Design of Optimal Precoders for Multiuser OFDM Systems with MMSE Equalization

DESIGN OF OPTIMAL PRECODERS FOR MULTIUSER OFDM SYSTEMS WITH MMSE EQUALIZATION

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Dedications:

To my dear parents and sister

Abstract

In this thesis, we consider a multiuser downlink OFDM system for which the channel state information (CSI) is known to both the transmitter and the receiver.

For such a system, we design an optimal precoder that minimizes the total mean square error (MSE) subject to a total power constraint for which a minimum MSE (MMSE) equalizer is employed. We show that, the MMSE precoder can be obtained by optimally allocating the subcarriers and optimally allocating the power. This problem can be solved by a two-stage process, in which we minimize the lower bound of the MSE to obtain the optimal power for each subcarrier, followed by seeking an optimal precoder to achieve this minimized lower bound. Specifically, our subcarrier allocation strategy states that, each subcarrier should be allocated to only one user that has the largest subchannel gain in that subcarrier.

Moreover, based on this subcarrier allocation strategy, we perform an optimal power loading and design the corresponding optimal precoder that minimizes the average bit error rate (BER). Here, the MMSE equalizer is also employed. This optimization problem is solved by two stages. In the first stage, we derive the lower bound of the average BER and minimize this lower bound. After we employ the MMSE subcarrier allocation strategy, the optimal power loading problem can be efficiently solved by interior point methods. In order to reduce computation complexity, an alternative, efficient power loading method is proposed here, which is much more efficient when the number of subcarriers is large. In the second stage, to achieve

the minimized lower bound, we seek a design of an optimal precoder. Simulation results show that for moderate to high signal-to-noise ratio (SNR), the performance of the minimum BER (MBER) precoder employed with the MMSE equalizer design is superior to several other design methods, including the MMSE precoder design.

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Acronyms

BER	Bit Error Rate
CP	Cyclic Prefix
CSI	Channel State Information
dB	Decibels
DFT	Discrete Fourier Transform
DMT	Discrete Multitone
FFT	Fast Fourier Transform
FIR	Finite Impulse Response
IBI	Interblock Interference
ISI	Intersymbol Interference
MBER	Minimum Bit Error Rate
MMSE	Minimum Mean Square Error
MSE	Mean Square Error
MUI	Multiuser Interference
OFDM	Orthogonal Frequency Division Multiplexing
QAM	Quadrature Amplitude Modulation
QPSK	Quaternary Phase Shift Keying
SINR	Signal to Interference Plus Noise Ratio

SNR	Signal to Noise Ratio
ZF	Zero Forcing
ZP	Zero Padding

Notations

$(\cdot)^H$	Conjugate transpose of a complex matrix or a vector
$(\cdot)^*$	Conjugate of a complex variable
$\mathrm{Diag}(\mathbf{A})$	Diagonal matrix consisting of the diagonal elements of matrix $\mathbf{A}$
$\mathrm{E}(\cdot)$	Expectation
$\mathrm{Im}(\cdot)$	Imaginary part of a complex value or a vector or a matrix
$\mathbf{I}_L$	$L \times L$ identity matrix
$\lvert \cdot \rvert$	Magnitude of a complex number
$\mathbf{A}$	Matrix $\mathbf{A}$
$\mathbf{X} \succ \mathbf{0}$	Matrix $\mathbf{X}$ is positive definite
$\mathbf{X} \succeq \mathbf{0}$	Matrix $\mathbf{X}$ is positive semidefinite
$(x)^+$	max(x,0)
max(m,n)	Maximum value of m and n
$[\cdot]_{mn}$	(m, n)th element of a matrix
$\mathrm{Re}(\cdot)$	Real part of a complex value or a vector or a matrix
s.t.	Subject to
$b_{i,j}$	the jth diagonal element of a diagonal matrix $\mathbf{B}_i$
$\mathrm{tr}(\cdot)$	Trace
$(\cdot)^T$	Transpose of a matrix or a vector
$\mathbf{a}$	Vector $\mathbf{a}$

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Chapter 1

Introduction

1.1 ISI and Block-by-Block Transmission

In digital communication systems, multipath transmission happens when the transmitted signal reaches the receiver over different paths. Multipath transmission will cause the received symbols to overlap and hence, produces the so-called intersymbol interference (ISI). Due to its introduction of bit errors at the receiver, the presence of ISI has been considered as a major obstacle to high speed data transmission. It is desirable for ISI to be mitigated.

One practical method is to use equalization to compensate for ISI at the receiver. The corresponding compensator for the ISI is called an equalizer. In the literature, there are two basic equalization methods: linear equalization and non-linear equalization. It is well known that for detection applications, maximum-likelihood sequence detection (MLSE) is an optimum detection method in terms of sequence error probability. However, it is computationally expensive and is often impractical to implement [11]. In this thesis, we will focus our attention on linear equalization since it is easier to implement than non-linear equalization. We know that zero-forcing (ZF) equalization and MMSE equalization are two basic forms of linear equalization.

ZF equalization eliminates the ISI by inverting the frequency response of the channel. However, if the frequency response of the channel is small in magnitude or approaches to zero at some particular frequencies, ZF equalization will produce severe noise enhancement. Another kind of linear equalization, MMSE equalization, was developed to alleviate this effect by minimizing the MSE between the transmitted and equalized signal [11].

It has been shown that transmitting the data in blocks can combat ISI [10]. In block transmission, the data stream is divided into consecutive blocks with equal size; between two blocks, redundant symbols are inserted to avoid interblock interference (IBI) at the receiver. Examples of block-data communication include orthogonal frequency division multiplexing (OFDM) [15] and discrete multitone (DMT) modulation [1]. The former one has been selected as a standard modulation scheme for terrestrial digital audio and video broadcasting, while the latter one has been chosen for high-bit-rate digital subscriber line (HDSL) and asymmetric digital subscriber line (ADSL) systems.

1.2 Precoding and Motivation

There is an increasing interest towards block-by-block transmission in recent years. For example, in [2] and [3], the authors have proposed a general class of linear block-based transmission scheme, in which linear MMSE precoders have been determined when the linear ZF and MMSE equalizers are both employed [2]. For uplink multiple access transmission employing a linear MMSE equalizer, the optimal design of the linear block precoder based on MMSE has been proposed [6] [26]. However, MMSE does not necessarily result in MBER. The problem of MBER, can be quite hard to deal with since the BER expression is non-linear. In the design of the MBER linear block precoders with linear equalizers for a single user case, previous work showed that, at moderate to high SNR region, the MBER linear block precoders are superior to the standard MMSE precoders by several decibels [5] [18] [19] [24] [25]. However, extending their work to multiuser case is not straightforward due to the following two main difficulties. One difficulty arises from multi-user interference (MUI). This causes the BER function for each user to be nonconvex with respect to the design parameters. The other difficulty is that the objective involving a linear combination of all complementary error functions is also nonconvex and hard to handle.

In this thesis, we consider a multiuser downlink OFDM system. We derive the optimal MMSE precoder and the corresponding equalizer. Specifically, we obtain the MMSE subcarrier allocation strategy. Then, based on this kind of subcarrier allocation strategy, the optimal power loading and the corresponding linear block precoder that minimizes the average BER are derived when a linear MMSE equalizer is employed.

1.3 Contributions of the Thesis

In this thesis, we focus on the design of linear transmitters for a multiuser downlink OFDM communication system given a block-based linear MMSE equalizer structure. Here we assume that the noise is white and the perfect channel knowledge is available at both the transmitter and the receiver. Our main contributions are as follows.

First, from the viewpoint of minimizing the total MSE, we design the optimal precoder and the corresponding equalizer. We show that, the problem of minimizing the MSE to obtain the optimal precoder can be solved by two stages. In the first stage, we derive and minimize the lower bound of the MSE to obtain the optimal total power for each subcarrier; in the second stage, we seek an optimal design to achieve this lower bound.

Second, we derive the lower bound of the average BER, and further minimize this lower bound for the MMSE equalizer. With the MMSE subcarrier allocation strategy, we obtain the convex power loading optimization problem and solve it by an interior point method. Specifically, we provide an alternative, more efficient power loading method to reduce the computation complexity. To achieve the minimized lower bound, we seek a design of an optimal precoder [22] [23]. We conclude, based on the simulation results, that the performance of the MBER precoder fitted with the MMSE equalizer design is superior to several other design methods, including the MMSE precoder we have designed in this thesis.

1.4 Organization of the Thesis

The thesis is organized as follows. In Chapter 2, a multiuser downlink OFDM transmission system model with linear MMSE equalizer is described. The assumptions are made in the thesis and the formula of the linear MMSE equalizer is provided. In Chapter 3, we design the optimal precoder fitted with the MMSE equalizer under the MMSE criterion. The MMSE subcarrier allocation strategy and the optimal power loading are derived in this chapter. In Chapter 4, we derive and minimize the lower bound of the BER. With the subcarrier allocation strategy proposed in Chapter 3, we develop the optimal precoder for which the MMSE equalizer is employed. We also derive an alternative, efficient power loading method to reduce the complexity. Performance comparisons of the MBER precoder with several other designs, including the MMSE precoder are shown in this chapter. Finally, the conclusions and discussion for future work are given in Chapter 5.

Chapter 2

Multiuser OFDM System with Linear MMSE Equalization

In this chapter, we provide an overview of a multiuser downlink OFDM system. We summarize the assumptions made in this thesis and derive the formula of the MMSE equalizer.

2.1 OFDM Modulation

In recent years, multicarrier, especially OFDM, is considered an important technology. Its applications in many fields, such as digital audio broadcasting (DAB) and digital video broadcasting (DVB), are rapidly developing [7] [8].

OFDM modulation has been regarded as a basic modulation mode on severe ISI channels. OFDM modulation carries out redundant block transmissions and one of its great advantages is the very simple equalization it requires. The scheme of OFDM modulation can be illustrated in Figure 2.1. The S/P and P/S boxes denote serial-to-parallel and parallel-to-serial conversion, respectively. As seen from Figure 2.1,

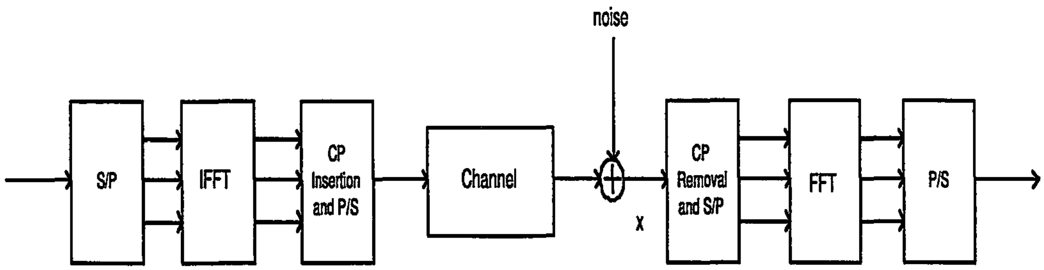

Figure 2.1: OFDM communication system.

at the transmitter, the data blocks are precoded by the inverse fast Fourier transform (IFFT) matrix, followed by the insertion of the cyclic prefix (CP); then the lengthened data blocks are sequentially transmitted through the channel. For the receiver, the CP is removed to avoid IBI. After that, a fast Fourier transform (FFT) matrix is used to process each truncated block. Now the frequency-selective channel is transformed into parallel independent subchannels which correspond to different orthogonal subcarriers [1] [4].

2.2 Multiuser OFDM System

Based on OFDM modulation, we consider a multiuser downlink scheme with N users (e.g. [12]), and let

$$\mathbf{s}_i(n) = [s_i(nM), s_i(nM+1), ..., s_i(nM+M-1)]^T, \; i = 1, 2..., N \tag{2.1}$$

denote the nth data block to be transmitted for User i. After precoding, the precoded signal can be expressed as

$$\mathbf{b}_i(n) = \mathbf{F}_i\mathbf{s}_i(n) = [b_i(nM), b_i(nM+1), ..., b_i(nM+M-1)]^T, \tag{2.2}$$

where $\mathbf{F}_i$ is an $M \times M$ precoder matrix. The scheme can be illustrated in Figure 2.2. From Figure 2.2, if we let $\mathbf{b}(n)$ denote the sum of the precoded signal for all the users,

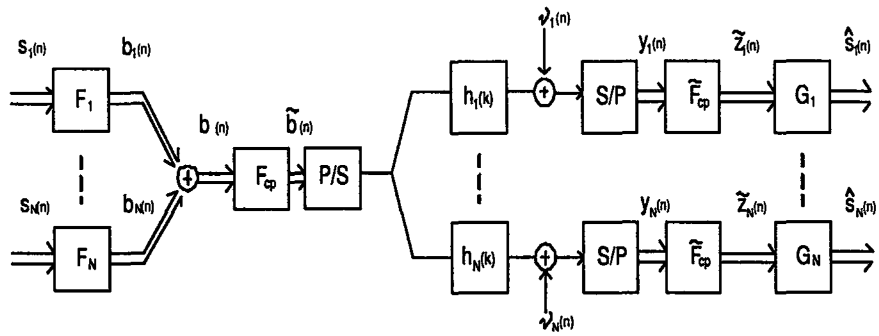

Figure 2.2: A block-based multiuser downlink OFDM communication system.

then

$$\mathbf{b}(n) = \sum_{i=1}^{N} \mathbf{b}_i(n). \tag{2.3}$$

This symbol vector $\mathbf{b}(n)$ is now processed by an $M \times M$ IFFT matrix $\mathcal{F}^H$ to yield the "time domain" block vector, where $(.)^H$ denotes conjugate transposition and the (m, k)th entry of matrix $\mathcal{F}$ is given by

$$[\mathcal{F}]_{mk} = \frac{1}{\sqrt{M}} e^{-j2\pi(m-1)(k-1)/M}, \ \forall m, k \in [1, M]. \tag{2.4}$$

For each block of M data symbols, redundancy is inserted such that more than M symbols (say Q, $Q > M$) are transmitted across the channel. We will show later that this redundancy is important for avoiding the IBI at the receiver. Now we have the following transmitted symbol vector:

$$\tilde{\mathbf{b}}(n) = \mathbf{F}_{cp}\mathbf{b}(n), \tag{2.5}$$

where the $Q \times M$ matrix $\mathbf{F}_{cp}$ represents the combination of IFFT and redundancy inserting processing. If we denote

$$\mathbf{y}_i(n) = [y_i(nQ), y_i(nQ+1), ..., y_i(nQ+Q-1)]^T \tag{2.6}$$

as the corresponding nth block of receiver inputs for the ith user, we have

$$y_i(n) = \sum_{l=-\infty}^{\infty} h_i(l)\tilde{b}(n-l) + \nu_i(n), \tag{2.7}$$

which are the results of convolution of symbols $\tilde{b}(n)$ with the channel impulse response $h_i(l)$, corrupted by the additive white Gaussian noise. Equation (2.7) can be written in a vector form as

$$\mathbf{y}_i(n) = \sum_{l=-\infty}^{\infty} \mathbf{H}_{i,l}\tilde{\mathbf{b}}(n-l) + \boldsymbol{\nu}_i(n), \tag{2.8}$$

where the $Q \times Q$ matrices $\mathbf{H}_{i,l}$ have the following structure

$$\mathbf{H}_{i,l} = \begin{pmatrix} h_i(lQ) & h_i(lQ-1) & \dots & h_i(lQ-Q+1) \\ h_i(lQ+1) & h_i(lQ) & \dots & h_i(lQ-Q+2) \\ \vdots & \ddots & \ddots & \vdots \\ h_i(lQ+Q-1) & h_i(lQ+Q-2) & \dots & h_i(lQ) \end{pmatrix}. \tag{2.9}$$

Now at the receiver, after the elimination of the redundancy symbols and the demodulation with the $M \times M$ FFT matrix $\mathcal{F}$, the "frequency domain" received signals are obtained. We use $\tilde{\mathbf{z}}_i(n)$ to represent them. Therefore, the vector of data symbols at the output of equalizer filter is represented as

$$\hat{\mathbf{s}}_i(n) = \mathbf{G}_i\tilde{\mathbf{z}}_i(n) = \mathbf{G}_i\tilde{\mathbf{F}}_{cp}\mathbf{y}_i(n), \tag{2.10}$$

where $\mathbf{G}_i$ is an $M \times M$ equalizer matrix, and the $M \times Q$ matrix $\tilde{\mathbf{F}}_{cp}$ denotes the matrix operation corresponding to the combined redundancy removal and FFT processing. Substituting (2.8) into equation (2.10), we have the following equalized symbol vector for User i,

$$\begin{aligned} \hat{\mathbf{s}}_i(n) &= \mathbf{G}_i\tilde{\mathbf{F}}_{cp}\sum_{l=-\infty}^{\infty} \mathbf{H}_{i,l}\tilde{\mathbf{b}}(n-l) + \mathbf{G}_i\tilde{\mathbf{F}}_{cp}\boldsymbol{\nu}_i(n) \\ &= \mathbf{G}_i\tilde{\mathbf{F}}_{cp}\mathbf{H}_{i,0}\tilde{\mathbf{b}}(n) + \mathbf{G}_i\tilde{\mathbf{F}}_{cp}\sum_{l=-\infty,l\neq 0}^{\infty} \mathbf{H}_{i,l}\tilde{\mathbf{b}}(n-l) + \mathbf{G}_i\tilde{\mathbf{F}}_{cp}\boldsymbol{\nu}_i(n). \end{aligned} \tag{2.11}$$

In (2.11), the first term of the right-hand-side models the ISI within the symbols of a block for the ith user plus the corresponding ISI from other users, and the second term denotes the IBI for User i and the corresponding IBI from other users.

2.3 Assumptions

The following assumptions are made throughout the thesis in deriving the optimal linear block precoders:

A1. The channels are quasi-static such that they are assumed constant within one block transmission of data symbols.

A2. Each channel is an Lth-order finite impulse response (FIR) filter. The channel state information (CSI) is known to both the transmitter and the receiver.

A3. $Q = M + L$, where Q denotes the length of transmitted symbols in one block, and M denotes the length of data blocks before precoding. Namely, the number of redundant symbols is chosen to be L.

A4. Linear MMSE equalization is used at the receiver.

A5. Threshold detection is applied on the equalized data block.

A6. The elements of the transmitted data vector $\mathbf{s}_i$ are uncorrelated equiprobable QPSK or 4-QAM symbols with $E(\mathbf{s}_i\mathbf{s}_i^H) = \mathbf{I}$, $i = 1, ...N$.

A7. The receiver noise $\boldsymbol{\nu}_i$ are zero mean, white, circularly symmetric complex Gaussian noise vectors which are uncorrelated with the transmitted symbols; i.e., $E(\boldsymbol{\nu}_i\mathbf{s}_i^H) = \mathbf{0}$, and have a common covariance matrix $E(\boldsymbol{\nu}_i\boldsymbol{\nu}_i^H) = \sigma^2\mathbf{I}$.

2.4 Interblock Interference and Cyclic Prefix

According to Assumptions A2) and A3), and using (2.2), (2.3), (2.5), we simplify equation (2.11) as follows:

$$\hat{\mathbf{s}}_i(n) = \mathbf{G}_i\tilde{\mathbf{F}}_{cp}\mathbf{H}_{i,0}\mathbf{F}_{cp}\sum_{k=1}^{N}\mathbf{F}_k\mathbf{s}_k(n) + \mathbf{G}_i\tilde{\mathbf{F}}_{cp}\mathbf{H}_{i,1}\mathbf{F}_{cp}\sum_{k=1}^{N}\mathbf{F}_k\mathbf{s}_k(n-1) + \mathbf{G}_i\tilde{\mathbf{F}}_{cp}\boldsymbol{\nu}_i. \quad (2.12)$$

From equation (2.12), we see that the IBI in the nth received data block only comes from the previous block. Therefore, the IBI can be completely eliminated if we choose $\tilde{\mathbf{F}}_{cp}\mathbf{H}_{i,1}\mathbf{F}_{cp} = \mathbf{0}$. From equation (2.9), we observe that for the channel matrix $\mathbf{H}_{i,1}$, only the entries in its $L \times L$ upper right sub-matrix are non-zero. Recalling the definition of $\mathbf{F}_{cp}$ and $\tilde{\mathbf{F}}_{cp}$ in Section 2.2, we let $\mathbf{F}_{cp} = \mathbf{T}\mathcal{F}^H$ and $\tilde{\mathbf{F}}_{cp} = \mathcal{F}\bar{\mathbf{T}}$, where the matrices $\mathbf{T}$ and $\bar{\mathbf{T}}$ represent the operation of redundancy inserting and removing, respectively. Then the condition of zero IBI, $\tilde{\mathbf{F}}_{cp}\mathbf{H}_{i,1}\mathbf{F}_{cp} = \mathbf{0}$, is equivalent to having $\bar{\mathbf{T}}\mathbf{H}_{i,1}\mathbf{T} = \mathbf{0}$. Two popular methods can be used to choose $\bar{\mathbf{T}}$ and $\mathbf{T}$ such that the IBI can be eliminated: zero padding (ZP) and cyclic prefix (CP) [27]. For the ZP, the block signal is transmitted with the last L samples in one block forced to be zeros. In this thesis, we focus on cyclic prefix transmission, which is a common method for OFDM modulation. For the CP transmission, the redundant symbols are inserted in each transmitted block and the first L samples of each received block are discarded in the receiver. This can be done by choosing

$$\mathbf{T} = \mathbf{T}_{cp} = \begin{bmatrix} \mathbf{0}_{L\times(Q-2L)} & \mathbf{I}_L \\ \multicolumn{2}{c}{\mathbf{I}_{(Q-L)}} \end{bmatrix}, \tag{2.13}$$

$$\bar{\mathbf{T}} = \bar{\mathbf{T}}_{cp} = \begin{bmatrix} \mathbf{0}_{(Q-L)\times L} & \mathbf{I}_{(Q-L)} \end{bmatrix}. \tag{2.14}$$

Using the above two equations, the equalized signal can be written as the following form (We remove the block index for notation simplicity):

$$\hat{\mathbf{s}}_i = \mathbf{G}_i\boldsymbol{\mathcal{F}}\bar{\mathbf{H}}_i\boldsymbol{\mathcal{F}}^H\sum_{k=1}^{N}\mathbf{F}_k\mathbf{s}_k + \mathbf{G}_i\mathbf{n}_i, \tag{2.15}$$

where

$$\bar{\mathbf{H}}_i = \bar{\mathbf{H}}_{i,cp} = \bar{\mathbf{T}}\mathbf{H}_{i,0}\mathbf{T} = \begin{bmatrix} h_i(0) & 0 & \dots & 0 & h_i(L) & \dots & h_i(1) \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & h_i(L) \\ h_i(L) & \dots & \dots & h_i(0) & 0 & \dots & 0 \\ 0 & h_i(L) & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & h_i(L) & \dots & \dots & h_i(0) \end{bmatrix} \tag{2.16}$$

is an $M \times M$ circulant matrix, and $\mathbf{n}_i = \tilde{\mathbf{F}}_{cp}\boldsymbol{\nu}_i$. Moreover, from the property of circulant matrices [27], we see that matrix $\bar{\mathbf{H}}_i$ can be decomposed as

$$\bar{\mathbf{H}}_i = \boldsymbol{\mathcal{F}}^H\mathbf{H}_i\boldsymbol{\mathcal{F}}, \tag{2.17}$$

where matrix $\mathbf{H}_i$ is diagonal with its mth diagonal element given by

$$[\mathbf{H}_i]_{mm} = \sum_{n=0}^{L} h_i(n)e^{-j2\pi(m-1)n/M}. \tag{2.18}$$

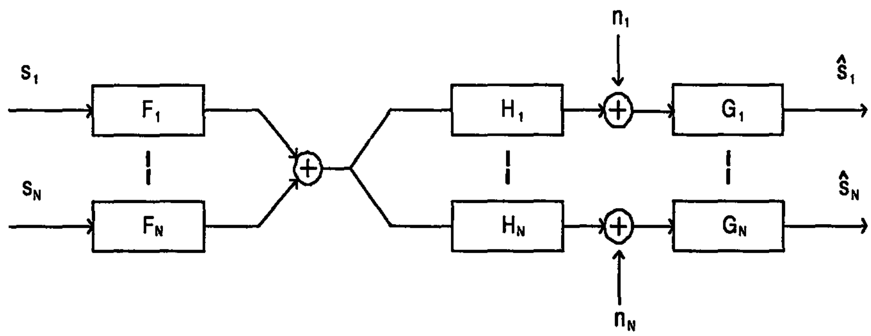

Figure 2.3: An equivalent drawing of an N user downlink OFDM scheme.

Hence, the circulant matrix $\bar{\mathbf{H}}_i$ can be diagonalized by IFFT and FFT processing, with the diagonal elements being considered as the frequency responses of the channel [27] [18]. Then (2.15) can be written as

$$\hat{\mathbf{s}}_i = \mathbf{G}_i \mathbf{H}_i \sum_{k=1}^{N} \mathbf{F}_k \mathbf{s}_k + \mathbf{G}_i \mathbf{n}_i. \tag{2.19}$$

The above multiuser OFDM system scheme can be illustrated in Figure 2.3.

For mathematical tractability, we assume that the structure of the precoder we design has the form as $\mathbf{F}_i = \bar{\mathbf{F}}_i \mathbf{V}_i$, where $\mathbf{V}_i$ is a unitary matrix and $\bar{\mathbf{F}}_i$ is a diagonal matrix. The advantage of this structure is its design simplicity since we only need to design the diagonal elements of matrix $\bar{\mathbf{F}}_i$ and the unitary matrix rather than the full precoder matrix. Interestingly, this precoder structure was proved to be an optimal structure that minimizes the total MSE for uplink multiple access OFDM systems [26].

2.5 Linear Block MMSE Equalizer

In this section, we present a formula for the linear MMSE equalizer. (The MMSE equalizer in the corresponding uplink system can be obtained in [26].)
We rewrite the equalized signal for User i (2.19) here:

$$\hat{\mathbf{s}}_i = \mathbf{G}_i\mathbf{H}_i \sum_{k=1}^{N} \mathbf{F}_k\mathbf{s}_k + \mathbf{G}_i\mathbf{n}_i. \tag{2.20}$$

Let $\mathbf{e}_i$ be the error vector for the estimated ith signal ($i = 1, 2, ...N$) :

$$\mathbf{e}_i = \hat{\mathbf{s}}_i - \mathbf{s}_i. \tag{2.21}$$

Substituting (2.20) into (2.21), we obtain

$$\begin{aligned} \mathbf{e}_i &= \mathbf{G}_i(\mathbf{H}_i \sum_{k=1}^{N} \mathbf{F}_k\mathbf{s}_k + \mathbf{n}_i) - \mathbf{s}_i \\ &= (\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i - \mathbf{I})\mathbf{s}_i + \mathbf{G}_i(\mathbf{H}_i \sum_{k=1, k\neq i}^{N} \mathbf{F}_k\mathbf{s}_k + \mathbf{n}_i). \end{aligned} \tag{2.22}$$

Then, its error covariance matrix is given by the following equation using Assumptions A6 and A7,

$$E(\mathbf{e}_i\mathbf{e}_i^H) = (\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i - \mathbf{I})(\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i - \mathbf{I})^H + \sum_{k=1, k\neq i}^{N} \mathbf{G}_i\mathbf{H}_i\mathbf{F}_k\mathbf{F}_k^H\mathbf{H}_i^H\mathbf{G}_i^H + \sigma^2\mathbf{G}_i\mathbf{G}_i^H. \tag{2.23}$$

Let

$$\mathbf{W}_i = (\sum_{k=1}^{N} \mathbf{H}_i\mathbf{F}_k\mathbf{F}_k^H\mathbf{H}_i^H + \sigma^2\mathbf{I})^{-1}, \tag{2.24}$$

we can rewrite the error covariance matrix for User i as

$$E(\mathbf{e}_i\mathbf{e}_i^H) = \mathbf{G}_i\mathbf{W}_i^{-1}\mathbf{G}_i^H - \mathbf{G}_i\mathbf{H}_i\mathbf{F}_i - (\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i)^H + \mathbf{I}. \tag{2.25}$$

Our goal in this section is to design the equalizer matrix $\mathbf{G}_i$ such that the total mean square error

$$\begin{aligned}\mathrm{MSE} &= \sum_{i=1}^{N} \mathrm{MSE}_i = \sum_{i=1}^{N} \mathrm{tr}(E(\mathbf{e}_i\mathbf{e}_i^H)) \\ &= \sum_{i=1}^{N} \mathrm{tr}\left(\mathbf{G}_i\mathbf{W}_i^{-1}\mathbf{G}_i^H - \mathbf{G}_i\mathbf{H}_i\mathbf{F}_i - (\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i)^H + \mathbf{I}\right) \end{aligned} \tag{2.26}$$

is minimized. In order to obtain the MMSE equalizer by minimizing (2.26), we first assume that the precoder $\mathbf{F}_i$ is fixed. Since $\mathrm{tr}(E(\mathbf{e}_j\mathbf{e}_j^H))$ $(j \neq i)$ is independent of $\mathbf{G}_i$, we minimize $\mathrm{tr}(E(\mathbf{e}_i\mathbf{e}_i^H))$ to obtain the linear MMSE equalizer for User i. Differentiating $\mathrm{tr}(E(\mathbf{e}_i\mathbf{e}_i^H))$ with respect to $\mathbf{G}_i$, we have

$$\begin{aligned}\frac{d}{d\mathbf{G}_i}\mathrm{MSE}_i &= \frac{d}{d\mathbf{G}_i}\mathrm{tr}(\mathbf{G}_i\mathbf{W}_i^{-1}\mathbf{G}_i^H - \mathbf{G}_i\mathbf{H}_i\mathbf{F}_i - (\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i)^H + \mathbf{I}) \\ &= \mathbf{W}_i^{-1}\mathbf{G}_i^H - \mathbf{H}_i\mathbf{F}_i. \end{aligned} \tag{2.27}$$

Letting the above function equal zero, we see

$$\mathbf{W}_i^{-1}\mathbf{G}_i^H - \mathbf{H}_i\mathbf{F}_i = \mathbf{0}, \tag{2.28}$$

which can be solved to obtain the MMSE equalizer

$$\mathbf{G}_i = \mathbf{F}_i^H\mathbf{H}_i^H\mathbf{W}_i. \tag{2.29}$$

From equation (2.29), we see that the MMSE equalizer can be obtained once we obtain the precoder matrix $\mathbf{F}_i$.

Chapter 3

MMSE Precoder and Equalizer

In this chapter, we design the optimal precoder that minimizes the total MSE in a downlink multiuser OFDM system which employs an MMSE equalizer. The optimal precoder can be obtained by optimally allocating the subcarriers and optimally allocating the power. This problem can be solved by two stages. In the first stage, we minimize the lower bound of the MSE to obtain the optimal total power for each subcarrier; in the second stage, we seek an optimal precoder to achieve this minimized lower bound. We show that, according to the MMSE subcarrier allocation strategy, each subcarrier should be allocated to only one user that has the largest subchannel gain in that subcarrier.

3.1 Formulation of the Problem

In this section, we formulate the optimization problem of designing the MMSE precoder. (The corresponding optimization problem of designing the MMSE precoder for the uplink multiple access system can be seen in [26].) Rewriting the error covariance matrix expression in (2.25) and the MMSE equalizer expression in (2.29), respectively, as follows:

$$E(\mathbf{e}_i\mathbf{e}_i^H) = \mathbf{G}_i\mathbf{W}_i^{-1}\mathbf{G}_i^H - \mathbf{G}_i\mathbf{H}_i\mathbf{F}_i - (\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i)^H + \mathbf{I}, \tag{3.1}$$

$$\mathbf{G}_i = \mathbf{F}_i^H\mathbf{H}_i^H\mathbf{W}_i, \; i = 1, 2, ...N, \tag{3.2}$$

we obtain the following MSE expression:

$$\begin{aligned} \mathrm{tr}\left(E(\mathbf{e}_i\mathbf{e}_i^H)\right) &= \mathrm{tr}(\mathbf{I} - \mathbf{G}_i\mathbf{H}_i\mathbf{F}_i) \\ &= \mathrm{tr}(\mathbf{I} - \mathbf{F}_i^H\mathbf{H}_i^H\mathbf{W}_i\mathbf{H}_i\mathbf{F}_i), \end{aligned} \tag{3.3}$$

where matrix $\mathbf{W}_i$ is defined as

$$\mathbf{W}_i = (\sum_{k=1}^{N}\mathbf{H}_i\mathbf{F}_k\mathbf{F}_k^H\mathbf{H}_i^H + \sigma^2\mathbf{I})^{-1}. \tag{3.4}$$

The detailed proof of equation (3.3) is shown in Appendix A. From equation (3.3), we see that the total MSE for all the users can be expressed by

$$\begin{aligned} \mathrm{MSE} &= \sum_{i=1}^{N}\mathrm{tr}(E(\mathbf{e}_i\mathbf{e}_i^H)) \\ &= -\sum_{i=1}^{N}\mathrm{tr}(\mathbf{F}_i^H\mathbf{H}_i^H\mathbf{W}_i\mathbf{H}_i\mathbf{F}_i - \mathbf{I}). \end{aligned} \tag{3.5}$$

Commonly, transmitter power is defined as the power which is used to transmit the data block (see [18], [24], [27] for details). In addition, with the use of Assumption A6 in Section 2.3, we have

$$\mathrm{tr}(E(\boldsymbol{\mathcal{F}}^H\mathbf{F}_i\mathbf{s}_i(\boldsymbol{\mathcal{F}}^H\mathbf{F}_i\mathbf{s}_i)^H)) = \mathrm{tr}(\mathbf{F}_iE(\mathbf{s}_i\mathbf{s}_i^H)\mathbf{F}_i^H) = \mathrm{tr}(\mathbf{F}_i\mathbf{F}_i^H), \tag{3.6}$$

where $\boldsymbol{\mathcal{F}}$ denotes the $M \times M$ FFT matrix (see Section 2.2). The transmitted power constraint can be expressed as

$$\sum_{i=1}^{N}\mathrm{tr}(\mathbf{F}_i\mathbf{F}_i^H) \leq P. \tag{3.7}$$

Thus, the optimization problem in which the optimal precoder can be designed is stated as

Formulation 1 *Find the optimal precoder* $\mathbf{F}_i$ *such that the total MSE (3.5) is minimized subject to a total power constraint; i.e.,*

$$\min_{\mathbf{F}_i} \quad -\sum_{i=1}^{N} \mathrm{tr}(\mathbf{F}_i^H \mathbf{H}_i^H \mathbf{W}_i \mathbf{H}_i \mathbf{F}_i - \mathbf{I}) \tag{3.8}$$

subject to the power constraint

$$\sum_{i=1}^{N} \mathrm{tr}(\mathbf{F}_i \mathbf{F}_i^H) \leq P.$$

Notice that

$$\begin{aligned} \mathrm{MSE} &= -\sum_{i=1}^{N} \mathrm{tr}(\mathbf{F}_i^H \mathbf{H}_i^H \mathbf{W}_i \mathbf{H}_i \mathbf{F}_i - \mathbf{I}) \\ &= -\sum_{i=1}^{N} \mathrm{tr}(\mathbf{H}_i \mathbf{F}_i \mathbf{F}_i^H \mathbf{H}_i^H \mathbf{W}_i - \mathbf{I}) \\ &= -\sum_{i=1}^{N} \mathrm{tr}(\mathbf{H}_i \mathbf{U}_i \mathbf{H}_i^H \mathbf{W}_i - \mathbf{I}), \end{aligned} \tag{3.9}$$

where

$$\mathbf{U}_i = \mathbf{F}_i \mathbf{F}_i^H. \tag{3.10}$$

Now equation (3.4) is equivalent to

$$\mathbf{W}_i = (\sum_{k=1}^{N} \mathbf{H}_i \mathbf{U}_k \mathbf{H}_i^H + \sigma^2 \mathbf{I})^{-1}. \tag{3.11}$$

We see that the matrix $\mathbf{U}_i$ and hence, $\mathbf{W}_i$, are both diagonal since the structure of the precoder to be designed is $\mathbf{F}_i = \bar{\mathbf{F}}_i \mathbf{V}_i$, where $\bar{\mathbf{F}}_i$ is a diagonal matrix and $\mathbf{V}_i$ is a unitary matrix (see Section 2.4 for details). Therefore, in order to obtain the optimal precoder matrix $\mathbf{F}_i$, we only need to find $\mathbf{U}_i$. Then one optimal matrix $\bar{\mathbf{F}}_i$ can be specifically determined by the square root of $\mathbf{U}_i$; i.e., $\bar{\mathbf{F}}_i = \mathbf{U}_i^{\frac{1}{2}}$. Choosing an arbitrary unitary matrix $\mathbf{V}_i$, we can obtain the optimal precoder $\mathbf{F}_i$. The corresponding optimal

MMSE equalizer $\mathbf{G}_i$ can then be calculated by equation (2.29). Now let us first design the optimal matrix $\mathbf{U}_i$. Let $u_{i,j}$, $|h_{i,j}|^2$, and $w_{i,j}$ denote the jth diagonal elements of matrices $\mathbf{U}_i$, $\mathbf{H}_i\mathbf{H}_i^H$ and $\mathbf{W}_i$, respectively, then, (3.9) can be rewritten as

$$\text{MSE} = -\sum_{i=1}^{N}\sum_{j=1}^{M}(|h_{i,j}|^2 u_{i,j} w_{i,j} - 1), \tag{3.12}$$

where

$$w_{i,j} = (\sum_{k=1}^{N} |h_{i,j}|^2 u_{k,j} + \sigma^2)^{-1}. \tag{3.13}$$

Substituting (3.13) into equation (3.12) yields

$$\text{MSE} = -\sum_{i=1}^{N}\sum_{j=1}^{M}(\frac{|h_{i,j}|^2 u_{i,j}}{\sum_{k=1}^{N} |h_{i,j}|^2 u_{k,j} + \sigma^2} - 1). \tag{3.14}$$

In light of equation (3.10), the power constraint (3.7) can be expressed as

$$\sum_{i=1}^{N}\sum_{j=1}^{M} u_{i,j} \leq P. \tag{3.15}$$

Since for an OFDM system and the diagonal structure of $\mathbf{U}_i$, the diagonal elements $u_{i,j}$ denote the power allocated to the corresponding subcarrier of User i. Now, the optimization problem which minimizes the MSE to obtain the optimal power $u_{i,j}$ can be stated as

Formulation 2 *Find the optimal power $u_{i,j}$ such that the MSE (3.14) is minimized subject to the power constraint; i.e.,*

$$\min_{u_{i,j}} \quad -\sum_{i=1}^{N}\sum_{j=1}^{M}(\frac{|h_{i,j}|^2 u_{i,j}}{\sum_{k=1}^{N} |h_{i,j}|^2 u_{k,j} + \sigma^2} - 1) \tag{3.16}$$

subject to the following constraints

$$u_{i,j} \geq 0,$$
$$\sum_{i=1}^{N}\sum_{j=1}^{M} u_{i,j} \leq P.$$

In the next section, we will show how to solve this optimization problem.

3.2 Solution for the Optimal Precoder

In this section, the optimal diagonal matrix $\mathbf{U}_i$ and the corresponding MMSE precoder and equalizer are derived. Solving Formulation 2 can yield the optimal $\mathbf{U}_i$. To do that, we perform the following subsections.

3.2.1 The Lower Bound of the MSE

In this subsection, we derive the lower bound of the MSE in equation (3.14). Rewriting the objective function of Formulation 2, we have

$$\text{MSE} = -\sum_{i=1}^{N}\sum_{j=1}^{M}\left(\frac{|h_{i,j}|^2 u_{i,j}}{\sum_{k=1}^{N}|h_{i,j}|^2 u_{k,j}+\sigma^2}-1\right). \tag{3.17}$$

Notice that

$$\sum_{i=1}^{N}\frac{|h_{i,j}|^2 u_{i,j}}{\sigma^2+\sum_{k=1}^{N}|h_{i,j}|^2 u_{k,j}}=\sum_{i=1}^{N}\frac{u_{i,j}}{\frac{\sigma^2}{|h_{i,j}|^2}+\sum_{k=1}^{N}u_{k,j}}. \tag{3.18}$$

Let i_j^* denote the user that has the largest subchannel gain among all the users for the jth subcarrier; [1] i.e., the subchannel gain of the i_j^*th user satisfies

$$|h_{i_j^*,j}|^2 = \max(|h_{1,j}|^2, |h_{2,j}|^2, ..., |h_{N,j}|^2). \tag{3.19}$$

Let p_j denote the total power allocated to the jth subcarrier from all the users:

$$p_j = \sum_{k=1}^{N} u_{k,j}. \tag{3.20}$$

[1]For simplicity, in this thesis, we presume that i_j^* is unique and exclude the discussion on cases when more than one user has the same maximum subchannel gain in one subcarrier.

Then, we have

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{u_{i,j}}{\frac{\sigma^2}{|h_{i,j}|^2} + \sum_{k=1}^{N} u_{k,j}} \\
= \quad & \frac{u_{i_j^*,j}}{\frac{\sigma^2}{|h_{i_j^*,j}|^2} + p_j} + \sum_{i=1, i \neq i_j^*}^{N} \frac{u_{i,j}}{\frac{\sigma^2}{|h_{i,j}|^2} + p_j} \\
\leq \quad & \frac{u_{i_j^*,j}}{\frac{\sigma^2}{|h_{i_j^*,j}|^2} + p_j} + \sum_{i=1, i \neq i_j^*}^{N} \frac{u_{i,j}}{\frac{\sigma^2}{|h_{i_j^*,j}|^2} + p_j} \\
= \quad & \frac{p_j}{\frac{\sigma^2}{|h_{i_j^*,j}|^2} + p_j}. \qquad (3.21)
\end{aligned}$$

From (3.21), we know that the equality in the second step holds if and only if

$$\sum_{i=1, i \neq i_j^*}^{N} \frac{u_{i,j}}{\frac{\sigma^2}{|h_{i,j}|^2} + p_j} = \sum_{i=1, i \neq i_j^*}^{N} \frac{u_{i,j}}{\frac{\sigma^2}{|h_{i_j^*,j}|^2} + p_j}. \qquad (3.22)$$

Since we assume that in the jth subcarrier, $|h_{i_j^*,j}|^2 > |h_{i,j}|^2$ for $i \neq i_j^*$, the equality (3.22) is tenable only when

$$u_{i,j} = 0, \;\; i = 1, 2, ...N, \, i \neq i_j^*. \qquad (3.23)$$

Combining (3.23) with (3.20), we obtain the following condition which holds the equality in equation (3.21):

$$u_{i,j} = \begin{cases} p_j & i = i_j^* \\ 0 & otherwise \end{cases}. \qquad (3.24)$$

Therefore, combining equation (3.17) with equations (3.18) and (3.21) yields

$$\begin{aligned}
\text{MSE} &= -\sum_{j=1}^{M}\sum_{i=1}^{N} \frac{u_{i,j}}{\frac{\sigma^2}{|h_{i,j}|^2} + \sum_{k=1}^{N} u_{k,j}} + NM \\
&\geq -\sum_{j=1}^{M} \frac{p_j}{\frac{\sigma^2}{|h_{i_j^*,j}|^2} + p_j} + NM \\
&= \text{MSE}_{LB}, \qquad (3.25)
\end{aligned}$$

where the equality holds if and only if condition (3.24) is satisfied for all subcarriers.

3.2.2 Minimizing and Achieving the MSE Lower Bound

Upon obtaining the lower bound of the MSE, in this subsection, we minimize this lower bound and try to achieve it. Then, we can obtain the optimal $u_{i,j}$ for Formulation 2.

First, we minimize the lower bound of the MSE to obtain the optimal power p_j. From equations (3.15) and (3.20), the power constraint can be rewritten as

$$\sum_{j=1}^{M} p_j \leq P. \tag{3.26}$$

This optimization problem can be stated as follows

Formulation 3 *Find the optimal power p_j such that the lower bound in (3.25) is minimized subject to the power constraint; i.e.,*

$$\min_{p_j} \quad -\sum_{j=1}^{M} \frac{p_j}{\frac{\sigma^2}{|h_{i_j^*,j}|^2} + p_j} \tag{3.27}$$

subject to the following constraints

$$\begin{aligned} & p_j \geq 0, \\ & \sum_{j=1}^{M} p_j \leq P. \end{aligned}$$

Since

$$\begin{aligned} & -\sum_{j=1}^{M} \frac{p_j}{\frac{\sigma^2}{|h_{i_j^*,j}|^2} + p_j} \\ = & -\sum_{j=1}^{M} \frac{|h_{i_j^*,j}|^2 p_j}{\sigma^2 + |h_{i_j^*,j}|^2 p_j} \\ = & -\sum_{j=1}^{M} \frac{|h_{i_j^*,j}|^2 p_j + \sigma^2 - \sigma^2}{\sigma^2 + |h_{i_j^*,j}|^2 p_j} \\ = & \sum_{j=1}^{M} \frac{\sigma^2}{\sigma^2 + |h_{i_j^*,j}|^2 p_j} - M, \end{aligned} \tag{3.28}$$

the above Formulation 3 can be simplified to the following form

Formulation 4 *Find the optimal power p_j such that (3.28) is minimized subject to the power constraint; i.e.,*

$$\min_{p_j} \quad \sum_{j=1}^{M} \frac{\sigma^2}{\sigma^2 + |h_{i_j^*,j}|^2 p_j} \tag{3.29}$$

subject to the following constraints

$$p_j \geq 0,$$
$$\sum_{j=1}^{M} p_j \leq P.$$

The convexity of Formulation 4 is determined by both the objective function and the constraints. The constraints are convex since they are linear. For the objective function, its convexity can be checked by calculating its Hessian matrix. Let $g(p_1, p_2, ..., p_M) = \sum_{j=1}^{M} \frac{\sigma^2}{\sigma^2 + |h_{i_j^*,j}|^2 p_j}$. Differentiating $g(p_1, p_2, ..., p_M)$ with respect to p_j, we see

$$\frac{\partial^2 g(p_1, p_2, ..., p_M)}{\partial p_j^2} = \frac{2\sigma^2 |h_{i_j^*,j}|^4}{(\sigma^2 + |h_{i_j^*,j}|^2 p_j)^3}, \tag{3.30}$$

$$\frac{\partial^2 g(p_1, p_2, ..., p_M)}{\partial p_j \partial p_k} = 0, \quad j \neq k. \tag{3.31}$$

From the above, the Hessian matrix of the objective function is diagonal with the positive diagonal elements; namely, it is positive definite. Hence, the objective function is convex. Therefore, we conclude that Formulation 4 is convex with respect to the parameters p_j.

From Formulation 4, we notice that this problem is similar to the single user power loading problem [18]. Similar to [18], the method of Lagrange multipliers is chosen to find the optimal p_j in Formulation 4. Here the channels $\mathbf{H}_i$ are rearranged such that the subchannel gains $|h_{i_j^*,j}|^2$ $(j = 1, 2, ...M)$ are in a descending order. Considering that the p_j are non-negative, the optimal p_j can be found to be (see Appendix B for details)

$$p_j^* = \left(\frac{P + \sigma^2 \sum_{m=1}^{\bar{M}} |h_{i_m^*,m}|^{-2}}{\sum_{m=1}^{\bar{M}} |h_{i_m^*,m}|^{-1}} |h_{i_j^*,j}|^{-1} - \sigma^2 |h_{i_j^*,j}|^{-2} \right)^+, \tag{3.32}$$

where $(x)^+ \triangleq \max(x, 0)$. Here, $\bar{M} \leq M$; $p_j^* > 0$ for all $j \in [1, \bar{M}]$, and $p_j^* = 0$ for all $j \in [\bar{M}+1, M]$.

Once the optimal power p_j^* has been decided, as we have mentioned in the beginning of this subsection, we seek a design to achieve this minimized lower bound. Rewrite the condition (3.24) as follows

$$u_{i,j} = \begin{cases} p_j^* & i = i_j^* \\ 0 & otherwise \end{cases}, \tag{3.33}$$

where i_j^* denotes the user that has the largest subchannel gain in the jth subcarrier. We see the minimized lower bound of the MSE can be achieved if and only if the condition (3.33) is satisfied for all subcarriers. According to (3.33), we can obtain the optimal $u_{i,j}$ for each user. Now by minimizing and achieving the lower bound of the MSE, we obtain the optimal $\mathbf{U}_i$ which minimizes the total MSE.

3.2.3 Subcarrier Allocation and Power Loading

In the above two subsections, we have obtained the optimal matrix $\mathbf{U}_i$ from the mathematical point. From another point of view, we have mentioned in Section 3.1 that $u_{i,j}$ denotes the power allocated to the jth subcarrier of User i. Since some $u_{i,j}$ are zero when we design the optimal matrix $\mathbf{U}_i$ under the MMSE criterion, which indicates that there is no power allocated to this subcarrier of the corresponding user. For example, if $u_{1,j} = 0$, then the power allocated to the jth subcarrier of User 1 is zero. So we should not transmit the data along the jth subcarrier for User 1. That means, this subcarrier will not be allocated to User 1. Thus, to design the optimal $u_{i,j}$, we should decide two factors: how to allocate the subcarriers and how to perform the power loading in each allocated subcarrier [26]. Therefore, we conclude that, minimizing and achieving the lower bound of the MSE to obtain $\mathbf{U}_i$ in the previous two subsections are equivalent to determining the above two factors under the MMSE criterion: *optimally allocating the subcarriers for all the users* and *optimally performing the power loading for each allocated subcarrier.* In the following, we give a detailed discussion of them.

For the first factor, we obtain the subcarrier allocation strategy from condition (3.33). In (3.33), we see that, for the jth subcarrier, there is no power allocated to any other users except for the i_j^*th user. So we should not transmit information symbols of other users along the jth subcarrier. Therefore, Condition (3.33) indicates that to achieve the minimized lower bound of the MSE, we should allocate one subcarrier to only one user that has the largest subchannel gain in that subcarrier. In other words, we have the following subcarrier assignment strategy with MMSE criterion in a multiuser downlink OFDM system:

MMSE Subcarrier Allocation Strategy: *We compare the gains of all corresponding subchannels for different users and allocate the one with the highest gain to the respective user. This allocation continues until all the subcarriers have been allocated.*

This strategy can be described by the following algorithm

Algorithm 1 *Let M denote the number of subcarriers, and $|h_{i,j}|^2$ denote the subchannel gains of User i for the jth subcarrier, respectively.*

Initialize $j = 1$.

while $j \leq M$ ***do***

1. *Find User i_j^* satisfying $|h_{i_j^*,j}|^2 > |h_{i,j}|^2$, $i = 1, 2, ...N, i \neq i_j^*$.*
2. *Allocate Subcarrier j to User i_j^**
3. *Repeat the above two steps by setting $j = j + 1$.*

end while

We give an example to show the subcarrier allocation result from the above strategy.

Example 1 Consider 2 users and 4 subcarriers. The channel matrices of these two users are defined as follows

$$\mathbf{H_1} = \begin{bmatrix} h_{1,1} & 0 & 0 & 0 \\ 0 & h_{1,2} & 0 & 0 \\ 0 & 0 & h_{1,3} & 0 \\ 0 & 0 & 0 & h_{1,4} \end{bmatrix}, \mathbf{H_2} = \begin{bmatrix} h_{2,1} & 0 & 0 & 0 \\ 0 & h_{2,2} & 0 & 0 \\ 0 & 0 & h_{2,3} & 0 \\ 0 & 0 & 0 & h_{2,4} \end{bmatrix}. \tag{3.34}$$

Assume that the subchannel gains, $|h_{i,j}|^2\,(i = 1, 2;\ j = 1, \ldots 4)$, satisfy the following relations

$$|h_{1,1}|^2 > |h_{2,1}|^2,\ |h_{1,3}|^2 > |h_{2,3}|^2,\ |h_{2,2}|^2 > |h_{1,2}|^2,\ |h_{2,4}|^2 > |h_{1,4}|^2. \tag{3.35}$$

Then according to the MMSE subcarrier allocation strategy, we allocate the first and the third subcarriers to User 1, and allocate the second and the fourth subcarriers to User 2.

According to this kind of allocation strategy, each subcarrier is allowed to be used by only one user to transmit the data. Interestingly, this strategy coincides with the one in [12], where the criterion is based on the maximum sum mutual information rate. Our allocation strategy is performed in order to obtain the MMSE, but in the special case when one user does not have a highest subchannel gain, this user may not be assigned any subcarriers. This will cause a low transmission data rate for that user and become a drawback of the scheme. Finding a tradeoff between the transmission data rate of that user and the MMSE in such a special case is the subject of future research.

Now, we can consider the second factor. We perform optimal power loading for each allocated subcarrier of each user. By solving Formulation 4; i.e., minimizing the lower bound of the MSE, we can obtain the optimal total power p_j^* for each subcarrier in equation (3.32). Since one subcarrier is allocated to only one user, the power p_j^* will be completely allocated to that corresponding user.

After we have designed the optimal diagonal matrix $\mathbf{U}_i$, we derive the MMSE precoder. As we have mentioned in Section 3.1, the optimal matrix $\bar{\mathbf{F}}_i$ can be determined by the square root of $\mathbf{U}_i$. Then, the MMSE precoder $\mathbf{F}_i$ and the corresponding equalizer $\mathbf{G}_i$ in equation (2.29) can be found.

3.3 Simulations

To show the performance of our MMSE design, in this section, we carry out simulation for a two user downlink OFDM system. We compare our design with the equal power allocation design. In the equal power allocation design, the total transmission power is equally distributed among the subcarriers with the same subcarrier assignment proposed in this chapter. In both designs, the unitary matrices $\mathbf{V}_i$ are chosen to be identity matrices.

Example 2: In this example, the total number of subcarriers M is assumed to be 32, and the power budget P equals 10. The noise vector at each receiver is assumed to be white Gaussian with a common covariance matrix; i.e., $E(\boldsymbol{\nu}_i\boldsymbol{\nu}_i^H) = \sigma^2\mathbf{I}$. The impulse response of the channel for User 1 is $[-0.6755 + \jmath 0.3915, -0.0445 + \jmath 1.1265, 0.7361 - \jmath 0.2874]$, and for User 2 is $[0.0842 - \jmath 0.9134, -0.3010 + \jmath 0.2139, 1.0701 + \jmath 0.3898]$. The input vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ are chosen as random 4-QAM vectors. Figure 3.1 shows the simulation results of the performance comparison between the MMSE and uniform power loading designs. We see from Figure 3.1 that the performance of the MMSE design proposed in this chapter is better than that of the equal power allocation with identity matrix design especially from middle to high SNR region.

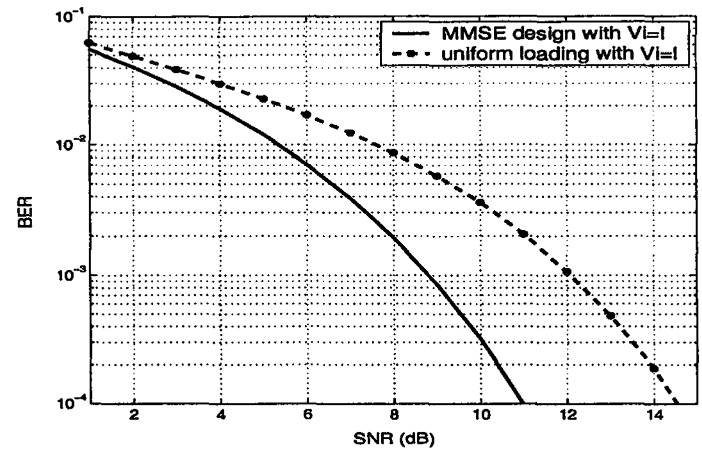

Figure 3.1: Performance comparison between MMSE and uniform power loading designs.

Example 3: In order to evaluate statistical average performance of our design methods, we perform this example. We show the average performance comparison for a random channel. The average is calculated over 1000 independent channel realizations. For each realization, we assume that both the transmitter and the receiver have the perfect knowledge of the channel. The taps of channel matrices are obtained from IID complex Gaussian distributions with zero mean and variance 0.5 per dimension. The total number of subcarriers and the power budget are identical to those used in Example 2. The simulation results are shown in Figure 3.2.

3.4 Summary

In this chapter, from the viewpoint of minimizing the total MSE, we designed the optimal precoder for which the MMSE equalizer is employed. We showed that, minimizing the MSE to obtain the optimal precoder can be done mainly in two stages.

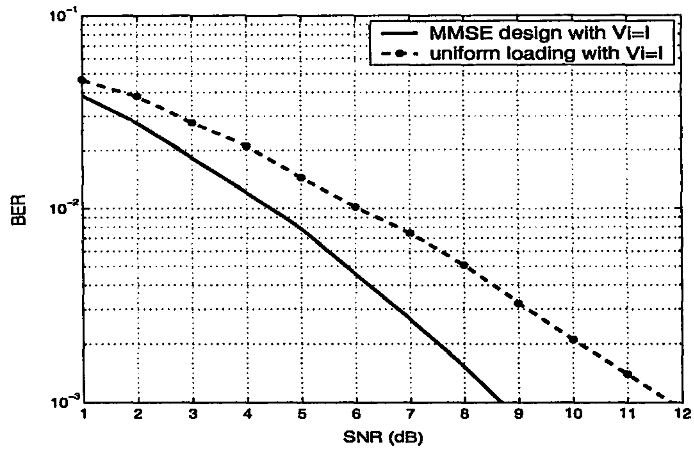

Figure 3.2: Average performance comparison between MMSE and uniform power loading designs.

First, we minimize the lower bound of the MSE to obtain the optimal total power for each subcarrier; and second, we seek the subcarrier allocation strategy to achieve this lower bound. Particularly, in the second stage, our subcarrier allocation strategy states that, each subcarrier is allowed to be used by only one user to transmit the data. Simulation results showed that the performance of our MMSE design is better than that of the equal power allocation design.

Chapter 4

MBER Precoder with MMSE Equalizer

In Chapter 3, we have designed an optimal precoder which minimizes the MSE. In this chapter, from the view point of minimizing the average BER, we discuss the corresponding precoder when 4-QAM or QPSK transmission, and MMSE equalization are employed. We choose the subcarrier allocation strategy which is proposed in Chapter 3. Then, different from the MMSE power loading problem, we perform optimal power loading and derive the corresponding optimal precoder by minimizing the average BER. This optimization problem can be done by solving a two-stage process. In the first stage, we derive the lower bound of the average BER and minimize this lower bound. After allocating the subcarriers, the formulation of the power loading optimization problem can be solved by interior point methods. Furthermore, to reduce computation complexity, an alternative, efficient power loading method is proposed here. In the second stage, we choose the special unitary matrix to achieve this lower bound and obtain the corresponding optimal precoder. Simulation results show that, the performance of the MBER design is superior to several other design methods, including the MMSE precoder design.

4.1 Problem Description

4.1.1 Average BER Expression

For 4-QAM signals, we detect a complex equalized signal vector by detecting its real and imaginary parts, separately, and quantize the elements in both parts of the detected signal vector to be ± 1. (The result in this chapter is equally valid for QPSK signals.)

With Assumption A6 (The real and imaginary parts of the signal are independent of each other) and similar to [18] [24], we have the following definition

Definition 1 *The average BER of the detected signal for User i, $P_e^{(i)}$, is defined to be the arithmetic average error probabilities for the real and imaginary parts of each symbol of the block, i.e.,*

$$P_e^{(i)} = \frac{1}{2M}\sum_{j=1}^{M}(P_{e,R}^{(i,j)} + P_{e,I}^{(i,j)}), \tag{4.1}$$

where $P_{e,R}^{(i,j)}$ and $P_{e,I}^{(i,j)}$ denote the error probability for the real and imaginary parts of the jth symbol of the data block for User i, respectively.

In order to calculate the average BER for User i, we calculate the BER of the real and imaginary parts of one 4-QAM symbol. This involves the following two steps [11]:
1) Consider the SINR of the real and imaginary parts of this symbol,
2) Consider the Gaussian distribution of the interference plus noise added on it.
First we define the SINR of the real and imaginary parts of the jth symbol in the data block for the ith user.

Definition 2 *In a receiver that makes decisions on sequences* $\mathrm{Re}([\mathbf{s}_i]_j)$ *and* $\mathrm{Im}([\mathbf{s}_i]_j)$, *the signal to interference (including ISI and MUI) plus noise ratios of these two parts at the decision points* $\mathrm{SINR}_{\mathrm{Re}([\mathbf{s}_i]_j)}$ *and* $\mathrm{SINR}_{\mathrm{Im}([\mathbf{s}_i]_j)}$ *are defined as the power of the signal divided by the power of the interference plus noise of the two parts, respectively.*

According to Appendix C, for User i, the SINR of the real and imaginary parts of the jth symbol in the data block can be written as:

$$\mathrm{SINR}_{\mathrm{Re}([\hat{\mathbf{s}}_i]_j)} = \mathrm{SINR}_{\mathrm{Im}([\hat{\mathbf{s}}_i]_j)} = \frac{[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}{1-[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}. \tag{4.2}$$

Here, $[\mathbf{s}_i]_j$ and $[\hat{\mathbf{s}}_i]_j$ represent the jth element of vectors $\mathbf{s}_i$ and $\hat{\mathbf{s}}_i$; $[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}$ represents the jth diagonal element of matrix $\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i$. If we can show that the interference plus noise added on the signal is Gaussian distributed, then we can express the BER function by using the complementary error function $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-z^2)dz$ [11]. Rewrite the received signal in (2.19) as follows:

$$\hat{\mathbf{s}}_i = \mathrm{Diag}(\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i)\mathbf{s}_i + (\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i - \mathrm{Diag}(\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i))\mathbf{s}_i + \mathbf{G}_i\mathbf{H}_i\sum_{k=1,k\neq i}^{N}\mathbf{F}_k\mathbf{s}_k + \mathbf{G}_i\mathbf{n}_i, \tag{4.3}$$

where the first term of the right-hand-side represents the desired signal, the second and the third term represent the ISI and the MUI, and the last term represents Gaussian noise. Here, $\mathrm{Diag}(\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i)$ is an $M \times M$ diagonal matrix consisting of the diagonal elements of $\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i$.

We know that the MUI obeys a circularly symmetric complex Gaussian distribution when the system has a large number of users [9] [13]. By applying this result to the block-based communication system with linear MMSE equalization, it was previously claimed [18] that the distribution of the ISI of the transmitting signal also converges to a Gaussian distribution as the block size, M, increases. Using these results, we can claim that for each received signal, the noise plus interference (ISI and MUI) tend to be Gaussian distributed when the number of users and/or the block size is sufficiently large.

Now by following the standard procedures for calculating the error probability of an antipodal symbol [11], we can efficiently compute the error probability for the real and imaginary parts of the jth element of the detected signal vector. We have

$$P_{e,R}^{(i,j)} = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{1}{2}\text{SINR}_{\text{Re}([\hat{\mathbf{s}}_i]_j)}}\right), \tag{4.4}$$

$$P_{e,I}^{(i,j)} = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{1}{2}\text{SINR}_{\text{Im}([\hat{\mathbf{s}}_i]_j)}}\right). \tag{4.5}$$

Substituting (4.2) into the above two equations, we obtain

$$P_{e,R}^{(i,j)} = P_{e,I}^{(i,j)} = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}{1-[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}\right)}\right), \tag{4.6}$$

which is equivalent to

$$P_{e,R}^{(i,j)} = P_{e,I}^{(i,j)} = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{1}{1-[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}-1\right)}\right), \tag{4.7}$$

where $1-[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}$ denotes the MSE for the jth symbol of User i since the error covariance matrix in equation (3.3) is

$$E(\mathbf{e}_i\mathbf{e}_i^H) = \mathbf{I} - \mathbf{G}_i\mathbf{H}_i\mathbf{F}_i. \tag{4.8}$$

So far, we have obtained the BER expression for the real and imaginary parts of one symbol. In order to simplify this BER expression, we resort to the following matrix inversion lemma [16]

Lemma 1 *(Matrix Inversion Lemma) Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be arbitrary square nonsingular matrices, then*

$$(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{DA}^{-1}\mathbf{B}+\mathbf{C}^{-1})^{-1}\mathbf{DA}^{-1}. \tag{4.9}$$

Applying the matrix inversion lemma to the error covariance matrix (4.8) yields (see Appendix A for details)

$$E(\mathbf{e}_i\mathbf{e}_i^H) = (\mathbf{I}+\mathbf{A}_i)^{-1}, \tag{4.10}$$

which results in the MSE for the jth symbol of User i as

$$\text{MSE}_{i,j} = [(\mathbf{I} + \mathbf{A}_i)^{-1}]_{jj}. \tag{4.11}$$

Here, matrix $\mathbf{A}_i = \mathbf{F}_i^H \mathbf{H}_i^H \mathbf{C}_i^{-1} \mathbf{H}_i \mathbf{F}_i$, and

$$\mathbf{C}_i = \sum_{k=1, k \neq i}^{N} \mathbf{H}_i \mathbf{F}_k \mathbf{F}_k^H \mathbf{H}_i^H + \sigma^2 \mathbf{I}. \tag{4.12}$$

Combining equation (4.7) with (4.11), we see

$$P_{e,R}^{(i,j)} = P_{e,I}^{(i,j)} = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{1}{[(\mathbf{I} + \mathbf{A}_i)^{-1}]_{jj}} - 1\right)}\right). \tag{4.13}$$

Now the bit error probability of User i can be calculated by

$$\begin{aligned} P_e^{(i)} &= \frac{1}{2M}\sum_{j=1}^{M}(P_{e,R}^{(i,j)} + P_{e,I}^{(i,j)}) \\ &= \frac{1}{2M}\sum_{j=1}^{M}\text{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{1}{[(\mathbf{I} + \mathbf{A}_i)^{-1}]_{jj}} - 1\right)}\right). \end{aligned} \tag{4.14}$$

Since the average performance of the system is our main concern, our goal is to design the optimal precoder that minimizes the average BER of all users. Therefore, we introduce the following definition.

Definition 3 *Let P_e denote the average bit error rate for all N users; i.e.,*

$$P_e = \frac{1}{N}\sum_{i=1}^{N} P_e^{(i)}. \tag{4.15}$$

Substituting (4.14) into (4.15), we obtain the expression of the average BER

$$P_e = \frac{1}{2NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\text{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{1}{[(\mathbf{I} + \mathbf{A}_i)^{-1}]_{jj}} - 1\right)}\right). \tag{4.16}$$

4.1.2 The Lower Bound of the BER

In this subsection, we try to obtain the lower bound of the BER using Jensen's inequality [20]. A key element is considering the convexity in the BER expression. Once the BER expression is convex, we can compute the lower bound of (4.14). Further, the lower bound of (4.16) can then be obtained at the same time. Therefore, we need to consider the convexity of the following function first,

$$f(x) = \operatorname{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{1}{x}-1\right)}\right).$$

The second derivative of $f(x)$ is given by (refer to Appendix D for details)

$$\frac{d^2}{dx^2}f(x) = \frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x^{-1}-1)\right)(x^{-1}-1)^{-\frac{1}{2}}x^{-4}\frac{(2x-1)^2}{1-x}. \tag{4.17}$$

From the above equation (4.17), we see that the second derivative of $f(x)$ with respect to x is non-negative when x is chosen in the region $0 < x < 1$. Applying this result to the BER expression $P_e^{(i)}$, we know that if $0 < [(\mathbf{I}+\mathbf{A}_i)^{-1}]_{jj} < 1$ for all $j \in [1, M]$, then $P_e^{(i)}$ is a convex function. Furthermore, for a convex function $f(x)$ with the constraints $q_j \geq 0$ and $\sum_{j=1}^{M} q_j = 1$, Jensen's inequality [20] states that $\sum_{j=1}^{M} q_j f(x_j) \geq f\left(\sum_{j=1}^{M} q_j x_j\right)$, in which the equality holds if and only if x_j are the same for all $j \in [1, M]$. By using this result to $P_e^{(i)}$, a lower bound of it can be obtained. We see

$$\begin{aligned} P_e^{(i)} &= \frac{1}{2M}\sum_{j=1}^{M}\operatorname{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{1}{[(\mathbf{I}+\mathbf{A}_i)^{-1}]_{jj}}-1\right)}\right) \\ &\geq \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{M}{\sum_{j=1}^{M}[(\mathbf{I}+\mathbf{A}_i)^{-1}]_{jj}}-1\right)}\right) \\ &= \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{M}{\operatorname{tr}((\mathbf{I}+\mathbf{A}_i)^{-1})}-1\right)}\right) \\ &= P_{eLB}^{(i)}, \end{aligned} \tag{4.18}$$

where in the second step, the equality holds if and only if $[(\mathbf{I}+\mathbf{A}_i)^{-1}]_{jj}$ are the same for all $j \in [1, M]$.

Now let us consider whether this convex region $0 < [(\mathbf{I}+\mathbf{A}_i)^{-1}]_{jj} < 1$ can be satisfied. The left inequality of convex region $[(\mathbf{I}+\mathbf{A}_i)^{-1}]_{jj} > 0$ is true because of the positive definite matrix $(\mathbf{I}+\mathbf{A}_i)^{-1}$. For the right inequality, we need the following lemma [16].

Lemma 2 *For any positive semi-definite matrices A and B, if*

$$\mathbf{A} \succeq \mathbf{B}, \textit{ then } \mathbf{A}^{-1} \preceq \mathbf{B}^{-1}.$$

By this lemma, we have the following matrix inequality

$$(\mathbf{I}+\mathbf{A}_i)^{-1} \prec \mathbf{I} \tag{4.19}$$

since $\mathbf{I}+\mathbf{A}_i \succ \mathbf{I}$. This implies the inequality $[(\mathbf{I}+\mathbf{A}_i)^{-1}]_{jj} < 1$. Therefore, the convex region can be satisfied for all j in a data block.

According to Definition 3 and the lower bound in (4.18), we have

$$\begin{aligned} P_e &\geq \frac{1}{2N}\sum_{i=1}^{N} \operatorname{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{M}{\operatorname{tr}((\mathbf{I}+\mathbf{A}_i)^{-1})} - 1\right)}\right) \\ &= P_{eLB}, \end{aligned} \tag{4.20}$$

where the equality in (4.20) holds if and only if all the diagonal entries of matrix $(\mathbf{I}+\mathbf{A}_i)^{-1}$ are equal, $\forall i \in [1, N]$.

4.1.3 Formulation of the Problem

In this subsection, we formulate the problem of designing the optimal precoder to achieve the MBER for downlink multiuser OFDM systems, subject to a transmitted power constraint. Adding a power constraint (3.7), the design problem for our system can be written as

$$\begin{aligned} \min_{\mathbf{F}_i} \quad & P_e \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{N} \operatorname{tr}(\mathbf{F}_i \mathbf{F}_i^H) \leq P. \end{aligned} \tag{4.21}$$

This optimization problem is difficult to solve directly. However, since we have shown that the lower bound of the BER obtained by Jensen's inequality can be achieved if and only if $\forall i \in [1, N]$, all the diagonal entries of matrix $(\mathbf{I} + \mathbf{A}_i)^{-1}$ are equal; therefore, instead of solving problem (4.21) directly, we will use the following two stages to find a solution [18] [24].

- Stage 1: Minimize the lower bound of the BER, subject to the constraint on transmission power.
- Stage 2: Seek a design which achieves this minimized lower bound.

For the first stage, when we try to minimize the lower bound of the BER to obtain the optimal precoder, the problem can be expressed as:

Formulation 5 *Find precoder $\mathbf{F}_i$ such that the lower bound of (4.20) is minimized subject to a power constraint; i.e.,*

$$\min_{\mathbf{F}_i} \quad \frac{1}{2N} \sum_{i=1}^{N} \operatorname{erfc}\left(\sqrt{\frac{1}{2} \left(\frac{M}{\operatorname{tr}((\mathbf{I} + \mathbf{A}_i)^{-1})} - 1 \right)} \right) \tag{4.22}$$

subject to a power constraint

$$\sum_{i=1}^{N} \operatorname{tr}(\mathbf{F}_i \mathbf{F}_i^H) \leq P.$$

In Section 2.4, we have mentioned that the structure of the precoder which we will design is $\mathbf{F}_i = \bar{\mathbf{F}}_i \mathbf{V}_i$, in which matrix $\bar{\mathbf{F}}_i$ is a diagonal matrix and $\mathbf{V}_i$ is a unitary matrix. It further follows that

$$\begin{aligned}
& \operatorname{tr}\left((\mathbf{I}+\mathbf{A}_i)^{-1}\right) \\
= & \operatorname{tr}\left((\mathbf{I}+\mathbf{F}_i^H\mathbf{H}_i^H\mathbf{C}_i^{-1}\mathbf{H}_i\mathbf{F}_i)^{-1}\right) \\
= & \operatorname{tr}\left((\mathbf{I}+(\bar{\mathbf{F}}_i\mathbf{V}_i)^H\mathbf{H}_i^H\mathbf{C}_i^{-1}\mathbf{H}_i\bar{\mathbf{F}}_i\mathbf{V}_i)^{-1}\right) \\
= & \operatorname{tr}\left(\mathbf{V}_i^H(\mathbf{I}+\bar{\mathbf{F}}_i^H\mathbf{H}_i^H\mathbf{C}_i^{-1}\mathbf{H}_i\bar{\mathbf{F}}_i)^{-1}\mathbf{V}_i\right) \\
= & \operatorname{tr}\left((\mathbf{I}+\bar{\mathbf{F}}_i^H\mathbf{H}_i^H\mathbf{C}_i^{-1}\mathbf{H}_i\bar{\mathbf{F}}_i)^{-1}\right). \qquad (4.23)
\end{aligned}$$

If we let $\bar{\mathbf{A}}_i$ denote $\bar{\mathbf{F}}_i^H\mathbf{H}_i^H\mathbf{C}_i^{-1}\mathbf{H}_i\bar{\mathbf{F}}_i$, then we have the following property:

$$\operatorname{tr}\left((\mathbf{I}+\mathbf{A}_i)^{-1}\right) = \operatorname{tr}\left((\mathbf{I}+\bar{\mathbf{A}}_i)^{-1}\right). \qquad (4.24)$$

Using (4.24), we can transform Formulation 5 to the following equivalent formulation:

Formulation 6 *Find $\bar{\mathbf{F}}_i$ such that the lower bound of the BER is minimized subject to a power constraint; i.e.,*

$$\min_{\bar{\mathbf{F}}_i} \quad \frac{1}{2N}\sum_{i=1}^{N}\operatorname{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{M}{\operatorname{tr}((\mathbf{I}+\bar{\mathbf{A}}_i)^{-1})}-1\right)}\right) \qquad (4.25)$$

subject to a power constraint

$$\sum_{i=1}^{N}\operatorname{tr}(\bar{\mathbf{F}}_i\bar{\mathbf{F}}_i^H) \leq P.$$

Obviously, it is easier to solve Formulation 6 than Formulation 5 since all matrices $\bar{\mathbf{F}}_i$ and $\bar{\mathbf{A}}_i$ are diagonal. In the next section, we will show how to solve the optimization problem in the context of Formulation 6.

4.2 Solution for the Optimal Precoder

Formulation 6 is still hard to be solved directly since the objective function is non-convex with respect to the parameters $\bar{\mathbf{F}}_i$, $i = 1, 2, \ldots N$, hence we should seek another method. Since we have mentioned in Section 3.1 that, in order to obtain the optimal matrix $\bar{\mathbf{F}}_i$, we only need to obtain $\mathbf{U}_i = \bar{\mathbf{F}}_i\bar{\mathbf{F}}_i^H$. Then matrix $\bar{\mathbf{F}}_i$ can be obtained by the square root of $\mathbf{U}_i$. As we have stated in Section 3.2.3 that, solving the optimal diagonal matrix $\mathbf{U}_i$ is equivalent to determining the following two factors: *optimal subcarrier allocation* and *optimal power allocation for each allocated subcarrier*. We discuss them in the following subsections.

4.2.1 Subcarrier Allocation

The first factor, optimally allocating the subcarriers with the MBER criterion, is a difficult problem because Formulation 6 is nonconvex and the objective involving the linear combination of all complementary error functions is hard to handle. Since exhaustive search is very impractical, in this chapter, we adopt the subcarrier allocation strategy obtained in Chapter 3 which achieves the MMSE (see Section 3.2.3). Then we briefly explain why this kind of allocation strategy should be good for error performance. Since MUI is a major obstacle to system performance, allocating a subcarrier to only one user can eliminate MUI. For one subcarrier, we allocate it to the user that has the largest subchannel gain in that subcarrier, since a larger subchannel gain should achieve the better performance for the system if we transmit the data along that subcarrier.

4.2.2 Power Loading

Based on the MMSE subcarrier allocation strategy, we then need to design the power added on each subcarrier. Let $u_{i,j}$ denote the power allocated to the jth subcarrier for User i. We try to derive the optimal variables $u_{i,j}$ in this subsection. Let $|h_{i,j}|^2$ denote the corresponding subchannel gain, and r_i denote the number of subcarriers allocated to User i. Notice that for each user i, the diagonal matrix

$\mathbf{C}_i = \sum_{k=1,k\neq i}^{N} \mathbf{H}_i\mathbf{F}_k\mathbf{F}_k^H\mathbf{H}_i^H + \sigma^2\mathbf{I}$ defined in equation (4.12) reduces to $\sigma^2\mathbf{I}$ because of the elimination of MUI. The length of useful data in one block transmitted for User i is r_i after we allocate the subcarriers. Hence, we have the following equation

$$\begin{aligned} & \operatorname{tr}((\mathbf{I}+\bar{\mathbf{A}}_i)^{-1}) \\ = \; & \operatorname{tr}\left((\mathbf{I}+\bar{\mathbf{F}}_i^H\mathbf{H}_i^H\mathbf{C}_i^{-1}\mathbf{H}_i\bar{\mathbf{F}}_i)^{-1}\right) \\ = \; & \operatorname{tr}\left((\mathbf{I}+\mathbf{H}_i^H\mathbf{H}_i\mathbf{U}_i\sigma^{-2}\mathbf{I})^{-1}\right) \\ = \; & \sum_{j=1}^{r_i} \frac{\sigma^2}{\sigma^2+|h_{i,j}|^2u_{i,j}}. \end{aligned} \tag{4.26}$$

Now Formulation 6 can be transformed to the following formulation:

Formulation 7 *Find the optimal power loading such that the lower bound of the BER is minimized subject to the power constraint; i.e.,*

$$\min_{u_{i,j}} \quad \frac{1}{2N}\sum_{i=1}^{N} \operatorname{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{r_i}{\sum_{j=1}^{r_i}\frac{\sigma^2}{\sigma^2+|h_{i,j}|^2u_{i,j}}}-1\right)}\right) \tag{4.27}$$

subject to the following constraints:

$$\begin{aligned} & \sum_{i=1}^{N}\sum_{j=1}^{r_i} u_{i,j} \leq P \\ & u_{i,j} \geq 0. \end{aligned}$$

Reformulating the above problem to a simpler form is done by introducing new variables τ_i such that $\tau_i = \sum_{j=1}^{r_i}\frac{\sigma^2}{\sigma^2+|h_{i,j}|^2u_{i,j}}$. Thus, we can rewrite Formulation 7 in the following alternative form:

Formulation 8 *Find the optimal power loading such that the lower bound of the BER is minimized, subject to the power constraint; i.e.,*

$$\min_{\tau_i,u_{i,j}} \quad \frac{1}{2N}\sum_{i=1}^{N} \operatorname{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{r_i}{\tau_i}-1\right)}\right) \tag{4.28}$$

subject to the following constraints:

$$\tau_i \geq \sum_{j=1}^{r_i} \frac{\sigma^2}{\sigma^2 + |h_{i,j}|^2 u_{i,j}}$$
$$\sum_{i=1}^{N} \sum_{j=1}^{r_i} u_{i,j} \leq P$$
$$u_{i,j} \geq 0.$$

Since the complementary error function is monotonically decreasing, we have $\mathrm{erfc}\left(\sqrt{\frac{1}{2}(\frac{r_i}{\tau_i}-1)}\right) \geq \mathrm{erfc}\left(\sqrt{\frac{1}{2}(\frac{r_i}{\sum_{j=1}^{r_i}\frac{\sigma^2}{\sigma^2+|h_{i,j}|^2u_{i,j}}}-1)}\right)$ whenever $\tau_i \geq \sum_{j=1}^{r_i}\frac{\sigma^2}{\sigma^2+|h_{i,j}|^2u_{i,j}}$. We know that the equality $\tau_i = \sum_{j=1}^{r_i}\frac{\sigma^2}{\sigma^2+|h_{i,j}|^2u_{i,j}}$ must hold at the optimality by a monotonicity argument when we minimize $\frac{1}{2N}\sum_{i=1}^{N}\mathrm{erfc}\left(\sqrt{\frac{1}{2}(\frac{r_i}{\tau_i}-1)}\right)$. This argument completes the equivalence of Formulations 7 and 8 [26].

Formulation 8 is convex with respect to the variables τ_i and $u_{i,j}$. Here we give a brief explanation. The convexity of the problem is determined not only by the objective function, but also by the constraints. The linear constraints $\sum_{i=1}^{N}\sum_{j=1}^{r_i} u_{i,j} \leq P$ and $u_{i,j} \geq 0$ are convex. For the objective function, we have shown that it is convex in Section 4.1.2. So all that remains is to show that the constraint $\tau_i \geq \sum_{j=1}^{r_i}\frac{\sigma^2}{\sigma^2+|h_{i,j}|^2u_{i,j}}$ is convex. Let $f_i(\tau_i, u_{i,1}, ...u_{i,r_i}) = \sum_{j=1}^{r_i}\frac{\sigma^2}{\sigma^2+|h_{i,j}|^2u_{i,j}} - \tau_i$. Then this constraint becomes $f_i(\tau_i, u_{i,1}, ...u_{i,r_i}) \leq 0$ of which the convexity is determined by the Hessian matrix of $f_i(\tau_i, u_{i,1}, ...u_{i,r_i})$. The Hessian can be calculated by differentiating $f_i(\tau_i, u_{i,1}, ...u_{i,r_i})$ with respect to $u_{i,j}$ and τ_i in the following forms,

$$\frac{\partial^2 f_i(\tau_i, u_{i,1}, ...u_{i,r_i})}{\partial u_{i,j}^2} = \frac{2\sigma^2|h_{i,j}|^4}{(|h_{i,j}|^2u_{i,j}+\sigma^2)^3},$$
$$\frac{\partial^2 f_i(\tau_i, u_{i,1}, ...u_{i,r_i})}{\partial \tau_i^2} = 0,$$
$$\frac{\partial^2 f_i(\tau_i, u_{i,1}, ...u_{i,r_i})}{\partial u_{i,j}\partial u_{i,k}} = 0,\ j \neq k,$$
$$\frac{\partial^2 f_i(\tau_i, u_{i,1}, ...u_{i,r_i})}{\partial u_{i,j}\partial \tau_i} = 0.$$

From the above, it is concluded that the Hessian matrix of $f_i(\tau_i, u_{i,1}, ...u_{i,r_i})$ is diagonal and positive semi-definite; thus this constraint is convex.

Formulation 8 can be solved efficiently by using interior point methods [17], and we will briefly describe them in Section 4.2.3.3.

4.2.3 Alternative Efficient Method for Optimal Power Loading

The solution of Formulation 8 yields the optimal matrix $\mathbf{U}_i$, then the optimal matrix $\bar{\mathbf{F}}_i$ is also determined. As we see, when we employ optimal power loading by Formulation 8, there are totally $N + M$ variables which are τ_i and $u_{i,j}$ ($j = 1, 2, ...r_i$, $i = 1, 2, ...N$). So solving it directly may be a computation burden particularly when the number of users N or the number of subcarriers M is large. In this section, we try to find an alternative, efficient method to conduct optimal power loading by the following two subsections.

4.2.3.1 Power Loading for Single User

Upon allocating the subcarriers, if we fix the total power of each user, then the power loading problem of each user will be changed to a single user problem. Therefore, we can use the single user power distribution strategy [18] here by fixing the total power of each user first.

Rewriting the lower bound of the BER for the ith user from (4.27), we have

$$P_{eLB}^{(i)} = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{r_i}{\sum_{j=1}^{r_i}\frac{\sigma^2}{\sigma^2+|h_{i,j}|^2 u_{i,j}}} - 1\right)}\right). \tag{4.29}$$

Since the complementary error function is monotonically decreasing, minimizing the lower bound of the BER for User i in equation (4.29) is equivalent to minimizing

$\sum_{j=1}^{r_i} \frac{\sigma^2}{\sigma^2+|h_{i,j}|^2 u_{i,j}}$. Letting P_i denote the total power for User i, we then need to solve the following formulation to obtain optimal power loading for each user.

Formulation 9 *Find optimal power loading for single user such that (4.26) is minimized, subject to the power constraint; i.e.,*

$$\min_{u_{i,j}} \quad \sum_{j=1}^{r_i} \frac{\sigma^2}{\sigma^2 + |h_{i,j}|^2 u_{i,j}} \tag{4.30}$$

subject to the constraints

$$\sum_{j=1}^{r_i} u_{i,j} \leq P_i,$$
$$u_{i,j} \geq 0.$$

Employing the result in [18], we obtain optimal power loading for User i from the above problem such that

$$u_{i,j} = \left(\frac{P_i + \sigma^2 \Sigma_{m=1}^{\bar{r}_i} |h_{i,m}|^{-2}}{\Sigma_{m=1}^{\bar{r}_i} |h_{i,m}|^{-1}} |h_{i,j}|^{-1} - \sigma^2 |h_{i,j}|^{-2} \right)^+, \tag{4.31}$$

where $(x)^+ \triangleq \max(x, 0)$. Here we rearrange the channel of each user such that the channel gains, $|h_{i,j}|^2$, are in a descending order. In equation (4.31), $\forall i \in [1, N]$, $\bar{r}_i \leq r_i$; $u_{i,j} > 0$ for all $j \in [1, \bar{r}_i]$, and $u_{i,j} = 0$ for all $j \in [\bar{r}_i + 1, r_i]$.

For simplicity, in this subsection, we consider the power allocated to each subcarrier to be positive, that is, $u_{i,j} > 0$. To satisfy it, we must add some constraints to the power P_i. In the following, the condition under which all the $u_{i,j}$ are positive is analyzed [18]. From (4.31), for all $j \in [1, r_i]$, we see $u_{i,j}$ is positive when

$$\frac{P_i + \sigma^2 \Sigma_{m=1}^{r_i} |h_{i,m}|^{-2}}{\Sigma_{m=1}^{r_i} |h_{i,m}|^{-1}} |h_{i,j}|^{-1} - \sigma^2 |h_{i,j}|^{-2} > 0, \tag{4.32}$$

which results in

$$P_i > \sigma^2 (|h_{i,j}|^{-1} \sum_{m=1}^{r_i} |h_{i,m}|^{-1} - \sum_{m=1}^{r_i} |h_{i,m}|^{-2}). \tag{4.33}$$

Let $|h_{i,min}|$ denote the minimum value among $|h_{i,j}|$. Then from inequality (4.33), we know that if

$$P_i > \sigma^2(|h_{i,min}|^{-1}\sum_{m=1}^{r_i}|h_{i,m}|^{-1} - \sum_{m=1}^{r_i}|h_{i,m}|^{-2}) \tag{4.34}$$

is satisfied, each value $u_{i,j}$ will be positive. This will be satisfied from moderate to high SNR region. (In case $|h_{i,min}|$ is very small, from (4.34), we know that P_i has to be very large to satisfy $u_{i,j} > 0$, which is not preferred in practice. Hence, in this case, we still resort to the power loading method discussed earlier in Section 4.2.2.)

4.2.3.2 Power Allocation among Users

In this subsection, we consider an optimal power allocation among users for moderate to high SNR region where each $u_{i,j}$ is positive. Once the power P_i is known, optimal power loading for each user can be determined by equation (4.31). Substituting equation (4.31) into (4.26), we see

$$\begin{aligned} & \sum_{j=1}^{r_i}\frac{\sigma^2}{\sigma^2+|h_{i,j}|^2u_{i,j}} \\ = & \sum_{j=1}^{r_i}\frac{\sigma^2\Sigma_{m=1}^{r_i}|h_{i,m}|^{-1}}{(P_i+\sigma^2\Sigma_{m=1}^{r_i}|h_{i,m}|^{-2})|h_{i,j}|} \\ = & \frac{\sigma^2(\mathrm{tr}(\Lambda_i^{-\frac{1}{2}}))^2}{P_i+\sigma^2\mathrm{tr}(\Lambda_i^{-1})}, \end{aligned} \tag{4.35}$$

where the jth diagonal element of matrix Λ_i is $|h_{i,j}|^2$.

From equation (4.35), the lower bound equation for User i (4.29) can be expressed as

$$P_{eLB}^{(i)} = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\alpha_i P_i + \beta_i}\right), \tag{4.36}$$

where

$$\alpha_i = \frac{1}{2}\frac{r_i}{\sigma^2(\mathrm{tr}(\mathbf{\Lambda}_i^{-\frac{1}{2}}))^2}, \tag{4.37}$$

$$\beta_i = \frac{1}{2}\frac{r_i \mathrm{tr}(\mathbf{\Lambda}_i^{-1})}{(\mathrm{tr}(\mathbf{\Lambda}_i^{-\frac{1}{2}}))^2} - \frac{1}{2}. \tag{4.38}$$

In this case, the lower bound function of the average BER (4.27) can be rewritten as

$$P_{eLB} = \frac{1}{2N}\sum_{i=1}^{N} \mathrm{erfc}\left(\sqrt{\alpha_i P_i + \beta_i}\right). \tag{4.39}$$

Now, we perform optimal power allocation among users by solving the following optimization problem:

Formulation 10 *Find optimal power allocation among users such that (4.39) is minimized, subject to the power constraint; i.e.,*

$$\min_{P_i} \quad \frac{1}{2N}\sum_{i=1}^{N} \mathrm{erfc}\left(\sqrt{\alpha_i P_i + \beta_i}\right) \tag{4.40}$$

subject to the following constraints:

$$\sum_{i=1}^{N} P_i \leq P,$$
$$P_i \geq 0.$$

Formulation 10 is convex with respect to P_i. We show its convexity by the following procedure. The convexity of the problem is only dependent on the objective function since the linear constraints $\sum_{i=1}^{N} P_i \leq P$ and $P_i \geq 0$ are convex. Thus, the Hessian matrix of the objective function should be found. Differentiating equation (4.40) with respect to P_i, we have,

$$\frac{\partial^2 P_{eLB}}{\partial P_i^2} = \frac{\alpha_i^2}{2N\sqrt{\pi}}\exp\left(-(\alpha_i P_i + \beta_i)\right)(\alpha_i P_i + \beta_i)^{-\frac{1}{2}}\left(1 + \frac{1}{2}(\alpha_i P_i + \beta_i)^{-1}\right), \tag{4.41}$$

$$\frac{\partial^2 P_{eLB}}{\partial P_i \partial P_j} = 0, \quad i \neq j. \tag{4.42}$$

From the condition

$$\alpha_i P_i + \beta_i \geq 0, \tag{4.43}$$

we see that the Hessian matrix of P_{eLB} is a diagonal matrix with positive diagonal elements, i.e., it is positive definite. Thus the objective function is a convex function. Consequently, this problem is indeed a convex optimization problem.

In order to solve Formulation 10, we also resort to interior point methods [17]. In Formulation 10, we only need to solve an N-variable problem. Compared to $N + M$ variables in Formulation 8, the complexity is greatly reduced especially when the number of subcarriers, M, is large.

4.2.3.3 Algorithms to Solve the Problems

In this subsection, we will briefly discuss the algorithms for solving Formulation 10. Here we will only focus on Formulation 10 instead of Formulation 8 since the former one is much simpler.

Since our problem has the inequality constraints, the popular optimization algorithms used for unconstrained convex problems, such as gradient method, steepest decent method, and Newton's method, cannot be used directly [17]. For our problem, we use an interior point method [17].

Define the logarithmic barrier functions as

$$\phi_1(P_i) = \begin{cases} -\sum_{i=1}^{N} \log(P_i) & P_i > 0 \\ +\infty & \text{otherwise} \end{cases}, \tag{4.44}$$

$$\phi_2(P_i) = \begin{cases} -\log(P - \sum_{i=1}^{N} P_i) & P > \sum_{i=1}^{N} P_i \\ +\infty & \text{otherwise} \end{cases}. \tag{4.45}$$

Now we consider a new unconstrained optimization problem

$$\min \quad tP_{eLB} + \phi_1(P_i) + \phi_2(P_i). \tag{4.46}$$

From problem (4.46), when $t \longrightarrow \infty$, $\phi_1(P_i)$ and $\phi_2(P_i)$ are very small compared to tP_{eLB}. Then, minimizing $tP_{eLB} + \phi_1(P_i) + \phi_2(P_i)$ can achieve almost the same result as minimizing P_{eLB}. Therefore, the solution of Formulation 10 can be approximated by that of (4.46) when t is sufficiently large. Two popular methods [17] can be chosen to approximate Formulation 10 by (4.46). One is the Unconstrained Minimization Method (UMM). In this method, a large t satisfying the desired accuracy is chosen. By solving one unconstrained minimization problem (4.46), we obtain the approximate solution for the original constrained problem. The disadvantage of this method is that it may converge slowly. In our simulation, we use another method known as the Sequential Unconstrained Minimization Technique (SUMT) or the Barrier Method.

- Barrier Method: given an initial $t = t_o$ and a feasible P_i, repeat

  1. Compute $P_i^*(t)$ from equation (4.46), starting at P_i;
  2. Update P_i by $P_i^*(t)$;
  3. Increase t.

  Repeat until the stopping criterion is satisfied.

The barrier method can converge rather fast if an appropriate scheme to update t is chosen. From the above, we see problem (4.46) should be solved in step 1 at each iteration. It is an unconstrained convex optimization problem. Many descent methods can be used to solve problem (4.46) [17]. Here we choose Newton's method. For a fixed t, let $f(P_i) = tP_{eLB} + \phi_1(P_i) + \phi_2(P_i)$. The algorithm is described in following steps: For the starting feasible point P_i,

1. Compute Newton direction $v = -[\nabla^2 f(P_i)]^{-1} \nabla f(P_i)$;
2. Line search: Choose a step size $t' > 0$;
3. Update P_i by $P_i + t'v$.

Repeat until the stopping criterion is satisfied.

Upon achieving the convergence of Newton's method and the barrier method, we can obtain the optimal power P_i. Once P_i has been derived, the power loading for each user can be determined by equation (4.31). Then, the optimal matrix $\bar{\mathbf{F}}_i$ can be obtained.

4.2.4 Achieving the BER Lower Bound

As indicated in Section 4.1.3, after we obtain the matrix $\bar{\mathbf{F}}_i$ by minimizing the lower bound of the BER, we should perform the second stage that shows how to achieve this BER lower bound. From Section 4.1.2, P_{eLB} can be achieved if and only if all the diagonal entries of matrix $(\mathbf{I} + \mathbf{A}_i)^{-1}$ are equal $\forall i \in [1, N]$, where $\mathbf{A}_i = \mathbf{F}_i^H \mathbf{H}_i^H \mathbf{C}_i^{-1} \mathbf{H}_i \mathbf{F}_i$. Notice that the structure of the precoder to be designed is $\mathbf{F}_i = \bar{\mathbf{F}}_i \mathbf{V}_i$ (see Section 2.4). So we have the freedom to choose a unitary matrix $\mathbf{V}_i$. According to equation (4.23), we have that $(\mathbf{I} + \mathbf{A}_i)^{-1} = \mathbf{V}_i^H (\mathbf{I} + \bar{\mathbf{A}}_i)^{-1} \mathbf{V}_i$, where $\bar{\mathbf{A}}_i = \bar{\mathbf{F}}_i^H \mathbf{H}_i^H \mathbf{C}_i^{-1} \mathbf{H}_i \bar{\mathbf{F}}_i$. If we let $\mathbf{\Gamma}_i = (\mathbf{I} + \bar{\mathbf{A}}_i)^{-1}$, then the key of the problem is how to choose $\mathbf{V}_i$ such that the diagonal elements of matrix $\mathbf{V}_i^H \mathbf{\Gamma}_i \mathbf{V}_i$ are identical. Since

$$[\mathbf{V}_i^H \mathbf{\Gamma}_i \mathbf{V}_i]_{mm} = \sum_{j=1}^{r_i} |[\mathbf{V}_i]_{jm}|^2 \gamma_{i,j}, \tag{4.47}$$

where $[\mathbf{V}_i]_{jm}$ represents the (j, m)th element of matrix $\mathbf{V}_i$, and $\gamma_{i,j}$ represents the jth diagonal element of matrix $\mathbf{\Gamma}_i$. In light of (4.47), $[\mathbf{V}_i^H \mathbf{\Gamma}_i \mathbf{V}_i]_{mm}$ can be identical if $\mathbf{V}_i$ is chosen to be a normalized DFT matrix such that $|[\mathbf{V}_i]_{jm}|^2 = \frac{1}{r_i}$. (Actually, matrix $\mathbf{V}_i$ can be chosen to be any unitary matrix which has the property $|[\mathbf{V}_i]_{jm}|^2 = \frac{1}{r_i}$ for all $m, j \in [1, r_i]$). Now, the diagonal elements of $\mathbf{V}_i^H \mathbf{\Gamma}_i \mathbf{V}_i$ reduce to $\mathrm{tr}(\mathbf{\Gamma}_i)/r_i$, and as a result, the lower bound is achieved [18] [24].

So far, we have designed the optimal precoder by choosing the optimal matrix $\bar{\mathbf{F}}_i$ and the unitary matrix $\mathbf{V}_i$. We should mention that, although we choose the MMSE subcarrier allocation strategy in this chapter, the power loading problem of the MBER design is different from that of the MMSE design. Moreover, unlike the arbitrary unitary matrix in the MMSE precoder, the unitary matrix can be chosen to be a DFT matrix to achieve the lower bound of the BER in the MBER precoder. The performance comparison of these two designs will be shown in Section 4.3.

4.3 Simulations

In this section, to show the performance of our MBER design, we compare it with several other design methods, including the MMSE design which is obtained in Chapter 3 by simulations. Here we carry out simulations for a two user downlink OFDM system. From the simulation results, we see that the MBER precoder fitted with the MMSE equalizer has better performance.

In Example 1 and Example 2, we compare our MBER design with two methods. One is the equal power allocation method with unitary matrix chosen as a DFT matrix. In this design, we also choose the MMSE subcarrier allocation strategy. Another is the optimal power allocation proposed in this chapter with unitary matrix chosen to be identity matrix.

Example 1: In this example, the parameters are the same as that of Example 2 in Section 3.3. The performance comparison is depicted in Figure 4.1. From the simulation results, we can tell that the MBER precoder has better performance than the other two designs.

Example 2: In Example 2, we show the average performance comparison among these three design methods. The parameters used in this example is chosen as those used in Example 3 of Section 3.3. The simulation results are shown in Figure 4.2.

In the following example, we show the average performance comparison between our two designs: MMSE design in Chapter 3 and MBER design.

Example 3: In this example, likewise, the parameters are also chosen as those

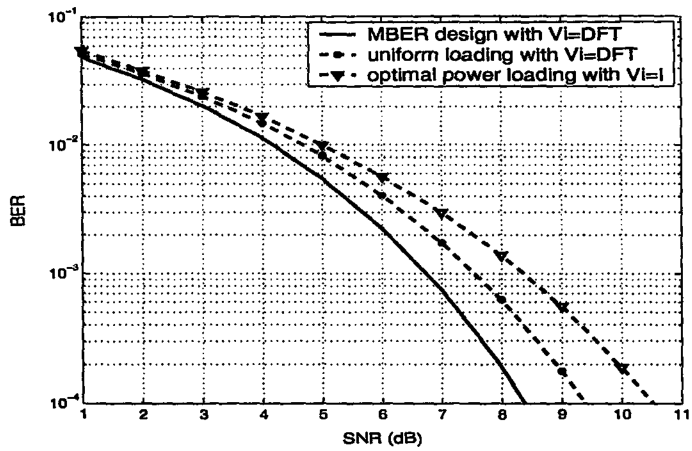

Figure 4.1: Performance comparison of MBER and the other two designs.

used in Example 3 of Section 3.3. For MMSE precoder, the unitary matrices $\mathbf{V}_i$ are chosen as a DFT matrix and an identity matrix, respectively. Figure 4.3 shows the comparison result. From the simulation, we see that the precoder based on the MBER criterion is superior to that based on the MMSE criterion; the superiority is especially pronounced in the moderate to high SNR range. For the MMSE design method, we should mention that although its performance is not as good as the MBER design, it has its own advantage: it is much easier to be designed since the associated power loading problem is much simpler than that of the MBER design.

4.4 Summary

In this chapter, from the viewpoint of minimizing arithmetic average BER, we performed the optimal power loading and designed the precoder which employs an MMSE equalizer.

This problem can be solved by a two-stage optimization processing. In the first

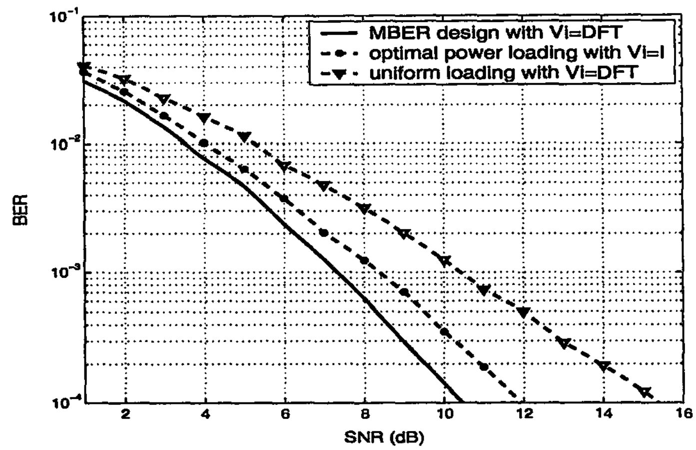

Figure 4.2: Average performance comparison of MBER and the other two design methods.

stage, we derive the lower bound of the average BER and minimize this lower bound. After allocating the subcarriers according to the strategy proposed in Chapter 3, the optimal power loading problem can then be solved by interior point methods. Furthermore, an alternative, efficient power loading method was proposed to reduce the computation complexity. In the second stage, we choose a unitary matrix to achieve this lower bound and derive the corresponding optimal precoder. In this chapter, to solve our problems, we also briefly discussed one of interior point methods, the barrier method. The simulation results showed that the MBER design has better performance than several other designs, including the MMSE precoder design which we have derived in Chapter 3.

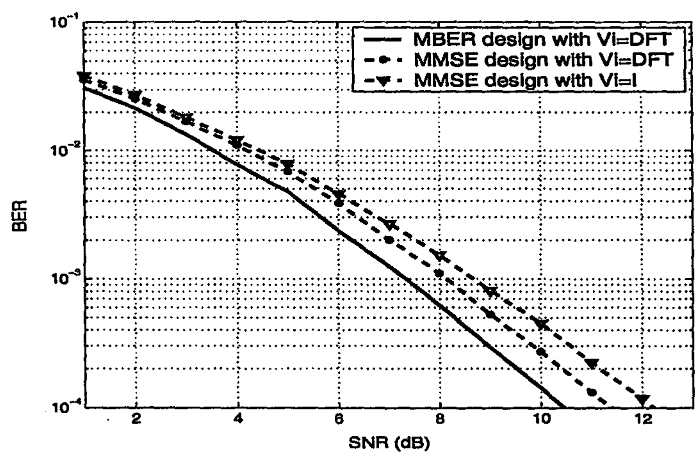

Figure 4.3: Average performance comparison between MBER and MMSE designs.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

In this thesis, we considered a multiuser downlink OFDM system for which the CSI is known to both the transmitter and the receiver. Here, we only focused on the linear MMSE equalization since it requires low computation cost.

For this system, we have designed an optimal precoder which minimizes the MSE subject to a total power constraint when the MMSE equalizer is employed. We showed that, this problem can be solved by first minimizing the lower bound of the MSE to obtain the optimal power and then seeking an optimal precoder to achieve this minimized lower bound. Specifically, according to the MMSE subcarrier allocation strategy, each subcarrier can be used by only one user.

Moreover, based on the MMSE subcarrier allocation strategy, we have performed an optimal power loading and designed the corresponding optimal precoder that minimizes the average BER, subject to the same power constraint. This problem was solved by a two-stage optimization process. In the first stage, we derive the lower bound of the average BER and minimize this lower bound. We showed that the optimal power loading problem is convex, which can be efficiently solved by interior point methods. In order to reduce computation complexity, we proposed an alternative, efficient power loading method. In the second stage, we choose a unitary matrix to achieve this lower bound and derive the corresponding optimal precoder. We concluded, in light of simulation results, that the performance of the MBER precoder fitted with the MMSE equalizer design is superior to other design methods, including the MMSE precoder design.

In this thesis, for mathematical tractability, we chose the special structure of the precoder. This precoder structure was proved to be an optimal structure that minimizes the total MSE for uplink multiple access OFDM systems [26]. However, for our multiuser downlink OFDM system, whether this structure results in the loss of globally optimal solution or not needs to be further investigated. In addition, the optimal BER precoder design based on the MMSE subcarrier allocation strategy may also lose the global optimal solution and needs to be improved.

5.2 Future Work

Based on the studies in this thesis, some interesting research issues open up:

- In this thesis, we proposed an optimal subcarrier allocation method according to the largest subchannel gain. However, since the method proposed here is designed with a total power constraint, in the special case when one user does not have a highest subchannel gain, then this user may not be assigned any subcarrier. This will cause a low transmission data rate for this user and will be a drawback for the scheme. Finding a tradeoff between the transmission data rate of this user and the performance in this special case is the subject of future research.

- In practice, the channels information cannot be estimated precisely. The optimal MBER and MMSE precoders fitted with partial channel knowledge are worth further investigating.

Appendix A

Error Covariance Matrix

Rewriting the error covariance matrix (2.25) here, we see

$$E(\mathbf{e}_i\mathbf{e}_i^H) = \mathbf{G}_i\mathbf{W}_i^{-1}\mathbf{G}_i^H - \mathbf{G}_i\mathbf{H}_i\mathbf{F}_i - (\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i)^H + \mathbf{I}. \tag{A.1}$$

Knowing that the linear MMSE equalizer is $\mathbf{G}_i = \mathbf{F}_i^H\mathbf{H}_i^H\mathbf{W}_i$ and $\mathbf{W}_i = \mathbf{W}_i^H$, we can obtain:

$$\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i = (\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i)^H, \tag{A.2}$$

and

$$\mathbf{G}_i\mathbf{W}_i^{-1}\mathbf{G}_i^H = \mathbf{F}_i^H\mathbf{H}_i^H\mathbf{W}_i\mathbf{H}_i\mathbf{F}_i = \mathbf{G}_i\mathbf{H}_i\mathbf{F}_i. \tag{A.3}$$

So

$$E(\mathbf{e}_i\mathbf{e}_i^H) = -\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i + \mathbf{I}, \tag{A.4}$$

which implies that the MSE for User i is $\mathrm{tr}(-\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i + \mathbf{I})$. (The corresponding MSE equation for single user case can be seen in [3] [5] [18].) This completes the proof of equation (3.3).

The following is the proof of equation (4.10).

Substituting the formulation of $\mathbf{G}_i$ from equation (2.29) into the error covariance matrix (A.4), we have

$$\begin{aligned} E(\mathbf{e}_i\mathbf{e}_i^H) &= -\mathbf{F}_i^H\mathbf{H}_i^H\mathbf{W}_i\mathbf{H}_i\mathbf{F}_i + \mathbf{I} \\ &= \mathbf{I} - \mathbf{F}_i^H\mathbf{H}_i^H(\mathbf{H}_i\mathbf{F}_i\mathbf{F}_i^H\mathbf{H}_i^H + \sum_{k=1,k\neq i}^{N}\mathbf{H}_i\mathbf{F}_k\mathbf{F}_k^H\mathbf{H}_i^H + \sigma^2\mathbf{I})^{-1}\mathbf{H}_i\mathbf{F}_i. \end{aligned} \tag{A.5}$$

If we define $\mathbf{C}_i = \sum_{k=1,k\neq i}^{N}\mathbf{H}_i\mathbf{F}_k\mathbf{F}_k^H\mathbf{H}_i^H + \sigma^2\mathbf{I}$, then,

$$E(\mathbf{e}_i\mathbf{e}_i^H) = \mathbf{I} - \mathbf{F}_i^H\mathbf{H}_i^H(\mathbf{H}_i\mathbf{F}_i\mathbf{F}_i^H\mathbf{H}_i^H + \mathbf{C}_i)^{-1}\mathbf{H}_i\mathbf{F}_i. \tag{A.6}$$

According to Matrix Inversion Lemma, (A.6) is equivalent to

$$E(\mathbf{e}_i\mathbf{e}_i^H) = (\mathbf{I} + \mathbf{F}_i^H\mathbf{H}_i^H\mathbf{C}_i^{-1}\mathbf{H}_i\mathbf{F}_i)^{-1}. \tag{A.7}$$

Denoting $\mathbf{F}_i^H\mathbf{H}_i^H\mathbf{C}_i^{-1}\mathbf{H}_i\mathbf{F}_i$ as matrix $\mathbf{A}_i$, then, we obtain equation (4.10) as

$$E(\mathbf{e}_i\mathbf{e}_i^H) = (\mathbf{I} + \mathbf{A}_i)^{-1}. \tag{A.8}$$

Appendix B

Optimal Variables p_j that Minimize the MSE Lower Bound

Rewrite the problem in Formulation 4 here, we have

$$\begin{aligned} \min_{p_j} \quad & \sum_{j=1}^{M} \frac{\sigma^2}{\sigma^2 + |h_{i_j^*,j}|^2 p_j} \\ \text{s.t.} \quad & \sum_{j=1}^{M} p_j \leq P, \end{aligned} \tag{B.1}$$

where the solution is valid if and only if $p_j \geq 0$. The Lagrangian function of problem (B.1) is given by

$$L(p_j, \mu) = \sum_{j=1}^{M} \frac{\sigma^2}{\sigma^2 + |h_{i_j^*,j}|^2 p_j} + \mu(\sum_{j=1}^{M} p_j - P), \tag{B.2}$$

where $\mu > 0$. Differentiating it with respect to p_j and setting the derivative to 0 yields

$$\frac{\partial L}{\partial p_j} = -\frac{\sigma^2 |h_{i_j^*,j}|^2}{(\sigma^2 + |h_{i_j^*,j}|^2 p_j)^2} + \mu = 0, \;\; \forall j \in [1, M]. \tag{B.3}$$

From (B.3), we have

$$p_j = \frac{\sigma}{|h_{i_j^*,j}|\sqrt{\mu}} - \frac{\sigma^2}{|h_{i_j^*,j}|^2}. \tag{B.4}$$

Since equality $\sum_{j=1}^{M} p_j = P$ must hold at the optimality when we minimize the lower bound of the MSE, by substituting (B.4) into $\sum_{j=1}^{M} p_j = P$, we obtain

$$\frac{\sigma}{\sqrt{\mu}} = \frac{P + \sigma^2 \sum_{j=1}^{M} |h_{i_j^*,j}|^{-2}}{\sum_{j=1}^{M} |h_{i_j^*,j}|^{-1}}. \tag{B.5}$$

From (B.5) and (B.4), the optimal p_j can be obtained to be

$$p_j^* = \frac{P + \sigma^2 \sum_{m=1}^{M} |h_{i_m^*,m}|^{-2}}{\sum_{m=1}^{M} |h_{i_m^*,m}|^{-1}} |h_{i_j^*,j}|^{-1} - \sigma^2 |h_{i_j^*,j}|^{-2}. \tag{B.6}$$

We have mentioned that p_j has to be non-negative. When p_j^* given by (B.6) is negative, the corresponding solution is found to be

$$p_j^* = \left(\frac{P + \sigma^2 \sum_{m=1}^{\bar{M}} |h_{i_m^*,m}|^{-2}}{\sum_{m=1}^{\bar{M}} |h_{i_m^*,m}|^{-1}} |h_{i_j^*,j}|^{-1} - \sigma^2 |h_{i_j^*,j}|^{-2} \right)^+, \tag{B.7}$$

where $(x)^+ \triangleq \max(x, 0)$. Here, $\bar{M} \leq M$; $p_j^* > 0$ for all $j \in [1, \bar{M}]$, and $p_j^* = 0$ for all $j \in [\bar{M}+1, M]$. To find the integer $\bar{M}$, we set $\bar{M} = M$ initially, then iteratively decrease $\bar{M}$ by 1 until all p_j^* are non-negative [18].

Appendix C

SINR of the Real and Imaginary Parts of One Symbol

In this section, the SINR of the real and imaginary parts of one estimated symbol is derived. Recalling from (2.19), the equalized signal can be expressed as

$$\hat{\mathbf{s}}_i = \mathbf{G}_i\mathbf{H}_i \sum_{k=1}^{N} \mathbf{F}_k\mathbf{s}_k + \mathbf{G}_i\mathbf{n}_i. \tag{C.1}$$

It results in that the jth symbol in vector $\hat{\mathbf{s}}_i$ can be written as

$$[\hat{\mathbf{s}}_i]_j = [\mathbf{G}_i]^j \cdot (\mathbf{H}_i \sum_{k=1}^{N} \mathbf{F}_k\mathbf{s}_k + \mathbf{n}_i), \tag{C.2}$$

where $[\mathbf{G}_i]^j$ denotes the jth row vector of equalizer $\mathbf{G}_i$. Since $(\mathbf{H}_i\mathbf{F}_k)\mathbf{s}_k = \sum_{l=1}^{M} [\mathbf{H}_i\mathbf{F}_k]_l[\mathbf{s}_k]_l$, where $[\mathbf{H}_i\mathbf{F}_k]_l$ denotes the lth column vector of matrix $\mathbf{H}_i\mathbf{F}_k$, and $[\mathbf{s}_k]_l$ denotes the lth element of vector $\mathbf{s}_k$; then, equation (C.2) becomes

$$[\hat{\mathbf{s}}_i]_j = [\mathbf{G}_i]^j \cdot (\sum_{l=1}^{M} [\mathbf{H}_i\mathbf{F}_i]_l[\mathbf{s}_i]_l) + [\mathbf{G}_i]^j (\sum_{k=1,k\neq i}^{N} \sum_{l=1}^{M} [\mathbf{H}_i\mathbf{F}_k]_l[\mathbf{s}_k]_l) + [\mathbf{G}_i]^j\mathbf{n}_i. \tag{C.3}$$

If we separate the real and imaginary parts of symbol $[\hat{\mathbf{s}}_i]_j$, then we have

$$\begin{aligned}
\mathrm{Re}([\hat{\mathbf{s}}_i]_j) &= \sum_{l=1}^{M}\mathrm{Re}([\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_i]_l))\mathrm{Re}([\mathbf{s}_i]_l) - \sum_{l=1}^{M}\mathrm{Im}([\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_i]_l))\mathrm{Im}([\mathbf{s}_i]_l) \\
&+ \sum_{k=1,k\neq i}^{N}\sum_{l=1}^{M}\mathrm{Re}([\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_k]_l))\mathrm{Re}([\mathbf{s}_k]_l) \\
&- \sum_{k=1,k\neq i}^{N}\sum_{l=1}^{M}\mathrm{Im}([\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_k]_l))\mathrm{Im}([\mathbf{s}_k]_l) \\
&+\mathrm{Re}([\mathbf{G}_i]^j)\mathrm{Re}(\mathbf{n}_i) - \mathrm{Im}([\mathbf{G}_i]^j)\mathrm{Im}(\mathbf{n}_i),
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
\mathrm{Im}([\hat{\mathbf{s}}_i]_j) &= \sum_{l=1}^{M}\mathrm{Re}([\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_i]_l))\mathrm{Im}([\mathbf{s}_i]_l) + \sum_{l=1}^{M}\mathrm{Im}([\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_i]_l))\mathrm{Re}([\mathbf{s}_i]_l) \\
&+ \sum_{k=1,k\neq i}^{N}\sum_{l=1}^{M}\mathrm{Re}([\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_k]_l))\mathrm{Im}([\mathbf{s}_k]_l) \\
&+ \sum_{k=1,k\neq i}^{N}\sum_{l=1}^{M}\mathrm{Im}([\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_k]_l))\mathrm{Re}([\mathbf{s}_k]_l) \\
&+\mathrm{Re}([\mathbf{G}_i]^j)\mathrm{Im}(\mathbf{n}_i) + \mathrm{Im}([\mathbf{G}_i]^j)\mathrm{Re}(\mathbf{n}_i).
\end{aligned} \tag{C.5}$$

First, we compute the SINR of the real part of the jth symbol for User i; the imaginary part essentially follows the same process. We consider the power of the signal now. Since Assumption A6 implies $E(\mathrm{Re}(\mathbf{s}_i)\mathrm{Re}(\mathbf{s}_i)^T) = E(\mathrm{Im}(\mathbf{s}_i)\mathrm{Im}(\mathbf{s}_i)^T) = \frac{1}{2}\mathbf{I}$, we have the power of the signal

$$P_{\mathrm{S}} = \frac{1}{2}\left(\mathrm{Re}([\mathbf{G}_i]^j \cdot [\mathbf{H}_i\mathbf{F}_i]_j)\right)^2 + \frac{1}{2}\left(\mathrm{Im}([\mathbf{G}_i]^j \cdot [\mathbf{H}_i\mathbf{F}_i]_j)\right)^2. \tag{C.6}$$

It is known that matrix $\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i$ is a Hermitian symmetric matrix. We can conclude that $[\mathbf{G}_i]^j \cdot [\mathbf{H}_i\mathbf{F}_i]_j = [\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}$ is a real value. Therefore, we have

$$P_{\mathrm{S}} = \frac{1}{2}\left(\mathrm{Re}([\mathbf{G}_i]^j \cdot [\mathbf{H}_i\mathbf{F}_i]_j)\right)^2 = \frac{1}{2}[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}^2. \tag{C.7}$$

Now we compute the power of the inference plus noise. From equation (C.4), the power of the inference plus noise is

$$\begin{aligned}
P_{\mathrm{I+N}} &= \frac{1}{2}\sum_{l=1,l\neq j}^{M}\left(\mathrm{Re}([\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_i]_l)\right)^2+\frac{1}{2}\sum_{l=1,l\neq j}^{M}\left(\mathrm{Im}([\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_i]_l)\right)^2\\
&+\frac{1}{2}\sum_{k=1,k\neq i}^{N}\sum_{l=1}^{M}\left(\mathrm{Re}([\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_k]_l)\right)^2+\frac{1}{2}\sum_{k=1,k\neq i}^{N}\sum_{l=1}^{M}\left(\mathrm{Im}([\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_k]_l)\right)^2\\
&+\frac{1}{2}\sigma^2\mathrm{Re}([\mathbf{G}_i]^j)\mathrm{Re}([\mathbf{G}_i]^j)^T+\frac{1}{2}\sigma^2\mathrm{Im}([\mathbf{G}_i]^j)\mathrm{Im}([\mathbf{G}_i]^j)^T\\
&= \frac{1}{2}\sum_{l=1,l\neq j}^{M}\left([\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_i]_l([\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_i]_l)^*\right)\\
&+\frac{1}{2}\sum_{k=1,k\neq i}^{N}\sum_{l=1}^{M}\left([\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_k]_l([\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_k]_l)^*\right)+\frac{1}{2}\sigma^2[\mathbf{G}_i]^j([\mathbf{G}_i]^j)^H, \qquad \text{(C.8)}
\end{aligned}$$

where * denotes conjugate. Using the fact that

$$[\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_i]_l([\mathbf{G}_i]^j[\mathbf{H}_i\mathbf{F}_i]_l)^* = [\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_i]_l[\mathbf{H}_i\mathbf{F}_i]_l^H)([\mathbf{G}_i]^j)^H, \qquad \text{(C.9)}$$

equation (C.8) can be written to the following expression

$$\begin{aligned}
P_{\mathrm{I+N}} &= \frac{1}{2}[\mathbf{G}_i]^j\sum_{l=1,l\neq j}^{M}([\mathbf{H}_i\mathbf{F}_i]_l[\mathbf{H}_i\mathbf{F}_i]_l^H)([\mathbf{G}_i]^j)^H\\
&+\frac{1}{2}[\mathbf{G}_i]^j\sum_{k=1,k\neq i}^{N}\sum_{l=1}^{M}([\mathbf{H}_i\mathbf{F}_k]_l[\mathbf{H}_i\mathbf{F}_k]_l^H)([\mathbf{G}_i]^j)^H\\
&+\frac{1}{2}\sigma^2[\mathbf{G}_i]^j([\mathbf{G}_i]^j)^H. \qquad \text{(C.10)}
\end{aligned}$$

To simplify the above equation, we do the following process. Since $\mathbf{H}_i\mathbf{F}_i(\mathbf{H}_i\mathbf{F}_i)^H = \sum_{l=1}^{M}[\mathbf{H}_i\mathbf{F}_i]_l([\mathbf{H}_i\mathbf{F}_i]_l)^H$, equation (C.10) is equivalent to

$$\begin{aligned}
P_{\mathrm{I+N}} &= \frac{1}{2}[\mathbf{G}_i]^j(\mathbf{H}_i\mathbf{F}_i\mathbf{F}_i^H\mathbf{H}_i^H-[\mathbf{H}_i\mathbf{F}_i]_j[\mathbf{H}_i\mathbf{F}_i]_j^H+\sum_{k=1,k\neq i}^{N}\mathbf{H}_i\mathbf{F}_k\mathbf{F}_k^H\mathbf{H}_i^H+\sigma^2\mathbf{I})([\mathbf{G}_i]^j)^H\\
&= \frac{1}{2}[\mathbf{G}_i]^j(\mathbf{W}_i^{-1}-[\mathbf{H}_i\mathbf{F}_i]_j[\mathbf{H}_i\mathbf{F}_i]_j^H)([\mathbf{G}_i]^j)^H, \qquad \text{(C.11)}
\end{aligned}$$

where

$$\mathbf{W}_i = (\sum_{k=1}^{N} \mathbf{H}_i\mathbf{F}_k\mathbf{F}_k^H\mathbf{H}_i^H + \sigma^2\mathbf{I})^{-1}.$$

From Appendix A, we know

$$\mathbf{G}_i\mathbf{W}_i^{-1}\mathbf{G}_i^H = \mathbf{G}_i\mathbf{H}_i\mathbf{F}_i, \tag{C.12}$$

which implies

$$[\mathbf{G}_i]^j\mathbf{W}_i^{-1}([\mathbf{G}_i]^j)^H = [\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}. \tag{C.13}$$

According to (C.13), equation (C.11) can be simplified to the following expression

$$\begin{aligned} P_{\mathrm{I+N}} &= \frac{1}{2}\left([\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj} - [\mathbf{G}_i]^j([\mathbf{H}_i\mathbf{F}_i]_j[\mathbf{H}_i\mathbf{F}_i]_j^H)([\mathbf{G}_i]^j)^H\right) \\ &= \frac{1}{2}\left([\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj} - [\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}^2\right). \end{aligned} \tag{C.14}$$

From (C.7) and (C.14), we obtain that for User i, the SINR of the real part of symbol j is

$$\mathrm{SINR}_{\mathrm{Re}([\mathbf{s}_i]_j)} = \frac{[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}{1 - [\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}. \tag{C.15}$$

In the same way, we can calculate the SINR for the corresponding imaginary part of one symbol. First, we try to obtain the signal power which is equivalent to $P_{\mathrm{S}} = \frac{1}{2}[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}^2$, then, the power of inference plus noise equals to $P_{\mathrm{I+N}} = \frac{1}{2}([\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj} - [\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}^2)$. So

$$\mathrm{SINR}_{\mathrm{Im}([\mathbf{s}_i]_j)} = \frac{[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}{1 - [\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}. \tag{C.16}$$

Now, let us give a conclusion: the SINR for the real and imaginary parts of the jth symbol for the ith user is

$$\mathrm{SINR}_{\mathrm{Re}([\mathbf{s}_i]_j)} = \mathrm{SINR}_{\mathrm{Im}([\mathbf{s}_i]_j)} = \frac{[\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}{1 - [\mathbf{G}_i\mathbf{H}_i\mathbf{F}_i]_{jj}}. \tag{C.17}$$

(The equations of SINR for single user case and some other systems have been obtained in [5] [18] [21] [24].)

Appendix D

Derivatives of $f(x) = \operatorname{erfc}\left(\sqrt{\frac{1}{2}\left(\frac{1}{x}-1\right)}\right)$

The first and second derivatives of $f(x)$ with respect to x are found as follows

$$
\begin{aligned}
\frac{d}{dx}f(x) &= \frac{2}{\sqrt{\pi}}\exp\left(-\frac{1}{2}(x^{-1}-1)\right)\frac{1}{2}\left(\frac{1}{2}(x^{-1}-1)\right)^{-\frac{1}{2}}\frac{1}{2x^2} \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x^{-1}-1)\right)(x^{-1}-1)^{-\frac{1}{2}}x^{-2}, && \text{(D.1)} \\
\frac{d^2}{dx^2}f(x) &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x^{-1}-1)\right)\frac{1}{2\pi}(x^{-1}-1)^{-\frac{1}{2}}x^{-2} \\
&\quad +\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x^{-1}-1)\right)\left(\frac{1}{2}(x^{-1}-1)^{-\frac{3}{2}}x^{-4}-2(x^{-1}-1)^{-\frac{1}{2}}x^{-3}\right) \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x^{-1}-1)\right)\frac{1}{2}(x^{-1}-1)^{-\frac{1}{2}}x^{-4}\left(1+(x^{-1}-1)^{-1}-4x\right) \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x^{-1}-1)\right)\frac{1}{2}(x^{-1}-1)^{-\frac{1}{2}}x^{-4}\frac{(2x-1)^2}{1-x}. && \text{(D.2)}
\end{aligned}
$$

Bibliography

[1] A. Ruiz, J. M. Cioffi and S. Kasturia, "Discrete multiple tone modulation with coset coding for the spectrally shaped channel", *IEEE Transactions on Communications*, vol. 40, no. 6, pp. 1012-1029, June 1992.

[2] A. Scaglione, G. B. Giannakis and S. Barbarossa, "Redundant filterbank precoders and equalizers Part I: unification and optimal designs", *IEEE Transactions on Signal Processing*, vol. 47, no. 7, pp. 1988-2006, July 1999.

[3] A. Scaglione, S. Barbarossa and G. B. Giannakis, "Filterbank transceivers optimizing information rate in block transmissions over dispersive channels", *IEEE Transactions on Information Theory*, vol. 45, no. 3, pp. 1019-1032, Apr. 1999.

[4] B. Muquet, Z. D. Wang, G. B. Giannakis, M. D. Courville and P. Duhamel, "Cyclic prefixing or zero padding for wireless multicarrier transmissions?" *IEEE Transactions on Communications*, vol. 50, no. 12, pp. 2136-2148, Dec. 2002.

[5] D. P. Palomar, J. M. Cioffi and M. A. Lagunas, "Joint Tx-Rx beamforming design for multicarrier MIMO channels: a unified framework for convex optimization", *IEEE Transactions on Signal Processing*, vol. 51, no. 9, pp. 2381-2401, Sept. 2003.

[6] E. A. Jorswieck and H. Boche, "Transmission strategies for the MIMO MAC with MMSE receiver: average MSE optimization and achievable individual MSE

region", *IEEE Transactions on Signal Processing*, vol. 51, no. 11, pp. 2872-2881, Nov. 2003.

[7] ETSI Normalization Committee, "Digital broadcasting systems for television, sound and data services; framing structure, channel coding and modulation for digital terrestrial television", Norme ETSI, Sophia-Antipolis, France, Doc. pr ETS 300 744, 1996.

[8] ETSI Normalization Committee, "Radio broadcasting systems, digital audio broadcasting (DAB) to mobile, portable and fixed receiver", Norme ETSI, Sophia-Antipolis, France, Doc. ETS 300 401, 1995-1997.

[9] H. V. Poor and S. Verdú, "Probability of error in MMSE multiuser detection", *IEEE Transactions on Information Theory*, vol. 43, no. 3, pp. 858-871, May 1997.

[10] J. A. C. Bingham, "Multicarrier modulation for data transmission: an idea whose time has come", *IEEE Communications Magazine*, pp. 5-14, May 1990.

[11] J. G. Proakis, *Digital Communications*, McGraw Hill, New York, Fourth edition, 2001.

[12] J. Jang and K. B. Lee, "Transmit power adaptation for multiuser OFDM systems", *IEEE Journal on Selected Areas in Communications*, vol. 21, no. 2, pp. 171-178, Feb. 2003.

[13] J. Zhang, E. K. P. Chong and D. N. C. Tse, "Output MAI distributions of linear MMSE multiuser receivers in DS-CDMA systems", *IEEE Transactions on Information Theory*, vol. 47, no. 3, pp. 1128-1144, Mar. 2001.

[14] L. Vandenberghe and S. Boyd, "Semidefinite programming", *SIAM Review*, vol. 44, pp. 2796-2815, 1996.

[15] N. J. Fliege, "Orthogonal multiple carrier data transmission", *European Transactions on Telecommunications*, vol. 3, no. 3, pp. 255-264, May 1992.

[16] R. A. Horn and C. R. Johnson, *Matrix Analysis*, New York, Cambridge University Press, 1985.

[17] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[18] S. S. Chan, Design of asyptotically minimum bit error rate linear block precoders for minimum mean square error equalization, Master's thesis, McMaster University, Department of Electrical and Computer Engineering, Hamilton, Ontario, Canada, Mar. 2003.

[19] S. S. Chan, T. N. Davidson and K. M. Wong, "Asymptotically minimum bit error rate block precoders for minimum mean square error equalization", *In IEEE Sensor Array and Multichannel Signal Processing Workshop Proceedings*, Rosslyn, VA, Aug. 2002.

[20] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley and Sons, New York, 1991.

[21] U. Madhow and M. L. Honig, "MMSE interference suppression for direct-sequence spread-spectrum CDMA", *IEEE Transactions on Commununications*, vol. 42, no. 12, pp. 3178-3188, Dec. 1994.

[22] X. Wang, J.-K. Zhang and K. M. Wong, "Design of optimum precoders for multiple access OFDM systems", *In the 22nd Biennial Symposium on Communications Proceedings*, Kingston, ON, June 2004.

[23] X. Wang, J.-K. Zhang and K. M. Wong, "Design of minimum bit error rate precoders for mutiuser OFDM systems fitted with MMSE receivers", *In IEEE Sensor*

Array and Multichannel Signal Processing Workshop Proceedings, Barcelona, Spain, July 2004.

[24] Y. W. Ding, Minimum BER block precoders for zero-forcing equalization, Master's thesis, McMaster University, Department of Electrical and Computer Engineering, Hamilton, Ontario, Canada, Oct. 2001.

[25] Y. W. Ding, T. N. Davidson, Z.-Q. Luo and K. M. Wong, "Minimum BER block precoders for zero-forcing equalization", *IEEE Transactions on Signal Processing*, vol. 51, no. 9, pp. 2410-2423, Sept. 2003.

[26] Z.-Q. Luo, T. N. Davidson, G. B. Giannakis and K. M. Wong, "Transceiver optimization for multiple access", *IEEE Transactions on Signal Processing*, vol. 52, no. 4, pp. 1037-1052, Apr. 2004.

[27] Z. Wang and Giannakis, "Wireless multicarrier communications", *IEEE Signal Processing Magazine*, vol. 17, no. 3, pp. 29-48, May 2000.