ON MULTI-SCALE REFINEMENT OF DISCRETE DATA

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By

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To my parents

Abstract

It is possible to interpret multi-resolution analysis from both Fourier-domain and temporal/spatial domain stand-points. While a Fourier-domain interpretation helps in designing a powerful machinery for multi-resolution refinement on regular point-sets and lattices, most of its techniques cannot be directly generalized to the case of irregular sampling. Therefore, in this thesis we provide a new definition and formulation of multi-resolution refinement, based on a temporal/spatial-domain understanding, that is general enough to allow multi-resolution approximation of different spaces of functions by processing samples (or observations) that can be irregularly distributed or even obtained using different sampling methods. We then continue to provide a construction for designing and implementing classes of refinement schemes in these general settings. The framework for multi-resolution refinement that we discuss includes and extends the existing mathematical machinery for multi-resolution analysis; and the suggested construction unifies many of the schemes currently in use, and, more importantly, allows designing schemes for many new settings.

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Preface

IT IS POSSIBLE to interpret multi-resolution analysis from both frequency or z-domain (FzD) and temporal or spatial domain (TSD) stand-points. Traditionally, it has been common in signal processing to rely on FzD notions and techniques in designing and interpreting multi-resolution schemes. Based on the notions of filters and filter-banks, these interpretations provide valuable insight into the whole process of multi-resolution approximation; and using FzD techniques, different classes of multi-resolution schemes that are optimal in different senses have been successfully devised in the past. Nevertheless, there are limits on what can be accomplished by relying solely on FzD techniques: z-transforms and discrete Fourier transforms depend on translation-invariance, and are suitable only for analyzing data associated with regularly spaced points on the line or in several dimensions (i.e. on lattices). In many instances, no obvious extension of FzD notions to irregular and arbitrary settings exists.

On the other hand, results obtained using the FzD machinery can often be translated to, and derived in, the TSD as well. And what makes TSD interpretations more attractive is that there is essentially no difference in

the TSD tools used for analyzing regular versus irregular data-sets: in both cases, we are concerned with *weighted summations*: the TSD counterparts of FzD filters. Translation-variance and any irregularities in the distribution of data samples can simply be taken care of by suitably modifying the weights for approximation at different positions.

While TSD designs and interpretations of multi-resolution systems relating multi-resolution approximations to splines and polynomial interpolation have existed from the very early days (the works of Deslauriers and Dubuc, Donoho, and Unser, among others come to mind), the introduction of the lifting scheme by Sweldens (1996), and his successive 1997 introduction of second generation multi-resolution constructions, were fundamental in many generalizations of multi-resolution signal processing to multi-dimensional, irregular, and translation-variant (for example involving bounded domains) settings. These contributions helped deepen our understanding of the connection between multi-resolution analysis (MRA), and works on subdivision and sequential refinement schemes, which date back to at least the 1950s and '60s, when Paul de Faget de Casteljau devised his algorithm for sequentially subdividing a piecwise linear curve, so that in the limit it would converge to a smooth curve that we know today by the name of Bézier.[1]

Works of Schröder, Sweldens, Daubechies, Kovačević, and others, that shortly preceded or followed the presentation of the lifting scheme, concen-

[1] Bézier and de Casteljau worked for two competing companies: Bézier worked for Renault and de Casteljau for Citroën. At about the same time, with two different approaches, they both discovered a class of smooth curves that are particularly useful in geometrical design. de Casteljau was not allowed by Citroën to publish his discovery and the family became known as Bézier curves. Today, de Casteljau's innovative algorithm for finding a point on a Bézier curve commemorates his name.

trated on applying this TSD understanding to refinement on multi-dimensional and irregularly-distributed point-sets. In one of the later works, Daubechies et al. (1999) provided a description of multi-resolution refinement on irregular multi-dimensional point-sets, and suggested a refinement scheme that uses Lagrange interpolation to insert values for new points at each level.

In this thesis I focus on a more general TSD framework for multi-scale refinement, and also study designing new classes of multi-resolution refinement schemes in this framework. The hope is to help in developing and extending a coherent and applicable understanding of multi-resolution refinement in a broad sense, and to provide a general machinery that can be utilized to construct multi-resolution refinement schemes suitable for a wide variety of settings.

Towards this aim, the main matter of thesis is divided into three chapters. Chapter 1 is concerned with the theory of multi-resolution refinement. In the first part of that chapter, I provide an alternate interpretation of multi-resolution refinement, similar to but more general than that of Daubechies et al. (1999), that is based exclusively on a TSD understanding of signals and signal processing; and prove parallels to several fundamental results, in particular providing a link between discrete representations and spaces of signals on the continuum. As I was following the path set by Sweldens and his colleagues, what is discussed in section 1.1 is conceptually very similar to the work of Daubechies et al. (1999). This section can therefore be viewed as an introduction to, and review of, their theory. I have not however intended an exact presentation of their construction; and the exposition is certainly different in ways, to prepare the stage for the subsequent presentation of a

new and more general theory of multi-resolution refinement in section 1.2.

Chapter 2 discusses some examples for the theoretical framework provided in chapter 1. I start by discussing several relevant constructions for multi-resolution refinement, before going on to provide a general construction for families of refinement schemes suited to any given space of discrete signals, as defined in 1.2. It will then be shown that the other discussed schemes are examples of this general construction.

A few concluding remarks and suggested directions for further investigation form the final chapter.

In appendix A, I have tried to summarize some mathematical definitons that the reader may wish to review, but the inclusion of which in the main text would further deflect the already not-so-straight course of discussion. For some other mathematical definitions and results I could find a place within the main text. A summary of the used notation is provided as a second appendix.

Footnotes have been used extensively. They are intended to provide explanations, side remarks, and/or reservations, that might be of interest, but which may not directly fit within the course of the thesis. They are aimed to provide a side note, or to serve as a friendly chat with the reader about a secondary point; and as such, I have not always attempted for a complete evaluation of the concept in consideration. Some of the footnote discussions have therefore been left open-ended.

This thesis addresses, and extends, certain aspects of the theory of multi-resolution analysis from a particular perspective. Many excellent reference texts on the general theory exist. Mallat's (1999) *A Wavelet Tour of Signal*

Processing is one of the most comprehensive.

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Through the almost two years that I have been studying and researching at McMaster as a graduate student, I have had the chance to learn and benefit from the guidance, support, knowledge, and friendship of many people. I am indebted to my supervisors, Drs Shirani and Wu, who supported me during this time and kindly offered me the freedom to study and research a topic that was of interest to me, while providing valuable advice along the path. Dr Protas of the Department of Mathematics and Statistics kindly spent hours of his valuable time to listen to my not-so-clear ideas, and I benefited greatly from his knowledge and encouragements.

The members of the defence committee, my supervisors and Drs S. Dumitrescu and S. Sirouspour, provided very interesting comments. I am particularly thankful to Dr Dumitrescu for her deep and though-provoking remarks.

Outside McMaster, I was able to learn from people who kindly provided answers to my questions on the Wavelet Digest discussion forum[2] and the `sci.math.*`, `comp.graphics.algorithms`, and `sci.geo.meteorology` newsgroups. In particular I thank Dr Lutz Lehmann for his many helpful posts on the Wavelet Digest discussion forum and Dr Robert Israel of UBC for his answers on `sci.math` and `sci.math.research`. I was first introduced to the basic theory of multi-resolution signal processing by Amir F. Dana, now at Caltech, and Dr M.B. Shamsollahi of Sharif University of Technology, who was my B.Sc. project adviser.

[2] At the address: `http://www.wavelet.org`.

In implementing the refinement schemes described in the second and third chapters of of this study I have used the MVP package for de Boor-Ron interpolation, which Dr Grandine has generously put in public domain. To evaluate the regularity of the functions resulting from the scheme discussed in subsection 2.3.3, the results of which have appeared in our ICASSP paper, Tafti et al. (2005), Ron et al. kindly provided an implementation of their method for computing the Sobolev regularity of refinable functions detailed in Ron et al. (2001).

There are many non-technical, but nonetheless important, aspects to graduate student life. People in our Multimedia Signal Processing Lab have been considerate and helpful. I have been very lucky to have as friends Navid and Peyman, with whom I have shared accommodation. My friends in the department and across the campus help make McMaster a happier place. Our department staff are exceptional. I particularly thank Cheryl Gies, Terry Greenlay, Cosmin Coroiu, and Helen Jachna who are always ready to offer all their help. The people at McMaster University's Centre for Leadership in Learning introduced me to many of the joys and challenges of teaching. This has been a continuous aspiration in my graduate studies, for the hope that someday I may have something to offer.

Finally, and most importantly, none of this, and none of what I can ever hope for, would have been possible without the kind, ever-present, and unlimited support and countless sacrifices of my parents and family. This thesis is dedicated to them.

'Then you should say what you mean,' the March Hare went on.
'I do,' Alice hastily replied; 'at least—at least I mean what I say—that's the same thing, you know.'
'Not the same thing a bit!' said the Hatter. 'Why, you might just as well say that "I see what I eat" is the same thing as "I eat what I see"!'

Alice's Adventures in Wonderland
CHARLES LUTWIDGE DODGSON (LEWIS CARROLL),
ENGLISH MATHEMATICIAN AND WRITER
(1832–1898)

P.D. Tafti: On Multi-Scale Refinement of Discrete Data.
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1

Multi-Resolution Refinement Revisited

> All generalizations, with the possible exception of this one, are false.
>
> *Paraphrased words of*
> KURT GÖDEL, MATHEMATICIAN AND LOGICIAN
> (1906–1978)

ONE OF THE FIRST notions that someone learning to read a map comes across is that of *scale*. As (s)he learns, on a $1:1,000,000$ scale map only the locations of cities and primary inter-city motorways can be marked. To see city roads one has to turn to a medium-scale map, for example one at $1:50,000$ scale. At $1:10,000$, it is even possible to clearly mark details such as by-ways and major buildings.

The full range of scales that man has explored is even wider—and much so. He has studied sub-micron and sub-atomic phenomena, but has also looked at stellar systems hundreds of light-years wide. Philip and Phylis Morrison, and the Office of Charles and Ray Eames, take us to an expedition

through the sweep of scales in *Powers of Ten: A Book about the Relative Size of Things in the Universe and the Effect of Adding Another Zero* (Morrison et al., 1982).

It may be difficult, or even impossible, at smaller scales to separate some of the features that are clearly distinguishable at finer *resolutions*. At the same time, wide-scale presentations help us notice and better appreciate general patterns and properties which might otherwise be missed in the limited scope of highly-detailed microscopic views.

In this context, the word *resolution* refers to 'the act, process, or capability of rendering distinguishable the component parts of an object or closely adjacent optical or photographic images, or of separating measurements of similar magnitude of any quantity in space or time', and also, to 'the smallest quantity which is measurable by such a process' (Oxford English Dictionary, 1989). From the above discussion, one immediately sees an appeal for *multi-resolution* representations. And while in our examples we have thus far focused on spatial resolution, the concept of resolution may be, and has been, extended and applied to almost anything that can be perceived.

When dealing with multi-resolution representations, the ability to switch between different resolutions is of prime importance. Through a *refinement* process, we are able to add details to and thus *refine* an initial representation, thereby creating a more detailed and more complex image at a higher-resolution.

As engineers usually find it useful to work with mathematical abstractions of concepts, in this chapter we study and formalize such an abstraction of multi-resolution refinement, for which we provide a new formulation based

on our definition of spaces of discrete signals (first introduced in subsection 1.1.1 and later generalized in subsection 1.2.1). This formulation then allows us to generalize and extend the notions of multi-resolution refinement, based on a temporal/spatial domain understanding.

It is important to emphasize here that while the classical development of multi-resolution analysis has depended extensively on Fourier- and z-domain techniques, these techniques often do not readily—or at all—extend to irregular settings. Therefore, in the first part of this chapter we adopt a different approach, structurally similar to that of Daubechies et al. (1999), which we later generalize in section 1.2. Our treatment, which is based on *refinement operators* and hierarchies of *sampling procedures*, is more general than that of Daubechies et al. (1999), who study subdivision operators in one and two dimensions.[1]

This new approach makes an abstract and general treatment of multi-resolution in wide ranges of domains and functional spaces possible.

1.1 Multi-Resolution Refinement

1.1.1 Discrete Signals. Of the simplest discrete signals that one may imagine are sequences of numbers (called *samples*) that are associated with equally-spaced points along a single axis (e.g. integral points on the real line). In a more general setting, a discrete signal may be thought of as an association of values to—not necessarily uniformly distributed—points in a

[1] A brief review of the development of these schemes was provided in the preface.

countable[2] subset of the d-dimensional Euclidean space, $\mathbb{R}^d$.

Specifically, if $\Theta \subset \mathbb{R}^d$ is such a countable point-set, $\ell^p(\Theta)$ for a choice of p, $1 \leq p \leq \infty$, will be a possible space of signals.[3] For a choice of p, a signal a is then an element of $\ell^p(\Theta)$ and maps each $\theta \in \Theta$ to a value, denoted by $a[\theta]$, from a field[4] $\mathbb{F}$ of scalars. The classic theory of multi-resolution analysis originated with the study of the setting where $d = 1$ and $\mathbb{F}$ is the field of reals, $\mathbb{R}$ (see e.g. Meyer, 1992; Daubechies, 1992; Cohen and Ryan, 1995; Mallat, 1999).

We like to have a basis for our spaces of signals. One such basis for $\ell^p(\Theta)$ consists of the signals

$$\delta_\phi[\theta] := [\theta = \phi]$$

for all $\theta \in \Theta$ (cf. A.3.5). A signal a can be decomposed in this basis as

$$a = \sum_{\phi \in \Theta} a[\phi]\delta_\phi. \tag{1.1}$$

1.1.2 Multi-Resolution Representations. Now, to have multi-resolution representations, one could think of a sequence of point-sets with decreasing spacings, and signals defined on these point-sets, which form a sequence of representations at increasing resolutions. For (Θ_i) to be such a sequence, the spacing between points in Θ_i should become smaller in some sense as $i \to \infty$. We formalize this by requiring that all these point-sets belong to a domain $\Omega \subseteq \mathbb{R}^d$, and that in the limit, this sequence become *dense* (see

2 By *countable* we mean either finite or denumerable.
3 $\ell^p(I)$ spaces for a general index set I are defined in A.3.5. Here Θ is the index set.
4 Examples of fields include the reals, $\mathbb{R}$, and the complex numbers, $\mathbb{C}$. For the definition of a field, see A.1.1.

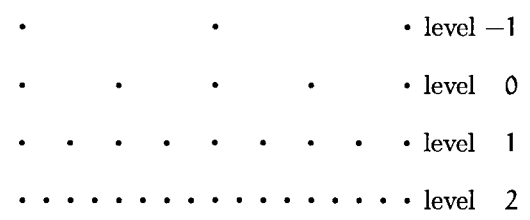


Figure 1.1: Dyadic point-sets are constructed by inserting mid-points between adjacent points.

A.2.2) in Ω, i.e.

$$\overline{\lim_{i\to\infty} \Theta_i} = \Omega. \tag{1.2}$$

Our discrete multi-resolution representations will then be a sequence (a_i) of signals $a_i \in \ell^p(\Theta_i)$.

1.1.3 Example. In the uni-dimensional case ($d = 1$) the simplest example of such a sequence of point-sets is perhaps the one obtainable by beginning with an infinite and equally-spaced point-set Θ_0, and for $i \geq 0$, recursively constructing Θ_{i+1} by adding to Θ_i the mid-points between each two adjacent points. For $i < 0$, Θ_i is constructed from Θ_{i+1} by removing every other point (see fig. 1.1). The resulting (Θ_i) sequence may be called dyadic because the spacing between adjacent points changes by a factor of $1/2$ at each level.

Along the same line, it is possible to subdivide the interval between two adjacent points into $M \geq 2$ sub-intervals. The scale would then change by a factor of $1/M$ at each stage. The following lemma shows that this scheme leads in the limit to a point-set that is dense in $\mathbb{R}$.

1.1.4 Lemma. The limit as $i \to \infty$ of a so-constructed sequence (Θ_i), which we denote by Θ_∞, is dense in $\mathbb{R}$. Every $x \in \mathbb{R}$ is a *limit point* (for a definition seeA.2.2) of Θ_∞.

Proof. Let h be the distance between adjacent points in Θ_0. For an arbitrary point $x \in \mathbb{R}$, denote by x_k its closest point in Θ_k that is distinct from all x_is for $i < k$. Clearly $d(x, x_k) \leq h/M^k$. Also since the sequence (Θ_i) is *nested*[5], $x_k \in \lim_{i\to\infty} \Theta_i := \Theta_\infty$ for all k. The sequence (x_k) consists of distinct points and converges to x. Therefore, by theorem 5 of section II.9 of Kolmogorov and Fomin (1998), x is a limit point of Θ_∞ and Θ_∞ is dense in $\mathbb{R}$. □

1.1.5 Example. Let G be a non-singular $d \times d$ matrix, and let D be a non-singular $d \times d$ *integer* matrix with $\rho(D) > 1$.[6] The following lattice construction provides a sequence satisfying (1.2) in the multi-dimensional ($d \geq 2$) setting (for discussion and some applications see Kovačević and Sweldens, 2000; Tafti et al., 2005; Gibson and Sayood, 1988):

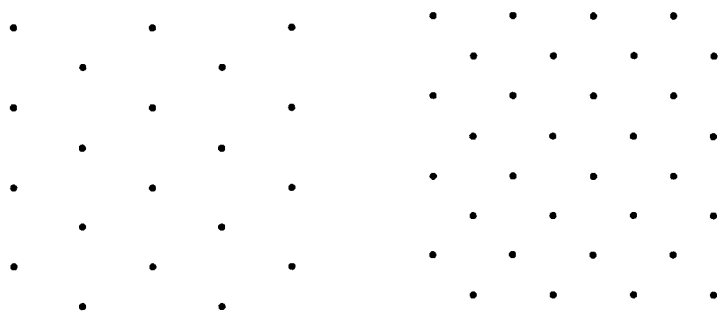
$$\Theta_i = GD^{-i}\mathbb{Z}^d = \left(GD^{-1}G^{-1}\right)\Theta_{i-1}.$$

That the limit of this sequence is dense in $\mathbb{R}^d$ can be proved in a manner similar to the proof of lemma 1.1.4, based on the understanding that the distance between adjacent points in Θ_i approaches 0 as $i \to \infty$ because of the $\rho(D) < 1$ condition.

---

[5] A sequence (A_i) of sets is *nested* if $A_i \subset A_{i+1}$ for all i.

[6] $\rho(D)$ denotes the *spectral radius* of D, which is equal to $\max_i |\lambda_i|$, where λ_is are the eigenvalues.



(a) A triangular lattice

(b) A checker-board (a.k.a. quincunx) lattice

Figure 1.2: Choosing $\Theta_i = GD^{-i}\mathbb{Z}^d$ leads to structures known as lattices.

1.1.6 Example: The Quincunx Lattice. An example of the described lattice structure which has found frequent applications in image processing, is the quincunx (a.k.a. checker-board or red-black) lattice. One of the possible G, D pairs for this lattice is

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

The quincunx lattice is depicted in fig. 1.2.

1.1.7 Linear Approximations. To improve the resolution, one may form an approximate of the higher-resolution signal to which details can then be added. For this, a way should be devised to approximate a level $i+1$ signal $a_{i+1} \in \ell^p(\Theta_{i+1})$ from the level i signal $a_i \in \ell^p(\Theta_i)$.

In the simplest case this approximation will be linear, meaning that the value of the level $i+1$ approximate, a_{i+1}, is calculated at each point $\theta \in \Theta_{i+1}$

as a weighted summation of $a_i[\phi]$s, $\phi \in \Theta_i$; i.e.

$$a_{i+1}[\theta] := \sum_{\phi \in \Theta_i} s_{i,\theta,\phi}\, a_i[\phi], \tag{1.3}$$

where $s_{i,\theta,\phi}$s are weights. If necessary, this approximation can then be adjusted by adding a correction signal, $d_{i+1} \in \ell^p(\Theta_{i+1})$, representing the details:[7]

$$d_{i+1}[\theta] + a_{i+1}[\theta] = d_{i+1}[\theta] + \sum_{\phi \in \Theta_i} s_{i,\theta,\phi}\, a_i[\phi] \quad \text{for each } \theta \in \Theta_{i+1}. \tag{1.4}$$

d_{i+1} in effect contains the new information that could not be, or simply was not, extracted from the low-resolution signal in the linear approximation process.

1.1.8 Refinement Operators. The above formulation can be re-stated using the notion of *linear operators* (A.3.4). If the sum in eqn (1.3) converges for all $\theta \in \Theta_{i+1}$, a linear *refinement operator* $\mathfrak{S}_i$—so named because it maps low-resolution signals to their *refined* high-resolution associates—may be defined for each i by specifying its operation on an arbitrary $a_i \in \ell^p(\Theta_i)$:

$$(\mathfrak{S}_i a_i)[\theta] := \sum_{\phi \in \Theta_i} s_{i,\theta,\phi}\, a_i[\phi] \quad \text{for all } \theta \in \Theta_{i+1}. \tag{1.5}$$

The following proposition holds:

[7] We always assume that the summation weights are chosen such that the infinite summation converges for all $a_i \in \ell^p(\Theta)$ for the chosen p (cf. 1.1.9).

1.1.9 Proposition. For $a_i \in \ell^p(\Theta_i)$, $1 \leq p \leq \infty$, let q be such that $1/p + 1/q = 1$. Then if

$$\left(\sum_{\phi \in \Theta_i} |s_{i,\theta,\phi}|^q \right)^{1/q}$$

converges and is bounded for all $\theta \in \Theta_{i+1}$, the sumimations in eqns (1.3) and (1.5) converge.

The proof follows directly from Hölder's inequality. For a more general result see theorem 1.2.7. □

Let us now revisit eqns (1.3) and (1.5). Each weight $s_{i,\theta,\phi}$ bears three indices: The first index, i, indicates the level or resolution; the second one, θ, informs us that this weight is being used to calculate a new approximation at point θ; and finally, the third index, ϕ, over which the summation is being performed, tells us which sample of the low-resolution signal is being weighted by this coefficient. We therefore see that each sample in a_i is given different weights in the calculation of different samples in a_{i+1}.

1.1.10 Example. We will see general classes of refinement operators in the next chapter. Here, as a simple example of an operator related to the uni-dimensional point-sets introduced in example 1.1.3, suppose that at each refinement step we approximate the value associated with a newly inserted mid-point by averaging the values for the two neighbouring points in the coarse point-set; and for points that exist in both sets we simply copy the value. This scheme and its associated weights are depicted in figure 1.3. Informally, we may represent $\mathfrak{S}_i$ by an infinite matrix (cf. Strang and Nguyen (1997)):

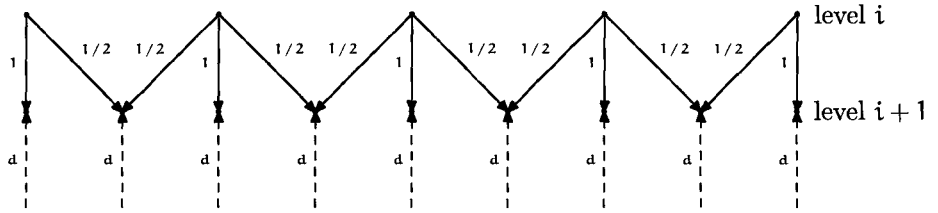


Figure 1.3: A simple uni-dimensional refinement scheme consists in copying original values and inserting averages between them. Details (corrections) may then be added.

$$\left(s_{i,\theta,\phi}\right)_{\substack{\theta\in\Theta_{i+1}\\ \phi\in\Theta_i}} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1/2 & 1/2 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1/2 & 1/2 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

1.1.11 Choosing Refinement Operators. Upon introducing the detail term $d_{i+1}[\theta]$ in eqn (1.4), the reader may have questioned the need for the approximation term (i.e. the summation). After all, no matter how inaccurate this approximation may be, it can be nonetheless adjusted by adding a larger correction term. So why not dismiss the approximation step altogether and save ourselves the trouble?

The answer lies in our wish to find a compact representation for the high-resolution signal. To this end we would like to be able to construct an acceptable approximate to our higher-resolution signal based on its coarser representation, thereby reducing the norm of the difference signal d_{i+1}.

Hence in practice we intend to choose the refinement weights such that $\mathfrak{S}_i a_i$ approximates a_{i+1} closely and therefore the (norm of the) adjustment d_{i+1} becomes small. The choice shall thus depend on the properties of the discrete signal, which in turn depend on the properties of the underlying distribution on the domain Ω—if such a distribution in fact exists—and our sampling procedure.

We will describe a novel approach to designing refinement operators, that makes explicit use of our model for the sampling procedure, in the second part of chapter 2.

1.1.12 Example. When dealing with *point-wise evaluations* of a *linear function*, we immediately see that, using the weights given in the previous example, mid-point samples can be exactly computed and there will be no need at all to consider detail coefficients. And yet for some other function, the 2-point average might not be a good estimate. This difference is visible in fig. 1.4.

The 2-point average provides in fact the value of the first-order polynomial interpolant at the mid-point. As we will later see, this idea can be extended by using higher-order polynomial approximations.

1.1.13 The Cascade Algorithm. We are now in a position to establish a link between our discrete scheme and spaces of functions. This connection also underlies the definition of *multi-resolution analysis.*[8]

[8] A *multi-resolution analysis* (MRA) consists of a sequence $(\mathcal{V}_i)$ of nested functional spaces satisfying several axioms that are detailed for example in Meyer (1992) and Mallat (1999). Sweldens (1997) generalized this definition and introduced *second generation* constructions. In his definition, $\mathcal{V}_i$s should satisfy the following properties (he uses a slightly different notation):

- $\mathcal{V}_i \subset \mathcal{V}_{i+1}$;

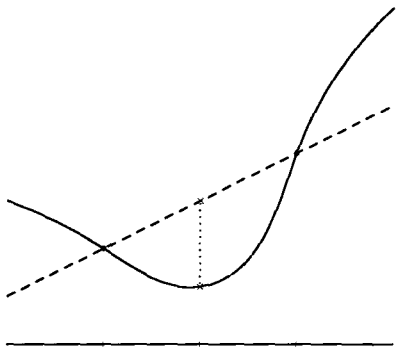


Figure 1.4: Approximation with the 2-point average recovers samples of a linear function without error, but introduces error in approximating a higher-order curve.

Fix p such that $1 \leq p \leq \infty$ and consider an $a_k \in \ell^p(\Theta_k)$, $k \in \mathbb{Z}$. For $i \geq k$ we define the signals $a_i \in \ell^p(\Theta_i)$ by[9]

$$a_i := \mathfrak{S}_{i-1} a_{i-1} \quad \text{for } i > k. \tag{1.6}$$

- $\overline{\bigcup_i \mathcal{V}_i} = L^2$;

- There exists a Riesz basis for each $\mathcal{V}_i$, given by scaling functions $\{g_i^k | k \in \mathcal{K}(i)\}$, where $\mathcal{K}(i) \subset \mathcal{K}(i+1)$ is an index set.

In this study we will not limit ourselves to L^2 functions. The reader will see later that multi-resolution spaces, as we will define them in this thesis, satisfy the first of the above properties. The second property is replaced by a requirement guaranteeing the unique representation of any function in our space of functions with the set of its samples as $i \to \infty$. Our multi-resolution spaces are also defined in terms of a basis consisting in scaling functions. We do not consider Riesz bases as they are relevant for a Hilbert space structure, which may not always exist in our choices of functional spaces. Instead, as a stability condition, we, after fixing p, require that for any initial sequence $a_k \in \ell^p(\Theta_k)$ (and later, in section 1.2, any $a_k \in \ell^p(\Lambda_k)$), the cascade algorithm converge to a function $f_\infty^{a_k}$ in the functional space in consideration.

The rest of this section concerns multi-resolution functional spaces and their relation to spaces of discrete signals $\ell^p(\Theta_i)$. In the next section we suggest a more general definiton of discrete signals, and then construct multi-resolution spaces based on a generalized understading, of which the discussions of this section will be a special case.

[9] That $a_i \in \ell^p(\Theta_i)$ (for finite i) follows from proposition 1.1.9.

As mentioned earlier, $\Theta_\infty := \lim_{i\to\infty} \Theta_i$ is dense in a domain Ω. If in the limit,

$$a_\infty := \cdots \mathfrak{S}_{k+2}\mathfrak{S}_{k+1}\mathfrak{S}_k a_k := \lim_{j\to\infty} \mathfrak{S}_{k+j}\cdots\mathfrak{S}_k a_k \tag{1.7}$$

exists in $\ell^p(\Theta_\infty)$, under certain conditions we can extend (or interpolate if you may) the function $a_\infty := \lim_{i\to\infty} a_i$—which is defined on Θ_∞—to a $\mathbf{C}^0(\Omega)$ function[10] $f_\infty^{a_k}$, that is defined on Ω and satisfies

$$f_\infty^{a_k}(\theta) = a_\infty[\theta] \quad \text{for } \theta \in \Theta_\infty. \tag{1.8}$$

This approach to defining a function over Ω through refinement *ad infinitum* is known as the *cascade algorithm* (Daubechies, 1992) or sometimes as *subdivision* (Cavaretta et al., 1991; Daubechies et al., 1999). Some conditions for the convergence of the cascade algorithm have been previously studied for stationary schemes (i.e. schemes in which the refinement coefficients remain the same across resolutions).[11]

[10] $\mathbf{C}^N(\Omega)$ is the space of functions on Ω that are at least N times continuously differentiable (Al-Gwaiz, 1992, pp. 16–17).

[11] Most of the research in this context has addressed regular and one-dimensional settings. See for example Daubechies and Lagarias (1991, 1992) for $L^1(\mathbb{R})$ solutions. Micchelli and Prautzsch (1989); Dyn and Levin (1990) study the problem for interpolating schemes. Heil (1992) surveys several approaches. The work of Daubechies and Lagarias (1992) is extended in Colella and Heil (1994). Cavaretta et al. (1991, ch. 2) prove (with a different notation) the following condition to be necessary for the convergence of *stationary* refinement schemes with finite masks in the uniform case:

$$\sum_{\phi\in\Theta_i} s_{i,\psi,\phi} = 1 \quad \text{for } \phi \in \Theta_i, \psi \in \Theta_{i+1}, i \in \mathbb{Z}. \tag{1.9}$$

The regularity of the solution to refinement equations in multi-dimensional settings linked to lattices introduced in example 1.1.5 has been studied in Ron and Shen (2000); Cohen et al. (1999); Jia (1999).

1.1.14 Scaling Functions. We will assume from now on that the refinement operators are such that the cascade algorithm converges to a continuous function $f_\infty^{a_k}$ for any initial $a_k \in \ell^p(\Theta_k)$ at any level i. Then, for a choice of a_k, $f_\infty^{a_k}$ can be represented as a linear combination of *scaling* functions g_k^θ with $\theta \in \Theta_k$, which are introduced in the following.

Let us fix $\theta \in \Theta_k$ and assume the notation $e_{m,\theta} := \delta_\theta \in \ell^p(\Theta_m)$. Furthermore, let $e_{m\to n,\theta}$, $m < n$, denote the signal in $\ell^p(\Theta_n)$ defined by the equation:

$$e_{m\to n,\theta} := \mathfrak{S}_{n-1}\cdots\mathfrak{S}_m e_{m,\theta} = \mathfrak{S}_{n-1}\cdots\mathfrak{S}_m\delta_\theta,$$

with $\delta_\theta \in \ell^p(\Theta_m)$.

It then follows from eqn (1.1) that

$$e_{k\to k+1,\theta} = \sum_{\phi\in\Theta_{k+1}} e_{k\to k+1,\theta}[\phi]\delta_\phi \quad (\text{with } \delta_\phi \in \ell^p(\Theta_{k+1}))$$

and therefore, from eqns (1.5) and (1.6),

$$\begin{aligned} e_{k\to\infty,\theta} &:= \lim_{j\to\infty} \mathfrak{S}_{k+j}\cdots\mathfrak{S}_k e_{k,\theta} \\ &= \lim_{j\to\infty} \mathfrak{S}_{k+j}\cdots\mathfrak{S}_{k+1}\left(\mathfrak{S}_k e_{k,\theta}\right) \\ &= \lim_{j\to\infty} \mathfrak{S}_{k+j}\cdots\mathfrak{S}_{k+1}\left(e_{k\to k+1,\theta}\right) \end{aligned}$$

Consideration of the continuity of the solution falls under the more general study of the regularity of solutions to refinement equations. We will not discuss convergence and regularity issues any further in this thesis (except briefly on one other occasion: see footnote 18 on p. 25), and will from now on assume that the refinement operators are such that for any initial signal $a_k \in \ell^p(\Theta_k)$ the cascade algorithm converges to a continuous function $f_\infty^{a_k}$ in the limit.

$$= \lim_{j\to\infty} \mathfrak{S}_{k+j} \cdots \mathfrak{S}_{k+1} \sum_{\phi\in\Theta_{k+1}} e_{k\to k+1,\theta}[\phi]\delta_\phi$$

$$= \sum_{\phi\in\Theta_{k+1}} e_{k\to k+1,\theta}[\phi] \lim_{j\to\infty} (\mathfrak{S}_{k+j} \cdots \mathfrak{S}_{k+1}\delta_\phi)$$

$$= \sum_{\phi\in\Theta_{k+1}} e_{k\to k+1,\theta}[\phi] e_{k+1\to\infty,\phi}.$$

We also have

$$e_{k\to k+1,\theta}[\phi] = \mathfrak{S}_k e_{k,\theta} = \sum_{\psi\in\Theta_k} s_{k,\phi,\psi}\ \delta_\theta[\psi] = s_{k,\phi,\theta}.$$

Thus,

$$e_{k\to\infty,\theta} = \sum_{\phi\in\Theta_{k+1}} s_{k,\phi,\theta} e_{k+1\to\infty,\phi};$$

and as we have assumed that the cascade algorithm for arbitrary $a_i \in \ell^p(\Theta_i)$ converges to a unique continuous function $f_\infty^{a_i}$,

$$f_\infty^{e_{k,\theta}} = \sum_{\phi\in\Theta_{k+1}} e_{k\to k+1,\theta}[\phi] f_\infty^{e_{k+1,\phi}}$$

$$= \sum_{\phi\in\Theta_{k+1}} s_{k,\phi,\theta} f_\infty^{e_{k+1,\phi}}. \tag{1.10}$$

Eqn (1.10) is known as a *two-scale* or *refinement equation,* as it relates functions from two scales or resolutions. The weights $s_{k,\phi,\theta}$, with $\phi \in \Theta_{k+1}$, are sometimes referred to as *refinement coefficients* or *weights*, or collectively as the *refinement mask*. The solution to eqn (1.10) is called a *scaling* or *refinable* function, the reason being that it is a linear combination of refinable functions of the next finer resolution.

For simplicity in the following we will use g_k^θ to denote $f_\infty^{e_{k,\theta}}$ as defined above. With this new notation, what was just proved can be re-written as the following proposition:

1.1.15 Proposition.

$$g_k^\theta = \sum_{\phi \in \Theta_{k+1}} s_{k,\phi,\theta} g_{k+1}^\phi. \tag{1.11}$$

1.1.16 Multi-Resolution Spaces. Now for arbitrary $a_i \in \ell^p(\Theta_i)$, we have

$$a_i = \sum_{\phi \in \Theta_i} a_i[\phi] \delta_\phi.$$

Refining both sides of this relation *ad infinitum* and looking for the limiting continuous functions leads us to

$$f_\infty^{a_i} = \sum_{\phi \in \Theta_i} a_i[\phi] g_i^\phi. \tag{1.12}$$

Thus, if we define spaces $\mathcal{V}_i$ as

$$\mathcal{V}_i := \mathrm{span}\{g_i^\phi | \phi \in \Theta_i\} \quad \text{for } i \in \mathbb{Z}, \tag{1.13}$$

it follows from eqn (1.12) that

$$f_\infty^{a_i} \in \mathcal{V}_i.$$

Also, as a result of eqns (1.11) and (1.13) we have the following corollary:

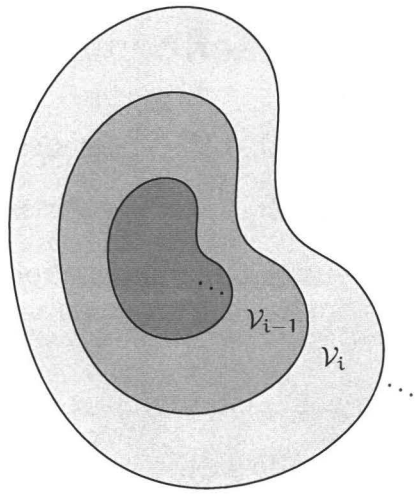


Figure 1.5: Multi-resolution spaces are nested.

1.1.17 Corollary. If for any initial discrete signal at any resolution the cascade algorithm converges to a unique continuous function in the limit, the spaces $\mathcal{V}_i$ defined by eqn (1.13) are nested, i.e.

$$\cdots \subset \mathcal{V}_{i-1} \subset \mathcal{V}_i \subset \mathcal{V}_{i+1} \subset \cdots.$$

Proof. From eqns (1.11) and (1.13),

$$g_\theta^i \in \mathcal{V}_{i+1} \quad \text{for all } \theta \in \Theta_i. \tag{1.14}$$

The conclusion then follows from (1.13). □

This nestedness property is depicted in fig. 1.5. We call the spaces $\mathcal{V}_i$ *multi-resolution spaces*, as they represent families of functions at different resolutions.

1.2 Multi-Resolution Refinement Revisited

1.2.1 Discrete Signals Revisited. The generalization of the definition of a discrete signal in 1.1.1 makes the study of classical (i.e. point-wise) regular and irregular sampling (using delta distributions) possible in the most general case. Nevertheless, this representation is still limiting, and also, it is arguable that when dealing with discrete representations of functions over the continuum, rarely in practice do we actually come across signals sampled in such manner.

Sampling is the task of performing measurements on an observable. Now, first of all, measurements in practice are often not truly point-wise evaluations, as the use of delta distributions would suggest. This causes not much difficulty when all measurements are identical except for them happening at different temporal or spatial instances; as in this case again a natural connection between measurements and point-sets in $\mathbb{R}^d$ can be established. However, it is not difficult to convince oneself that not all different measurements can be simply linked to, and represented by, points in $\mathbb{R}^d$.[12]

Secondly, it is desirable to be able to consider multi-resolution approximations of functions in spaces that may not have a Hilbert-space structure, or those which may be defined over arbitrary domains equipped with different measures.

To overcome these limitations, we propose the following disposition: Let the original observable be an element of a *Banach space*[13] $\mathcal{F}$ with a *separable*

[12] Think for example of samples that represent averages of a function over irregular domains of different shapes and sizes in the d-dimensional Euclidean space. How is one going to link each of these domains to a single point in $\mathbb{R}^d$?

[13] A Banach space is a complete space equipped with a norm. Examples of separable Ba-

dual $\mathcal{F}^*$.[14] Every *continuous linear functional* (hereafter *functional*; cf. A.2.4) on $\mathcal{F}$ is a mapping from $\mathcal{F}$ to a field $\mathbb{F}$ of scalars, and therefore represents one type of measurement on elements of $\mathcal{F}$. We define a *sampling procedure* Λ as a countable collection of functionals, that is, a countable subset of the *dual* (A.2.4) $\mathcal{F}^*$ of $\mathcal{F}$. Next, we define a discrete signal as an element of the space $\ell^p(\Lambda)$ for some pre-specified p.

Thus, in this more general definition of a discrete signal, each sample value is no longer associated with a point in the Euclidean $\mathbb{R}^d$ space, but is rather paired with a point in the dual $\mathcal{F}^*$ of the space $\mathcal{F}$ of observables.

1.2.2 Multi-Resolution Representations. We suppose a sequence (Λ_i) of countable subsets of $\mathcal{F}^*$ with the property that the linear span of

$$\Lambda_\infty := \lim_{i\to\infty} \Lambda_i$$

is weak*ly[15] dense in $\mathcal{F}^*$. It then follows from the next theorem that a set of evaluations over Λ_∞ identify exactly one element of $\mathcal{F}$.[16]

1.2.3 Theorem. Let $x \in \mathcal{F}$. Having λx for all $\lambda \in \Lambda_\infty$ identifies exactly one $x \in \mathcal{F}$ iff span Λ_∞ is weak*ly dense in $\mathcal{F}^*$.

Proof of sufficiency. We will prove that if $\lambda x = \lambda y$ for some $x, y \in \mathcal{F}$ and all

nach spaces include L^p and ℓ^p spaces with $1 \le p < \infty$ (cf. A.3.5 and A.3.6). See A.3.3 for further discussion.

14 cf. A.2.2. From the separability of $\mathcal{F}^*$ the separability of $\mathcal{F}$ also follows (Megginson, 1998, theorem 1.2.11).

15 Weakly* and weak* are more common, and are pronounced weakly-star and weak-star respectively; however, weak-star-ly is perhaps more grammatically precise. The adjective weak* refers to properties that are true with respect to the weak* topology (A.2.5).

16 This condition was communicated to the author (without proof) by Dr Robert Israel of UBC in answer to his question on Usenet (Israel, 2005). I here provide an original proof.

$\lambda \in \Lambda_\infty$, then $x = y$.

As span Λ_∞ is dense in $\mathcal{F}^*$, for any $x^* \in \mathcal{F}^*$ there exists a sequence (x_n^*) in span Λ_∞ that weak*ly converges to x^*.

Now, from the assumption of $\lambda x = \lambda y$ for all $\lambda \in \Lambda_\infty$ it follows that $x^*x = x^*y$ for all $x^* \in \mathcal{F}^*$. This is because:

The natural map defined by

$$F_f x^* := x^* f$$

for arbitrary $f \in \mathcal{F}$ is by definition continuous in the weak* topology. Consequently,

$$F_{x-y} x_n^* \to F_{x-y} x^* \quad \text{as } n \to \infty. \tag{1.15}$$

Now, as for all n, x_n^* are chosen to be in span Λ_∞, we have

$$F_{x-y} x_n^* := x_n^*(x - y) = x_n^* x - x_n^* y = 0.$$

The left-hand-side of (1.15) is therefore identically zero, and we have

$$F_{x-y} x^* := x^*(x - y) = 0 \quad \text{for all } x^* \in \mathcal{F}^*. \tag{1.16}$$

We will now proceed to show that (1.16), together with the assumption that $x \neq y$ lead to contradiction. Specifically, if $x \neq y$, let $z := x - y$. $\mathcal{Z} := \text{span}\{z\} = \{az | a \in \mathbb{F}\}$ is a subspace of $\mathcal{F}$. We define the map $\eta : \mathcal{Z} \longrightarrow \mathbb{F}$

by

$$\eta(az) := a.$$

η is clearly linear and therefore a functional on $\mathcal{Z}$. By the Hahn-Banach Extension Theorem (cf. A.3.7) we can extend η to a continuous linear functional over $\mathcal{F}$ (i.e. a functional in $\mathcal{F}^*$). But then,

$$\eta(x - y) := \eta z = 1,$$

which contradicts (1.16). □

Proof of necessity. We will prove this direction again by contradiction. Assume that the set of values λx for all $\lambda \in \Lambda_\infty$ identifies a unique $x \in \mathcal{F}$ while $\operatorname{span} \Lambda_\infty$ is *not* dense in $\mathcal{F}^*$. We can therefore find an $x_0^* \in \mathcal{F}^*$ such that $x_0^* \notin \overline{\operatorname{span} \Lambda_\infty}$. From lemma 2.10.1 of Hille and Phillips (1957), there exists an $x_0 \in \mathcal{F}$ for which we have $x_0^* x_0 = 1$, but $x^* x_0 = 0$ for all $x^* \in \operatorname{span} \Lambda_\infty$.

x_0 is obviously non-zero. It follows that for any $\lambda \in \operatorname{span} \Lambda_\infty$,

$$\lambda x = \lambda(x + \alpha x_0), \tag{1.17}$$

for any scalar α, which contradicts the initial uniqueness assumption. □

1.2.4 Remark. Let $\mathcal{F} = C_0(\Omega)$ (i.e. the space of contiuous functions with compact support). Then choosing Λ_i to be the set of functionals corresponding to point-wise evaluations on a point-set Θ_i satisfying the requirements of section 1.1, leads to all possible muti-resolution representations of section 1.1. We therefore see that this new approach includes all cases discussed

in the previous section. That the linear span of $\lim_{i\to\infty} \Lambda_i$ is weak*ly dense in $\mathcal{F}^*$ for $\mathcal{F} = \mathbf{C}_0(\Omega)$ follows from the following lemma.

1.2.5 Lemma. With the definitions of remark 1.2.4, $\operatorname{span} \Lambda_\infty$ is weak*ly dense in the dual of the space $\mathbf{C}_0(\Omega)$ of continuous functions with compact support.

Proof. As a continuous function is uniquely identified by its value over a dense subset of its domain (in this case $\Theta_\infty \subset \Omega$), from theorem 1.2.3 we know that $\operatorname{span} \Lambda_\infty$ is weak*ly dense in the dual of $\mathbf{C}_0(\Omega)$. □

The space $\mathbf{C}_0(\Omega)$ is itself dense in $\mathbf{L}^p(\Omega)$ for $1 \leq p < \infty$ (Adams and Fournier, 2003, theorem 2.19). Therefore, these discrete representations of continuous functions in the limit ($i \to \infty$) identify a dense subset of $\mathbf{L}^p(\Omega)$.

1.2.6 Refinement Operators. Now that our spaces of discrete signals have been defined and we have formally described multi-resolution representations, we are in the position to address the issue of travelling between these spaces. Similar to the definitions of 1.1.8, the means for this change of resolution will be *refinement operators,* which are mappings from spaces of lower-resolution signals to spaces of finer representations.

Formally, for a sequence (Λ_i) satisfying the properties stated in 1.2.2, we define refinement operators as bounded linear operators $\mathfrak{S}_i : \ell^p(\Lambda_i) \longrightarrow \ell^p(\Lambda_{i+1}) : a \mapsto (\mathfrak{S}_i a)$ with

$$(\mathfrak{S}_i a)[\lambda] := \sum_{\mu\in\Lambda_i} s_{i,\lambda,\mu} a[\mu] \quad \text{for } \lambda \in \Lambda_{i+1}. \tag{1.18}$$

The following theorem provides conditions on the weights $s_{i,\lambda,\mu}$ assuring that $\mathfrak{S}_i$ is bounded.

1.2.7 Theorem. $\mathfrak{S}_i : \ell^p(\Lambda_i) \longrightarrow \ell^p(\Lambda_{i+1})$, defined in 1.18, is bounded for $1 < p \leq \infty$ if for $q := p/(p-1)$,

$$\left(\sum_{\mu \in \Lambda_i} |s_{i,\lambda,\mu}|^q \right)^{1/q}$$

converges and is bounded for all $\lambda \in \Lambda_{i+1}$. Also, $\mathfrak{S}_i$ is bounded for $p = 1$ if $\sup_{\mu \in \Lambda_i} s_{i,\lambda,\mu}$ is bounded for all $\lambda \in \Lambda_{i+1}$.

Proof. Consider an arbitrary $a \in \ell^p(\Lambda_i)$. For all valid i, and for $\lambda \in \Lambda_{i+1}$, form the mapping $s_{i,\lambda} : \Lambda_i \longrightarrow \mathbb{F} : \mu \mapsto s_{i,\lambda,\mu}$. The stated conditions are equivalent to

$$s_{i,\lambda} \in \ell^q(\Lambda_i).$$

The conclusion then follows from Hölder's inequality:[17]

$$\|a s_{i,\lambda}\|_1 \leq \|a\|_p \|s_{i,\lambda}\|_q,$$

that is,

$$\sum_{\mu \in \Lambda_i} |a[\mu] s_{i,\lambda}[\mu]| \leq \left(\sum_{\mu \in \Lambda_i} |a[\mu]|^p \right)^{1/p} \cdot \left(\sum_{\mu \in \Lambda_i} |s_{i,\lambda}[\mu]|^q \right)^{1/q}.$$

As the right-hand-side is bounded when the conditions of the theorem are

[17] $\|xy\|_1 \leq \|x\|_p \|y\|_q$.

satisfied, and since

$$\sum_{\mu\in\Lambda_i} a[\mu]s_{i,\lambda}[\mu] \leq \sum_{\mu\in\Lambda_i} |a[\mu]s_{i,\lambda}[\mu]|,$$

$\mathfrak{S}_i$ is bounded. □

1.2.8 Adding Details. At the i-th stage, new details may be added in the form of a signal $d_i \in \ell^p(\Lambda_i)$ (cf. 1.1.7). These details account for the difference between the information that can be represented by an $\ell^p(\Lambda_i)$ signal, and that representable as $\mathfrak{S}_{i-1}a_{i-1}$ for some $a_{i-1} \in \ell^p(\Lambda_{i-1})$.

1.2.9 The Cascade Algorithm. The cascade algorithm in this new setting is quite similar to that mentioned in the previous section; and, when convergent to a function in $\mathcal{F}$, leads to a similar definition of refinable functions.

Specifically, let $a_k \in \ell^p(\Lambda_k)$ for some chosen p, and for a sequence of refinement operators $(\mathfrak{S}_i)$, define:

$$a_i := \mathfrak{S}_{i-1}a_{i-1}.$$

We have already shown in theorem 1.2.3 that any $x \in \mathcal{F}$ can be uniquely identified by the values λx, $\lambda \in \Lambda_\infty$. Now if for *any* initial choice of $a_k \in \ell^p(\Lambda_k)$ (for a fixed p), as $i \to \infty$, the sequence (a_i) converges to a mapping a_∞ that identifies a function $f_\infty^{a_k} \in \mathcal{F}$ through

$$\lambda f_\infty^{a_k} = a_\infty[\lambda] \quad \text{for all } \lambda \in \Lambda_\infty, \tag{1.19}$$

we say that the cascade algorithm converges.[18] We will assume the convergence of the cascade algorithm in the sequel.

1.2.10 Refinable Functions. Similar to the previous section, when the cascade algorithm converges, we will denote by $e_{m\to n,\lambda}$ (with $n > m$, $\lambda \in \Lambda_m$) the mapping iteratively defined through

$$e_{m\to n,\lambda} := \mathfrak{S}_{n-1} e_{m\to n-1,\lambda},$$

with $e_{m\to m,\lambda} := e_{m,\lambda} := \delta_\lambda \in \ell^p(\Lambda_m)$.

For $\lambda \in \Lambda_k$, let g_k^λ be the (unique) function in $\mathcal{F}$ satisfying

$$\mu g_k^\lambda = e_{k\to\infty,\lambda}[\mu] := \left(\lim_{j\to\infty} e_{k\to k+j,\lambda}\right)[\mu] \quad \text{for } \mu \in \Lambda_\infty; \tag{1.20}$$

i.e. $g_k^\lambda = f_\infty^{e_{k,\lambda}} = f_\infty^{\delta_\lambda}$ in the notation of eqn (1.19) (with $\delta_\lambda \in \ell^p(\Lambda_i)$). The following proposition holds:

1.2.11 Proposition.

$$g_k^\lambda = \sum_{\mu\in\Lambda_{k+1}} e_{k\to k+1,\lambda}[\mu] g_{k+1}^\mu = \sum_{\mu\in\Lambda_{k+1}} s_{k,\mu,\lambda} g_{k+1}^\mu. \tag{1.21}$$

The proof is similar to that of proposition 1.1.15. □

[18] In practice we often choose $\mathcal{F}$ to be a space of sufficiently smooth functions (e.g. $C_0^N(\Omega)$ for some N). In this thesis we will not further study conditions for the convergence of the cascade algorithm to a smooth function, as these can be quite involved, especially in the irregular case. The interested reader may wish to review, among others, Daubechies et al. (2001) for one possible analysis. (cf. footnote 11 on p. 14.)

1.2.12 Multi-Resolution Spaces. Again, multi-resolution approximation spaces[19] $\mathcal{V}_i$ can be defined as the linear span of functions g_i^λ, $\lambda \in \Lambda_i$. That is,

$$\mathcal{V}_i := \mathrm{span}\{g_i^\lambda | \lambda \in \Lambda_i\}.$$

It then follows from proposition 1.2.11 that these spaces are nested. Also, following a discussion similar to that of 1.1.16, we see that when starting with an arbitrary signal $a_k \in \ell^p(\Lambda_k)$, the limit function of the cascade algorithm will be

$$f_\infty^{a_k} = \sum_{\lambda \in \Lambda_k} a_k[\lambda] g_k^\lambda, \tag{1.22}$$

which resides in $\mathcal{V}_k$.

[19] For a discussion of how this definition relates to and extends the classical definitions of multi-resolution analysis, see footnote 8 on p. 11.

P.D. Tafti: On Multi-Scale Refinement of Discrete Data.
M.A.Sc. thesis. Department of Electrical and Computer Engineering,
McMaster University, Hamilton, Ontario, Canada, 2005.

2

A New Construction for Refinement Operators

> A theory has only the alternative of being right or wrong. A model has a third possibility: it may be right, but irrelevant.
>
> *In* The Physicist's Conception of Nature, *edited by Jagdish Mehra (p. 618). Dordrecht, 1973.*
> MANFRED EIGEN, 1967 NOBEL LAUREATE IN CHEMISTRY (1927–)

IN THE PREVIOUS CHAPTER we saw that multi-resolution representations and related multi-resolution spaces of functions could be described in terms of a sequence of sampling procedures (for different resolutions), which define spaces of discrete signals; and refinement operators that link these signal spaces. A fundamental question that we did not answer then, was that of how to choose the refinement coefficients that define those refinement operators. We briefly mentioned that properties of the underlying function and

the sampling procedure should be taken into consideration, but did not clarify what we exactly meant by them; nor did we indicate how these properties could be considered.

In this chapter we will propose a novel construction for refinement operators, that is naturally linked to the sampling procedures underlying our discrete signal spaces, and also allows us to account for the geometrical relationship between the samples. But first, we will review some related examples of multi-resolution on uni-dimensional domains, that have been previously studied in the literature (section 2.1). We will follow by introducing some useful mathematical notions and results. Then, after revisiting two examples of multi-resolution on lattices, we will continue to introduce our new construction, that is based on local functional interpolation.

2.1 Multi-Resolution on Uni-Dimensional Domains

As the idea of associating signal samples with points in space is quite common (see 1.1), not surprisingly, *interpolating* multi-resolution schemes—schemes that in the limit converge to functions *passing through* the original samples—have received considerable attention in the past. A central tool in many of the devised schemes, both single- and multi-dimensional, is *Lagrange interpolation* (see e.g. Deslauriers and Dubuc, 1989; Kovačević and Sweldens, 2000; Daubechies et al., 1999).

Interest has also existed in *average interpolation,* where samples are not

considered as point-wise evaluations of a function, but rather as integrals or averages on short intervals in the uni-dimensional case (Donoho, 1994) or on partitions in multi-dimensional settings (Tafti et al., 2005).

Multi-resolution refinement schemes based on Lagrange and average interpolation were first studied for regular uni-dimensional point-sets, and later generalized to multi-dimensional and/or irregular settings. To provide an *entrée en matière* for our ultimate discussion of new families of multi-resolution schemes, in this section we review uni-dimensional constructions for Lagrange and average interpolating refinement.

2.1.1 Lagrange Interpolation. Lagrange interpolation deals with finding polynomial solutions to point-wise interpolation problems: Let Θ be a finite set of *distinct* points from $\mathbb{R}$, and let $a \in \ell^p(\Theta)$, for some p, represent a set of values associated with these points. In Lagrange interpolation, one then tries to find the lowest degree polynomial π_a that satisfies the conditions

$$\pi_a(\theta) = a[\theta] \quad \text{for } \theta \in \Theta.$$

In the univariate case, i.e. when $\Theta \subset \mathbb{R}$, it is well understood that the Lagrange interpolation problem has a unique solution, that can be found by solving the following linear algebra equation:[1,2]

$$\left[\theta^j\right]_{\substack{\theta\in\Theta \\ 0\le j<n}} \mathbf{c} = \mathbf{a}, \tag{2.1}$$

[1] For the rest of this chapter, it is easier to assume an order on point-sets Θ (and later on sampling procedures Λ), and write the equations in the vector form. Then, for an $a \in \ell^p(\Theta)$, $\mathbf{a}$ is used to denote the vector $\left[a[\theta]\right]_{\theta\in\Theta}$.

[2] We use the convention that $0^0 := 1$.

where $\mathbf{c} = \left[c_j\right]_{0\le j<n}$ is the vector of interpolant coefficients (i.e. c_j is the coefficient of $()^j$), $\mathbf{a} := \left[a[\theta]\right]_{\theta\in\Theta}$, and $n := \#\Theta$. $\left[\theta^j\right]_{\substack{\theta\in\Theta\\0\le j<n}}$ is known as a *Vandermonde* matrix. For an arbitrary point ϕ we then have

$$\pi_a(\phi) = c_0 + \sum_{0<j<n} c_j\phi^j = \left[\phi^j\right]'_{0\le j<n}\mathbf{c} = \left[\phi^j\right]'_{0\le j<n}\left(\mathrm{inv}\left[\theta^j\right]_{\substack{\theta\in\Theta\\0\le j<n}}\right)\mathbf{a}, \tag{2.2}$$

which shows that for each map $a \in \ell^p(\Theta)$, $\pi_a(\phi)$ can be evaluated by taking a weighted average of the sample values (represented by the vector $\mathbf{a}$, with weights that depend on Θ and ϕ (represented by $\left[\phi^j\right]'_{0\le j<n}\left(\mathrm{inv}\left[\theta^j\right]_{\substack{\theta\in\Theta\\0\le j<n}}\right)$).

The above equation shows that Lagrange interpolation is linear; that is, given two different maps a and b on the same point-set, and a scalar β, the polynomial π_{a+b} that interpolates $\beta a + b$ is equal to $\beta\pi_a + \pi_b$. Therefore, the solutions to all interpolation problems on Θ form a *sub-space* Π_Θ of Π. We say Π_Θ is *correct* for Θ to mean that the interpolation problem has a *unique* solution in Π_Θ for any $a \in \ell^p(\Theta)$.

Eqn (2.2) also shows that the set $\{1, ()^1, \ldots, ()^{n-1}\}$ is a basis for all sub-spaces Π_Θ with $\#\Theta = n$,[3] which means that all these sub-spaces are in fact the same: the space $\Pi_{<n}$ of all polynomials of degree less than n.

2.1.2 Deslauriers and Dubuc's Lagrange Iterative Interpolation. A little less than twenty years ago, Deslauriers and Dubuc introduced their now famous iterative interpolation scheme (see Deslauriers and Dubuc, 1989). This interpolation scheme consists in iterative insertion of new sample values at mid-points of a discrete point-set—thus forming a sequence of dyadic point-sets (1.1.3)—by interpolating neighbour sample values at each stage.

[3] $()^j : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto x^j$ is a monomial map (see also 2.2.2).

To be specific, suppose that (Θ_i) is a sequence of dyadic point-sets, and that we are given a map $a_0 \in \ell^p(\Theta_0)$. Let a_i be the ith level data map. The values associated with points in Θ_{i+1} are then calculated iteratively from values for points in Θ_i:

For each new mid-point $\theta \in \Theta_{i+1} \backslash \Theta_i$, let N_θ be the subset of Θ_i containing the n points (n even) in Θ_i closest to θ. The new sample value at θ, which is $a_{i+1}[\theta]$, is calculated by interpolating the values $a_i[\phi]$ for $\phi \in N_\theta$ with a Lagrange interpolant of degree $n-1$. For points θ that exist both in Θ_i and in Θ_{i+1}, $a_i[\theta]$ and $a_{i+1}[\theta]$ will be equal.

Using the formulation of section 1.1, the Deslauriers-Dubuc iterative Lagrange interpolation scheme is equivalent to applying refinement operators $\mathfrak{S}_i$ defined by the following formula:

$$(\mathfrak{S}_i a_i)[\theta] = \begin{cases} a_i[\theta] & \text{for } \theta \in \Theta_i, \\ \sum_{\phi \in N_\theta} s_{i,\theta,\phi} a_i[\phi] & \text{for } \theta \in \Theta_{i+1} \backslash \Theta_i, \end{cases} \tag{2.3}$$

where from eqn (2.2) we have

$$\left[s_{i,\theta,\phi}\right]_{\phi \in N_\theta} = \left[\theta^j\right]'_{0 \le j < n} \operatorname{inv}\left[\phi^j\right]_{\substack{\phi \in N_\theta \\ 0 \le j < n}}.$$

Furthermore, from the uniqueness of the interpolant (which follows from that the Vandermonde matrix is not singular), with a simple change of variables we can see that the interpolation scheme is shift and scaling invariant, meaning that the refinement weights do not change across scales, and also that the vectors of refinement weights used to calculate values for any two

Table 2.1: Deslauriers and Dubuc's iterative interpolation scheme may be represented by a vector of refinement weights. These weights are used to compute $a_{i+1}[\theta]$ from $a_i[\phi]$ for $\phi \in N_\theta$. In this table (from Kovačević and Sweldens (2000)) points from $N_\theta \subset \Theta_i$ are marked with crosses and the new mid-point from Θ_{i+1} is marked with a dot. n is the size of the neighbourhood.

n	×	×	×	×	·	×	×	×	×
2	0	0	0	$1/2$		$1/2$	0	0	0
4	0	0	$-1/2^2$	$9/2^2$		$9/2^2$	$-1/2^2$	0	0
6	0	$3/2^8$	$-25/2^8$	$150/2^8$		$150/2^8$	$-25/2^8$	$3/2^8$	0
8	$-5/2^{11}$	$49/2^{11}$	$-245/2^{11}$	$1225/2^{11}$		$1225/2^{11}$	$-245/2^{11}$	$49/2^{11}$	$-5/2^{11}$

points in $\Theta_{i+1}\backslash\Theta_i$ have the same elements.

The example we studied in 1.1.10 is the simplest case of Deslauriers-Dubuc interpolation, with $n = 2$. Table 2.1, from Kovačević and Sweldens (2000), summarizes refinement coefficients for some other values of n.

2.1.3 Extension to Non-Uniform Settings. The idea behind Deslauriers-Dubuc interpolation is easily extensible to non-uniform settings. Again, supposing that we have a nested sequence $(\Theta_i)_{i\geq 0}$ of point-sets, for each new point $\theta \in \Theta_{i+1}\backslash\Theta_i$, the value $a_{i+1}[\theta]$ is calculated by locally interpolating the $a_i[\phi]$ values for ϕs in a certain set N_θ (which can be of different sizes for different θs) of points in Θ_i neighbouring θ. For points $\theta \in \Theta_{i+1} \cap \Theta_i$,[4] the value of $a_i[\theta]$ is copied to $a_{i+1}[\theta]$.[5]

Even in the uniform setting, this approach provides a natural answer for multi-resolution refinement on bounded domains; since for points near the boundaries, the neighbours can be chosen from one side. This results in different refinement weight vectors for central versus near-the-boundary

[4] Since (Θ_i) is nested, we actually have $\Theta_{i+1} \cap \Theta_i = \Theta_i$.

[5] This is in fact a second generation construction (Sweldens, 1997), and was provided in Sweldens and Schröder (1996) as an example (with a different formulation).

points.

It follows from eqn (2.2) that this generalization of Deslauriers-Dubuc interpolation also results in linear operators which can be realized similar to (2.3). But in this case the refinement weights are in general no longer similar for different locations and resolutions.

2.1.4 Average Interpolation. Donoho in 1994 suggested that a multi-resolution refinement scheme could be based on the idea of *average interpolation.* Unlike in Lagrange interpolation, where we are looking for a polynomial that takes given values at given points, in average interpolation one tries to find a polynomial which has given *averages* on given *intervals.*

In average interpolating refinement, one begins with a set Υ_0 of non-overlapping equi-length intervals υ that partition the real axis. A dyadic sequence of interval-sets, (Υ_i), may be iteratively constructed based on the following rule: Υ_{i+1}, $i \geq 0$, is formed by subdividing each interval υ in Υ_i into two equi-length intervals, υ_L and υ_R.[6] In this scheme we start with a signal a_0 on Υ_0—which we assume indicates averages of a function on the intervals $\upsilon \in \Upsilon_0$—and iteratively construct a multi-resolution sequence (a_i) of signals. A description of the scheme for refining the ith level representation follows:

First, for $\upsilon \in \Upsilon_i$, let N_υ be the subset of Υ_i containing the n closest neighbours of υ in Υ_i (υ included). The values $a_{i+1}[\upsilon_L]$ and $a_{i+1}[\upsilon_R]$, with $\upsilon_L, \upsilon_R \in \Upsilon_{i+1}$ being the subdivisions of υ, are then computed by first finding

[6] L and R are for *left* and *right.*

the lowest degree polynomial π_υ satisfying

$$\frac{\int_\omega \pi_\upsilon(t)\,dt}{\int_\omega 1\,dt} = a_i[\omega] \quad \text{for all } \omega \in N_\upsilon;$$

and subsequently letting

$$a_{i+1}[\upsilon_L] := \frac{\int_{\upsilon_L} \pi_\upsilon(t)\,dt}{\int_{\upsilon_L} 1\,dt},$$
$$a_{i+1}[\upsilon_R] := \frac{\int_{\upsilon_R} \pi_\upsilon(t)\,dt}{\int_{\upsilon_R} 1\,dt}.$$

One question remains, and that is how the average interpolating polynomial can be found. In this case the answer lies in solving a modified version of (2.1) that involves a modified Vandermonde matrix, containing averages of monomials on different intervals rather then their evaluations at different points:

$$\left[\frac{\int_\omega t^j\,dt}{\int_\omega 1\,dt}\right]_{\substack{\omega\in N_\upsilon \\ 0\le j<n}} \mathbf{c}_\upsilon = \mathbf{a}_i. \tag{2.4}$$

Also, similar to (2.2), we can see that $\int_{\upsilon_L} \pi_\upsilon(t)\,dt$ and $\int_{\upsilon_R} \pi_\upsilon(t)\,dt$ (and therefore $a_{i+1}[\upsilon_L]$ and $a_{i+1}[\upsilon_R]$), can be calculated from a weighted average of $a_i[\omega]$, $\omega \in N_\upsilon$. This shows that average interpolating refinement also fits in the framework of 1.2.

2.1.5 Extensions. It is not necessary to divide each partition into two at each level—any other number will do. Also, extension to the case where we have an irregular partitioning of the real line is straight-forward. We can choose neighbourhoods N_υ of different sizes for different intervals $\upsilon \in \Upsilon_i$

that we want to subdivide, and find the values $a_{i+1}[v_L]$ and $a_{i+1}[v_R]$ by averaging, over v_L and v_R respectively, the polynomial that average-interpolates $a_i[\omega]$, $\omega \in N_v$ (cf. Sweldens and Schröder, 1996).

2.2 A Few Mathematical Notions of Subsequent Utility

Generalization of the previously described schemes to multivariate cases is in general not straight-forward.[7] Simple as the interpolation problem may seem in one variable, it is much more involved in the multivariate case. First of all, unlike the single-variable case, no obvious basis of monomials of degree less than n exists for arbitrary n; therefore, it is not clear how an invertible Vandermonde matrix can be formed.[8] Secondly, due to a situation known as the *loss of Haar,* no n-dimensional subspace of Π^d, the space of d-variate polynomials, is correct for all sets of n points.[9]

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7 Of course in the case of *separable* lattices one may construct a multi-dimensional scheme by applying uni-dimensional refinement masks along different dimensions. But this case is quite limiting, as most lattices are not separable, and moreover, this approach is not generalizable to irregular settings. Furthermore, we will later introduce more general constructions for multi-resolution refinement, based on local functional interpolation, that would not have been possible if we were limiting ourselves to separable lattice structures.

8 For example, in the bivariate case, when $n = 5$ how should we—or indeed should we—choose a subset of $\{1, x, y, x^2, y^2, xy\}$ in forming the Vandermonde matrix?

9 One of the challenges in multivariate interpolation is that, unlike the univariate case, no n-dimensional subspace of the space of polynomials (or any other space of continuous functions) is correct for all point-sets Θ with cardinality n. In other words, in the multivariate case in addition to their number, the geometrical relationship of the points also becomes important. This can be seen for example from the following argument by de Boor (1992):

Consider an n-dimensional ($n > 1$) subspace P of $C(\mathbb{R}^d)$ ($d > 1$) with a basis $\{\hat{m}_0, \ldots, \hat{m}_{n-1}\}$, and a set Θ of n distinct points in $\mathbb{R}^d$. For P to be correct for Θ,

Nevertheless, Lagrange and average interpolation are only examples of a more general conception of interpolation that we are just about to introduce.

2.2.1 Functional Interpolation. The definition of an interpolation problem may be generalized to that of finding a function q in a space $\mathcal{Q}$ that satisfies a set of functional equations of the form

$$\lambda q = a[\lambda] \quad \text{for } \lambda \in \Lambda, \tag{2.7}$$

with $\Lambda \subset \mathcal{Q}^*$ being a set of functionals on $\mathcal{Q}$, and $a \in \ell^p(\Lambda)$ for some p.

Lagrange interpolation will then be concerned with the special case where $\mathcal{Q}$ is the space of polynomials of degree $< \#\Lambda$, and λs are point-wise evalu-

---

the (generalized) Vandermonde matrix

$$\left[\hat{m}_j(\theta)\right]_{\substack{\theta\in\Theta \\ 0\le j<n}}$$

should be non-singular, as the interpolation problem is equivalent to solving the linear system (cf. eqn (2.1))

$$\left[\hat{m}_j(\theta)\right]_{\substack{\theta\in\Theta \\ 0\le j<n}} \mathbf{c} = \mathbf{a}. \tag{2.5}$$

Now consider a continuous curve

$$\gamma : [0,1] \longrightarrow (\mathbb{R}^d)^n : t \mapsto (\gamma_0(t), \ldots, \gamma_{n-1}(t)),$$

and let us define the function

$$g : [0,1] \longrightarrow \mathbb{R} : t \mapsto \det\left[\hat{m}_j(\gamma_i(t))\right]_{\substack{i \\ j}}.$$

g is continuous because it is a composition of continuous functions. As $n, d > 1$, we can choose γ such that

$$\gamma(1) = (\gamma_1(0), \gamma_0(0), \ldots, \gamma_n(0)), \tag{2.6}$$

while $\gamma_i(t)$s remain distinct for each value of t. (This is not true in the case of $d = 1$ because there (2.6) implies that $\gamma_0(t)$ and $\gamma_1(t)$ meet for at least one value of t.) We consequently have $g(1) = -g(0)$, which implies that g vanishes at some t_0 in $[0,1]$ (due to continuity). $\{\gamma_0(t_0), \ldots, \gamma_{n-1}(t_0)\}$ is then a set of n distinct points for which the matrix described in (2.5) is rank-deficient. □

ation functionals.

2.2.2 Multi-Indices and d-Variate Polynomials. We introduce the following notation for further usage. In the d-variate case, a *multi-index* $\alpha := [\alpha_i]_{0\le\alpha<d}$ is an element of $\mathbb{Z}_{0+}^d$ (the set of non-negative integer d-tuples). For $X := [x_i]_{0\le i<d} \in \mathbb{R}^d$, we then define:

$$X^\alpha := \prod_{0\le i<d} x_i^{\alpha_i}.$$

Also,

$$|\alpha| := \sum_{0\le i<d} \alpha_i.$$

When π is a polynomial in d variables (or more generally, a *formal power series*[10] in d indeterminates), it is convenient to denote the normalized coefficient of the αth-degree term in π by $\alpha(\pi)$; i.e.

$$\pi(X) = \sum \frac{\alpha(\pi)}{\alpha!} X^\alpha.$$

We also take this opportunity to introduce de Boor and Ron's (1992a) notation:

$$()^\alpha : \mathbb{R}^d \longrightarrow \mathbb{R} : X \mapsto X^\alpha.$$

2.2.3 de Boor and Ron's Least Solution to the Interpolation Problem. To address the ambiguity that we have already encountered in finding the interpolant in the multivariate setting, de Boor and Ron in 1990 provided

[10] A *formal power series* f is an infinite sequence, here represented in the form $\sum_{\alpha\in\mathbb{Z}_{0+}^d} \frac{\alpha(f)}{\alpha!} X^\alpha$; however, as Weisstein (2005a) indicates, with the understanding that no value is assigned to X.

a solution with many interesting properties. They generalized this solution in de Boor and Ron (1992b) to include functional interpolation using multivariate polynomials, and addressed its computational aspects, via Gaussian elimination on generalized Vandermonde matrices, in de Boor and Ron (1992a) and de Boor (1994).

Briefly speaking, their solution uses the following duality pairing between polynomials and formal power series:[11]

$$\langle \pi, f \rangle = \sum_{\alpha \in \mathbb{Z}^d_{0+}} \alpha(\pi)\alpha(f)/\alpha! = \sum_{\alpha \in \mathbb{Z}^d_{0+}} \alpha(f)\, D^\alpha \pi(0).$$

Here $D^\alpha \pi(0)$ denotes the αth partial derivative of π, evaluated at 0.

For any formal power series f, de Boor and Ron introduce the notation $f_\downarrow$ to denote the *least* term of f, i.e. the unique homogeneous polynomial for which the least-degree term of $f - f_\downarrow$ is of a higher degree than that of f (de Boor and Ron, 1992b). They then go on to prove that the space

$$\Lambda_\downarrow := \operatorname{span}\{\lambda_\downarrow | \lambda \in \operatorname{span} \Lambda\},$$

is correct for Λ, and call it the least solution to the interpolation problem.[12] Gaussian elimination by segments, on the generalized Vandermonde matrix,

$$\left[\lambda()^\alpha\right]_{\substack{\lambda \in \Lambda \\ \alpha \in \mathbb{Z}^d_{0+}}}, \tag{2.8}$$

[11] This duality pairing shows that every linear functional on Π^d can be identified with a unique power series f, and vice versa. Thus, we can identify the dual of Π^d with the space of formal power series.

[12] There is a minor difference in notation: de Boor and Ron (1992b) actually use Λ to denote what in our notation would be span Λ.

is suggested as a method for systematically finding the interpolant to given data (de Boor and Ron, 1992a; de Boor, 1994). In Gaussian elimination by segments, all columns of the Vandermonde matrix that are related to power maps of the same total degree (i.e. $()^{\alpha}$s with identical $|\alpha|$) are considered together, resulting in a *block* row-echelon matrix. The details of the interpolation are technical, and could be the subject of a monograph in their own right. Nevertheless, it is agreeable that this scheme is a justified generalization to the discussion given in 2.1.1.

In realizing our construction for refinement operators, that we will introduce in the sequel, we have used Grandine's ISO C implementation of de Boor and Ron's algorithm for finding the least solution (Grandine).

2.3 Multi-Resolution on Multi-Dimensional Lattices

Multi-dimensional lattices were introduced in example 1.1.5. In this section we will review how, with proper generalization, the Lagrange and average interpolating refinement schemes that were discussed in 2.1 can also be applied to these multi-dimensional structures.

2.3.1 Kovačević and Sweldens's Interpolating Multi-Resolution on Lattices. Kovačević and Sweldens (2000) used the machinery of de Boor and Ron for Lagrange interpolation and introduced a multi-resolution refinement scheme on multi-dimensional lattices. Their refinement scheme can be viewed as a generalization of Deslauriers-Dubuc interpolation. Using the

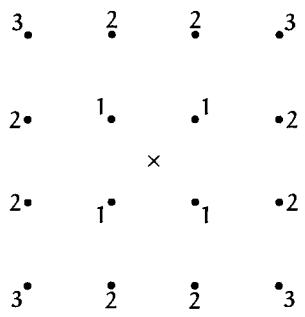


Figure 2.1: Neighbours of a point on the quincunx lattice form rings (labeled by numbers). An × marks the location of the newly inserted point at which we are interpolating.

fact that a Lattice Θ_{i+1}, as defined in 1.1.5, is a union of translates of Θ_i,[13] they suggested the following procedure:[14]

for each $\theta \in \Theta_{i+1} \backslash \Theta_i$, a neighbourhood $N_\theta \in \Theta_i$ of a fixed pre-chosen size, of closest points to θ in Θ_i are considered. $a_{i+1}[\theta]$ is computed by Lagrange interpolation of the values $a_i[\phi]$ for $\phi \in N_\theta$. For $\theta \in \Theta_{i+1} \cap \Theta_i$, the value of $a_i[\theta]$ is copied to $a_{i+1}[\theta]$.

Similar to Deslauriers and Dubuc's iterative interpolation, Kovačević and Sweldens's scheme can be represented by linear operators and therefore falls within the framework of 1.2. We will prove this in more generality in proposition 2.4.4. For example, for the quincunx lattice (see 1.1.6) and the neighbourhood *rings* shown in fig. 2.1, this scheme can be represented by refinement weights of table 2.2 (from Kovačević and Sweldens, 2000).

2.3.2 Average-Interpolating Refinement on Lattices. As a variation of the above-mentioned scheme, and also a generalization of Donoho (1994), in Tafti et al. (2005) we suggested how Donoho's average interpolating re-

[13] Or the other way around: Θ_i is a subsampled version of Θ_{i+1}.
[14] To be consistent with the rest of this thesis, we have changed the formulation.

Table 2.2: Refinement weights of Kovačević and Sweldens's Lagrange interpolating multi-resolution scheme, for neighbourhoods of sizes 4 and 12.

neighbourhood size	ring (no. points)	
	1(4)	2(8)
4	$1/2^2$	—
12	$10/2^5$	$-1/2^5$

finement scheme could be extended to arbitrary multi-dimensional lattices. There we introduced the notion of a *partitioning* $\Diamond$ of a domain Ω, as a collection of disjoint subsets $\Diamond_\theta$ of Ω, indexed by points of a lattice Θ,[15] whose union covers Ω (except possibly for a set of measure zero).

Then, considering a sequence (Θ_i) of point-sets in $\mathbb{R}^d$ with the lattice structure of 1.1.5, we specified a corresponding sequence $(\Diamond_i)$ of partitionings: provided with a partitioning $\Diamond_0$ for level 0,[16] one can construct $\Diamond_i$s by transforming $\Diamond_0$ with the same lattice matrices, G, D (as we did for Θ_0 and Θ_is in 1.1.5).

Next, for a chosen neighbourhood size n, in Tafti et al. (2005) we introduced a refinement scheme resulting in a sequence of signals (a_i), $a_i \in \ell^p(\Theta_i)$. This scheme consists in the sequential application of the following two steps, here formulated for the ith level:

1. **Insertion:** For each $\theta \in \Theta_{i+1} \backslash \Theta_i$, we first find the minimum degree polynomial solution $\pi_{i,\theta}$ to the functional interpolation problem:

$$\frac{\int_{\Diamond_{i,\phi}} \pi_{i,\theta}}{\int_{\Diamond_{i,\phi}} 1} = a_i[\phi] \quad \text{for } \phi \in N_\theta,$$

[15] We have here simplified the unnecessarily complex notation that was used in Tafti et al. (2005).

[16] This partitioning may be, for example, given by Voronoi regions of the lattice points.

where $N_\theta \subset \Theta_i$ is the set of n closest points to θ in Θ_i. $a_{i+1}[\theta]$ is then defined:

$$a_{i+1}[\theta] := \frac{\int_{\Diamond_{i+1,\theta}} \pi_{i,\theta}}{\int_{\Diamond_{i+1,\theta}} 1}.$$

This step amounts to resampling, by locally averaging at a finer scale, a polynomial with given local averages on nearby partitions. It is dubbed 'insertion' because the points $\theta \in \Theta_{i+1} \backslash \Theta_i$ and their associated values are inserted into Θ_i.

2. **Update:** In the second step, the original values $a_i[\theta]$ for $\theta \in \Theta_i$ are updated to give us $a_{i+1}[\theta]$ for $\theta \in \Theta_i$. This is necessary because $a_i[\theta]$s correspond to averages on coarser resolution (i.e. bigger) partitions in $\Diamond_i$, and should be changed to match local averages on the finer partitions in $\Diamond_{i+1}$.

   To find the new values we first find the minimum degree polynomial $\tilde{\pi}_{i,\theta}$ satisfying

$$\frac{\int_{\Diamond_{i+1,\phi}} \tilde{\pi}_{i,\theta}}{\int_{\Diamond_{i+1,\phi}} 1} = a_{i+1}[\phi] \quad \text{for } \phi \in N_\theta.$$

   This time, N_θ is a subset of $\Theta_{i+1} \backslash \Theta_i$, and we are average interpolating values calculated in the previous step. Next, $a_{i+1}[\theta]$ is calculated:

$$a_{i+1}[\theta] := \frac{\int_{\Diamond_{i+1,\theta}} \tilde{\pi}_{i,\theta}}{\int_{\Diamond_{i+1,\theta}} 1}.$$

   This step is labeled 'update' because we change the values for samples at $\theta \in \Theta_i$ in this step—unlike Lagrange interpolating refinement

where, as we saw in 2.3.1, these values were simply copied.[17]

Again, these two steps define a *linear refinement operator,* as detailed in 1.2. Linearity follows from the linearity of the solution to the interpolation problem, which again consits in solving a linear system.

Not knowing about de Boor and Ron's solution to the functional interpolation problem, in Tafti et al. (2005) we suggested that the average interpolating polynomial satisfying

$$\frac{\int_{\Diamond_\phi} \pi}{\int_{\Diamond_\phi} 1} = a[\phi] \quad \text{for } \phi \in \Theta,$$

could be found by choosing proper columns of the (semi-infinite) modified Vandermonde matrix

$$\left[\frac{\int_{\Diamond_\phi} X^\alpha}{\int_{\Diamond_\phi} 1}\right]_{\substack{\phi\in\Theta \\ \alpha\in\mathbb{Z}^d_{0+}}},$$

to form an invertible submatrix.

More specifically, we suggested choosing $A := \{\alpha_0, \ldots, \alpha_{n-1}\}$ (where $n = \#\Theta$), with $|\alpha_0| \le |\alpha_1| \le \cdots \le |\alpha_{n-1}|$, such that the matrix

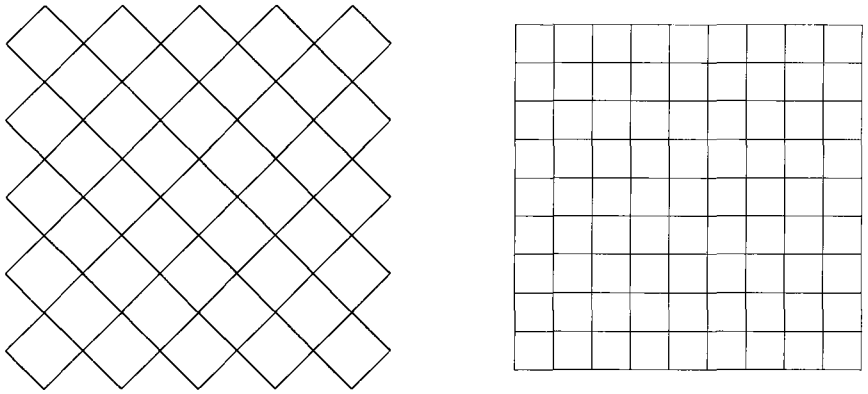
$$V := \left[\frac{\int_{\Diamond_\theta} X^\alpha}{\int_{\Diamond_\theta} 1}\right]_{\substack{\theta\in\Theta \\ \alpha\in A}}$$

would be invertible, and $|\alpha_{n-1}|$ would be minimum among all possible choices of such n αs.[18] Then, our suggested solution to the interpolation problem

[17] Note that the inputs to this step are sample values $a_{i+1}[\theta]$ for $\theta \in \Theta_{i+1}\backslash\Theta_i$, themselves obtained at the insertion step from $a_i[\phi]$s. Alternatively, we could have calculated the outputs of this step directly from $a_i[\phi]$s.

[18] We then did not address the uniqueness problem. However, it turns out that due to the degree-reducing property of de Boor and Ron's solution (see de Boor and Ron, 1992b,

Figure 2.2: Partitionings $\Diamond_i, \Diamond_{i+1}$ for subsequent levels are related by the same transformation matrices G, D that relate lattice point-sets Θ_i, Θ_{i+1} of the two levels (depicted here for the quincunx lattice).



(a) Level i (b) Level $i+1$

would be obtained by solving the linear system

$$Vc = a.$$

2.3.3 Example: Average-Interpolating Refinement on the Quincunx Lattice. As an example for this scheme, we may consider average-interpolating refinement on the quincunx lattice (1.1.6). The matrix $D = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$, that relates Θ_i to Θ_{i+1}, for this lattice corresponds to a rotation by 45° and a scaling by a factor of $\sqrt{2}$. Accordingly, each partition in $\Diamond_{i+1}$ is obtained by a $-45°$ rotation and $1/\sqrt{2}$ scaling of a partition in $\Diamond_i$ (fig. 2.2).

The neighbourhood rings for insertion and update steps are shown in fig. 2.3. Weights for insertion and update steps for two neighbourhood sizes are summarized in table 2.3.

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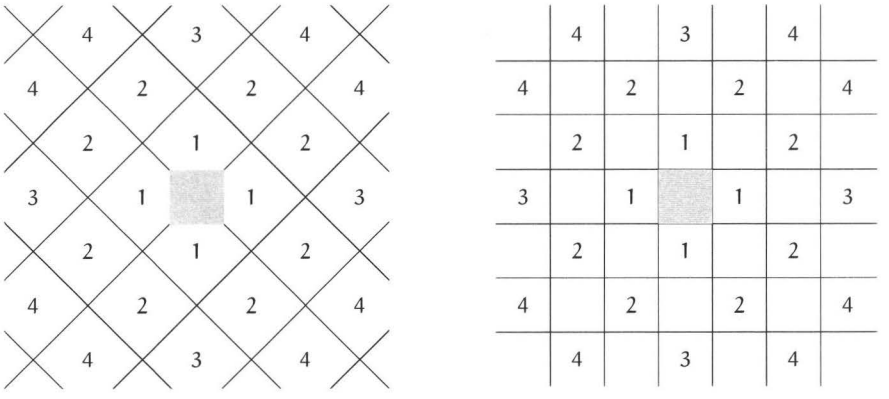
theorem 5.10), their solution satisfies the requirements set in Tafti et al. (2005).

Figure 2.3: Neighbourhoods for average interpolation on the quincunx lattice have different shapes for insertion and update steps. Here neighbourhood rings are labeled by numbers and the partition at which we are interpolating is shaded.



(a) Insertion (b) Update

Table 2.3: Insertion and update weights for average-interpolating refinement of Tafti et al. (2005) on the quincunx lattice.

neighbourhood size	Insertion		Update	
	on ring 1	on ring 2	on ring 1	on ring 2
4	0.2500	—	0.2500	—
12	0.3229	−0.0365	0.3125	−0.0313

2.4 Refinement Based on Functional Interpolation

The schemes mentioned in the previous sections all shared several key features. They all depended on interpolation with polynomials (Lagrange or average interpolation) and, as a result, could be represented by *linear* refinement operators. Also, the properties of the underlying signal, in the sense of the correlation between nearby samples (which itself depends on the smoothness of the underlying function), could be taken into account by

choosing neighbourhoods of different sizes.

As we have already implied, Lagrange and average interpolation are examples of functional interpolation (2.2.1). And while the focus in previous sections of this chapter has been only on the special cases of Lagrange and average interpolation, and then again only in one dimension or on multi-dimensional lattices, basing our refinement scheme directly on functional interpolation, as we will see, allows us to consider refinement on arbitrary domains, or in different functional spaces, in much more generality. It also provides us with the means to naturally consider the underlying sampling procedure. Furthermore, the resulting class of refinement schemes can be directly implemented, since as we saw, a powerful machinery for polynomial functional interpolation already exists (cf. 2.2.3).

When sampling a function on the continuum, the correlation between nearby sample values depends on the properties of this function. The smoother the function is, the more correlated these sample values become. In designing a multi-resolution scheme, this could be taken into consideration by choosing *neighbourhoods* of samples, on which samples at the next finer resolution would depend. Introducing the notion of neighbourhoods also allows us to deal with arbitrary domains and different boundary conditions in a consistent manner.

2.4.1 Reminder: Multi-Resolution Representations and Signals. The reader will recall that in 1.2 we discussed how a multi-resolution sequence of discrete signals (a_i) could be represented by elements of a sequence of discrete signal spaces: $(\ell^p(\Lambda_i))$ for some p; where Λ_is (the sampling procedures for different resolutions) are subsets of $\mathcal{F}^*$, the dual of the Banach

space $\mathcal{F}$ that our discrete signal spaces approximate. We further required that $\mathsf{span}(\Lambda_\infty)$ be weak*ly dense in $\mathcal{F}^*$, thus allowing any function in $\mathcal{F}$ to be represented uniquely by a collection of sample values on Λ_∞ (see theorem 1.2.3).

We will attend to this relationship between multi-resolution representations and sets of functionals in our upcoming construction for refinement operators.

2.4.2 Neighbourhoods. We have already argued the utility of the notion of neighbourhoods. This notion will now be formally defined. With each $\lambda \in \Lambda_{i+1}$, we associate a finite set $N_\lambda \subset \Lambda_i$ and call it the neighbourhood of λ. As we will see, the values $a_i[\mu]$, $\mu \in N_\lambda$, are those that will be considered when computing $a_{i+1}[\lambda]$.

2.4.3 Refinement Based on Functional Interpolation. We are now in a position to finally define our refinement operators $\mathfrak{S}_i$. This task, against the provided background, is now pleasantly simple: Given $a_i \in \ell^p(\Lambda_i)$, for each $\lambda \in \Lambda_{i+1}$ with a neighbourhood $N_\lambda \subset \Lambda_i$ we first find the polynomial π_λ satisfying

$$\mu\pi_\lambda = a_i[\mu] \quad \text{for } \mu \in N_\lambda, \tag{2.9}$$

using de Boor and Ron's algorithm for functional interpolation (see 2.2.3 and de Boor and Ron (1992b)).[19] Next, the value $a_{i+1}[\lambda] := (\mathfrak{S}_i a_i)[\lambda]$ is

[19] Note however that this approach is not restricted to de Boor-Ron interpolation, or even to *polynomial* interpolation. For special functional spaces, other spaces of interpolating functions may be more suitable.

assigned:

$$a_{i+1}[\lambda] := (\mathfrak{S}_i a_i)[\lambda] := \lambda\pi_\lambda.$$

We will undertake the trouble to prove that such defined, the operator $\mathfrak{S}_i$ can be represented by weighted averages as detailed in 1.2; and will also show how the refinement weights can be computed.

2.4.4 Proposition. Linear refinement operators $\mathfrak{S}_i$, as defined above, correspond to weighted averagings of the form

$$(\mathfrak{S}_i a_i)[\lambda] = \sum_{\mu \in N_\lambda} s_{i,\lambda,\mu} a_i[\mu],$$

with weights $s_{i,\lambda,\mu}$ satisfying

$$s_{i,\lambda,\mu} = (\mathfrak{S}_i \delta_\mu)[\lambda]. \tag{2.10}$$

Proof. From A.4 we have:

$$a_i = \sum_{\mu \in \Lambda_i} a_i[\mu]\delta_\mu,$$

and therefore,

$$\mathfrak{S}_i a_i = \sum_{\mu \in \Lambda_i} a_i[\mu]\,(\mathfrak{S}_i \delta_\mu)$$

or equivalently,

$$(\mathfrak{S}_i a_i)[\lambda] = \sum_{\mu \in \Lambda_i} a_i[\mu]\,(\mathfrak{S}_i \delta_\mu)[\lambda]. \tag{2.11}$$

Now, $(\mathfrak{S}_i a_i)[\lambda]$ is zero for $\mu \notin N_\lambda$. This is because, with $\mu \notin N_\lambda$, all the val-

ues that are being interpolated (i.e. $\delta_\mu[\nu]$ for $\nu \in N_\lambda$) are zero, and therefore the interpolant is also identically zero. Hence, we can rewrite (2.11) as

$$(\mathfrak{S}_i a_i)[\lambda] = \sum_{\mu \in N_\lambda} a_i[\mu] (\mathfrak{S}_i \delta_\mu)[\lambda].$$

We may now denote $(\mathfrak{S}_i \delta_\mu)[\lambda]$ by $s_{i,\lambda,\mu}$ to have

$$(\mathfrak{S}_i a_i)[\lambda] = \sum_{\mu \in N_\lambda} s_{i,\lambda,\mu} a_i[\mu]. \tag{2.12}$$ □

The following lemma also holds. We will use it to prove a subsequent result.

2.4.5 Lemma. Let $\mathbf{1} \in \mathcal{F} : x \mapsto 1$ be the constant unity function. Let also μs be normalizable such that $\mu\mathbf{1} = 1$. Then, for a signal $a_i \in \ell^p(\Lambda_i)$ that is constantly equal to $c \in \mathbb{F}$ on N_λ, we have

$$(\mathfrak{S}_i a_i)[\lambda] = c.$$

Proof. From $\mu\mathbf{1} = 1$ for linear functionals $\mu \in N_\lambda$, it follows that

$$\mu(c\mathbf{1}) = c = a_i[\mu] \quad \text{for } \mu \in N_\lambda.$$

$(c\mathbf{1})$ therefore satisfies the interpolation conditions at λ (2.9). $(c\mathbf{1})$ is the constant polynomial, and is the lowest degree polynomial that satisfies our interpolation conditions. Therefore $\pi_\lambda = c\mathbf{1}$. Then, as by the normalization

assumption we also have $\lambda(c\mathbf{1}) = c$, it follows that

$$(\mathfrak{S}_i a_i)[\lambda] := \lambda\pi_\lambda = \lambda(c\mathbf{1}) = c. \quad \square$$

2.4.6 Corollary. With the same conditions as those of lemma 2.4.5, we have

$$\sum_{\mu \in N_\lambda} s_{i,\lambda,\mu} = 1 \quad \text{for all } \lambda \in \Lambda_{i+1}. \tag{2.13}$$

Proof. Follows directly from lemma 2.4.5 and eqn (2.12) by setting $c = 1$ and for each λ, considering a function in $\ell^p(\Lambda_i)$ that is constantly equal to 1 on N_λ. $\square$

The above result shows that, with the suggested normalization, our refinement operators satisfy the conditions that Daubechies et al. (1999) require of their *subdivision operators*.[20]

2.4.7 Remark. Constructions for interpolating and average-interpolating multi-resolution refinement that we have discussed in 2.1 and 2.3 are in fact special cases of the construction introduced above:[21]

[20] Daubechies et al.'s subdivision operators are very similar to the refinement operators we introduced in section 1.1, where we considered discrete functions on point-sets (rather than on sets of functionals). While in general, (2.13) is neither strictly necessary nor sufficient for the convergence of a *non-stationary* cascade algorithm (the cascade algorithm is defined in 1.1.13 and 1.2.9), the reader may wish to consult footnote 11 on p. 13 for a brief discussion of the relevance of this condition for *stationary* refinement schemes. (For stationary schemes refinement masks do nat change across resolutions.) As emphasized earlier, in this thesis we do not intend to investigate conditions for the convergence of the cascade algorithm.

[21] Actually, in the multi-dimensional average interpolating scheme of 2.3.2, rather than the scheme itself, each of the two *insertion* and *update* steps can be realized by our new refinement operators.

Simply, for the uni- and multi-dimensional schemes based on Lagrange interpolation (see 2.1.2, 2.1.3, and 2.3.1), it is sufficient to compose Λ_i of point-wise evaluation functionals at points in Θ_i. Extending the same idea to irregular multi-dimensional point-sets directs us to the operators Daubechies et al. (1999) suggest.

Similarly, for average interpolation, Λ_i should consist of functionals corresponding to averaging over intervals in Υ_i (in the univariate case) or over partitions in $\Diamond_i$ (in the multivariate case).

2.4.8 Remark. Notice how this refinement scheme is naturally linked to the sampling procedures, as defined in 1.2, that lead to our discrete signals. Properties of the underlying function, and the domain, can also be taken into consideration when choosing the neighbourhoods for each functional. Additionally, basing our definitions on topological notions such as functionals and dual spaces allows us to collectively consider refinement schemes for different spaces of functions.

2.4.9 Example. As discussed above, we have in fact already encountered several special cases of this new construction. Here we will produce a more complex situation. Let $\mathcal{F}$ be a space of locally integrable functions including bivariate polynomials, defined on a domain Ω in $\mathbb{R}^2$; and suppose that each functional $\lambda \in \Lambda_i \subset \mathcal{F}^*$ corresponds to calculating the average of its argument on the area enclosed by an arbitrary and probably irregular polygone, say D_λ, in Ω. That is,

$$\lambda f := \frac{\iint_{D_\lambda} f \, dx \, dy}{\iint_{D_\lambda} 1 \, dx \, dy}.$$

The construction we have just introduced then provides us with a multi-resolution refinement scheme for data maps associated with Λ_is.

In practice, in order to find the multivariate polynomial satisfying (2.9), we initially need to be able to form the generalized Vandermonde matrix of eqn (2.8), upon which we may then exercise Gaussian elimination by segments (cf. 2.2.3). We should therefore find a way to compute

$$\lambda()^\alpha := \frac{\iint_{D_\lambda} x^m y^n \, dx\, dy}{\iint_{D_\lambda} 1 \, dx\, dy}.$$

where $\alpha := \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$.[22] The denominator is simply the area of D_λ, and is a special case of the numerator integral,

$$\iint_{D_\lambda} x^m y^n \, dx\, dy, \tag{2.14}$$

with $m = n = 0$. It is therefore sufficient to be able to compute (2.14) numerically, for an arbitrary M-gone D_λ. This computation is made possible by an application of Green's theorem:[23,24]

Let the polygone D_λ be defined by its M vertices, $\left[\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right]$, $0 \leq i < M$. Also define $\left[\begin{smallmatrix} x_M \\ y_M \end{smallmatrix}\right] := \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]$. For $f(x,y) = \frac{-1}{n+1} x^m y^{n+1}$ and $g(x,y) = 0$, the right-hand-side of (2.15) will be equal to the desired integral (eqn (2.14)). Evalu-

22 Recall that $()^\alpha : \mathbb{R}^d \longrightarrow \mathbb{R} : X \mapsto X^\alpha$ is a function in $\mathcal{F}$.

23 A version of Green's theorem states that over a region D in the plane with boundary ∂D,

$$\int_{\partial D} f(x,y)\, dx + g(x,y)\, dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\, dy. \tag{2.15}$$

(From Weisstein, 2005b).

24 The method for computing the integral of a bivariate monomial over a polygonal domain, that we discuss here, is known to the numerical computation community.

ation of the left-hand-side involves calculating $\int f(x,y)\,dx$ over the edges of D_λ. It can be easily shown that after elementary integrations we have:[25]

$$\text{L.h.s.} = \frac{-1}{n+1} \sum_{k\in K} \sum_{0\le r\le n+1} \frac{\binom{n+1}{r}}{m+r+1} a_k^r b_k^{n+1-r} \left(x_{k+1}^{m+r+1} - x_k^{m+r+1}\right);$$

where K is the subset of $\{0,1,\ldots,M\}$, such that for $k \in K$ the edge connecting $\left[\begin{smallmatrix} x_k \\ y_k \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_{k+1} \\ y_{k+1} \end{smallmatrix}\right]$ is not vertical and can be parametrized as $y(x) = a_k x + b_k$. This also defines a_k, b_k.

For this setting, we have implemented our refinement scheme in ISO C, and have used Grandine's implementation of de Boor and Ron's algorithm (Grandine). Two of the inputs to our programme are the two sets of functionals, Λ_i and Λ_{i+1}, members of which are identified with the coordinates of their corresponding polygones. (These polygones may have different shapes and numbers of corners.) Another input is the neighbourhood database, that indicates the functionals $\mu \in \Lambda_i$ belonging to the neighbourhood N_λ of λ, for each $\lambda \in \Lambda_{i+1}$. Finally, the last input is the ith level signal, $a_i[\lambda]$ for $\lambda \in \Lambda_i$. Our software then outputs the $i+1$th level signal, $a_{i+1} := \mathfrak{S}_i a_i$.[26]

As an example, suppose that the solid polygones of fig. 2.4 form a neighbourhood N_λ for the dashed polygone λ. As we saw in proposition 2.4.4, the refinement operator $\mathfrak{S}_i$ may be identified with weighted summations of the

[25] To see this, simply write the line equation for non-vertical edges connecting $\left[\begin{smallmatrix} x_k \\ y_k \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} x_{k+1} \\ y_{k+1} \end{smallmatrix}\right]$ as $y(x) = a_k x + b_k$, and replace this in the integral $\int_{x_k}^{x_{k+1}} f(x, y(x))\,dx$, which now becomes:

$$\int_{x_k}^{x_{k+1}} \frac{-1}{n+1} x^m (a_k x + b_k)^{n+1}\,dx.$$

Then use the binomial expansion for $(a_k x + b_k)^n$. (For vertical edges the integral is trivially zero.)

[26] The source code and Linux x86 binaries can be found on the accompanying CD-ROM, and are also available from the author upon request.

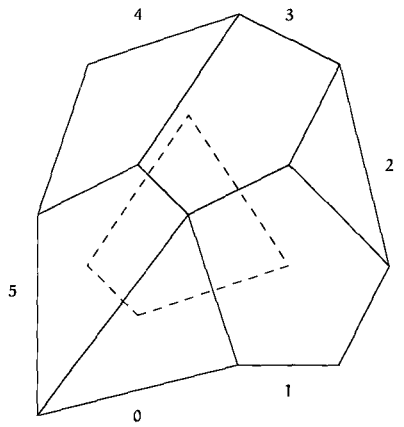


Figure 2.4: Several irregular polygones (solid) form a neighbourhood for another irregular polygone (dashed).

form

$$(\mathfrak{S}_i a_i)[\lambda] = \sum_{\mu \in N_\lambda} s_{i,\lambda,\mu} a_i[\mu],$$

and the same proposition also provides us with a way to calculate the weights (by passing δ_μ signals as the input to the refinement scheme). We have summarized the weights for $\mu \in N_\lambda$ (here indexed by numbers) in table 2.4. As predicted by corollary 2.4.6, these weights add up to 1.

Table 2.4: The new value associated with the dashed polygone is equal to a weighted summation of the values given for the solid polygones, using the following weights.

Neighbour's Index	Weight
0	−0.722514
1	1.008270
2	−0.857378
3	1.175449
4	−0.616545
5	1.012718

3

Conclusion

> A conclusion is the place where you got tired of thinking.
>
> ANON.

THE MAIN PURPOSE of this thesis was to provide a definition and formulation of multi-resolution refinement, general enough to allow multi-resolution approximation of different spaces of functions based on samples (or observations) that could be irregularly distributed or even differently obtained (chapter 1). We also provided a construction for designing and implementing refinement schemes in these general settings (chapter 2). The framework for multi-resolution refinement that we discussed in chapter 1 includes and extends (within a new formulation) the existing mathematical machinery for multi-resolution analysis. And the structure suggested in chapter 2 provides a unified formulation for many of the schemes currently in use, and allows us to design schemes for many new settings. The approach we have proposed also gives rise to many questions that we have not

fully addressed here, and would benefit from further study. In this concluding chapter some of these will be outlined. In addition, several questions need to be answered when designing a refinement scheme for a particular application. While the answeres to these questions are application-specific, we provide some general remarks and considerations.

3.1 Suggestions for Further Investigation

3.1.1 The Stability and Convergence of the Cascade Algorithm. As we pointed out on many occasions throughout this thesis, a question that is of critical importance but which we did not investigate is that under what conditions on the refinement coefficients the cascade algorithm is stable, and converges to a continuous solution; and when this happens, how the regularity of the solution can be characterized.

We briefly mentioned in footnote 11 on page 14 that a necessary (but insufficient) condition for the convergence of *stationary* refinement schemes is that the refinement coefficients used for calculating each new sample value from lower resolution values sum up to 1. However, this condition is not strictly necessary and may be weakened in the non-stationary case.

Another important aspect of the problem is the characterization of the regularity of the solution. In the uniform one-dimensional case, as well as in the multi-dimensional cases where we deal with lattice structures, Fourier-domain techniques may be used to answer this question, as has been done for example in Cohen et al. (1999), Jia (1999), and Ron and Shen (2000) for refinement on multi-dimensional lattices. Yet, when dealing with the

more general case of irregular sampling, frequency-domain methods are not applicable, and temporal/spatial techniques should be sought. Moreover, it should be kept in mind that in these cases our refinement schemes have different properties at different locations. We should therefore look for methods for evaluating *local* regularity of the solution. A logical approach is to study differences between values associated with nearby samples: the smoother the solution is, the faster the sample values converge as a function of distance between their related sample points or functionals.[1] Daubechies et al. (2001) follow a similar path.

Apart from these general analyses that use the refinement weights, one may choose to study the regularity of solutions to the schemes described in 2.4 by studying the construction itself. Since those schemes are based on functional interpolation, perfect reconstruction of functions that fall in the intersection of the subspaces of solutions to the interpolation problem is guaranteed. Also, as we showed for example in corollary 2.4.6, known properties of the interpolant can help in proving results about the solutions.

3.1.2 Spaces of Interpolating Functions. In the construction for refinement operators that was suggested in section 2.4, we considered subspaces of the space of polynomials, provided by de Boor and Ron's solution, as our spaces of interpolating functions. This use is justified by the many interesting properties of the de Boor-Ron interpolant, which are listed for example in de Boor and Ron (1992b). For all that, still other polynomial solution spaces, and more generally, other spaces of interpolating functions may be

[1] Of course, the notion of *distance* between functionals, in the sense that we intend here, should itself be suitably defined.

considered, and may prove to be more suitable for certain classes of functions.

3.1.3 Formation of Multi-Resolution Sampling Procedures. A question that needs to be answered when designing a multi-resolution refinement scheme for a particular setting is that of how to form multi-resolution sampling procedures[2] that satisfy the requirements set forth in 1.2.2, keeping in mind that for many applications we may not actually need an infinite sequence of sampling procedures.

In some cases, the layout of these multi-level sets of functionals may be dictated by the application. (For example when a natural subsampling or clustering of the samples exists, as may be the case when processing sensor-network measurements.) In others, we may have the freedom to define the sampling procedures. When dealing with point-wise evaluation functionals, a common practice is to impose a mesh structure on the sampling points, and add new sampling points at midpoints of the edges connecting mesh points, or at mid-points of the faces bounded by these edges. Similarly, when working with averages on a partitioning of the domain (as in example 2.4.9), we may choose to exploit the duality between Voronoi regions and Delaunay triangluations, subdividing the Delaunay triangluation to insert new vertices for our Voronoi partitioning.

3.1.4 Forming Neighbourhoods. Our construction in 2.4 required that functionals in two subsequent sampling procedures be related through the notion of *neighbourhoods*. In general, the formation of these neighbourhoods

[2] The reader will remember that we defined sampling procedures as sets Λ_i of functionals on our space $\mathcal{F}$, that define our spaces $\ell^p(\Lambda_i)$ of discrete signals.

should reflect the properties of the underlying function, the samples of which we are processing. The neighbourhood for each higher-resolution functional determines which lower-resolution samples will be considered when calculating a value for this functional. When the underlying function is smooth, the correlation between sample values for nearby functionals is higher and as such, we may choose a larger neighbourhood size. Also, in some applications (such as sensor-network signal processing) a multi-level clustering of samples may already exist.

> I remember one occasion when I tried to add a little seasoning to a review, but I wasn't allowed to. The paper was by Dorothy Maharam, and it was a perfectly sound contribution to abstract measure theory. The domains of the underlying measures were not sets but elements of more general Boolean algebras, and their range consisted not of positive numbers but of certain abstract equivalence classes. My proposed first sentence was: 'The author discusses valueless measures in pointless spaces'.
>
> *I want to be a mathematician, Springer-Verlag, 1985.*
> PAUL R. HALMOS, MATHEMATICIAN (1916–)

A

Some Mathematical Concepts

> For the things of this world cannot be made known without a knowledge of mathematics.
>
> *Opus Majus*
> ROGER BACON (C. 1214–1292)

THIS APPENDIX is a compilation of some mathematical definitions and concepts cited throughout this thesis. It is not intended to be exhaustive, but is to serve as a brief reference only.

A.1 Preliminaries

A.1.1 Fields. Let $\mathbb{F}$ be a set for which two operations, called *addition* and *multiplication* and denoted respectively by $+$ and $\cdot$, are defined. The system $\langle \mathbb{F}, +, . \rangle$ is called a *field* if the following are satisfied:[1]

1. Addition is associative and commutative.

[1] Quoting Webber (1966, pp. 128–129) begins.

2. There is an element, denoted by z, such that

$$u + z = u = z + u, \quad \text{for all } u \in \mathbb{F}.$$

3. For each $u \in \mathbb{F}$, there is an element denoted by $-u$ such that

$$u + (-u) = z = -u + u.$$

4. Multiplication is associative and commutative.

5. There is an element $\neq z$, denoted by e, such that

$$u \cdot e = u = e \cdot u, \quad \text{for all } u \in \mathbb{F}.$$

6. For each $u \in \mathbb{F}$, $u \neq z$, there exists an element in $\mathbb{F}$, denoted by u^{-1}, for which

$$u \cdot u^{-1} = e = u^{-1} \cdot u. \tag{A.1}$$

7. Multiplication is distributive over addition.[2]

Informally, we usually refer to $\mathbb{F}$ itself, with its known addition and multiplication operations, as the field. $\langle \mathbb{R}, +, . \rangle$ and $\langle \mathbb{C}, +, . \rangle$, the systems of real and complex numbers, are familiar examples of fields.

[2] Quoting Webber (1966, pp. 128–129) ends.

A.2 Topological Spaces

A.2.1 Topological Spaces. A *topological space* is a pair $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ is a set and $\mathcal{O}$ is a collection of subsets of $\mathcal{X}$ satisfying the following axioms (Adams and Fournier, 2003; Wikipedia, 2005):

1. $\varnothing, \mathcal{X} \in \mathcal{O}$.
2. The union of any collection of elements of $\mathcal{O}$ is also in $\mathcal{O}$.
3. For all $U, V \in \mathcal{O}$, $U \cap V$ is also in $\mathcal{O}$.

$\mathcal{O}$ is called a *topology* on $\mathcal{X}$. Elements of $\mathcal{O}$ are *open sets*, and their complements in $\mathcal{X}$ are *closed sets*. Any open set including an $x \in \mathcal{X}$ is a *neighbourhood* of x. For the sake of brevity we may also refer to $\mathcal{X}$ itself as a topological space.

A.2.2 More Definitions. Consider topological spaces $(\mathcal{X}, \mathcal{O})$ and $(\mathcal{Y}, \mathcal{T})$. Then:[3]

- $\mathcal{X}$ is a *Hausdorff topological space* if for any $x, y \in \mathcal{X}$ where $x \neq y$ disjoint neighbourhoods for x and y exist.
- The *closure* of $S \subset \mathcal{X}$, denoted by $\overline{S}$, is the smallest closed set including S, i.e. the intersection of all closed sets including S.
- $D \subset \mathcal{X}$ is *dense in* $S \subset \mathcal{X}$ if $D \subset S \subset \overline{D}$.[4]
- $\mathcal{X}$ is *separable* if it has a countable dense subset.

[3] See Megginson (1998, pp. 109,139–140,142), Lang (1969, p. 22) and Kolmogorov and Fomin (1998, p. 24).

[4] For example, the closure of $\mathbb{Q}$ (the set of rationals) in $\mathbb{R}$ with its familiar topology is equal to $\mathbb{R}$. Therefore $\mathbb{Q}$ is dense in $\mathbb{R}$.

- x is a *limit point* of $V \subset \mathcal{X}$ if every neighbourhood of x includes an infinite number of points in V.

- A mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is *continuous* if for each open subset V of $\mathcal{Y}$, $f^{-1}(V)$ is open.

A.2.3 Topological Vector Spaces. A vector space $\mathcal{X}$ over a field $\mathbb{F}$ with a topology $\mathcal{O}$ is a *topological vector space* (TVS) or a *linear topological space* if the addition of vectors and multiplication of vectors by scalars are continuous operations.

A.2.4 Functionals and the Dual Space. A *functional* is a mapping from a topological space $\mathcal{X}$ to a field $\mathbb{F}$ of scalars. λ is a *linear functional* if for all $x, y \in \mathcal{X}$ and all $\alpha \in \mathbb{F}$,

$$\lambda(\alpha x + y) = \alpha\lambda x + \lambda y.$$

The vector space of all continuous linear functionals on a TVS $\mathcal{X}$ is called its *continuous dual space* or quite often simply its *dual space,*[5] and is denoted by $\mathcal{X}^*$ (Adams and Fournier, 2003; Megginson, 1998). Equipped with a topology, $\mathcal{X}^*$ itself becomes a TVS.

A.2.5 The Weak* Topology. Let $\mathcal{X}$ be a TVS. The *weak* topology* on $\mathcal{X}^*$ is the smallest topology on $\mathcal{X}^*$ with respect to which the functional

$$F_x : \mathcal{X}^* \longrightarrow \mathbb{F} : x^* \mapsto x^*x, \tag{A.2}$$

[5] In the latter case, the space of *all* linear functionals on $\mathcal{X}$ is referred to as the *algebraic* dual space.

known as the *natural* mapping, is continuous for each $x \in \mathcal{X}$ (cf. Adams and Fournier (2003, p. 4) and Megginson (1998, sec. 2.6)). The adjective and adverb *weak** and *weak*ly* then refer to properties that hold with respect to this topology.

A.3 Normed Vector Spaces

A.3.1 Normed Vector Spaces. A *normed vector space* is a pair $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, where $\mathcal{X}$ is a vector space and $\|\cdot\|_{\mathcal{X}}$ is a *norm* on $\mathcal{X}$, that is, a real-valued function $\|\cdot\|_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathbb{R} : x \mapsto \|x\|_{\mathcal{X}}$ satisfying the following conditions (Megginson, 1998, p. 9):

For $x, y \in \mathcal{X}$ and scalar α,

1. $\|x\|_{\mathcal{X}} \geq 0$, with equality iff $x = 0$;
2. $\|\alpha x\|_{\mathcal{X}} = |\alpha| \|x\|_{\mathcal{X}}$;
3. $\|x + y\|_{\mathcal{X}} \leq \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{X}}$.

Where not necessary to distinct different normed spaces, the subscript of $\|\cdot\|_{\mathcal{X}}$ will usually be dropped. Also informally we may refer to $\mathcal{X}$ itself as the normed space.

A.3.2 Norm Induced Metric. The norm of $\mathcal{X}$ induces a *metric* on $\mathcal{X}$ through the formula $d(x, y) := \|x - y\|$. With this definition $\mathcal{X}$ is a *metric space*.

A.3.3 Banach Spaces. A normed vector space is a *Banach space* or *B-space* or *complete normed space* if it is a *complete* metric space with the metric

induced by its norm. A *complete* metric space is a metric space in which every sequence (x_n) satisfying the Cauchy criterion (see below) converges. (x_n) satisfies the Cauchy criterion if for all $\epsilon > 0$ there exists N_ϵ, such that for all $n', n'' \geq N_\epsilon$, $d(x_{n'}, x_{n''}) < \epsilon$ (Kolmogorov and Fomin, 1998).

A.3.4 Linear Operators. A *linear operator* $\mathfrak{T} : \mathcal{V} \longrightarrow \mathcal{W}$ is a mapping from a first vector space $\mathcal{V}$ (the *domain*) to a second vector space $\mathcal{W}$ (the *codomain*), both over the same field $\mathbb{F}$, whereby for $x, y \in \mathcal{V}$ and $\alpha \in \mathbb{F}$,

$$\mathfrak{T}(x + y) = \mathfrak{T}x + \mathfrak{T}y,$$

$$\mathfrak{T}(\alpha x) = \alpha \mathfrak{T}x. \tag{A.3}$$

The notations $\mathfrak{T}x$ and $\mathfrak{T}(x)$ may be interchangeably used.

When $\mathcal{V}$ and $\mathcal{W}$ are normed spaces, *boundedness* of operators may be studied. $\mathfrak{T} : \mathcal{V} \longrightarrow \mathcal{W}$ is *bounded* if a scalar $c \in \mathbb{F}$ exists such that

$$\|\mathfrak{T}x\|_{\mathcal{W}} \leq c\|x\|_{\mathcal{V}}, \quad \text{for all } x \in \mathcal{V}.$$

The norm of a bounded linear operator can then be defined:

$$\|\mathfrak{T}\| := \sup_{\substack{x \in \mathcal{V}, \\ \|x\|_{\mathcal{V}} \neq 0}} \frac{\|\mathfrak{T}x\|_{\mathcal{W}}}{\|x\|_{\mathcal{V}}}.$$

Bounded linear operators extend the concept of linear transformations—which are represented by matrices in the finite-dimensional case—to possibly infinite-dimensional spaces.

A.3.5 $\ell^p(I)$ **Spaces.** For a countable set I of indices, called the *index set,* and a field $\mathbb{F}$, $\ell^p(I)$ is defined for $1 \leq p \leq \infty$ as the normed vector space of all mappings $a : I \longrightarrow \mathbb{F} : \iota \mapsto a[\iota]$ for which the norm defined as

$$\|a\|_p = \begin{cases} \left(\sum_{\iota \in I} |a[\iota]|^p\right)^{1/p} & \text{for } 1 \leq p < \infty, \\ \sup_{\iota \in I} |a[\iota]| & \text{for } p = \infty, \end{cases}$$

is finite.

One possible basis for $\ell^p(I)$, $1 \leq p < \infty$, is formed by functions δ_ι (defined below[6]) for all $\iota \in I$.

$$\delta_\iota[\kappa] := [\iota = \kappa].$$

Any $a \in \ell^p(I)$ can be decomposed as

$$a = \sum_{\kappa \in I} a[\kappa]\delta_\kappa. \tag{A.4}$$

The spaces $\ell^p(I)$, $1 \leq p \leq \infty$, are examples of Banach spaces.

A.3.6 $L^p(\Omega)$ **Spaces.** Let Ω be a σ-finite[7] positive measure space with a measure μ. For $1 \leq p \leq \infty$, the Lebesgue space $L^p(\Omega)$ is the space of

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[6] Using Iverson's convention (see Graham et al. (1994, p. 24) or Knuth (1992)), in which a true-or-false statement enclosed in square brackets is equal to one if true and equal to zero if false. (Iverson himself used parantheses instead of square brackets. See Iverson (1962, p. 11).)

[7] A measure μ on Ω is σ-finite if Ω is a countable union of sets of finite measure.

functions f with domain Ω and codomain $\mathbb{F}$ for which the norm

$$\|f\|_p = \begin{cases} \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} & \text{for } 1 \leq p < \infty, \\ \text{ess sup}_\Omega |f| & \text{for } p = \infty, \end{cases}$$

is finite. $L^p(\Omega)$, $1 \leq p \leq \infty$, for each choice of p, is an example of a Banach space. For $1 \leq p < \infty$ the dual of $L^p(\Omega)$ is isomorphic to $L^q(\Omega)$, where q satisfies $1/p + 1/q = 1$.

A.3.7 The Hahn-Banach Extension Theorem (Normed Space Version). Let λ_0 be a bounded linear functional on a subspace $\mathcal{X}_0$ of a normed vector space $\mathcal{X}$. Then there exists a bounded linear functional λ on all of the space $\mathcal{X}$ that agrees with λ_0 on $\mathcal{X}_0$, and satisfies $\|\lambda\|_{\mathcal{X}^*} = \|\lambda_0\|_{\mathcal{X}_0^*}$ (Megginson, 1998, p. 75).

> 'And you do Addition?' the White Queen asked. 'What's one and one and one and one and one and one and one and one and one and one?'
> 'I don't know,' said Alice. 'I lost count.'
> 'She can't do Addition...'
>
> *Through the Looking-Glass*
> CHARLES LUTWIDGE DODGSON (LEWIS CARROLL), ENGLISH MATHEMATICIAN AND WRITER (1832–1898)

B

Summary of Used Notation

By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and, in effect, increases the mental power of the race.

In P.J. Davis and R. Hersh: The Mathematical Experience. *Birkhäuser, 1981.*
ALFRED NORTH WHITEHEAD, BRITISH MATHEMATICIAN, LOGICIAN AND PHILOSOPHER (1861–1947)

Special Sets and Spaces

$\mathbb{R}$	The field of reals.
$\mathbb{C}$	The field of complex numbers.
$\mathbb{Z}$	The set of integers.
$\mathbb{R}^d$	The d-dimensional Euclidean space.
$\mathbb{Z}^d_{0+}$	The set of non-negative integer d-tuples.
$L^p(\Omega)$	The space of functions on Ω with finite L^p norm (A.3.6).

$\ell^p(I)$	The space of maps on the countable set I with finite ℓ^p norm (A.3.5).
$\mathbf{C}^N(\Omega)$	The space of N-times continuously differentiable functions on Ω.
$\mathbf{C}_0^N(\Omega)$	The space of finitely supported N-times continuously differentiable functions on Ω.
Π	The space of univariate polynomials.
Π^d	The space of d-variate polynomials.

Miscellaneous

$A = B$	A is equal to B.
$A := B$	A is by definition equal to B.
[statement]	Iverson's convention: The bracketed statement is equal to 1 if the statement is true, is equal to 0 otherwise (footnote 6 on p. 67).
$\#X$	The cardinality of the set X.
$\mathfrak{T} : \mathcal{V} \longrightarrow \mathcal{W} : v \mapsto w$	$\mathfrak{T}$ maps (the space) $\mathcal{V}$ to (the space) $\mathcal{W}$. The element $v \in \mathcal{V}$ is mapped to $w \in \mathcal{W}$ (A.3.4).
$\langle a, b \rangle$	(Denotes a duality pairing between a and b.)
$(\cdot)$	(Used to denote a sequence.)
$[\cdot]_{\text{index 1}}$	(Indicates a column vector with elements indexed by index 1.)
$[\cdot]_{\substack{\text{index 1} \\ \text{index 2}}}$	(Indicates a matrix with rows indexed by index 1 and columns indexed by index 2.)
X^α	$\prod x_i^{\alpha_i}$ (with $X := [x_i]$ and $\alpha \in \mathbb{Z}_{0+}^d$.)

$D^{\alpha} f$	The αth partial derivative of f ($\alpha \in \mathbb{Z}_{0+}^{d}$).
$()^{\alpha}$	(For $\alpha \in \mathbb{Z}_{0+}^{d}$) the mapping $\mathbb{R}^{d} \longrightarrow \mathbb{R} : X \mapsto X^{\alpha}$ (2.2.2).
$\alpha(f)$	(Where f is a polynomial or a formal power series, and $\alpha \in \mathbb{Z}_{0+}^{d}$) the coefficient of $()^{\alpha}$ in f (2.2.2).

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