On transverse stability of periodic waves in the Kadomtsev–Petviashvili equation

ON TRANSVERSE STABILITY OF PERIODIC WAVES IN THE KADOMTSEV–PETVIASHVILI EQUATION

BY

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Abstract

This thesis is devoted to the proof of linear stability of the one-dimensional periodic waves in the Kadomtsev–Petviashvili (KP-II) equation with respect to two-dimensional bounded perturbations. The method of the proof is based on the construction of a self-adjoint operator K such that the operators JL and JK commute, where $J = \partial_x$ expresses a symplectic structure for the KP-II equation and L is a self-adjoint Hessian operator of the energy function at the periodic wave. In the situation when K is strictly positive except for a finite-dimensional kernel included in the kernel of L, the operator JL has no unstable eigenvalues and the associated time evolution is globally bounded.

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Chapter 1

Introduction

This thesis gives a proof of transverse spectral stability of one-dimensional periodic traveling waves in a model equation, which was derived by B. Kadomtsev and V. Petviashvili (KP) in 1970 [20]. By using scaling transformations, we can take the KP equation in the following normalized form

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0, \tag{1.1}$$

where the subscript denote partial derivatives with respect to the spatial variables (x, y) and the temporal variable t. We refer to this equation as to the KP-II equation, which the index II means the version relevant to the case of negative transverse dispersion, and as a specific model for surface water waves in the case of small surface tension. The KP-I equation is achieved by replacing the positive sign of the term u_{yy} by a negative sign, and it refers to the case of positive transverse dispersion and the large surface tension for surface water waves. Both versions of the KP equation generalize the one-dimensional Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.2}$$

The KP equation quickly became very popular due to its integrability properties [32], including a wealthy family of exact solutions, a bi-Hamiltonian structure and the recursion operator, a countable set of conserved quantities and symmetries, as well as the inverse scattering transform techniques. It also became very popular in the analysis of the stability of nonlinear waves, both relying upon functional-analytic methods and integrability techniques. As a model equation for surface water waves, some of the obtained results were extended to the Euler equations describing the full hydrodynamic problem [5, 11, 16, 36].

Stability properties of traveling waves are quite different between the two versions of the KP equation. While both periodic and solitary waves are transversely unstable under general bounded perturbations in the KP-I equation (e.g., see recent works [8, 13, 19, 34, 35] and the references therein), it is expected that they are transversely stable in the KP-II equation [1, 20]. Numerical evidences of these stability properties can be found for instance in [24, 25]. For the case of solitary waves, the transverse nonlinear stability has been recently proved for periodic transverse perturbations in [28], and for fully localized perturbations in [27]. In contrast, there are few analytical results for periodic waves for which, in particular, the question of transverse nonlinear stability is open.

By using a linearized version of the dressing method, explicit eigenfunction of the spectral stability problem associated with the periodic waves of the KP-II equation (1.1) were constructed in [26]. Completeness of the eigenfunctions and generalizations to the case of oblique transverse perturbations were elaborated few year later [37]. The results achieved by this method depend on explicit computations involving Jacobian elliptic functions for the periodic waves and the associated Jost functions, which are hard to check or confirm.

Linearized operators arising in stability studies for Hamiltonian systems such as the KP equation have a typical product structure JL in which J is a skew-adjoint operator and L is a self-adjoint operator. Well-known results show that, under suitable conditions, the number of unstable eigenvalues (i.e., the eigenvalues with positive real part) of the operator JL is bounded by the number of nonpositive eigenvalues of the self-adjoint operator L (e.g., see [6, 17, 21] and the references therein). In particular, if the operator L is positive-definite this immediately implies that JL has no unstable spectrum. Since typically L is related to the Hessian operator of an energy functional that is conserved in the time evolution of the Hamiltonian system, besides spectral stability, one can also conclude on nonlinear, orbital stability in the case of positive and coercive L. Such results have been extensively used in the analysis of the stability of nonlinear waves (e.g., see the books [2, 22, 33]).

While these arguments work very well for solitary waves, for periodic waves they allow, so far, to only understand stability with respect to co-periodic perturbations (i.e., which have the same period as that of the wave). The main difficulty in the case of periodic waves, is the fact that the number of negative eigenvalues of the operator L increases when the period of the perturbations is an increasing multiple of the period of the wave, and that L has negative essential spectrum when the perturbations are localized. These are serious obstacles in controlling unstable eigenvalues and then proving stability of periodic waves for arbitrary bounded perturbations.

In the context of the KP-II equation, the additional obstacle on the classical counting result for the unstable eigenvalues is due to the unboundedness of the operator L in the transverse (y) direction [15]. It turns out that in the case of the periodic waves of the KP-II equation, the self-adjoint operator L has unbounded spectrum both from below and above. Consequently, this eigenvalue counting only allows to obtain a partial result, showing spectral stability of small-amplitude periodic waves with respect to perturbations which are co-periodic in the direction of propagation x, and have long wavelengths in the transverse direction y [15].

This thesis is based on the joint paper with M. Haragus [14], where we have generalized the classical eigenvalue counting result by showing that the operator L can be replaced by another self-adjoint operator K, provided the operators JL and JK commute. More precisely, under suitable assumptions, we have proven in [14] that if K is positive with a finite-dimensional kernel contained in the kernel of L, the operator JL has no unstable eigenvalues.

Very recently, the idea of using a positive definite operator K has been exploited in [3, 7, 31] and [4, 9] and allowed the authors to show the orbital stability of periodic waves with respect to subharmonic perturbations (i.e., the period of the perturbations is an integer multiple of the period of the wave) for the Korteweg-de Vries (KdV) and the cubic nonlinear Schrödinger (NLS) equations, respectively. The self-adjoint operator L is related to the Hessian operator of the standard energy functional expanded at the periodic traveling wave. Similarly, the self-adjoint operator K can be found from the Hessian operator of a higher-order energy functional, due to the integrability properties of the KdV and NLS equations, Because the phase flows of the two Hamiltonian systems commute, the linearized operators JL and JK commute. For the KdV and NLS equations, neither L and K are positive operators, but a suitable linear combination of these operators is positive [4, 7, 9].

Since the KP-II equation is integrable, there exists a higher-order energy functional, which was used in the proof of global well-posedness for the KP-I equation [29, 30]. It is rather surprising that when K is constructed from the higher-order energy functional neither K nor a linear combination of K and L is positive.

In order to avoid this obstacle, we shall start with the operator K obtained for the KdV equation and find an operator K for the KP-II equation by a direct search from the commutativity relation of JL and JK. Then we show that a suitable linear combination of L and K is indeed a positive operator. However, this self-adjoint operator K constructed directly from the commutativity relation does not seem to be related to the Hessian operator of some higher-order conserved quantity of the KP-II equation. In particular, we cannot use this construction to conclude on the nonlinear, orbital stability of these periodic waves, which remains an open problem.

The following sections represent the main results and organization of the thesis.

1.1 Transverse spectral stability

One-dimensional periodic traveling waves of the KP-II equation (1.1) are solutions of the KdV equation (1.2) of the form $u(x,t) = v(x+ct)$, where v is a periodic function and c is a constant speed of propagation. Without loss of generality, we can restrict v to be 2π-periodic and even.

In a coordinate system moving with the speed c of the periodic traveling wave, the corresponding linearization of the KP-II equation (1.1) is given by

$$(w_t + w_{xxx} + cw_x + 6(v(x)w)_x)_x + w_{yy} = 0, \tag{1.3}$$

in which, for notational simplicity, we denoted by x the variable $x + ct$ from the KP-II equation (1.1). Following the transverse spectral stability approach in [15], we consider solutions of the form

$$w(x,y,t) = e^{\lambda t + ipy} W(x),$$

with W satisfying the differential equation

$$\lambda W_x + W_{xxxx} + cW_{xx} + 6(v(x)W)_{xx} - p^2 W = 0. \tag{1.4}$$

The spectral problem (1.4) can be formulated as $\mathcal{A}_{c,p}W = 0$, where

$$\mathcal{A}_{c,p}(\lambda) = \lambda\partial_x + \partial_x^4 + c\partial_x^2 + 6\partial_x^2(v(x)\,\cdot) - p^2 \tag{1.5}$$

is a linear differential operator with 2π-periodic coefficients. The spectral stability problem is concerned with the invertibility of $\mathcal{A}_{c,p}(\lambda)$ for certain values of p and in a suitable function space.

Definition 1.1 *We say that the periodic wave v is spectrally stable with respect to general bounded two-dimensional perturbations if $\mathcal{A}_{c,p}(\lambda)$ is invertible in $C_b(0,2\pi)$, the Banach space of uniformly bounded continuous functions on $\mathbb{R}$, for every $p \in \mathbb{R}$ and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.*

Let us first consider the spectral problem (1.4) in $L^2_{\mathrm{per}}(0,2\pi)$, the space of 2π-periodic square-integrable functions. The operator ∂_x is not invertible in $L^2_{per}(0,2\pi)$, since

$$\partial_x : H^1_{\mathrm{per}}(0,2\pi) \subset L^2(0,2\pi) \mapsto L^2(0,2\pi)$$

has a nontrivial kernel: $\partial_x(1) = 0$. However, if $p \neq 0$, then W in the spectral problem (1.4) belongs to the space $\dot{L}^2_{\mathrm{per}}(0,2\pi)$ defined by

$$\dot{L}^2(0,2\pi) = \left\{ u \in L^2(0,2\pi) : \quad \int_0^{2\pi} u(x)dx = 0 \right\}.$$

Then,

$$\partial_x : H^1_{\mathrm{per}}(0,2\pi) \subset \dot{L}^2(0,2\pi) \mapsto \dot{L}^2(0,2\pi)$$

is invertible and the inverse operator $\partial_x^{-1} : \dot{L}^2(0,2\pi) \to H^1_{\mathrm{per}}(0,2\pi) \subset \dot{L}^2(0,2\pi)$ is bounded.

Thanks to the fact that $\dot{L}^2_{per}(0,2\pi)$ is an invariant subspace of the spectral problem (1.4), the operator $\mathcal{A}_{c,p}(\lambda)$ is invertible if and only if λ belongs to the resolvent set of the operator

$$\mathcal{B}_{c,p} = -\partial_x^3 - c\partial_x - 6\partial_x v(x) + p^2\partial_x^{-1}, \tag{1.6}$$

which is a closed operator in $\dot{L}^2_{\text{per}}(0, 2\pi)$ with domain $H^3_{per}(0, 2\pi) \subset \dot{L}^2_{\text{per}}(0, 2\pi)$. Consequently, the spectral problem (1.4) is reduced to that of studying the spectrum of $\mathcal{B}_{c,p}$, which is an operator with compact resolvent, hence with point spectrum consisting of isolated eigenvalues with finite algebraic multiplicities, only. Moreover, $\mathcal{B}_{c,p}$ has the JL product structure given by

$$\mathcal{B}_{c,p} = JL_{c,p}, \tag{1.7}$$

with

$$J = \partial_x, \quad L_{c,p} = -\partial_x^2 - c - 6v(x) + p^2\partial_x^{-2}. \tag{1.8}$$

We point out that $\dot{L}^2_{per}(0, 2\pi)$ is an invariant subspace for the operators $\mathcal{B}_{c,p}$ and J but not for $L_{c,p}$. Therefore, in the subsequent analysis, we replace, when needed, the operator $L_{c,p}$ by the projected operator $\Pi_0 L_{c,p}$, where $\Pi_0 : \ L^2_{\text{per}}(0, 2\pi) \mapsto \dot{L}^2_{\text{per}}(0, 2\pi)$ is the standard orthogonal projection on the nonzero Fourier modes. For notational simplicity, we denote the projected operator $\Pi_0 L_{c,p}$ also by $L_{c,p}$, and refer to it as the restriction of $L_{c,p}$ to $\dot{L}^2_{\text{per}}(0, 2\pi)$. Thus, we rewrite the spectral problem (1.4) for $p \neq 0$ in the equivalent form:

$$\mathcal{B}_{c,p}W = JL_{c,p}W = \lambda W. \tag{1.9}$$

Similarly, one can consider the linearized operator $\mathcal{B}_{c,p} = JL_{c,p}$ for the subharmonic perturbations in space $\dot{L}^2_{\text{per}}(0, 2\pi N)$ with an integer N. Our main result is the following theorem.

Theorem 1 *Consider a periodic traveling wave v of the KdV equation* (1.2) *with $c > 1$. For every $p \neq 0$, the linear operator $\mathcal{B}_{c,p} = JL_{c,p}$ acting in $\dot{L}^2_{\text{per}}(0, 2\pi N)$ for every $N \in \mathbb{N}$ has a purely imaginary spectrum.*

By using the Floquet theorem, the Bloch transform, one can then extend the proof of Theorem 1 to the linearized operator $\mathcal{B}_{c,p} = JL_{c,p}$ defined for general bounded perturbations in space $L^2(\mathbb{R})$. Consequently, Theorem 1 allows us to state that the periodic traveling wave v is spectrally stable with respect to general bounded two-dimensional perturbations according to Definition 1.1. See further details in our paper [14].

1.2 Transverse linear stability

The way to prove Theorem 1 is to construct a positive operator $K_{c,p}$ in $\dot{L}^2_{\text{per}}(0, 2\pi)$ with a finite-dimensional kernel such that

$$JK_{c,p}JL_{c,p} = JL_{c,p}JK_{c,p} \tag{1.10}$$

and the kernel of $K_{c,p}$ in $\dot{L}^2_{\text{per}}(0, 2\pi)$ is included in the kernel of $L_{c,p}$. Positivity properties of the operator $K_{c,p}$ also allows us to prove a transverse linear stability result. Due to the lack of coercivity of $K_{c,p}$ for general bounded perturbations, the transverse linear stability result is restricted to doubly periodic perturbations, which are subharmonic with zero mean in the direction of propagation x and have an arbitrary, but fixed, period in the transverse direction y.

Restricting to periodic perturbations which have zero mean in x, we rewrite the linearized equation (1.3) as an evolutionary problem

$$\frac{dw}{dt} = \mathcal{B}_c w, \tag{1.11}$$

where

$$\mathcal{B}_c = JL_c, \quad J = \partial_x, \quad L_c = -\partial_x^2 - c - 6v(x) - \partial_x^{-2}\partial_y^2. \tag{1.12}$$

Here, the operator $\mathcal{B}_c$ is well-defined and closable in the space of locally square-integrable functions on $\mathbb{R}^2$ which are $2\pi N$-periodic and have zero mean in x, for some $N \in \mathbb{N}$, and are $2\pi/p$-periodic in y, for some fixed wave number p. We denote this space by $\dot{L}^2(N,p)$. In this space, the operators J and L_c are skew- and self-adjoint operators, respectively.

Similarly to the operator $K_{c,p}$, we can find a positive self-adjoint operator K_c satisfying the commutativity property

$$JK_cJL_c = JL_cJK_c. \tag{1.13}$$

Let us show that the associated quadratic form $\langle K_c\cdot,\cdot\rangle$ is constant along suitable solutions to the linearized equation (1.11), hence the quadratic form acts as a

Lyapunov functional. Indeed, a simple formal calculation gives

$$\frac{d}{dt}\langle K_c w, w\rangle = \langle K_c J L_c w, w\rangle + \langle K_c w, J L_c w\rangle = \langle K_c J L_c w, w\rangle - \langle L_c J K_c w, w\rangle = 0.$$

This calculation becomes rigorous for appropriately regular solutions. Thus, for suitable solutions $w(t)$ to the linearized equation (1.11), we have

$$\langle K_c w(t), w(t)\rangle = \langle K_c w(0), w(0)\rangle, \quad \forall\ t \in \mathbb{R}. \tag{1.14}$$

If the operator K_c is coercive in some norm, then the solutions $w(t)$ to the linearized equation (1.11) stay bounded in this norm for all times, which then implies linear stability.

The transverse linear stability result is obtained in the energy space for the quadratic form (1.14), which coincides with the Hilbert space $H^{2,1}(N,p)$ defined by

$$H^{2,1}(N,p) = \{w \in \dot{L}^2(N,p) : \quad w_x, w_{xx}, w_y \in \dot{L}^2(N,p)\}$$

and equipped with the standard norm denoted by $\|\cdot\|_{2,1}$. The following theorem represents the second main result of this thesis.

Theorem 2 *Consider a periodic traveling wave v of the KdV equation* (1.2) *with $c > 1$. For any $N \in \mathbb{N}$ and any positive $p \in \mathbb{R}$, there exists a constant $C_{c,N,p}$ such that any solution $w \in C^1(\mathbb{R}, H^{2,1}(N,p))$ of the linearized equation* (1.11) *satisfies the inequality*

$$\|w(t) - a(t)\partial_x v\|_{2,1} \leqslant C_{c,N,p}\|w(0)\|_{2,1}, \quad |a'(t)| \leqslant C_{c,N,p}, \tag{1.15}$$

where $a(t)$ represents the orthogonal projection of the solution on the derivative $\partial_x v$ of the periodic wave,

$$a(t) = \frac{\langle w(t), \partial_x v\rangle}{\|\partial_x v\|^2}.$$

Remark 1.2 *Due to the translation invariance of the KP-II equation, the derivative $\partial_x v$ of the periodic wave v belongs to the kernel of L_c and K_c. As a result, the operator K_c is only coercive on the subspace orthogonal to $\partial_x v$. This explains the presence of the term $a(t)\partial_x v$ in the first estimate in* (1.15)*. Furthermore, the linearized operator $\mathcal{B}_c$ has a generalized kernel with the 2×2 Jordan block. This explains a possible linear growth of $a(t)$, as indicated by the second inequality in* (1.15)*. The estimates in* (1.15) *are the linear counterpart of a standard nonlinear orbital stability result claiming that, as expected in the presence of translational invariance, solutions stay close to the orbit $\{v(\cdot + x_0)\}_{x_0 \in \mathbb{R}}$ of the periodic traveling wave v.*

Remark 1.3 *We do not discuss here the initial value problem for the linearized equation* (1.11)*, and hence the question of existence of solutions $w \in C^1(\mathbb{R}, H^{2,1}(N,p))$. However, on the basis of semigroup theory, one expects that for initial data $w(0) \in H^{5,3}(N,p)$ a unique solution of* (1.11) *exists which satisfies $w \in C^1(\mathbb{R}, H^{2,1}(N,p)) \cap C^0(\mathbb{R}, H^{5,3}(N,p))$, where the space $H^{5,3}(N,p)$ is defined similarly to $H^{2,1}(N,p)$.*

1.3 Organization of the thesis

We first consider the KdV equation (1.2) and obtain results on existence of periodic wave solutions, construction of Hessian operators L_c and M_c, and positivity of a linear combination of Hessian operators. Based on these results, we then prove the transverse spectral and linear stability of periodic waves in the KP-II equation (1.1).

The thesis is organized as follows. Chapter 2 consists of the following sections:

- In Section 2.1, we obtain the traveling periodic wave solutions to the KdV equation (1.2) with the speed c near $c = 1$ by means of the Lyapunov - Schmidt reduction method.

- In Section 2.2, we use conserved quantities of the KdV equation (1.2) and derive the Hessian operators L_c and M_c which are not strictly positive definite.

- In Section 2.3, we prove positivity of the mixed Hessian operator $K_{c,b} = M_c - bL_c$ for some $b \in (b_-, b_+)$, where $0 < b_- < b_+ < \infty$ exist for every $c > 1$.

- In Section 2.4, we present the explicit formulas for the periodic wave solution and the interval (b_-, b_+) in terms of the Jacobian elliptic functions.

Sections 2.1, 2.2, and 2.3 contain original results obtained in the thesis. Section 2.4 is deduced from paper [7].

Chapter 3 contains the following sections:

- In Section 3.1, we prove the transverse spectral stability of periodic waves stated in Theorem 1.

- In Section 3.2, we prove the transverse linear stability of periodic waves stated in Theorem 2.
- In Section 3.3, we discuss conserved quantities and commuting operators of the KP-II equation (1.1).

Results of Chapter 3 are obtained together with M. Haragus (University of Besancon, France) and have been published in paper [14].

Chapter 2

Periodic waves in the KdV equation

Here we consider the KdV equation (1.2) rewritten again in the form

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{2.1}$$

Periodic traveling waves of the KdV equation (2.1) are solutions of the form $u(x,t) = v(x+ct)$, where v is a periodic function in its argument. Due to the Galilean invariance, one can integrate the resulting third-order differential equation for v with zero integration constant and obtain v from the second-order differential equation

$$\frac{d^2 v}{dx^2} + cv + 3v^2 = 0. \tag{2.2}$$

Without loss of generality, due to the scaling transformation and the translation invariance, we can scale the period of the periodic traveling wave to 2π and translate the wave profile v to be even in x. Hence we consider 2π-periodic even solutions to the differential equation (2.2).

This chapter is organized as follows. In Section 2.1 we prove existence of the periodic wave v near $c = 1$ by using the method of Lyapunov–Schmidt reductions. The two energy functions and the associated Hessian operators are introduced in Section 2.2. Positivity of a certain linear combination of the two Hessian operators is proved by using the perturbation theory in Section 2.3. Exact solutions for the existence of the periodic wave v and the positivity result for the Hessian operators are reviewed in Section 2.4.

2.1 Existence of periodic waves

Here we represent the periodic solution v of the differential equation (2.2) in the small-amplitude limit, when the solution can be expanded in the power series in terms of the wave amplitude a. This is achieved with the method of Lyapunov–Schmidt reductions.

As is explained before, we work with 2π-periodic even functions v. The space of 2π-periodic squared integrable functions is denoted by L^2_{per}, whereas the space of 2π periodic even squared integrable functions is denoted by $L^2_{per,even}$. In both cases, we use the norm given by

$$\|u\|_{L^2_{per}} = \left(\int_{-\pi}^{\pi} u^2 dx \right)^{\frac{1}{2}}.$$

The Sobolev space of 2π periodic functions with squared integrable derivatives up to the second order is denoted by H^2_{per} and we use the corresponding norm given by

$$\|u\|_{H^2_{per}} = \left(\int_{-\pi}^{\pi} \left(u_{xx}^2 + u_x^2 + u^2 \right) dx \right)^{\frac{1}{2}}.$$

The notation $u = \mathcal{O}_{H^2_{\rm per}}(a^p)$ with $p > 0$ indicate that u is small in $H^2_{\rm per}$-norm if a is small such that $\|u\|_{H^2_{per}} \leqslant C|a|^p$ for some a-independent positive constant C.

The main result of the Lyapunov–Schmidt reduction method is given by the following proposition.

Proposition 2.1 *For every $c > 1$ such that $|c-1|$ is sufficiently small, there exists a unique 2π-periodic even solution v to the differential equation (2.2), which can be parameterized by the amplitude parameter a, such that*

$$v(x) = a\cos(x) + \frac{a^2}{2}\left[\cos(2x) - 3\right] + \frac{3a^3}{16}\cos(3x) + \mathcal{O}_{H^2_{\rm per}}(a^4) \tag{2.3}$$

and

$$c = 1 + \frac{15}{2}a^2 + \mathcal{O}(a^4). \tag{2.4}$$

Proof. Since zero is a trivial solution of (2.2) in space $H^2_{\rm per}$, we can analyze bifurcation from the zero solution by using the linearized version of (2.2):

$$\frac{d^2v}{dx^2} + cv = 0. \tag{2.5}$$

The general solution of (2.5) is

$$v = c_1 \cos(\sqrt{c}x) + c_2 \sin(\sqrt{c}x)$$

where c_1, c_2 are arbitrary constants. The general solution v is a 2π periodic function if and only if $\sqrt{c}$ is a non-negative integer. If v is also an even function, then $c_2 = 0$.

Bifurcations of periodic waves may happen for each non-negative integer $\sqrt{c}$. However, if $c = 1$, then $\cos(\sqrt{c}x)$ has only one maximum on $[-\pi, \pi]$. This profile defines the "basic" periodic wave of the differential equation (2.2), which is described in the proposition. Therefore, we set $c = 1$ at the bifurcation point and define the linear operator $L_0 := 1 + \partial_x^2$ in $L^2_{per,even}$ with the domain H^2_{per}.

Since $\mathrm{Ker}(L_0) = \mathrm{span}\{\cos(\cdot)\}$ in $L^2_{per,even}$, we consider the decomposition:

$$v = a\cos(x) + V(x; a) \tag{2.6}$$

and

$$c = 1 + C(a), \tag{2.7}$$

where $V(x; a)$ and $C(a)$ are to be found. In order to determine parameter a uniquely, we will use the orthogonality constraint,

$$\langle \cos(\cdot), V(\cdot; a)\rangle_{L^2_{per}} = 0, \tag{2.8}$$

hence $V(x; a)$ can be be expressed as Fourier cosine series without the first Fourier mode $\cos(x)$. The parameter a in the decomposition (2.6) is the projection of v to the first Fourier mode.

When we substitute (2.6) and (2.7) into (2.2), we get

$$\frac{d^2V}{dx^2} + (1 + C(a))V + 3V^2 + 6a\cos(x)V = -aC(a)\cos(x) - 3a^2\cos^2(x) \quad (2.9)$$

Multiplying both sides of (2.9) by $\cos(x)$ and integrating in x between $-\pi$ and π yield the following constraint on $C(a)$

$$C(a) := -\frac{1}{a\pi}\int_{-\pi}^{\pi} \cos(x)\left[3V(x;a)^2 + 6a\cos(x)V(x;a)\right] dx, \quad (2.10)$$

where we have used

$$\int_{-\pi}^{\pi} \cos^3(x)dx = 0 \qquad \int_{-\pi}^{\pi} \cos^2(x)dx = \pi.$$

Let P_0 be the orthogonal projection operator in L^2_{per} to the complement of $\cos(\cdot)$. We prove that for every $f \in L^2_{per,even}$ we have

$$\|P_0(1 + \partial_x^2)^{-1}P_0 f\|_{L^2_{per}} \leqslant \|f\|_{L^2_{per}}. \quad (2.11)$$

Indeed, writing

$$P_0 f(x) = \frac{a_0}{2} + \sum_{n=2}^{\infty} a_n cos(nx)$$

and $L_0\phi = P_0 f$, we obtain

$$\phi(x) = \frac{a_0}{2} + c_1\cos(x) + \sum_{n=2}^{\infty}\frac{a_n}{1-n^2}cos(nx),$$

where the value c_1 is not determined. By Parseval's equality, we obtain

$$\frac{1}{\pi}\|P_0\phi\|^2_{L^2_{per}} = \frac{1}{2}a_0^2 + \sum_{n=2}^{\infty}\frac{a_n^2}{(1-n^2)^2} \leqslant \frac{1}{2}a_0^2 + \sum_{n=2}^{\infty}a_n^2 = \frac{1}{\pi}\|f\|^2_{L^2_{per}},$$

which yields (2.11). Similarly, we can show that for every $f \in H^2_{per,even}$, we have

$$\left\|P_0(1+\partial_x^2)^{-1}P_0 f\right\|_{H^2_{per}} \leqslant \|f\|_{H^2_{per}}. \tag{2.12}$$

We can now invert L_0 and rewrite (2.9) in the equivalent form

$$V + L_a V + N_a(V) = H_a \tag{2.13}$$

where

$$\begin{aligned}
L_a V &= P_0(1+\partial_x^2)^{-1}P_0\left[6a\cos(x)V + C(a)V\right], \\
N_a(V) &= 3P_0(1+\partial_x^2)^{-1}P_0 V^2, \\
H(a) &= -3a^2 P_0(1+\partial_x^2)^{-1}P_0\cos^2(x).
\end{aligned}$$

We claim that there exist positive constants c_1,c_2,c_3, such that

$$\|L_a V\|_{H^2_{per}} \leqslant c_1(|a| + |C(a)|)\,\|V\|_{H^2_{per}} \tag{2.14}$$

$$\|N_a V\|_{H^2_{per}} \leqslant c_2 \|V\|^2_{H^2_{per}} \tag{2.15}$$

$$\|H_a\|_{H^2_{per}} \leqslant c_3 a^2. \tag{2.16}$$

The proof of (2.16) follows from (2.12). The proof of (2.14) and (2.15) follows from (2.12) and the Banach algebra property of the Sobolev space H^2_{per}. Indeed, there is a positive constant c_b such that,

$$\|UW\|_{H^2_{per}} \leqslant c_b \|U\|_{H^2_{per}} \|W\|_{H^2_{per}} \quad \forall U, W \in H^2_{per}. \tag{2.17}$$

It follows from (2.10) that $C(a) = \mathcal{O}(a^{-1}\|V(\cdot;a)\|^2_{H^2_{per}} + \|V(\cdot;a)\|_{H^2_{per}})$ as $a \to 0$ and $\|V(\cdot;a)\|^2_{H^2_{per}} \to 0$. By using (2.14), (2.15), (2.16), and the implicit function theorem for small a, we obtain a unique solution $V \in H^2_{per,even}$ of the implicit equation (2.13) in the neighborhood of the zero solution satisfying the bound $\|V\|_{H^2_{per}} \leqslant c_V a^2$ for some $c_V > 0$. Consistently, the unique expression for $C(a)$ in (2.10) yields the bound $|C(a)| \leqslant c_C a^2$ for some $c_C > 0$.

The explicit expression for $V(x;a)$ and $C(a)$ are derived from the formal expansions

$$V(x;a) = a^2 V_2(x) + a^3 V_3(x) + \mathcal{O}(a^4) \qquad C(a) = a^2 C_2 + a^3 C_3 + \mathcal{O}(a^4).$$

Substituting these expansions to (2.9), we collect the linear inhomogeneous equations:

$$\mathcal{O}(a^2) \qquad \frac{d^2V_2}{dx^2} + V_2 = -3\cos^2(x), \tag{2.18}$$

$$\mathcal{O}(a^3) \qquad \frac{d^2V_3}{dx^2} + V_3 + 6\cos(x)V_2(x) = -C_2\cos(x), \tag{2.19}$$

and the constraint

$$C_2 = \frac{-6}{\pi}\int_{-\pi}^{\pi}\cos^2(x)V_2 dx. \tag{2.20}$$

Equation (2.18) is solved with $V_2 = A\cos(2x) + B$, where $A = \frac{1}{2}$ and $B = -\frac{3}{2}$ are uniquely computed. Substituting V_2 into the constraint (2.20), we obtain $C_2 = \frac{15}{2}$. Then, equation (2.19) is solved uniquely with $V_3 = \frac{3}{16}\cos(3x)$. From these explicit computations, expressions (2.3) and (2.4) are obtained. ■

2.2 Energy functionals and Hessian operators

The KdV equation (2.1) is globally well-posed in H^m_{per} for every $m \in \mathbb{N}$ [23]. Global solutions satisfy several basic conserved quantities, in particular, the mass

$$C(u) = \int_{-\pi}^{\pi} u dx, \tag{2.21}$$

the momentum

$$Q(u) = \frac{1}{2}\int_{-\pi}^{\pi} u^2 dx, \tag{2.22}$$

and the energy

$$E(u) = \frac{1}{2} \int_{-\pi}^{\pi} (u_x^2 - 2u^3) dx. \tag{2.23}$$

Conservation of C follows from integration of the KdV equation (2.1) in x subject to the periodic boundary conditions for strong solutions $u(t)$ defined in $C(\mathbb{R}, H^3(\mathbb{R}))$.

Conservation of Q follows from multiplying (2.1) by u and integrating over x:

$$\frac{d}{dt} Q(u) + \left[2u^3 + u \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 \right] \Bigg|_{x=-\pi}^{x=\pi} = 0.$$

Periodic boundary conditions for strong solutions $u(t)$ defined in $C(\mathbb{R}, H^3(\mathbb{R}))$ imply that $Q(u)$ is constant in t.

Conservation of $E(u)$ follows from the formulation of the KdV equation (2.1) in the Hamiltonian form:

$$\frac{\partial u}{dt} = \frac{\partial}{\partial x} \mathrm{grad} E(u), \quad \text{where} \quad \mathrm{grad} E(u) = -\frac{\partial^2 u}{\partial x^2} - 3u^2.$$

From here, we obtain

$$\frac{d}{dt} E(u) = \left\langle \mathrm{grad} E(u), \frac{\partial u}{dt} \right\rangle_{L^2} = \langle \mathrm{grad} E(u), \partial_x \mathrm{grad} E(u) \rangle_{L^2} = 0,$$

again for strong solutions $u(t)$ defined in $C(\mathbb{R}, H^3(\mathbb{R}))$.

Proposition 2.2 *The periodic wave v satisfying* (2.2) *is a critical point of the energy functional $S_c(u) := E(u) - cQ(u)$ in H^1_{per}.*

Proof. The Euler–Lagrange equation for $S_c(u)$ yields the second-order differential equation

$$\mathrm{grad}E(u) - c\mathrm{grad}Q(u) = -\frac{d^2u}{dx^2} - 3u^2 - cu = 0, \tag{2.24}$$

which coincides with equation (2.2). ∎

Convexity of $S_c(u)$ at $u = v$ is defined by Hessian operator $L_c := S_c''(v)$. In order to derive the Hessian operator L_c, we set $u = v + w$, where v is the periodic wave and w is a small perturbation. Then, we obtain

$$\begin{aligned}
S_c(v+w) &= \frac{1}{2}\int_{-\pi}^{\pi}\left[(v_x+w_x)^2 - 2(v+w)^3 - c(v+w)^2\right]dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi}\left(v_x^2 - 2v^3 - cv^2\right)dx \\
&+ \frac{1}{2}\int_{-\pi}^{\pi}\left(2v_xw_x - 6v^2w - 2cvw\right)dx \\
&+ \frac{1}{2}\int_{-\pi}^{\pi}(w_x^2 - 6vw^2 - cw^2)dx - \frac{1}{2}\int_{-\pi}^{\pi}w^3dx
\end{aligned}$$

Since v is a critical point of S_c, the linear term is identically zero after integration by parts, so that we have

$$S_c(v+w) - S_c(v) = \frac{1}{2}\left\langle S_c''(v)w, w\right\rangle_{L^2_{per}} + \mathcal{O}\left(\|w\|^3\right)_{H^1_{per}}$$

with

$$L_c := S_c''(v) = -\frac{\partial^2}{\partial x^2} - 6v(x) - c. \tag{2.25}$$

Remark 2.3 *The Hessian operator L_c is not positive in L^2_{per} for $c > 1$. Indeed,* *for $c = 1$, we have $v = 0$ by Proposition 2.1, so that $L_{c=1} = -\partial_x^2 - 1$, which is not positive.*

Since L_c is not positive for $c > 1$, it is impossible to conclude on stability of the periodic wave v from the convexity argument of the energy functional S_c. However, in order to remedy the difficulty, we follow the approach of [7] and consider additional conserved quantities of the KdV equation (2.1). Indeed, since the KdV equation is an integrable system [32], it has infinitely many conserved quantities, in particular, the higher-order energy defined in H^2_{per}:

$$H(u) = \frac{1}{2}\int_{-\pi}^{\pi}\left(u_{xx}^2 - 10uu_x^2 + 5u^4\right)dx. \tag{2.26}$$

Proposition 2.4 *The periodic wave v satisfying* (2.2) *is a critical point of the energy functional $R_c(u) := H(u) - c^2Q(u) + I_cC(u)$ in H^2_{per}, where*

$$I_c = \left(\frac{dv}{dx}\right)^2 + cv^2 + 2v^3, \tag{2.27}$$

is the first-order invariant for the second-order equation (2.2).

Proof. Consider the modified energy functional $H(u) + \mu Q(u) + \nu C(u)$, where μ and ν are Lagrange multipliers. The Euler–Lagrange equation for the modified energy functional is equivalent to the fourth-order differential equation

$$\begin{aligned}&\mathrm{grad}H(u) + \mu\mathrm{grad}Q(u) + \nu\mathrm{grad}C(u)\\ &= u_{xxxx} + 10uu_x + 5u_x^2 + 10u^3 + \mu u + \nu = 0.\end{aligned} \tag{2.28}$$

Let v be the periodic wave satisfying (2.2) and (2.27). Substituting these expressions into (2.28) yields

$$\begin{aligned} 0 &= -cu_{xx} - 6uu_x - 6u_x^2 + 10u(-cu - 3u^2) + 5u_x^2 + 10u^3 + \mu u + \nu \\ &= c(cu + 3u^2) + (cu^2 + 2u^3 - I_c) - 4cu^2 - 12u^3 + 10u^3 + \mu u + \nu \\ &= (c^2 + \mu)u - I_c + \nu. \end{aligned}$$

Thus, v is a critical point of $H(u) + \mu Q(u) + \nu C(u)$ if $\mu = -c^2$ and $\nu = I_c$, which yields the higher-order energy functional in the form $R_c(u) := H(u) - c^2 Q(u) + I_c C(u)$. ■

We compute the Hessian operator $M_c := R_c''(v)$ at the periodic wave v by using the same procedure as above. The straightforward computations yield

$$M_c = \frac{\partial^4}{\partial x^4} + 10\frac{\partial}{\partial x}v(x)\frac{\partial}{\partial x} + 10v''(x) + 30v^2(x) - c^2. \tag{2.29}$$

Remark 2.5 *The Hessian operator M_c is not positive in L^2_{per} for $c > 1$. Indeed, for $c = 1$, we have $v = 0$ by Proposition 2.1, so that $M_{c=1} = \partial_x^4 - 1$, which is not positive.*

Since the periodic wave v is a critical point of both S_c and R_c, it is also a critical point of a linear combination of the energy functionals $R_c(u) - bS_c(u)$ for an arbitrary parameter $b \in \mathbb{R}$. The Hessian operator of the combined energy functionals is a linear combination of the operators M_c and L_c:

$$K_{c,b} := M_c - bL_c. \tag{2.30}$$

If neither L_c nor M_c is positive for $c > 1$, we shall try to find $b \in \mathbb{R}$ such that $K_{c,b}$ is positive in L^2_{per} for every $c > 1$, for which the periodic wave v exists.

Remark 2.6 *For $c = 1$, we have*

$$K_{c=1,b} = \partial_x^4 - 1 + b\left(\partial_x^2 + 1\right) = \left(\partial_x^2 + \frac{b}{2}\right)^2 - \left(1 - \frac{b}{2}\right)^2,$$

which is positive for $b = 2$.

2.3 Positivity of Hessian operators

Here we prove that there exists a choice of parameter b in (2.30) such that $K_{c,b}$ is positive in L^2_{per} for $c > 1$ such that $|c-1|$ is sufficiently small. The spectrum of $K_{c,b}$ in $L^2_{\rm per}$ is purely discrete and consists of isolated eigenvalues of finite algebraic multiplicity. Zero is always an eigenvalue of $K_{c,b}$ for any $c > 1$ and $b \in \mathbb{R}$, according to the following result.

Proposition 2.7 *For every $c > 1$, we have*

$$L_c \partial_x v = 0, \quad M_c \partial_x v = 0$$

Proof. The result $L_c \partial_x v = 0$ is due to the translational symmetry of the KdV equation (2.1). Let us show that $M_c \partial_x v = 0$. Indeed, v is a critical point of R_c, hence it satisfies the fourth-order differential equation

$$v_{xxxx} + 10 v v_{xx} + 5 v_x^2 + 10 v^3 - c^2 v + I_c = 0.$$

Taking derivative of this equation yields $M_c \partial_x v = v_{xxxxx} + 10 v v_{xxx} + 20 v_x v_{xx} + 30 v^2 v_x - c^2 v_x = 0$. ■

For $c = 1$ and $b = 2$, $K_{c=1,b=2} = (\partial_x^2 + 1)^2$ is a positive operator in L^2_{per} with the double zero eigenvalue, which corresponds to the two eigenfunctions $\{\cos(x), \sin(x)\}$. The odd eigenfunction corresponds to the leading order of the eigenfunction $\partial_x v(x)$ in Proposition 2.7. The even eigenfunction does not correspond to the translational symmetry of the KdV equation (2.1). Therefore, we expect that the zero eigenvalue splits for every $c > 1$. The following result shows that the splitting generates a positive eigenvalue if $b = 2$.

Proposition 2.8 *Consider the spectrum of $K_{c,b=2}$ in L^2_{per} for $c > 1$ such that $|c-1|$ is sufficiently small. Then, $K_{c,b=2}$ has a simple zero eigenvalue, one positive small eigenvalue, and all other eigenvalues are strictly positive, bounded away from zero as $c \to 1$.*

Proof. We construct perturbation expansions to solutions of

$$K_{c,b=2} W = \lambda W, \quad W \in H^4_{per}, \tag{2.31}$$

near $c = 1$ and $\lambda = 0$. The other eigenvalues of $K_{c=1,b=2}$ are strictly positive and remain so for $c > 1$ and $|c-1|$ sufficiently small, by the perturbation theory.

By Proposition 2.1, we expand $K_{c,b=2}$ as $P_0 + aP_1 + a^2P_2 + \mathcal{O}(a^3)$, where

$$\begin{aligned} P_0 &= (\partial_x^2 + 1)^2, \\ P_1 &= 10\partial_x \cos(x)\partial_x + 2\cos(x) \\ P_2 &= 5\partial_x \left(\cos(2x) - 3\right)\partial_x + \cos(2x) - 3. \end{aligned}$$

We are looking for solutions of the eigenvalue problem (2.31) in the form of the expansion

$$\begin{cases} W = W_0 + aW_1 + a^2W_2 + \mathcal{O}(a^3) \\ \lambda = a\lambda_1 + a^2\lambda_2 + \mathcal{O}(a^3). \end{cases} \tag{2.32}$$

Since $\mathrm{Ker}(P_0) = \mathrm{span}\{e^{i\cdot}, e^{-i\cdot}\}$ in L^2_{per}, we set

$$W_0 = d_1 e^{ix} + d_{-1} e^{-ix}, \tag{2.33}$$

where (d_1, d_{-1}) are arbitrary coefficients at this point. To define (d_1, d_{-1}) uniquely, we impose the orthogonality constraints $\langle e^{\pm i\cdot}, W_{1,2}\rangle_{L^2_{per}} = 0$ on the correction terms of the expansion (2.32). Collecting terms in powers of a, we obtain

$$\mathcal{O}(a): \qquad P_0W_1 + P_1W_0 = \lambda_1 W_0, \tag{2.34}$$

$$\mathcal{O}(a^2): \qquad P_0W_2 + P_1W_1 + P_2W_0 = \lambda_1 W_1 + \lambda_2 W_0. \tag{2.35}$$

By solving (2.34) under the orthogonality constraints $\langle e^{\pm i\cdot}, W_1\rangle_{L^2_{per}} = 0$, we obtain the unique choice $\lambda_1 = 0$ and the unique solution

$$W_1 = d_1 e^{2ix} - d_1 - d_{-1} + d_{-1} e^{-2ix}. \tag{2.36}$$

To solve (2.35), one needs first to remove the resonant terms e^{ix} and e^{-ix} by equaling the corresponding coefficients to zero. This is done by the two equations

$$\begin{cases} e^{ix}: & 2d_1 + 2d_{-1} = \lambda_2 d_1, \\ e^{-ix}: & 2d_1 + 2d_{-1} = \lambda_2 d_{-1}. \end{cases} \tag{2.37}$$

If λ_2 and (d_1, d_{-1}) satisfy (2.37), a unique solution to the linear equation (2.35) exists for W_2 under the orthogonality constraints $\langle e^{\pm i\cdot}, W_2 \rangle_{L^2_{per}} = 0$. The linear system (2.37) can be viewed as the matrix eigenvalue problem for eigenvalues λ_2 and eigenvectors (d_1, d_{-1}):

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_{-1} \end{bmatrix} = \lambda_2 \begin{bmatrix} d_1 \\ d_{-1} \end{bmatrix}.$$

The two eigenvalues are $\lambda_2 = 0$ and $\lambda_2 = 4$. The first zero eigenvalue corresponds to the zero eigenvalue $\lambda = 0$ of the eigenvalue problem (2.31) with the translational mode $W(x) = \partial_x v(x)$ by Proposition 2.7. The second positive eigenvalue corresponds to the small positive eigenvalue λ, which remains positive for small a by the perturbation theory arguments. ∎

The spectrum of $K_{c,b}$ considered in $L^2(\mathbb{R})$ is purely continuous and consists of bands that may be separated from each other by gaps. One band touches zero because $K_{c,b}\partial_x v = 0$ for every $c > 1$ and $b \in \mathbb{R}$. To prove positivity of $K_{c,b}$ in L^2_{per}, we need to show that this band is non-negative and all other bands are strictly positive.

In order to overcome the difficulty of working with the continuous spectrum of $K_{c,b}$ in $L^2(\mathbb{R})$, we use the Bloch decomposition, based on Floquet theory, and consider the operator

$$\begin{aligned} P_{c,b}(k) &= e^{-ikx}K_{c,b}e^{ikx} \\ &= (\partial_x + ik)^4 + 10(\partial_x + ik)v(x)(\partial_x + k) - 10cv(x) - c^2 \\ &\quad +b\left[(\partial_x + ik)^2 + c + 6v(x)\right], \end{aligned}$$

where $k \in \mathbb{T} := [-1/2, 1/2]$ is the Bloch quasi-momentum. The spectrum of $P_{c,b}(k)$ can now be defined in L^2_{per}, where it is purely discrete and consists of isolated eigenvalues of finite algebraic multiplicity. We are now investigating eigenvalues of $P_{c,b}(k)$ near $c = 1$, $b = 2$, and $k = 0$ by the perturbation theory, The following result derives the constraint on $|b - 2|$ under which the operator $P_{c,b}(k)$ is positive near $c = 1$.

Proposition 2.9 *Consider the spectrum of $P_{c,b}(k)$ in L^2_{per} for $c > 1$ such that $|c - 1|$ is sufficiently small. For every $b \in (b_-, b_+)$, where $b_\pm$ are given by the asymptotic expansion*

$$b_\pm = 2 \pm 2a + \mathcal{O}(a^2), \tag{2.38}$$

all eigenvalues of the operator $P_{c,b}(k)$ for $k \in \mathbb{T}$ are strictly positive except for the simple zero eigenvalue that corresponds to $P_{c,b}(0)\partial_x v = K_{c,b}\partial_x v = 0$.

Proof. We construct perturbation expansions to solutions of

$$P_{c,b}(k)W = \lambda W, \quad W \in H^4_{per}, \tag{2.39}$$

near $c = 1$, $b = 2$, $k = 0$, and $\lambda = 0$. The other eigenvalues of $K_{c=1,b=2}(0)$ are strictly positive and remain so for $c > 1$ and $|c - 1|$ sufficiently small, by the perturbation theory. To simplify the presentation, we introduce the formal scaling

$$b = 2 + a\beta, \quad k = a^2\kappa \tag{2.40}$$

and obtain expansions for a-independent β and κ. Justification of such expansions can be developed in the same way as it was done in [9], [18].

By Proposition 2.1, we expand $P_{c,b}(k)$ with the scaling (2.40) as

$$P_{c,b}(k) = P_0 + a(P_1 + \beta Q_1 + \kappa R_1) + a^2(P_2 + \beta Q_2 + \kappa R_2 + \beta\kappa S_2 + \kappa^2 T_2) + \mathcal{O}(a^3),$$

where P_0, P_1, and P_2 are the same as in the expansions of Proposition 2.8, whereas

$$\begin{aligned}
Q_1 &= \partial_x^2 + 1, \\
R_1 &= 4i\partial_x(\partial_x^2 + 1), \\
Q_2 &= 6\cos(x), \\
R_2 &= 10i\left(\cos(x)\partial_x + \partial_x\cos(x)\right), \\
S_2 &= 2i\partial_x, \\
T_2 &= -2(1 + 3\partial_x^2).
\end{aligned}$$

By using expansions (2.32) with the leading order (2.33), we collect the powers of a and obtain in the order of $\mathcal{O}(a)$ that $\lambda_1 = 0$ and W_1 is given by the same expression (2.36). At the order of $\mathcal{O}(a^2)$, we remove the resonant terms e^{ix} and e^{-ix} by equaling the corresponding coefficients to zero, according to the following equations

$$\begin{cases} e^{ix}: & 2d_1 + 2d_{-1} - 2\kappa\beta d_1 + 4\kappa^2 d_1 = \lambda_2 d_1, \\ e^{-ix}: & 2d_1 + 2d_{-1} + 2\kappa\beta d_{-1} + 4\kappa^2 d_{-1} = \lambda_2 d_{-1}. \end{cases}$$

The two constraints can be viewed as the matrix eigenvalue problem for eigenvalues λ_2 and eigenvectors (d_1, d_{-1}):

$$\begin{bmatrix} 2 - 2\kappa\beta + 4\kappa^2 & 2 \\ 2 & 2 + 2\kappa\beta + 4\kappa^2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_{-1} \end{bmatrix} = \lambda_2 \begin{bmatrix} d_1 \\ d_{-1} \end{bmatrix}.$$

The two eigenvalues are

$$\lambda_2 = 2 + 4\kappa^2 \pm 2\sqrt{1 + \beta^2\kappa^2}.$$

The plus sign gives a strictly positive eigenvalue, whereas the minus sign gives the positive eigenvalue if $\sqrt{1 + \beta^2\kappa^2} \leqslant 1 + 2\kappa^2$, that is, if $\beta^2 \leqslant 4$. This computation and the perturbation theory argument proves the assertion of the proposition. ■

2.4 Exact results

Here we collect together the exact results on the existence of periodic waves v and the positivity of the Hessian operator $K_{c,b}$. Although these results are available in the literature, e.g. [7], the method used to prove positivity of $K_{c,b}$ is indirect and computationally challenging. We will verify, however, that the exact results produce in the small-amplitude limit the same outcomes as the conditions obtained in Propositions 2.1 and 2.9.

The following result gives the exact expression for the periodic wave v satisfying the second-order differential equation (2.2).

Proposition 2.10 *For every $c > 1$, the 2π-periodic even solution v to the differential equation* (2.2) *is expressed by the Jacobi elliptic function* cn *in the form*

$$v(x) = \frac{2K^2(k)}{3\pi^2}\left[1 - 2k^2 - \sqrt{1-k^2+k^4} + 3k^2\mathrm{cn}^2\left(\frac{K(k)}{\pi}x; k\right)\right], \tag{2.41}$$

where $K(k)$ is the complete elliptic integral of the first kind and the elliptic modulus parameter $k \in (0,1)$ defines uniquely the speed parameter c in the form

$$c = \frac{4K^2(k)}{\pi^2}\sqrt{1-k^2+k^4}. \tag{2.42}$$

Proof. Recall from [10] that the function

$$u(\xi) := 2k^2\mathrm{cn}^2(\xi; k), \quad k \in (0,1) \tag{2.43}$$

is a $2K(k)$-periodic even solution to the following second-order differential equation:

$$\frac{d^2u}{d\xi} + 4(1-2k^2)u + 3u^2 = 4k^2(1-k^2). \tag{2.44}$$

We will connect the differential equation (2.44) with the original equation (2.2). By using the transformation

$$v(x) = \frac{K^2(k)}{\pi^2}\left[A + u\left(\frac{K(k)}{\pi}x\right)\right], \quad \xi = \frac{K(k)}{\pi}x, \tag{2.45}$$

where A is not defined yet, we realize that v is a 2π-periodic even solution to the differential equation

$$\begin{aligned}\frac{\pi^4}{K(k)^4}\left[\frac{d^2v}{dx^2} + 3v^2\right] + \frac{\pi^2}{K(k)^2}\left[4(1-2k^2) - 6A\right]v \\ = 4k^2(1-k^2) + 4A(1-2k^2) - 3A^2.\end{aligned} \tag{2.46}$$

By comparing (2.46) and (2.2), we define A from the quadratic equation

$$4k^2(1-k^2) + 4A(1-2k^2) - 3A^2 = 0, \tag{2.47}$$

and the speed parameter c from the condition

$$c = \frac{K^2(k)}{\pi^2}\left[4(1-2k^2) - 6A\right]. \tag{2.48}$$

We shall solve (2.47) with the condition that $A = 0$ for $k = 0$, which ensures that $v \to 0$ in the small-amplitude limit $k \to 0$. The unique root of the quadratic equation (2.47) that satisfies these requirements is given by

$$A = \frac{2}{3}\left[1 - 2k^2 - \sqrt{1 - k^2 + k^4}\right]. \tag{2.49}$$

Substituting (2.49) into (2.43) and (2.45) yields (2.41). Substituting (2.49) into (2.48) yields (2.42). ■

Let us show that the exact result of Proposition 2.10 reduces to the asymptotic result of Proposition 2.1 as $k \to 0$. Recall the asymptotic expansion from [10]:

$$K(k) = \frac{\pi}{2}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \mathcal{O}(k^6)\right]$$

and

$$\mathrm{cn}(\xi;k) = \frac{2\pi}{kK(k)}\left[\frac{q^{1/2}}{1+q}\cos\left(\frac{\pi\xi}{2K(k)}\right) + \frac{q^{3/2}}{1+q}\cos\left(\frac{3\pi\xi}{2K(k)}\right) + \mathcal{O}(q^{5/2})\right]$$

where

$$q := e^{-\frac{K'(k)\pi}{K(k)}} = \frac{1}{16}k^2 + \frac{1}{32}k^4 + \mathcal{O}(k^6).$$

Substituting these expansions to (2.41) and expanding the result in powers of k^2, we obtain

$$v(x) \;=\; \frac{1}{4}k^2\cos(x) + \frac{1}{32}k^4\left[\cos(2x) - 3\right] + \frac{1}{8}k^4\cos(x) + \mathcal{O}(k^6).$$

If the wave amplitude a is defined by $a = \frac{k^2}{4} + \mathcal{O}(k^4)$, the expansion above is equivalent to the asymptotic expansion (2.3) in Proposition 2.1.

Similarly, we expand (2.42) in powers of k^2 and obtain

$$\begin{aligned} c &= \left(1+\frac{1}{2}k^2+\frac{11}{32}k^4+\mathcal{O}(k^6)\right)\left(1-\frac{1}{2}k^2+\frac{3}{8}k^4+\mathcal{O}(k^6)\right) \\ &= 1+\frac{15}{32}k^4+\mathcal{O}(k^6), \end{aligned}$$

which yields the asymptotic expansion (2.4) if $a=\frac{k^2}{4}+\mathcal{O}(k^4)$.

The following result gives the exact expression on the end points b_+ and b_- that depend on the wave speed $c>1$ such that the operator $K_{c,b}$ defined by (2.30) is positive in $L^2(\mathbb{R})$ for $b\in(b_-,b_+)$. The exact expression is obtained in [7] by using an indirect and computationally challenging method.

Proposition 2.11 *For every $c>1$, the operator $K_{c,b}$ considered in $L^2(\mathbb{R})$ is positive except for the translation mode $\partial_x v$ if $b\in(b_-,b_+)$, where*

$$b_- := \left(\frac{5}{3}+\frac{1-2k^2}{3\sqrt{1-k^2+k^4}}\right)c, \quad b_+ := \left(\frac{5}{3}+\frac{1+k^2}{3\sqrt{1-k^2+k^4}}\right)c, \tag{2.50}$$

where $k\in(0,1)$ is the elliptic modulus of the exact expression (2.41) for the periodic wave v and the speed c is related to k by the expression (2.42).

Proof. The function u in (2.43) that solves the differential equation (2.44) is a critical point of the energy function

$$\tilde{S}(u)=\frac{1}{2}\int_{-K(k)}^{K(k)}\left[\left(\frac{du}{d\xi}\right)^2-2u^3-4(1-2k^2)u^2+8k^2(1-k^2)u\right]d\xi.$$

By using the scaling transformation from u to v given by (2.45), (2.48), and (2.49), we transform $\tilde{S}(u)$ to the energy function $S_c(v)=E(v)-cQ(v)$, where Q

and E are given by (2.22) and (2.23) respectively,

$$\tilde{S}(u) = \frac{\pi^5}{2K(k)^5} S_c(v) + 2A(k)\left(1 - 2k^2 - A(k)\right) \int_{-K(k)}^{K(k)} d\xi$$

Since the last term is independent of v, the Euler–Lagrange equation for $S_c(v)$ recovers the differential equation (2.2) for the 2π-periodic wave v.

Similarly, we define the higher-order energy function for the function u in (2.43) in the form used in [7]:

$$\begin{aligned} \tilde{R}(u) &= \frac{1}{2}\int_{-K(k)}^{K(k)} \left[\left(\frac{d^2u}{d\xi^2}\right)^2 - 10u\left(\frac{du}{d\xi}\right)^2 + 5u^4 - (16 - 56k^2 + 56k^4)u^2 \right] d\xi \\ &\quad + c_{21}\tilde{S}(u), \end{aligned}$$

where c_{21} is a numerical constant. It was defined in [7] that the higher-order energy function $\tilde{R}(u)$ is positive except for the translation mode $u'(\xi)$ if

$$4(3k^2 - 2) < c_{21} < 4(4k^2 - 2). \tag{2.51}$$

By using the scaling transformation from u to v given by (2.45), (2.48), and (2.49), we transform $\tilde{R}(u)$ to the linear combination of energy functions $R_c(v) = H(v) - c^2 Q(v) + I_c C(u)$ and $S_c(v) = E(v) - cQ(v)$, where Q, E, C, and H are given by (2.22), (2.23), (2.21), and (2.26) respectively. After transformation, we obtain

$$\begin{aligned}\tilde{R}(u) &= \frac{\pi^7}{2K(k)^7}R_c(v) + \frac{\pi^5}{2K(k)^5}\left(c_{21} + 10A\right)S_c(v) \\ &+\frac{1}{2}\int_{-K(k)}^{K(k)}\left[5A^4 - A^2\left(16 - 56k^2 + 56k^4\right) + 4A\left(1 - 2k^2 - A\right)\right]d\xi.\end{aligned}$$

Again, the Euler–Lagrange equation for $R_c(v)$ and $S_c(v)$ recovers the differential equation (2.2) for the 2π-periodic wave v, whereas the last term is independent of v. By comparing the expressions for $\tilde{R}(u)$ and $R_c(v) - bS_c(v)$, we obtain the following relationship between b and c_{21}

$$b = -\frac{K(k)^2}{\pi^2}\left(c_{21} + 10A\right),$$

where A is defined by (2.49). We use the constraint (2.51) on c_{21} and arrive to the constraint on b given by

$$\frac{4K(k)^2}{3\pi^2}\left(5\sqrt{1-k^2+k^4} + 1 - 2k^2\right) < b < \frac{4K(k)^2}{3\pi^2}\left(5\sqrt{1-k^2+k^4} + 1 + k^2\right).$$

Substituting c from (2.42), we arrive to the interval (b_-, b_+), where b_- and b_+ are defined by (2.50). ■

Let us show that the exact result of Proposition 2.11 reduces to the asymptotic result of Proposition 2.9 as $k \to 0$. Since $c = 1 + \mathcal{O}(k^4)$, we obtain from (2.50),

$$b_\pm = 2 \pm \frac{1}{2}k^2 + \mathcal{O}(k^4),$$

and since $a = \frac{1}{4}k^2 + \mathcal{O}(k^4)$, the expansion yields $b_\pm = 2 \pm 2a + \mathcal{O}(a^2)$, which coincides with the asymptotic expansion (2.38).

Chapter 3

Proof of transverse stability of the periodic waves

Following a standard terminology, for a closed linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ in a Hilbert space $\mathcal{H}$, we denote by $\sigma_s(A)$, $\sigma_c(A)$, and $\sigma_u(A)$, the subsets of the spectrum $\sigma(A)$ lying in the open left-half complex plane, on the imaginary axis, and in the open right-half complex plane, respectively:

$$
\begin{aligned}
\sigma_s(A) &= \{\lambda \in \sigma(A)\ ;\ \operatorname{Re}\lambda < 0\},\\
\sigma_c(A) &= \{\lambda \in \sigma(A)\ ;\ \operatorname{Re}\lambda = 0\},\\
\sigma_u(A) &= \{\lambda \in \sigma(A)\ ;\ \operatorname{Re}\lambda > 0\}.
\end{aligned}
$$

The spectral sets are referred to as the stable, central, and unstable spectra of A, respectively.

Let us consider closed linear operators J, L, and K acting in $\mathcal{H}$ with the following properties:

(i) J is a skew-adjoint operator ($J^* = -J$) with bounded inverse.

(ii) L and K are self-adjoint operators ($L^* = L$ and $K^* = K$) such that the operators JL and JK commute, i.e., the operators $(JL)(JK)$ and $(JK)(JL)$ have the same domain $\mathcal{D} \subset \mathcal{H}$, and

$$(JL)(JK)u = (JK)(JL)u, \quad \forall\ u \in \mathcal{D}. \tag{3.1}$$

(iii) The spectrum $\sigma_u(JL) \cup \sigma_s(JL)$ of the operator JL consists, at most, of isolated eigenvalues with finite algebraic multiplicities, and the generalized eigenvectors associated to these eigenvalues belong to the domain of the operator JK.

The following lemma states that the quadratic form associated with K is identically zero for every invariant subspace of JL associated with $\sigma_u(JL) \cup \sigma_s(JL)$.

Lemma 3.1 *Suppose assumptions (i)–(iii) are satisfied. If $\lambda_0 \in \sigma_u(JL) \cup \sigma_s(JL)$, then*

$$\langle K\varphi_0, \varphi_0\rangle = 0 \quad \text{for every} \ \ \varphi_0 \in \ker(JL - \lambda_0 I). \tag{3.2}$$

Proof. It is sufficient to show the statement (3.2) for a simple eigenvalue λ_0. There exists $\varphi_0 \in D(JL)$ so that $JL\varphi_0 = \lambda_0\varphi_0$. By using (3.1) and assuming $\varphi_0 \in D(JK)$, we obtain

$$\lambda_0 JK\varphi_0 = JKJL\varphi_0 = JLJK\varphi_0$$

By the assumption that J^{-1} is bounded, we then have

$$LJK\varphi_0 = \lambda_0 K\varphi_0,$$

from which we obtain

$$\langle LJK\varphi_0, \varphi_0\rangle = \lambda_0\langle K\varphi_0, \varphi_0\rangle \tag{3.3}$$

Since operator J is skew-adjoint and operator K is self-adjoint, we also obtain

$$\langle LJK\varphi_0, \varphi_0\rangle = -\langle K\varphi_0, JL\varphi_0\rangle = -\bar{\lambda}_0\langle K\varphi_0, \varphi_0\rangle \tag{3.4}$$

Combining (3.3) and (3.4) together, we obtain,

$$\left(\lambda_0 + \bar{\lambda}_0\right)\langle K\varphi_0, \varphi_0\rangle = 0$$

Since $\mathrm{Re}(\lambda_0) \neq 0$ if $\lambda_0 \in \sigma_u(JL) \cup \sigma_s(JL)$, we finally obtain (3.2). ■

The following theorem gives the main tool for the proof of transverse stability of periodic waves in the KP-II equation (1.1).

Theorem 3 *Let assumptions (i)–(iii) be satisfied. Assume that K is positive and the finite-dimensional kernel of K is contained in the kernel of JL. Then, $\sigma_u(JL) \cup \sigma_s(JL)$ is empty.*

Proof. Let $E_u \cup E_s$ be the spectral subspace of $\mathcal{H}$ associated to $\sigma_u(JL) \cup \sigma_s(JL)$. From equation (3.1), we obtain $\langle Ku, u\rangle = 0$ for every $u \in E_u \cup E_s$. Since K is positive and $\ker(K)$ is finite-dimensional, this implies that $Ku = 0$ for every $u \in E_u \cup E_s$, so that $u \in ker(K) \subset ker(JL)$. However, this would give a contradiction $JLu = 0$, $u \in E_c$. Hence, $E_u \cup E_s$ is empty. ■

3.1 Proof of Theorem 1

To adopt Theorem 3 to the proof of Theorem 1, we need to compute the self-adjoint operators L and K and the skew-adjoint operator J in Assumptions (i)–(iii) in connection to the transverse stability problem for the periodic waves in the KP-II equation (1.1). As is explained in the context of the KP-II equation, the operators J and L are posed in $\dot{L}^2_{per}(0, 2\pi)$ and have the form (1.8), which is rewritten again as

$$J = \partial_x, \quad D(J) = H^1_{per}(0, 2\pi) \tag{3.5}$$

and

$$L_{c,p} = -\partial_x^2 - c - 6v(x) + p^2\partial_x^{-2}, \quad D(L_{c,p}) = H^2_{per}(0, 2\pi). \tag{3.6}$$

Proposition 3.2 *The operator $J = \partial_x$ is invertible in $\dot{L}^2_{per}(0, 2\pi)$ with a bounded inverse.*

Proof. Let us represent $f \in \dot{L}^2_{per}(0, 2\pi)$ by the Fourier series

$$f(x) = \sum_{n \in \mathbb{Z}\backslash\{0\}} c_n e^{inx},$$

where the coefficients are square summable. Since $\ker(J)$ is empty in $\dot{L}^2_{per}(0, 2\pi)$, the inverse operator $J^{-1} : \dot{L}^2_{per}(0, 2\pi) \mapsto \dot{L}^2_{per}(0, 2\pi)$ exists. The unique solution of $Ju = f$ for $u \in H^1_{per}(0,2\pi) \subset \dot{L}^2_{per}(0,2\pi)$ is given explicitly as

$$u(x) = \sum_{n \in \mathbb{Z}\backslash\{0\}} \frac{c_n}{in} e^{inx}.$$

Hence, the bound on the inverse operator is obtained by the Parseval's identity:

$$\|u\|^2_{L^2_{per}} = 2\pi \sum_{n \in \mathbb{Z}\backslash\{0\}} \frac{|c_n|^2}{n^2} \leqslant 2\pi \sum_{n \in \mathbb{Z}\backslash\{0\}} |c_n|^2 = \|f\|^2_{L^2_{per}},$$

so that $\|u\|_{L^2_{per}} \leqslant \|f\|_{L^2_{per}}$. ■

After applying J^{-1} from Proposition 3.2 to the commutativity condition (3.1) (meaning that we integrate (3.1) in x with the zero mean in $\dot{L}^2_{per}(0,2\pi)$), we rewrite the commutativity condition in the form

$$L_{c,p} \partial_x M_{c,p} = M_{c,p} \partial_x L_{c,p}, \tag{3.7}$$

where $M_{c,p}$ is a self-adjoint operator to be found. The following proposition presents one candidate for $M_{c,p}$.

Proposition 3.3 *The self-adjoint operator*

$$M_{c,p} = \partial_x^4 + 10\partial_x v(x) \partial_x - 10cv(x) - c^2 + \frac{5}{3} p^2 \left(1 + c\partial_x^{-2}\right), \tag{3.8}$$

with the domain $D(M_{c,p}) = H^4_{per}(0,2\pi)$ satisfies (3.7) in $\dot{L}^2_{per}(0,2\pi)$ with J and $L_{c,p}$ given by (3.5) and (3.6).

Proof. For the purpose of symbolic computation, we can write

$$L_{c,p} = L_{\rm KdV} + p^2 L_{\rm KP}, \tag{3.9}$$

with

$$L_{\rm KdV} = -\partial_x^2 - c - 6v(x), \quad L_{\rm KP} = \partial_x^{-2}.$$

Similarly, we represent the operator $M_{c,p}$ in the form

$$M_{c,p} = M_{\rm KdV} + p^2 M_{\rm KP}, \tag{3.10}$$

with

$$M_{\rm KdV} = \partial_x^4 + 10\partial_x v(x)\partial_x - 10cv(x) - c^2$$

since $M_{\rm KdV}$ follows from the operator obtained in (2.29). The part $M_{\rm KP}$ is to be found. By straightforward calculation, one can check that the commutativity relation (3.7) is satisfied at the order $\mathcal{O}(p^0)$:

$$L_{\rm KdV}\partial_x M_{\rm KdV} - M_{\rm KdV}\partial_x L_{\rm KdV} = 0. \tag{3.11}$$

Indeed, the relation (3.11) follows from the fact that the energy functions E and H in (2.23) and (2.26) define two commuting Hamiltonian flows related to the KdV equation (2.1), for which $L_{\rm KdV}$ and $M_{\rm KdV}$ are Hessian operators at the periodic wave v.

Now, we proceed at the order $\mathcal{O}(p^2)$,

$$L_{\mathrm{KdV}}\partial_x M_{\mathrm{KP}} - M_{\mathrm{KP}}\partial_x L_{\mathrm{KdV}} + L_{\mathrm{KP}}\partial_x M_{\mathrm{KdV}} - M_{\mathrm{KdV}}\partial_x L_{\mathrm{KP}} = 0 \tag{3.12}$$

where

$$\begin{aligned} L_{\mathrm{KP}}\partial_x M_{\mathrm{KdV}} - M_{\mathrm{KdV}}\partial_x L_{\mathrm{KP}} &= \partial_x^{-1} M_{\mathrm{KdV}} - M_{\mathrm{KdV}}\partial_x^{-1} \\ &= 10v(x)\partial_x - 10c\partial_x^{-1}v(x) - 10\partial_x v(x) + 10cv(x)\partial_x^{-1} \\ &= 10c\left(v(x)\partial_x^{-1} - \partial_x^{-1}v(x)\right) - 10v'(x) \end{aligned}$$

By using symbolic computation, we find that

$$M_{\mathrm{KP}} = \frac{5}{3}(1 + c\partial_x^{-2}) \tag{3.13}$$

satisfies

$$L_{\mathrm{KdV}}\partial_x M_{\mathrm{KP}} - M_{\mathrm{KP}}\partial_x L_{\mathrm{KdV}} = 10(v' + c\partial_x^{-1}v - cv\partial_x^{-1})$$

Since L_{KP} and M_{KP} are operators with constant coefficients, then the commutativity relation (3.7) is satisfied at the order $\mathcal{O}(p^4)$:

$$L_{\mathrm{KP}}\partial_x M_{\mathrm{KP}} - M_{\mathrm{KP}}\partial_x L_{\mathrm{KP}} = 0.$$

Hence, the commutability condition (3.7) is satisfied at all orders in p^2 with the operator $M_{c,p}$ given by (3.8). ■

Remark 3.4 *Since $v = 0$ for $c = 1$ by Proposition 2.1, we simplify operators*

$L_{c,p}$ and $M_{c,p}$ for $c = 1$:

$$L_{c=1,p} = -\partial_x^2 - 1 + p^2 \partial_x^{-2}.$$

and

$$M_{c=1,p} = \partial_x^4 - 1 + \frac{5}{3} p^2 (1 + \partial_x^{-2}).$$

Since the operators have constant coefficients, we can explicitly compute the spectrum of these operators in $\dot{L}^2_{per}(0, 2\pi)$:

$$\sigma(L_{c=1,p}) = \left\{k^2 - 1 - p^2 k^{-2}, \quad k \in \mathbb{N}\right\}$$

and

$$\sigma(M_{c=1,p}) = \left\{k^4 - 1 + \frac{5}{3} p^2 (1 - k^{-2}), \quad k \in \mathbb{N}\right\}.$$

Neither $L_{c=1,p}$ nor $M_{c=1,p}$ is positive if $p \neq 0$. However, the linear combination $M_{c=1,p} - 2L_{c=1,p}$ is positive because the spectrum is

$$\sigma(M_{c=1,p} - 2L_{c=1,p}) = \left\{(k^2 - 1)^2 + \frac{5}{3} p^2 + \frac{1}{3} p^2 k^{-2}, \quad k \in \mathbb{N}\right\}. \tag{3.14}$$

We can now complete the proof of Theorem 1.

Proof of Theorem 1. We construct a linear combination of the operators $M_{c,p}$ and $L_{c,p}$.

$$K_{c,p,b} = M_{c,p} - b L_{c,p},$$

where $b \in \mathbb{R}$ is to be defined. We can rewrite explicitly

$$K_{c,p,b} = K_{c,0,p} + \frac{5}{3}p^2 - \left(b - \frac{5}{3}c\right) p^2 \partial_x^{-2}. \tag{3.15}$$

By Proposition 2.11, the operator $K_{c,0,p}$ is positive in $L^2_{per}(0, 2\pi N)$ for every $b \in (b_-, b_+)$ defined by (2.50) and for every $N \in \mathbb{N}$. On the other hand, the last term in (3.15) is positive for every $b > \frac{5}{3}c$. Since $2c \in (b_-, b_+)$ and $2c > \frac{5}{3}c$, we can fix $b = 2c$ for every $c > 1$. Hence, the operator $K_{c,p,b=2c}$ is positive for every $c > 1$ and $p \in \mathbb{R}$. When $p \neq 0$, $\ker(K_{c,p,b=2c})$ is trivial in $L^2_{per}(0, 2\pi N)$ for every $N \in \mathbb{N}$. By Theorem 3, the spectrum $\sigma_u(JL_{c,p}) \cup \sigma_s(JL_{c,p})$ in $\dot{L}^2_{per}(0, 2\pi N)$ is empty for every $c > 1$, $N \in \mathbb{N}$, and $p \neq 0$. Theorem 1 is proven. ■

Remark 3.5 *For $p = 0$, the operator $K_{c,p=0,b=2c}$ has a one-dimensional kernel in $L^2_{per}(0, 2\pi N)$ spanned by $\partial_x v$, which coincides with the kernel of $JL_{c,p=0}$ for every $c > 1$ and $N \in \mathbb{N}$. By Theorem 3, the spectrum $\sigma_u(JL_{c,p=0}) \cup \sigma_s(JL_{c,p=0})$ in $L^2_{per}(0, 2\pi N)$ is empty for every $c > 1$ and $N \in \mathbb{N}$. This result coincides with the one considered in [3, 7] for the KdV equation (1.2).*

3.2 Proof of Theorem 2

Thanks to Proposition 2.11 and the representation (3.15), we have coercivity of the quadratic form associated with the operator $K_{c,p,b=2c}$ for every $c > 1$ and $p \neq 0$. The quadratic form is defined in space $H^2_{per}(0, 2\pi N) \cap \dot{L}^2_{per}(0, 2\pi N)$ for every $N \in \mathbb{N}$ due to the definition of the operators $L_{c,p}$ and $M_{c,p}$ in (3.6) and (3.8). Hence, we state the following proposition without proof.

Proposition 3.6 *For every $c > 1$, $N \in \mathbb{N}$, and $p \neq 0$, there exists a positive constant $C_{c,N,p}$ such that*

$$\langle K_{c,p,b=2c} W, W \rangle \geqslant C_{c,N,p} \|W\|^2_{H^2_{per}}, \quad \forall\, W \in H^2_{per}(0, 2\pi N) \cap \dot{L}^2_{per}(0, 2\pi N).$$

In order to prove Theorem 2, we extend definitions of operators $L_{c,p}$ and $M_{c,p}$ to the two-dimensional double-perturbation perturbations defined in space $\dot{L}^2(N, p)$, which is the space of locally square-integrable functions on $\mathbb{R}^2$ which are $2\pi N$-periodic and have zero mean in x, for some $N \in \mathbb{N}$, and are $2\pi/p$-periodic in y, for some fixed wave number p. In other words, we define

$$L_c = -\partial_x^2 - c - 6v(x) - \partial_x^{-2}\partial_y^2, \tag{3.16}$$

and

$$M_c = \partial_x^4 + 10\partial_x v(x)\partial_x - 10cv(x) - c^2 - \frac{5}{3}\left(1 + c\partial_x^{-2}\right)\partial_y^2, \tag{3.17}$$

with the domains $D(L_c) = H^{2,2}(N, p)$ and $D(M_c) = H^{4,2}(N, p)$ respectively. We also define

$$K_c := M_c - 2cL_c, \tag{3.18}$$

with the domain $D(K_c) = D(M_c) = H^{4,2}(N, p)$. Since the quadratic form generated by K_c is defined in $H^{2,1}(N, p)$, Theorem 2 is formulated and proven in space $H^{2,1}(N, p) \subset \dot{L}^2(N, p)$. The result of Proposition 3.6 is reinstated as follows.

Proposition 3.7 *For every $c > 1$, $N \in \mathbb{N}$, and $p \neq 0$, there exists a positive* *constant $C_{c,N,p}$ such that*

$$\langle K_c w, w\rangle \geqslant C_{c,N,p}\|w\|^2_{H^{2,1}(N,p)}, \quad \forall\ w \in H^{2,1}(N,p): \quad \langle w, \partial_x v\rangle = 0 \tag{3.19}$$

where the inner product is defined in $L^2(N,p)$.

Proof. The kernel of the operator K_c is one-dimensional in $\dot{L}^2(N,p)$ and is spanned by the translational mode $\partial_x v$. With the orthogonality condition $\langle w, \partial_x v\rangle = 0$ for every $w \in \dot{L}^2(N,p)$, the smallest eigenvalue of K_c in $\dot{L}^2(N,p)$ is strictly positive and we have the coercivity

$$\langle K_c w, w\rangle \geqslant C_{c,N,p}\|w\|^2_{L^2(N,p)}, \quad \forall\ w \in H^{2,1}(N,p): \quad \langle w, \partial_x v\rangle = 0.$$

Since the quadratic form is bounded in $H^{2,1}(N,p) \subset \dot{L}^2(N,p)$, Gårding's inequality further implies the coercivity (3.19) in the $H^{2,1}(N,p)$ norm. ■

We can now complete the proof of Theorem 2.

Proof of Theorem 2. Let us consider the Cauchy problem for the linear equation:

$$\begin{cases} \frac{dw}{dt} = \partial_x L_c w, & t > 0, \\ w(0) = w_0, \end{cases} \tag{3.20}$$

and assume existence of a solution $w \in C^1(\mathbb{R}, H^{2,1}(N,p))$. We set the orthogonal decomposition

$$w(t) = a(t)\partial_x v + \tilde{w}(t), \quad a(t) = \frac{\langle w(t), \partial_x v\rangle}{\|\partial_x v\|^2}, \quad \langle \tilde{w}(t), \partial_x v\rangle = 0. \tag{3.21}$$

Since the operator K is self-adjoint and $\partial_x v$ is the kernel of K, we obtain

$$\langle K_c w, w\rangle = a^2 \langle K_c \partial_x v, \partial_x v\rangle + 2a\langle K_c \partial_x v, \tilde{w}\rangle + \langle K_c \tilde{w}, \tilde{w}\rangle = \langle K_c \tilde{w}, \tilde{w}\rangle.$$

By the coercivity bound (3.19), the energy conservation (1.14) and the boundedness of the quadratic form $\langle K_c w, w\rangle$ in $H^{2,1}(N,p)$, we obtain

$$\begin{aligned} C_{c,N,p}\|\tilde{w}(t)\|^2_{H^{2,1}(N,p)} &\leqslant \langle K_c \tilde{w}(t), \tilde{w}(t)\rangle = \langle K_c w(t), w(t)\rangle \\ &= \langle K_c w_0, w_0\rangle \leqslant C_0 \|w_0\|^2_{H^{2,1}(N,p)}. \end{aligned}$$

This yields the first inequality in (1.15).

In order to prove the second inequality in (1.15), we substitute (3.21) to (3.20) and obtain

$$\frac{da}{dt}\partial_x v + \frac{d\tilde{w}}{dt} = \partial_x L_c \tilde{w}. \tag{3.22}$$

Projecting (3.22) to $\partial_x v$, we obtain

$$\frac{da}{dt} = \frac{\langle \partial_x L_c \tilde{w}, \partial_x v\rangle}{\|\partial_x v\|^2} = -\frac{\langle \tilde{w}(t), L_c \partial_x^2 v\rangle}{\|\partial_x v\|^2},$$

where we note that $L_c \partial_x^2 v \in L^2(N,p)$. Therefore, there is a positive constant C such that

$$\left|\frac{da}{dt}\right| \leqslant C\,\|\tilde{w}(t)\| \leqslant C\|\tilde{w}(t)\|_{H^{2,1}(N,p)} \leqslant C\|w_0\|_{H^{2,1}(N,p)},$$

which yields the second inequality in (1.15). Theorem 2 is proven. ∎

3.3 Energy functions for the KP-II equation

The KP-II equation (1.1) can be formulated in the Hamiltonian form

$$\frac{\partial u}{dt} = \frac{\partial}{\partial x}\mathrm{grad}\widetilde{E}(u),$$

associated with the energy

$$\widetilde{E}(u) = \frac{1}{2}\int\int \left[u_x^2 - 2u^3 - (\partial_x^{-1}u_y)^2\right] dxdy. \tag{3.23}$$

The y-independent part of $\widetilde{E}(u)$ is equivalent to $E(u)$ for the KdV equation (1.2) given by (2.23). The energy function $\widetilde{E}(u)$ is constant in t.

A higher-order energy function is constructed for the global well-posedness analysis of the KP-I equation in [29, 30]. After transforming this quantity to the variables used in the KP-II equation (1.1), it can be written in the form

$$\begin{aligned}\widetilde{H}(u) &= \frac{1}{2}\int\int \Bigg[u_{xx}^2 - 10uu_x^2 + 5u^4 - \frac{10}{3}u_y^2 + \frac{5}{9}(\partial_x^{-2}u_{yy})^2 \\ &\quad + \frac{10}{3}u^2\partial_x^{-2}u_{yy} + \frac{10}{3}u\left(\partial_x^{-1}u_y\right)^2\Bigg] dxdy.\end{aligned} \tag{3.24}$$

The y-independent part of $\widetilde{H}(u)$ is equivalent to $H(u)$ for the KdV equation (1.2) given by (2.26). The higher-order energy function $\widetilde{H}(u)$ is constant in t.

Similarly to the KdV equation, the KP-II equation (1.1) has the conserved mass $\widetilde{C}(u) = \int\int u dxdy$ and the conserved momentum $\widetilde{Q}(u) = \frac{1}{2}\int\int u^2 dxdy$.

The periodic traveling wave v with speed c is a critical point of the two variational problems

$$\widetilde{S}_c(u) = \widetilde{E}(u) - c\widetilde{Q}(u)$$

and

$$\widetilde{R}_c(u) = \widetilde{H}(u) - c^2\widetilde{Q}(u) + 2I_c\widetilde{C}(u),$$

where I_c is the same first-order invariant (2.27).

Computing the Hessian operator at the periodic wave v with speed c from the first variational problem $\widetilde{S}_c(u)$ and performing the Fourier transform in variable y with parameter p, we obtain $L_{c,p}$ given by (3.6). Repeating the same computation for $\widetilde{R}_c(u)$, we obtain the Hessian operator

$$\begin{aligned} \widetilde{M}_{c,p} &= \partial_x^4 + 10\partial_x v(x)\partial_x - 10cv(x) - c^2 \\ &\quad -\frac{10}{3}p^2\left(1 + v(x)\partial_x^{-2} + \partial_x^{-1}v(x)\partial_x^{-1} + \partial_x^{-2}v(x)\right) + \frac{5}{9}p^4\partial_x^{-4}. \end{aligned} \tag{3.25}$$

A long, but straightforward, symbolic computation shows that the commutativity condition (3.7) is satisfied with the two linear operators $L_{c,p}$ and $\widetilde{M}_{c,p}$ given by (3.6) and (3.25), respectively.

Remark 3.8 *The expression (3.25) for the operator $\widetilde{M}_{c,p}$ is different from the expression (3.8) for the operator $M_{c,p}$. The difference between these two operators,*

$$M_{c,p} - \widetilde{M}_{c,p} = \frac{5}{3}p^2\left(3 + c\partial_x^{-2} + 2v(x)\partial_x^{-2} + 2\partial_x^{-1}v(x)\partial_x^{-1} + 2\partial_x^{-2}v(x)\right) - \frac{5}{9}p^4\partial_x^{-4},$$

also satisfies the commutativity condition (3.7). Hence, the operator equation (3.7) admits multiple solutions, and the most general form for a solution $M_{c,p}$ is *unknown.*

In contrast to the operator $M_{c,p}$ given by (3.8), the operator $\widetilde{M}_{c,p}$ given by (3.25) cannot be used to construct the positive operator

$$\widetilde{K}_{c,p,b} = \widetilde{M}_{c,p} - bL_{c,p}, \tag{3.26}$$

for every $b \in \mathbb{R}$ and $p \neq 0$. The following result illustrates the nonpositivity of $\widetilde{K}_{c,p,b}$ for $c = 1$, contrary to the positivity of $K_{c=1,p,b=2}$ in Remark 3.4.

Proposition 3.9 *For $c = 1$ and $v = 0$, the spectrum of $\widetilde{K}_{c=1,p,b}$ in $\dot{L}^2_{per}(0, 2\pi)$ is given by*

$$\sigma\left(\widehat{K}_{c=1,p,b}\right) = \left\{ \left(k^2 - \frac{b}{2}\right)^2 - \left(1 - \frac{b}{2}\right)^2 + \frac{p^2(5p^2 - 30k^4 + 9bk^2)}{9k^4}, \quad k \in \mathbb{N} \right\}$$

For every $b \in \mathbb{R}$, there exist values of $p \neq 0$ such that $\sigma\left(\widehat{K}_{c=1,p,b}\right)$ includes negative eigenvalues.

Proof. Let us consider the function

$$F_b(k,p) := \left(k^2 - \frac{b}{2}\right)^2 - \left(1 - \frac{b}{2}\right)^2 + \frac{p^2(5p^2 - 30k^4 + 9bk^2)}{9k^4}.$$

Since $F_b(k,p)$ is bi-quadratic in p, it is easy to find the sets of points on the (k,p)-plane, where $F_b(k,p) = 0$. Due to the symmetry, we express the sets on the (k^2, p^2)-plane. The zero level of $F_b(k,p)$ is located along the two curves given explicitly by

$$p_{\pm}^2(k^2) = \frac{3k^2}{10}\left[10k^2 - 3b \pm \sqrt{80k^4 - 40bk^2 + 9b^2 - 20b + 20}\right]. \tag{3.27}$$

The value of $F_b(k,p)$ is negative between the curves. Both curves extend to infinity along the asymptotic behavior

$$p_{\pm}^2(k^2) = \frac{3(5 \pm 2\sqrt{5})}{5}k^4 + \mathcal{O}(k^2) \quad \text{as} \quad k^2 \to \infty.$$

Thus, independently of the value of $b \in \mathbb{R}$ and $k \in \mathbb{N}$, there exists a nonzero interval in the values of p such that $F_b(k,p) < 0$. ■

Figure 3.1 illustrates the curves (3.27) for the zero level of $F_b(k,p)$ for $b = 2.1$. The curves look similar for other values of b.

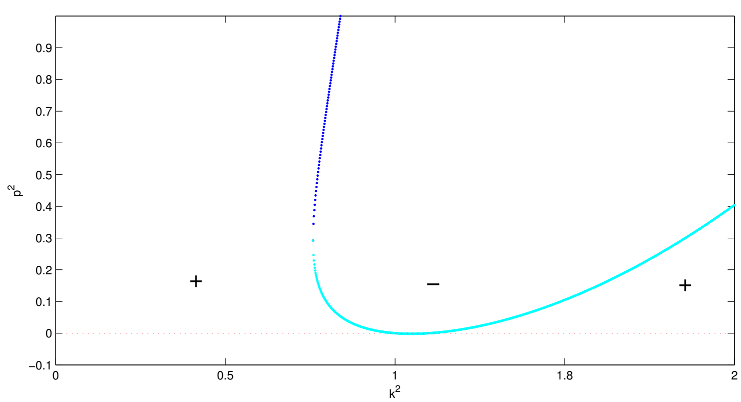

Figure 3.1: The curves (3.27) on the (k^2, p^2) plane for $b = 2.1$.

It is tempting to construct a higher-order energy functional with the linear operator $M_{c,p}$ defined by (3.8), which can also be used for a nonlinear stability proof of the periodic waves in the KP-II equation (1.1). Since the KdV part in $M_{c,p}$ is the Hessian operator for $R_c(u)$, which is constructed from the high-order energy functional $H(u)$ in (2.26), whereas the KP part in $M_{c,p}$ has constant coefficients, a higher-order energy functional can be thought in the following form

$$\widetilde{Z}(u) = \int\int \left[u_{xx}^2 - 10uu_x^2 + 5u^4 + \frac{5}{3}u_y^2 - \frac{5c}{3}(\partial_x^{-1}u_y)^2 \right] dxdy. \tag{3.28}$$

However, the function $\widetilde{Z}(u)$ has a speed parameter c in front of the last term, which is also the last term of the conserved energy $\widetilde{E}(u)$ in (3.23). Since c is an independent parameter, the quantity $\widetilde{Z}(u)$ in (3.28) is not related to a conserved quantity of the KP-II equation (1.1). Therefore, the commuting operator K_c in (3.18) is not the Hessian operator for a higher-order conserved quantity of the KP-II equation (1.1).

Summarizing, the existence of a Lyapunov functional for the KP-II equation (1.1) which could be used for a transverse nonlinear stability proof for periodic waves is not known, and this nonlinear stability problem remains open. We point out that the analytical difficulty of using the higher-order energy functional $\widetilde{H}(u)$ for a nonlinear stability proof seems to be the same as the one arising in the proof of global well-posedness of the KP-II equation in the energy space (see [12] and the references therein).

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