ON THE EXISTENCE OF NON-ZERO-LINEAR

CONTINUOUS FUNCTIONALS ON FRÉCHET SPACES

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CONTINUOUS FUNCTIONALS ON FRÉCHET SPACES

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SCOPE AND CONTENTS:

This thesis is concerned with a necessary and sufficient condition for the existence of non-zero linear continuous functionals on Fréchet or more general topological vector spaces. The main idea is based on the famous Hahn-Banach theorem. Since the connection between Hahn-Banach theorem and separation theorems is well known, here we study some separation theorems as well.

PREFACE

We investigate the concepts of B_0-space, locally convex topological vector space, Fréchet space in detail and prove the Hahn-Banach extension and Banach separation theorems. Furthermore, we give a necessary and sufficient condition for the existence of non-zero linear continuous functionals on certain topological vector spaces. These results are given in sections 1 to 6.

In the last section 7, we extend the Banach separation theorem from normed linear spaces to certain topological vector spaces.

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TABLE OF CONTENTS

1. Introduction

We know (M. M. Day [1]) that there exists a topological vector space, namely $L^p(0, 1)$, $0 < p < 1$, on which there is no non-zero continuous linear functional.

In this paper, we will give a necessary and sufficient condition for the existence of non-zero continuous linear functionals on any real Fréchet space - a vector space with a Fréchet norm abbreviated F-norm which is due to S. Mazur and W. Orlicz [4]. Our basic idea is based on the famous Hahn-Banach extension theorem. We use the method of T. Husain [2].

We collect our notations and terminology in Section 2. In Section 4, we discuss a subclass of Fréchet spaces which is called the class of B_0-spaces, and then investigate the existence of non-zero continuous linear functionals. Essentially, a B_0-space is a locally convex topological vector space. If a topological vector space is locally convex, then there exists a non-zero continuous linear functional (see A. Wilansky [7]). But the converse is not necessarily true as we show by an example in Section 5. Furthermore, S. Mazur and W. Orlicz [4] found a necessary and sufficient condition for linear metric space to be B_0-normable. A. Kolmogoroff [3] (also see A. Taylor [6]), proved the necessary and sufficient condition for topological vector spaces to be B-normable. Using theorem 6.1, we derive two theorems of S. Mazur and A. Kolmogoroff from theorems 6.2 and 6.3. Moreover, we obtain our desired result, namely theorem 6.4, and then extend it to general topological vector spaces in theorem 6.5.

In connection with the existence of non-zero continuous linear functionals on a topological vector space (TVS). We give Banach separation theorem and Eidelheit theorem in Section 3, and then extend them to general topological vector spaces (theorem 7.3 and theorem 7.4).

2. Notations and terminology

Let X be a linear space with more than one element, and the scalars are real numbers unless otherwise specified.

If X is furnished with a topology T such that addition and scalar multiplication are continuous, we say that X is a topological vector space abbreviated TVS. If each neighborhood of 0 in a TVS X contains a convex neighborhood of 0, we say that X is a locally convex topological vector space, abbreviated LCTVS.

Definition 2.1

Let X be a linear space. We say $\| \|_F$ is a Fréchet norm on X if it satisfies,

(1) $\|x\|_F = 0$ if and only if $x = 0$, and $\|x\|_F \geq 0$ for all x in X

(2) $\|x + y\|_F \leq \|x\|_F + \|y\|_F$ for all x, y in X

(3) $\|-x\|_F = \|x\|_F$, and

(4) If $t_n \longrightarrow t_0$, $\|x_n - x_0\|_F \longrightarrow 0$ then $\|t_n x_n - t_0 x_0\|_F \longrightarrow 0$ where t_n, t_0 scalars and x_n, x in X.

Definition 2.2

Let X be a linear space. We say $\| \|$ is a seminorm on X, if

(1) $\|x\| \geq 0$ for all x in X

(2) $\|tx\| = |t| \cdot \|x\|$ for all x in X and all scalars t, and

(3) $\|x - y\| \leq \|x\| + \|y\|$ for all x, y in X.

If the condition (1) is changed to "$\|x\| = 0$ if and only if $x = 0$ for each x in X, and $\|x\| \geq 0$ for all x in X", then $\| \ \|$ is called a norm on X.

Definition 2.3

Let X be a linear space, and $\{\| \ \|_i,\ i = 1, 2, \ldots, n, \ldots\}$ be a sequence of seminorms defined on X in which $\|x\|_i = 0$ for each i then x = 0. We say $\| \ \|_{B_0}$ is a B_0-norm, if

$$\|x\|_{B_0} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x\|_i}{1 + \|x\|_i}, \quad x \in X.$$

It is easy to see that Fréchet norm, norm and B_0-norm define a metric. Hence spaces with these norms are metric spaces.

Definition 2.4

If $(X, \| \ \|)$ is a complete normed linear space, then we call $(X, \| \ \|)$ a Banach space.

If $(X, \| \ \|_{B_0})$ is a complete B_0-norm space, then we call $(X, \| \ \|_{B_0})$ a B_0-space.

If $(X, \| \ \|_F)$ is a complete Fréchet normed space, then we call $(X, \| \ \|_F)$ a Fréchet space.

Definition 2.5

If X is a linear space, we say f is a linear functional on X provided f is scalar-valued and

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all x, y in X and all scalars α, β.

Definition 2.6

Let (X, d) be a metric space with metric d and f a function on X. We say that f is continuous if it is continuous with respect to this metric d. (i.e. if x_n converges to x_0 with respect to the metric d, then $f(x_n) \longrightarrow f(x_0)$).

Definition 2.7

A set X with binary relation ($\leq$) is said to be a partially ordered set if the following conditions hold:

(1) $x \leq x$ for all $x \in X$.

(2) If $x \leq y$ and $y \leq x$ then $x = y$.

(3) If $x \leq y$ and $y \leq z$ then $x \leq z$.

A partially ordered set with binary relation $\leq$ is denoted by $(X, \leq)$.

Definition 2.8

A partially ordered set $(X, \leq)$ is said to be a linearly ordered set if for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

Definition 2.9

b is said to be a maximal element in a partially ordered set X if there exists no element x,

$$x \geq b \text{ where } x \in X,\ x \neq b.$$

Definition 2.10

Let A be a subset of a partially ordered set $(X, \leq)$. b is said to be an upper bound if $b \in X$, and

$$b \geq x \text{ for all } x \in A.$$

Zorn's Lemma. Let $(X, \leqslant)$ be a partially ordered set. If every linearly ordered subset of $(X, \leqslant)$ has an upper bound, then X contains a maximal element.

Definition 2.11

A set D is said to be a directed set if D is a partially ordered set with partially ordered relation $\geq$, and for every $\alpha, \beta \in D$ there exists $\gamma \in D$ such that $\gamma \geq \alpha$, $\gamma \geq \beta$.

Definition 2.12

A net in a topological space is a function defined on a directed set D with values in that topological space and *is* denoted by $(x_\alpha, \alpha \in D)$.

Definition 2.13

A net $(x_\alpha, \alpha \in D)$ is said to be eventually in a set B if there exists $\alpha_0 \in D$ such that $x_\alpha \in B$ for all $\alpha \geq \alpha_0$. It is said to be frequently in B if for any $\alpha_0 \in D$ there exist some α such that $\alpha \geq \alpha_0$ and $x_\alpha \in B$.

Definition 2.14

Let $(x_\alpha, \alpha \in D)$ be a net in a topological space X. x_α is said to converge to a (i.e. x_α is a net converging to a) if for any neighborhood U_a of a, x_α is eventually in U_a, *and* denoted by $x_\alpha \longrightarrow a$.

Example 2.1. Let X be a topological space, and $a \in X$. Let D be the set of all neighborhoods α of a, ordered by inclusion ($\alpha \geq \beta$ iff $\alpha \subset \beta$, $\alpha, \beta \in D$). It is easy to see that D is a directed set. For each $\alpha \in D$ let x_α be a point in α then (x_α, D) is a net in X and $x_\alpha \longrightarrow a$. In fact, let U_a be an arbitrary neighborhood of a then $U_a \in D$ (by definition of D). It remains to prove that x_α is eventually in U_a. Take $\alpha_0 = U_a$ there is an α such that $\alpha \subset \alpha_0 = U_a$. Hence $\alpha \geq \alpha_0$. It follows that $x_\alpha \in \alpha \subset U_a$ which shows that x_α is eventually in U_a.

Definition 2.15

Let $(X, \mathcal{T})$ be a TVS with a topology T. We say that f is continuous if f is continuous with respect to T (i.e. whenever x_α is a net converging to a, $f(x_\alpha) \longrightarrow f(a)$).

3. The Hahn-Banach extension theorem and Separation theorem

First we prove a lemma and then apply Zorn's lemma to prove the Hahn-Banach extension theorem.

Lemma 3.1. Let X be a linear space, X_0 a proper subspace of X, $X_1 = \{ X_0 + \gamma x_0 : \ x_0 \in X \setminus X_0,\ \gamma \text{ scalars} \}$.

Suppose P is a seminorm on X_1, and f a linear functional on X_0 such that

$$f(x) \leq P(x) \qquad x \in X_0.$$

Then there exists a linear functional $\hat{f}$ on X_1 such that

$$\hat{f}(x) = f(x) \qquad \text{for all } x \in X_0,$$

and

$$\hat{f}(x) \leq P(x) \qquad \text{for all } x \in X_1.$$

Proof. By hypothesis $f(x) \leq P(x)$ for all $x \in X_0$, and we notice that for any $x_1,\ x_2 \in X_0$, $x_1 - x_2 \in X_0$ and

$$f(x_1) - f(x_2) \leq P(x_1 + x_0) + P(x_0 + x_2),$$

$$\text{i.e. } -P(x_0 + x_2) - f(x_2) \leq P(x_1 + x_0) - f(x_1).$$

Let

$$m = \sup_{x_2 \in X_0} \{ -P(x_0 + x_2) - f(x_2) \} \text{ and}$$

$$M = \inf_{x_1 \in X_0} \left\{ P(x_1 + x_0) - f(x_1) \right\}.$$

Clearly $m \leq M$. Choose c_0 such that $m \leq c_0 \leq M$.

Let $\hat{f}(y) = f(x) + c_0 \gamma$ where $y = x + \gamma x_0$, $y \in X_1$, $x \in X_0$.

Obviously the representation of $y = x + \gamma x_0$ is unique, and it is quite clear that $\hat{f}$ is a linear functional on X_1, and

$$\hat{f}(y) = f(y) \leq p(y) \quad \text{if } y \in X_0.$$

It remains to show that $\hat{f}(y) \leq p(y)$ for each $y \in X_1$.

If $\gamma = 0$, $y \in X_0$, and the result is trivial.

If $\gamma > 0$, $\hat{f}(y) = f(x) + c_0 \gamma \leq f(x) + M\gamma$ where $y \in X_1$, $x \in X_0$, and $y = \gamma\, x_0 + x$.

Therefore $\hat{f}(\frac{y}{\gamma}) \leq f(\frac{x}{\gamma}) + M \leq f(\frac{x}{\gamma}) + p(x_0 + \frac{x}{\gamma}) - f(\frac{x}{\gamma})$

Hence $\hat{f}(y) \leq \gamma p(x_0 + \frac{x}{\gamma}) = p(\gamma x_0 + x) = p(y)$, for each $y \in X_1$.

If $\gamma < 0$, $\hat{f}(y) = f(x) + c_0 \gamma \leq f(x) + m\gamma$,

$$\hat{f}(\tfrac{y}{\gamma}) \geq f(\tfrac{x}{\gamma}) + m \geq f(\tfrac{x}{\gamma}) - p(x_0 + \tfrac{x}{\gamma}) - f(\tfrac{x}{\gamma})$$

then $\hat{f}(\frac{y}{\gamma}) \geq - p(x_0 + \frac{x}{\gamma})$, and

$$\hat{f}(y) \leq - \gamma p(x_0 + \tfrac{x}{\gamma}) = p(\gamma x_0 + x) = p(y) \text{ for each } y \in X_1.$$

Hence $\hat{f}(y) \leq p(y)$ for all $y \in X_1$. The proof is complete.

Theorem 3.1 (Hahn-Banach). Let X be a TVS, and p a seminorm defined on X. If X_0 is a subspace of X, and f a linear functional on X_0 such that

$$f(x) \leq p(x) \qquad \text{for all } x \in X_0,$$

then there exists a linear functional $\hat{f}$ on X such that

$$\hat{f}(x) = f(x) \qquad \text{for all } x \in X_0,$$

and $$\hat{f}(x) \leq p(x) \qquad \text{for all } x \in X.$$

Proof. Let $\mathcal{F}$ be the set of all pairs $(F, \tilde{f})$ such that F is a subspace containing X_0, and $\tilde{f}$ an extension of f (i.e. $\tilde{f}$ is defined on F and $\tilde{f}(x) = f(x), x \in X_0$), and

$$\tilde{f}(x) \leq p(x) \qquad \text{for all } x \in F.$$

We define the partially ordered relation $\leq$ in $\mathcal{F}$ as follows:

$$(F_\alpha, \tilde{f}_\alpha) \leq (F_\beta, \tilde{f}_\beta) \text{ if } F_\alpha \subset F_\beta, \text{ and } \tilde{f}_\alpha(x) = \tilde{f}_\beta(x),\ x \in F_\alpha.$$

Every linearly ordered subset of $\mathcal{F}$ has an upper bound. In fact, suppose $A = \{(F_\alpha, \tilde{f}_\alpha), \alpha \in \Lambda\}$ is an arbitrary linearly ordered subset of $\mathcal{F}$. Let $\bar{F} = \bigcup_{\alpha \in \Lambda} F_\alpha$. Since Λ is linearly order $\forall \alpha, \beta \in \Lambda$, either $F_\alpha \subseteq F_\beta$ or $F_\beta \subseteq F_\alpha$. It follows $\bar{F}$ is a subspace of X. We define a functional $\bar{f}$ on $\bar{F}$ such that

$$\bar{f}(x) = \tilde{f}_\alpha(x),\ x \in F_\alpha.$$

Since $\tilde{f}_\alpha$ is a linear functional, and $\tilde{f}_\alpha(x) \leq p(x)$ for each α, $\bar{f}$ is a linear functional on $\bar{F}$, and $\bar{f}(x) \leq p(x)$ for all $x \in \bar{F}$. Furthermore, $(\bar{F}, \bar{f})$ is an upper bound of A. By Zorn's lemma, there exists a maximal element in $\mathcal{F}$, $(F, \hat{f})$ say.

It remains to show that F = X. If not, then $X \supsetneq F$. Let $x_1 \in X \setminus F$, $X_1 = \{F + \gamma x_1, \gamma \in R\}$. Clearly X_1 is a subspace containing F. By above lemma 3.1, there exists a linear functional f_1 on X_1 such that

$$f_1(x) = \hat{f}(x), \text{ for all } x \in F \text{ and } f_1(x) \leqslant p(x) \text{ for all } x \in X_1.$$

Thus

$(F, \hat{f}) < (X_1, f_1)$ and $(X_1, f_1) \in \mathcal{F}$ which contradicts the fact that $(F, \hat{f})$ is a maximal element in $\mathcal{F}$.

Hence $F = X$, and $\hat{f}$ satisfies the required conditions.

Q.E.D.

Remark. In a normed linear space X if f is continuous we can obtain that $\|f\|_{X_0} = \|\hat{f}\|_X$. In fact, take p as in theorem 3.1 to be

$$p(x) = \|f\|_{X_0} \cdot \|x\|.$$

Then

$$|\hat{f}(x)| \leqslant p(x) = \|f\|_{X_0} \cdot \|x\|,$$

and

$$\|\hat{f}\|_X \leqslant \|f\|_{X_0}.$$

Clearly

$$\|f\|_{X_0} \leqslant \|\hat{f}\|_X, \text{ and hence } \|\hat{f}\|_X = \|f\|_{X_0}.$$

Corollary 3.1. Given conditions of the above theorem 3.1, and given f on X_0 such that $|f(x)| \leqslant p(x)$, then there exists a linear functional $\hat{f}$ on X which is an extension of f and

$$|\hat{f}(x)| \leqslant p(x), \text{ for all } x \in X.$$

Proof. Since $\hat{f}(x) \leqslant p(x)$ for all $x \in X$ and

$$\hat{f}(-x) \leqslant p(-x) = p(x),$$

this shows that

$$\hat{f}(x) \leq p(x) \text{ and } -\hat{f}(x) \leq p(x) \text{ for all } x \in X.$$

Q.E.D.

Theorem 3.2 (Banach separation theorem). Let X be a normed space, V_1 a non-empty open convex set, and V_2 be a linear manifold such that $V_2 \cap V_1 = \emptyset$. Then there exists a continuous linear functional f and a real number c such that

$$f(x) = c \text{ for all } x \in V_2 \text{ and } f(x) < c \text{ for all } x \in V_1,$$

furthermore, there exists a closed hyperplane H such that

$$H \supset V_2 \text{ and } H \cap V_1 = \emptyset.$$

Proof. Without loss of generality, we can assume that $0 \in V_1$. Let p be a Minkowski functional defined on X with respect to V_1 i.e.,

$$p(x) = \inf\left\{\lambda > 0: \quad x \in \lambda V_1\right\}.$$

Let Y be the subspace generated by V_2 (i.e. Y is the smallest subspace of X containing V_2). We observe that V_2 is a hyperplane in Y. Therefore there exists a linear functional f_0 on Y such that

$$V_2 = \left\{x \in Y: \quad f_0(x) = c\right\}.$$

There is no loss of generality in assuming that

$$V_2 = \left\{x \in Y: f_0(x) = 1\right\}.$$

Since V_1 is open, for each $x \in V_1$, $p(x) < 1$ and $V_2 \cap V_1 = \emptyset$ which implies that

$$f_0(x) = 1 \leq p(x) \qquad \text{for all } x \in V_2 .$$

On the other hand, since for every $y \in Y$

$$y = \alpha x \text{ where } \alpha \text{ real, } x \in V_2,$$

we obtain

$$\alpha \geq 0, \; f_0(\alpha x) = \alpha \; f_0(x) \leq \alpha \; p(x) = p(\alpha x) = p(y)$$

$$\alpha < 0, \; f_0(\alpha x) = \alpha \; f_0(x) \leq 0 \leq p(\alpha x).$$

Thus $\qquad f_0(y) \leq p(y) \quad$ for all $y \in Y$.

Hence by the Hahn-Banach extension theorem 3.1, there exists a linear functional f on X such that

$$f(x) = f_0(x) \qquad \text{for all } x \in Y,$$

$$f(x) \leq p(x) \qquad \text{for all } x \in X, \text{ and } \quad \|f\| = \|f_0\|.$$

Clearly $\|f_0\| = 1$, and then f is continuous. Furthermore let $H = \{x \in X: \; f(x) = 1\}$, then H is a hyperplane in X, $H \supset V_2$, and $H \cap V_1 = \emptyset$ because $f(x) \leq p(x) < 1$ for all $x \in V_1$. The continuity of f implies that H is closed. Q.E.D.

<u>Corollary 3.2</u> (support theorem) Let X be a normed linear space and $x_0 \neq 0$. Then there exists a continuous linear functional f such that $f(x_0) = \|x_0\|$, $\|f\| = 1$.

<u>Proof</u>. If the dimension of X is equal to one, $X = \{\alpha \; x_0: \; \alpha$ all scalars$\}$ and we can define a continuous linear functional f on X, $f(\alpha \; x_0) = \alpha \; \|x_0\|$, and then $f(x_0) = \|x_0\|$, $\|f\| = 1$. Assume that the dimension of X is greater than one. Let $V_2 = \{\alpha \; x_0: \; \alpha$ all scalars$\}$ and V_1 a convex neighborhood of $x_1 (x_1 \neq x_0)$ such that $x_0 \notin V_1$ and $V_2 \cap V_1 = \emptyset$, and then apply theorem 3.2.

The result follows immediately. Q.E.D.

Corollary 3.3 (Eidelheit theorem). Let V_1, V_2 be convex sets in a normed linear space X with $V_1 \cap V_2 = \emptyset$, and V_2 is open. Then there exists a continuous linear functional f such that

$$\sup_{y \in V_1} f(y) \leq \inf_{x \in V_2} f(x).$$

Proof. Let $V = V_2 - V_1 = \bigcup_{y \in V_1} (V_2 - y)$. Since $V_2 - y$ is an open set for each $y \in V_1$, V is open. Clearly V is convex. Furthermore $0 \notin V$ because $V_1 \cap V_2 = \emptyset$. Since $\{0\}$ is a linear manifold, and V is an open convex set which is not empty, by the above separation theorem 3.2, we obtain a continuous linear functional f, such that $f \neq 0$ and $f(x) \geq 0$ for all $x \in V$.

Put $x = x_2 - x_1$ for all $x_2 \in V_2$, $x_1 \in V_1$ which implies that

$$f(x) = f(x_2) - f(x_1) \geq 0 \text{ for all } x_2 \in V_2,\ x_1 \in V_1.$$

Hence $\sup_{y \in V_1} f(y) \leq \inf_{x \in V_2} f(x)$. Q.E.D.

4. B_0-space and its non-zero continuous linear functionals

In this section, we shall show that there always exists a non-zero continuous linear functional on a B_0-space.

Example of B_0-space 4.1. The linear space $C_0(-\infty, \infty)$ consists of all continuous, bounded real functions on $(-\infty, \infty)$. Define the B_0-norm by

$$\|x\|_{B_0} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x\|_i}{1 + \|x\|_i}, \text{ where } \|x\|_i = \sup_{t \in [-i,i]} |x(t)|.$$

Then $C_0(-\infty, \infty)$ becomes a B_0-space.

In fact, $\|x\|_i$ is a norm for each i, and

$$\|x\|_{B_0} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x\|_i}{1 + \|x\|_i} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \|x\|_i \leq M \sum_{i=1}^{\infty} \frac{1}{2^i} = M < \infty,$$

where $|x(t)| \leq M$ for all $t \in (-\infty, \infty)$. The completeness of C_0 is due to the fact that C_0 is complete with respect to every $\| \ \|_i$ for each i.

Lemma 4.1. Every B_0-space is a Fréchet space, and $\|x_n\|_{B_0} \longrightarrow 0$ if and only if $\|x_n\|_i \longrightarrow 0$ for each i.

Proof. It is sufficient to prove that $\|x\|_{B_0}$ is a Fréchet norm.

(a) $\|0\|_{B_0} = 0$ since $\|0\|_i = 0$ for each i, and if $\|x\|_{B_0} = 0$ then

$\|x\|_i = 0$ for each i. Hence $x = 0$.

(b) $\|x\|_{B_0} = \|-x\|_{B_0}$, since $\|x\|_i = \|-x\|_i$ for each i.

(c) $\|x + y\|_{B_0} \leq \|x\|_{B_0} + \|y\|_{B_0}$.

Without loss of generality, let $\|x + y\|_{B_0} \neq 0$ and

$$\|x + y\|_{B_C} = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{1}{\|x + y\|_i} + 1 \right)^{-1}.$$

We notice that

$$\frac{1}{\|x + y\|_i} \geqslant \frac{1}{\|x\|_i + \|y\|_i}, \text{ and } \left(\frac{1}{\|x + y\|_i} + 1\right)^{-1} \leqslant \left(\frac{1}{\|x\|_i + \|y\|_i} + 1\right)^{-1},$$

therefore

$$\|x + y\|_{B_0} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x + y\|_i}{1 + \|x + y\|_i} \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i}\left(\frac{1}{\|x\|_i + \|y\|_i} + 1\right)^{-1}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x\|_i + \|y\|_i}{1 + \|x\|_i + \|y\|_i}$$

$$\leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x\|_i}{1 + \|x\|_i} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|y\|_i}{1 + \|y\|_i} = \|x\|_{B_0} + \|y\|_{B_0}$$

(d) It is easy to see that if $t_n \longrightarrow t_0$ as $n \longrightarrow \infty$ and if

$$\|x_n - x_0\|_{B_0} \longrightarrow 0, \text{ then } \|t_n x_n - t_0 x_0\|_{B_0} \longrightarrow 0,$$

where t_n, t_0 are scalars.

By the similar argument as above, we obtain

$\|x_n\|_{B_0} \longrightarrow 0$ if and only if $\|x_n\|_i \longrightarrow 0$ for each i. Q.E.D.

<u>Lemma 4.2.</u> Every B_0-space is a locally convex TVS.

<u>Proof.</u> Let $U_\varepsilon(0) = \{ x: \|x\|_{B_0} < \varepsilon \}$. There exists a convex

neighborhood $V_\varepsilon(0)$ of 0 such that $V_\varepsilon(0) \subset U_\varepsilon(0)$. Consider

$$V_\varepsilon(0) = \{x: \|x\|_i < \varepsilon \quad \text{for each } i\}.$$

Then

$$V_\varepsilon(0) \subset U_\varepsilon(0),$$

since for every $x \in V_\varepsilon(0)$, $\|x\|_{B_0} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x\|_i}{1 + \|x\|_i} \leqslant$

$$\leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \|x\|_i < \varepsilon .$$

The convexity of $V_\varepsilon(0)$ follows from the fact that $\|x\|_i$ is a seminorm for each i. Q.E.D.

Lemma 4.3. In every B_0-space X, there exists $x_0 \neq 0$ and a positive integer n such that

$$\operatorname*{Max}_{n} (\|x_0\|_1, \ldots, \|x_0\|_n) \neq 0.$$

Proof. The proof of this lemma is a direct consequence of the definition of B_0-space.

Theorem 4.1. On every B_0-space, there exists a non-zero continuous linear functional.

Proof. By lemma 4.3, we can define a seminorm p on X such that

$$p(x) = \operatorname*{Max}_{n} (\|x\|_1, \ldots, \|x\|_n) \text{ and for some } x_0,\ p(x_0) \neq 0.$$

We consider the linear subspace $X_0 = \{t\, x_0: \ t \text{ scalars}\}$, and define a linear functional f on X_0 by

$$f(x) = p(x_0)t \qquad \text{for all } x \in X_0.$$

Clearly $f(x_0) = p(x_0)$, and $f(x) \leq p(x)$ for all $x \in X_0$. By the Hahn-Banach extension theorem, there exists a linear functional $\hat{f}$ on X such that

$$\hat{f}(x) = f(x) \quad x \in X_0, \text{ and } \hat{f}(x) \leqslant p(x), \text{ for all } x \in X.$$

Using the Corollary 3.1, then

$$|\hat{f}(x)| \leqslant p(x) \quad \text{for all } x \in X.$$

Clearly $\hat{f} \neq 0$ since $\hat{f}(x_0) = f(x_0) = p(x_0) \neq 0$.

It remains to show that f is continuous. If

$$\|x_j - x_0\|_{B_0} \longrightarrow 0 \text{ as } j \longrightarrow \infty \text{, then}$$

$$\|x_j - x_0\|_i \longrightarrow 0 \text{ for each } i. \text{ Therefore}$$

$$\operatorname*{Max}_n (\|x_j - x_0\|_1, \ldots, \|x_j - x_0\|_n) \longrightarrow 0. \text{ That means}$$

$$p(x_j - x_0) \longrightarrow 0 \text{ for } \|x_j - x_0\|_{B_0} \longrightarrow 0 \text{ as } j \longrightarrow \infty .$$

Therefore $\hat{f}(x_n - x_0) \longrightarrow 0$. This shows that $\hat{f}$ is continuous.

Q.E.D.

Example 4.2. A non-zero continuous linear functional on $C(-\infty, \infty)$ can be represented by

$$\hat{f}(x) = \int_{-1}^{1} x(t)dg_1(t) + \ldots + \int_{-n}^{n} x(t)dg_n(t),$$

where g_i is a real function of bounded variation for each

$i = 1, \ldots, n$, and $\hat{f}(x) \neq 0$.

5. The existence of non-zero continuous linear functional on LCTVS.

Definition 5.1. A set C is convex if $tx + (1-t)y \in C$ for any $x, y \in C$, and for all scalars t, $0 \leq t \leq 1$.

Definition 5.2. A TVS X is said to be locally convex if for any neighborhood $U(0)$ of 0, there exists a convex neighborhood $V(0)$ of 0 which is contained in $U(0)$.

Definition 5.3. A set C is said to be symmetric if $-C = C$ i.e. $\{-x: x \in C\} = \{x: x \in C\}$.

Definition 5.4. A set C is said to be balanced if $tx \in C$ for any $x \in C$ and all scalars t, $|t| \leq 1$.

Definition 5.5. A set C in a TVS X is said to be absorbing at a point b, if for any $x \in X$, there exists a $\varepsilon > 0$ such that

$$b + tx \in C \text{ where } |t| \leq \varepsilon .$$

Furthermore, a set C is said to be absorbing if it is absorbing at 0.

Definition 5.6. A function p defined by a convex, balanced, absorbing set C is said to be a Minkowski functional p, where $p(x) = \inf\{t > 0: x \in tC\}$.

Lemma 5.1. Let C be a convex, balanced, absorbing set, p its Minkowski functional, and μ a scalar. If $\mu > p(a)$, then $a \in \mu C$.

Proof. If $\mu > p(a)$, then by definition of p,

$$\mu > \inf \left\{ t > 0: \quad a \in tC \right\},$$

or

$$1 > \inf \left\{ \frac{t}{\mu} > 0: \quad a \in tC \right\}.$$

This means that there exists a scalar $t_0 > 0$ such that

$$0 < \frac{t_0}{\mu} \leqslant 1 \text{ and } a \in t_0 C.$$

Since C is balanced, $\frac{t_0}{\mu} \cdot a \in t_0 C$ i.e. $a \in \mu C$. Q.E.D.

Lemma 5.2. Suppose p satisfies the same conditions as those in lemma 5.1, then p is a seminorm.

Proof. It is sufficient to prove $p(a + b) \leqslant p(a) + p(b)$ for any a, $b \in C$. For arbitrary a, $b \in C$, and $\varepsilon > 0$,

$$p(a) < p(a) + \varepsilon \ , \ p(b) < p(b) + \varepsilon \ .$$

By lemma 5.1, it follows that

$$a \in (p(a) + \varepsilon)C, \ b \in (p(b) + \varepsilon)C.$$

Clearly, $sC + tC = (s + t)C$ for all scalars s and t, since C is a convex set. Thus we obtain

$$a + b \in (p(a) + \varepsilon)C + (p(b) + \varepsilon)C = (p(a) + p(b) + 2\varepsilon)C$$

and

$$p(a + b) \leqslant p(a) + p(b) + 2\varepsilon \ .$$

Letting $\varepsilon \longrightarrow 0$, $p(a + b) \leqslant p(a) + p(b)$. Q.E.D.

Lemma 5.3. Let X be a TVS, and V a convex, symmetric neighborhood of 0. Then V is convex, balanced, and absorbing.

Proof. Since V is convex and symmetric, $tx \in V$ for $x \in V$, $|t| \leq 1$. This shows that V is balanced.

We are left to show that V is absorbing. Let $x \in X$, $t_n \longrightarrow 0, t_n > 0$, which implies that $t_n x \longrightarrow 0$. That means there exists $t_0 > 0$ such that $t_0 x \in V$.

Hence $tx = \frac{t}{t_0} \cdot t_0 x \in V$ for $0 \leq t \leq t_0$. Q.E.D.

Theorem 5.1. Let X be a LCTVS with Hausdorff topology. Then there exists a family of seminorms $\{p_\alpha\}_{\alpha \in \Lambda}$, $x_0 \neq 0$, and $\alpha_0 \in \Lambda$ such that

$$p_{\alpha_0}(x_0) \neq 0.$$

Proof. Observe that if V(0) is a convex neighborhood of 0 in X, then so is -V(0). Therefore $V(0) \bigcap -V(0)$ becomes a convex, symmetric neighborhood of 0. There exists a fundamental system $\{V_\alpha, \alpha \in \Lambda\}$ of neighborhoods of 0, such that each V_α is convex, balanced, and absorbing by Lemma 5.3. Define p_α as follows

$$p_\alpha(x) = \inf\{t > 0: \ x \in tV_\alpha\}.$$

By lemma 5.2, p_α is a seminorm for each $\alpha \in \Lambda$.

It remains to prove that there exists $\alpha_0 \in \Lambda$ such that, for some $x_0 \neq 0$, $p_{\alpha_0}(x_0) \neq 0$.

If not, $p_\alpha(x_0) = 0$ for every $\alpha \in \Lambda$ which implies $x_0 \in V_\alpha$ for each $\alpha \in \Lambda$. Hence $x_0 = 0$, because X is a Hausdorff space. But this is a contradiction because $x_0 \neq 0$. Q.E.D.

Theorem 5.2. Every locally convex Hausdorff TVS has a non-zero continuous linear functional.

Proof. Suppose X is a given locally convex Hausdorff TVS. By theorem 5.1, we obtain $x_0 \neq 0$ and $\alpha_0 \in \Lambda$, $p_{\alpha_0}(x_0) \neq 0$. For the linear subspace $X_0 = \{tx_0: \text{ all scalars } t\}$, define

$$f_0(x) = tp_{\alpha_0}(x_0) \text{ where } x = tx_0.$$

It is easy to see that f_0 is a linear functional on X_0. Clearly $f_0(x) \leq p_{\alpha_0}(x)$, $x \in X_0$. Therefore, by the Hahn-Banach extension theorem 3.1 and corollary 3.1, there exists a linear functional $\hat{f}$ on X such that

$$\hat{f}(x) = f(x) \qquad x \in X_0$$

and

$$|\hat{f}(x)| \leq p_{\alpha_0}(x) \qquad x \in X.$$

$\hat{f}$ is continuous. In fact, we see that $\hat{f}$ is bounded on $\{x \in X: \ p_{\alpha_0}(x) < 1\}$ which is a neighborhood of 0. (see Lemma 7.8).

(i.e. If $(x_\alpha, \alpha \in D)$ is a net in X, and $x_\alpha \longrightarrow x$ (see Example 2.1), then $p_{\alpha_0}(x_\alpha - x) \longrightarrow 0$, hence $\hat{f}(x_\alpha - x) \longrightarrow 0$).

Theorem 5.3. Every LCTVS X has a non-zero continuous linear functional *except X is indiscrete*.

Proof. We can define

$$p(x) = \underset{n}{\text{Max}}\ (p_{\alpha_1}(x), \ldots, p_{\alpha_n}(x)) \text{ such that}$$

$$p(x_0) = \underset{n}{\text{Max}}\ (p_{\alpha_1}(x_0), \ldots, p_{\alpha_n}(x_0)) \neq 0 \text{ for some } x_0 \in X,$$

where $p_{\alpha_1}, \ldots, p_{\alpha_n}$ are seminorms.

Clearly, p is a seminorm. We recall the theorem 5.2, then there exists a linear functional $\hat{f}$ such that

$$|\hat{f}(x)| \leqslant p(x), \text{ and } \hat{f}(x_0) \neq 0.$$

The proof of continuity of f is the same as the proof of the above theorem 5.2. Q.E.D.

The following example shows that there exists a TVS which has a non-zero continuous linear functional, but it is not locally convex.

Example 5.1. Let ℓ^p, $0 < p < 1$, be the set of all sequences

such that $\|x\| = \sum_{i=1}^{\infty} |x_i|^p < \infty$, where $x = (x_1, x_2, \ldots, x_n, \ldots)$.

ℓ^p, $0 < p < 1$, is not locally convex. But

$$f(x) = \sum_{i=1}^{\infty} t_i x_i, \text{ where } \sup_i |t_i| < \infty,\ t_i \not\equiv 0 \text{ is a}$$

non-zero continuous linear functional (see [7] p.180.).

6. Non-zero continuous linear functionals on Fréchet spaces

Let F be a Fréchet space with the Fréchet norm $\| \ \|_F$, denoted by $(F, \| \ \|_F)$. Then we have the following:

Theorem 6.1. If $(F, \| \ \|_F)$ has a convex, symmetric neighborhood $U(0)$ of 0, then there exists a seminorm p_U such that $\|x_n\|_F \to 0$ implies $p_U(x_n) \to 0$.

Proof. By lemma 5.3, the convex, symmetric neighborhood $U(0)$ of 0 is essentially a convex, balanced, absorbing neighborhood of 0. Let p_U be the Minkowski functional,

$$p_U(x) = \inf\{ t > 0: \ x \in tU(0)\} .$$

Lemma 5.2 implies that p_U is a seminorm.

It remains to show that if $\|x_n\|_F \to 0$, then $p_U(x_n) \to 0$. Since $U(0)$ is a neighborhood of 0, for each scalar ε, $\varepsilon U(0)$ a neighborhood of 0. In fact, let $O_U(0)$ be an open set which is contained in $U(0)$. It is easy to see that $\varepsilon O_U(0)$ is an open set which is contained in $\varepsilon U(0)$ because $(F, \| \ \|_F)$ is a TVS.

Therefore, for any $\varepsilon > 0$, $x_n \in \varepsilon U(0)$ for sufficiently large n, and then $p_U(x_n) \leq \varepsilon$ for sufficiently large n. Hence $p_U(x_n) \to 0$. Q.E.D.

Theorem 6.2. A Fréchet space F is B_0-normable if and only if each $A_m = \{ x: \|x\|_F < \frac{1}{m} \}$, m = 1, 2, ..., contains a convex, symmetric neighborhood $U_m(0)$ of 0.

Proof. (necessity) Since F is B_0-normable, for each A_m there exists $B_{\varepsilon_m} = \left\{x: \|x\|_{B_0} < \varepsilon_m\right\}$ such that $B_{\varepsilon_m} \subset A_m$.

We notice that every B_0-space is a LCTVS (section 4). Hence there exists a convex, symmetric neighborhood $U_m(0)$ such that

$$U_m \subset B_{\varepsilon_m} \subset A_m.$$

(Sufficiency). By hypothesis, for each A_m, there exists a convex, symmetric neighborhood $U_m(0)$ of 0 such that

$$U_m(0) \subset A_m.$$

Define a sequence of seminorms p_{U_m} as follows:

$$p_{U_m}(x) = \inf\left\{t > 0: \; x \in t\, U_m(0)\right\}, \; m = 1, 2, \ldots .$$

Clearly if $p_{U_m}(x) = 0$ for each m, then $x = 0$. For, if not, then there exists $x_0 \neq 0$, $p_{U_m}(x_0) = 0$ for all m. But $\|x_0\|_F \neq 0$,

so that, $x_0 \notin A_m$ for some m. But $p_{U_m}(x_0) = 0$ for all m implies $x_0 \in \varepsilon U_m(0) \subset U_m(0)$ for all m, where $0 < \varepsilon < 1$ (since $U_m(0)$ is balanced). This shows that $x_0 \in U_m(0) \subset A_m$ for all m, a contradiction. Obviously, by Theorem 6.1, $\|x_n\|_F \longrightarrow 0$ implies $p_{U_m}(x_n) \longrightarrow 0$. Let $\|x\|_{B_0}$ be the B_0-norm defined by $\left\{p_{U_m}\right\}_{m=1}^{\infty}$. Now $p_{U_m}(x_n) \longrightarrow 0$ as $n \longrightarrow \infty$ for each m implies

$$\|x_n\|_{B_0} \longrightarrow 0.$$

We are left to prove that $\|x_n\|_{B_0} \longrightarrow 0$ implies $\|x_n\|_F \longrightarrow 0$. In fact, $\|x_n\|_{B_0} \longrightarrow 0$ implies $p_{U_m}(x_n) \longrightarrow 0$ for all m. So

$x_n \in U_m(0)$ for sufficiently large n and each m. Therefore $x_n \in A_m$, and $\|x_n\|_F < \frac{1}{m}$ for all m and large n. Thus we obtain $\|x_n\|_F \longrightarrow 0$ as $n \longrightarrow \infty$. Q.E.D.

Definition 6.1. Let X be a TVS. A subset S of X is said to be bounded if for every neighborhood V(0) of 0, there exists a non-zero real number ε, such that $\varepsilon S \subset V(0)$.

Theorem 6.3. A Fréchet space F can be B-normable if and only if there exists a convex, symmetric neighborhood $U(0)$ of 0, and for any scalar t_n, $t_n \longrightarrow 0$, and $x_n \in U(0)$, $\|t_n x_n\|_F \longrightarrow 0$.

Proof. In view of the fact that if a linear space is B-normable then there exists such a neighborhood as the one described in the theorem, we need only to show the other part.

First we claim that if $U(0)$ is described as in the theorem then $U(0)$ is bounded. For otherwise, there exists a V(0) of 0 such that $\varepsilon U(0) \not\subset V(0)$ for any ε. i.e. there exists x, $x \in U(0)$ such that $\varepsilon x \notin V(0)$ for any ε, which gives rise to a contradiction, since $t_n \longrightarrow 0$, $x_n \in U(0)$ implies $\|t_n x_n\|_F \longrightarrow 0$.

Define a seminorm p_U as follows:

$$p_U(x) = \inf\{t > 0: \quad x \in t \cdot U(0)\}.$$

We want to prove that p_U is a norm. Let $x_0 \neq 0$. By the condition that $t_n \longrightarrow 0$, $x_n \in U(0)$ implies $\|t_n x_n\|_F \longrightarrow 0$, there exists a scalar $b > 0$ such that $bx_0 \notin U(0)$.

For all scalars $a > 0$ if $\frac{1}{a} x_0 \in U(0)$ then $ab \geq 1$. Indeed, if $ab < 1$, it follows that

$a.b. \frac{1}{a} \cdot x_0 \in ab\, U(0) \subset U(0)$ i.e. $b\, x_0 \in U(0)$, since $U(0)$ is balanced. This contradicts the fact that $bx_0 \notin U(0)$. Hence

$$p_U(x_0) = \inf\left\{a > 0 : x_0 \in a\, U(0)\right\} \geq \frac{1}{b} > 0, \text{ and}$$

p_U becomes a norm. By theorem 6.1, we know that $\|x_n\|_F \longrightarrow 0$ which implies $p_U(x_n) \longrightarrow 0$.

It remains to show that $p_U(x_n) \longrightarrow 0$ implies $\|x_n\|_F \longrightarrow 0$. Indeed, $U(0)$ is bounded. That means for any neighborhood $V(0)$ of 0 in F, there exists ε such that $\varepsilon U(0) \subset V(0)$. $U(0)$ is symmetric. Without loss of generality, we can assume that $\varepsilon > 0$ with $\varepsilon U(0) \subset V(0)$. Since $p_U(x_n) \longrightarrow 0$, we can obtain that $p_U(x_n) < \varepsilon$ for sufficiently large n which implies $x_n \in \varepsilon U(0)$, and then $x_n \in V(0)$ for sufficiently large n. Hence $\|x_n\|_F \longrightarrow 0$. Q.E.D.

From the above theorems we have the following:

Theorem 6.4. A Fréchet space X has a non-zero continuous linear functional if and only if there exists a convex, symmetric neighborhood $U(0)$ of 0 such that $U(0) \neq X$.

Proof. (necessity) Suppose f is a non-zero continuous linear functional on X. Let $U(0) = \left\{x : |f(x)| < 1\right\}$. Clearly,

$U(0)$ is an open convex, symmetric set which contains 0.

It is enough to show that $U(0) \neq X$. If $U(0) = X$, then by continuity of f we obtain $f(x) = 0$ for each $x \in X$ which leads to a contradiction.

(Sufficiency) $p_U(x) = \inf\{t > 0: \; x \in tU(0)\}$ is a seminorm, and

$$\{x: p_U(x) < 1\} \subset U(0),$$
$$\{x: p_U(x) \leqslant 1\} \supset U(0).$$

We want to show that there exists a $x_0 \neq 0$ such that $p_U(x_0) \neq 0$. If $p_U(x) = 0$ for all $x \in X$, then

$$X \subset \{x: \; p_U(x) < 1\} \subset U(0) \quad \text{i.e.} \quad U(0) = X$$

which contradicts the hypothesis that $U(0) \neq X$. By the Hahn-Banach extension theorem, there exists a linear functional $\hat{f}$ on X (see theorems 4.1 and 5.2) and $\hat{f}(x_0) = p_U(x_0) \neq 0$, such that $|\hat{f}(x)| \leqslant p_U(x)$ for all $x \in X$.

The continuity of $\hat{f}$ follows from theorem 6.1 that $\|x_n - x_0\|_F \longrightarrow 0$ implies $p_U(x_n - x_0) \longrightarrow 0$, then $\hat{f}(x_n - x_0) \longrightarrow 0$.
Q.E.D.

Example 6.1. (M. M. Day). $L^p = L^p(0, 1)$, $0 < p < 1$, consists of all Lebesgue measurable real-valued functions on $(0,1)$ such that $\int_0^1 |f|^p < \infty$. $\|f\| = \int_0^1 |f|^p$ defines a Fréchet norm on L^p. Let M be the subspace of all bounded functions in L^p. We shall show that M has no non-zero continuous linear functionals. Since M is dense in L^p, this will imply the conclusion for L^p.

Let U be a convex neighborhood of 0. Then $\{ f : \|f\| < \varepsilon \} \subset U$ for some $\varepsilon > 0$. Let f be an arbitrary function in M, say $|f(t)| < N$. Since $0 < p < 1$ we can choose an integer n such that $\frac{N^p}{n^{1-p}} < \varepsilon$. Subdivide (0, 1) into n disjoint intervals $I_1, I_2, \ldots, I_n$ of equal length. For each k = 1, 2, ..., n define a function f_k by

$$f_k(x) = \begin{cases} nf(x) & \text{if } x \in I_k \\ 0 & \text{otherwise} . \end{cases}$$

Then

$$\|f_k\| = \int_0^1 |f_k(t)|^p \, dt \leq n^p . N^p . \frac{1}{n} = \frac{N^p}{n^{1-p}} < \varepsilon \quad \text{so that}$$

each f_k is in U. Since $f = \frac{1}{n} \sum_{k=1}^{n} f_k$ is a convex combination of f_k, f is in U. But f is arbitrary, we have $U = M$. By the above theorem 6.4, M has no non-zero continuous linear functional.

<u>Example 6.2</u>. Let S(0, 1) be the set of all measurable real-valued functions on (0, 1) with a Fréchet norm defined by

$$\|f\| = \int_0^1 \frac{|f(x)|}{1 + |f(x)|} \, dx.$$

Let U be a convex neighborhood of 0 containing the sphere $\{ f : \|f\| < \varepsilon \}$. Choose an integer $n > \frac{1}{\varepsilon}$. Subdivide (0, 1) into n disjoint intervals $I_1, I_2, \ldots, I_n$ of equal length.

For each $k = 1, 2, \ldots, n$ define a function f_k by

$$f_k(x) = \begin{cases} nf(x) & \text{if } x \in I_k \\ 0 & \text{otherwise,} \end{cases}$$

where f is an arbitrary but fixed function in $S(0, 1)$.

Then

$$\| f_k \| = \int_0^1 \frac{|f_k|}{1 + |f_k|} \leq \frac{1}{n} < \varepsilon ,$$

so that f_k is in U . Since U is convex, $f = \frac{1}{n} \sum_{k=1}^{n} f_k$ is in U

and so $U = S$. Hence $S(0, 1)$ has no non-zero continuous linear functionals.

Theorem 6.5. Let F be a TVS in Theorem 6.4. Then the statement given in theorem 6.4 is true.

Proof. The proof of necessary part being the same as that of theorem 6.4. (Sufficiency). Let $U(0)$ be a convex, symmetric neighborhood of 0, $U(0) \neq F$, then $\varepsilon U(0)$ is a neighborhood of 0, for any non-zero scalars ε .

The existence of a non-zero linear functional f on F is guaranteed by theorem 6.4. The proof of continuity of f is the same as the proof of theorem 5.2. Q. E. D.

7. A generalization of the Separation theorem.

In this section, we shall generalize the Separation theorem given in Section 3.

Definition 7.1. A set S in a TVS X is said to be an affine set if there exists a subspace S_0 of X and x_0 in X such that $S = x_0 + S_0$, where $x_0 + S_0 = \{ x_0 + y: \ y \in S_0 \}$.

Definition 7.2. A set C in a TVS X is said to be a cone if $tx \in C$ for all $t \geq 0$, $x \in C$.

It is easy to see that a cone C is convex if and only if $C + C \subset C$ where $C + C = \{ x + y: \ x, y \in C \}$.

We shall make use of the following lemmas:

Lemma 7.1. Let V be a convex set in a TVS X absorbing at a, and let $b \in V$. Then V is absorbing at each point of the line segment $[a, b)$.

Proof. For any $c \in [a, b)$, $c = ta + (1-t)b$ for some t, $0 < t \leq 1$. Let $x \in X$ be given. Since V is absorbing at a, there exists $\gamma > 0$ such that $a + sx \in V$ for $|s| \leq \gamma$. Then, for $|s| \leq t\gamma$, $c + sx = t(a + (\frac{s}{t}) \ x) + (1 - t)b \in V$ because V is convex and $a + (\frac{s}{t}) \ x \in V$, $b \in V$. Hence V is absorbing at c. Q. E. D.

Lemma 7.2. Let V be a convex set in a TVS X absorbing at a. Let x be an arbitrary vector in X. Then V is absorbing at $a + tx$ for sufficiently small $t \geq 0$.

Proof. In fact, since V is absorbing at a, we can choose $\gamma > 0$ such that $y = a + \gamma x \in V$. For $0 \leq t < \gamma$, $a + tx = (1 - \frac{t}{\gamma})a + (\frac{t}{\gamma})y \in [a, y)$. By virtue of Lemma 7.1, V is absorbing at $a + tx$,

$0 \leq t < \gamma$. Q. E. D.

Lemma 7.3. Let V be a convex cone in a TVS X which is absorbing at each point of $V_0 = V \setminus (-V) = \{ x: x \in V \text{ and } x \notin (-V)\}$, $V \cup (-V) = X$. Then V_0 is absorbing at each of its points.

Proof. If $V = -V$, then $V_0 = \emptyset$. The result is obviously true. Suppose that $V_0 \neq \emptyset$, and the conclusion is false (i.e. there exists $a \in V_0$ such that for every $\varepsilon > 0$ there is $x \neq 0$, $a + t_0 x \notin V_0$ for some t_0, $0 < t_0 < \varepsilon$). By hypothesis, V is absorbing at $a \in V_0$ and by Lemma 7.2, V is absorbing at $a + tx$ for sufficiently small $t \geq 0$ and any $x \in X$. Thus we can choose $t_1 > 0$ such that $a + t_1 x \notin V_0$ (i.e. $-(a + t_1 x) \in V$) and V is absorbing at $a + t_1 x$. Furthermore, Lemma 7.1 implies that V is absorbing at each point of the segment $(-(a + t_1 x), (a + t_1 x)]$. We notice that $x \neq 0$, $t_1 > 0$, $0 \in (-(a + t_1 x), (a + t_1 x)]$. Therefore V is absorbing. Since V is a convex and absorbing cone,

$$V = \bigcup_{n=1}^{\infty} nV = X.$$ Hence $V = -V = X$ and therefore $V_0 = V \setminus (-V) = \emptyset$, a contradiction.

Lemma 7.4. Let X be a TVS, and V a convex cone in X. If

(a) $V \cup (-V) = X$,

and (b) V is absorbing at each point of $V \setminus (-V)$,

then $V \cap (-V)$ is a maximal subspace of X or the space X itself.

Proof. Let $S = V \cap (-V)$. It is easy to see that S is a linear subspace of X. If $S \neq X$, we claim that S is a maximal subspace of X. First of all, the condition (a) implies that

$V \setminus (-V) \neq \emptyset$, otherwise $S = X$. Let $b \in V \setminus (-V)$. Clearly $b \notin S$. We want to prove that $Rb + S = \{\alpha b + \beta x\colon\ x \in S \text{ and } \alpha, \beta \text{ scalars}\} = X$ (i.e. b and S span X). Since $S \cup (V \setminus S) \cup ((-V) \setminus S) = X$, it suffices to show that $Rb + S \supset S$, $Rb + S \supset V \setminus S$, and $Rb + S \supset ((-V) \setminus S)$. Obviously $Rb + S \supset S$. But if $Rb + S \supset (-V) \setminus S$, then $-(Rb + S) = Rb + S \supset -((-V) \setminus S) = V \setminus S$. Thus, it is enough to show that $Rb + S \supset (-V) \setminus S$. For a given $x \in (-V) \setminus S$, let

$$B = \{ tb + (1 - t)x\colon \text{ all scalars } t \} .$$

Lemma 7.3, and condition (b) imply that $V \setminus (-V)$ and $(-V) \setminus V$ are absorbing at each of their points. Therefore, $C = B \cap (V \setminus (-V))$ and $D = B \cap ((-V) \setminus V)$ are two non-empty open subsets of B, and $b \in C$, $x \in D$, $C \cap D = \emptyset$. From the fact that $(V \setminus (-V))' \cap ((-V) \setminus V)' = S$, where $(V \setminus (-V))'$ is the complement of $(V \setminus (-V))$, it follows that $(C \cup D) \cap S = \emptyset$.

Furthermore, $X \supset B$, $C \cup D \subset (V \setminus (-V)) \cup ((-V) \setminus V)$, and $\emptyset \neq B \setminus (C \cup D) \subset X \setminus \{ (V \setminus (-V)) \cup ((-V) \setminus V)\} = S$, therefore

$\emptyset \neq B \setminus (C \cup D) \subset (B \setminus (C \cup D)) \cap S = B \cap S$ because $(C \cup D) \cap S = \emptyset$.

This shows that $B \cap S \neq \emptyset$. Then there exists $g \in B \cap S$, $g = tb + (1 - t)x$ for some $t \neq 1$ (since $b \notin S$) and $x \in (-V) \setminus S$ (given.). Therefore

$x \in \frac{1}{1 - t}(g - tb) \in Rb + S$, and hence $Rb + S \supset (-V) \setminus S$. Q.E.D.

<u>Lemma 7.5</u>. Let B be a convex subset of a TVS X which is not absorbing at 0, but is absorbing at a certain point a of B. Then there exists a linear functional f such that

$S = \{ x: f(x) = 0 \}$ is a maximal subspace and $f(b) \geqslant 0$ for every b in E (i.e. S lies on one side of B).

Proof. In order to prove this lemma, we shall construct a maximal subspace S of X and then apply the Hahn-Banach extension theorem. The proof of this lemma runs through the following steps:

(a) We claim that $-a \notin tB$ for all $t > 0$. Suppose $-a \in tB$ for some $t > 0$ (i.e. $-\frac{a}{t} \in B$). Since $a \in B$ and B is absorbing at a, and B is convex, B is absorbing at every point of the line segment $(-\frac{a}{t}, a]$ (see Lemma 7.1). Moreover, since $0 \in (-\frac{a}{t}, a]$, B is absorbing at 0 which is a contradiction.

(b) Let Q be the family of convex cones such that

$$Q = \{ C: \ C \supset B, \ C \text{ is a convex cone, and } -a \notin C \} ,$$

clearly, $Q \neq \emptyset$ because $\bigcup_{t \geqslant 0} \{ tB \} \in Q$. We order Q by inclusion. If D is a chain in Q, let $K = \bigcup_{E \in D} E$ then K is a convex cone. Hence, by Zorn's lemma, there exists a maximal element K, say.

(c) $K \cup (-K) = X$. It suffices to show that for $b \in X$ and if $b \notin K$, then $b \in -K$. Indeed, let $P = \{ tb + x: \ t \geqslant 0, \ x \in K \}$.

Clearly, p is a convex cone. Also $-a \in p$, for otherwise $-a \notin p$. Since $p \supsetneqq K$, this contradicts the fact that K is maximal. Therefore $-a = tb + x$ for some $t > 0$ and $x \in K$ ($t \neq 0$ because $-a \notin K$) so that $b = \frac{1}{t}(-(a + x))$. Furthermore, since $K \supset B$ $x \in K$, and K is a convex cone, we have $a + x \in K + K \subset K$, and hence $b = \frac{1}{t}(-(a + x)) \in \frac{1}{t}(-K) \subset -K$ i.e. $b \in -K$.

(d) K is absorbing at each point of $K \setminus (-K)$. For every $b \in K \setminus (-K)$, clearly $-b \notin K$. By step (c),

$$-a = t(-b) + x \text{ for some } t > 0 \text{ and } x \in K.$$

Therefore $\frac{1}{2}\, tb = \frac{1}{2}\,(a + x) \in K$. Since $K \supset B$, B is absorbing at a (by hypothesis), so is K. On the other hand, $x \in K$ then K is absorbing at each point of the line segment $[a, x)$, in particular, K is absorbing at $\frac{1}{2}\,(a + x) = \frac{1}{2}\, tb$. This shows that $\frac{2}{t}\, K$ is absorbing at b. Since $\frac{2}{t}\, K \subset K$, K is absorbing at b.

(e) From Lemma 7.4, $S = K \cap (-K)$ is a maximal subspace of X. Since $-a \notin K$, $-a \notin S$, $S \neq X$, by the Hahn-Banach theorem, there exists a linear functional f such that $S = \{ x\colon f(x) = 0\}$ and $f(a) = 1$.

It remains to prove that $f(b) \geq 0$ for every $b \in B$. Suppose $f(b) < 0$ for some $b \in B$. Since $-a = \left(\frac{b}{f(b)} - a\right) + \frac{b}{(-f(b))}$, $b \in B \subset K$, $-f(b) > 0$, and K is a convex cone, $\frac{b}{(-f(b))} \in K$. Moreover, $f\left(\frac{b}{f(b)} - a\right) = 0$, $\frac{b}{f(b)} - a \in S \subset K$, and hence $-a \in K + K \subset K$ i.e. $-a \in K$, a contradiction. Q.E.D.

<u>Definition 7.3</u>. V_2, V_1 are said to be <u>separated</u> if there exists a linear functional $f \neq 0$ and real t such that

$$f(a) \geq t \geq f(b) \text{ or } f(a) \leq t \leq f(b) \text{ for all } a \in V_2,\ b \in V_1.$$

The following theorem 7.1 is a generalization of Eidelheit theorem.

<u>Theorem 7.1</u>. Let V_1, V_2 be two convex sets in a TVS X, V_2 is absorbing at some point a of X but is not absorbing at any point of V_1. Then V_1, V_2 are separated by a linear functional f.

Proof. First of all, $V_2 - V_1 = \{a - b: a \in V_2, b \in V_1\}$ is a convex set because V_1, V_2 are convex, and $V_2 - V_1$ is absorbing at each point of $a - V_1$ because V_2 is absorbing at a by hypothesis. Furthermore, $V_2 - V_1$ is not absorbing at 0. For otherwise V_2 would be absorbing at some point of V_1. By Lemma 7.5, it follows that there exists a linear functional $f \neq 0$ such that $H = \{x: f(x) = 0\}$ is a maximal subspace and $f(x) \geqslant 0$ for all $x \in V_2 - V_1$. Then we obtain that

$$\inf_{a \in V_2} f(a) \geqslant \sup_{b \in V_1} f(b).$$

Letting $t = \inf_{a \in V_2} f(a)$, we see that $f(a) \geqslant t \geqslant f(b)$

for all $a \in V_2$ and $b \in V_1$ (i.e. V_2, V_1 are separated by f).

Corollary 7.1. Let X be a TVS, V_1, V_2 be convex sets with $V_1 \cap V_2 = \emptyset$ and at least one of which has an interior point. Then there exists a linear functional f and real t such that $f(b) \leqslant t \leqslant f(a)$, for all $b \in V_1$, $a \in V_2$.

Proof. Suppose V_2 has an interior point. Now $V_1 \cap V_2 = \emptyset$ implies that V_2 is not absorbing at any point of V_1. Indeed, if V_2 is absorbing at some point $b \in V_1$, then there exists $\varepsilon > 0$ such that $b + tx \in V_2$ for $|t| \leqslant \varepsilon$. Hence for $t = 0$, $b \in V_2$, $V_1 \cap V_2 \neq \emptyset$, a contradiction because $V_1 \cap V_2 = \emptyset$ (by hypothesis). Hence the result follows from Theorem 7.1. Q.E.D.

Lemma 7.6. A maximal subspace S of a TVS X is either closed or dense.

Proof. First, we notice that the closure of S is also a linear subspace. Now if $\overline{S} \neq X$, then $S \subset \overline{S}$ and the maximality of S imply $S = \overline{S}$ i.e. S is closed. Q.E.D.

Lemma 7.7. If a linear functional f on a TVS X is continuous at a certain point a, then it is continuous everywhere.

Proof. Let $x \in X$, and $x_\alpha \longrightarrow x$, then $a + x_\alpha - x \longrightarrow a$, and $f(x_\alpha) - f(x) = f(a + x_\alpha - x) - f(a) \longrightarrow 0$. Q.E.D.

Lemma 7.8. Let f be a linear functional on a TVS X. If f is bounded on some neighborhood of 0, then f is continuous.

Proof. Let V be a neighborhood of 0 in R, and let $f(U)$ be bounded for a given neighborhood U of 0 in X, then $tV \supset f(U)$ for some $t > 0$, or $f^{-1}(V) \supset \frac{1}{t}U$. Hence $f^{-1}(V)$ is a neighborhood of 0, and f is continuous at 0, and therefore f is continuous everywhere by Lemma 7.7. Q.E.D.

Theorem 7.2. A linear functional f is continuous if and only if $S = \{x: f(x) = 0\}$ is closed.

Proof. (Necessity) Since $\{0\}$ is closed, and f is continuous then $S = f^{-1}(\{0\})$ is closed.

(Sufficiency) We notice that X/S is an one dimensional TVS and hence $\tilde{f}: X/S \longrightarrow R$ is a continuous linear mapping. But $X \longrightarrow X/S$ is a continuous canonical mapping. Hence $f: X \longrightarrow R$ is continuous. Q.E.D.

Corollary 7.2. A linear functional f on a TVS X is continuous if and only if $S = \{ x: f(x) = 0 \}$ is closed.

Proof. The result follows directly from the theorem 7.2 and Lemma 7.6.

Lemma 7.9. Let V_1, V_2 be two convex sets in TVS X with at least one of which has an interior point. If V_1 and V_2 are separated by a linear functional f. Then f is continuous.

Proof. Suppose V_2 has an interior point. Let V_2^0 be the set of all interior points of V_2, and for some real t, $H = \{x: f(x) = t\}$, which is a hyperplane in X. Then $H \cap V_2^0 = \emptyset$. In fact, $f(b) \leq t \leq f(a)$ for all $a \in V_2$, $b \in V_1$ (by Definition 7.3). We want to show that $f(a) \neq t$ for all $a \in V_2^0$. Suppose $f(a) = t$ for some $a \in V_2^0$. Choose $x \in X$ such that $f(x) = 1$ (this is possible, since $f \neq 0$ and V_1, V_2 can be separated by f). Since $a \in V_2^0$, V_2 is absorbing at a, $a + \alpha x \in V_2$ for some $\alpha < 0$, and $f(a + \alpha x) = t + \alpha f(x) = t + \alpha < t$. But $f(b) \leq t \leq f(a)$ for all $a \in V_2$, $b \in V_1$. Thus $f(a) \neq t$ and hence $H \cap V_2^0 = \emptyset$. It follows that H is not dense in X and therefore $S = \{x: f(x) = 0\}$ is not dense either. In view of Corollary 7.2, f is continuous. Q.E.D.

Theorem 7.3. Let X be a TVS and V_1, V_2 be two convex sets with $V_1 \cap V_2 = \emptyset$ and at least one of which has an interior point. Then V_1 and V_2 can be separated by a continuous linear functional.

Proof. By Corollary 7.1, V_1, V_2 can be separated by a linear functional f. The continuity of f follows from Lemma 7.9.

Remark 1. Theorem 7.3 shows that if there exists two convex sets in a TVS X with $V_1 \cap V_2 = \emptyset$, and at least one of which has an interior point, then there exists a non-zero continuous linear functional.

Remark 2. The condition of "at least one of which has an interior point" in theorem 7.3 is essential. For example, $V_1 = \{b\}$, $V_2 = \{a\}$ $(a \neq b)$ be two disjoint convex sets in $L^p(0, 1)$, $0 < p < 1$. We know that there does not exist a non-zero continuous linear functional on $L^p(0 < p < 1)$ (see Example 6.1).

Lemma 7.10. Let B be a convex subset of a TVS X, which is not absorbing at 0; but it is absorbing at some point a of B. Let $S = K \cap (-K)$ with K as defined in the proof of Lemma 7.5, and f a linear functional such that $S = \{x: f(x) = 0\}$, $f(a) = 1$. Then $f(b) > 0$ for each b at which B is absorbing.

Proof. Lemma 7.5 shows that $f(b) \geq 0$ for all $b \in B$. We shall show that $f(b) \neq 0$ for each b at which B is absorbing. Suppose $f(b) = 0$. For $a \neq b$, there exists $\varepsilon > 0$ such that $b + ta \in B$ for $|t| \leq \varepsilon$, since B is absorbing at b. Choosing $t < 0$, $|t| \leq \varepsilon$, we have $f(b + ta) = tf(a) = t < 0$, a contradiction. Q.E.D.

Lemma 7.11. Let V be a convex subset of a TVS X. If V is absorbing at each of its points, and $0 \notin V$. Then there exists a maximal subspace S of X such that $V \cap S = \emptyset$.

Proof. V is not absorbing at 0, for otherwise $0 \in V$. Therefore, by Lemma 7.5, there exists a maximal subspace $S = \{x: f(x) = 0\}$, such that $f(y) \geq 0$ for all $y \in V$. Since V is absorbing at each of its points, Lemma 7.10 implies that $f(y) > 0$ for all $y \in V$. Hence $V \cap S = \emptyset$. Q.E.D.

Lemma 7.12. Let X and B be topological vector spaces, and $f: X \longrightarrow B$ a linear map. Suppose that H is a set in X satisfying $f(H) = B$. Then $X = H + f^{-1}(\{0\})$.

Proof. For any a in X, by hypothesis, $f(a) \in f(H)$. Then $f(a) = f(h)$, for some $h \in H$ i.e. $f(a - h) = 0$, $a - h \in f^{-1}(\{0\})$. Hence $a = (a - h) + h \in f^{-1}(\{0\}) + H$. Q.E.D.

Lemma 7.13. Let X, B be topological vector spaces, and $f: X \longrightarrow B$ a linear onto map. Let H be a maximal subspace of B. Then $f^{-1}(H)$ is a maximal subspace of X.

Proof. Let $S = f^{-1}(H)$. Then $S \neq X$, since f is onto. Let H_1 be a subspace with $H_1 \supsetneqq S$. Then $f(H_1) \supsetneqq H$ and so $f(H_1) = B$. By Lemma 7.12,

$$X = H_1 + f^{-1}(\{0\}) \subset H_1 + S \subset H_1 + H_1 = H_1.$$

Hence $H_1 = X$ and so S is maximal.

Theorem 7.4. Let V be a convex subset of a TVS X and V is absorbing at each of its points, and S be an affine set in X such that $S \cap V = \emptyset$. Then there exists a hyperplane H such that $H \supset S$ and $H \cap V = \emptyset$.

Proof. Since S is an affine set (see definition 7.1), $S = a + S_0$ where $a \in X$ and S_0 a subspace of X. First of all, we consider S to be a subspace of X. We wish to show that there exists a maximal subspace H such that $H \supset S$ and $H \cap V = \emptyset$. Let $f: X \longrightarrow X/S$ be defined by $f(a) = a + S$, for each $a \in X$. Clearly, f is a linear onto map. Since V is convex, absorbing at each of its points, so is $f(V)$, and $\{0\} \notin f(V)$(because $f(S) = \{0\} \in X/S$, $S \cap V = \emptyset$, $f(S) \cap f(V) = \emptyset$). By Lemma 7.11, there exists a maximal subspace H_1 of X/S such that $H_1 \cap f(V) = \emptyset$. Let $H = f^{-1}(H_1)$. By Lemma 7.13, H is a maximal subspace of X, $H \supset S$ because $f(S) = \{0\} \subset H_1$, and $f^{-1}(f(V)) \cap f^{-1}(H_1) = \emptyset$. Since $V \subset f^{-1}(f(V))$, $V \cap f^{-1}(H_1) = V \cap H = \emptyset$.

Now in general, if S is an affine set, then $S = a + S_0$ where $a \in X$, S_0 a subspace of X. Clearly $(S - a) \cap (V - a) = \emptyset$ i.e. $S_0 \cap (V - a) = \emptyset$, and $V - a$ is convex and absorbing at each of its points. By the above argument, there exists a maximal subspace H_0 such that $H_0 \supset S_0$ and $H_0 \cap (V - a) = \emptyset$. Let $H = H_0 + a$, then we have $H \supset S$ and $H \cap V = \emptyset$. Q.E.D.

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