only in the Ψ term; it is ignored in the addition of both Ω_r terms. This formulation presents an interesting interpretation of the scenario: the high-cost refiner examines when the low-cost refiners prefer to have it active in the petroleum market and when or if they would prefer to have it inactive. Its own profits do not factor into the decision, which might be anticipated given the assumed game theoretic definitions of rationality.

Solution Procedure

Since the high-cost refiner's profits are not factored into the elimination scenario potential function defined by Eqs. (18) to (21) this potential function cannot be used to determine the high-cost refiner's production plan. We thus use a two stage solution process with the rationale that the high-cost refiner selects as its production plan the game theoretically optimal scheme as though it is not facing elimination (i.e., its competitive production scheme from the planning problem in which there is no elimination option) and then holds to that production scheme in the elimination scenario. Mathematically, this corresponds to first solving the potential function corresponding to Eqs. (8) to (11) to calculate the high-cost refiner's Nash equilibrium profile, and then holding these values constant for only the high-cost refiner in the solution of the elimination scenario potential function defined by Eqs. (18) to (21). Thus the low-cost refiners may alter their own production schemes in response to the high cost refiner's fixed choice, but *ceteris paribus* it is anticipated that any changes in behaviour from the first stage to the second are attributable only to whether and when the high-cost refiner is eliminated.

Refinery Models

The refinery production planning model implemented in this work is that of Tominac and Mahalec[1] defined in the supplementary material. We make one change to the model in that we do not limit the high-cost refiner's production throughput to a fixed value in the elimination scenario. We use only the regular gasoline and type 2 diesel products in this model for two reasons: Tominac and Mahalec[1] observed mainly small values of the other four products whose production they modelled; and to reduce the overall model size and complexity in the dynamic framework.

Results and Discussion

On avoiding elimination

Results are interpreted in the context of the high-cost refiner and its objective of remaining an active market participant. To reiterate, we assume that the high-cost refiner is not willing to modify its current market share in order to disincentivize its own elimination. Instead, the high-cost refiner's survival is dependent on the amount by which it maintains higher prices than would exist following its elimination and whether those prices are high enough to incentivize the low-cost refiners to desire the high-cost refiner's continued participation in the market; in the alternative, the high-cost refiner is shut down and its market share seized by the low-cost refiners. As the high-cost refiner attempts to upgrade its processes and reduce its costs, the amount by which it requires high prices is reduced, and the market prices fall to match. The interaction between prices and competitors is complex in real cases; we attempt to isolate price changes in the facet of upgrades through the notion of the high-cost refiner's higher price requirement.

Using the standard data set to construct a dynamic game theoretic refinery production planning model over an eight month time horizon discretized into ten periods, the Nash equilibrium product price trajectories in Figure 2 are obtained. The standard data set of Tominac and Mahalec[1] assumes a value of A_{pw}^{HC} equal to 5% of A_{pw} in domestic markets, and zero elsewhere. In this example the high-cost refiner is eliminated in the fourth period; the price trajectories corresponding to the case in which no elimination occurs are included in the figure for comparison. Domestic market product prices actually rise following elimination due to an overall decrease in market supply. Figure 3 presents refiner and market supply volume trajectories for domestic regular gasoline sales. Market supply opens at its upper limit and remains there for three periods; following elimination of the high-cost refiner market supply drops below its upper limit, causing prices to rise in subsequent periods, though both low-cost refiners are able to increase their production rates. Since the high-cost refiner is able to export, this change influences global market prices as well. Thus while prices are high low-cost refiners prefer the additional marginal profit obtained by allowing the high-cost refiner to persist. As prices drop, they prefer the market share that is made available by eliminating the high-cost competitor. In this example, the high-cost refiner thus has three periods in which to implement upgrades that would enable it to compete with the low-cost refiners and prevent its elimination in period four. As this corresponds to less than three months of time, it is unlikely the high-cost refiner would be able to prevent this outcome.

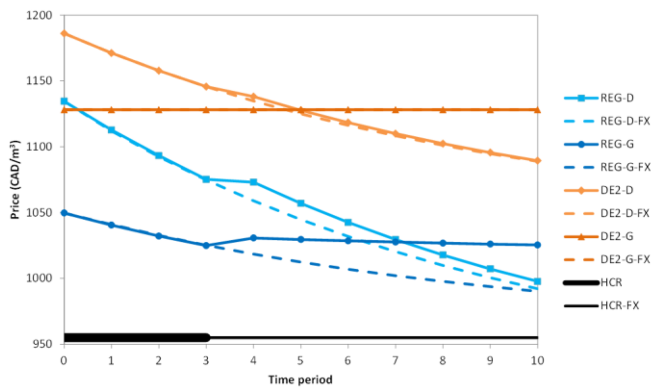

Figure 2. Nash equilibrium price trajectories with normal high-cost refiner A_{pw}^{HC} value. REG and DE2 are used to denote regular gasoline and diesel prices; D and G are used to denote domestic and global markets. The line labelled HCR indicates whether and when the high-cost refiner has been eliminated. The dashed FX lines indicate price trajectories in the case no elimination occurs.

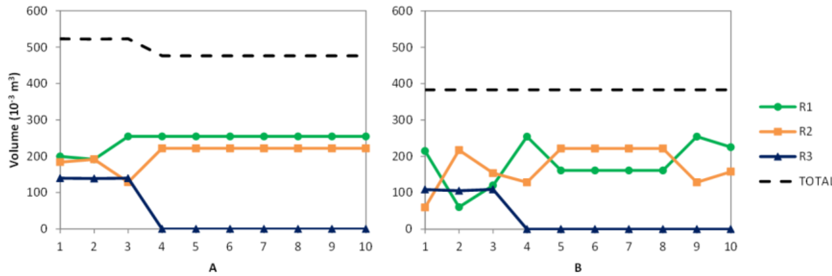

Figure 3. Domestic market supply of regular gasoline including (A) and excluding (B) imports of finished product.

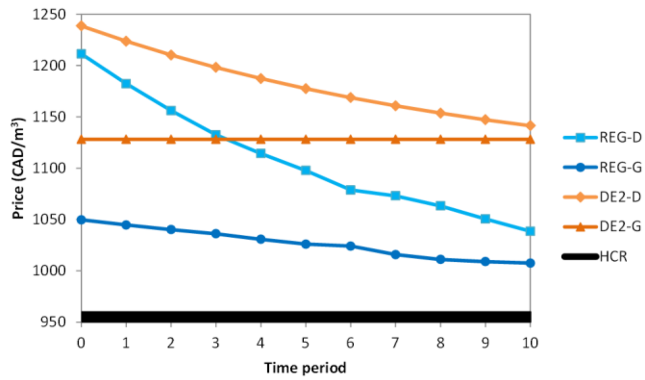

Figure 4. Nash equilibrium price trajectories with large high-cost refiner A_{pw}^{HC} value. REG and DE2 are used to denote regular gasoline and diesel prices; D and G are used to denote domestic and global markets. The line labelled HCR indicates whether and when the high-cost refiner has been eliminated.

A second scenario is examined in which the high-cost refiner is extremely inefficient and requires market prices to be in significant excess of their competitive values. This scenario is represented using A_{pw}^{HC} equal to A_{pw} in domestic markets. The Nash equilibrium price trajectories are presented in Figure 4. In this scenario the high-cost refiner is never eliminated; even as prices decrease from their initial values. Since the high-cost refiner is active in the terminal period, we interpret this scenario to represent a market in which the high-cost refiner is never eliminated; the low-cost refiners desire it to remain active indefinitely. Despite its inefficiencies there is thus no external incentive to the high-cost refiner to upgrade its facilities. Optimization of its own internal costs independent of the greater market context in which it operates may suggest capacity expansion as a viable and profitable endeavor for the operators of this refinery;

nevertheless they run the risk of becoming too competitive. If the high-cost refiner in this scenario upgrades its facilities (following which we assume that market prices decrease via A_{pw}^{HC} adapt to the new market context) it may find itself in the initial scenario presented wherein its benefit to its competitors has decreased below the value of its market share in a market that no longer includes itself, and is subsequently eliminated by its competitors. The high-cost refiner thus must be able to upgrade its facilities to the extent that it becomes a competitive low-cost refiner, or upgrade only so far as to improve its operating costs without risking elimination. Either of these options entails careful evaluation of the greater market context during and following its upgrade procedures. Of course, in this case the high-cost refiner may elect to do nothing at all and anticipate its continued market participation; any risk of elimination emerges only when and if it takes action to improve its efficiency.

Influence of time discretization

Dynamic games are differential optimal control problems; discretization of the differential form of the dynamic game facilitates the use of NLP and MINLP solvers in the solution of games of finite length. The formulation of the dynamic Cournot game is such that the terminal period of the dynamic game is interpreted as an equivalent static Cournot game; i.e., the terminal point of the finite dynamic Cournot game Nash equilibrium trajectory is the Nash equilibrium of a static Cournot game. A fixed length time horizon of eight months is considered in this work: thus changing the number of discretization points (or time periods) in the dynamic model changes the duration of each period, and by the interpretation of discrete dynamic games, the static game solution corresponding to the terminal period. The terminal point of a Nash equilibrium trajectory is not expected to remain constant in alternative time discretizations; it is of interest whether the quantitative pattern observed over the horizon is altered by the selection of a different time discretization as well.

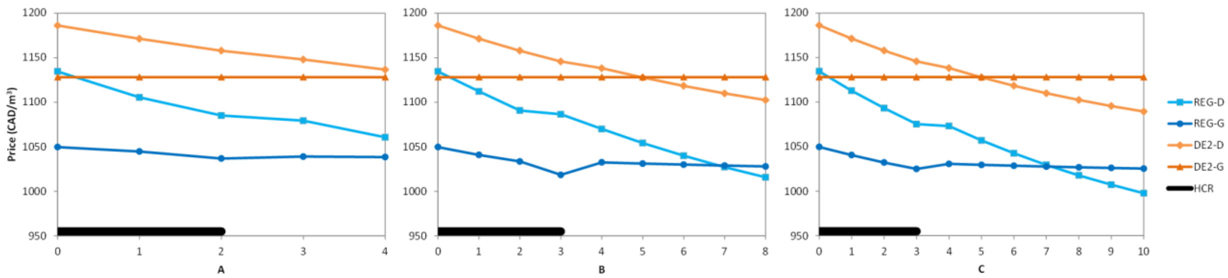

Figure 5. Nash equilibrium price trajectories obtained for discretizations of four (A) eight (B) and ten (C) periods of an eight month planning horizon.

Maintaining the horizon length of eight months, discretizations of four, eight, and ten time periods are applied to the dynamic production planning model and the Nash equilibrium price trajectories obtained are presented in Figure 5. It is observed that the terminal points of the trajectories vary with time discretization as anticipated. The qualitative paths of the trajectories are similar in each case, though in the case of the four point discretization we observe that resolution is lost. Importantly, the elimination trend does not seem to be adversely impacted by the selection of horizon discretization; it is interpreted from this that the elimination decision is a function of profit and the total duration of the time horizon, but not discretization.

Existence of multiple equilibria

While price trajectories represent (generally) unique dynamic Nash equilibrium solutions to the problems examined, the underlying refiner decision variables yielding these trajectories are not unique. Refiner in-house production levels (i.e., excluding imports) of domestic market regular gasoline are presented in Figure 3B. The production trajectories corresponding to the refiners are linked: even excluding the import volumes, the collective regular gasoline total is the same at every point on the horizon.

The game constraints limit the refiners' strategy space and force them to operate in concert with one another. The resulting solution is a generalized Nash equilibrium[34,35,36]. The existence of non-unique solutions is addressed by Rosen through the process of normalization[37]. We do not attempt to apply normalization procedures to our solutions; we are interested in the macroscopic behaviour of the refiners: knowing that the underlying planning model is feasible (and that multiple solutions exist) demonstrates the connections between the physical capabilities of the plants and organization-level decision making. Since the optimum is identical in these equivalent cases, the qualitative and quantitative interpretations of the results with regard to strategy are not altered. Proper application of normalization would presumably yield smooth in-house as well as import trajectories for individual refiners, which would be desirable from an implementation perspective.

Model solution statistics

Model size is a particular challenge in the solution of discretized dynamic game theoretic problems; the growth of bilinear terms in the presented formulation increases rapidly with each additional time point added to the horizon. Price at any point in time is a function of all preceding prices and the original price at time zero; the original price is asymptotically rejected in successive time points, but nevertheless complicates the model equations. The solution procedure defined for this framework applies CONOPT 3.17A[38], IPOPT 3.12[39], and ANTIGONE 1.1[40] in the first stage NLP to determine the high-cost refiner's production trajectory, and then DICOPT[41] and ANTIGONE 1.1 to solve the MINLP which determines whether and when the high-cost refiner is eliminated. CONOPT provides a warm start for IPOPT, which in turn provides a warm start for ANTIGONE; in this way solution times are significantly reduced. The same warm start strategy is used with DICOPT and ANTIGONE in the solution of the MINLP. Maximum solution times of 1000 seconds were applied for the warm start solvers, but were not realized in the solution procedure. Solution statistics for the models presented in this work are collected in Table 1, and were generated on a Dell Optiplex 9010 computer with Intel Core-i7-3770 CPU and a 3.40 GHz processor operating on Windows 10 64-bit system.

Table 1. Solution statistics

	Standard 4 period	Standard 8 period	Standard 10 period	Inefficient 10 period
Model statistics (GAMS)				
Single equations	2732	5802	7487	7487
Single variables	2042	2534	5972	5972

Nonlinear entities	304	1120	1720	1720
Second Stage ANTIGONE Output				
Solve summary				
Objective value	764.7155	612.1062	549.6476	772.8618
Model status	1	1	1	1
After pre-processing				
Variables	579	1263	1653	1653
Continuous	575	1255	1643	1643
Binary	4	8	10	10
Equations	1122	2484	3267	3267
Linear	969	1931	2418	2418
Nonconvex nonlinear	153	553	849	849
Nonlinear terms	1264	5216	8200	8200
Bilinear/quadratic	1232	5152	8120	8120
Sigmoidal	32	64	80	80
Solve statistics				
Relative gap	1.00E-09	1.00E-09	1.00E-09	1.00E-09
Total time (CPU s)	4.25	23.46	57.95	61.69
Total combined first and second stage solve time (CPU s)				
	1212.10	184.27	1021.48	555.16

Data are presented for the second stage ANTIGONE solve. The numbers of bilinear terms increase quickly with the number of time periods, and results in large nonconvex nonlinear problems overall. In all four cases, global optimality is obtained, indicated by the model status of 1, with a relative optimality gap of 1×10^{-9}. The solution times are on the scale of seconds to minutes for the final second stage solve; more time is spent obtaining feasible solutions in the first stage problem and in the DICOPT phase of the second stage. The multistage solution approach is such that the solution at which ANTIGONE starts its search possesses an optimality gap within stopping tolerance. Once ANTIGONE initializes the solution and completes its pre-solve operations, only a single iteration is required before the algorithm concludes optimality has been obtained. Without the two stage approach, a global solver has yet to be found which closes the gap in reasonable time.

Conclusions

A framework has been presented for dynamic game theoretic refinery production planning problems in which refiners operate in multiple markets with pricing functions based on a modified dynamic Cournot oligopoly model with sticky prices. This framework is used to examine competitor elimination scenarios from the perspective of a refiner facing potential elimination by its competitors. Our results indicate that there exist scenarios in which a refiner's best strategy is to upgrade its facilities in order to avoid elimination, while in other cases the same refiner would be better off taking no action at all. Where upgrades are required in order to avoid closure, the available time before closure is obtained and presents a deadline on upgrade procedures. These decisions are the result of a complex trade-off between a refiner's own cost structures and its interactions with its competitors. Planning scenarios accounting for these types of interactions have not been examined in prior production planning literature, and can identify market forces whose impacts are otherwise not possible to quantify in an optimal planning framework. Process plants in large markets may face these complex types of decisions wherein internal costs and planning have market-wide impacts and consequences; the presented game theoretic planning framework aids in exploring these problems.

Notation

Sets

N	(n) time periods
R	(r) refiners
R_L	(r) low-cost refiners
R_H	(r) high-cost refiners
P	(p) products
W	(w) markets

W_D	(w) domestic markets
W_G	(w) global markets
RW_D	(r, w) refiner r is located in domestic market w
Ξ	(ξ) strategy set

Parameters

s	Price stickiness factor
A_{pw}	Price decline rate
A_{pw}^{HC}	Price increase due to high-cost refiner activity
B_{pw}	Price of product p in market w corresponding to supply of exactly D_{pw}
D_{pw}	Nominal market supply of product p in market w
π_{0pw}	Initial price of product p in market w corresponding to the zeroth time period
i	Profit discounting factor
$\bar{q}_{nrpw}$	Maximum production of product p in period n to rmarket w by refiner r including both in-house production and imports

Continuous Variables

J_r	Profit for refiner r
π_{npw}	Price of product p in market w in period n
$\tilde{\pi}_{npw}$	Static Cournot price of product p in market w in period n
q_{nrpw}	Total volume of product p delivered to market w in period n by refiner r accounting for in-house production and imports
C_{nr}	General cost function of refiner r in period n
Z	Potential function value
Ω_r	Potential function individual term
Ω_r^S	Profit component of Ω_r
Ω_r^C	Cost component of Ω_r
Ψ	Potential function collective term

$\gamma_{nrpww'}$ Exact bilinear reformulation variable

Binary Variables

y_{nw} Determines at which period n high-cost refiners are eliminated from market w

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Acknowledgements

This work has been funded by the Natural Sciences and Engineering Research Council of Canada, the Government of Ontario, and the McMaster Advanced Control Consortium whom we thank for their support. Philip Tominac would like to dedicate this paper in memory of Anica Vukelic.

Chapter 4

Conjectures regarding the existence and properties of mixed integer potential games

The content of this chapter is revision of the manuscript submitted for publication under the following title:

Tominac P, Mahalec V. Conjectures regarding the existence and properties of mixed integer potential games.

Conjectures regarding the existence and properties of mixed integer potential games

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Category: Game theory

Keywords –potential game; mixed-integer programming; expansion planning; strategic planning

Abstract

A novel mixed integer nonlinear programming framework is investigated in which game theoretic capacity expansion and production planning problems formulated as potential games yield pure strategy Nash equilibria (PSNE) to finite games implied by the enumeration of integer solutions. The theoretical properties of and existence criteria for PSNE are presented for two types of upgrade games. These games represent the first instances of a class of games that we term mixed integer potential games (MIPG) and which are solved using mixed integer nonlinear programming (MINLP) methods without enumeration of the implied finite game payoff matrix. Conjectures are made regarding MIPG properties, and an industrially relevant example problem is solved to a PSNE solution in which two refiners compete in multiple markets and products where each has available for purchase four potential unit upgrades. The problem is formulated and solved as a MIPG, and the implied finite game payoff matrix is enumerated to verify that the solution is a PSNE.

Introduction

Mixed-integer programming models form the basis for a number of important problems in chemical engineering, as have been cited by Grossmann (2002), Lin and Floudas (2005), and Trespalacios and Grossmann (2014). These models are of particular importance to supply chain planning and scheduling; a subject for which there exists a wide body of literature and well-researched theoretical background. Tominac and Mahalec (2017) draw on theoretical aspects of game theory to solve competitive production planning problems modelled as nonlinear static potential games. As a framework in which to solve production planning problems, the potential game structure formalized by Monderer and Shapley (1996) is attractive: the interests (notionally, profits) of multiple competing entities can be collected under a single objective which is then optimized to determine the resulting Nash equilibrium occurring as each individual entity attempts to maximize its profits. Many relevant supply chain optimization problems can be modelled under this framework as competitive problems with multiple interested parties.

Certain problems cannot be modelled as potential games due to a gap in the theory which exists, to the best of our knowledge, between the solution concepts to finite and infinite games. This gap in the theory has been addressed by other authors outside the context of the potential game. Morrow, Mineroff, and Whitefoot (2014) solve mixed integer design problems by simulation. Liu, Du, Jiao, and Xia (2017) use a bilevel programming formulation for product design problems. Shiau and Michalek (2009) solve mixed game problems as nonlinear programs with explicit Nash constraints. The concept of the Nash equilibrium exists similarly for games with only finitely-valued variables and for games with purely continuous decision variables, but with important differences; Webb (2007) provides definitions and detailed information regarding Nash equilibria. To the best of the authors' knowledge, no theorem exists at the time of writing defining the corresponding Nash equilibrium for a mixed integer potential game (MIPG) i.e., a game containing both continuous and integer-valued variables and whose equilibrium solution satisfies both finite and continuous definitions of a Nash equilibrium. A theory of mixed integer potential games is not required to solve all competitive mixed integer problems, but where both integer and continuous variables represent game decisions, such problems form MIPGs and cannot be solved as potential games to a guaranteed Nash equilibrium. This problem is relevant to a number of engineering supply chain problems, in particular to the classes of capacity expansion, unit upgrade, and facility location problems such as those in Neiro and Pinto (2008) and You and Grossmann (2008).

We present theoretical arguments for the existence of mixed integer potential games possessing PSNE the mathematical conditions under which a PSNE exists, and numerical experiments demonstrating our arguments. The primary benefit resulting from these mathematical arguments is that it is unnecessary to obtain the entire game payoff matrix in order to determine a Nash equilibrium; mixed integer programming algorithms are capable of obtaining mixed integer game PSNE with significantly reduced computational burden. Based on our results, we conjecture that it is possible to demonstrate that a class of mixed integer potential games exists which under reasonable assumptions will possess PSNE. The novel elements of this work are:

- A definition for the class of mixed integer games as ordinal potential games,
- Conditions under which two player, two strategy games will be ordinal potential games and possess PSNE (termed ordinality conditions),
- A numerical example of a two-by-two MIPG, and verification of the ordinality conditions for this example,
- A numerical example of refinery planning model as a large scale MIPG with a PSNE which is solved using the MINLP potential game approach, and the PSNE verified through enumeration of the implied game payoff matrix,
- Conjectures regarding the existence and properties of PSNE in higher dimension MIPGs.

Background

Brief definitions of discrete and continuous Nash equilibria as are necessary for later arguments are presented here, as well as a brief discussion of the differences between pure and mixed strategy equilibria which arise in finite games. Relevant theory regarding Nash equilibria in potential games is also presented.

Nash Equilibrium

As a solution concept the Nash equilibrium is defined similarly for discrete and continuous games. A game in either form is generally referred to as G and played by a finite set of players R, each whom has a strategy set which at this point may be finite or infinitely valued Ξ_r, with strategies defined as ξ_r. Player profits, or more generally, payoffs, are $J_r(\xi_r, \xi_{-r})$ and are a function of each player's individually selected strategy ξ_r as well as all the strategies of its competitors ξ_{-r}. The Nash equilibrium is the set of strategies yielding the result in Eq. (1) where ξ_r^* is a player's optimal strategy, and no deviation from this strategy improves that player's payoff.

$$J_r(\xi_r^*, \xi_{-r}^*) \geq J_r(\xi_r, \xi_{-r}^*) \quad \forall \xi_r \in \Xi_r, r \in R \tag{1}$$

Nash (1951) proved the existence of at least one equilibrium in either pure or mixed strategies in every finite game. Continuously valued games possess one or more equilibria in the equivalent of a pure strategy, guaranteed by theorems due to Debreu (1952), Glicksberg (1952) and Fan (1952). The mixed strategy case applies to games of perfect information only in the case of finite games.

Pure and mixed strategy Nash equilibria in finite games

In a finite game a player is confined to a choice among limited strategies, and in certain games no PSNE may result. Nash's method of reconciling this result is the mixed strategy in which one or more players selects two or more strategies each with some probability. The game outcome is thus an expected value in the long run subject to realization. The mathematical definition of a mixed strategy Nash equilibrium (MSNE) in the general case of a finite game with $|R| \geq 3$ players and possessing strategy sets Ξ_r is NP-hard and not easily solved, although algorithms exist for determining such equilibria such as those in McKelvey and McLennan (1996). The PSNE is a case of the MSNE in which each player selects a single strategy with probability one. While every finite game possesses at least one Nash equilibrium, there is in general no guarantee that said equilibrium is of the pure strategy type due to Nash (1952).

Nash equilibria in continuous potential games

Formalized by Monderer and Shapley (1996) potential games are a subclass of games for which a potential exists and defines the Nash equilibria of the game. We address continuous potential games first, and finite potential games in the section that follows. The Nash equilibrium of a continuous game is defined as the solution to the set of differential equations defined by Eq. (2) interpreted to mean that each player maximizes its profit with respect to the variables it controls.

$$\frac{\partial J_r(\xi_r, \xi_{-r})}{\partial \xi_r} = 0 \quad \forall \xi_r \in \Xi_r, r \in R \tag{2}$$

A continuous game is a potential game if a function $Z(\xi_r, \xi_{-r})$ exists with the property that its derivatives with respect to each player's strategy variables are exactly the derivatives of that player's profit functions, as in Eq. (3). Thus, maximizing the potential function becomes equivalent to solving the set of differential equations, and the potential function maximum is defined as a Nash equilibrium. This is true even if the potential function is maximized subject to constraints, although in the event constraints are active at equilibrium the equilibrium is referred to as a generalized Nash equilibrium as defined by Rosen (1965) and discussed by Ghosh, Cottatellucci, and Altman (2015).

$$\frac{\partial Z(\xi_r, \xi_{-r})}{\partial \xi_r} = \frac{\partial J_r(\xi_r, \xi_{-r})}{\partial \xi_r} \quad \forall \xi_r \in \Xi_r, r \in R \tag{3}$$

Nash equilibria in finite potential games

A finite potential game is a game for which a potential exists, analogous to the definition of the continuous game potential function. If the matrix of player rewards for a finite game has dimension $|\Xi_1| \times \ldots \times |\Xi_r| \times \ldots \times |\Xi_R| \times |R|$, i.e., matrix dimensions $|\Xi_1| \ldots |\Xi_R|$ corresponding to players' strategies and $|R|$ representing a vector of payoff values $j_r(\xi_r, \xi_{-r})$ to each player resulting in each realization, then the game potential is a matrix of dimension $|\Xi_1| \times \ldots \times |\Xi_r| \times \ldots \times |\Xi_R|$ containing scalar values $z(\xi_r, \xi_{-r})$ which satisfy the definition of an exact potential as defined by Monderer and Shapley (1996) in Eq. (4).

$$j_r(\xi_r, \xi_{-r}) - j_r(\xi'_r, \xi_{-r}) = z(\xi_r, \xi_{-r}) - z(\xi'_r, \xi_{-r}) \quad \forall \xi_r, \xi'_r \in \Xi_r, r \in R \tag{4}$$

The interpretation of this definition is that the difference between a player's rewards $j_r(\xi_r, \xi_{-r}) - j_r(\xi'_r, \xi_{-r})$ obtained by selection of two strategies ξ_r and ξ'_r is exactly the same as the difference between the potential values corresponding to those strategies $z(\xi_r, \xi_{-r}) - z(\xi'_r, \xi_{-r})$, and holds true for every player. Less stringent potential definitions include the weighted potential Eq. (5), and the ordinal potential Eq. (6).

$$j_r(\xi_r, \xi_{-r}) - j_r(\xi'_r, \xi_{-r}) = w_r\big(z(\xi_r, \xi_{-r}) - z(\xi'_r, \xi_{-r})\big) \quad \forall \xi_r, \xi'_r \in \Xi_r, r \in R \tag{5}$$

$$j_r(\xi_r, \xi_{-r}) - j_r(\xi'_r, \xi_{-r}) > 0 \Leftrightarrow z(\xi_r, \xi_{-r}) - z(\xi'_r, \xi_{-r}) > 0 \quad \forall \xi_r, \xi'_r \in \Xi_r, r \in R \tag{6}$$

The weighted potential allows the difference between the player profits to be a weighting by w_r of the exact potential difference; the ordinal potential only requires that the signs of these differences be the same. In addition to these definitions, Monderer and Shapley (1996) prove the following theorem regarding the definition of a potential game: G is a finite game, then G has the finite improvement property if and only if G has a generalized ordinal potential[6]. The finite improvement property, as it relates to a finite game, states that if players are sequentially allowed to select from among their available strategies one which improves their payoff, this process terminates following a finite number of steps. It is axiomatic that this termination point is a Nash equilibrium. Monderer and Shapley (1996) also prove the following corollary, which will be essential in this work: let G be a finite game possessing the finite improvement property; in addition, let G have the property $j_r(\xi_r, \xi_{-r}) \neq j_r(\xi'_r, \xi_{-r}) \quad \forall \xi_r, \xi'_r \in \Xi_r, r \in R$, then G possesses an ordinal potential. In summary, if a game possessing the finite improvement property also has the property that each player's strategic choice results in a unique payoff, then the game possesses an ordinal potential, or equivalently, if the finite game possesses a pure strategy Nash equilibrium and the payoffs to each player are unique in every strategic realization, then the game is an ordinal potential game.

Problem statement

We examine the static game theoretic refinery model of Tominac and Mahalec (2017) in the context of a capacity expansion and unit upgrade problem. The model includes multiple refiners operating in competitive markets seeking to produce multiple products in a profile maximizing their individual profits. For a full description of the original model, refinery schematic, assumptions, equations, and the parameter values and data sources used in its development, see Tominac and Mahalec (2017) and the associated supplementary material. In this work, we consider the case where the refiners are looking for the most competitive means of which to expand their operations, and so we add to the model a menu of unit upgrades of two types: unit capacity upgrades and cost reduction upgrades. Refiners select and purchase upgrades from the menu corresponding to various unit operation upgrades. The modified game theoretic refinery production planning and upgrade model used in this work is included as supplementary material. Our work is differentiated from similar capacity expansion and upgrade problems in that the objective function is a game theoretic one, and the resulting optimal upgrade selections are a Nash equilibrium of a game theoretic problem of mixed integer and continuous variables, hence the termed a mixed integer game (MIPG) problem. The resulting upgrade selections made by each player are representative of a competitive upgrade plan, satisfying the definitions of a Nash equilibrium in both the continuous and discrete aspects of the game.

In order to verify that the solutions obtained to these MIPG problems are indeed Nash equilibria, a two-player version of the Cournot game of Tominac and Mahalec (2017) is analyzed and the conditions under which a discrete potential game is defined and a PSNE is obtained in the implied matrix game are determined. The resulting conditions provide a numerical test which can be applied to determine whether the solution to a given MIPG problem will possess the correct interpretation as a PSNE; if the conditions are satisfied, the MIPG solution will not be a PSNE as none exists in the implied finite game.

Models and formulation

For the purpose of deriving the necessary existence conditions to first qualify the MIPG as a proper discrete potential game and to second verify that a PSNE will exist, we consider (without loss of generality with respect to two-player game theoretic problems) a simplified two-player Cournot game with the same pricing structure as that of the refinery production planning game, but with only a single arbitrary product, one market, and no underlying process constraints which would add significant complexity to the derivation. The pricing structure is such that the continuous Cournot game problem will be an exact potential game, satisfying the definitions in Eqs. (2) and (3), and also that in Eq. (4), which can be easily verified. The pricing structure for the two-player game is defined in Eq. (7). Player costs functions are taken as quadratic functions of production and are assumed to be convex such that there exists an efficient production point for each player wherein cost is minimized, as well as a single unique Nash equilibrium. The cost functions are included as Eqs. (8) and (9).

$$\pi(q_1, q_2) = A + B - \frac{A}{D}(q_1 + q_2) \tag{7}$$

$$C_1(q_1) = a_1 q_1^2 + b_1 q_1 + c_1 \tag{8}$$

$$C_2(q_2) = a_2 q_2^2 + b_2 q_2 + c_2 \tag{9}$$

Player profits are determined by Eqs. (10) and (11), and are functions of both players' production volumes due to the dependence of price upon market supply. The profit functions yield the potential function in Eq. (12).

$$J_1(q_1, q_2) = \left(A + B - \frac{A}{D}(q_1 + q_2)\right) q_1 - a_1 q_1^2 - b_1 q_1 - c_1 \tag{10}$$

$$J_2(q_1, q_2) = \left(A + B - \frac{A}{D}(q_1 + q_2)\right) q_2 - a_2 q_2^2 - b_2 q_2 - c_2 \tag{11}$$

$$Z(q_1, q_2) = (A + B - b_1)q_1 + (A + B - b_2)q_2 - \frac{A}{D}q_1q_2 - \left(\frac{A}{D} + a_1\right)q_1^2 - \left(\frac{A}{D} + a_2\right)q_2^2 - c_1 - c_2 \tag{12}$$

The Nash equilibrium defined by this potential function is determined by the solution to the set of equations defined by Eqs. (13) and (14), and is equivalent to the potential function maximum. The solution (q_1^*, q_2^*) is defined by Eqs. (15) and (16), and represents the production volumes resulting in a Nash equilibrium in the continuous Cournot game.

$$\frac{\partial J_1(q_1, q_2)}{\partial q_1} = \frac{\partial Z(q_1, q_2)}{\partial q_1} = A + B - \frac{A}{D}(2q_1 + q_2) - 2a_1q_1 - b_1 = 0 \tag{13}$$

$$\frac{\partial J_2(q_1, q_2)}{\partial q_2} = \frac{\partial Z(q_1, q_2)}{\partial q_2} = A + B - \frac{A}{D}(q_1 + 2q_2) - 2a_2q_2 - b_2 = 0 \tag{14}$$

$$q_1^* = \frac{\frac{A}{D}(A + B - 2b_1 + b_2) + 2a_2(A + B - b_1)}{3\left(\frac{A}{D}\right)^2 + 4\frac{A}{D}(a_1 + a_2) + 4a_1a_2} \tag{15}$$

$$q_2^* = \frac{\frac{A}{D}(A + B + b_1 - 2b_2) + 2a_1(A + B - b_2)}{3\left(\frac{A}{D}\right)^2 + 4\frac{A}{D}(a_1 + a_2) + 4a_1a_2} \tag{16}$$

PSNE existence conditions

We are interested in how the Cournot solution behaves as the generator of a finite potential game when a single parameter is varied; i.e., under what conditions the set of optimal solutions to Cournot games of the form defined in the previous section results in a potential game with a pure strategy Nash equilibrium. In this derivation we consider the case in which each player has a single available upgrade, and thus two strategies: either to purchase the upgrade, or not. The purchase cost of the upgrade is assumed to be fixed and thus its magnitude does not impact the qualitative interpretation of the derivation; for this reason upgrade cost is not included in the derivation. An upgrade is interpreted as a change in a single parameter resulting in a different game with new equilibrium.

Enumeration of all solutions to all possible combinations of upgrade purchase decisions defines a finite game payoff matrix in which the Nash equilibrium solution to the underlying continuous game defines the payoffs in the finite game, and to which we argue that the potential function values associated with the underlying Cournot game are also a potential to the finite game implicitly defined. To illustrate, consider an upgrade to each player's linear cost term b_r defined as Δb_r such that each player's resulting linear cost term following the upgrade is $b_r + \Delta b_r$. As apparent from Eqs. (15) and (16), the optimal solution (q_1^*, q_2^*) is dependent on b_r, and thus is expected to change in strategic realization of the game. The implied finite game payoff matrix defined by the solutions to the Cournot game in each realization of upgrade purchase is presented as Eq. (17), where the 'no upgrade' strategy is represented as $\emptyset_r$ and the upgrade strategy as Δb_r. The corresponding potential function values are shown in Eq. (18).

$$\begin{array}{cc} & \begin{array}{cc} \quad\emptyset_2 \quad\quad\quad\quad\quad\quad\quad & \Delta b_2 \end{array} \\ \begin{array}{c} \emptyset_1 \\ \Delta b_1 \end{array} & \begin{bmatrix} J_1(q_1, q_2), J_2(q_1, q_2) & J_1(q_1'', q_2'), J_2(q_1'', q_2') \\ J_1(q_1', q_2''), J_2(q_1', q_2'') & J_1(q_1''', q_2'''), J_2(q_1''', q_2''') \end{bmatrix} \end{array} \tag{17}$$

$$\begin{array}{cc} & \begin{array}{cc} \quad\emptyset_2 \quad\quad\quad & \Delta b_2 \end{array} \\ \begin{array}{c} \emptyset_1 \\ \Delta b_1 \end{array} & \begin{bmatrix} Z(q_1, q_2) & Z(q_1'', q_2') \\ Z(q_1', q_2'') & Z(q_1''', q_2''') \end{bmatrix} \end{array} \tag{18}$$

Together, these two matrices demonstrate an important feature of the implied finite game: it cannot be guaranteed to be an exact potential game. This assertion is easily verified using the definition of an exact potential game from Eq. (4), which is applied to player 1 in Eq. (19) across player 1's strategies while player 2 holds to its $\emptyset_2$ strategy. Verification arises from comparison of the difference between the Cournot profit values with the difference between the potential function values, which is not guaranteed to be equal as required.

$$\begin{aligned} j_1(\emptyset_1, \emptyset_2) - j_1(\Delta b_1, \emptyset_2) &= J_1(q_1, q_2) - J_1(q_1', q_2'') \\ &\neq Z(q_1, q_2) - Z(q_1', q_2'') \end{aligned} \tag{19}$$

Player 1's finite game payoffs under comparison are $j_1(\emptyset_1, \emptyset_2)$ and $j_1(\Delta b_1, \emptyset_2)$, which are equal to $J_1(q_1, q_2)$ and $J_1(q_1', q_2'')$, respectively. The requirement however is that $J_1(q_1, q_2) - J_1(q_1', q_2'') = Z(q_1, q_2) - Z(q_1', q_2'')$ in order for the game to be an exact potential, but this is not guaranteed to be true, because $q_2 \neq q_2''$, as is known from the solution derivation, and therefore $J_1(q_1, q_2) - J_1(q_1', q_2'') \neq \ Z(q_1, q_2) - Z(q_1', q_2'')$. From the potential function definition, it is guaranteed only that $\frac{\partial Z(\xi_r, \xi_{-r})}{\partial \xi_r} = \frac{\partial J_r(\xi_r, \xi_{-r})}{\partial \xi_r}$ which no longer holds because both r player's and the $-r$ player's strategies have changed simultaneously. Thus, we cannot guarantee that the finite game arising from the changes to the underlying Cournot exact potential game will itself be an exact potential game.

While the implied finite game does not possess an exact potential, Monderer and Shapley's theorems regarding finite potential games include the definition of an ordinal potential game. It can be demonstrated that the continuous Cournot potential game will, under the correct numerical conditions, emit a finite game possessing a PSNE and which is guaranteed to be an ordinal potential game. These two properties are related, and both are demonstrated by an examination of the conditions under which the emitted finite potential game possesses no PSNE and only a MSNE.

The conditions under which a given finite game possesses only a MSNE are related to the payoffs that each player receives in the realization of a set of strategies; these conditions must be such that in every realization of the game, at least one player is incentivized to change their strategy due to the existence of a better payoff. Equivalently, there can be no realization of the finite game matrix in which players are content with their payoffs. Functionally, the conditions disallowing a PSNE state that the first player's column maximum payoffs do not coincide with the second player's row maximum payoffs. The general case of this circumstance is expressed logically in terms of strategies in Eq. (20).

$$\begin{gathered} \forall r \in R, \xi_r \in \Xi_r, \xi_{-r} \in \Xi_{-r} \quad \exists \xi_r' \in \Xi_r \\ s.t. \quad j_r(\xi_r', \xi_{-r}) > j_r(\xi_r, \xi_{-r}) \quad \wedge \quad j_{-r}(\xi_r', \xi_{-r}) < j_{-r}(\xi_r, \xi_{-r}) \end{gathered} \tag{20}$$

Some notation is introduced in order to make subsequent arguments more compact; rather than use the $\Delta b_r, \emptyset_r$ notations for upgrade purchasing, we use the more general 0,1 notation in which 0 represents the strategy 'not upgrading,' and 1 represents the 'upgrade' strategy. These strategy indicators are included as a subscript tuple in player order in square brackets with the profit variable; thus $J_{1,[0,1]}$ represents the profit resulting to player 1 in the case where player 1 does not upgrade and player 2 upgrades. The important factor in the determination of whether a PSNE exists is the relative payoff to a player, not the absolute value, and thus the notation $\Delta J_{r,[\xi_1,\xi_2]}$ is introduced to indicate the difference is profit obtained by a player relative to the profit that would be obtained in the $J_{r,[0,0]}$ case; for example, $\Delta J_{2,[1,1]} = J_{2,[1,1]} - J_{2,[0,0]}$ and of course $\Delta J_{r,[0,0]} = 0$. Under this notation, the requirements necessary in order for a two-by-two finite game to possess only a MSNE and to fail to exhibit a PSNE are presented collectively in Eq. (21). Note that these conditions remain true if all inequalities are reversed; the indicated form will be used in this work, with the understanding the reverse-sign conditions are equivalent, and must also be considered in evaluating the existence of a PSNE.

$$\left.\begin{array}{c} \Delta J_{1,[1,0]} < 0 \\ \Delta J_{1,[0,1]} < 0 \\ \Delta J_{1,[1,1]} > \Delta J_{1,[1,0]} \\ \Delta J_{1,[1,1]} > \Delta J_{1,[0,1]} \\ \Delta J_{2,[1,0]} > 0 \\ \Delta J_{2,[0,1]} > 0 \\ \Delta J_{2,[1,1]} < \Delta J_{2,[1,0]} \\ \Delta J_{2,[1,1]} < \Delta J_{2,[0,1]} \end{array}\right\} \tag{21}$$

With the conditions under which a given two-by-two game will possess only a MSNE established, we return to the concept of an ordinal potential game and to the verification that the finite game emitted by the Cournot upgrade game will possess an ordinal potential. Monderer and Shapley (1996) first note that an ordinal potential game must possess the finite improvement property; possessing the finite improvement property implies that the game also possesses a PSNE, thus the conditions in Eq. (21) define all possible cases in which the resulting game would not have the finite improvement property. To guarantee that the game possesses an ordinal potential, each strategic realization of the game must result in a unique payoff for a given player. This additional requirement is easily added to the MSNE conditions in Eq. (21) by replacing the strict inequality signs with non-strict signs; the new conditions are presented in Eq. (22), and define cases in which the game would lack a PSNE and additionally cases in which a PSNE exists where a given player's payoffs are not unique. These new conditions are referred to as ordinal conditions, and the finite game is an ordinal potential game if it cannot simultaneously satisfy all eight conditions in either the given or reversed-inequality form.

$$\left.\begin{array}{c} \Delta J_{1,[1,0]} \leq 0 \\ \Delta J_{1,[0,1]} \leq 0 \\ \Delta J_{1,[1,1]} \geq \Delta J_{1,[1,0]} \\ \Delta J_{1,[1,1]} \geq \Delta J_{1,[0,1]} \\ \Delta J_{2,[1,0]} \geq 0 \\ \Delta J_{2,[0,1]} \geq 0 \\ \Delta J_{2,[1,1]} \leq \Delta J_{2,[1,0]} \\ \Delta J_{2,[1,1]} \leq \Delta J_{2,[0,1]} \end{array}\right\} \tag{22}$$

Two types of upgrade are given specific consideration; upgrades which influence the linear cost parameter Δb_r, and upgrades which increase available production capacity. In order to investigate the latter case, we consider the Cournot potential game subject to a constraint of the form $q_r \leq \bar{q}_r$ where $\bar{q}_r$ is a parameter representing the maximum product output from a given player r. Upgrades to this amount are changes to the constraint bound $\Delta\bar{q}_r$ such that the modified constraint is of the form $q_r \leq \bar{q}_r + \Delta\bar{q}_r$. The ordinal conditions are evaluated as functions of each of these two upgrades in order to obtain numerical tests which indicate whether a given two-player upgrade game will be a MIPG with an ordinal potential; in doing so enumeration of the implied finite game matrix is avoided, and in fact, is not necessary to ensure that the MIPG optimum is a Nash equilibrium. These games are referred to as ordinal MIPGs of cost and capacity, with respect to the type of upgrade.

Ordinal mixed integer games of cost

In an ordinal cost game, two players have access to an upgrade which changes the linear term of their cost functions by an amount Δb_r. Each player has two strategies; either to purchase the upgrade or not; the magnitude of the upgrade is given (different upgrade magnitudes would correspond to additional strategies) and these two strategies yield a two-by-two finite game defined by the optima to the Cournot oligopoly potential game in each of the four cases. The ordinal conditions are based on differences in profit values, which can be determined from changes in the optimal solution $(\Delta q_1^*, \Delta q_2^*)$ relative to the base case $[0,0]$ corresponding to no upgrades. To simplify the following algebraic expressions the parameter θ is defined in Eq. (23).

$$\theta = \left(3\left(\frac{A}{D}\right)^2 + 4\frac{A}{D}(a_1 + a_2) + 4a_1a_2\right)^{-1} \tag{23}$$

The relative changes to the solutions $(\Delta q_1^*, \Delta q_2^*)$ which occur in each case are defined in Eq. (24); the expressions for $\Delta q_{r,[0,0]}$ are both zero.

$$\left.\begin{array}{c}
\Delta q_{1,[1,0]} = -2\left(\frac{A}{D} + a_2\right)\theta\Delta b_1 \\
\Delta q_{1,[0,1]} = \frac{A}{D}\theta\Delta b_2 \\
\Delta q_{1,[1,1]} = -2\left(\frac{A}{D} + a_2\right)\theta\Delta b_1 + \frac{A}{D}\theta\Delta b_2 \\
\Delta q_{2,[1,0]} = \frac{A}{D}\theta\Delta b_1 \\
\Delta q_{2,[0,1]} = -2\left(\frac{A}{D} + a_1\right)\theta\Delta b_2 \\
\Delta q_{2,[1,1]} = -2\left(\frac{A}{D} + a_1\right)\theta\Delta b_2 + \frac{A}{D}\theta\Delta b_1
\end{array}\right\} \tag{24}$$

These changes in production as a function of Δb_r are used to determine the relevant $\Delta J_{r[\xi_1,\xi_2]}$ values, which are presented in Eq. (25).

$$\left.\begin{array}{c}
\Delta J_{1,[1,0]} = \left(A + B + \frac{A}{D}\theta\left(\frac{A}{D} + 2a_2\right)\Delta b_1\right)\left(-2\frac{A}{D} - 2a_2\right)\theta\Delta b_1 - 4a_1\theta^2\left(\frac{A}{D} + a_2\right)^2\Delta b_1^2 + 2\theta\left(\frac{A}{D} + a_2\right)\Delta b_1^2 \\
\Delta J_{1,[0,1]} = \left(A + B + \frac{A}{D}\theta\left(\frac{A}{D} + 2a_1\right)\Delta b_2\right)\left(\frac{A}{D}\right)\theta\Delta b_2 - a_1\theta^2\left(\frac{A}{D}\right)^2\Delta b_2^2 - \frac{A}{D}\theta b_1\Delta b_2 \\
\Delta J_{1,[1,1]} = \left(A + B + \frac{A}{D}\theta\left(\left(\frac{A}{D} + 2a_2\right)\Delta b_1 + \left(\frac{A}{D} + 2a_1\right)\Delta b_2\right)\right)\left(-2\left(\frac{A}{D} + a_2\right)\Delta b_1 + \frac{A}{D}\Delta b_2\right)\theta \\
-a_1\theta^2\left(-2\left(\frac{A}{D} + a_2\right)\Delta b_1 + \frac{A}{D}\Delta b_2\right)^2 - \Delta b_1\theta\left(-2\left(\frac{A}{D} + a_2\right)\Delta b_1 + \frac{A}{D}\Delta b_2\right) \\
\Delta J_{2,[1,0]} = \left(A + B + \frac{A}{D}\theta\left(\frac{A}{D} + 2a_2\right)\Delta b_1\right)\left(\frac{A}{D}\right)\theta\Delta b_1 - a_2\theta^2\left(\frac{A}{D}\right)^2\Delta b_1^2 - \frac{A}{D}\theta b_2\Delta b_1 \\
\Delta J_{2,[0,1]} = \left(A + B + \frac{A}{D}\theta\left(\frac{A}{D} + 2a_1\right)\Delta b_2\right)\left(-2\frac{A}{D} - 2a_1\right)\theta\Delta b_2 - 4a_2\theta^2\left(\frac{A}{D} + a_1\right)^2\Delta b_2^2 + 2\theta\left(\frac{A}{D} + a_1\right)\Delta b_2^2 \\
\Delta J_{2,[1,1]} = \left(A + B + \frac{A}{D}\theta\left(\left(\frac{A}{D} + 2a_2\right)\Delta b_1 + \left(\frac{A}{D} + 2a_1\right)\Delta b_2\right)\right)\left(-2\left(\frac{A}{D} + a_1\right)\Delta b_2 + \frac{A}{D}\Delta b_1\right)\theta \\
-a_2\theta^2\left(-2\left(\frac{A}{D} + a_1\right)\Delta b_2 + \frac{A}{D}\Delta b_1\right)^2 - \Delta b_2\theta\left(-2\left(\frac{A}{D} + a_1\right)\Delta b_2 + \frac{A}{D}\Delta b_1\right)
\end{array}\right\} \tag{25}$$

Applying to these the ordinal conditions in Eq. (22), the algebraic expressions in Eq. (26) are obtained which define the circumstances in which a given MIPG of cost will not be an ordinal potential game. Thus as a numerical test, a given MIPG of cost is an ordinal potential game if not all of these conditions can be simultaneously satisfied for the selected game parameters.

$$\left.\begin{array}{c}
\Delta J_{1,[1,0]} \leq 0 \Rightarrow \Delta b_1 \leq \dfrac{-(A+B)\theta^{-1}}{\left(\frac{A}{D}\right)^2 + 2\frac{A}{D}(a_1+a_2) + 2a_1a_2 - 1} \\
\Delta J_{1,[0,1]} \leq 0 \Rightarrow \Delta b_2 \leq \dfrac{-(A+B-b_1)\theta^{-1}}{\left(\frac{A}{D}\right)^2 + a_1\frac{A}{D}} \\
\Delta J_{1,[1,1]} \geq \Delta J_{1,[1,0]} \Rightarrow (A+B)\theta^{-1} - \left(\left(\frac{A}{D}\right)^2 + \theta^{-1}\right)\Delta b_1 + \left(\left(\frac{A}{D}\right)^2 + \frac{A}{D}a_1\right)\Delta b_2 \geq 0 \\
\Delta J_{1,[1,1]} \geq \Delta J_{1,[0,1]} \Rightarrow \left(2\frac{A}{D} + 2a_2 - 2\left(\frac{A}{D}\right)^3 - 6\left(\frac{A}{D}\right)^2 a_2 - 4\frac{A}{D}a_2^2 + 4\left(\frac{A}{D}\right)^2 a_1 + 8\frac{A}{D}a_1a_2 + 4a_1a_2^2\right)\Delta b_1^2 - (A+B)\left(2\frac{A}{D} + 2a_2\right)\Delta b_1 \\
+b_1\left(3\left(\frac{A}{D}\right)^3 + 4\left(\frac{A}{D}\right)^2(a_1+a_2) + 4\frac{A}{D}a_1a_2\right)\Delta b_2 - \left(\left(\frac{A}{D}\right)^3 + 8\left(\frac{A}{D}\right)^2 a_1 + 8\frac{A}{D}a_1a_2 + \frac{A}{D}\right)\Delta b_1\Delta b_2 \geq 0 \\
\Delta J_{2,[1,0]} \geq 0 \Rightarrow \Delta b_1 \geq \dfrac{-(A+B-b_2)\theta^{-1}}{\left(\frac{A}{D}\right)^2 + a_2\frac{A}{D}} \\
\Delta J_{2,[0,1]} \geq 0 \Rightarrow \Delta b_2 \geq \dfrac{-(A+B)\theta^{-1}}{\left(\frac{A}{D}\right)^2 + 2\frac{A}{D}(a_1+a_2) + 2a_1a_2 - 1} \\
\Delta J_{2,[1,1]} \leq \Delta J_{2,[1,0]} \Rightarrow \left(2\frac{A}{D} + 2a_1 - 2\left(\frac{A}{D}\right)^3 - 6\left(\frac{A}{D}\right)^2 a_1 - 4\frac{A}{D}a_1^2 + 4\left(\frac{A}{D}\right)^2 a_2 + 8\frac{A}{D}\mathrm{a}_1a_2 + 4a_1^2a_2\right)\Delta b_2^2 - (A+B)\left(2\frac{A}{D} + 2a_1\right)\Delta b_2 \\
+b_2\left(3\left(\frac{A}{D}\right)^3 + 4\left(\frac{A}{D}\right)^2(a_1+a_2) + 4\frac{A}{D}a_1a_2\right)\Delta b_1 - \left(\left(\frac{A}{D}\right)^3 + 8\left(\frac{A}{D}\right)^2 a_2 + 8\frac{A}{D}a_1a_2 + \frac{A}{D}\right)\Delta b_1\Delta b_2 \leq 0 \\
\Delta J_{2,[1,1]} \leq \Delta J_{2,[0,1]} \Rightarrow (A+B)\theta^{-1} - \left(\left(\frac{A}{D}\right)^2 + \theta^{-1}\right)\Delta b_2 + \left(\left(\frac{A}{D}\right)^2 + \frac{A}{D}a_2\right)\Delta b_1 \leq 0
\end{array}\right\} \quad (26)$$

Ordinal Mixed Integer Games of Capacity

Determining the ordinal conditions for a capacity upgrade game follows a similar logic to the cost-based game, but requires an additional first step: since the capacity game is subject to constraints, the potential function in question is also subject to those constraints, and the Karush-Kuhn-Tucker (KKT) conditions must be applied in order to determine (q_1^*, q_2^*). The potential function for the MIPG of capacity is presented as Eq. (27), and its Lagrangian as Eq. (28), which leads to the KKT conditions presented in Eq. (29).

$$\left.\begin{array}{c} Z = (A + B - b_1)q_1 + (A + B - b_2)q_2 - \dfrac{A}{D}q_1q_2 - \left(\dfrac{A}{D} + a_1\right)q_1^2 - \left(\dfrac{A}{D} + a_2\right)q_2^2 - c_1 - c_2 \\ q_1 \leq \bar{q}_1 \\ q_2 \leq \bar{q}_2 \end{array}\right\} \tag{27}$$

$$\begin{aligned} L = {} & (A + B - b_1)q_1 + (A + B - b_2)q_2 - \frac{A}{D}q_1q_2 - \left(\frac{A}{D} + a_1\right)q_1^2 - \left(\frac{A}{D} + a_2\right)q_2^2 - c_1 - c_2 \\ & + \mu_1(q_1 - \bar{q}_1) + \mu_2(q_2 - \bar{q}_2) \end{aligned} \tag{28}$$

$$\left.\begin{array}{c} A + B - \dfrac{A}{D}(2q_1 + q_2) - 2a_1q_1 - b_1 + \mu_1 = 0 \\ A + B - \dfrac{A}{D}(q_1 + 2q_2) - 2a_2q_2 - b_2 + \mu_2 = 0 \\ q_1 \leq \bar{q}_1 \\ q_2 \leq \bar{q}_2 \\ \mu_1(q_1 - \bar{q}_1) = 0 \\ \mu_2(q_2 - \bar{q}_2) = 0 \\ \mu_1 \leq 0 \\ \mu_2 \leq 0 \end{array}\right\} \tag{29}$$

The possible combinations of the complementarity constraints imply that there exist four individual resolutions to this problem: both capacity constraints inactive, one player's constraint active and the other's inactive, the opposite of the previous, and both constraints active, all four of which would need to be accounted for in each case of the two-by-two game. The simplest case which can result in the finite capacity game is that in which it is assumed that players begin the game operating at capacity, and the capacity added (we will assume that the upgrade does not result in diminished capacity) leaves both players operating at an optimum where their capacity constraints are inactive. This means that in either case that one of, or both players, upgrade their capacity, that following the upgrade, both will be operating below the level of their capacity constraint. This includes a player who does not upgrade when its competitor has done so, implying that such a player reduces its output. That this will result will be made apparent by the $(\Delta q_1^*, \Delta q_2^*)$ value calculation, from which it will be seen that when one player upgrades and the other does not, the non-upgrade player reduces its production level from its constraint bound to a lesser amount. Operating under these assumptions the optimal solutions $q_{r,[\xi_1,\xi_2]}^*$ are defined in Eq. (30).

$$
\left.\begin{aligned}
q^*_{1,[0,0]} &= \left(\frac{A}{D}(A+B-2b_1+b_2+2\mu_1-\mu_2)+2a_2(A+B-b_1+\mu_1)\right)\theta \\
q^*_{2,[0,0]} &= \left(\frac{A}{D}(A+B+b_1-2b_2-\mu_1+2\mu_2)+2a_1(A+B-b_2+\mu_2)\right)\theta \\
q^*_{1,[1,0]} &= \left(\frac{A}{D}(A+B-2b_1+b_2-\mu_2)+2a_2(A+B-b_1)\right)\theta \\
q^*_{1,[0,1]} &= \left(\frac{A}{D}(A+B-2b_1+b_2+2\mu_1)+2a_2(A+B-b_1+\mu_1)\right)\theta \\
q^*_{1,[1,1]} &= \left(\frac{A}{D}(A+B-2b_1+b_2)+2a_2(A+B-b_1)\right)\theta \\
q^*_{2,[1,0]} &= \left(\frac{A}{D}(A+B+b_1-2b_2+2\mu_2)+2a_1(A+B-b_2+\mu_2)\right)\theta \\
q^*_{2,[0,1]} &= \left(\frac{A}{D}(A+B+b_1-2b_2-\mu_1)+2a_1(A+B-b_2)\right)\theta \\
q^*_{2,[1,1]} &= \left(\frac{A}{D}(A+B+b_1-2b_2)+2a_1(A+B-b_2)\right)\theta
\end{aligned}\right\} \tag{30}
$$

The corresponding $(\Delta q_1, \Delta q_2)$ values are collected in Eq. (31). Note that since the Lagrange multipliers (μ_1, μ_2) are nonpositive in value, a negative Δq coefficient indicates positive change; also note that in the single player upgrade cases, the non-upgrading player reduces its production in response to the upgrading player's increased output. Substitution of the Δq values into the profit functions is qualitatively similar to the cost-based game; the resulting algebraic conditions define circumstances under which a MIPG of capacity will not be an ordinal potential game, and are presented in Eq. (32).

$$
\left.\begin{aligned}
\Delta q_{1,[1,0]} &= -2\left(\frac{A}{D}+a_2\right)\theta\mu_1 \\
\Delta q_{1,[0,1]} &= \frac{A}{D}\theta\mu_2 \\
\Delta q_{1,[1,1]} &= -2\left(\frac{A}{D}+a_2\right)\theta\mu_1+\frac{A}{D}\theta\mu_2 \\
\Delta q_{2,[1,0]} &= \frac{A}{D}\theta\mu_1 \\
\Delta q_{2,[0,1]} &= -2\left(\frac{A}{D}+a_1\right)\theta\mu_2 \\
\Delta q_{2,[1,1]} &= -2\left(\frac{A}{D}+a_1\right)\theta\mu_2+\frac{A}{D}\theta\mu_1
\end{aligned}\right\} \tag{31}
$$

$$\left.\begin{array}{c}
\Delta J_{1,[1,0]} \leq 0 \Rightarrow \mu_1 \geq -\frac{(A+B-b_1)\theta^{-1}}{\left(\frac{A}{D}\right)^2 + 2\frac{A}{D}(a_2-a_1) + 2a_1a_2} \\
\Delta J_{1,[0,1]} \leq 0 \Rightarrow \mu_2 \leq -\frac{(A+B-b_1)\theta^{-1}}{\left(\frac{A}{D}\right)^2 + \frac{A}{D}a_1} \\
\Delta J_{1,[1,1]} \geq \Delta J_{1,[1,0]} \Rightarrow (A+B-b_1) - \left(\frac{A}{D}\right)^2\theta\mu_1 + \left(\left(\frac{A}{D}\right)^2 + \frac{A}{D}a_1\right)\theta\mu_2 \geq 0 \\
\Delta J_{1,[1,1]} \geq \Delta J_{1,[0,1]} \Rightarrow (A+B-b_1)\left(-2\left(\frac{A}{D}+a_2\right)\right) + \left(-2\left(\frac{A}{D}\right)^3 - 4\left(\frac{A}{D}\right)^2 a_1 - 6\left(\frac{A}{D}\right)^2 a_2 - 4\frac{A}{D}a_2^2 - 8\frac{A}{D}a_1a_2 - 4a_1a_2^2\right)\theta\mu_1 - \left(\frac{A}{D}\right)^3\theta\mu_2 \geq 0 \\
\Delta J_{2,[1,0]} \geq 0 \Rightarrow \mu_1 \geq -\frac{(A+B-b_2)\theta^{-1}}{\left(\frac{A}{D}\right)^2 + \frac{A}{D}a_2} \\
\Delta J_{2,[0,1]} \geq 0 \Rightarrow \mu_2 \leq -\frac{(A+B-b_2)\theta^{-1}}{\left(\frac{A}{D}\right)^2 + 2\frac{A}{D}(a_2-a_1) + 2a_1a_2} \\
\Delta J_{2,[1,1]} \leq \Delta J_{2,[1,0]} \Rightarrow (A+B-b_2)\left(-2\left(\frac{A}{D}+a_1\right)\right) - \left(\frac{A}{D}\right)^3\theta\mu_1 + \left(-2\left(\frac{A}{D}\right)^3 - 6\left(\frac{A}{D}\right)^2 a_1 - 4\frac{A}{D}a_1^2 - 4\left(\frac{A}{D}\right)^2 a_2 - 8\frac{A}{D}a_1a_2 - 4a_1^2a_2\right)\theta\mu_2 \leq 0 \\
\Delta J_{2,[1,1]} \leq \Delta J_{2,[0,1]} \Rightarrow (A+B-b_2) - \left(\frac{A}{D}\right)^2\theta\mu_2 + \left(\left(\frac{A}{D}\right)^2 + \frac{A}{D}a_2\right)\theta\mu_1 \leq 0
\end{array}\right\} \tag{32}$$

Numerical Examples

Illustrative example – MIPG of cost

A simple game of the form defined by Eqs. (7)-(9) is presented which is solved as a MINLP to a Nash equilibrium solution satisfying both continuous game and discrete game equilibrium definitions. The implied finite game matrix is enumerated and presented for comparison with the MINLP solution matrix, and the ordinal game conditions are evaluated in order to verify that the game is indeed an ordinal cost game. Parameter values for this example problem are as follows: $A = 2$, $B = 8$, $D = 10$, $a_1 = 1$, $a_2 = 2$, $b_1 = 5$, $b_2 = 4$, $c_1 = 3$, $c_2 = 2$, and $\Delta b_1 = \Delta b_2 = -1$. The MINLP formulation is as in Eq. (33). This problem is indicative of two competing producers of a single product operating in a single Cournot-type market, with each producer having the option to implement a single upgrade, the cost of which is irrelevant to the qualitative interpretation of the result.

$$\begin{aligned}
\max_{\substack{q_1,q_2 \geq 0 \\ y_1,y_2 \in \{0,1\}}} \ & Z(q_1, q_2, y_1, y_2) \\
&= \left(A + B - (b_1 + y_1\Delta b_1)\right)q_1 \\
&+ \left(A + B - (b_2 + y_2\Delta b_2)\right)q_2 - \frac{A}{D}q_1q_2 - \left(\frac{A}{D} + a_1\right)q_1^2 \\
&- \left(\frac{A}{D} + a_2\right)q_2^2 - c_1 - c_2
\end{aligned} \tag{33}$$

As a pre-solve measure, the ordinal conditions corresponding to the game of cost in Eq. (26) are checked; it is determined that conditions 4 and 8 are not satisfied by the chosen parameter values in the given inequality case, and conditions 2, 5, 6, and 7 are all unsatisfied in the reverse-sign case. It is thus assured that the game is an ordinal potential game and that the solution to the MINLP formulation of the problem in Eq. (33) will be a PSNE. Solution of this problem yields an objective value of 9.787, corresponding to profit values of 5.723 and 3.963 to players 1 and 2 resulting from the PSNE strategy $(y_1, y_2) = (1,1)$ in which both players implement their upgrade.

Enumeration of all strategy sets gives the potential function matrix in Eq. (34), and the corresponding profit values in Eq. (35). Verification of the existence of a PSNE in these enumerated matrices is accomplished by finding that the potential maximum of 9.787 coincides with the profit matrix column (player 1) and row (player 2) maxima corresponding to the strategy $(1,1)$. The matrix of production volumes is presented in Eq. (36); it is noted that production changes due to an opposing player changing its strategy are small, and thus that the example presented is close to having an exact potential.

$$\begin{array}{ccc} Z & 0 & 1 \\ \begin{array}{c} 0 \\ 1 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} 5.511 & 7.107 \\ 8.192 & 9.787 \end{bmatrix}} \end{array} \tag{34}$$

$$\begin{array}{ccc} j & 0 & 1 \\ \begin{array}{c} 0 \\ 1 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} 3.055,2.383 & 3.043,3.980 \\ 5.737,2.369 & 5.723,3.963 \end{bmatrix}} \end{array} \tag{35}$$

$$\begin{array}{ccc} q & 0 & 1 \\ \begin{array}{c} 0 \\ 1 \end{array} & \multicolumn{2}{c}{\begin{bmatrix} 2.437,1.473 & 2.434,1.721 \\ 2.927,1.471 & 2.924,1.718 \end{bmatrix}} \end{array} \tag{36}$$

In this example, enumeration is a computationally feasible means of determining the PSNE, however, increasing cardinality of players' strategy sets results in the rapidly increasing size of the game matrix. Assuming a general game with R players each possessing $|\Xi_r|$ strategies (which shall be assumed to be of equal number Ξ) then the number of matrix elements is Ξ^R. To complicate this scaling, a strategy set must include all possible player decisions; thus in a set of players each having a set of potential upgrades U_r (where again it is assumed here that all $|U_r| = U$) each player will possess

$\Xi = 2^U$ strategies, corresponding to a matrix with 2^{UR} entries. Problems of meaningful size and scale quickly become inconvenient to solve by enumeration; the ability to solve MIPG problems as MINLPs takes advantage of decades of developments in branching algorithms to reduce the computational effort required to obtain a PSNE. The solutions to competitive expansion problems as Nash equilibria are competitively optimal and are desirable for their potential industrial applications.

Application example – competitive refinery expansion planning

We now turn to an application of MIPG modelling using an industrially motivated example in which two petroleum refiners compete in the same markets and seek to expand their facilities in order to establish a competitive advantage. This scenario is formulated as a static multiproduct multimarket game of the form in Tominac and Mahalec[4] with six products and two markets. Refiners have four upgrades available which may be purchased and implemented in any combination. These are: a hydrocracker processing upgrade, a hydrotreater processing upgrade, a CDU capacity upgrade, and a gasoline blender capacity upgrade. Upgrade costs are estimated based on data in the Refining Processes Handbook (2008) and Meyers (2004) Handbook of Petroleum Refining Processes. The complexity of this formulation prohibits the use of the ordinality conditions derived in this work; in part because the case of multiple products and markets have not been addressed in those derivations, and in addition because this example includes strategies in which multiple upgrades are selected, and such cases were not considered due to the additional algebraic complexity introduced. For these reasons, the refinery upgrade MINLP is solved using the MIPG potential game approach to find the PSNE, and the result is verified by enumeration. The patterns in the results lead to several conjectures regarding the existence of PSNE in MIPGs which will be formalized in the section that follows. The enumerated potential function values and player profit values are presented in Figure 1 as matrices in which refiner 1 is the row player and refiner 2 is the column player, which have been overlaid with heat maps for readability. Rows and columns are labeled with strategy numbers corresponding to the binary variable values in the legend included in the figure. Both refiners in this example possess that same upgrade options, and so identical upgrade labels are used for both players. The values in this figure represent optimal solutions to the enumerated cases, and are obtained using ANTIGONE 1.1 in GAMS 24.7.1 with the relative gap set to 1×10^{-9}, and warm started with DICOPT on a Dell Optiplex 9010 computer with Intel Core-i7-3770 CPU, a 3.40 GHz processor, and Windows 10 64-bit operating system. The model has 820 continuous and 16 binary variables with 695 equations following ANTIGONE preprocessing. Of these equations, 39 are nonconvex nonlinear and the remainder are linear. The model contains 130 nonlinear terms, 128 are bilinear and the remaining two are the sigmoidal cost functions. All cases are solved to global optimality with model and solver status both reported as 1 in all cases, indicating optimality and normal completion, respectively. The results obtained under the specified gap are taken to be sufficiently close to true global optimality for the purposes of this example. It can be verified that the profit matrices and potential matrix independently define the same pure strategy equilibrium. The PSNE occurs in the case that both players purchase all upgrades except the hydrocracker processing upgrade, corresponding to row and column eight in the matrices.

(A) Potential function values

	S01	S02	S03	S04	S05	S06	S07	S08	S09	S10	S11	S12	S13	S14	S15	S16
S01	-60	-61	-44	84	104	103	150	361	-67	-68	-51	77	97	96	143	358
S02	-56	-57	-40	87	109	108	154	364	-63	-64	-47	80	102	101	147	361
S03	-123	-124	-106	22	42	41	88	299	-130	-131	-113	15	35	34	81	296
S04	25	24	28	128	190	189	222	401	18	17	21	121	183	182	215	395
S05	134	133	150	278	298	297	345	555	127	126	143	271	291	290	338	552
S06	134	133	151	279	299	298	345	556	127	126	144	272	292	291	338	552
S07	72	71	88	217	237	236	283	494	65	64	81	210	230	229	276	490
S08	305	304	306	402	471	470	501	675	298	297	299	395	464	463	494	668
S09	-67	-68	-51	77	97	96	143	354	-74	-75	-58	70	90	89	136	351
S10	-63	-64	-47	80	102	101	147	357	-70	-71	-54	73	95	94	140	354
S11	-130	-131	-113	15	35	34	81	292	-137	-138	-120	8	28	27	74	289
S12	18	17	21	121	183	182	215	394	11	10	14	114	176	175	208	388
S13	127	126	143	271	291	290	338	548	120	119	136	264	284	283	331	545
S14	127	126	144	272	292	291	338	549	120	119	137	265	285	284	331	545
S15	65	64	81	210	230	229	276	487	58	57	74	203	223	222	269	483
S16	321	320	317	401	487	486	512	672	314	313	310	394	480	479	505	666

(B) Refiner 1 profit values

	S01	S02	S03	S04	S05	S06	S07	S08	S09	S10	S11	S12	S13	S14	S15	S16
S01	-265	-265	-252	-239	-265	-265	-252	-238	-265	-265	-252	-239	-265	-265	-252	-238
S02	-264	-264	-249	-237	-264	-264	-249	-236	-264	-264	-249	-237	-264	-264	-249	-235
S03	-327	-327	-314	-301	-327	-327	-314	-300	-327	-327	-314	-301	-327	-327	-314	-300
S04	-202	-202	-203	-215	-203	-203	-203	-217	-202	-202	-203	-215	-203	-203	-203	-218
S05	-70	-70	-57	-45	-70	-70	-57	-44	-70	-70	-57	-45	-70	-70	-57	-43
S06	-71	-71	-57	-45	-71	-71	-57	-44	-71	-71	-57	-45	-71	-71	-57	-43
S07	-132	-132	-119	-107	-132	-132	-119	-106	-132	-132	-119	-107	-132	-132	-119	-105
S08	76	76	73	58	75	75	73	55	76	76	73	58	75	75	73	53
S09	-272	-272	-259	-246	-272	-272	-259	-245	-272	-272	-259	-246	-272	-272	-259	-245
S10	-271	-271	-256	-244	-271	-271	-256	-243	-271	-271	-256	-244	-271	-271	-256	-242
S11	-334	-334	-321	-308	-334	-334	-321	-307	-334	-334	-321	-308	-334	-334	-321	-307
S12	-209	-209	-210	-222	-210	-210	-210	-224	-209	-209	-210	-222	-210	-210	-210	-225
S13	-77	-77	-64	-52	-77	-77	-64	-51	-77	-77	-64	-52	-77	-77	-64	-50
S14	-78	-78	-64	-52	-78	-78	-64	-51	-78	-78	-64	-52	-78	-78	-64	-50
S15	-139	-139	-126	-114	-139	-139	-126	-113	-139	-139	-126	-114	-139	-139	-126	-112
S16	83	83	75	51	83	83	75	48	83	83	75	51	83	83	75	48

(C) Refiner 2 profit values

	S01	S02	S03	S04	S05	S06	S07	S08	S09	S10	S11	S12	S13	S14	S15	S16
S01	-287	-288	-288	-183	-122	-123	-94	91	-294	-295	-295	-190	-129	-130	-101	86
S02	-283	-284	-288	-183	-118	-119	-93	91	-290	-291	-295	-190	-125	-126	-100	85
S03	-287	-288	-288	-183	-122	-123	-94	91	-294	-295	-295	-190	-129	-130	-101	86
S04	-268	-269	-276	-195	-101	-102	-81	76	-275	-276	-283	-202	-108	-109	-88	69
S05	-287	-288	-288	-183	-122	-123	-94	91	-294	-295	-295	-190	-129	-130	-101	86
S06	-286	-287	-288	-183	-122	-123	-94	91	-293	-294	-295	-190	-129	-130	-101	86
S07	-287	-288	-288	-183	-122	-123	-94	91	-294	-295	-295	-190	-129	-130	-101	86
S08	-267	-268	-275	-197	-100	-101	-80	73	-274	-275	-282	-204	-107	-108	-87	66
S09	-287	-288	-288	-183	-122	-123	-94	91	-294	-295	-295	-190	-129	-130	-101	86
S10	-283	-284	-288	-183	-118	-119	-93	91	-290	-291	-295	-190	-125	-126	-100	85
S11	-287	-288	-288	-183	-122	-123	-94	91	-294	-295	-295	-190	-129	-130	-101	86
S12	-268	-269	-276	-195	-101	-102	-81	76	-275	-276	-283	-202	-108	-109	-88	69
S13	-287	-288	-288	-183	-122	-123	-94	91	-294	-295	-295	-190	-129	-130	-101	86
S14	-286	-287	-288	-183	-122	-123	-94	91	-293	-294	-295	-190	-129	-130	-101	86
S15	-287	-288	-288	-183	-122	-123	-94	91	-294	-295	-295	-190	-129	-130	-101	86
S16	-261	-262	-272	-204	-95	-96	-78	67	-268	-269	-279	-211	-102	-103	-85	60

(D) Legend of strategies

S01	0	0	0	0	S09	1	0	0	0
S02	0	0	0	1	S10	1	0	0	1
S03	0	0	1	0	S11	1	0	1	0
S04	0	0	1	1	S12	1	0	1	1
S05	0	1	0	0	S13	1	1	0	0
S06	0	1	0	1	S14	1	1	0	1
S07	0	1	1	0	S15	1	1	1	0
S08	0	1	1	1	S16	1	1	1	1

Figure 1. Potential and profit values in refining example MIPG

The value in the MIPG approach to competitive expansion planning problems lies in computational efficiency; by solving finite game problems as MINLPs branching algorithms dictate the number of matrix entries which must be calculated, which in the worst case results in complete enumeration. In addition, the existence of MIPG implies that the complete reward matrix is not required in order to determine whether the outcome of a game will be a PSNE; as long as the ordinality conditions are satisfied it is assured that the outcome will be a PSNE and this solution can be obtained using standard mixed integer nonlinear optimization approaches. This result is novel at time of writing; no other game-type problems of mixed integer and continuous decision variables have been solved satisfying both continuous and discrete definitions of Nash equilibria, and this example represents the first practical MIPG solved to equilibrium by optimization. While a single example is presented, no parameter combinations tested – including those with unreasonably large, small, or negative values – defining a feasible game was observed to yield a matrix without a pure strategy equilibrium.

Conjectures

We have presented algebraic conditions defining the existence of mixed integer games of two players possessing two strategies. Analysis of these games is nontrivial even in the two-by-two case, and the algebraic analysis required scales poorly as the dimension of the game in question increases both in strategy set cardinality and in the number of interested players. Based on the analytical and numerical results obtained, and because we have not been able to conclusively prove the existence of the observed results, we offer the following conjectures regarding the existence and properties of mixed integer games based on continuous Cournot potential games:

1) MIPG games exist possessing PSNE,
2) The existence of PSNE in a MIPG is determined by parameters associated with the continuous game which define the optima to each strategic realization of the implied finite game which collectively define the implied finite game payoff matrix,

3) Conjectures 1 and 2 generalize to games of arbitrary strategic size and number of players,
4) In the event it exists, a PSNE is obtained as the solution to a mixed integer nonlinear program; enumeration of the implied finite game payoff matrix is not required to achieve such equilibria.

MIPG are interesting and relevant problems, and proof of the conjectures made in this work merits additional research. In particular, the ordinal conditions defining when a MIPG possesses a PSNE must be generalized for cases of arbitrary numbers of players and player strategies, including cases in which multiple finite changes are applied in a single strategy (i.e., the multiple active upgrade case addressed in the refinery problem) and influence the objective value of the game. In addition, it may be possible to prove that the ordinality conditions guarantee the existence of PSNE in MIPGs under reasonable assumptions. The conditions that have been presented in this work are useful as metrics, but they have not been resolved in such a way as to make generalized guarantees with respect to game parameters; our conditions are useful only in specific instances of a game to determine whether a PSNE exists.

Conclusions

We have presented conditions defining parameter instances of two-player two-strategy Cournot based mixed integer ordinal potential games which guarantee that the defined game will possess a pure strategy Nash equilibrium for two types of discrete strategies which have been interpreted as upgrades in a competitive refinery capacity expansion and production planning problem. Based on results presented in which games of higher dimension possessed PSNE, in particular a refinery expansion model in which two refiners each possess sixteen strategies, we conjecture that the properties we have defined for small MIPGs extend to games of higher dimension. The MIPG as a framework for industrially motivated competitive problems is useful and provides insight into organizational level planning; such a framework allows planning problems to be cast across the height and breadth of an organization, optimizing both high and low-level decisions of both discrete and continuous types. MIPGs merit additional research, in part to lay out their theoretical properties, and to explore their applications and utility in difficult industrial problems where competitive interactions interplay with technological decisions.

Notation

Sets

R	(r) refiners or players
Ξ_R	(ξ_r) strategy set for refiner r

Parameters

A	Price decline rate
B	Price of product p in market w corresponding to supply of exactly D_{pw}
D	Nominal market supply of product p in market w
$\bar{q}_r$	Maximum production rate by refiner r
a_r	Refiner r quadratic cost term
b_r	Refiner r linear cost term
c_r	Refiner r constant cost term

Continuous Variables

J_r	Profit function for refiner r
j_r	A profit value for refiner r; used to indicate scalar values
π	Price
q_r	Product volume produced by refiner r
C_r	General cost function of refiner r
Z	Potential function value
L	Lagrangian function value

μ_r Lagrange multiplier

Binary Variables

y_r Refiner r decision to purchase an upgrade

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Acknowledgements

This work has been funded by the Natural Sciences and Engineering Research Council of Canada, the Government of Ontario, and the McMaster Advanced Control Consortium whom we thank for their support.

Chapter 5

Conclusions and recommendations

Conclusions

This thesis makes significant progress towards the development of frameworks for the analysis of competitive behaviour in strategic production planning. The game theoretic concept of the potential function has a natural synergy with mathematical formulations of process systems, effectively allowing economic models to build directly onto complex process models and to have the entire program optimized under a single objective function. The static and dynamic competitive models presented in this thesis take strides in the direction of creating enterprise optimization models in which both enterprise level objectives and detailed process constraints can be included in a single program. In addition, a new class of game theoretic problems has been identified in the mixed integer game, and progress has been made in characterizing these games.

Enterprise optimization

The competitive production planning frameworks developed in Chapters 2 and 3, and the expansion planning framework of Chapter 4 represent means by which the combined goals of profit maximization, cost minimization, and process optimization may be incorporated into a single mathematical program optimizing an entire enterprise. Process details from the operational to the strategic can be incorporated with models of markets and delivery channels, including competitors in those channels, into a single problem which has as its objective the economic success of the enterprise and a rational, game theoretic objective by which to achieve it. The Cournot potential game objective used in this work is based on the fundamental link between the behaviours of market price and supply. The refinery models allow rigorous engineering models to regulate production, and offer insight to competitive and mechanical limitations in process operations. The primary limitation to the achievement of this objective is the size and scaling of such models, the solutions to which are limited by available computation power.

Mixed integer games

The class of mixed integer games is of interest from an academic perspective, representing the game theoretic analogue to the mixed integer nonlinear program. Of primary interest is the result determined in Chapter 4: that enumeration of all payoffs is not required to determine a Nash equilibrium. A knowledge of the mathematical nature of the game in question and the parameters which define the form of the solution is sufficient. Determination of the conditions under which an equilibrium in pure strategies is guaranteed to exist is a challenging problem becoming significantly more difficult with increasing problem sizes. Nevertheless, conjectures have been made regarding the form of these problems and the suspected features of the class of games if it can be defined.

Concluding thoughts

The driving force initiating this thesis was a suspicion that existing production planning techniques offer suboptimal recommendations since the influence of competitors on strategic decision making is ignored. The work in this thesis demonstrates that by accounting for the presence of industrial opponents enterprise strategy is significantly altered. In many ways the results presented are reflective of a more intuitive planning solution. Game theoretic production planning avoids some of the common pitfalls of standard planning approaches without the requirement of additional constraints; product portfolios are diversified according to the behaviour of market prices and demands, and take into account the optimal behaviour of competitors, meaning that the resulting plans are conservative rather than being best case scenarios. The computation results achieved in these frameworks provide logical reasons for intuitive behaviours, bringing computational solutions closer to human solutions.

Recommendations for further work

The work in this thesis opens up several avenues for investigation into game theoretic solutions for process engineering problems. Some of these are based on validating assumptions made in the modelling process, while others represent novel lines of inquiry that extend the work. In addition, the opportunity to explore the class of mixed integer games is an interesting one, especially if it is possible to mathematically prove the existence or properties of either specific mixed integer games or for the class as a whole. With consideration of these options, three research projects are recommended.

Market entry and denial

A primary assumption in the production planning framework is that refiners outside a domestic market cannot sell product there; this has been justified in several ways, but rather than work with this assumption, it can be reformulated into a new research question, namely: is it viable for a refiner outside a domestic market to attempt to enter that domestic market by constructing refining resources there? This question should be posed as a game theoretic market entry problem factoring in rigorous models of refinery building costs and construction duration, and consider the options possessed by domestic refiners that might be pursued in order to discourage such entries, such as price management, or in keeping with the work in Chapter 3 and 4, whether domestic refiners can initiate counter upgrade procedures to make market entry potentially unprofitable for the outsider. The problem has industrial significance and further links enterprise management decisions with operational constraints.

Uncertainty and Bayesian games

A particular problem in the game theoretic framework is the assumption that competitors possess perfect knowledge of each other and can plan accordingly. The Bayesian game structure allows uncertainty of opposing players' strategies to be included in the framework, and should have a form similar to that of a stochastic program. Bayesian game structure is such that opponents have beliefs about one another and formulate their own responses based on the distribution of those beliefs. Thus a scenario-based formulation of a refinery production planning problem could be cast in which each refiner has associated with it opponents beliefs as a set of discrete cases; the solution to such a model is a Bayesian Nash equilibrium and will be indicative of the optimal strategy under uncertainty.

Extensions to mixed integer games

This thesis includes a demonstration of the existence of mixed integer games, but no proof that the structures behave in the way that they appear to. A mathematical exercise undertaken to prove conclusively that games of the form in Chapter 4 are either guaranteed to possess a PSNE in all cases, or else are guaranteed to possess such an equilibrium under a defined set of conditions would represent a large step towards the understanding of this new class of games. This project is perhaps the most difficult of the three, but is of significant academic interest.

Additional Considerations

Other factors which might be addressed in future work could contribute to the realism of the market models. Tariffs and transportation costs are relatively simple to implement, and introduce geographical dependencies into the market models. Tariffs and transportation costs both impact the price realized by refiners in the models. Transport costs give refiners geographically near their markets an asymmetric advantage. Tariffs would decrease prices in specific global markets for certain domestic refiners. The interplay of such factors with the motivation of profit will differentiate refiners based on geographical location, even if such refiners are identical in capacity and cost structure.

The game theoretic concept of the Stackelberg game has largely been avoided in this thesis under the assumption that refinery behaviours are visible on the requisite time scales and are thus adequately modelled using Nash equilibrium concepts; i.e., refiners can observe and react to changes in competitors' production, and similarly can observe and respond to upgrade plans. The Stackelberg game provides a useful solution concept when this assumption is not true, and could provide insights into elements of refinery interactions and competition. In the Stackelberg optimization framework, one player moves first and is able to maximize its objective directly while its competitors are constrained to Nash equilibrium strategies. By analogy, such a game could be solved in the frameworks proposed in this thesis by maximizing the first-mover's profit, with all other players subject to the KKT conditions of a corresponding Nash Cournot potential game. The development of such KKT conditions presents the primary boundary to such pursuits.

Appendix

Supplementary materials

This section contains the supplementary materials included for online publication with the manuscripts in Chapters 2 and 4. Lists of figure and tables included in the online document have been omitted here.

Supplementary material to: A game theoretic framework for petroleum refinery strategic production planning

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A. The Cournot oligopoly in brief

The Cournot oligopoly is a classic economic model used to examine market competition, which we use to structure game theoretic interactions between competitors. The Cournot oligopoly defines a game in which a set N of producers of a good each must decide how much of that good to sell to a market[21,40]. The realized price for the good is a function of the collective amount the players deliver to the market. This scenario results in a game in which each player's only strategic decision is a production volume. We focus on a static Cournot game assuming complete information and homogeneous product[21].

Each market participant attempts to maximize its profits according to a function $J_n = Pr \cdot q_n - c_n(q_n)$ where Pr represents market price, q_n is the amount supplied by producer n, each of whom has a production cost $c_n(q_n)$ which is some function of its production level. The solution to this game theoretic model is a Nash equilibrium in terms of the quantities of product q_n that each of the N producers supply. The market price Pr is a function of q_n and the most common interpretation is that in Eq. (A1).

$$Pr = \begin{cases} A - \sum_n q_n & \sum_n q_n < A \\ 0 & \sum_n q_n \geq A \end{cases} \tag{A1}$$

A is a parameter indicative of the marginal value of the first unit sold on the market. This interpretation is presented in similar form in[21,32,40]. Player objective functions can be rewritten as functions of total production of the form in Eq. (A2).

$$J_n = (A - \textstyle\sum_{n'} q_{n'})q_n - c_n(q_n) \tag{A2}$$

The Nash equilibrium in terms of q_n for this oligopoly problem is defined by the solution to the set of best response equations in Eq. (A3).

$$\frac{\partial J_n}{\partial q_n} = 0 \quad \forall n \in N \tag{A3}$$

The Cournot oligopoly presented here is a potential game[33]. The corresponding potential function has the form in Eq. (A4).

$$Z = \sum_n \big(A \cdot q_n - q_n^2 - c_n(q_n)\big) - \sum_n \sum_{\substack{n' \\ n'<n}} (q_n q_{n'}) \tag{A4}$$

B. Refinery production planning model

The purpose of the production planning model equations is to determine the volumes of products that the refinery should produce in order to satisfy the model objective and what crude oil stocks, intermediate products, and blending strategies must be used in order to satisfy quality constraints associated with each product. Flow of material between process units is defined based on inclusion of set elements in equation definitions. The total volumetric inlet flow to each unit in the refinery is defined by Eq. (B1). Inlet flow is broken down by mode for those units which have multiple operating modes in Eq. (B2). The minimum and maximum total inlet flows into a unit are defined by Eqs. (B3) and (B4).

$$FA(t,u,n) = \sum_{\substack{m \\ m \in UM}} FVM(m,t,u,n) \quad \forall t \in T, u \in U, n \in N \tag{B1}$$

$$FVM(m,t,u,n) = \sum_{\substack{i \\ i \in U_{IN}}} FVM_{IN}(m,t,i,n) \quad \forall t \in T, n \in N, (u,m) \in UM \tag{B2}$$

$$FA(t,u,n) \geq MinProd(u,n)\tau(t) \quad \forall t \in T, u \in U_C, n \in N \tag{B3}$$

$$FA(t,u,n) \leq MaxProd(u,n)\tau(t) \quad \forall t \in T, u \in U_C, n \in N \tag{B4}$$

Volumetric flow rates of streams exiting a unit are defined using a similar set of equations. Streams entering a unit have a corresponding stream or streams leaving that unit which are defined by specific yield values. Yield relationships are governed by Eq. (B5). The total volume leaving each unit is defined by Eq. (B6).

$$FVM_{OUT}(i,m,t,u,n) = Y(i,m,u)FVM(m,t,u,n) \\ \forall t \in T, n \in N, (i,u,m) \in UM_{OUT,C} \tag{B5}$$

$$FA(t,u,n) = \sum_{i \in U_{OUT}} FV(t,i,n) \quad \forall t \in T, u \in U_C, n \in N \tag{B6}$$

Unit outlet and inlet volumetric flow rates are calculated on a stream basis using Eqs. (B7) and (B8).

$$FV(t,i,n) = \sum_{m \in UM} FVM_{OUT}(i,m,t,u,n) \quad \forall t \in T, n \in N, (i,u) \in U_{OUT} \tag{B7}$$

$$FV(t,i,n) = \sum_{m \in UM} FVM_{IN}(m,t,i,n) \quad \forall t \in T, n \in N, (i,u) \in U_{IN} \tag{B8}$$

The CDU uses Eq. (B9) to compute intermediate yields based on the crude streams entering the unit.

$$FVM_{OUT}(i,m,t,u,n) = \sum_{i' \in U_{IN}} X(i,m,i')FVM_{IN}(m,t,i',n) \\ \forall m \in M, i \in I, t \in T, n \in N, u = CDU \tag{B9}$$

The holdings of refining intermediates are defined by balance equations around the intermediate tanks and the initial tank content in Eqs. (B10) and (B11), and by constraints which maintain the tank level between its maximum and minimum values in Eqs. (B12) and (B13).

$$\sum_{i \in TK_{IN}} FV(t,i,n) - \sum_{i \in TK_{OUT}} FV(t,i,n) + V_{ini}(tk,n) - V(tk,t,n) = 0 \\ \forall tk \in TK, t = 1, n \in N \tag{B10}$$

$$\sum_{i \in TK_{IN}} FV(t,i,n) - \sum_{i \in TK_{OUT}} FV(t,i,n) + V(tk,t-1,n) - V(tk,t,n) = 0 \quad \forall tk \in TK, t > 1, n \in N \tag{B11}$$

$$V(tk,t,n) \geq \underline{V}(tk) \quad \forall tk \in TK, t \in T, n \in N \tag{B12}$$

$$V(tk,t,n) \leq \overline{V}(tk) \quad \forall tk \in TK, t \in T, n \in N \tag{B13}$$

The process of blending refining intermediates into products is governed by a number of equations and constraints which dictate blend volumes and quality specifications. The volume of a stream to be blended into a particular product is defined by Eq. (B14). The blended volume of a product is defined by Eq. (B15).

$$\sum_{p \in BL_{OUT}} VB(i,p,t,n) = FV(t,i,n) \quad \forall t \in T, n \in N, (bl,i) \in BL_{IN} \tag{B14}$$

$$\sum_{i \in BL_{IN}} VB(i,p,t,n) = VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,p) \in BL_{OUT} \tag{B15}$$

The minimum and maximum fractions of an intermediate allowed in the blending of a product are defined by Eqs. (B16) and (B17).

$$VB(i,p,t,n) \geq RMin(bl)VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,i,p) \in BLIP \tag{B16}$$

$$VB(i,p,t,n) \leq RMax(bl)VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,i,p) \in BLIP \tag{B17}$$

The total volume processed in a blender is defined by Eq. (B18). This volume must be within the lower and upper capacity values for each blender, reflected through the constraints in Eqs. (B19) and (B20).

$$\sum_{(i,p) \in BLIP} VB(i,p,t,n) = VBlendT(t,bl,n) \quad \forall bl \in BL, t \in T, n \in N \tag{B18}$$

$$VBlendT(t,bl,n) \geq BlendMin(bl)\tau(t) \quad \forall bl \in BL, t \in T, n \in N \tag{B19}$$

$$VBlendT(t,bl,n) \leq BlendMax(bl)\tau(t) \quad \forall bl \in BLt \in T, n \in N \tag{B20}$$

Quality properties are divided into three groups: properties based on volume, based on weight, and based on nonlinear relationships. The upper and lower bounds for each property are defined by Eqs. (B21) to (B26).

$$\sum_{i \in BL_{IN}} VB(i,p,t,n) qq(i,q) \geq \underline{Q}(q,p) VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,p,q) \in BL_{OUT,VOL} \tag{B21}$$

$$\sum_{i \in BL_{IN}} VB(i,p,t,n) qq(i,q) \leq \overline{Q}(q,p) VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,p,q) \in BL_{OUT,VOL} \tag{B22}$$

$$\sum_{i \in BL_{IN}} VB(i,p,t,n) qq(i,q) qq(i,q') \geq \underline{Q}(q,p) \sum_{i \in BL_{IN}} VB(i,p,t,n) qq(i,q') \quad \forall t \in T, n \in N, q' = SG, (bl,p,q) \in BL_{OUT,WT} \tag{B23}$$

$$\sum_{i \in BL_{IN}} VB(i,p,t,n) qq(i,q) qq(i,q') \leq \overline{Q}(q,p) \sum_{i \in BL_{IN}} VB(i,p,t,n) qq(i,q') \quad \forall t \in T, n \in N, q' = SG, (bl,p,q) \in BL_{OUT,WT} \tag{B24}$$

$$\sum_{i \in BL_{IN}} VB(i,p,t,n) qq(i,q)^{1.25} \geq \underline{Q}(q,p)^{1.25} VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,p,q) \in BL_{OUT,NL} \tag{B25}$$

$$\sum_{i \in BL_{IN}} VB(i,p,t,n) qq(i,q)^{1.25} \leq \overline{Q}(q,p)^{1.25} VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,p,q) \in BL_{OUT,NL} \tag{B26}$$

The products produced by blending are either stored in product tanks or delivered to a market for sale. The product tank balances for the initial tank condition and for subsequent time periods take the form of Eqs. (B27) and(B28). The maximum and minimum product tank levels are defined by Eqs. (B29) and (B30).

$$VBlend(t,p,n) + VP_{ini}(p) - VP(t,p,n) - Prc(t,p,n) = 0 \quad \forall t = 1, p \in P, n \in N \tag{B27}$$

$$VBlend(t,p,n) + VP(t-1,p,n) - VP(t,p,n) - Prc(t,p,n) = 0 \quad \forall t > 1, p \in P, n \in N \tag{B28}$$

$$VP(t,p,n) \geq \underline{VP}(p) \quad \forall t \in T, p \in P, n \in N \tag{B29}$$

$$VP(t,p,n) \leq \overline{VP}(p) \quad \forall t \in T, p \in P, n \in N \tag{B30}$$

At the end of the planning horizon all tank levels should return to their minimum levels. Equations (B31) and (B32) enforce this constraint for the sets of intermediate and product tanks.

$$V(tk,t,n) = \underline{V}(tk) \quad \forall tk \in TK, t = T, n \in N \tag{B31}$$

$$VP(t,p,n) = \underline{VP}(p) \quad \forall t = T, p \in P, n \in N \tag{B32}$$

The total amount of each product produced by a refiner is delivered to a market for sale. Eq. (B33) defines the balance between the products produced and those delivered to a market.

$$Prc(t,p,n) - \sum_w Dlv(t,p,n,w) = 0 \quad \forall t \in T, p \in P, n \in N \tag{B33}$$

The costs of crude oil, unit operation, and blender operation are defined by Eqs. (B34), (B35), and (B36), respectively.

$$CrudeOilCost(n) = \sum_t \left[(1 + 0.01t) \sum_{\substack{i \\ i \in IC}} Cost(i) FV(t,i,n)\right] \quad \forall n \in N \tag{B34}$$

$$UnitOpCost(n) = \sum_{m,n,u} OpCost(u,m) FVM(m,t,u,n) \quad \forall n \in N \tag{B35}$$

$$BlendOpCost(n) = \sum_{bl,n} BLcost(bl)\, VBlendT(t,bl,n) \quad \forall n \in N \tag{B36}$$

Refiners are able to import products from another seller located elsewhere whose prices are fixed at values of $CI(p,w)$ for refiners in domestic markets. Buyers in domestic and global markets do not have access to this purchasing channel; refiners may purchase imports at a price $CI(p,w)$ and sell them in their domestic market at the market price $\Pr(p,w)$. Imports cannot be sold in global markets and are limited to an amount of

1.589×10^6 m^3 per year of each product by each refiner as a reasonable upper limit. The cost of imports incurred by a refiner is defined by Eq. (B37).

$$ImpCost(n) = \sum_{p,w} CI(p,w)\, Imp(p,n,w) \quad \forall n \in N \tag{B37}$$

Refiners also incur time-based costs which are calculated based on the total amount produced in a given time period and which decrease in each subsequent time period in the planning horizon. Eq. (B38) defines this cost value which serves, all else being equal, to make production near the end of the planning horizon more efficient.

$$TimeCost(n) = TC \sum_{t,p} (1 - 0.01 \cdot t) Produce(t,p,n) \quad \forall n \in N \tag{B38}$$

For convenience of equation writing we define the variable $TotCost(n)$ as in Eq. (B39).

$$TotCost(n) = CrudeOilCost(n) + UnitOpCost(n) + BlendOpCost(n) + TEC(n) + ImpCost(n) + TimeCost(n)\ \forall n \in N \tag{B39}$$

Deliveries to global markets are unrestricted and are driven purely by competition, but deliveries to domestic markets by the refiners situated in those markets face contracts stipulating that neither too low a supply of any one product, nor more than the market can absorb, be collectively produced. In domestic markets the collective supply from domestic refiners is constrained to fall within upper and lower bounds. Since the model is formulated as a deterministic static game refiners are capable of making competitive plays guaranteed to satisfy these constraints, which take the form in Eqs. (B40) and(B41).

$$\sum_{t,n} Dlv(t,p,n,w) \geq \underline{D}(p,w) \quad \forall p \in P, w \in W_L \tag{B40}$$

$$\sum_{t,n} Dlv(t,p,n,w) \leq \overline{D}(p,w) \quad \forall p \in P, w \in W_L \tag{B41}$$

C. Table of set elements and indices

Set	Indices	Elements
BL	(bl)	GB, DB
I	(i)	crude1, crude2, crude3, lpg, srln, srhn, kero, lgo, hgo, rsd, rft, srds, hclf, hchf, hcln, hckero, hcds, hchn, fccf, fccln, fcchn, fcclco, fcchco, srln_tk, rft_tk, hcln_tk, fccln_tk, fcchn_tk, srds_tk, hcds_tk, fcclco_tk
IC	(i)	crude1, crude2, crude3
M	(m)	1, 2
N	(n)	R1, R2, R3
P	(p)	REG, MID, PRE, DE1, DE2, DE4
Q	(q)	RON, MON, ARO, FLS, CNU, SUL, SG, RVP
W	(w)	LM1, EM1
T	(t)	1, 2
TK	(tk)	tk1, tk2. Tk3, tk4, tk5, tk6, tk7, tk8
U	(u)	CDU, CR, HC, FCC, HT1, HT2
UPG	(upg)	uHCproc, uHTproc, uCDUcap, uCRcap, uHCcap, uFCCcap, uGBcap, uDBcap
BL_{BLEND}	(i)	(GB).(srln_tk, rft_tk, hcln_tk, fccln_tk, fcchn_tk), (DB).(srds_tk, hcds_tk, fcclco_tk)
BL_{IN}	(bl, i)	srln_tk, rft_tk, hcln_tk, fccln_tk, fcchn_tk, srds_tk, hcds_tk, fcclco_tk
BL_{OUT}	(bl, p)	(GB).(REG, MID, PRE), (DB).(DE1, DE2, DE4)
$BL_{OUT,VOL}$	(bl, p, q)	(GB).(REG, MID, PRE).(RON, MON, ARO, SG), (DB).(DE1, DE2, DE4).(FLS, CNU, SG)
$BL_{OUT,WT}$	(bl, p, q)	(GB).(REG, MID, PRE).(RVP)
$BL_{OUT,NL}$	(bl, p, q)	(DB).(DE1, DE2, DE4).(SUL)
$BLIP$	(bl, i, p)	$BL_{IN} \cdot BL_{OUT}$

DWN	(p', p)	(REG).(REG), (MID).(REG, MID), (PRE).(REG, MID, PRE), (DE1).(DE1), (DE2).(DE1, DE2), (DE4).(DE1, DE2, DE4)
LCN	(n)	R1, R2
HCN	(n)	R3
$NPrest$	(n, p)	(R1, R2, R3).(REG)
P_g	(p)	REG, MID, PRE
P_d	(p)	DE1, DE2, DE4
Q_g	(q)	RON, MON, ARO, SG, RVP
Q_d	(q)	FLS, CNU, SUL, SG
Q_{VOL}	(q)	RON, MON, ARO, FLS, CNU, SG, RVP
Q_{WT}	(q)	SUL
W_E	(w)	EM1
W_L	(w)	LM1
WLN	(n, w)	(R1, R2, R3).(LM1)
WLE	(w, w')	(LM1).(EM1)
WN	(n, w)	(R1, R2, R3).(LM1, EM1)
$WLCN$	(n, w)	(R1, R2).(LM1)
$WHCN$	(n, w)	(R3).(LM1)
TK_{IN}	(tk, i)	(tk1).(srln), (tk2).(rft), (tk3).(hcln), (tk4).(fccln), (tk5).(fcchn), (tk6).(srds), (tk7).(hcds), (tk8).(fcclco)
TK_{OUT}	(tk, i)	(tk1).(srln_tk), (tk2).(rft_tk), (tk3).(hcln_tk), (tk4).(fccln_tk), (tk5).(fcchn_tk), (tk6).(srds_tk), (tk7).(hcds_tk), (tk8).(fcclco_tk)
U_{IN}	(u, i)	(CDU).(crude1, crude2, crude3), (CR).(srhn, hchn), (HC).(hclf, hchf), (FCC).(fccf), (HT1).(lgo), (HT2).(hgo)
U_{OUT}	(u, i)	(CDU).(lpg, srln, srhn, kero, lgo, hgo, rsd), (CR).(rft), (HC).(hcln, hchn, hckero, hcds), (FCC).(fccln, fcchn,

		fcclco, fcchco), (HT1).(srds, hclf), (HT2).(hchf, fccf)
U_C	(u)	CDU, CR, HC, FCC
UM	(u,m)	(CDU, CR, HC, FCC).(1, 2),(HT1, HT2).(1)
UM_C	(u,m)	(CR, HC, FCC).(1, 2), (HT1).(1)
$UM_{OUT,C}$	(i,u,m)	$U_{OUT} \cdot UM_C$
$ProcUp$	(upg)	uHCproc, uHTproc
$CapUp$	(upg)	uCDUcap, uCRcap, uHCcap, uFCCcap, uGBcap, uDBcap
SUD	(i,upg)	(crude2).(uHTproc), (crude3).(uHCproc, uHTproc)
UUD	(u,upg)	(CDU.uCDUcap), (CR.uCRcap), (HC.uHCcap), (FCC.uFCCcap)
BUD	(bl,upg)	(GB).(uGBcap), (DB).(uDBcap)

D. Tables of parameter values

Table 1. $A(p,w)$ (CAD/m^3)

w	p					
	REG	MID	PRE	DE1	DE2	DE4
LM1	245.30	226.43	232.72	163.54	163.54	163.54
EM1	163.54	150.96	157.24	113.22	113.22	113.22

Table 2. $AHC(p,w)$ (CAD/m^3)

w	Scale factor	p					
		REG	MID	PRE	DE1	DE2	DE4
LM1	5%	4.04	3.82	3.90	2.76	2.76	2.76
LM1	15%	12.13	11.46	11.71	8.29	8.29	8.29
LM1	25%	20.21	19.11	19.52	13.81	13.81	13.81
LM1	35%	28.30	26.75	27.33	19.34	19.34	19.34
LM1	45%	36.38	34.39	35.14	24.87	24.87	24.87
LM1	55%	44.47	42.04	42.94	30.39	30.39	30.39

Table 3. $\boldsymbol{B(p,w)}$ **(CAD/m^3)**

w	p					
	REG	MID	PRE	DE1	DE2	DE4
LM1	886.86	956.05	1006.37	1075.56	1018.95	962.34
EM1	886.86	956.05	1006.37	1075.56	1018.95	962.34

Table 4. $\boldsymbol{CI(p)}$ **(CAD/m^3)**

p					
REG	MID	PRE	DE1	DE2	DE4
1065.51	1144.89	1205.63	1287.38	1221.07	1154.76

Table 5. $\boldsymbol{D(p,w)}$ **(10^6 m^3/year)**

w	p					
	REG	MID	PRE	DE1	DE2	DE4
LM1	6.38	0.48	0.71	0.08	3.19	0.17
EM1	7.45	0.56	0.83	0.10	3.73	0.20

Table 6. $\boldsymbol{\underline{D}(p,w)}$ **(10^6 m^3/year)**

w	p					
	REG	MID	PRE	DE1	DE2	DE4
LM1	5.75	0.43	0.64	0.07	2.87	0.16

Table 7. $\boldsymbol{\overline{D}(p,w)}$ **(10^6 m^3/year)**

w	p					
	REG	MID	PRE	DE1	DE2	DE4
LM1	7.66	0.57	0.86	0.10	3.83	0.21

Table 8. $F(p, w)$ (CAD/m^3)

w	p					
	REG	MID	PRE	DE1	DE2	DE4
LM1	1132.16	1182.48	1239.09	1239.09	1182.48	1125.87
EM1	1050.40	1107.00	1163.61	1182.48	1125.87	1075.56

Table 9. $HCNset$ (10^6 m^3/year)

n	
R3	2.86

Table 10. Efficiency cost curve parameters

	n		
	R1	R2	R3
$EC_H(n)$ (10^6 m^3)	5.56	4.79	4.12
$EC_K(n)$ (CAD/m^3)	6.04	5.98	6.16
$EC_P(n)$ ((m^3)^3/CAD)	2.009×10^{11}	2.010×10^{11}	2.011×10^{11}
$EC_A(n)$ (CAD/(m^3)^3)	1.24×10^{-12}	1.24×10^{-12}	1.24×10^{-12}
$EC_B(n)$ (CAD/(m^3)^2)	-1.38×10^{-5}	-1.19×10^{-5}	-1.02×10^{-5}
$EC_C(n)$ (CAD/ m^3)	44.45	34.49	27.27

Table 11. Cap

Cap	0.65

Table 12. $Cost(i)$ (CAD/m^3)

i		
crude 1	crude 2	crude 3
610.20	577.30	535.04

Table 13. $\boldsymbol{MaxProd}(\boldsymbol{u},\boldsymbol{n})$ **(10^3 m^3/day)**

u	n		
	R1	R2	R3
CDU	18.28	15.90	13.51
CR	5.30	5.30	5.30
HC	10.60	10.60	10.60
FCC	10.60	10.60	10.60
GB	10.60	10.60	10.60
DB	9.54	9.54	9.54

Table 14. $\boldsymbol{MinProd}(\boldsymbol{u},\boldsymbol{n})$ **(10^3 m^3/day)**

u	n		
	R1	R2	R3
CDU	9.54	7.95	7.15
CR	1.06	1.06	1.06
HC	0.53	0.53	0.53
FCC	0.53	0.53	0.53

Table 15. Intermediate tank capacity data (10^3 m^3)

tk	$\overline{V}(tk)$	$\underline{V}(tk)$	$V_{ini}(tk)$
tk1	47.70	0	0
tk2	47.70	0	0
tk3	47.70	0	0
tk4	47.70	0	0
tk5	47.70	0	0
tk6	47.70	0	0
tk7	47.70	0	0
tk8	47.70	0	0

Table 16. Product tank capacity data (10^3 m^3)

p	$\overline{VP}(p)$	$\underline{VP}(p)$	$VP_{ini}(p)$
REG	159	1.59	1.59
MID	159	1.59	1.59
PRE	159	1.59	1.59
DE1	159	1.59	1.59
DE2	159	1.59	1.59
DE4	159	1.59	1.59

Table 17. Blender capacity data (10^3 m^3/month)

bl	$BlendMax(bl)$	$BlendMin(bl)$
GB	318	4.70
DB	286	4.70

Table 18. $\boldsymbol{BLcost(bl)}$ (CAD/m^3)

bl	
GB	6.29×10^{-2}
DB	6.29×10^{-2}

Table 19. $\boldsymbol{\tau(t)}$ (months)

t	$\tau(t)$
1	6
2	6

Table 20. $\boldsymbol{OpCost(u,m)}$ (CAD/m^3)

u	m	
	1	2

CDU	1.95	1.41
CR	2.61	5.43
HC	3.37	2.62
FCC	2.12	2.07
GB	0.21	0.21
DB	2.20	2.20

Table 21. $\boldsymbol{qq(i,q)}$

i	q							
	RON	MON	ARO	FLS	CNU	SUL	SG	RVP
srln	69.4	64.2	0	0	0	0	0.694	2.378
rft	103	90.8	74.9	0	0	0	0.818	2.378
hcln	93.2	81.6	18	0	0	0	0.751	12.335
hcds	0	0	0	56	50	0.008	0.832	0
fccln	87.7	75.8	25	0	0	0	0.713	13.876
fcchn	82.3	73.5	20	0	0	0	0.764	19.904
fcclco	0	0	0	53	50	0.009	0.802	0
srds	0	0	0	46	40	0.008	0.852	0

Table 22. $\boldsymbol{\overline{Q}(q,p)}$

q	p					
	REG	MID	PRE	DE1	DE2	DE4
RON	200	200	200	200	200	200
MON	200	200	200	200	200	200
ARO	60	50	45	200	200	200
FLS	200	200	200	200	200	200
CNU	200	200	200	200	200	200
SUL	0.01	0.01	0.01	0.01	0.01	0.05

SG	0.81	0.81	0.81	0.85	0.87	0.9
RVP	15.6	15.6	15.6			

Table 23. $\underline{\boldsymbol{Q}}(\boldsymbol{q},\boldsymbol{p})$

q	p					
	REG	MID	PRE	DE1	DE2	DE4
RON	88	91	94	0	0	0
MON	75	78	81	0	0	0
ARO	0	0	0	0	0	0
FLS	0	0	0	40	45	55
CNU	0	0	0	40	40	30
SUL	0	0	0	0	0	0
SG	0.7	0.7	0.7	0.81	0.81	0.81
RVP	0	0	0	0	0	0

Table 24. $\overline{\boldsymbol{R}}(\boldsymbol{i},\boldsymbol{p})$

i	p					
	REG	MID	PRE	DE1	DE2	DE4
srln	1	1	1	0	0	0
rft	1	1	1	0	0	0
hcln	1	1	1	0	0	0
hcds	1	1	1	0	0	0
fccln	1	1	1	0	0	0
fcchn	0	0	0	1	1	1
fcclco	0	0	0	1	1	1
srds	0	0	0	1	1	1

Table 25. $\underline{\boldsymbol{R}}(\boldsymbol{i},\boldsymbol{p})$

i	p					
	REG	MID	PRE	DE1	DE2	DE4
srln	0	0	0	0	0	0
rft	0	0	0	0	0	0
hcln	0	0	0	0	0	0
hcds	0	0	0	0	0	0
fccln	0	0	0	0	0	0
fcchn	0	0	0	0	0	0
fcclco	0	0	0	0	0	0
srds	0	0	0	0	0	0

Table 26. $\boldsymbol{TC}$ **(CAD/m^3)**

TC	0.314

Table 27. $\boldsymbol{Y}(\boldsymbol{i},\boldsymbol{m},\boldsymbol{u})$ **(yield fraction)**

$i.m$	u			
	CR	HC	FCC	HT1
rft.1	0.8			
rft.2	0.9			
hcln.1		0.5		
hchn.1		0.3		
hckero.1		0.1		
hcds.1		0.1		
hcln.2		0.3		
hchn.2		0.2		
hckero.2		0.2		
hcds.2		0.3		

fccln.1			0.5	
fcchn.1			0.3	
fcclco.1			0.1	
fcchco.1			0.1	
fccln.2			0.3	
fcchn.2			0.2	
fcclco.2			0.2	
fcchco.2			0.3	
srds.1				0.072
hclf.1				0.928

Table 28. $\boldsymbol{X(i, m, i')}$ **(%)**

m	i	i'		
		Crude 1	Crude 2	Crude 3
1	lpg	2.18	1.45	0.86
	srln	6.37	5.91	12.21
	srhn	17.14	16.19	8.00
	kero	15.83	15.21	5.87
	lgo	13.25	13.60	6.73
	hgo	29.87	30.60	29.99
	rsd	16.57	17.05	36.33
2	lpg	1.97	1.23	0.76
	srln	5.76	5.30	10.79
	srhn	15.50	14.51	7.07
	kero	12.12	11.49	2.16
	lgo	25.16	25.51	18.64
	hgo	26.17	26.91	27.39
	rsd	14.52	14.99	33.18

Supplementary material to: Conjectures regarding the existence and properties of mixed integer potential games

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A. Refinery expansion and production planning model

The refinery model used for the capacity expansion and production planning model is a modification of that used by Tominac and Mahalec[4] and uses the same notation and data. Data values have been reproduced here for consistency and convenience. The model is an MINLP, with integer variables used to represent upgrade purchase decisions within a competitive nonlinear potential game model. The purpose of each equation is detailed briefly for the purpose of reproducibility.

Volumetric flows are determined by Eqs. (A1) to (A5); inlet unit flow by Eq. (A1), flow to units operating in multiple modes by Eq. (A2), Eq. (A3) defines minimum production rates, and Eqs. (A4) and (A5) define maximum flow rates under available upgrade schemes.

$$FA(t,u,n) = \sum_{m \in UM}^{m} FVM(m,t,u,n) \quad \forall t \in T, u \in U, n \in N \tag{A1}$$

$$FVM(m,t,u,n) = \sum_{i \in U_{IN}}^{i} FVM_{IN}(m,t,i,n) \quad \forall t \in T, n \in N, (u,m) \in UM \tag{A2}$$

$$FA(t,u,n) \geq MinProd(u,n)\tau(t) \quad \forall t \in T, u \in U_C, n \in N \tag{A3}$$

$$\begin{gathered} FVM_{IN}(m,t,i,n) \leq UY(upg,n)\big(MaxProd(CDU) + UPGcap(uCDUcap)\big)\tau(t) \\ \forall m \in M, t \in T, n \in N, (i,upg) \in SUD \end{gathered} \tag{A4}$$

$$\begin{gathered} FA(t,u,n) \leq \left[MaxProd(u) + \sum_{upg \in UUD}^{upg} UY(upg,n)UPGcap(upg)\right]\tau(t) \\ \forall t \in T, u \in U_C, n \in N \end{gathered} \tag{A5}$$

Eqs. (A6) and (A7) define volumetric flow rates of material leaving process units; effluent rates are linked to influent rates by fixed yield relationships.

$$\begin{gathered} FVM_{OUT}(i,m,t,u,n) = Y(i,m,u)FVM(m,t,u,n) \\ \forall t \in T, n \in N, (i,u,m) \in UM_{OUT,C} \end{gathered} \tag{A6}$$

$$FA(t,u,n) = \sum_{i \in U_{OUT}} FV(t,i,n) \quad \forall t \in T, u \in U_C, n \in N \tag{A7}$$

Stream basis flow rates are determined using Eqs. (A8) and (A9).

$$FV(t,i,n) = \sum_{m \in UM} FVM_{OUT}(i,m,t,u,n) \quad \forall t \in T, n \in N, (i,u) \in U_{OUT} \tag{A8}$$

$$FV(t,i,n) = \sum_{m \in UM} FVM_{IN}(m,t,i,n) \quad \forall t \in T, n \in N, (i,u) \in U_{IN} \tag{A9}$$

Effluent rates in the CDU are determined by Eq. (A10) using an independent set of yield data.

$$\begin{gathered} FVM_{OUT}(i,m,t,u,n) = \sum_{i' \in U_{IN}} X(i,m,i') FVM_{IN}(m,t,i',n) \\ \forall m \in M, i \in I, t \in T, n \in N, u = CDU \end{gathered} \tag{A10}$$

Intermediate product tank levels are defined by Eqs. (A11) and (A12); Eqs. (A13) and (A14) define the maximum and minimum levels within these tanks.

$$\begin{gathered} \sum_{i \in TK_{IN}} FV(t,i,n) - \sum_{i \in TK_{OUT}} FV(t,i,n) + V_{ini}(tk,n) - V(tk,t,n) = 0 \\ \forall tk \in TK, t = 1, n \in N \end{gathered} \tag{A11}$$

$$\begin{gathered} \sum_{i \in TK_{IN}} FV(t,i,n) - \sum_{i \in TK_{OUT}} FV(t,i,n) + V(tk,t-1,n) - V(tk,t,n) = 0 \\ \forall tk \in TK, t > 1, n \in N \end{gathered} \tag{A12}$$

$$V(tk,t,n) \geq \underline{V}(tk) \quad \forall tk \in TK, t \in T, n \in N \tag{A13}$$

$$V(tk,t,n) \leq \overline{V}(tk) \quad \forall tk \in TK, t \in T, n \in N \tag{A14}$$

Stream dedication to product blenders is determined by Eq. (A15); Eq. (A16) defines total blend volume. Intermediate product fraction allowances in a blend are defined by Eqs. (A17) and (A18).

$$\sum_{p \in BL_{OUT}} VB(i,p,t,n) = FV(t,i,n) \quad \forall t \in T, n \in N, (bl,i) \in BL_{IN} \tag{A15}$$

$$\sum_{i \in BL_{IN}} VB(i,p,t,n) = VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,p) \in BL_{OUT} \tag{A16}$$

$$VB(i,p,t,n) \geq RMin(bl)VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,i,p) \in BLIP \tag{A17}$$

$$VB(i,p,t,n) \leq RMax(bl)VBlend(t,p,n) \quad \forall t \in T, n \in N, (bl,i,p) \in BLIP \tag{A18}$$

The total volume processed in a blender is defined by Eq. (A19), and Eqs. (A20) and (A21) define the upper and lower bounds on this volume.

$$\sum_{\substack{i,p \\ (i,p)\in BLIP}} VB(i,p,t,n) = VBlendT(t,bl,n) \quad \forall bl \in BL, t \in T, n \in N \tag{A19}$$

$$VBlendT(t,bl,n) \geq BlendMin(bl)\tau(t) \quad \forall bl \in BL, t \in T, n \in N \tag{A20}$$

$$VBlendT(t,bl,n) \leq \left[BlendMax(bl) + \sum_{\substack{upg \\ upg\in BUD}} UY(upg,n)UPGcap(upg)\right]\tau(t) \\ \forall bl \in BLt \in T, n \in N \tag{A21}$$

Eqs. (A22) to (A27) represent product quality bounds for all products.

$$\sum_{\substack{i \\ i\in BL_{IN}}} VB(i,p,t,n)qq(i,q) \geq \underline{Q}(q,p)VBlend(t,p,n) \\ \forall t \in T, n \in N, (bl,p,q) \in BL_{OUT,VOL} \tag{A22}$$

$$\sum_{\substack{i \\ i\in BL_{IN}}} VB(i,p,t,n)qq(i,q) \leq \overline{Q}(q,p)VBlend(t,p,n) \\ \forall t \in T, n \in N, (bl,p,q) \in BL_{OUT,VOL} \tag{A23}$$

$$\sum_{\substack{i \\ i\in BL_{IN}}} VB(i,p,t,n)qq(i,q)qq(i,q') \geq \underline{Q}(q,p)\sum_{\substack{i \\ i\in BL_{IN}}} VB(i,p,t,n)qq(i,q') \\ \forall t \in T, n \in N, q' = SG, (bl,p,q) \in BL_{OUT,WT} \tag{A24}$$

$$\sum_{\substack{i \\ i\in BL_{IN}}} VB(i,p,t,n)qq(i,q)qq(i,q') \leq \overline{Q}(q,p)\sum_{\substack{i \\ i\in BL_{IN}}} VB(i,p,t,n)qq(i,q') \\ \forall t \in T, n \in N, q' = SG, (bl,p,q) \in BL_{OUT,WT} \tag{A25}$$

$$\sum_{\substack{i \\ i\in BL_{IN}}} VB(i,p,t,n)qq(i,q)^{1.25} \geq \underline{Q}(q,p)^{1.25}VBlend(t,p,n) \\ \forall t \in T, n \in N, (bl,p,q) \in BL_{OUT,NL} \tag{A26}$$

$$\sum_{\substack{i \\ i\in BL_{IN}}} VB(i,p,t,n)qq(i,q)^{1.25} \leq \overline{Q}(q,p)^{1.25}VBlend(t,p,n) \\ \forall t \in T, n \in N, (bl,p,q) \in BL_{OUT,NL} \tag{A27}$$

Product balances are defined by Eqs. (A28) and(A29), and Eqs. (A30) and (A31) define the maximum and minimum product inventories in the balance equations.

$$VBlend(t,p,n) + VP_{ini}(p) - VP(t,p,n) - Prc(t,p,n) = 0 \quad \forall t = 1, p \in P, n \in N \tag{A28}$$

$$VBlend(t,p,n) + VP(t-1,p,n) - VP(t,p,n) - Prc(t,p,n) = 0 \quad \forall t > 1, p \in P, n \in N \tag{A29}$$

$$VP(t,p,n) \geq \underline{VP}(p) \quad \forall t \in T, p \in P, n \in N \tag{A30}$$

$$VP(t,p,n) \leq \overline{VP}(p) \quad \forall t \in T, p \in P, n \in N \tag{A31}$$

Product tank levels are constrained such that they return to their minimum (and also starting) levels at the end of the planning horizon by Eqs. (A32) and (A33).

$$V(tk,t,n) = \underline{V}(tk) \quad \forall tk \in TK, t = T, n \in N \tag{A32}$$

$$VP(t,p,n) = \underline{VP}(p) \quad \forall t = T, p \in P, n \in N \tag{A33}$$

Eq. (A34) allocates product volumes to various markets for delivery.

$$Prc(t,p,n) - \sum_w Dlv(t,p,n,w) = 0 \quad \forall t \in T, p \in P, n \in N \tag{A34}$$

Eqs. (A35) to (A38) represent costs of crude oil, unit operation, blend operation, and upgrade purchasing.

$$CrudeOilCost(n) = \sum_t \left[(1 + 0.01t) \sum_{\substack{i \\ i \in IC}} Cost(i) FV(t,i,n)\right] \quad \forall n \in N \tag{A35}$$

$$UnitOpCost(n) = \sum_{m,n,u} OpCost(u,m) FVM(m,t,u,n) \quad \forall n \in N \tag{A36}$$

$$BlendOpCost(n) = \sum_{bl,n} BLcost(bl)\, VBlendT(t,bl,n) \quad \forall n \in N \tag{A37}$$

$$UpgradesCost(n) = \sum_{upg} UY(upg,n) UPGcost(upg) \quad \forall n \in N \tag{A38}$$

Eq. (A39) defines the cost of imports, which serve primarily as a slack variable for domestic market production.

$$ImpCost(n) = \sum_{p,w} CI(p,w)\, Imp(p,n,w) \quad \forall n \in N \tag{A39}$$

Eq. (A40) defines a time cost which becomes lower in subsequent time periods, thus making production decisions non-degenerate.

$$TimeCost(n) = TC \sum_{t,p} (1 - 0.01 \cdot t) Produce(t,p,n) \quad \forall n \in N \tag{A40}$$

Eq. (A41) aggregates cost values.

$$TotCost(n) = CrudeOilCost(n) + UnitOpCost(n) + BlendOpCost(n) + UpgradesCost(n) + TEC(n) + ImpCost(n) + TimeCost(n)\ \forall n \in N \tag{A41}$$

Eqs. (A42) and(A43) define domestic market supply limitations.

$$\sum_{t,n} Dlv(t,p,n,w) \geq \underline{D}(p,w) \quad \forall p \in P, w \in W_L \tag{A42}$$

$$\sum_{t,n} Dlv(t,p,n,w) \leq \overline{D}(p,w) \quad \forall p \in P, w \in W_L \tag{A43}$$

Eqs. (A44) to (A46) define the overall potential function objective.

$$\max Z = \Psi + \sum_n \Omega(n) \tag{A44}$$

$$\Psi = \sum_p \left[-\frac{A(p)}{Demand(p)} \sum_{\substack{n,n' \\ n<n'}} \left(\sum_t Deliver(t,p,n)\right)\left(\sum_t Deliver(t,p,n')\right) \right] \tag{A45}$$

$$\Omega(n) = \sum_p \left[\left(A(p) + B(p) - \frac{A(p)}{Demand(p)} \left(\sum_t Deliver(t,p,n)\right) \right) \left(\sum_t Deliver(t,p,n)\right) \right] - TotCost(n) \quad \forall n \in N \tag{A46}$$

B. Table of set elements and indices

Set	Indices	Elements
BL	(bl)	GB, DB
I	(i)	crude1, crude2, crude3, lpg, srln, srhn, kero, lgo, hgo, rsd, rft, srds, hclf, hchf, hcln, hckero, hcds, hchn, fccf, fccln,

		fcchn, fcclco, fcchco, srln_tk, rft_tk, hcln_tk, fccln_tk, fcchn_tk, srds_tk, hcds_tk, fcclco_tk
IC	(i)	crude1, crude2, crude3
M	(m)	1, 2
N	(n)	R1, R2, R3
P	(p)	REG, MID, PRE, DE1, DE2, DE4
Q	(q)	RON, MON, ARO, FLS, CNU, SUL, SG, RVP
W	(w)	LM1, EM1
T	(t)	1, 2
TK	(tk)	tk1, tk2. Tk3, tk4, tk5, tk6, tk7, tk8
U	(u)	CDU, CR, HC, FCC, HT1, HT2
UPG	(upg)	uHCproc, uHTproc, uCDUcap, uCRcap, uHCcap, uFCCcap, uGBcap, uDBcap
BL_{BLEND}	(i)	(GB).(srln_tk, rft_tk, hcln_tk, fccln_tk, fcchn_tk), (DB).(srds_tk, hcds_tk, fcclco_tk)
BL_{IN}	(bl, i)	srln_tk, rft_tk, hcln_tk, fccln_tk, fcchn_tk, srds_tk, hcds_tk, fcclco_tk
BL_{OUT}	(bl, p)	(GB).(REG, MID, PRE), (DB).(DE1, DE2, DE4)
$BL_{OUT,VOL}$	(bl, p, q)	(GB).(REG, MID, PRE).(RON, MON, ARO, SG), (DB).(DE1, DE2, DE4).(FLS, CNU, SG)
$BL_{OUT,WT}$	(bl, p, q)	(GB).(REG, MID, PRE).(RVP)
$BL_{OUT,NL}$	(bl, p, q)	(DB).(DE1, DE2, DE4).(SUL)
$BLIP$	(bl, i, p)	$BL_{IN} \cdot BL_{OUT}$
DWN	(p', p)	(REG).(REG), (MID).(REG, MID), (PRE).(REG, MID, PRE), (DE1).(DE1), (DE2).(DE1, DE2), (DE4).(DE1, DE2, DE4)
LCN	(n)	R1, R2
HCN	(n)	R3

$NPrest$	(n,p)	(R1, R2, R3).(REG)
P_g	(p)	REG, MID, PRE
P_d	(p)	DE1, DE2, DE4
Q_g	(q)	RON, MON, ARO, SG, RVP
Q_d	(q)	FLS, CNU, SUL, SG
Q_{VOL}	(q)	RON, MON, ARO, FLS, CNU, SG, RVP
Q_{WT}	(q)	SUL
W_E	(w)	EM1
W_L	(w)	LM1
WLN	(n,w)	(R1, R2, R3).(LM1)
WLE	(w,w')	(LM1).(EM1)
WN	(n,w)	(R1, R2, R3).(LM1, EM1)
$WLCN$	(n,w)	(R1, R2).(LM1)
$WHCN$	(n,w)	(R3).(LM1)
TK_{IN}	(tk,i)	(tk1).(srln), (tk2).(rft), (tk3).(hcln), (tk4).(fccln), (tk5).(fcchn), (tk6).(srds), (tk7).(hcds), (tk8).(fcclco)
TK_{OUT}	(tk,i)	(tk1).(srln_tk), (tk2).(rft_tk), (tk3).(hcln_tk), (tk4).(fccln_tk), (tk5).(fcchn_tk), (tk6).(srds_tk), (tk7).(hcds_tk), (tk8).(fcclco_tk)
U_{IN}	(u,i)	(CDU).(crude1, crude2, crude3), (CR).(srhn, hchn), (HC).(hclf, hchf), (FCC).(fccf), (HT1).(lgo), (HT2).(hgo)
U_{OUT}	(u,i)	(CDU).(lpg, srln, srhn, kero, lgo, hgo, rsd), (CR).(rft), (HC).(hcln, hchn, hckero, hcds), (FCC).(fccln, fcchn, fcclco, fcchco), (HT1).(srds, hclf), (HT2).(hchf, fccf)
U_C	(u)	CDU, CR, HC, FCC
UM	(u,m)	(CDU, CR, HC, FCC).(1, 2),(HT1, HT2).(1)
UM_C	(u,m)	(CR, HC, FCC).(1, 2), (HT1).(1)
$UM_{OUT,C}$	(i,u,m)	$U_{OUT} \cdot UM_C$

$ProcUp$	(upg)	uHCproc, uHTproc
$CapUp$	(upg)	uCDUcap, uCRcap, uHCcap, uFCCcap, uGBcap, uDBcap
SUD	(i, upg)	(crude2).(uHTproc), (crude3).(uHCproc, uHTproc)
UUD	(u, upg)	(CDU.uCDUcap), (CR.uCRcap), (HC.uHCcap), (FCC.uFCCcap)
BUD	(bl, upg)	(GB).(uGBcap), (DB).(uDBcap)

C. Tables of parameter values

Table 1. $\boldsymbol{A(p,w)}$ (CAD/m^3)

w	p					
	REG	MID	PRE	DE1	DE2	DE4
LM1	245.30	226.43	232.72	163.54	163.54	163.54
EM1	163.54	150.96	157.24	113.22	113.22	113.22

Table 2. $\boldsymbol{AHC(p,w)}$ (CAD/m^3)

w	Scale factor	p					
		REG	MID	PRE	DE1	DE2	DE4
LM1	5%	4.04	3.82	3.90	2.76	2.76	2.76
LM1	15%	12.13	11.46	11.71	8.29	8.29	8.29
LM1	25%	20.21	19.11	19.52	13.81	13.81	13.81
LM1	35%	28.30	26.75	27.33	19.34	19.34	19.34
LM1	45%	36.38	34.39	35.14	24.87	24.87	24.87
LM1	55%	44.47	42.04	42.94	30.39	30.39	30.39

Table 3. $\boldsymbol{B(p,w)}$ (CAD/m^3)

w	p					
	REG	MID	PRE	DE1	DE2	DE4
LM1	886.86	956.05	1006.37	1075.56	1018.95	962.34

EM1	886.86	956.05	1006.37	1075.56	1018.95	962.34

Table 4. $\boldsymbol{CI(p)}$ **(CAD/m³)**

p					
REG	MID	PRE	DE1	DE2	DE4
1065.51	1144.89	1205.63	1287.38	1221.07	1154.76

Table 5. $\boldsymbol{D(p,w)}$ **(10^6 m³/year)**

w	*p*					
	REG	MID	PRE	DE1	DE2	DE4
LM1	6.38	0.48	0.71	0.08	3.19	0.17
EM1	7.45	0.56	0.83	0.10	3.73	0.20

Table 6. $\boldsymbol{\underline{D}(p,w)}$ **(10^6 m³/year)**

w	*p*					
	REG	MID	PRE	DE1	DE2	DE4
LM1	5.75	0.43	0.64	0.07	2.87	0.16

Table 7. $\boldsymbol{\overline{D}(p,w)}$ **(10^6 m³/year)**

w	*p*					
	REG	MID	PRE	DE1	DE2	DE4
LM1	7.66	0.57	0.86	0.10	3.83	0.21

Table 8. $\boldsymbol{F(p,w)}$ **(CAD/m³)**

w	*p*					
	REG	MID	PRE	DE1	DE2	DE4
LM1	1132.16	1182.48	1239.09	1239.09	1182.48	1125.87
EM1	1050.40	1107.00	1163.61	1182.48	1125.87	1075.56

Table 9. $\boldsymbol{HCNset}$ **(10^6 m^3/year)**

n	
R3	2.86

Table 10. Efficiency cost curve parameters

	n		
	R1	R2	R3
$EC_H(n)$ (10^6 m^3)	5.56	4.79	4.12
$EC_K(n)$ (CAD/m^3)	6.04	5.98	6.16
$EC_P(n)$ ((m^3)^3/CAD)	2.009×10^{11}	2.010×10^{11}	2.011×10^{11}
$EC_A(n)$ (CAD/(m^3)^3)	1.24×10^{-12}	1.24×10^{-12}	1.24×10^{-12}
$EC_B(n)$ (CAD/(m^3)^2)	-1.38×10^{-5}	-1.19×10^{-5}	-1.02×10^{-5}
$EC_C(n)$ (CAD/ m^3)	44.45	34.49	27.27

Table 11. $\boldsymbol{Cap}$

Cap	0.65

Table 12. $\boldsymbol{Cost(i)}$ **(CAD/m^3)**

i		
crude 1	crude 2	crude 3
610.20	577.30	535.04

Table 13. $\boldsymbol{MaxProd(u,n)}$ **(10^3 m^3/day)**

u	n		
	R1	R2	R3
CDU	18.28	15.90	13.51
CR	5.30	5.30	5.30

HC	10.60	10.60	10.60
FCC	10.60	10.60	10.60
GB	10.60	10.60	10.60
DB	9.54	9.54	9.54

Table 14. $\boldsymbol{MinProd(u,n)}$ **(10^3 m^3/day)**

u	n		
	R1	R2	R3
CDU	9.54	7.95	7.15
CR	1.06	1.06	1.06
HC	0.53	0.53	0.53
FCC	0.53	0.53	0.53

Table 15. Intermediate tank capacity data (10^3 m^3)

tk	$\overline{V}(tk)$	$\underline{V}(tk)$	$V_{ini}(tk)$
tk1	47.70	0	0
tk2	47.70	0	0
tk3	47.70	0	0
tk4	47.70	0	0
tk5	47.70	0	0
tk6	47.70	0	0
tk7	47.70	0	0
tk8	47.70	0	0

Table 16. Product tank capacity data (10^3 m^3)

p	$\overline{VP}(p)$	$\underline{VP}(p)$	$VP_{ini}(p)$
REG	159	1.59	1.59
MID	159	1.59	1.59

PRE	159	1.59	1.59
DE1	159	1.59	1.59
DE2	159	1.59	1.59
DE4	159	1.59	1.59

Table 17. Blender capacity data (10^3 m^3/month)

bl	$BlendMax(bl)$	$BlendMin(bl)$
GB	318	4.70
DB	286	4.70

Table 18. $\boldsymbol{BLcost(bl)}$ (CAD/m^3)

bl	
GB	6.29×10^{-2}
DB	6.29×10^{-2}

Table 19. $\boldsymbol{\tau(t)}$ (months)

t	$\tau(t)$
1	6
2	6

Table 20. $\boldsymbol{OpCost(u,m)}$ (CAD/m^3)

u	m	
	1	2
CDU	1.95	1.41
CR	2.61	5.43
HC	3.37	2.62
FCC	2.12	2.07
GB	0.21	0.21

DB	2.20	2.20

Table 21. $\boldsymbol{qq(i,q)}$

i	q							
	RON	MON	ARO	FLS	CNU	SUL	SG	RVP
srln	69.4	64.2	0	0	0	0	0.694	2.378
rft	103	90.8	74.9	0	0	0	0.818	2.378
hcln	93.2	81.6	18	0	0	0	0.751	12.335
hcds	0	0	0	56	50	0.008	0.832	0
fccln	87.7	75.8	25	0	0	0	0.713	13.876
fcchn	82.3	73.5	20	0	0	0	0.764	19.904
fcclco	0	0	0	53	50	0.009	0.802	0
srds	0	0	0	46	40	0.008	0.852	0

Table 22. $\boldsymbol{\overline{Q}(q,p)}$

q	p					
	REG	MID	PRE	DE1	DE2	DE4
RON	200	200	200	200	200	200
MON	200	200	200	200	200	200
ARO	60	50	45	200	200	200
FLS	200	200	200	200	200	200
CNU	200	200	200	200	200	200
SUL	0.01	0.01	0.01	0.01	0.01	0.05
SG	0.81	0.81	0.81	0.85	0.87	0.9
RVP	15.6	15.6	15.6			

Table 23. $\boldsymbol{\underline{Q}(q,p)}$

q	p

	REG	MID	PRE	DE1	DE2	DE4
RON	88	91	94	0	0	0
MON	75	78	81	0	0	0
ARO	0	0	0	0	0	0
FLS	0	0	0	40	45	55
CNU	0	0	0	40	40	30
SUL	0	0	0	0	0	0
SG	0.7	0.7	0.7	0.81	0.81	0.81
RVP	0	0	0	0	0	0

Table 24. $\overline{\boldsymbol{R}}(\boldsymbol{i},\boldsymbol{p})$

i	p					
	REG	MID	PRE	DE1	DE2	DE4
srln	1	1	1	0	0	0
rft	1	1	1	0	0	0
hcln	1	1	1	0	0	0
hcds	1	1	1	0	0	0
fccln	1	1	1	0	0	0
fcchn	0	0	0	1	1	1
fcclco	0	0	0	1	1	1
srds	0	0	0	1	1	1

Table 25. $\underline{\boldsymbol{R}}(\boldsymbol{i},\boldsymbol{p})$

i	p					
	REG	MID	PRE	DE1	DE2	DE4
srln	0	0	0	0	0	0
rft	0	0	0	0	0	0
hcln	0	0	0	0	0	0

hcds	0	0	0	0	0	0
fccln	0	0	0	0	0	0
fcchn	0	0	0	0	0	0
fcclco	0	0	0	0	0	0
srds	0	0	0	0	0	0

Table 26. $\boldsymbol{TC}$ **(CAD/m^3)**

TC	0.314

Table 27. $\boldsymbol{UPGcost(upg)}$ **(10^6 CAD)**

upg	
uHCproc	7.00
uHTproc	3.63
uCDUcap	8.40
uCRcap	2.475
uHCcap	6.00
uFCCcap	10.50
uGBcap	1.00
uDBcap	1.00

Table 28. $\boldsymbol{UPGcost(upg)}$ **(10^3 m^3/day)**

upg	
uCDUcap	9.27
uCRcap	2.65
uHCcap	5.30
uFCCcap	5.30
uGBcap	5.30
uDBcap	4.77

Table 29. $\boldsymbol{Y(i,m,u)}$ **(yield fraction)**

$i.m$	u			
	CR	HC	FCC	HT1
rft.1	0.8			
rft.2	0.9			
hcln.1		0.5		
hchn.1		0.3		
hckero.1		0.1		
hcds.1		0.1		
hcln.2		0.3		
hchn.2		0.2		
hckero.2		0.2		
hcds.2		0.3		
fccln.1			0.5	
fcchn.1			0.3	
fcclco.1			0.1	
fcchco.1			0.1	
fccln.2			0.3	
fcchn.2			0.2	
fcclco.2			0.2	
fcchco.2			0.3	
srds.1				0.072
hclf.1				0.928

Table 30. $\boldsymbol{X(i,m,i')}$ **(%)**

m	i	i'		
		Crude 1	Crude 2	Crude 3

1	lpg	2.18	1.45	0.86
	srln	6.37	5.91	12.21
	srhn	17.14	16.19	8.00
	kero	15.83	15.21	5.87
	lgo	13.25	13.60	6.73
	hgo	29.87	30.60	29.99
	rsd	16.57	17.05	36.33
2	lpg	1.97	1.23	0.76
	srln	5.76	5.30	10.79
	srhn	15.50	14.51	7.07
	kero	12.12	11.49	2.16
	lgo	25.16	25.51	18.64
	hgo	26.17	26.91	27.39
	rsd	14.52	14.99	33.18

Relationships with existing market structures

A primary scenario revisited throughout this work consists of a market in which a high-cost refiner competes with multiple low-cost refiners. The basis for distinguishing high-cost refiners from low-cost has been discussed, and in summary, may be due to a variety of economic factors which result in one refiner possessing a disadvantage relative to the others with which it competes. This scenario is intended to capture elements of the Canadian refining economy, which are exemplified in a government publication cited as reference number 37 in the second chapter of this thesis. This publication includes a map of western Canada showing the locations and capacities of refineries and the pipeline networks connecting them to domestic and foreign markets, and is included as Figure A1.

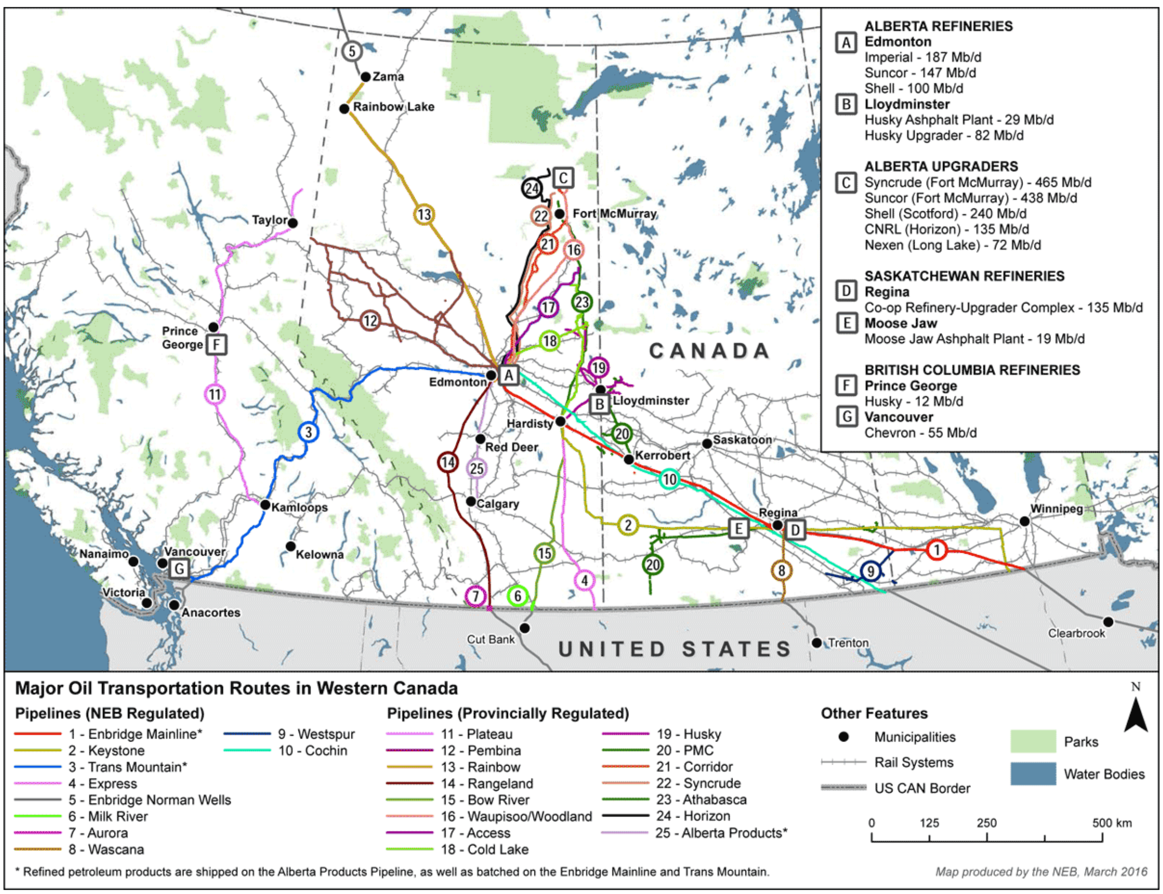

Figure A1. Map of refineries and pipelines in western Canada.

From the perspective of the frameworks developed in this thesis this map presents an interesting case study: refineries located in the city of Edmonton (labeled A) have a combined capacity exceeding 400 Mb/d (thousands of oil barrels per day) and have access to the trans-mountain pipeline (labeled 3) leading to Canada's west coast markets and shipping lanes. The city of Prince George hosts a single refinery with a capacity of 12 Mb/d and feeds into the same west coast markets; in the framework developed in this thesis it would be considered a high-cost refiner relative to those in Edmonton. Of course, this analysis does not consider the influence exerted by larger parent corporations on the sustainability of such market arrangements; the identified case study is based primarily on arguments of scale. Nevertheless, the question remains as to the factors which contribute to such asymmetrical competition in the Canadian petrochemical market, and what considerations motivate the structure of refining assets in this geographical region. Our work explores this question through a game theoretic production planning interpretation, which has shaped our answers to these questions.