# $N^{th}$ ORDER SELF ADAPTING CONTROL SYSTEMS

# N$^{th}$ ORDER SELF ADAPTING CONTROL SYSTEMS

By

VICTOR A. K. TEMPLE, B.Sc.Eng.Phys.

A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree
Master of Engineering

McMaster University
October 1967

MASTER OF ENGINEERING (1967)
(Electrical Engineering)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: $N^{th}$ Order Self Adapting Control Systems

AUTHOR: Victor A. K. Temple, B.Sc.Eng.Phys. (University of Manitoba)

SUPERVISOR: N. K. Sinha, B.Sc.Eng. (Banaras), Ph.D. (Manchester)

NUMBER OF PAGES: vii , 77

SCOPE AND CONTENTS: The very sophisticated control systems of today are built around computers. It is felt that an improved form of cost function in vector or matrix form is needed to fully and most easily utilize the computer's advantages. After defining a vector cost function $\underline{L}$ , the problem of adapting and learning simplifies to the solution of a partial difference equation. Total system properties are easily defined as matrix arrays which are "learned" in an adapting and "learning" control loop.

The relative merits of open and closed loop adaptive systems were investigated. The $N^{th}$ order adaptive control system was finally chosen to be closed loop after developing two criterion equations in two unknowns which, if satisfied guaranteed improved system sensitivity with the closed loop configuration.

Finally, several simple examples are given in experiment form to demonstrate the applicability of the proposed control system techniques.

## PREFACE

The basic function of the control engineer is to make a system perform in some specified way. Usually the specifications are stated mathematically and the total or overall performance of the system measured in a mathematical expression called a cost function. Minimization of this cost function is the aim of the optimal control system.

Only too frequently, however, the control engineer is met with the problems of the very complicated system, the system whose transfer function or whose parameters can only be guessed, or the system whose optimal control cannot be found. To this add the problem of including system sensitivity as a criterion in the cost function and one has the beginning of the control engineer's problems.

Not satisfied with the optimal control (if it can be found or reasonably guessed) it may be desirable or even necessary to think in terms of an adaptive controller, a sophistication of optimal control which undertakes to change the controllers to offset changes in plant parameters.

Notwithstanding the great difficulties, many partial solutions have been made or proposed. Basically two philosophies have evolved. In one a plant model and plant identification are required together with the ability to predict plant parameter changes. The other is more direct, requiring the prediction of the system's output at the next time interval. Hill climbing is one of the more significant theories using the direct approach.

The ultimate in control systems today is the learning and adapting system. In one sense it is a system which learns how to adapt its controllers to give the optimal output at all times. In a broader sense a learning system must, in addition, be able to develop its own cost function. The "$N^{th}$ Order Self Adapting System" discussed in the text of this thesis does the former and has thus been called 'self adapting' rather than 'learning and adapting', although it <u>does</u> learn how to adapt itself.

The major and only aim of this thesis is to provide a general method for minimizing the cost function. In attempting to do so it becomes first evident that a modified form of cost function is necessary and then clear that many of the problems of optimization are neatly solved by a simple form of learning. The particular cost function chosen is in vector form and is particularly defined in order to easily and naturally include sensitivity.

## ACKNOWLEDGMENTS

I wish to thank my supervisor, Dr. Sinha, for his encouragement and my wife and typist for her patience and hard work.

## TABLE OF CONTENTS

APPENDIX

# PART I THEORETICAL

# I DEFINITION OF THE PROBLEM

Consider the following system with $\underline{O} = \underline{\underline{S}}_{op}\,\underline{R}$

$$\underline{R} \longrightarrow \boxed{\underline{\underline{S}}_{op}(\underline{\alpha}, \underline{c})} \longrightarrow \underline{O}$$

where $\underline{\underline{S}}_{op}$ is a two dimensional array of operators, including both the control mechanism and the plant

in which $J(\underline{R},\underline{O},\underline{D},t)$ is to be minimized[1]. To optimize the system with $\underline{R}$ given it is necessary to devise a suitable controller such that $\underline{\underline{S}}_{op}\underline{R}$ gives the optimal output $\underline{O}^*$ and $J^*$ the optimal cost function.

An adaptive system is desirable when a number of parameters $\underline{\alpha}$ or $\alpha_i$ of $\underline{\underline{S}}_{op}$ are prone to change or when the system is in some manner noise contaminated. The adaptive system minimizes the change in $J$ from $J^*$ by changing the controllers (see block diagram on next page). It is the essence of the self adapting system that some rationale be acquired in the adaptor to generate the suitable controller changes. In general this will be shown to involve the learning of three matrices[2] and the prediction of what can be called the plant cost velocity vector.

Chapter II will deal with the derivation of a suitable cost function for learning or self adapting

---

1. $\underline{D}(t) \triangleq$ the desired output; $\underline{R}$, the input; and $J = \int_{t_0}^{t_f} H(\underline{R},\underline{O},\underline{D},t)\,dt$, the cost function.

2. These matrices are $\partial\underline{G}/\partial\underline{c}$, $\partial\underline{G}/\partial\underline{f}$ and $\partial\underline{G}/\partial\underline{R}$ where $\underline{c}$ is the forward loop control vector, $\underline{f}$ the feedback loop control vector, and $\underline{G}$ a derived vector cost function (to be dealt with in detail later).

systems while Chapter III will outline the mathematics of an algorithm suitable for computer solution.

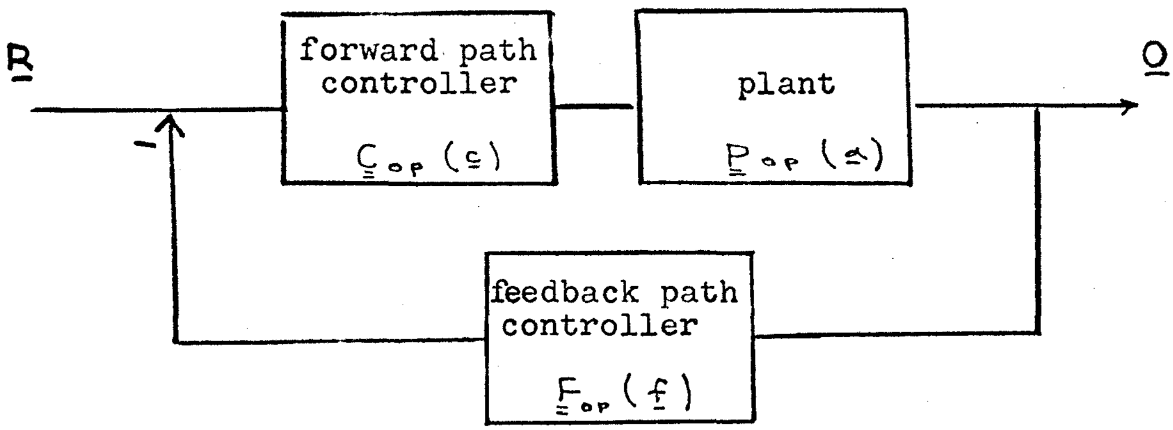



figure 1a.

## II DERIVATION OF A SUITABLE COST FUNCTION

This chapter is devoted to deriving a vector cost function suitable for the computer algorithm of Chapter IV and sufficiently general to cover almost all control problems.

### 1. The Vector Cost Function $\underline{J}$

The scalar cost function $J$ does not hold as much information about the system as desirable. First it is a scalar quantity and second it is only available as a parameter at the final time $t_f$ . Thus $J$ in this form is not of much use as a performance indicator in a system in which parameter changes are always taking place.

To obviate these shortcomings it is possible to redefine $J$ so that it has a value for all times $t$ .

$$J(t) = \int_{t_0}^{t} H(\underline{R}, \underline{D}, \underline{O}, t)\, dt \quad ^{1}$$

With $J$ so defined the quantities $\dot{J}$ , $\ddot{J}$ , .. $d^n J/dt^n$ are available (provided the derivatives of $H$ are continuous) and thus a vector $\underline{J}$ can be formed.

$$\underline{J} = [\, J_1 , J_2 , J_3 , \; \cdots \; J_N ]^T$$

where $J_1 = J(t)$ ; $J_n = \dfrac{d^{n-1} J(t)}{dt^{n-1}} \dfrac{\Delta t^{n-1}}{(n-1)!}$

The equivalent problem is now to minimize $J_1(t_f)$ by judicious adapting at intervals $\Delta t$ apart. The only way available is to minimize

---

1. $H$ is constrained to be positive.

$$\int_{t_0 + j\Delta t}^{t_0+(j+1)\Delta t} H \, dt \quad \left( \text{which is proportional to } J_2 \ \left( \triangleq \dot{J}_1 \Delta t \right) \right)$$

which in some cases does not necessarily minimize $J_1(t_f)$

$$J_1(t_f) = \int_{t_0}^{t_f} H \, dt \quad \left( \equiv J, \text{ the original cost function} \right).$$

That is, minimizing $J_2(t)$ does not necessarily minimize $J$ but merely drives the system along the path of steepest descent towards a local minimum in the $J_1$ surface. This possibility motivates the following theorem.

Theorem I

If constraints allow $J_2(t)$ to be zero for all $t \in (t_0, t_f)$ then $\| J_1(t) \|$ will remain at one of possibly many equivalent minima. Being zero, this will be a global minimum of which there will be more than one if there is more than one way to obtain $J_2 \equiv 0$.

Note that $$J_2 = \dot{J}_1 \Delta t = \int_t^{t+\Delta t} H \, dt$$

Realizing that $\underline{Q} = \underline{S}_{op} \underline{R}$, $J_1(t)$ can be written as $J_1(\underline{u})$ where $\underline{u}$ is a generalized vector composed of $\underline{R}$, $\underline{D}$, $t$ and the parameters of $\underline{S}_{op}$. Similarly $J_2 \ (\equiv \dot{J}_1 \Delta t)$ and all other derivatives of $J_1$ can be written as functions of $\underline{u}$ and its derivatives.

Picturing the $J_1(t)$ surface as a function of $\underline{u}$, that is, $J_1(\underline{u}(t))$, the shape of the $J_1 \ (\underline{u}^*(t))$ minima[1] become very important. Clearly if $\Delta\underline{u}$ is the

1. $\underline{u}^*(t)$ in this case the $\underline{u}(t)$ that gives the minimum $J_1(t)$.

maximum change from $\underline{u}^*$ in time $\Delta t$ then figure (2a) shows the ideal minimum to be prefered to that of figure (2b) which in turn is better than that of figure (2c).

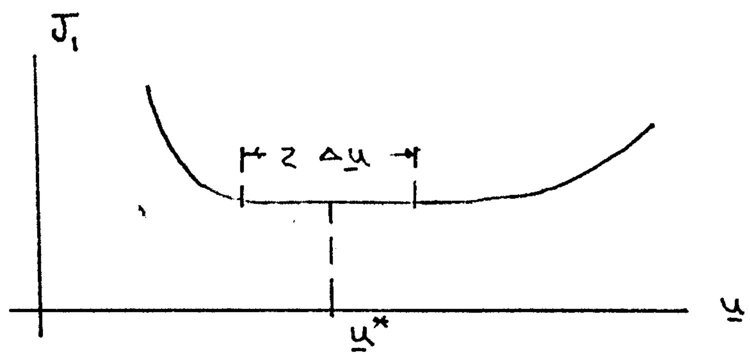



figure 2a.

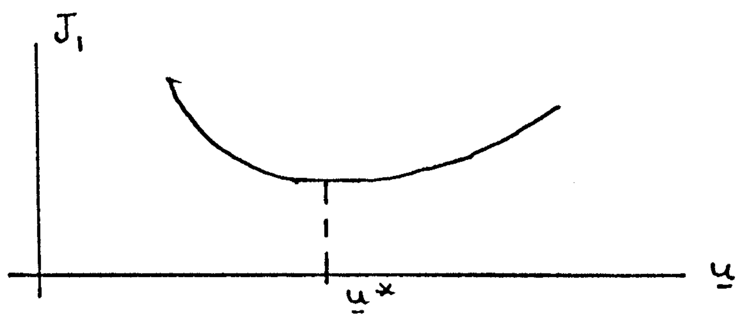



figure 2b.

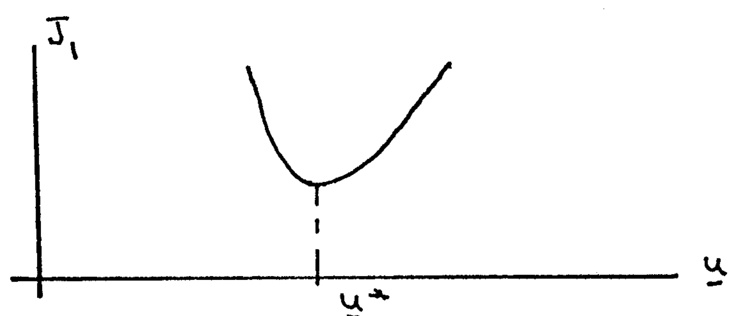
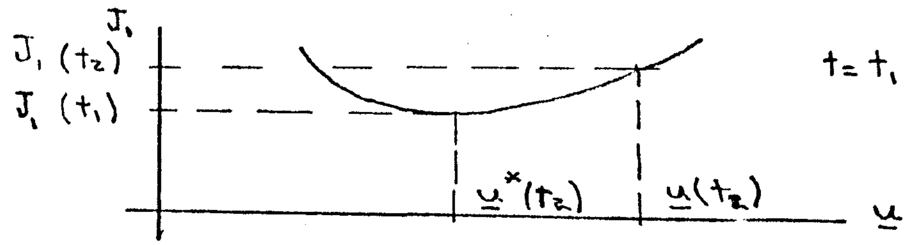



figure 2c.

Since $\underline{u}$ <u>does</u> change it is not only important to direct $J_1$ to a global minimum but also important to consider the shape of that minimum particularly in the $\Delta \underline{u}$ neighbourhood. It is here that the concept of a vector cost function $\underline{J}$ becomes useful since it is the higher derivatives of $J_1$ that indicate the shape of the $J_1$ surface.

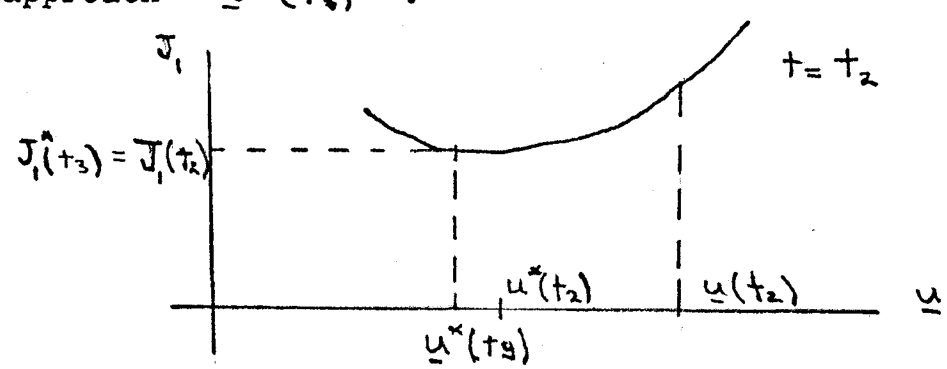
This is quickly illustrated by considering two consecutive $J_1(\underline{u})$ diagrams.




<u>figure 3a.</u>

Suppose $\underline{u}(t_1 + \Delta t) = \underline{u}(t_2)$ as above. Clearly since $J$ is additive $J_1(t_2)$ is as shown above and $\dot{J}_1 dt$ is then the same as $dJ_1/d\underline{u}\ \Delta\underline{u}$[1].

For $t_2 = t_1 + \Delta t$, the $J_1(\underline{u})$ shape may change slightly. The $J_1$ minimum will be $J_1(t_2)$ from figure 3a. at perhaps a slightly different $\underline{u}^*$ value $\underline{u}^*(t_3)$. Now the best one can hope for is that $\underline{u}(t_2 + \Delta t)$ will approach $\underline{u}^*(t_3)$.



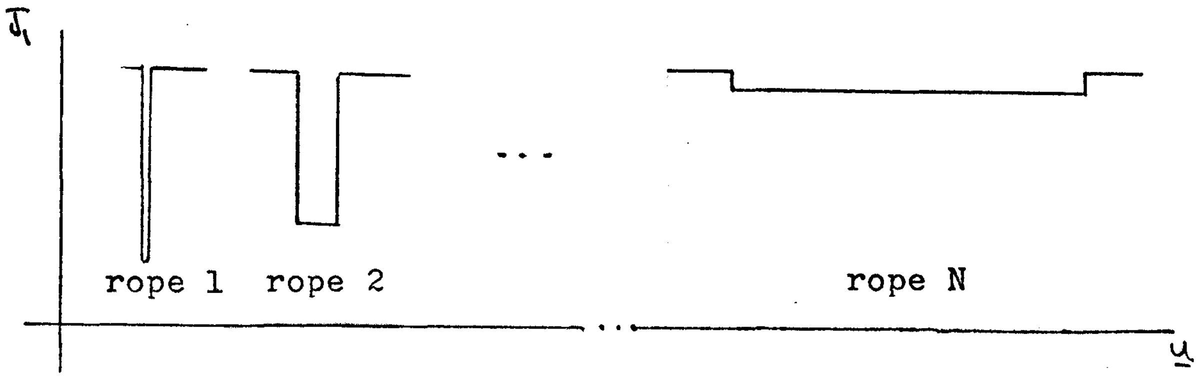

<u>figure 3b.</u>

1. This is restated in precise mathematical form in Chapter III, Section 1.

Clearly while a first order adapting system might continually attempt to set $\underline{u}(t_m+\Delta t) = \underline{u}^*(t_{m+1})$, a higher order adapting system would in addition attempt to alter the local shape of the $J_1(\underline{u})$ surface towards the ideal as exemplified in figure 2a . Mathematically this merely requires the minimization of the norm of the vector $\underline{J}$ rather than the component $J_1$ alone. An "$N^{th}$ Order Self Adapting System" will minimize the norm of an N dimensional $\underline{J}$ vector[1].

---

1. Such a method will not in general select at each step the $\underline{u}$ value which gives the smallest $J_1(t)$, contending that the risk be too high . . . . that is, $\dot{J}_1 \Delta t$ and $\ddot{J}_1 \Delta t^2/2$ are considered as well as $J_1$ in contributing to $J(t)$ . A suitable analogy is in the problem of two tightrope walkers racing each other across a chasm on a windy day. They must use ropes of the same material and both must walk at the same speed along whichever rope they choose. If they fall a certain amount of time is automatically lost climbing up a safety rope. Minimizing $J_1$ would entail taking the smallest diameter, which being the lightest, dips down the least. Minimizing $\|\underline{J}\|$ one would select the rope wide enough so that a change in the wind would at most give a $\Delta \underline{u}$ which would leave the racer still on the rope. In selecting the rope, one barters time for safety or equivalently the shape of the minimum (higher derivatives of $\underline{J}$ ) for the actual minimum value $J_1$ .




The particular rope chosen would depend on $\Delta t$ the reaction time of the racers and the expected maximum $\Delta \underline{u}$ which could occur in that time.

## 2. The Vector Cost Function $\underline{G}$

Let us first consider dealing with a first order adaptive system in which $J_1(t)$ is to be minimized. Since $J_1 = J_1(\underline{R}, \underline{D}, \underline{O}, t)$ or $J_1(\underline{u})$ is generally the integral of an integrand which is always positive, it is convenient and desirable to replace $J_1$ by a vector $\underline{G}_1$ whose norm $\|G'\|$ varies in a manner similar to the integrand $H(\underline{u})$.

Where $J_1 \ (\triangleq \int_{t_0}^{T} H\,dt)$ was once minimized at constant time intervals, it is essentially equivalent to minimize $\|\underline{G}'\|$ at the same time intervals. There are several advantages. Only the same components of $H$ need be measured but there is $n$ times the information in the $\underline{G}'$ cost indicator (if $\underline{G}'$ is an $n$ vector).

For example, given that $J = \int_{t_0}^{t_f} (\underline{R}^2 + (\underline{O} - \underline{D})^2)\,dt$ is to be minimized, for a first order self adapting system

$$\underline{J} = [\,J_1\,]^T = \left[\int_{t_0}^{t} (\underline{R}^2 + (\underline{O} - \underline{D})^2)\,dt\right]^T$$

then $\underline{G}'$ might be selected as

$$\underline{G}' = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_l \\ O_1 - D_1 \\ O_2 - D_2 \\ \vdots \\ O_m - D_m \end{bmatrix}$$

Here as in many other cases $\underline{G}'$ is best chosen such that $\|G'\| = H$ so that minimizing $\|\underline{G}'\|$ is identical to sending $J_1(t)$ to a local minimum (also a global minimum provided the conditions of Theorem I are met).

Having thus introduced the vector $\underline{G}$ superscript one, $\underline{G}^1$ , it remains only to define $\underline{G}^{n+1}$

$$\underline{G}^{n+1} = \frac{d\,\underline{G}^{n}}{d\,t}\,\Delta t / n$$

and thus $\underline{G}$ like $\underline{J}$ is defined as

$$\underline{G} = \begin{bmatrix} \underline{G}^1 \\ \underline{G}^2 \\ \vdots \\ \underline{G}^N \end{bmatrix}$$

where the higher orders of $\underline{G}^n$ , like $J_n$ correspond to the shape of the minimum (or more descriptively the 'risk').

By choice of $\underline{G}$ , step-wise minimization of $\|\underline{G}\|$ is equivalent to step-wise minimization of $\|\underline{J}\|$ . Note however that for each component of $\underline{J}$ , $J_n$ , there is a vector $\underline{G}^n$ in $\underline{G}$ . Also note that this growth of output information from $J = \int H\,dt$ to $\underline{G}$ is accomplished with no additional measuring leads and requires only $\underline{G}(t - \Delta t)$ be remembered in order to calculate the higher orders of $\underline{G}^n$ at $t = t$.

## 3. The Vector Cost Function $\underline{\underline{C}}$

Consider $\underline{C}^1$ the first component of $\underline{C}$ just as $\underline{G}^1$ was the first component of $\underline{G}$ .

For the meantime $\underline{C}$ will be defined by defining its first component vector. That is

$$\underline{C}^1 \triangleq \underline{G}^1 - \underline{G}^{1*}$$

where $\underline{G}^{1*}$ is the global optimum $\underline{G}^1$ having taken into account the constraints. In the cases where there are no constraints $\underline{G}'^*$ will generally be zero or some easily determined constant. Where there are detrimental constraints $\underline{G}'^*$ will often be difficult to calculate. However the motivation for introducing and dealing with $\underline{G}'$ as defined above is this . . . . Minimizing $\underline{G}'$ by driving it to any of its local minima drives the system to one of possibly many equivalent global minima. And it is this minimization which can be done by computer with the algorithm outlined in Chapter III, and in Part II of this thesis.

Rather than solve the problem of constraints the vector cost function $\underline{G}$ merely presents the problem in a different form. One must provide a means of calculating $\underline{G}'^*$. In the most difficult case this involves calculating the best possible (optimum) trajectory that the system could take under the conditions of the constraints. In such cases where $\underline{O}^*$ ( $\triangleq$ the optimal trajectory) involves a complicated precalculation, some of the advantages of the self adaptive system cannot be utilized and the self adaptive system works only as an adaptor. However, even so, the calculation of the optimal controller is unnecessary as the self adaptive control loop, as it adapts, provides the optimum controller.

Note that

$$\underline{G}'^*(t, \underline{O}, \underline{D}, \underline{R}) = \underline{G}'(t, \underline{O}^*, \underline{D}, \underline{R}^*)$$

where $\underline{R}^*$ is the best possible input out of the possible $\underline{R}$ values. (Note that if $\underline{R}$ were fixed to $\underline{R}_o$, $\underline{R}^*$ would then be $\underline{R}_o$ .)

Thus $\underline{G}'$ can alternately be defined as

$$\underline{G}' = \underline{G}'(t, \underline{O}, \underline{D}, \underline{R}) - \underline{G}'(t, \underline{O}^*, \underline{D}\ \underline{R}^*)$$

or $$\underline{G}' = \underline{G}'(t, \underline{O}, \underline{O}^*, \underline{R})$$

Obviously $\underline{E}'^{*}$ is zero (actually this has been achieved through definition) which is a global minimum for $\|\underline{E}'\|$ . Since $\underline{G}'^{*}$ is a global minimum of $\underline{G}'$ then $\underline{E}' = \underline{0}$ for all $t$ guarantees a global minimum for the system as a whole!

Where $N^{th}$ order self adapting is desired $\underline{E}'$ must be extended to vector $\underline{E}$ where

$$\underline{E} = \begin{bmatrix} \underline{E}' \\ \vdots \\ \underline{E}^{N} \end{bmatrix} \equiv \begin{bmatrix} \underline{E}' \\ \dot{\underline{E}} \Delta t \\ \vdots \\ \frac{d^{N-1}\underline{E}'}{dt^{N-1}} \frac{\Delta t^{N-1}}{(N-1)!} \end{bmatrix}$$

and

$$\underline{E}^{n} = \frac{d}{dt} \underline{E}^{n-1} \Delta t / _{n-1}$$

The computer algorithm of Chapter III and Part II will minimize $\|\underline{E}(t)\|$.

## 4. The Desired Trajectory $D(t)$

It is essential in the approach that will be taken to the porposed control system that a desired output must be at all times either known or calculable.

$\underline{D}(t) \triangleq$ the desired output. If $\underline{D}$ is to be calculated as some function of the present 'state', this 'state' must also be measurable or uniquely calculable from some physical quantities of the system. Let the required physical quantities for recalculating $\underline{D}$ be denoted as $\underline{d}(t)$ .

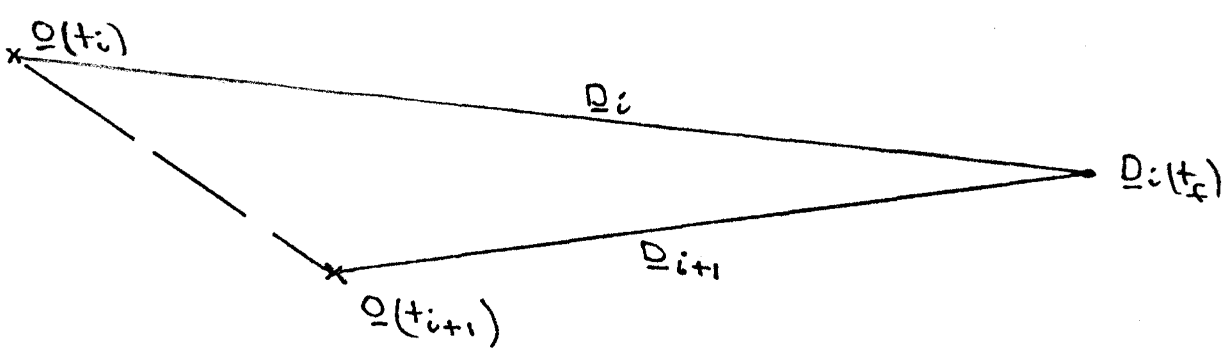
It may not be immediately apparent what the implications of $\underline{D}(\underline{d},t)$ are in practice. However if we regard $\underline{D}$ as an optimal trajectory of certain (or all) states of $\underline{O}$ and realize that in general $\underline{O} \neq \underline{D}$ then it is obvious that as frequently as we measure $\underline{Q}$ and $\underline{D}$ a new $\underline{D}(t)$ may be required. Since $\underline{Q}$ and $\underline{D}$ will be later evaluated at intervals $\Delta t$ apart, we define the set of functions

$$\{ \underline{D}_i(t) \} \equiv \{ \underline{D}(\underline{d}_i, t) \}$$

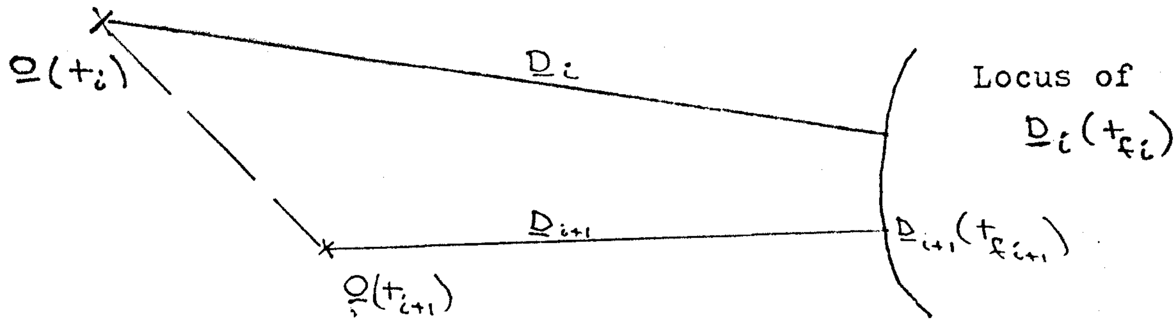
such that $t_{i+1} - t_i = \Delta t$

Trying to geometrically picture $\{ \underline{D}_i \}$ we can set up some sort of closed conical type bounding surface with apex at $\underline{D}_0(t)$. An example of the necessity of such a set $\{ \underline{D}_i \}$ is the rendezvous problem in which a ship at vector $\underline{Q}(t_i) \neq \underline{D}_i(t_i)$ at time $t = t_i$ immediately requires the setting of a new optimal trajectory from point $\underline{Q}(t_i)$ rather than point $\underline{D}_i(t_i) = \underline{Q}^*(t_i)$ with perhaps a new interception point and a new interception time. Note that our setting of a new $\underline{D}_i$ eliminates some of the unnecessary motion normal to $\underline{D}_{i-1}$, presumably saving fuel.

<u>Example 1</u> Fixed End Point, No Detrimental Constraints (that is, $\underline{Q}^* = \underline{D}$ )



Example 2  Partially Fixed End Point, No Detrimental Constraints




If we regard $\underline{Q} \neq \underline{D}$ as error then our recalculation of $\underline{D}$ is an essential optimizing step and hence the necessity that the function $\underline{D}(\underline{d}, t)$ be known is restrictive. Fortunately in many problems $\underline{D}_i(t)$ is easily obtained. For example:

(1) regulator problems,

(2) minimum energy problems with $\underline{D}_i$ recalculable in time $\Delta t$ ,

(3) problems with an entirely predetermined $\underline{D}$ given, and

(4) any problem in which $\underline{D}_o$ can be found and in which $\underline{D}_i$ in time $\Delta t$ can be recalculated.

Sample Problems

(1) Regulator ... with plant input $\underline{R}$ ; minimize

$$J = \int_0^T (\underline{Q} - \underline{R})^2 \, dt \qquad \text{obviously} \quad \underline{D} = \underline{R} \quad (\text{and } \underline{Q}^* \text{ as well})$$

and thus $\underline{D}_i$ is measurable at all times by measuring $\underline{R}$ .

(2) A parachutist jumps from a plane at a certain point $\underline{X}_o$ to land in a target area on the ground $\underline{X}_f$ . He has calculated where to jump ( $\underline{X}_o$ ) by knowing certain laws of physics. He carries a small compressed air cylinder

which is all he can use to guide his path. He is to minimize the amount of compressed air he uses up to land at $\underline{x}_f$ .

(3) Production from a constant number of machines is to be maximized.

(4) A constant speed vehicle is to cross a body of water with random currents minimizing the square of the time taken and the integral of the electric current squared used to drive the motor which turns the rudder.

Note that in the examples $\underline{D}(t)$ is readily available even though in problems (2) and (4) $\underline{D}_i$ changes. In problems (1) and (2) and (4) $\underline{O}_i^*(t) \doteq \underline{D}_i(t)$ while in problem (3), $\underline{O}^*$ is significantly different from $\underline{D}$ due to constraints. In problems (2) and (4) a global minimum cannot be assured since $\underline{O}^*$ (and thus $\underline{G}'^*(t, \underline{O}, \underline{D}, \underline{R})$ and $\underline{E}'(t, \underline{O}, \underline{O}^*, \underline{R})$ ) must be predicted.

## 5. Optimal and Suboptimal Alternative Solutions

Virtually in every problem the desired output is known or can be simply calculated. It is because of its inherent availability as opposed to the constraint complicated $\underline{O}^*(t)$ that trajectories, which are possibly suboptimal, will be tolerated as two of four alternative solutions.

Alternative 1

Where there are no constraints or where these constraints are such that $\underline{G}'^*$ is readily calculable, set

$$\underline{E} = \underline{G} - \underline{G}^* \qquad (2\text{-}5\text{-}1)$$

In this case one can expect a global minimum cost function.

Alternative 2

Where there are constraints, find in some way $\underline{O}^*$ .

Then set

$$\underline{E} = \underline{G}(t, \underline{O}, \underline{D}, \underline{R}) - \underline{G}(t, \underline{O}^{*}, \underline{D}, \underline{R}) \qquad (2\text{-}5\text{-}2)$$

or $\underline{E} = \underline{G}(t, \underline{O}, \underline{O}^{*}, \underline{R})$ (2-5-3)

In this case one can again expect a global minimum cost function.

<u>Alternative 3</u>

Where there are constraints but $\underline{O}^{*}$ is deemed too difficult to find, set

$$\underline{E} = \underline{G}(t, \underline{O}, \underline{D}, \underline{R}) \qquad (2\text{-}5\text{-}4)$$

In this case one can expect only some sort of local minimum as $\underline{O}$ attempts to follow $\underline{D}$ at each step minimizing $\| \underline{E}(t) \|$.

<u>Alternative 4</u>

It may often be the case that the effect of some of the constraints may be simple to calculate or that certain portions of the $\underline{O}^{*}$ trajectory may be available. In this case $\underline{E}$ may be set as (4a) and (4b) respectively.

(4a)

$$\underline{E} = \underline{G}(t, \underline{O}, \underline{D}', \underline{R}) \qquad (2\text{-}5\text{-}5)$$

(modifying $\underline{D}$ to $\underline{D}'$ to satisfy the simple constraints)

(4b)

$$\underline{E} = \underline{G}(t, \underline{O}, \underline{D}, \underline{R}) - \underline{G}(t, \underline{O}^{*}, \underline{D}, t) \qquad (2\text{-}5\text{-}6)$$

(for such $t$ that $\underline{O}^{*}$ is available)

In this case one can only expect some sort of a locally minimum cost function. Intuitively, Alternative 4 appears to be better than Alternative 3.

## 6. Summary

A vector cost function $\underline{J}(t)$ was developed which was in integral form. One equivalent cost function was formed from the integrand with each dimension of $\underline{J}$ providing several dimensions in $\underline{G}$ . Both $\underline{J}(t)$ and $\underline{G}(t)$ had drawbacks in that they could only go to local minima. A theorem was stated by which a local minimum could be recognized as a global minimum.

To generalize the solution to the problems with non-trivial constraints a vector cost function $\underline{\mathcal{G}}$ was defined with the advantage that its local minima were also global minima. However, recognizing that many such $\underline{\mathcal{G}}$ cost functions could not be calculated four alternative solutions were proposed. In two of these the difficulty in finding such $\underline{\mathcal{G}}$ functions was avoided by accepting a possibly non-optimal solution. In these $\underline{\mathcal{G}}$ was taken to be $\underline{G}$ and thus the local minimum of $\underline{G}$ accepted as a compromise.

# III $N^{th}$ ORDER SELF ADAPTING CONTROL LOOP LOGIC

Chapter III will develop the equations to be used in the $N^{th}$ order self adapting control logic.

In addition the relationship between sensitivity and adaptivity will be developed to an extent where two sets of system matrices $\underline{\underline{A}}'_i$ and $\underline{\underline{S}}'_i$ can be defined.

It will also be shown that in general, feedback is advantageous and can be effectively used to shape the $J_1(\underline{u})$ , $\underline{G}'(\underline{u})$ or $\underline{L}'(\underline{u})$ minima.

## 1. Adaptive Systems and Sensitivity

Consider minimizing the function $\underline{L}'(u(t))$ in order to send $J_1(t_f)$ or $J$ to its global minimum. Assume that at time $t$, $\underline{L}'(t)$ and all of its time derivatives are available. Since the system is to adapt at intervals $\Delta t$ apart, $\underline{L}'$ will be expanded in a Taylor series in time.

Thus

$$\underline{L}'(t+\Delta t) = \underline{L}'(t) + \dot{\underline{L}}'(t)\Delta t + \ddot{\underline{L}}'(t)\frac{\Delta t^2}{2} + \ldots \qquad (3\text{-}1\text{-}1)$$

Clearly $\|\underline{L}'(t+\Delta t)\|$ is minimized when the norm of the R. H. S. is minimized. If $\frac{d^{n-1}\underline{L}'}{dt^{n-1}}\Delta t^{n-1}/(n-1)!$ is recognized as $\underline{L}^n$ defined in Chapter II, Section 3, then equation (3-1-1) above can be written as simply

$$\underline{L}'(t+\Delta t) = \underline{L}'(t) + \underline{L}^2(t) + \underline{L}^3(t) + \ldots \qquad (3\text{-}1\text{-}2)$$

It is obvious that minimizing $\|\underline{L}(t)\|$ will thus minimize[1.] $\| \underline{E}'(t+\Delta t) \|$ . Thus the $N^{th}$ order self adapting system in minimizing $\| \underline{E}(t) \|$ actually minimizes the first N terms of a Taylor expansion of $\underline{E}'$ $t+\Delta t$ (and thus the first N terms of the Taylor expansion of $J_1(t+\Delta t)$ ).

If $\underline{\alpha}$ is defined as the vector of changing plant parameters then $\underline{E}'(t+\Delta t)$ can be developed as a Taylor series in the two variables $\underline{\alpha}$ and $t$ . Thus

$$\underline{E}'(t+\Delta t) = \underline{E}'(t+\Delta t, \underline{\alpha}+\Delta\underline{\alpha})$$

$$= \underline{E}'(t,\underline{\alpha}) + \left\{ \frac{\partial \underline{E}'}{\partial \underline{\alpha}} \Delta\underline{\alpha} + \frac{\partial \underline{E}'}{\partial t} \Delta t \right\}$$

$$+ \left\{ \frac{\partial^2 \underline{E}'}{\partial \underline{\alpha}^2} \frac{\Delta\underline{\alpha}^2}{2} + \frac{\partial^2 \underline{E}'}{\partial \underline{\alpha}\, \partial t} \Delta\underline{\alpha}\,\Delta t + \frac{\partial^2 \underline{E}'}{\partial t^2} \frac{\Delta t^2}{2} \right\}$$

$$+ \{ \; . \qquad etc. \qquad (3\text{-}1\text{-}3)$$

If at $(t, \underline{\alpha})$ , $\underline{E}' = \underline{E}'^{*}$ then it is guaranteed by $\underline{E}(t) = \underline{0}$ that the time derivatives of $\underline{E}$ at $t$ are zero whence equation (3-1-3) above can be written

$$\underline{E}'(\underline{\alpha}+\Delta\underline{\alpha}, t+\Delta t) = \underline{E}' + \frac{\partial \underline{E}'}{\partial \underline{\alpha}} \Delta\underline{\alpha} + \frac{\partial^2 \underline{E}'}{\partial \underline{\alpha}^2} \frac{\Delta\underline{\alpha}^2}{2} + etc. \qquad (3\text{-}1\text{-}4)$$

which merely expresses mathematically that $\underline{E}'(t+\Delta t)$ drifts away from $\underline{E}'^{*}$ only through the change of the plant parameters $\underline{\alpha}$ [2.] in the time interval $(t, t+\Delta t)$.

---

[1.] This can be proven as follows by noting that $\| \underline{E}' \| \leq \| \underline{E} \|$ is an identity (triangle inequality).

[2.] It is important to realize that this is true only if no new adaptive measures have taken place in $(t, t+\Delta t)$.

By comparing (3-1-1) and (3-1-4)

$$\frac{\partial^n \underline{L}'}{\partial \underline{\alpha}^n} \frac{\Delta \underline{\alpha}^n}{n!} = \frac{d^n \underline{L}'}{dt^n} \frac{\Delta t^n}{n!} \quad \forall\, n > 0 \tag{3-1-5}$$

This can be compared with the geometric interpretation of figures 3a and 3b in Chapter II, Section I which led to the equivalent expression

$$\frac{d^n J_1(\underline{y})}{d \underline{y}^n} \frac{\Delta \underline{y}^n}{n!} = \frac{d^n J_1(\underline{y}(t))}{dt^n} \frac{\Delta t^n}{n!}$$

It is at this point the sensitivity matrices $\underline{\underline{S}}'_n$ can be defined . . . . looking at equation (3-1-5) define the N^th order system sensitivity matrix as

$$\underline{\underline{S}}'_n \triangleq \frac{\partial^n \underline{L}'(t)}{\partial \underline{\alpha}^n}$$

the double bars under a quantity denotes a two dimensional array

For completeness define

$$S'_n = \| \underline{\underline{S}}'_n \|$$

and $\hat{\underline{\underline{S}}}'_n = \underline{\underline{S}}'_n / S'_n$

Now introduce the parameters $\underline{c}$ the set of controller parameters. These, like $\underline{\alpha}$, appear in the net system operator $\underline{S}_{op}$ (Chapter I, page ) and therefore, like $\underline{\alpha}$, are a part of the generalized coordinate vector $\underline{y}$ (Chapter II, page 2 ). In an adaptive system these must be calculated so as to cancel out the effects of $\underline{\underline{S}}'_n \frac{\Delta \underline{\alpha}^n}{n!}$ in $\underline{L}'(t+\Delta t)$. In writing $\frac{d \underline{L}'}{dt}$ one cannot merely write

$$\frac{d \underline{L}'}{dt} = \frac{\partial \underline{L}'}{\partial \underline{\alpha}} \frac{d \underline{\alpha}}{dt} + \frac{\partial \underline{L}'}{\partial \underline{c}} \frac{d \underline{c}}{dt} \tag{3-1-6}$$

Certainly the preceeding equation is correct. However, as a consequence of the finite time between adaptive decisions one is forced to use the following difference equation

$$\underline{\underline{G}}'(\underline{\alpha}+\Delta\underline{\alpha},\underline{c}+\Delta\underline{c}) - \underline{\underline{G}}'(\underline{\alpha},\underline{c})$$

$$= 2\left\{ \sum_{\substack{k+l=i,\\ i=1}}^{\infty} \frac{\partial^i \underline{\underline{G}}'}{\partial \underline{c}^k \, \partial \underline{\alpha}^l} \, \frac{\Delta\underline{c}^k \, \Delta\underline{\alpha}^l}{i\,!} \right\} \tag{3-1-7}$$

If $\underline{\underline{G}}'(\underline{\alpha},\underline{c}) = \underline{0}$ then $\Delta\underline{\underline{G}}'$ must be set to $\underline{0}$ also. This can be accomplished only by

$$\frac{\partial^i \underline{\underline{G}}'}{\partial \underline{\alpha}^i} \Delta\underline{\alpha}^i = - \frac{\partial^i \underline{\underline{G}}'}{\partial \underline{c}^i} \Delta\underline{c}^i \tag{3-1-8}$$

The $N^{th}$ order self adaptive system attempts to satisfy equation (3-1-8) for $i = 1, 2, \cdots N$.

Here it is useful to define the system's $i^{th}$ adaptivity matrix $\underline{\underline{A}}'_i$

$$\underline{\underline{A}}'_i \triangleq \frac{\partial^i \underline{\underline{G}}'}{\partial \underline{c}^i}$$

Introducing the symbols $\underline{\underline{A}}'_i$ and $\underline{\underline{S}}'_i$ into equation (3-1-8) one obtains

$$\underline{\underline{S}}'_i \, \Delta\underline{\alpha}^i = - \underline{\underline{A}}'_i \, \Delta\underline{c}^i \tag{3-1-9}$$

Unfortunately it is awkward to work with equation (3-1-8) or (3-1-9) except for $i = 1$. Satisfying (3-1-8) for $i = 1$ alone, however, gives only the first order adaptor. Fortunately the difficulty is avoided merely by extending the sensitivity $\underline{\underline{S}}'$ to $\underline{\underline{S}}$ just as $\underline{J}_1$, $\underline{G}'$ and $\underline{\underline{G}}'$ were extended to $\underline{J}$, $\underline{G}$ and $\underline{\underline{G}}$ respectively.

Thus in defining

$$\underline{\underline{S}} = \frac{\partial \underline{G}}{\partial \underline{\alpha}}$$

and

$$\underline{\underline{A}} = \frac{\partial \underline{G}}{\partial \underline{c}}$$

for

$$\underline{G} = [\underline{G}^1, \underline{G}^2, \ldots \underline{G}^N]^T$$

one can assure $N^{th}$ order insensitivity and $N^{th}$ order adaptivity by

$$\underline{\underline{S}} \, \Delta\underline{\alpha} = - \underline{\underline{A}} \, \Delta\underline{c} \qquad (3\text{-}1\text{-}10)$$

That equation (3-1-10) and equation (3-1-9) are indeed equivalent can be seen by merely noting that

$$\| \Delta\underline{G} \| \equiv \| \underline{\underline{S}} \, \Delta\underline{\alpha} + \underline{\underline{A}} \, \Delta\underline{c} \| \equiv \underline{0}$$

implies that

$$\Delta\underline{G}^n = \underline{0} \qquad \forall \; n = 1, 2 \ldots N$$

but the $i^{th}$ term of equation (3-1-7) can be recognized as $\Delta\underline{G}^n$ whence

$$\underline{\underline{S}}_i^{\,1} \, \Delta\underline{\alpha}^i = - \underline{\underline{A}}_i^{\,1} \, \Delta\underline{c}^i \qquad \forall \, i = 1, 2 \ldots N$$

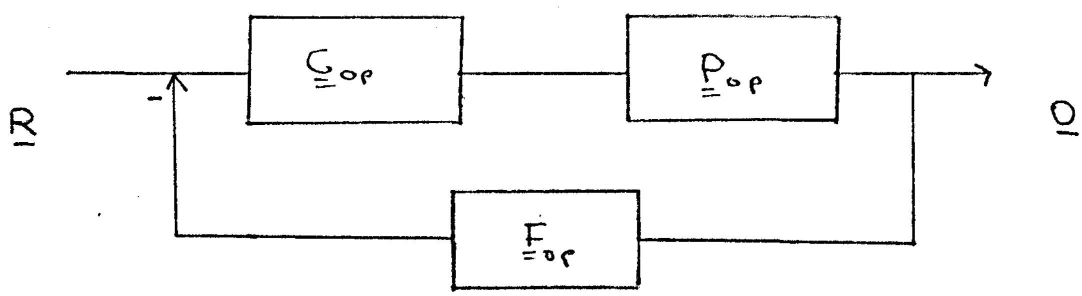
<u>Hence for $\underline{G}$ and $N$ vector and $\underline{G} = \underline{0}$ then $\underline{\underline{S}} \, \Delta\underline{\alpha} + \underline{\underline{A}} \, \Delta\underline{c} = \underline{0}$ implies $\underline{G}'(t + \Delta T) = \underline{0}$ to the $N+1^{st}$ term of its Taylor expansion!</u>

## 2. The Effect of Feedback

It is the purpose of this section to investigate the effect of feedback on the cost function $\underline{G}'$. The inclusion of a feedback path with operation matrix $\underline{\underline{F}}_{op}$ (Chapter I, page 1 ) will be justified on the basis of an improved system sensitivity $\underline{\underline{S}}$ and better adaptivity.

For simplicity the subscripts $F$ and $NF$ will be used to denote the feedback and no feedback cases respectively.

Recall the block diagram on page 3 of ChapterI which is the closed loop control.




By inspection

$$\left[ \underline{\underline{I}} + \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \right] \underline{O} = \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{R} \qquad (3\text{-}2\text{-}1)$$

Where $\left[ \underline{\underline{I}} + \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \right]^{-1}$ exists

$$\underline{O} = \underline{\underline{S}}_{op} \underline{R} = \left[ \underline{\underline{I}} + \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \right]^{-1} \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{R}$$

whence $\underline{\underline{S}}_{op}$ can alternately be defined

$$\underline{\underline{S}}_{op} = \left[ \underline{\underline{I}} + \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \right]^{-1} \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \qquad (3\text{-}2\text{-}2)$$

Recall that the sensitivity matrix $\underline{\underline{S}}$ is defined as $\partial \underline{G} / \partial \underline{\alpha}$ (Chapter III, page 22 ). Since $\underline{G}$ depends on $\underline{\alpha}$ only through $\underline{O}$, $\frac{\partial \underline{G}}{\partial \underline{\alpha}}$ can be expanded as

$$\mathbf{S} = \frac{\partial \underline{G}}{\partial \underline{\alpha}} = \frac{\partial \underline{G}}{\partial \underline{O}} \; \frac{\partial \underline{O}}{\partial \underline{\alpha}}$$

Since $\partial \underline{G} / \partial \underline{O}$ is independent of the type of feedback used, it is useful to define matrix $\mathbf{M}$

$$\mathbf{M} \triangleq \frac{\partial \underline{G}}{\partial \underline{O}} \qquad (3\text{-}2\text{-}3)$$

whence $\quad \mathbf{S} = \mathbf{M} \cdot \dfrac{\partial \underline{O}}{\partial \underline{\alpha}}$

To find $\partial \underline{O} / \partial \underline{\alpha}$ it is necessary to take the derivative of equation (3-2-1) with respect to $\underline{\alpha}$ .

$$\left[ \mathbf{I} + \mathbf{P}_{op} \mathbf{C}_{op} \mathbf{F}_{op} \right] \partial \underline{O} / \partial \underline{\alpha} + \frac{\partial \mathbf{P}_{op}}{\partial \underline{\alpha}} \mathbf{C}_{op} \mathbf{F}_{op} \underline{O} = \frac{\partial \mathbf{P}_{op}}{\partial \underline{\alpha}} \mathbf{C}_{op} \underline{R} \qquad (3\text{-}2\text{-}4)$$

Manipulating (3-2-4) in order to use (3-2-1) to remove $\underline{O}$ one finds finally that

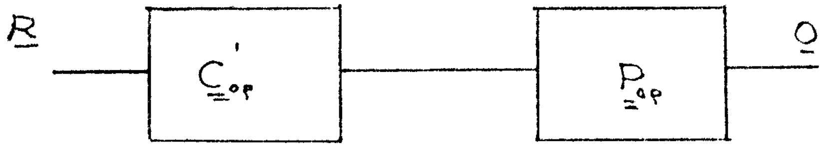
$$\left[ \mathbf{I} + \mathbf{P}_{op} \mathbf{C}_{op} \mathbf{F}_{op} \right] \left[ \frac{\partial \mathbf{P}_{op}}{\partial \underline{\alpha}} \mathbf{C}_{op} \mathbf{F}_{op} \right]^{-1} \left[ \mathbf{I} + \mathbf{P}_{op} \mathbf{C}_{op} \mathbf{F}_{op} \right] \frac{\partial \underline{O}_F}{\partial \underline{\alpha}} = \mathbf{F}_{op}^{-1} \underline{R} \qquad (3\text{-}2\text{-}5)$$

where $\partial \mathbf{P}_{op} / \partial \underline{\alpha}$ might be regarded as the fundamental expression of the plant sensitivity alone[1].

---

1. Note that $\partial \mathbf{P}_{op} / \partial \underline{\alpha}$ (which might be called the plant sensitivity) in some cases is calculable if $\underline{\alpha}$ is measurable. However such a definition of sensitivity is too limiting — first in that it is fixed, secondly in that it is not the system sensitivity (though related to it), thirdly because it yields no information whether or not feedback is to be preferred (and if so what type of feedback operator) and fourthly because in the adaptive system it is not $\partial \mathbf{P}_{op} / \partial \underline{\alpha}$ which is required but $\partial \mathbf{P}_{op} / \partial \underline{\alpha} \; \Delta \underline{\alpha}$ where $\Delta \underline{\alpha}$ is not known but must be predicted (—one may as well then predict $\partial \mathbf{P}_{op} / \partial \underline{\alpha} \; \Delta \underline{\alpha}$ in its entirety as predict $\Delta \underline{\alpha}$, both being a vector of the same dimension).

In the open loop case consider the following block diagram.




Since $\underline{O}$ must still equal $\underline{\underline{S}}_{op}\underline{R}$ , $\underline{\underline{C}}'_{op}$ cannot in general equal $\underline{\underline{C}}_{op}$ of the closed loop case and so is primed.

Differentiating $\underline{O} = \underline{\underline{P}}_{op}\underline{\underline{C}}'_{op}\underline{R}$ (3-2-6)

one quickly finds

$$\frac{\partial \underline{O}}{\partial \underline{\alpha}} = \frac{\partial \underline{\underline{P}}_{op}}{\partial \underline{\alpha}} \underline{\underline{C}}'_{op}\underline{R} \tag{3-2-7}$$

Using equation (3-2-1) to replace $\underline{\underline{C}}'_{op}$ in (3-2-6) by $\underline{\underline{C}}_{op}$ at length, one obtains

$$\left[\frac{\partial \underline{\underline{P}}_{op}}{\partial \underline{\alpha}}\underline{\underline{C}}_{op}\underline{\underline{F}}_{op}\right]^{-1}\left[\underline{\underline{I}} + \underline{\underline{P}}_{op}\underline{\underline{C}}_{op}\underline{\underline{F}}_{op}\right]\frac{\partial \underline{O}_{NF}}{\partial \underline{\alpha}} = \underline{\underline{F}}_{op}^{-1}\underline{R} \tag{3-2-8}$$

Define now $\underline{\underline{L}}$

$$\underline{\underline{L}} \triangleq \left[\frac{\partial \underline{\underline{P}}_{op}}{\partial \underline{\alpha}}\underline{\underline{C}}_{op}\underline{\underline{F}}_{op}\right]^{-1}\left[\underline{\underline{I}} + \underline{\underline{P}}_{op}\underline{\underline{C}}_{op}\underline{\underline{F}}_{op}\right] \tag{3-2-9}$$

With the combined difficulties of operator mathematics and the rules of matrix multiplication, the best that can be done with equations (3-2-5) and (3-2-8) is the following.[1]

$$\left.\begin{aligned} \underline{\underline{S}}_{NF} &= \underline{\underline{M}}\,\underline{\underline{L}}^{-1}\left[\underline{\underline{I}} + \underline{\underline{P}}_{op}\underline{\underline{C}}_{op}\underline{\underline{F}}_{op}\right]\underline{\underline{L}}\frac{\partial \underline{O}_F}{\partial \underline{\alpha}} \\ \underline{\underline{S}}_F &= \underline{\underline{M}}\frac{\partial \underline{O}_F}{\partial \underline{\alpha}} \end{aligned}\right| \tag{3-2-10}$$

---

1. These few equations contain a wealth of information and are certainly worth a study in more detail than will be gone into here.

or if $\underline{\underline{M}}^{-1}$ can be defined, then

$$\underline{\underline{S}}_{NF} = \underline{\underline{M}}\ \underline{\underline{L}}^{-1} [\underline{\underline{I}} + \underline{\underline{P}}_{op}\underline{\underline{C}}_{op}\underline{\underline{F}}_{op}]\ \underline{\underline{L}}\ \underline{\underline{M}}^{-1}\ \underline{\underline{S}}_{F} \tag{3-2-11}$$

Of more immediate interest and more easily understood is the relationship between the first order sensitivities $\underline{\underline{S}}'_{NF}$ and $\underline{\underline{S}}'_{F}$ . Recall that

$$\underline{\underline{S}}' = \frac{\partial \underline{G}'}{\partial \underline{\alpha}} = \frac{\partial \underline{G}'}{\partial \underline{O}} \frac{\partial \underline{O}}{\partial \underline{\alpha}} \triangleq \underline{\underline{M}}' \frac{\partial \underline{O}}{\partial \underline{\alpha}}$$

whence

$$\underline{\underline{S}}'_{NF} = \underline{\underline{M}}'\ \underline{\underline{L}}^{-1} [\underline{\underline{I}} + \underline{\underline{P}}_{op}\underline{\underline{C}}_{op}\underline{\underline{F}}_{op}]\ \underline{\underline{L}}\ \underline{\underline{M}}'^{-1}\ \underline{\underline{S}}'_{NF} \tag{3-2-12}$$

$\underline{\underline{S}}' = \partial \underline{G}'/\partial \underline{\alpha}$ relates the effect on the cost function $\underline{G}'$ and ultimately the effect on the original cost function of changes in the plant parameters $\underline{\alpha}$ .

If $\underline{G}'$ was selected as alternative one or two, (Chapter II, page 15 ) then $\underline{\underline{S}}'$ will relate to $\Delta\underline{\alpha}$ the change in $J$ $\left(J = \int_{t_0}^{t_f} H\,dt\right)$ from a global minimum. If $\underline{G}'$ was chosen as alternatives three or four, then $\underline{\underline{S}}'$ will relate to $\Delta\underline{\alpha}$ the change in the integrand of $J$ — that is, $H$ — from a local minimum.

What is lost in using $\underline{\underline{S}}'$ instead of $\underline{\underline{S}}$ is a little accuracy synonomous with truncating a Taylor series in $\Delta\underline{\alpha}$ after the first two terms. However, for the purposes of this section — comparing open loop and closed loop systems — such an approximation is a simplification which will not in the least bias the conclusion[1.] of this thesis and which will allow a little more light to be shed on the subject. The reason is simply that $\underline{\underline{M}}' \triangleq \partial \underline{G}'/\partial \underline{O}$

---

1. The higher order sensitivities $\partial^n \underline{G}'_{NF}/\partial \underline{\alpha}^n$ and $\partial^n \underline{G}'_{F}/\partial \underline{\alpha}^n$ are related by similar expressions but involve higher powers of $[\underline{\underline{I}} + \underline{\underline{P}}_{op}\underline{\underline{C}}_{op}\underline{\underline{F}}_{op}]$ .

is liable to be a square matrix with a unique inverse whereas $\underline{M} = \partial \underline{L} / \partial \underline{\alpha}$ is liable to be a rectangular array.

Returning to equation (3-2-12) note first that the bracket can be multiplied out leaving

$$\underline{\underline{S}}'_{NF} = \left[ \underline{\underline{I}} + \underline{\underline{M}}' \underline{\underline{L}}^{-1} \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \underline{\underline{L}} \, \underline{\underline{M}}'^{-1} \right] \underline{\underline{S}}'_{F} \qquad (3\text{-}2\text{-}13)$$

which in the term $\underline{\underline{M}}' \underline{\underline{L}}^{-1} \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \underline{\underline{L}} \, \underline{\underline{M}}'^{-1}$ shows clearly that the relative sensitivity depends on the cost function (through $\underline{\underline{M}}'$ ), on the plant sensitivity itself (through $\underline{\underline{L}}$ ), as well as on the presence or absence of feedback (through $\underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op}$ ).

While it would be nice to say that the relative sensitivity was due mainly to $\underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op}$ it must be pointed out that it is only in cases where $(\underline{\underline{M}}' \underline{\underline{L}}^{-1})(\underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op})$ is non-rotational[1] that $\underline{\underline{M}}' \underline{\underline{L}}^{-1} \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \underline{\underline{L}} \, \underline{\underline{M}}'^{-1}$ can be set equal to $\underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op}$ .

Where $[ \underline{\underline{I}} + \underline{\underline{M}}' \underline{\underline{L}}^{-1} \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \underline{\underline{L}} \, \underline{\underline{M}}'^{-1} ]$ operating on $\underline{\underline{S}}_{F}$ yields a matrix sensitivity $\underline{\underline{S}}_{NF}$ whose norm[2], $\| \underline{\underline{S}}_{NF} \|$ , is greater than $\| \underline{\underline{S}}_{F} \|$ then feedback is to be desired. This inequality can be often obtained by proper selection of $\underline{\underline{C}}_{op}$ and $\underline{\underline{F}}_{op}$ which are arbitrary to a certain extent. To ensure that feedback is desirable the following two equations must be satisfied.

$$\underline{O} = \underline{\underline{S}}_{op} ( \underline{\underline{C}}_{op}, \underline{\underline{F}}_{op}, \underline{\underline{P}}_{op} ) \; \underline{R} \qquad (3\text{-}2\text{-}14)$$

$$\| \underline{\underline{S}}_{F} \| < \| \underline{\underline{S}}_{NF} \| \qquad (3\text{-}2\text{-}15)$$

---

1. Non-rotational for the sake of this paper will be defined as: $\underline{\underline{A}} \, \underline{\underline{B}}$ is non-rotational iff $\underline{\underline{A}} \, \underline{\underline{B}} = \underline{\underline{B}} \, \underline{\underline{A}}$ .

2. $\| \underline{\underline{A}} \| \triangleq \sum_{i,j} | A_{ij} |$

Since these are merely two equations in the two unknowns $\underline{\underline{C}}_{op}$ and $\underline{\underline{F}}_{op}$ there are only the unstated physical realizability conditions and the few systems where $\underline{\underline{L}}$ and $\underline{\underline{M}}'$ are such that equations (3-2-14) and (3-2-15) above are inconsistent where nothing can be gained by using a closed loop control. Even in such cases the open loop can be treated as a special form of closed loop in which $\underline{\underline{F}}_{op}\,\underline{Q} \equiv \underline{0}$.

The following example is intended to illustrate the comparison between $\underline{\underline{S}}_F$ and $\underline{\underline{S}}_{NF}$ for an extremely simple case in which $\underline{\underline{M}}'$, $\underline{\underline{M}}'^{-1}$, $\underline{\underline{L}}$ and $\underline{\underline{L}}^{-1}$ are available and in which $\underline{\underline{P}}_{op}$, $\underline{\underline{C}}_{op}$ and $\underline{\underline{F}}_{op}$ are one by one arrays of constant operators. $\underline{\underline{F}}_{op}$ and $\underline{\underline{C}}_{op}$ will be chosen to satisfy (3-2-14) and (3-2-15), with the resultant choice of negative feedback for an inherently less sensitive system.

Example

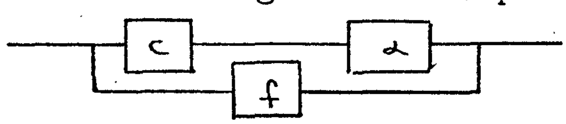
Given a plant with parameter vector $\underline{\alpha} = [\alpha]^T$ with given gain $\alpha = \alpha_0$ and $J = \int_{t_0}^{t_f} (\lambda \underline{R} - \underline{Q})^2 \, dt$, design the system least sensitive to changes in $\underline{\alpha}$. First select $\underline{G}' = \lambda \underline{R} - \underline{Q}$ ; $\underline{G}'^{*} = \underline{0}$

thus $\underline{\underline{L}}' = \lambda \underline{R} - \underline{Q}$

and $\underline{\underline{S}}' = \dfrac{\partial \underline{\underline{L}}'}{\partial \underline{\alpha}} = \dfrac{\partial \underline{\underline{L}}'}{\partial \underline{Q}} \dfrac{d\underline{Q}}{d\underline{\alpha}} = -1 \dfrac{\partial \underline{Q}}{\partial \underline{\alpha}}$

Hence $\underline{\underline{M}}' = -1$ and $M'^{-1} = -1$

With the following closed loop control

$\underline{R}$  $\underline{Q}$


$$\underline{\underline{P}}_{op}\,\underline{\underline{C}}_{op}\,\underline{\underline{F}}_{op} = \alpha\, c\, f$$

Furthermore

Thus
$$\frac{\partial \underline{\underline{P}}_{op}}{d\alpha} = \frac{d\alpha}{d\alpha} = 1$$

$$\underline{\underline{L}}_{op} = \left[ \frac{\partial \underline{\underline{P}}_{op}}{d\alpha} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \right]^{-1} \left[ \underline{\underline{I}} + \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \right] = \frac{1}{1cf} (1 + \alpha c f)$$

Whence
$$\underline{\underline{S}}'_{NF} = \left[ \underline{\underline{I}} + \underline{\underline{M}}' \underline{\underline{L}}^{-1} \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \underline{\underline{L}} \underline{\underline{M}}'^{-1} \right] \underline{\underline{S}}'_{F}$$
$$= \left[ \underline{\underline{I}} + \underline{\underline{P}}_{op} \underline{\underline{C}}_{op} \underline{\underline{F}}_{op} \right] \underline{\underline{S}}'_{F}$$
$$= \left[ 1 + \alpha c f \right] \underline{\underline{S}}'_{F}$$

With the one dimensional sensitivity $\underline{\underline{S}}'$ is $1 \times 1$ and thus (3-2-13) becomes $S_{NF} = (1 + \alpha c f) S_F$ . Clearly, provided $\alpha c f > 0$ or $\alpha c f < -2$ , $\| \underline{\underline{S}}'_F \| < \| \underline{\underline{S}}'_{NF} \|$ . The conditional equations,(3-2-14) and (3-2-15), can be seen to have simplified to selecting $c$ and $f$ to satisfy[1].

$$|1 + \alpha c f| > 1 \tag{3-2-16}$$

and

$$\alpha c / (1 + \alpha c f) = \lambda \tag{3-2-17}$$

and perhaps the additional constraints $|c|, |f| \leq 10$ For example with $\lambda = 10$ and $\alpha_{,} = 20$ the problem's solution calls for $c = 10$ and $f = .095$ which maximizes $|S_{NF}| / |S_F|$ and keeps $\underline{Q} = \underline{\underline{S}}_{op} \underline{R} = \lambda \underline{R}$ .

Thus with $|c|, |f| \leq 10$ , $\lambda = 10$, and the nominal value of $\alpha = 20$ , equations (3-2-14) and (3-2-15) yield a closed loop system with a gain of $10$ with $1/20$ the sensitivity of the required open loop system.

---

1. Note that $|1 + \alpha c f| > 1$ can be satisfied by $\alpha c f < -2$ which corresponds to positive feedback. In such cases, provided equation (3-3-2) can also be satisfied, even positive feedback is preferable to the open loop system.

To sum up this section, feedback systems are preferable to open loop systems provided the controllers $\underline{\underline{F}}_{op}$ and $\underline{\underline{C}}_{op}$ can be selected to satisfy

$$\left.\begin{aligned} &\| \underline{\underline{S}}'_F \| < \| \underline{\underline{S}}'_{NF} \| \\ &\underline{Q} = \underline{\underline{S}}_{op} ( \underline{\underline{C}}_{op}, \underline{\underline{P}}_{op}, \underline{\underline{F}}_{op} ) \, \underline{R} \\ &\underline{\underline{C}}_{op} \in \{ \text{ allowed set of } \underline{\underline{C}}_{op} \} \\ &\underline{\underline{F}}_{op} \in \{ \text{ allowed set of } \underline{\underline{F}}_{op} \} \end{aligned} \right| \qquad (3\text{-}2\text{-}18)$$

Furthermore the general block diagram of the self adapting system should be a feedback one in that the open loop is the special case . . . . $\underline{\underline{F}}_{op}\underline{Q} = \underline{0}$ .

In so far as adaptivity is concerned note only that a reduced $\| \underline{\underline{S}}' \|$ implies a reduced $\underline{\underline{S}}' \, \Delta\underline{\alpha}$ and a correspondingly reduced $\underline{\underline{A}}' \, \Delta\underline{c}$ mirrored in a practical sense in a smaller adaptive change $\Delta\underline{c}$ and a smaller Taylor series truncation error in $\Delta\underline{E}'$ , that is, a smaller difference between infinite order adapting and $N^{th}$ order adapting!

In terms of the geometric interpretation the meaning of a smaller $\underline{\underline{S}}'$ is clearly a smaller value of $\partial \underline{E}' / \partial \underline{\alpha}$ . That is, the slope of the $\underline{E}'(\underline{u})$ minima have been reduced in the $\underline{\alpha}$ directions to $\underline{\underline{S}}'_F$ .

## 3. The General Adaptor Equation

Having defined $\underline{E}$ and $\underline{u}$ and introduced the $\underline{E}(\underline{u})$ control surface it is a simple matter to write down the exact (continuous) adaptor equation. That is

$$d\underline{E} = \frac{\partial \underline{E}}{\partial \underline{\alpha}} d\underline{\alpha} + \frac{\partial \underline{E}}{\partial \underline{c}} d\underline{c} + \frac{\partial \underline{E}}{\partial \underline{f}} d\underline{f} + \frac{\partial \underline{E}}{\partial \underline{D}} d\underline{D} + \frac{\partial \underline{E}}{\partial \underline{R}} d\underline{R} + \frac{\partial \underline{E}}{\partial t} dt \qquad (3\text{-}3\text{-}1)$$

with $\underline{E}(t + \Delta t) = \underline{E}(t) + d\underline{E}$

Due to the finite adaptor logic decision time, $\Delta t$, $d\underline{G}$ must be written as $\Delta \underline{G}$ below

$$\Delta \underline{G} = \frac{\partial \underline{G}}{\partial \underline{\alpha}} \Delta \underline{\alpha} + \frac{\partial \underline{G}}{\partial \underline{c}} \Delta \underline{c} + \frac{\partial \underline{G}}{\partial \underline{f}} \Delta \underline{f} + \frac{\partial \underline{G}}{\partial \underline{D}} \Delta \underline{D} + \frac{\partial \underline{G}}{\partial R} \Delta \underline{R} + \frac{\partial \underline{G}}{\partial t} \Delta t \qquad (3\text{-}3\text{-}2)$$

and $\underline{G}(t+\Delta t) = \underline{G}(t) + \Delta \underline{G}$ (3-3-3)

Note that this is only a first order Taylor approximation to $\underline{G}$. However for $\underline{G} = [\underline{G}^1, \underline{G}^2, \ldots \underline{G}^N]^T$ it is equivalent to an $N^{th}$ order Taylor approximation to $\underline{G}'$. And it is $\underline{G}'$ which is related to the integrand of the original cost function $J$. Hence equations (3-3-2) and (3-3-3) above will be called the $N^{th}$ order adaptor equations as these provide for $N^{th}$ order adapting of $\underline{u}$ in the function $\underline{G}(\underline{u})$.

Consider separately the terms of (3-3-2). Clearly $\partial \underline{G}/\partial \underline{\alpha}\ \Delta \underline{\alpha}$ can only be predicted since $\Delta \underline{\alpha}$ is unknown. On the otherhand $\Delta \underline{f}$, $\Delta \underline{c}$ and $\Delta \underline{R}$ are at our disposal and the reaction of $\underline{G}$ to each of these can be separately found by setting $\Delta \underline{f}$, $\Delta \underline{c}$ or $\Delta \underline{R}$ and measuring $\Delta \underline{G}$. With $\underline{G} = \underline{G}(\underline{R}, \underline{O}, \underline{D}, t)$ given then $\partial \underline{G}/\partial \underline{D}$ is given and if $\partial \underline{G}/\partial t \neq 0$ then $\partial \underline{G}/\partial t$ is also obtainable from $\underline{G}(\underline{R}, \underline{O}, \underline{D}, t)$.

It can be seen then that the $N^{th}$ order self adapting system implies a first order adapting of the derived vector cost function $\underline{G}$. The method involves:

1. prediction of $\partial \underline{G}/\partial \underline{\alpha}\ \Delta \underline{\alpha}$,
2. updating the matrices $\partial \underline{G}/\partial \underline{R}$, $\partial \underline{G}/\partial \underline{f}$ and $\partial \underline{G}/\partial \underline{c}$,
3. calculation of $\partial \underline{G}/\partial \underline{D}$ and where applicable $\partial \underline{G}/\partial t$ from the given function $\underline{G}$, and
4. setting $\Delta \underline{f}$ and $\Delta \underline{c}$ or where $\underline{R}$ is not necessarily fixed setting $\Delta \underline{f}$, $\Delta \underline{c}$ and $\Delta \underline{R}$.

Putting equations (3-3-2) and (3-3-3) together and defining $\partial \underline{G}/\partial \underline{\alpha}\ \Delta \underline{\alpha} \triangleq \underline{B}$ then the self adapting system minimizes $\| \underline{G}(t+\Delta t) \|$ by trying to set $\underline{G}(t+\Delta t) = \underline{0}$

as below . . . .

$$\underline{G}(t+\Delta t) = \underline{G}(t) + \Delta\underline{G}$$

$$= \underline{G}(t) + \underline{B} + \frac{\partial\underline{G}}{\partial\underline{c}}\Delta\underline{c} + \frac{\partial\underline{G}}{\partial\underline{f}}\Delta\underline{f} + \frac{\partial\underline{G}}{\partial\underline{D}}\Delta\underline{D} + \frac{\partial\underline{G}}{\partial\underline{R}}\Delta\underline{R} + \frac{\partial\underline{G}}{\partial t}\Delta t \qquad (3\text{-}3\text{-}4)$$

. . . . by setting $\Delta\underline{f}$ and $\Delta\underline{c}$ and/or $\Delta\underline{R}$ .

One of many possible implementations is shown in the following section.

To sum up this section, the adaptor algorithm[1] requires:
1. measurement of $\underline{G}$ and $\underline{R}$ and $\underline{D}$ ,
2. prediction of $\underline{B}$ ,
3. updating (learning) of $\frac{\partial\underline{G}}{\partial\underline{R}}$ , $\frac{\partial\underline{G}}{\partial\underline{f}}$ and $\frac{\partial\underline{G}}{\partial\underline{c}}$ ,

and 4. setting of $\Delta\underline{c}$ and/or $\Delta\underline{f}$ and/or $\Delta\underline{R}$ to minimize $\|\underline{G}(t+\Delta t)\|$ .

The heart of the algorithm is the adaptor equation

$$\underline{G}(t+\Delta t) = \underline{G}(t) + \frac{\partial\underline{G}}{\partial\underline{f}}\Delta\underline{f} + \frac{\partial\underline{G}}{\partial\underline{c}}\Delta\underline{c} + \frac{\partial\underline{G}}{\partial\underline{R}}\Delta\underline{R} + \underline{B} + \frac{\partial\underline{G}}{\partial\underline{D}}\Delta\underline{D} + \frac{\partial\underline{G}}{\partial t}\Delta t \qquad (3\text{-}3\text{-}4)$$

## 4. A Computer Algorithm for Implementing the General Adaptor Equation

The adaptive system that was chosen has six different loops. In each the function $\underline{G}$ and $\underline{R}$ are measured, $\underline{D}$ calculated, and $\underline{B}$ predicted. In each

---

1. A specific way in which each of these can be done is outlined in the next section and given in detail in the actual computer program of Part II.

the adaptive controller changes[1] $\Delta \underline{c}_a$ and $\Delta \underline{f}_a$[2] are calculated. The loops differ in the actual changes in $\underline{c}$ and $\underline{f}$ that take place, $\Delta \underline{c}_r$ and $\Delta \underline{f}_r$.

Loop 1.

Learn $\partial \underline{G} / \partial \underline{R}$

set $\Delta \underline{f}$ and $\Delta \underline{c}$ to $\underline{0}$

that is $\Delta \underline{f}_r$ and $\Delta \underline{c}_r = \underline{0}$

This allows the adaptor to relate the change in $\underline{G}$ from the expected value $\langle \underline{G} \rangle$ to be related to $\Delta \underline{R}$. Each time the adaptor cycles through loop 1., the $j^{th}$ row of $\partial \underline{G} / \partial \underline{R}$ is updated. The particular equation to be used is

$$\frac{\partial G_j}{\partial R_i} \Delta R_i = \frac{|G_i - \langle G_i \rangle| - |\overline{B_i - \langle B_i \rangle}|}{|G_i - \langle G_i \rangle| + |\overline{B_i - \langle B_i \rangle}|} \left( G_i - \langle G_i \rangle \right) \qquad (3\text{-}4\text{-}1)$$

$$i = 1, 2, \ldots N$$

where $\partial G_j / \partial \underline{R}$ is the new value of the $j^{th}$ row of the matrix $\partial \underline{G} / \partial \underline{R}$

and $\langle \underline{A} \rangle$ expected value of Quantity $\underline{A}$

and $\overline{\underline{A}}$ average value of Quantity $\underline{A}$

Loop 2.

Adapt $\Delta \underline{f}$ then $\Delta \underline{c}$

set $\Delta \underline{f}_r$ to $\Delta \underline{f}_a$

then $\Delta \underline{c}_r$ to $\Delta \underline{c}_a$

---

1. Subscript '$a$' denotes adaptive.

2. Subscript '$r$' on $\Delta \underline{f}$ and $\Delta \underline{c}$ denotes the actual changes that are to be made in $\underline{f}$ and $\underline{c}$.

This means that the controller adapts completely to minimize $\| \underline{G}(t+\Delta t) \|$ .

Loop 3.

Learn $\partial \underline{G} / \partial \underline{c}$

$\Delta \underline{f}_r = \underline{0}$ ; $\Delta \underline{c}_r = j^{th}$ component of $\Delta \underline{c}_a$

This allows the adaptor to relate the changes in $\underline{G}$ from the expected value to be related to $\Delta c_{a_j}$ and allows the $j^{th}$ column of $\frac{\partial \underline{G}}{\partial \underline{c}}$ to be updated.

$$\frac{\partial G_i}{\partial c_j} \Delta c_j = \frac{|G_i - \langle G_i \rangle| - |\overline{B_i - \langle B_i \rangle}|}{|G_i - \langle G_i \rangle| + |\overline{B_i - \langle B_i \rangle}|} \left( G_i - \langle G_i \rangle \right)$$

$$i = 1, 2, \ldots N \qquad (3\text{-}4\text{-}2)$$

Loop 4.

Adapts $\Delta \underline{c}$ then $\Delta \underline{f}$

set $\Delta \underline{f}_r = \Delta \underline{f}_a$ and then $\Delta \underline{c}_r = \Delta \underline{c}_a$

Again the system is adapted.

Loop 5.

Learn $\partial \underline{G} / \partial \underline{f}$

$\Delta \underline{c}_r = \underline{0}$ ; $\Delta \underline{f}_r = j^{th}$ component of $\Delta \underline{f}_a$

This allows the adaptor to relate the change in $\underline{G}$ from the expected value to be related to $\Delta f_{r_j}$ and allows the $j^{th}$ column of $\frac{\partial \underline{G}}{\partial \underline{f}}$ to be updated.

$$\frac{\partial G_i}{\partial f_j} \Delta f_j = \frac{|G_i - \langle G_i \rangle| - |\overline{B_i - \langle B_i \rangle}|}{|G_i - \langle G_i \rangle| + |\overline{B_i - \langle B_i \rangle}|} \left( G_i - \langle G_i \rangle \right)$$

$$i = 1, 2, \ldots N \qquad (3\text{-}4\text{-}3)$$

Loop 6.

Adapt $\Delta \underline{f}$ then $\Delta \underline{c}$

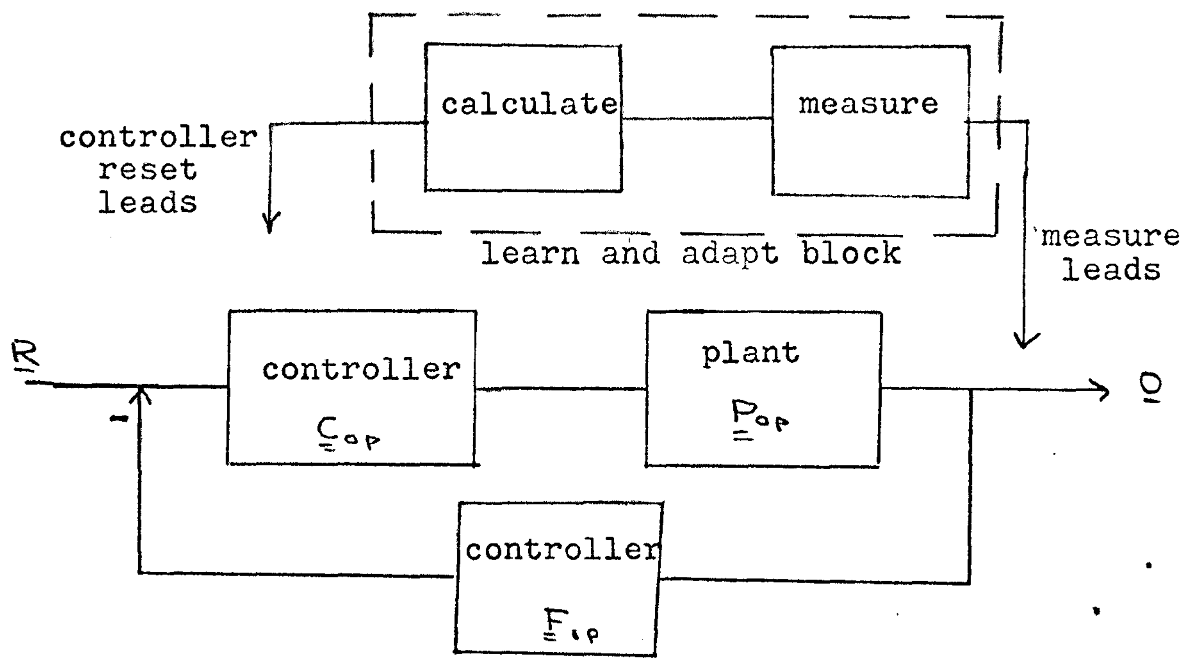
Set $\Delta \underline{f}_r = \Delta \underline{f}_a$ and then $\Delta \underline{c}_r = \Delta \underline{c}_a$

Again the adaptive loop is adapting.

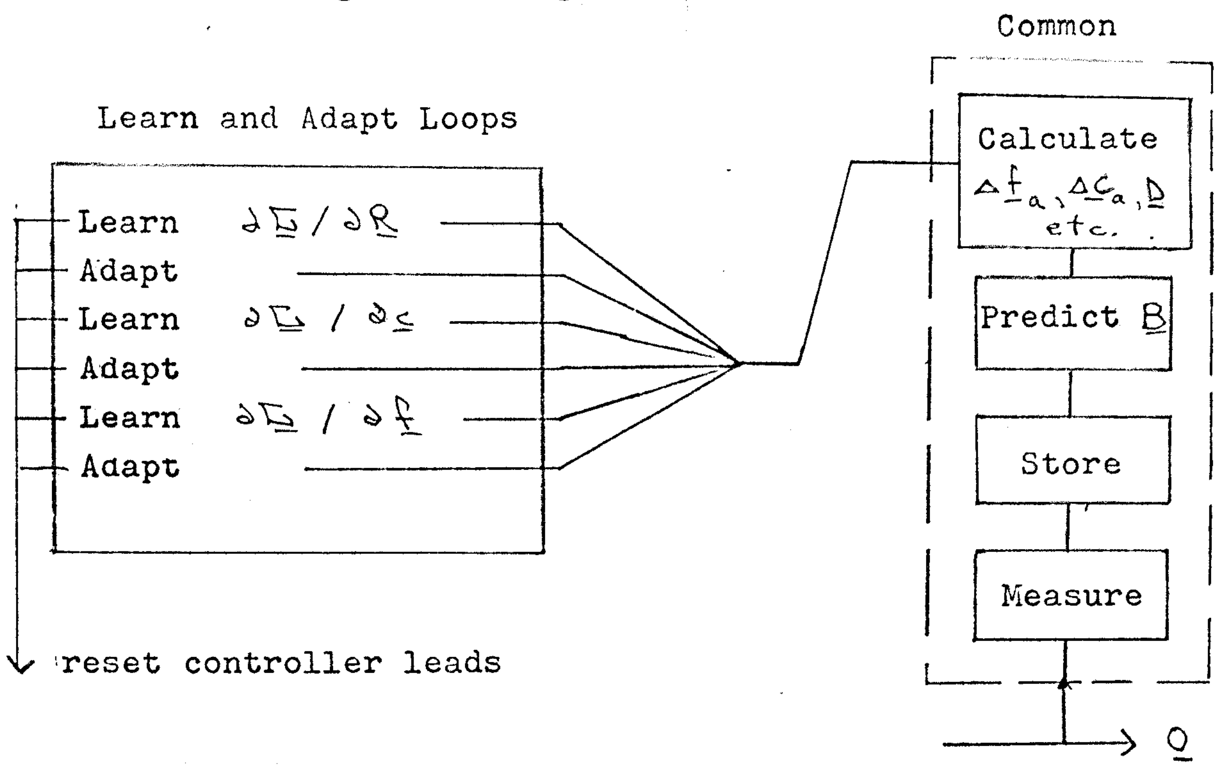
The adaptor recycles the loops each time incrementing $j$ , adapting at intervals of $2 \Delta t$ , and updating the entire matrices $\partial \underline{G} / \partial \underline{R}$ , $\partial \underline{G} / \partial \underline{c}$ and $\partial \underline{G} / \partial \underline{f}$ at intervals of the order of $6 N \Delta t$ seconds.

With parameter changes with a period of $6 N \Delta t$ or more the learned matrices, if stable, can be expected to be very accurate. Where $\underline{\alpha}$ changes with a period less than $6 N \Delta t$ these matrices, where stable, can be expected to have an accuracy better than $|\Delta \underline{\alpha}| / |\underline{\alpha}|$ leading to a second order error in $\underline{G}$ (of the order of $\Delta \underline{\alpha}^2$ ).

This results in the following block diagram.



The 'learning' and adapting block can be expanded into the following block diagram.




The controller section marked common, including storage, prediction, and responsible for the various required calculations and measurements is a problem in two ways. First the method of prediction which best suits a system is dependent on the way in which $\underline{\alpha}$ varies. Second, with any computer, there is often both a time factor and a memory factor in dealing with a large number of stored events. A simple method of predicting $\underline{B}$ and $\overline{\underline{B} - \langle \underline{B} \rangle}$ which is a compromise between statistically varying $\underline{\alpha}$ values and regular time variation $\underline{\alpha}$ values and which in addition eliminates the storage problem, is the following:

i. $\underline{L}(t) - \langle \underline{L}(t) \rangle \triangleq \underline{B}(t)$

$$\langle \underline{L}(t+\Delta t) \rangle = \underline{L}(t) + \langle \Delta \underline{L} \rangle$$

$$= \underline{L}(t) + \langle \underline{B}(t+\Delta t) \rangle + \frac{\partial \underline{L}}{\partial \underline{R}} \Delta \underline{R}$$

$$+ \frac{\partial \underline{L}}{\partial \underline{c}} \Delta \underline{c} + \text{etc.} \qquad (3\text{-}4\text{-}4)$$

ii. $\underline{EB}(t) \triangleq \underline{B}(t) - \langle \underline{B}(t) \rangle$

$$\overline{\underline{B}(t)}_K \triangleq \frac{K-1}{K} \overline{\underline{B}(t-\Delta t)}_K + \underline{B}(t)/K$$

$$\langle \underline{B}(t+\Delta t) \rangle = \frac{K-1}{K} \underline{B}(t) + \underline{EB}(t)/K$$

where $K$ is an integer $\geqslant 1$ (3-4-5)

iii. $\overline{\underline{EB}(t)}_K \triangleq \frac{K-1}{K} \overline{\underline{EB}(t-\Delta t)}_K + \underline{EB}(t)/K$ (3-4-6)

With the above definitions the R.H.S. of equations (3-4-1), (3-4-2) and (3-4-3) becomes

$$\frac{|B_i(t)| - |\overline{EB_i(t)}_K|}{|B_i(t)| + |\overline{EB_i(t)}_K|} B_i$$

Note that equations (3-4-4), (3-4-5) and (3-4-6) require only the storage of $\underline{L}(t)$, $\langle \underline{L}(t) \rangle$, $\underline{EB}(t)$, $\overline{\underline{EB}(t)}_K$, $\underline{B}(t)$, $\langle \underline{B}(t) \rangle$ and $\overline{\underline{B}(t)}_K$. Thus the memory problem and minipulation problem have been eliminated.

Note as well the introduction of the integer parameter $K$ which is used in such a manner as to weight the contributions of recent values of $\underline{B}(t)$ and $\underline{EB}(t)$ more heavily. In systems with slowly varying $\underline{\alpha}$ it is clearly best to use $K = 1$. Essentially what this does is to set

$$\langle \underline{B}(t+\Delta t) \rangle = \underline{B}(t) + \dot{\underline{B}}(t) \Delta t$$

(where $\dot{\underline{B}}(t) \Delta t \equiv \underline{EB}(t)$).

Where $K > 1$

$$< \underline{B}(t+\Delta t) > = \overline{\underline{B}(t)}_K + \overline{\dot{\underline{B}}(t)_K \Delta t}$$

$$\left( \text{where} \quad \overline{\dot{\underline{B}}(t)}_K \equiv \overline{\underline{E}\underline{B}(t)}_K \right)$$

Looking at expressions (3-4-5) and (3-4-6) $K \neq \infty$ has the effect of a non infinite memory average in which events at $t - T$ are weighted in the averages $\overline{\underline{B}}_K$ and $\overline{\underline{E}\underline{B}}_K$ by a factor $\gamma$,

$$\gamma = \left[ \frac{K-1}{K} \right]^{T/\Delta t}$$

Clearly $K$ values $> 1$ will be best suited[1] to random systems with changing values of $\underline{\alpha}$.

For totally random $\underline{\alpha}$ the self adaptive system can only give the optimum comtroller for the predicted value of $\underline{\alpha}$ $(\underline{\alpha} = \overline{\underline{\alpha}}_K)$.

This completes the outline of the $N^{th}$ order self adapting control logic. It must be pointed out that while the elements of storage, prediction, learning, measuring and calculating are necessary there are many ways of including them to perform the same overall function. The actual implementation of these functions shown in the block diagram and outlined roughly in the text is hoped to be close to optimal in so far as accuracy, general applicability, and computer decision time are concerned.

Part II, to follow, is intended to demonstrate the applicability in practice as well as in theory.

---

[1] The self adaptive controller is relatively less and less effective as the frequency of $\Delta\underline{\alpha}$ increases.

## 5. Summary

A general sensitivity in matrix form has been defined for the purpose of this thesis. It has been shown that the control system should be closed loop provided that the conditions of equation (3-2-18) can be met. The control logic has therefore been developed to accomplish $N^{th}$ order self adapting for a closed loop system with feedback path controller $\underline{\underline{F}}_{\sim P}$ and forward path controller, $\underline{\underline{G}}_{oP}$ .

It must be pointed out that there has only been heuristic argument for the actual implementation of the essentials learning, adapting, and predicting as represented in equations (3-4-1), (3-4-2), (3-4-3), (3-4-4), (3-4-5) and (3-4-6). Others, with as much justification, might decide to refine the equations given[1.] or develop entirely different ones to suit special known properties of the variation of $\underline{\Delta\alpha}$ in their particular system. If the computer memory is given or assumed infinite a best prediction method can be found as a function of the type of statistics that $\underline{\Delta\alpha}$ obeys. Most of these methods can be found in texts or papers dealing with prediction.

---

1. and to be used in Part II, Chapters IV and V.

# PART II EXPERIMENTAL

## IV ADAPTIVE SYSTEMS AND FEEDBACK

Chapter IV demonstrates the desirability of adaptive systems and in particular adaptive systems with feedback. The experiments were performed on an analog computer using the author as the 'learning' and adapting loop.

### 1. Experiment 1

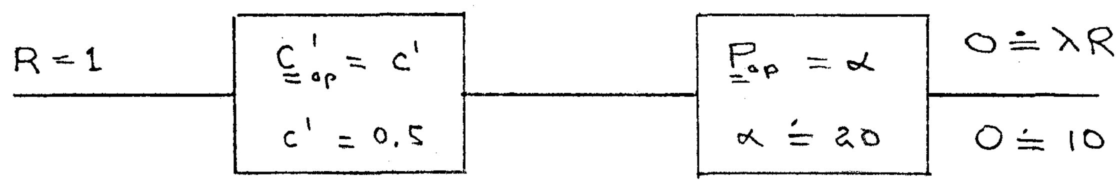
**Description**

Experiment 1, deals with the system described in Chapter III, Section 2. A plant with nominal transfer function $\underline{P}_{op} \doteq 20$ is to be regulated in such a way as to minimize

$$J = \int_{t_0}^{t_f} (\lambda R - O)^2 \, dt \quad .$$

$\lambda$ will be taken as 10 and $R$ will be taken as 1 .

The open loop nominal optimal controller is simply $\underline{C}'_{op} = c' = 1/2$ .




For the first order adaptive system it is obvious $\underline{C}'_{op}$ should be varied such that $\underline{C}'_{op} \underline{P}_{op} = 10$ . That is $\alpha \, \Delta c' = -c' \Delta \alpha$ . But this is just the first order adapter equation[2] developed in Chapter III, Section 1. (See any of equations (3-1-8), (3-1-9) or (3-1-10).)

In table 1, page 43 , are listed the $J$ values normalized to a 10 second interval for:

---

1. In this problem $\underline{G}' = \lambda \underline{R} - \underline{O}$ and $\underline{G}'^{*} = 0$ whence $\underline{E}' = \lambda \underline{R} - \underline{O}$ . With $\underline{R} = 1$ , $\partial \underline{E}' / \partial \underline{\alpha} = \underline{C}'_{op}$ and $\partial \underline{E}' / \partial \underline{c} = \underline{\alpha}$ . Thus (3-1-10) becomes $\alpha \, \Delta c' = c' \Delta \alpha$ .

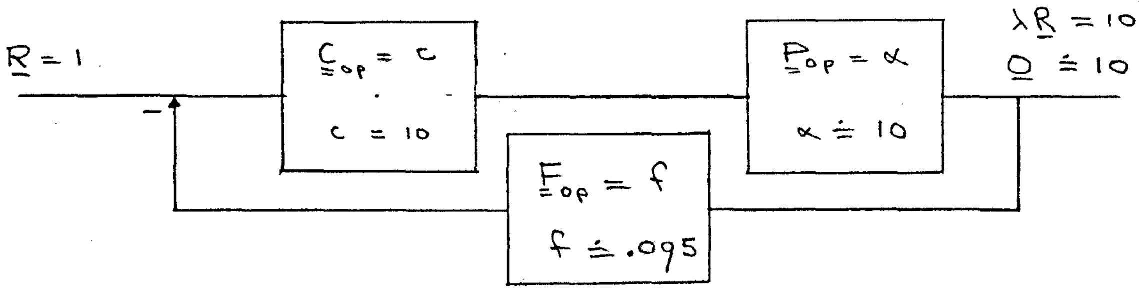
1a Open loop system with $\underline{C}'_{op}$ fixed at the nominal optimal value $.5$

1b Open loop system with $\underline{C}'_{op}$ adapted to minimize $\|\underline{E}\|$ $(=(10 - O)^2)$.

2a Closed loop system with $\underline{C}_{op}$ and $\underline{F}_{op}$ fixed at the nominal optimal values, $10$ and $.095$.

2b Closed loop system with $\underline{C}_{op}$ fixed at its optimal value and $\underline{F}_{op}$ adapted to minimize $\|\underline{E}\|$.

The optimal closed loop controller as found by equation (3-2-18) gives $\underline{C}_{op} = c = 10$, its maximum value, and $\underline{F}_{op} = f$ at $.095$.




The $J$ values for the above system were measured for $\underline{C}_{op} = 10$ and $\underline{F}_{op} = .095$ which is the optimal controller for the nominal value of $\alpha = 20$.

The system was then adapted. Equation (3-2-18) indicated that $\underline{C}_{op}$ should be kept at its maximum value and $\|\underline{E}\|$ minimized by changing $\underline{F}_{op}$.

## Conclusions

There were really two objectives to performing the text experiment. In table 1, two distinct comparissons can be made in order to demonstrate:

1. that the adaptive controller is significantly better than the nominally optimal controller, at least for a low frequency change in $\alpha$, and

2. that the closed loop optimal control yields significantly smaller $\bar{J}$ values than the open loop roughly by a factor[1] of $(1+\alpha f_c)^2$ .

Data

| Case | | $J = \int_{t_1}^{t_1+10} (\underline{Q} - 10\underline{R})^2 \, dt$ |
|---|---|---|
| 1a | Open Loop Nominal Optimal | 2.8 |
| 1b | Open Loop Adaptive | .04 |
| 2a | Closed Loop Nominal Optimal | .14 |
| 2b | Closed Loop Adaptive | .0014 |

table 1.

2. Experiment 2

Description

Experiment 2 was essentially the same as Experiment 1. The difference lay in the plant which varied in the following manner. . .

$$\boxed{\text{plant} \quad \underline{P}_{op} = \frac{10}{s+\alpha}} \qquad \text{where} \quad \alpha = 1 + \Delta\alpha(t)$$

1. The quantity $(1+\alpha c f)^2$ (related to $[\underline{I} + \underline{P}_{op}\underline{C}_{op}\underline{F}_{op}]$ of equation (3-2-18) ) is squared as J is proportional to the integral of $\underline{E}$ squared.

Experiment 2 used two values of $\Delta\alpha(t)$

(i) $\Delta\alpha(t) = .2 \sin t/2\pi$

(ii) $\Delta\alpha(t)$ = square wave with amplitude 0.2 and frequency of 1.0 radians per second.

## Conclusions

The results, tabulated in table 2, lead to the same conclusion as those in table 1. The order of the plant has not led to any difficulties. On the contrary the plant acts as a filter to its own high frequency parameter changes and enables the system to adapt reasonably well to a square wave parameter variation.

## Data

| $\Delta\alpha(t)$ [1] | Case | $J = \int_{t_1}^{t_1+10} (O - 10R)^2\,dt$ |
|---|---|---|
| .2 ∿ 1. | Open Loop Nominal Optimal | 2.5 |
| .2 ∿ 1. | Open Loop Adaptive | .05 |
| .2 ⊓⊔ 1. | Open Loop Nominal Optimal | 10.2 |
| .2 ⊓⊔ 1. | Open Loop Adaptive | 1.2 |
| .2 ∿ 1. | Closed Loop Nominal Optimal | .14 |
| .2 ∿ 1. | Closed Loop Adaptive | .0015 |
| .2 ⊓⊔ 1. | Closed Loop Nominal Optimal | 1.1 |
| .2 ⊓⊔ 1. | Closed Loop Adaptive | .13 |

table 2.

1. ".2 ∿ 1.0" indicates sine wave with fractional amplitude ( of $\alpha$ ) equal to .2 and radian frequency of $\omega = 1.0$. ".2 ⊓⊔ 1.0" indicates square wave with amplitude $.2\,\alpha$ and radian frequency $\omega = 1.0$.

## 3. Summary

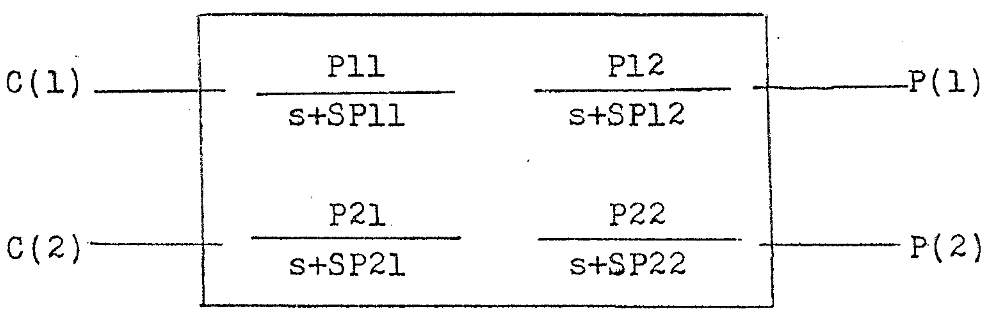
From Experiments 1 and 2, it could be concluded that adaptive systems (the examples were first order adaptive) were superior to nominally optimal systems. Furthermore both systems showed improved behaviour in the closed loop configurations. It was also interesting to note that a square wave variation in $\underline{\alpha}$ could be better adapted to than might have been expected.

# V COMPUTER IMPLEMENTATION OF THE GENERAL ADAPTOR EQUATION

A plant, which is given below, was chosen to test the theory of Part I for the case of first order self adapting. A number of different tests are made on this plant — $\underline{R}$ fixed, $\underline{R}$ varying, open and closed loop adaptive, open and closed loop nominal optimal. For each case the appropriate system matrices must be learned. The full power of the $N^{th}$ order self adapting system is shown in the way that the system starts out from extremely bad initial controllers, 'learns' the matrices $\partial \underline{G}/\partial \underline{c}$, $\partial \underline{G}/\partial \underline{f}$ and $\partial \underline{G}/\partial \underline{R}$, drives the controllers to their optimal values and then adapts to changes in either $\underline{\alpha}$, $\underline{R}$ or $\underline{D}$. Note, too, that this will be accomplished knowing nothing about the plant except that it has two inputs and two outputs! For this particular plant, with $\underline{R}$ fixed, it will be found that once the appropriate matrices are learned that the entire learn and adapt loop may be replaced by a passive network of only four potentiometers and two adders!

## 1. Statement of the Problem

Given, a plant with input $\underline{C}$ and output $\underline{P}$




P11=100.
P21=-1.+.5 sin 5.5 t
SP11=10.+2 sin 1.0 t
SP21=20.

P12=1+.5 sin 3.0 t
P22=100.
SP12=20.
SP22=10+2 sin 2.0 t

Required, that $J = \int_{t_1}^{t_2} (\underline{P}-\underline{D})\cdot(\underline{P}-\underline{D})\,dt$ be minimized where $D_1 = 3$ and $D_2 = 8$.

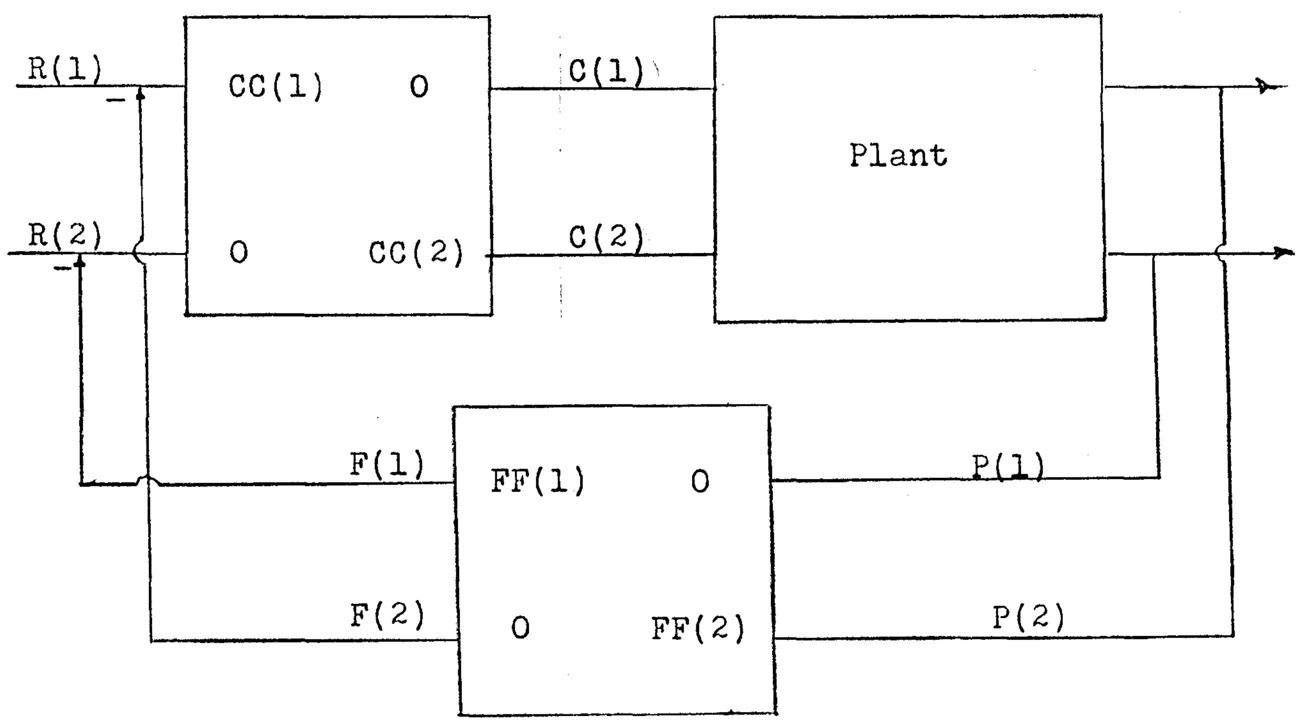
The entire problem was simulated and solved on an IBM 7040 computer.

## 2. Computer Simulation

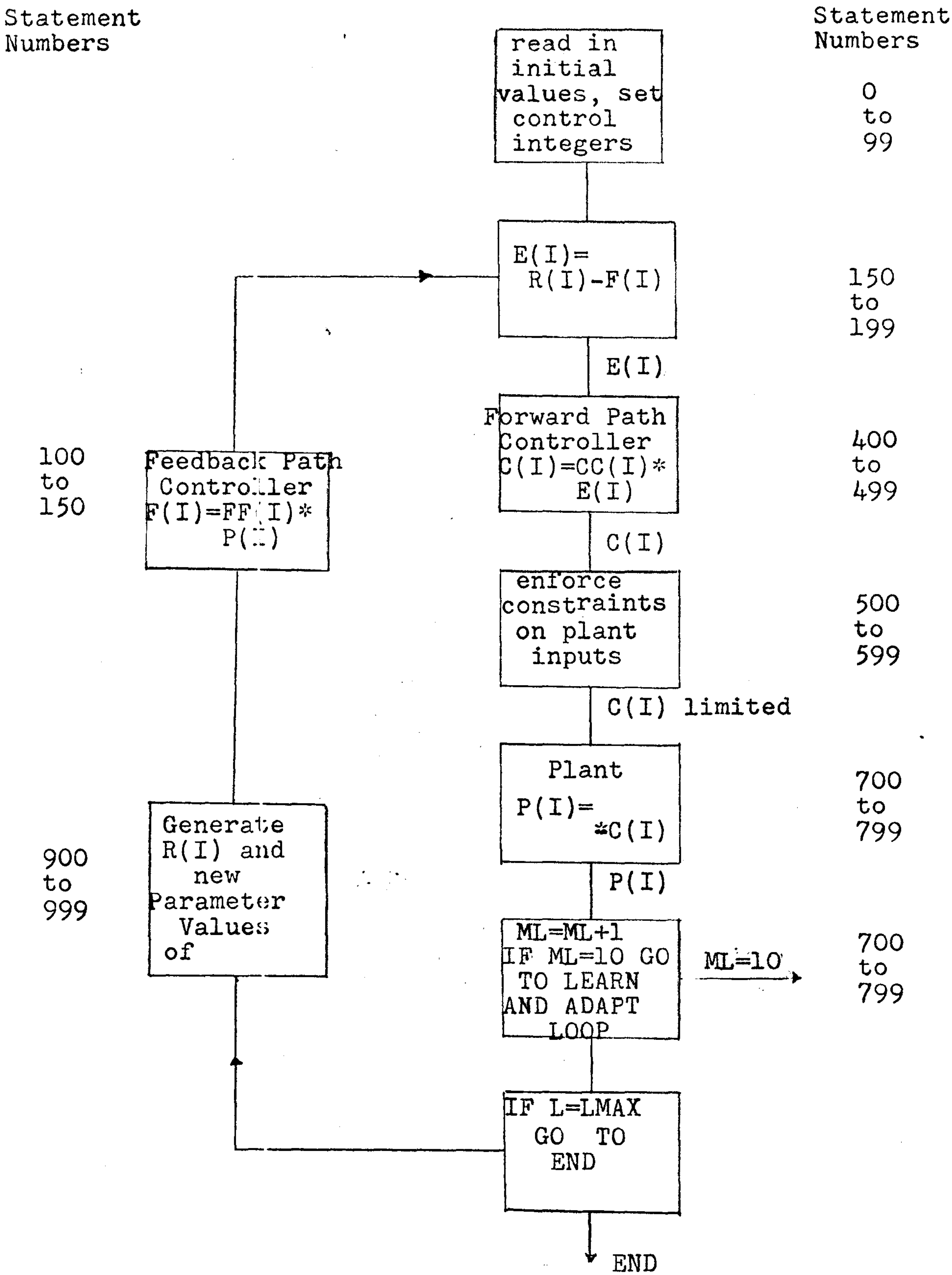
The computer simulation of the plant with its controllers, and the learning and adapting loop, can be considered separately.

The following block diagram and computer flow chart can be used to define the symbols used in the computer programs and to identify the blocks in the computer program.

### Control System Block Diagram

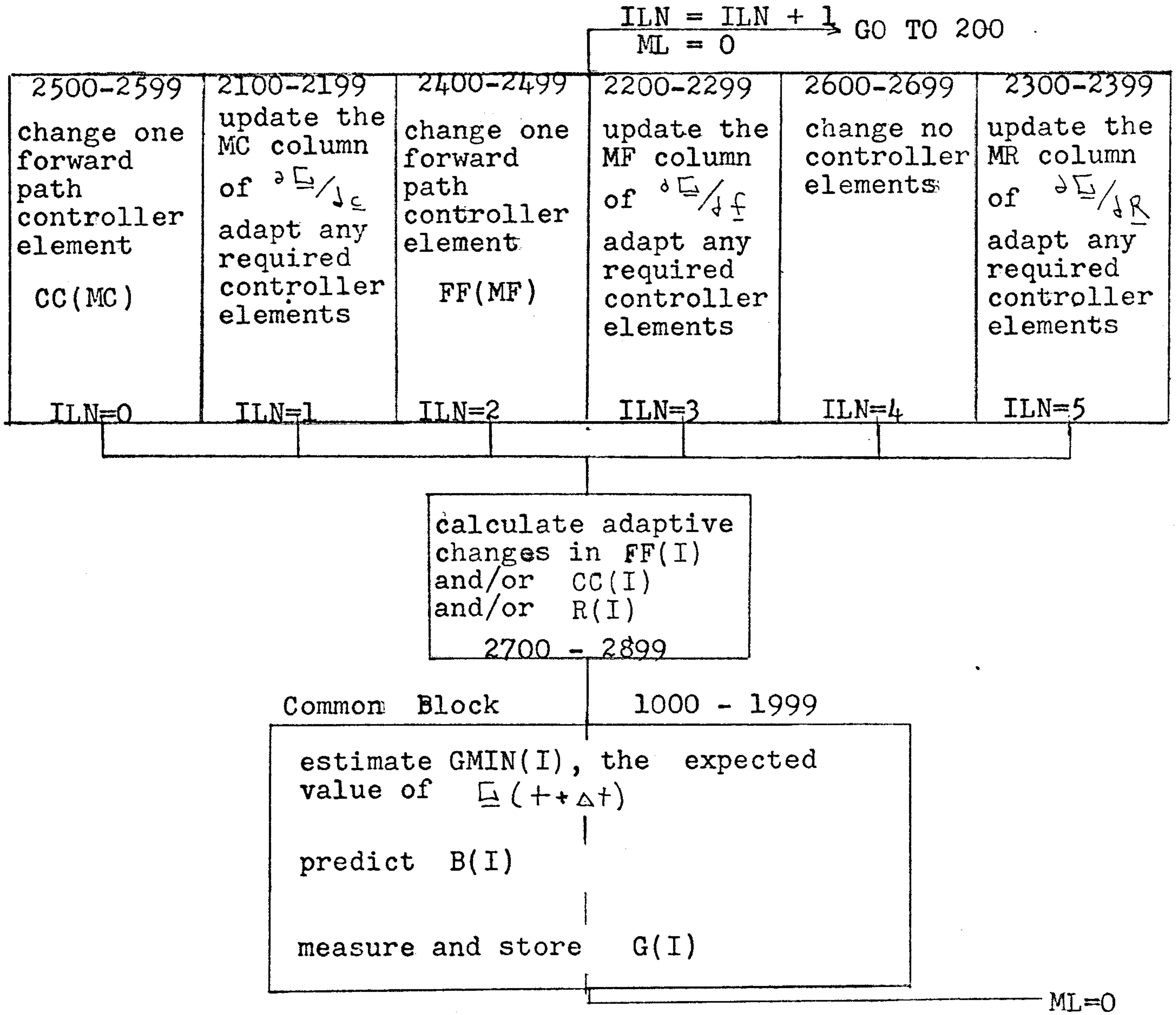


Flow Chart of Control System Simulation



The following is a flow chart of the computer learn and adapt loop. The complete computer program can be found in Appendix I, page 67.

<u>Flow Chart of the Learn and Adapt Loop</u>



## 3. Tests and Results

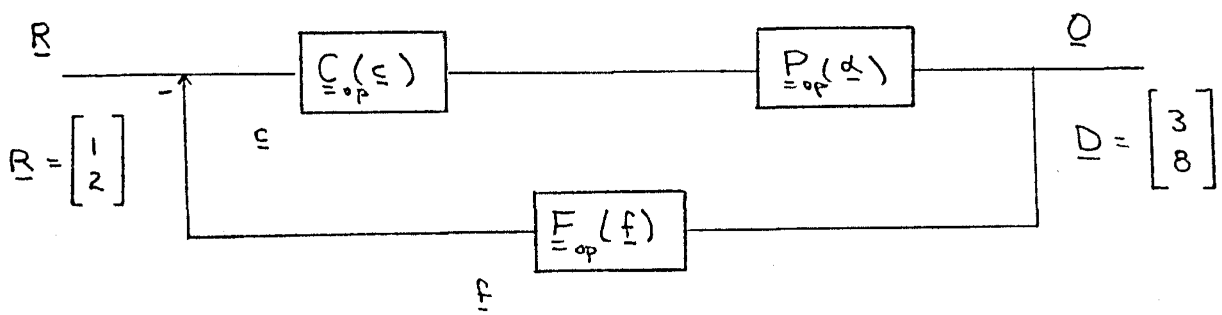
There were ten tests performed on the plant given in Section 1 of this chapter. The first four of these were done with the input $\underline{R}$ fixed and $\underline{D}$, the desired output a constant.

$$\text{That is,} \quad \underline{R} = \begin{bmatrix} R(1) \\ R(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \underline{D} = \begin{bmatrix} D(1) \\ D(2) \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

For these four cases $\Delta \underline{D}$ and $\Delta \underline{R}$ were zero and the learn and adapt loop had only to learn the matrices $\partial \underline{G} / \partial \underline{c}$ and/or $\partial \underline{G} / \partial \underline{f}$. Three cases were adaptive. The fourth was merely nominal optimal, that is, the controllers were set to the optimal value for the average value of $\underline{\alpha}$.

The results of tests one to four are as follows.

<u>Test 1</u> Closed Loop; learn $\partial \underline{G} / \partial \underline{c}$ and $\partial \underline{G} / \partial \underline{f}$ ; adapt $\underline{c}$ and $\underline{f}$




(a) $$\partial \underline{G} / \partial \underline{c} = \begin{bmatrix} -.8315 & 8.463 \\ -12.36 & 5.127 \end{bmatrix} \quad \text{at } t = .4 \text{ sec.}$$

$$\partial \underline{G} / \partial \underline{f} = \begin{bmatrix} -7.512 & -.0869 \\ .3087 & -23.25 \end{bmatrix} \quad \text{at } t = .4 \text{ sec.}$$

(b) typical values of $\underline{c}$ and $\underline{f}$ were

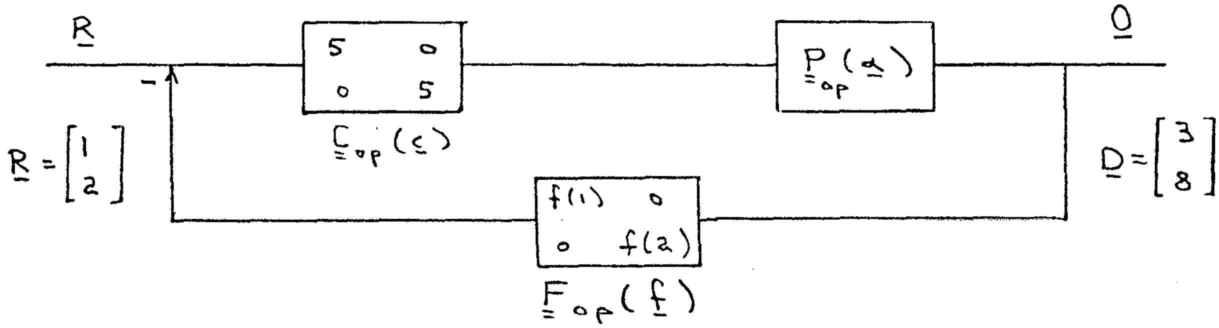
$$\underline{\underline{c}} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad ; \quad \underline{f} = \begin{bmatrix} .31 \\ .23 \end{bmatrix}$$

(c) J (t=5 sec) = .015227
J (t=1 sec) = .015211

$$J = \int (\underline{O} - \underline{D})^2 \, dt = .000016$$

<u>Test 2</u> Closed Loop; learn $\partial \underline{\underline{G}} / \partial \underline{f}$ ;
adapt only $\underline{f}$

That is the forward loop controller was fixed at $\underline{c} = [5, 5]^T$ and the system adapted by using only the feedback path controller variable $\underline{f}$ .



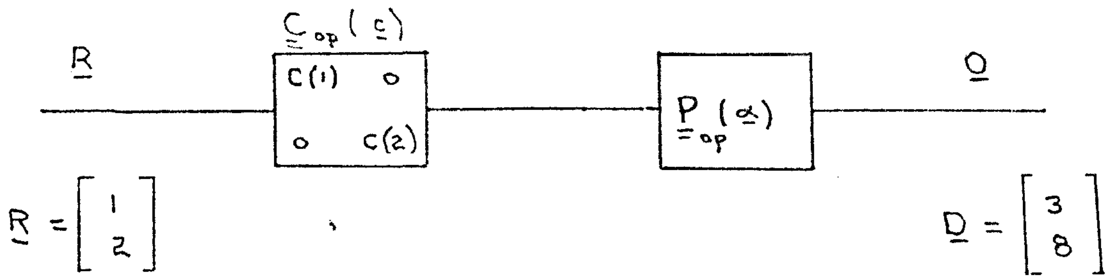

(a) $$\partial \underline{\underline{G}} / \partial \underline{f} = \begin{bmatrix} -9.331 & 7.590 \\ -.1939 & -36.16 \end{bmatrix} \text{ at } t = .4$$

(b) typical $\underline{f}$ values were
(c) J (t=5 sec) = 3.1586
J (t=1 sec) = 3.1586

$$J' = \int (\underline{O} - \underline{D})^2 \, dt < .00005$$

<u>Test 3</u> Open Loop; learn $\partial \underline{\underline{G}} / \partial \underline{c}$ ;
adapt $\underline{c}$

In the open loop configuration only $\partial \underline{G} / \partial \underline{c}$ need be learned ( if $\Delta \underline{R}$ is known to be zero) and the system is adapted by changing $\underline{c}$ .




(a) $\partial \underline{G} / \partial \underline{c} = \begin{bmatrix} 1.031 & .0842 \\ -.0180 & 1.534 \end{bmatrix}$ at t= 2.8 sec

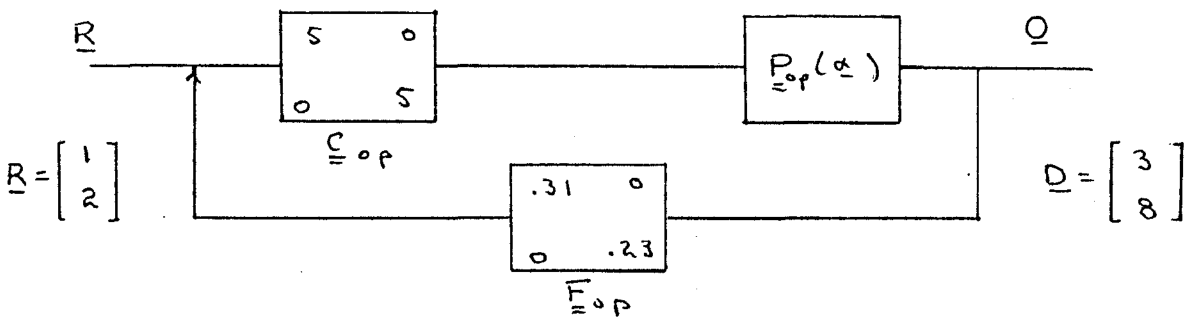
(b) typical $\underline{c}$ value was $\underline{c} = \begin{bmatrix} .3 \\ .4 \end{bmatrix}$

(c) J (t=5 sec) = 8.1635
J (t=3 sec) = 7.9703

$$J = \int_3^5 dt \, ( \underline{O} - \underline{D} )^2 = .1932$$

<u>Test 4</u> Closed Loop Nominal Optimal; $\underline{c}$ and $\underline{f}$ fixed

$\underline{c}$ and $\underline{f}$ were fixed at values found to be about the average of some of the values in Test 1.




(a) No learning.
(b) $\underline{c}$ fixed at $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ ; $\underline{f}$ fixed at $\begin{bmatrix} .3134 \\ .2995 \end{bmatrix}$

(c) $J\ (t=5 \text{ sec}) = .046049$

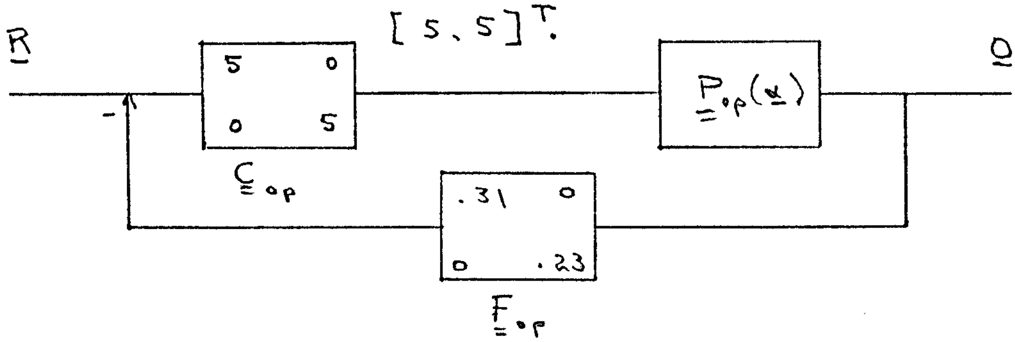
$\quad\ (t=1 \text{ sec}) = .013324$

$$J = \int_1^5 (\underline{O} - \underline{D})^2\, dt = .032725$$

Note that by comparing tests 1 and 4 the adaptive system yields about a 2000 times lower cost function.

Tests 5, 6, and 7 are closed loop adaptive, closed loop nominal optimal, and open loop respectively. $\underline{R}$ the input is allowed to fluctuate in addition to the plant parameters $\underline{\alpha}$ .

<u>Test 5</u> Closed Loop; learn $\partial \underline{G}/\partial \underline{R}$ and $\partial \underline{G}/\partial \underline{f}$ ; adapt $\underline{f}$ and keep $\underline{c}$ fixed at $[5, 5]^T$.




$$\underline{R} = \begin{bmatrix} 1.+ .30 \text{ sine } .04\, t \\ 2.+ .01 \sqcap .01\,\pi \end{bmatrix} \qquad \underline{D} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

(a) $$\partial \underline{G}/\partial \underline{R} = \begin{bmatrix} 3.103 & .0167 \\ -.0873 & .0365 \end{bmatrix} \text{ at } t = 9.6 \text{ sec}$$

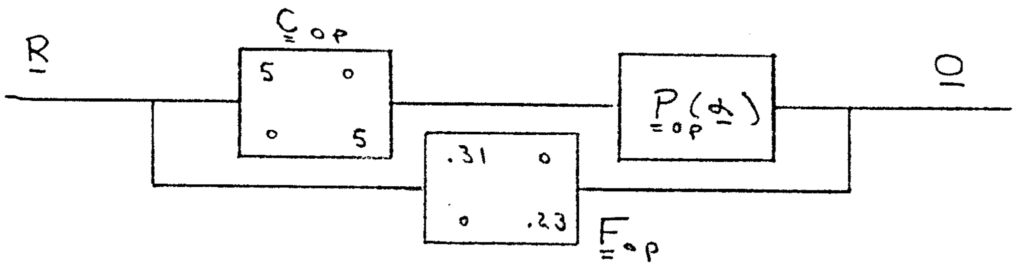
$$\partial \underline{G}/\partial \underline{f} = \begin{bmatrix} -10.29 & -.1459 \\ -1.265 & -27.51 \end{bmatrix} \text{at } t = 9.6 \text{ sec}$$

(b) $\underline{c} = [\,5\,,\,5\,]^T \qquad ; \qquad \underline{f} = [\,.3134\,,\,.2295\,]^T$

(c) J (t=10.) = 10.9107
J (t=6.0) = 10.9104

$$J = \int_6^{10} (\underline{O} - \underline{D})^2 \, dt = .0003$$

<u>Test 6</u> Closed Loop Nominal Optimal; no learning or adapting




$$R = \begin{bmatrix} 1 + .3 \sin .04\,t \\ 2 + .01 \sqcap .01\,\pi \end{bmatrix} \qquad \underline{D} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

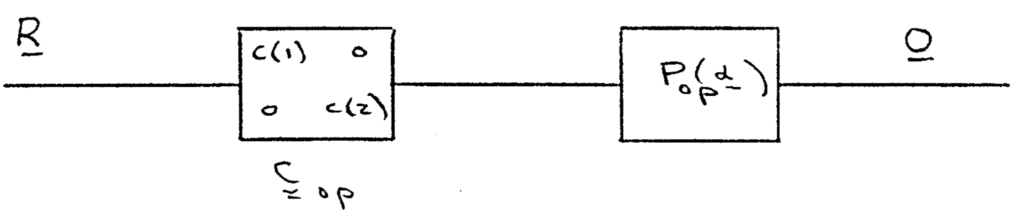
(a) No learning
(b) $\underline{c}$ fixed at $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ , $\underline{f}$ fixed at $\begin{bmatrix} .3134 \\ .2295 \end{bmatrix}$

(c) J (t=10) = .52054
J (t=6) = .19764

$$J = \int_6^{10} (\underline{O} - \underline{D})^2 \, dt = .32290$$

which is about 1000 times the J of Test 5, the adaptive counterpart.

<u>Test 7</u> Open Loop; learn $\partial \underline{G} / \underline{R}$ and $\partial \underline{G} / \partial \underline{c}$ ; adapt $\underline{c}$ .



$$\underline{R} = \begin{bmatrix} 1 + .3 \text{ sine } & .04\ t \\ 2 + .01 \sqcap & .01\ \pi \end{bmatrix} \qquad \underline{D} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

(a) $\partial \underline{G} / \partial \underline{R} = \begin{bmatrix} .689 & .0042 \\ 11.09 & .0851 \end{bmatrix}$ at t = 8.3 sec.

$\partial \underline{G} / \partial \underline{c} = \begin{bmatrix} .8124 & .0065 \\ 1.461 & 2.106 \end{bmatrix}$ at t = 8.3 sec.

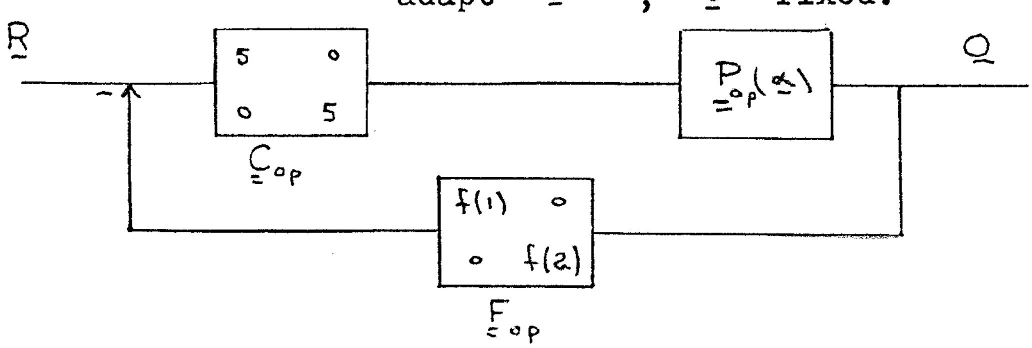
(b) $\underline{c} \sim \begin{bmatrix} .3 \\ .4 \end{bmatrix}$

(c) J (t=10 sec) = .81796
J (t=6 sec) = .81130

$$J = \int_6^{10} (\underline{Q} - \underline{D})^2\, dt = .00666$$

Tests 8, 9, and 10 are identical to tests 5, 6, and 7, except that in addition to varying $\underline{R}$ and $\underline{\alpha}$, $\underline{D}$ is varied as well.

<u>Test 8</u> Closed Loop; learn $\partial \underline{G}/\partial \underline{R}$ , $\partial \underline{G}/\partial \underline{f}$ ; adapt $\underline{f}$ , $\underline{c}$ fixed.




$$\underline{R} = \begin{bmatrix} 1. + .3 \text{ sine } & .04\ t \\ 2. + .01 & .01 \end{bmatrix} \qquad \underline{D} = \begin{bmatrix} 3 - \exp\ (-.01\ t) \\ 8 + .8 \text{ sine } .03\ t \end{bmatrix}$$

(a) $\partial \underline{G} / \partial \underline{R} = \begin{bmatrix} 2.535 & -.0057 \\ 39.74 & -.3992 \end{bmatrix}$ at t = 3.5 sec.

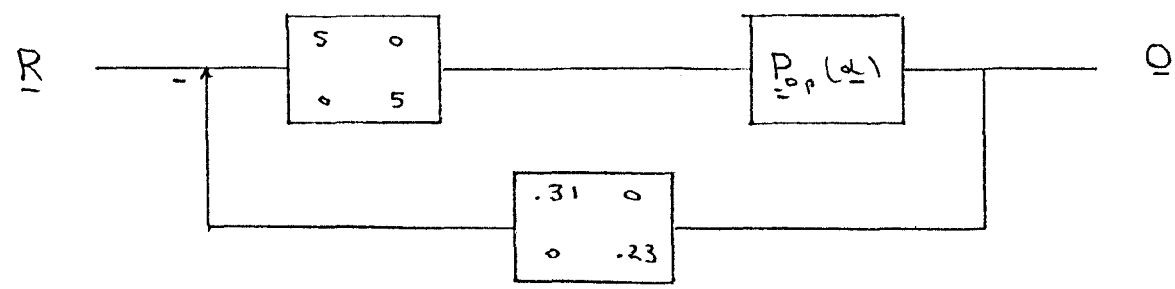
$\partial \underline{G} / \partial \underline{f} = \begin{bmatrix} -6.772 & .1698 \\ -38.17 & -23.03 \end{bmatrix}$

(b) $\underline{c} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ ; $\underline{f} = \begin{bmatrix} .3134 \\ .2295 \end{bmatrix}$

(c) J (t = 6 sec) = 11.8110
J (t = 2 sec) = 11.8080

$$J = \int_{2}^{6} (\underline{O} - \underline{D})^2 \, dt = .0030$$

<u>Test 9</u> Closed Loop Nominal Optimal; no learning or adapting .



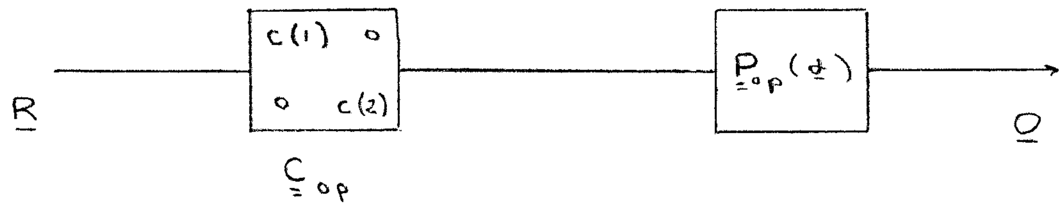

$\underline{R}$ $\underline{D}$

(a) no learning
(b) $\underline{c}$ fixed at $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$, $\underline{f}$ fixed at $\begin{bmatrix} .3134 \\ .2295 \end{bmatrix}$

(c) J (t=10 sec) = 3.9017
J (t=6 sec) = 2.4562

$$J = \int_{6}^{10} (\underline{O} - \underline{D})^2 \, dt = 1.4455$$

<u>Test 10</u> Open Loop; learn $\partial \underline{G} / \partial \underline{R}$ and $\partial \underline{G} / \partial \underline{c}$ ; adapt $\underline{c}$ .




$\underline{R}$ as in Test 8 $\underline{D}$ as in Test 8

(a) $$\partial \underline{G} / \partial \underline{R} = \begin{bmatrix} 2.499 & .0048 \\ -3.901 & .0097 \end{bmatrix}$$ at t = 3.5 sec.

$$\partial \underline{G} / \partial \underline{f} = \begin{bmatrix} 1.174 & .1091 \\ -.6705 & 1.859 \end{bmatrix}$$ at t = 3.5 sec.

(b) $$\underline{c} \sim \begin{bmatrix} .3 \\ .4 \end{bmatrix}$$

(c) J ( t = 7 sec ) = 4.2656

J ( t = 3 sec ) = 4.2641

$$J = \int_3^7 ( \underline{O} - \underline{D} )^2 \, dt = .0015$$

Table of Results of Experiment 3, Tests 1 to 10

| Test | Varying Quantities | Adaptive Quantities | Cost Function J Normalized to 4 Second Interval |
|---|---|---|---|
| 1. Closed Loop Self Adapting | $\underline{\alpha}$ | $\underline{c}, \underline{f}$ | .000016 |
| 2. Closed Loop Self Adapting | $\underline{\alpha}$ | $\underline{f}$ | .00005 |
| 3. Open Loop Self Adapting | $\underline{\alpha}$ | $\underline{c}$ | .3864 |
| 4. Closed Loop Nominal Optimal | $\underline{\alpha}$ | none | .032725 |
| 5. Closed Loop Self Adaptive | $\underline{\alpha}, \underline{R}$ | $\underline{f}$ | .0003 |
| 6. Closed Loop Nominal Optimal | $\underline{\alpha}, \underline{R}$ | none | .3229 |
| 7. Open Loop Self Adaptive | $\underline{\alpha}, \underline{R}$ | $\underline{c}$ | .00666 |
| 8. Closed Loop Self Adaptive | $\underline{\alpha}, \underline{R}, \underline{D}$ | $\underline{f}$ | .0030 |
| 9. Closed Loop Nominal Optimal | $\underline{\alpha}, \underline{R}, \underline{D}$ | none | 1.4455 |
| 10. Open Loop Self Adaptive | $\underline{\alpha}, \underline{R}, \underline{D}$ | $\underline{c}$ | .0015 |

table 3.

Tests 1 to 10 of Experiment 3 bear out the statements made, namely:

1. that the closed loop system, being less sensitive yields lower cost functions than the open loop system,

and 2. that the adaptive system is very much better than the nominal optimal system.

However, in performing this experiment it became clear that another problem, that of stability, became of major importance. The next section deals chiefly with the causes and effects of instability.

## 4. The Problem of Stability

Stability became a problem of importance with the advent of the feedback control system. In the $N^{th}$ order self adapting system, there is not one but three major feedback paths. The first is the normal feedback through $\underline{F}_{op}(\underline{f})$. The second and third are through the learn and adapt loop. Oscillations may occur in three different ways. Firstly there is the usual type of instability, where the feedback operator is such that the system's gain approaches infinity. Secondly one may find the adaptor loop driving $\underline{c}$ and $\underline{f}$ in a limit cycle of period $2\Delta t$. Thirdly one may find that a propagation of the noise error always present to some extent in $\underline{B}$ occurs causing the learn matrices to either oscilate or diverge from the true values.

It is felt that the stability of learning and adapting systems is of fundamental importance and deserves a thorough investigation.

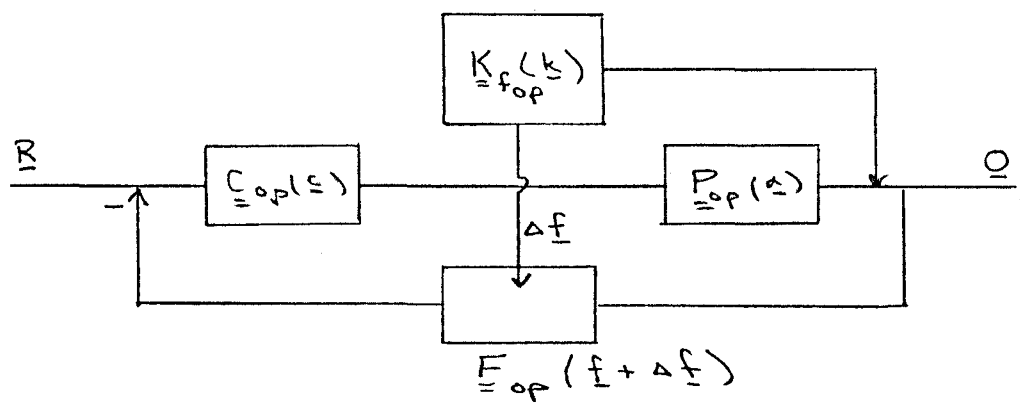
In Experiment 3, instability, present with high gains in the adaptor loop, could be suppressed by putting a reasonable limit on the quantities $\Delta\underline{f}$ and $\Delta\underline{c}$. These, limits prevented the system from entering regions of instability in $\underline{u}$ space[1] from which only accidental recovery appeared possible. On the other hand these limits were large enough to permit complete adaption so that their job was to keep the system stable during the first .1 seconds in which the first learning was occuring.

---

1. $\underline{u}$ space is defined in Chapter II, Section 1.

It must be pointed out that the adaptor equation (equation (3-3-4) ) was used to successively approximate each of the matrices which must be learned. Unfortunately there is no quarantee that the matrices will converge uniformly or otherwise to their true values. On the other hand if the system <u>does</u> adapt, then one can be assured that one has the true learn matrices and can be confident that higher order self adapting, better prediction, or faster operation ( $\Delta t$ smaller) will improve the performance even more.

The following diagram and equations represent a first look at the form of equations involved.




First note that $\underline{\underline{K}}_{f_{op}}$ is defined by that operator array operating on $\underline{O}$ which yields the correct $\Delta \underline{f}$

that is $\Delta \underline{f} = \underline{\underline{K}}_{f_{op}} \underline{O}$

whence

$$\underline{\underline{P}}_{op}(\underline{\alpha})\,\underline{\underline{C}}_{op}(\underline{c})\left[\underline{R} - \underline{\underline{F}}_{op}(\underline{f} + \underline{\underline{K}}_{f_{op}}\underline{O})\,\underline{O}\right] = \underline{O}$$

Now note that the form of $\underline{\underline{F}}_{op}$ and $\underline{G}$ must be known to proceed further.

## VI HIGHER ORDER ADAPTIVE SYSTEMS

For $N^{th}$ order adapting it is necessary not only to minimize $\|\underline{G}^1\|$ , but also $\|\underline{G}^i\|$, $i = 1, 2 . N$. This can only be accomplished if the derivatives of the output can be controlled independently.

For example, for second order adapting, $\underline{G} = [\underline{G}^1, \underline{G}^2]^T$. But $\underline{G}^2$ , being proportional to the time derivative of the output, introduces a dependence of $\underline{G}$ on the first derivatives of the system's outputs. To vary $\underline{G}^1$ and $\underline{G}^2$ then requires controllers which can independently change the outputs and their first derivatives. The least component controller is a matrix of first order operators rather than a matrix of constants as in the example of Chapter V. The smallest array which can possibly give second order adaption of an $M$ output plant is the following:

$$\underline{\underline{C}}_{op} = \begin{bmatrix} \frac{c_1}{c_{M+1}s+1} & 0 & & 0 \\ 0 & \frac{c_2}{c_{M+2}s+1} & & \\ \vdots & & \ddots & \vdots \\ 0 & & & \frac{c_M}{c_{M+M}s+1} \end{bmatrix}$$

These $2M$ controller variables $\{c_m\}$ are what one would expect since $\underline{G}^1$ is an $M$ vector as is $\underline{G}^2$ . $\partial \underline{G}/\partial \underline{c}$ is then a $2M$ by $2M$ matrix. $\underline{G}$ , a vector of $2M$ dimensions, is thus controlled by a $2M$ variable controller $\underline{\underline{C}}_{op}$ .

# VII CONCLUSIONS

This chapter contains a sumary of the entire thesis followed by a list of conclusions which can be made on the basis of the arguments presented.

The usual integral form of cost function is abandoned in favor of a generalized vector cost function $\underline{G}$ which allows $N^{th}$ order adapting at intervals $\Delta t$ apart and which, if $\|\underline{G}\|$ be minimized, guarantees the system reading a locally minimum cost function $J$. The vector cost function $\underline{\underline{G}}$ is defined, $\underline{G} - \underline{G}^*$ [1] such that minimizing the norm of $\underline{\underline{G}}$ takes the system to a globally minimum $J$ value. Note that there are as many controller variables as there are dimensions of $\underline{\underline{G}}$ and that the order of the operators in the operator arrays $\underline{\underline{C}}_{op}$ and $\underline{\underline{F}}_{op}$ need be at least of order $N$ in an $N^{th}$ order self adaptive system[2]. Further note that in general since $\underline{G}$ (and thus $\underline{\underline{G}}$) are formed from quantities in $H$ ($H$ the integrand of $J$) that no new variables need be measured to find $\underline{\underline{G}}$.

With vector cost function $\underline{\underline{G}}$ defined, it is possible to write a difference equation for $\Delta \underline{\underline{G}}$. Through the way in which $\underline{\underline{G}}$ has been defined it is possible to have an $N^{th}$ order self adaptive system while merely using the first term in the Taylor expansion for $\Delta \underline{\underline{G}}$. This expression for $\Delta \underline{\underline{G}}$ has been called the "general adaptor equation" and the partial derivatives (matrices, being partials of a

---

1. $\underline{G}^*$ and methods of finding $\underline{G}^*$ are discussed in Chapter II, Sections 3 and 5. Note also that if the conditions of Theorum I, Chapter II, Section 1 are met then $\underline{G}^* \equiv \underline{0}$.

2. See Chapter VI.

vector with respect to another vector) in the equation have been called sensitivities or more commonly "learned matrices" due to the fact that they must be learned while the system is in operation. The quantity $\partial \underline{E}/\partial \underline{\alpha}\, \Delta \underline{\alpha}$ is predicted and the system driven to and kept at $\underline{E} = \underline{0}$ by changing the controller parameters $\underline{c}$ and/or $\underline{f}$ .

In Chapter III, Section 2, $\partial \underline{E}/\partial \underline{\alpha}$ is discussed as the sensitivity of the system and equation (3-2-18) developed to differentiate between closed loop and open loop adaptive systems on the basis of improved sensitivity. The experiments of Part II bear out the prediction that closed loop systems are less sensitive.

The experiments of PART II prove too that the self adaptive system is feasible and useful as indicated by the results in Tables 1, 2, and 3. Difficulties initially encountered indicated, however, the serious problem of stability of multiloop systems particularly with high gain and inherent time delays.

The problem of instability was encountered in Experiment 3 of Chapter V when the learned matrices were the initial sets of random numbers and the system entered unstable $\underline{u}$ space before these random arrays could be corrected. The problem was overcome by limiting the gain; that is $\|\Delta \underline{c}\| < \Delta \underline{c}$ maximum and $\|\Delta \underline{f}\| < \Delta \underline{f}$ maximum. If this method gets no results it is likely that one can fall back on an open loop system of self adapting which, if combined with limited adaptor gains, will almost certainly be stable.

<u>List of Conclusions</u>

1. Neither the plant $\underline{P}_{op}$ nor the plant parameter change $\Delta\underline{\alpha}$ need be known to utilize $N^{th}$ order self adapting.

2. Noise and even totally random $\Delta\underline{\alpha}$ values can be handled. The high frequency noise components which appear at $\underline{Q}$ are smoothed and predicted as their average value. However the low frequency or band limited components are accurately predicted and their effect nullified by the adaptor.

3. Often it will be found that the learned martices are virtually constant in which case the entire learn and adapt loop may be replaced by a network of at most $K$ active elements where $K$ is the number of dimensions in $\underline{G}$ . This fact is extremely important because it releases the digital computer.

4. If there are no detrimental constraints ( the conditions of Theorem I are met) or if one is satisfied with a locally minimum J value, the calculations which are required to be done manually are nonexistant or at worst trivial.

5. The $N^{th}$ order self adaptive system in addition to adapting to changes in $\Delta\underline{\alpha}$ is able to adapt to changes in $\underline{R}$, the generalized input vector, and $\underline{D}$, the desired output.

6. The controllers and/or the plant may be nonlinear. The only limitation in this respect is that $\underline{G}$ be piecewise continuous at least and continuous if at all possible.

7. With a slight modification the plant and/or the controllers may have a time delay provided the time delay is measurable or is known in some fashion. The system in its present form is able however to adapt to plants which have a time delay $\tau \ll \Delta t$.

8. Even if the plant parameters do not vary the self adapting system will adapt to change in $\underline{D}$ and/or $\underline{R}$ .

9. Selecting the forms for $\underline{C}_{op}$ and/or $\underline{F}_{op}$ the adaptor loop will drive them to their best values. Removing the adaptor loop them leaves the controllers at their nominal optimal values -- values which might have been otherwise impossible to obtain. A whole range of optimal control problems and optimal filter problems are open to a solution free from manual calculation and more accurate as well in that the actual plant, not some model of it, may be used.

10. The vector cost function $\underline{G}$ has been defined in such a manner that the first omitted term in the Taylor series expansion of $\underline{G}'(t+\Delta t)$ is multiplied by a factor of $\Delta t^N / N!$ . This indicates the level of accuracy which the $N^{th}$ order self adapting system operates at. However there is no point in using a system of order $L$ where $\Delta t^L / L!$ is any smaller than the expected error in the predicted quantity $\underline{B}$ ( $\underline{B} = \partial \underline{G} / \partial \underline{\alpha} \, \Delta \underline{\alpha}$ plus other system noise).

11. No extra measurements need be introduced to find $\underline{G}$ than were necessary for finding $J$ and $H$ .

12. One can simply and naturally include the overall system sensitivity as a criterion in the cost function by merely using $N > 1$, that is, a higher than first order self adapting system. This apparently providential by-product is linked with the generalization of the vector cost function from first order to higher order self adapting which in turn is done to obtain any prespecified degree of accuracy in the cost function.

APPENDIX

# APPENDIX 1

The computer program of Experiment 3, Test 10 follows.

It is included as an aid for those who might wish to design more sophisticated $N^{th}$ order self adaptive systems. The program is built of various blocks and while the program is admittedly not particularly efficient, the blocks themselves can be considered basic. However as a first program it passes the most important test. — that is, it works.

```
$JOB   WATFOR  003511 TEMPLE
$IBJOB         NODECK
$IBFTC TEST10
C     00  TO  99     INPUT SECTION
C   DIMENSIONS
      MC=1
      ML=0
      MR=1
      MF=1
      DIMENSION GG(5)                 ,RRR(5)
      DIMENSION   DGD(5)
      DIMENSION DGCF(5)
      DIMENSION DGR(5),DGT(5),DGF(5),DGC(5)        ,PGPT(5)
      DIMENSION C(5) ,CC(5),DLC(5),DLF(5) ,PGPD(5,5)
      DIMENSION     F(5),FF(5),P(5),E(5),R(5)
      DIMENSION         BIGC(5),SMALLC(5)
      DIMENSION        G(5),B(5),PGPR(5,5),PGPF(5,5),PGPC(5,5)
      DIMENSION        DLR(5),DLD(5),D(5),ZK(5),GMIN(5)
      DIMENSION      EB(5)       ,PGT(5)
      DIMENSION      DELTB(5)
      DIMENSION          DVF(5),Z(5)
      DIMENSION           DVC(5)
      DIMENSION            CCU(5),CCM(5),FFU(5),FFM(5)
      DIMENSION A(25,25),AA(25,25)
C   INITIAL VALUES  AND  CONSTRAINTS
      READ(5,1)     SP11,SP12,SP21,SP22,P11,P12,P21,P22
      READ(5,2) (CCM(I),I=1,2),(CC(I),I=1,2),(CCU(I),I=1,2)
      READ(5,2) (FFM(I),I=1,2),(FF(I),I=1,2),(FFU(I),I=1,2)
      READ(5,3) ((PGPR(I,J),J=1,2),I=1,2),((PGPF(I,J),J=1,2),I=1,2),
     1((PGPC(I,J),J=1,2),I=1,2),((PGPD(I,J),J=1,2),I=1,2)
      READ(5,4) (R(I),I=1,2),(ZK(I),I=1,2)
      READ(5,5) (BIGC(I),I=1,2),(SMALLC(I),I=1,2)
1      FORMAT(8F10.4)
2      FORMAT(6F12.4)
3     FORMAT(4F20.8)
4     FORMAT(2F20.8)
5     FORMAT(4F20.8)
C    ' DISCRETIONAL '    CONSTANTS
      READ(5,10)MEXP,LMAX,ZN,DELT   ,DELT1
10      FORMAT(2I10,3F20.8)
          DO 20 I=1,2
      GG(I)=0.
      P(I)=0.
      RRR(I)=0.
      G(I)=0.
      DGD(I)=0.
      PGPT(I)=0.
      GMIN(I)=0.
      D(I)=R(I)*ZK(I)
      F(I)=0.
      PGT(I)=0.
      DLR(I)=0.
      DELTB(I)=0.
      DLF(I)=0.
20      DLC(I)=0.
      WRITE(6,15)
15    FORMAT(54H          COST FUNCTION      VECTOR COST     PREDICTED
      WRITE(6,16)
16    FORMAT(54H            .01 SEC. INT'LS   FUNCTION 'G'       ERROR 'B'
```

```
      WRITE(6,17)
17    FORMAT(1H0)
      ILN=5
      L=1
        ZL=L
      ZS=0.
      ZZS=1.
      ZJ=0.
      D1P1=0.
      D1P2=0.
      C1=0.
      C2=0.
      P(1)=3.
      PGPR(2,2)=0.
      P(2)=4.
      PGPF(1,1)=-4.
      PGPF(2,2)=-9.
      GO TO 150
C  100  TO  199  BLOCK
C SIMULATION OF FEEDBACK PATH CONTROLLER TRANSFER FUNCTION (OPERATOR
C MATRIX  F'OP' ).
100   CONTINUE
      DO 101 I=1,2
101   F(I)=FF(I)*P(I)
150   CONTINUE
      DO 199 I=1,2
199   E(I)=R(I)-F(I)
      GO TO 400
C  400  TO  500  BLOCK
C SIMULATION OF THE FORWARD PATH CONTROLLER TRANSFER FUNCTION ( OPERATOR
C  MATRIX  C'OP' )
400    CONTINUE
       DO 401 I=1,2
401   C(I)=CC(I)*R(I)
       GO TO 500
C   500  TO  599   BLOCK
C   LIMITING IMPOSED ON THE PLANT UNPUT 'C(I)' .
 500  CONTINUE
      DO 501 I=1,2
      IF(C(I).GT.BIGC(I))C(I)=BIGC(I)
      IF(C(I).LT.SMALLC(I))C(I)=SMALLC(I)
501   CONTINUE
      GO TO 700
C  700  TO  799   BLOCK      'P'OP''
C   PLANT  TRANSFER  FUNCTION
700    CONTINUE
       D1C1=(C(1)-C1)/DELT
       C1=C(1)
       D1C2=(C(2)-C2)/DELT
       C2=C(2)
       D2P1 =P11*SP12*C(1) +P11*D1C1 +P12*SP11*C(2) +P12*D1C2
     1  -(SP11+SP12)*D1P1  -SP11*SP12*P(1)
       D1P1=D2P1*DELT+D1P1
       P(1)=D1P1*DELT+P(1)
       D2P2= P22*SP21*C(2) +P22*D1C2 +P21*SP22*C(1) +P21*D1C1
     1   -(SP22+SP21)*D1P2 -SP22*SP21*P(2)
       D1P2=D2P2*DELT+D1P2
      P(2)=D1P2*DELT+P(2)
```

```
      ML = ML+1
      DO 701  I=1,2
      DGT(I)=P(I)-D(I)-GG(I)
      PGPT(I)=DGT(I)/DELT
701   GG(I)=P(I)-D(I)
      IF(ML.EQ.10)GO TO 1000
      GO TO 100
C  1000  TO  1999   BLOCK
C   MEASURE  AND  PREDICT  SECTION
1000  CONTINUE
      ML=0
      DO 1008 I=1,2
      DGCF(I)=0.
      DO 1008 J=1,2
1008  DGCF(I)=DGCF(I)+PGPF(I,J)*DLF(J)+PGPC(I,J)*DLC(J)
      DO 1001 I=1,2
C  MEASURE G(I)
      IF(L.LT.100)RRR(I)=0.
      DGC(I)=P(I)-D(I)-G(I) -PGT(I) *DELT1          -RRR(I)  -DGD(I)
      DGR(I)=P(I)-D(I)-G(I) -PGT(I) *DELT1 -DGCF(I)  -DGD(I)
      DGF(I)=P(I)-D(I)-G(I) -PGT(I) *DELT1        -RRR(I) -DGD(I)
      G(I) =P(I)-D(I)
1001  B(I)=G(I)-GMIN(I)
      DO 1005 I=1,2
      PGT(I)=PGPT(I)
1005  GMIN(I)=G(I)      +PGPT(I)*DELT1
      GO TO 800
C  800  TO  899   BLOCK
C  PRINTED  OUTPUT  SECTION
800   CONTINUE
      DO 801 I=1,2
801    ZJ=ZJ+G(I)**2*DELT1
      IF(L.GT.LMAX)GO TO 5000
      DO 803 I=1,2
      WRITE(6,802)  L,ZJ,G(I),B(I)
802   FORMAT(1H ,9X,I4,3X,E12.6,3X,E12.6,3X,E12.6)
803   CONTINUE
      GO TO 900
C  2100  TO  2199   BLOCK
C  UPDATE PARTIAL G(I)  PARTIAL   C(J)
2100   CONTINUE
       IF(ILN.NE.1)GO TO 2200
      IF(ABS(DLC(MC)).LT.1./10.**MEXP)GO TO 2110
       DO 2101 J=1,2
2101  PGPC(J,MC)=DGC(J)/DLC(MC)
      WRITE(6,2104)
2104  FORMAT(1H0,15X,45H PGPT                      PGPC
      DO 2106  I=1,2
      WRITE(6,2105)     PGPT(I)                      ,(PGPC(I,J),J=1,2)
2105  FORMAT(13X,E12.6,10X,E12.6,3X,E12.6   )
2106  CONTINUE
2110  MC=MC+1
      IF(MC.GT.2)MC=1
      GO TO 2700
C  2200  TO  2299
C  UPDATE PARTIAL G(I) PARTIAL F(J)
2200  CONTINUE
      IF(ILN.NE.3)GO TO 2300
```

```
      IF(ABS(DLF(MF)).LT.1./10.**MEXP)GO TO 2210
      DO 2201 J=1,2
2201  PGPF(J,MF)=DGF(J)/DLF(MF)
      WRITE(6,2204)
2204  FORMAT(1H0,15X,45H PGPT                          PGPF
      DO 2206  I=1,2
      WRITE(6,2205)     PGPT(I)                        ,(PGPF(I,J),J=1,2)
2205  FORMAT(13X,E12.6,10X,E12.6,3X,E12.6   )
2206  CONTINUE
2210  MF=MF+1
      IF(MF.GT.2)MF=1
      GO TO 2800
C  2500  TO  2599   BLOCK
C  CHANGE SINGLE FORWARD PATH CONTROLLER ELEMENT IN ORDER TO EVALUATE
C  THE COLUMN VECTOR PGPC(I,MC)
2500  CONTINUE
      IF(ILN.NE.0)GO TO    2600
      GO TO 2700
2501  CONTINUE
      DO 2505  I=1,2
      DLF(I)=0.
2505  DLC(I)=0.
      DLC(MC)=DVC(MC)
      DO 2508 I=1,2
      IF(DLC(I).LT.(-.15))DLC(I)=-.15
      IF(DLC(I).GT..15)DLC(I)=.15
2508  CONTINUE
      DO 2510  I=1,2
      Z(I)=0.
      DO 2509  J=1,2
2509  Z(I)=Z(I)+PGPC(I,J)*DLC(J)
2510   GMIN(I)=GMIN(I)+Z(I)
      GO TO 200
C  2600  TO  2699   BLOCK
C  ILN = 4 BLOCK  DLF(I)  AND  DLC(I) =0. TO UPDATE PGPR(J,MR)
2600  CONTINUE
      DO 2605 I=1,2
      DLF(I)=0.
2605  DLC(I)=0.
      GO TO 200
C  2300  TO  2399
C  UPDATE PARTIAL G(I) PARTIAL R(J)
2300  CONTINUE
      IF(L.LT.100)GO TO 2400
      IF (ILN.NE.5)GO TO 2400
      IF(ABS(DLR(MR)).LT.1./10.**MEXP) GO TO 2310
      DO 2301 J=1,2
2301  PGPR(J,MR)=DGR(J)/DLR(MR)
      WRITE(6,2450)((PGPR(I,J),J=1,2),I=1,2)
2450  FORMAT(60X,2E12.6)
2310  MR=MR+1
      IF(MR.GT.2)MR=1
      GO TO 2800
C  2400  TO  2499   BLOCK
C  CHANGE SINGLE FEEDBACK PATH CONTROLLER ELEMENT  IN ORDER TO EVALUATE
C THE COLUMN VECTOR  PGPF(I,MF)
2400  CONTINUE
      IF(ILN.NE.2)GO TO 2500
```

```
      GO TO 2800
2401  CONTINUE
      DO 2405 I=1,2
      DLC(I)=0.
2405   DLF(I)=0.
      DLF(MF)=DVF(MF)
      DO 2410 I=1,2
      Z(I)=0.
      DO 2409 J=1,2
2409  Z(I)=Z(I)+PGPF(I,J)*DLF(J)
2410  GMIN(I)=GMIN(I)+Z(I)
      GO TO 200
C  200  TO  299  BLOCK
C SET THE CONTROLLERS TO THE NEW OPORATOR MATRICES
200   CONTINUE
       L=L+1
       ILN=ILN+1
       IF(ILN.EQ.6)ILN=0
      IF(ILN.EQ.2)ILN=4
       DO 201 I=1,2
      IF(DLC(I).LT.(-.15))DLC(I)=-.15
      IF(DLC(I).GT..15)DLC(I)=.15
201    CC(I)=CC(I)+DLC(I)
      WRITE(6,222)(P(I),I=1,2)
222   FORMAT  (1H0,19X,4HF(I),3X,E12.6,3X,E12.6                 )
       GO TO 100
C   2700  TO  2799   BLOCK
C  CALCULATE THE ADAPTING CHANGE IN CC(I) TO GIVE G(T+DELT)=0. OR ITS
C  MINIMUM VALUE
2700  CONTINUE
      DO 2701 I=1,2
      DO 2701 J=1,2
2701   A(I,J)=PGPC(I,J)
      A1=A(1,1)
      DET=A(1,1)*A(2,2)-A(1,2)*A(2,1)
      A(1,1)=A(2,2)/DET
      A(2,2)=A1/DET
      A(1,2)=-A(1,2)/DET
      A(2,1)=-A(2,1)/DET
      DO 2705 I=1,2
      DVC(I)=0.
      DO 2705 J=1,2
2705  DVC(I)=DVC(I)-A(I,J)*GMIN(J)
      DO 2710 I=1,2
      IF((DVC(I)+CC(I)).GT.CCU(I))DVC(I)=CCU(I)-CC(I)
      IF((DVC(I)+CC(I)).LT.CCM(I))DVC(I)=CCM(I)-CC(I)
2710  CONTINUE
      IF(ILN.EQ.0)GO TO 2501
      DO 2712 I=1,2
      IF(DVC(I).GT..15)DVC(I)=.15
      IF(DVC(I).LT.(-.15))DVC(I)=-.15
2712  CONTINUE
      DO 2715 I=1,2
      Z(I)=0.
      DO 2714 J=1,2
2714  Z(I)=Z(I)+PGPC(I,J)*DVC(J)
      DLC(I)=DVC(I)
2715  GMIN(I)=GMIN(I)+Z(I)
```

```
      GO TO 200
C   2800   TO   2899   BLOCK
C CALCULATE THE ADAPTING CHANGE IN FF(I) TO GIVE G(T+DE T) A MINIMUM
2800  CONTINUE
      GO TO 2700
C   900   TO   999   BLOCK
C   STATIC   CALCULATIONS   AND   SIGNAL   CALCULATIONS.
900    CONTINUE
      ZL=L
      IF(ZS.LT.100.)GO TO 990
      ZS=0.
      ZZS=ZZS*(-1.)
990   ZS=ZS+1.
903   R1=       1.+.3*SIN(.04*ZL*DELT1)
      R2=  2.+ .01*ZZS
      DLR(2)=R2-R(2)
      DLR(1)=R1-R(1)
       R(1)=R1
      R(2)=R2
      DO 905 I=1,2
      RRR(I)=0.
      DO 905 J=1,2
905   RRR(I)=RRR(I)+PGPR(I,J)*DLR(J)
       SP11=10.+2.*SIN(ZL*DELT1)
       SP22= 10.+2.*SIN(2.*ZL*DELT1)
       P12=1.+.5*SIN(3.*ZL*DELT1)
       P21=-1.+.5*SIN(5.5*ZL*DELT1)
      D1=3.-EXP(-(.01*ZL))
      D2=8.+.8*SIN(ZL*.03    )
      DLD(1)=D1-D(1)
      DLD(2)=D2-D(2)
      DGD(1)=-DLD(1)
      DGD(2)=-DLD(2)
      D(1)=D1
      D(2)=D2
       GO TO 2100
5000   CONTINUE
       END
$ENTRY
10.       20.       20.       10.       100.       1.       -1.       1
-10000.      -10000.      .5               .5       10000.      10000.
-100.      -100.                                      100.         100.
10.                                                                  10.
    10.                                                           10.
-10.                                                                   -
  -1.                                                           -1.
1.                    2.
3.                        4.
200.                  200.                 -200.                -200.
         4     10000  20.                   .001                    .01
$IBSYS
```

# APPENDIX II

## Definition of Basic Symbols

### Vector Quantities

A single line or bar under a quantity denotes it as a vector or one dimensional array.

| | |
|---|---|
| $\underline{R}$ | the system's inputs |
| $\underline{O}$ | the system's outputs |
| $\underline{D}$ | the system's desired outputs |
| $\underline{\alpha}$ | the vector of nonconstant plant parameters |
| $\underline{c}$ | the vector of parameters of the forward loop controller |
| $\underline{f}$ | the vector of parameters of the feedback loop controller |
| $\underline{J}, \underline{G}, \underline{G}$ | vector cost functions |

Other important vector quantities are the following:

| | |
|---|---|
| $\underline{B}$ | the predicted vector $= \partial \underline{G}/\partial \underline{\alpha} \; \Delta \underline{\alpha}$ + noise |
| $\partial \underline{G}/\partial t$ | the explicit variation of $\underline{G}$ with time |

### Matrix Quantities

| | |
|---|---|
| $\underline{\underline{S}}_{op}$ | that array of operators which represents the transfer function of the entire system, that is, $\underline{O} = \underline{\underline{S}}_{op} \underline{R}$ |

$\underline{\underline{S}}$ with or without integral number subscripts, denotes the sensitivity of the vector cost function with respect to changes in plant parameters

$$\underline{\underline{S}} = \partial \underline{G} / \partial \underline{\alpha}$$

Other important matrix quantities are the following

$$\partial \underline{G} / \partial \underline{\alpha} \, , \; \partial \underline{G} / \partial \underline{f} \, , \; \partial \underline{G} / \partial \underline{c} \, , \; \partial \underline{G} / \partial \underline{D} \, , \; \partial \underline{G} / \partial \underline{R}$$

All of these denote the various sensitivities of the vector cost function $\underline{G}$ to important system variables.

Scalar Quantities

$J$ the original cost function often in the form $J = \int H \; dt$

$H$ the integrand, often of one sign only, of J.

$t, \Delta t$ time quantities

Miscellaneous

$\forall$ means "for all"

$\|Q\|$ If quantity is a vector, say $\| \underline{x} \|$ , then $\| \underline{x} \| = \underline{x} \cdot \underline{x}$

If quantity is a matrix, say $\| \underline{\underline{X}} \|$ , then

$$\| \underline{\underline{X}} \| = \sum | X_{ij} |$$

'op' The subscript 'op' on an array means that the quantities in that array may in general be operators.

## APPENDIX III

### References

Though no references were used in preparation and though none have since been found that are directly applicable it is felt that the following may be helpful. The two books deal with adaptive systems of different kinds. The paper referred to is one which supports the direct philosophy of adaptive control and may be helpful on that account.

Eveleigh, 1967, "Adaptive Control and Optimization Techniques", (McGraw Hill).

Mishkin and Braun, 1961, "Adaptive Control Systems" (McGraw Hill)

Zaborsky, J., and Humphrey, W. L., 1964, "Control Without Model or Plant Identification", I.E.E.E. Transactions on Applications and Industry, November 1964.