Computing the intersection of regular Hessenberg varieties with Schubert cells

COMPUTING THE INTERSECTION OF REGULAR HESSENBERG VARIETIES WITH SCHUBERT CELLS

BY

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Introduction

Hessenberg varieties are a family of subvarieties of the (complex) flag variety first studied by De Mari, Procesi, and Shayman (F. De Mari, 1992). We can parameterize Hessenberg varieties by two objects: a linear operator X which acts on $\mathbb{C}^n$ (usually written as an $n \times n$ matrix) and a Hessenberg function h, which is a special type of function on the set $\{1, 2, ..., n-1, n\}$. We denote a Hessenberg variety by $Hess(X, h)$ and say it's a regular Hessenberg variety when X is a regular matrix.

The goal of this thesis is to give a method of describing the intersection of regular Hessenberg varieties with parts of the flag variety which contain it. We will then analyse the smoothness of a point in one particular intersection.

Chapter 1 gives the basics on flags and Hessenberg varieties. We show how flags can be thought of as matrices in $GL_n(\mathbb{C})$, and how the entire flag variety is isomorphic to certain cosets of $GL_n(\mathbb{C})$. The Bruhat Decomposition Theorem is used to partition the flag variety into Schubert cells parametrized by a set of permutations W. We also introduce the matrix patch $\mathcal{M}_w$ corresponding to each Schubert cell. These matrix patches cover the flag variety and allow for direct computations of the intersection of a Hessenberg variety with any particular Schubert cell.

The goal of Chapter 2 is to provide an explicit algorithm for computing a set of polynomial generators for an ideal with vanishing locus equal to the intersection $Hess(X, h) \cap \mathcal{M}_w$. In his PhD thesis, Erik Insko gave a method of producing such a set of generators for $Hess(X, h) \cap \mathcal{M}_w$ where X is regular nilpotent (Insko, 2012, see Ch.4). Here we expand on Insko's method by allowing X to be any regular matrix in Jordan normal form – i.e. not just nilpotent ones. In addition, we give a way of expressing each computed polynomial generator as a degree-separated sum of monomials.

In Chapter 3 we focus on the intersection of a particular Schubert cell with the Hessenberg variety – $Hess(X, h) \cap \mathcal{M}_{w_0}$. This particular Schubert cell corresponds to the permutation $w_0(1, 2, ...n) = (n, n-1, ..., 2, 1)$ which reverses the letter ordering on a word of n-many letters. Using the results of Precup we derive a dimension formula and non-emptiness criteria for $Hess(X, h) \cap \mathcal{M}_{w_0}$. We then use Chapter 2's algorithm to compute a set of generators for an ideal with vanishing locus $Hess(X, h) \cap \mathcal{M}_{w_0}$. By analysing how these generators affect the Jacobian matrix we conclude that the origin of $Hess(X, h) \cap \mathcal{M}_{w_0}$ is a smooth point.

Abstract

In his PhD thesis Erik Insko gave the conditions in which the intersection of any Schubert cell with a regular nilpotent Hessenberg variety is smooth. In this thesis we relax the nilpotent condition and aim to extend his method to describe regular Hessenberg varieties without the nilpotent restriction. We conclude that one specific intersection $Hess(X, h) \cap \mathcal{M}_{w_0}$ is smooth at the origin.

Contents

Chapter 1

Background

The object of study in this paper will be Hessenberg varieties, which are subvarieties of the flag variety.

Definition 1.1. A (complete) *flag* $F_\bullet$ is a sequence of nested subspaces of $\mathbb{C}^n$ of the form

$$F_\bullet = (F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq F_n = \mathbb{C}^n), \forall i \ dim(F_i) = i. \tag{1.1}$$

The set of all such flags is called the *flag variety* of $\mathbb{C}^n$ and is denoted $\mathcal{F}lags(\mathbb{C}^n)$.

Definition 1.2. A *Hessenberg variety* is a subvariety of $\mathcal{F}lags(\mathbb{C}^n)$ parameterized by two objects:

- A function $h : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ which satisfies $h(i+1) \geq h(i)$ for $\forall i$ $1 \leq i < n$ and $h(i) \geq i$ for $\forall i$ $1 \leq i \leq n$. Such a function is called a *Hessenberg function*.

- A linear operator $X : \mathbb{C}^n \to \mathbb{C}^n$. We can represent X as a $n \times n$ matrix with respect to the standard basis of $\mathbb{C}^n$.

The Hessenberg variety associated to X and h is defined as

$$Hess(X, h) = \{F_\bullet \in \mathcal{F}lags(\mathbb{C}^n) \text{ such that } XF_i \subseteq F_{h(i)} \text{ for all } i\}. \tag{1.2}$$

To compute XF_i we need to be able to think of flags as matrices.

Lemma 1.3. *Each flag $F_\bullet \in \mathcal{F}lags(\mathbb{C}^n)$ can be associated to an element of $GL_n(\mathbb{C})$.*

Proof: Consider a flag

$$F_\bullet = (F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq F_n = \mathbb{C}^n).$$

By assumption, F_1 is a 1 dimensional subspace, so it has a basis element v_1. Since $F_1 \subseteq F_2$ we can extend v_1 to a set $\{v_1, v_2\}$ which is a basis of F_2 (by the basis extension theorem). Continuing this process we arrive at $\{v_1, v_2, ..., v_n\}$, a basis for $F_n = \mathbb{C}^n$.

Now consider the matrix A obtained by putting these basis elements into a matrix as column vectors:

$$A = \begin{bmatrix} v_1 & |v_2 & |... & |v_n \end{bmatrix}.$$

Since the set $\{v_1, v_2, ..., v_n\}$ is a basis, the vectors are linearly independent and so A is an invertible matrix, i.e. $A \in GL_n(\mathbb{C})$. ■

However, since bases are not unique, more than one matrix can be associated to each flag of $GL_n(\mathbb{C})$. To get a one-to-one correspondence we need the terminology and theorems of linear algebraic groups.

Definition 1.4. Let G be a linear algebraic group. In this paper we set $G = GL_n(\mathbb{C})$. Consider the following subgroups of G:

- Let $B \subset G$ be a Borel subgroup of G. In this paper we set B to be the group of invertible upper triangular complex $n \times n$ matrices.
- Let $B^- \subset G$ be the opposite Borel subgroup of B. In this paper we set B^- to be the group of invertible lower triangular complex $n \times n$ matrices.
- Let $U \subset B$ be a subgroup of invertible upper triangular complex $n \times n$ matrices with 1 on the diagonal.
- Let $U^- \subset B^-$ be a subgroup of invertible lower triangular complex $n \times n$ matrices with 1 on the diagonal.
- Let $W \subset G$ be the Weyl subgroup of G. Let S_n be the group of permutations on n letters. For each $s \in S_n$, associate a $n \times n$ permutation matrix $w \in W$ such that the ith row of w has its $s(i)$th entry equal to 1 and all other entries equal to 0. In this paper W will be this group of $n \times n$ permutation matrices.

Example 1.5. Consider the permutation on 4 letters $s(1234) = (4231)$. Denote this element of S_4 by s_{4231}. Then s_{4231} has the associated permuation matrix in W:

$$w_{4231} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 1.6 (Bruhat Decomposition Theorem)**.** *Let G be a linear algebraic group with Weyl subgroup W. For any Borel subgroup $B \subset G$, G admits the following partition:*

$$G = \bigcup_{w \in W} BwB \tag{1.3}$$

Passing to the quotient, G/B then admits the following partition:

$$G/B = \bigcup_{w \in W} BwB/B \tag{1.4}$$

Theorem 1.7. *Applying Definition 1.4 to Theorem 1.6 gives the following decomposition*

$$GL_n(\mathbb{C})/B = \bigcup_{w \in W} BwB/B \tag{1.5}$$

Definition 1.8. For each $w \in W$ we call the coset BwB/B the *Schubert cell* associated to w.

Theorem 1.9. *$\mathcal{F}lags(\mathbb{C}^n) \simeq GL_n(\mathbb{C})/B$ as sets.*

Proof: The group $GL_n(\mathbb{C})$ acts transitively on $\mathcal{F}lags(\mathbb{C}^n)$ by multiplication of matrices. The subgroup $B \subset GL_n(\mathbb{C})$ is the stabilizer of the flag corresponding to the identity matrix (the "standard flag"). This implies $\mathcal{F}lags(\mathbb{C}^n) \simeq GL_n(\mathbb{C})/B$ as sets.■

Immediate from Theorem 1.9 are the following observations:

Observation 1.10. *(i) Every flag $F_\bullet \in \mathcal{F}lags(\mathbb{C}^n)$ corresponds to a coset of the form gB in $GL_n(\mathbb{C})/B$.*

(ii) Any two matrices representing the same flag differ by multiplication by an element of B.

Definition 1.11. Let w_0 be the permutation matrix associated to the permutation $s(1,2,...,n-1,n) = (n,n-1,...,2,1)$. w_0 is called the longest word of the Weyl group.

Since $GL_n(\mathbb{C})/B$ is a smooth complex variety it is described by gluing together smooth affine patches. We will identify the patches in $GL_n(\mathbb{C})/B$ with translates of a particular Schubert cell.

Definition 1.12. Define Bw_0B/B to be the "big cell", the cell associated to w_0. We can translate the big cell by elements of W by writing $ww_0^{-1}Bw_0B/B$.

The patch $(id)w_0^{-1}Bw_0B/B$ is an affine open neighborhood of the flag $(id)B$. Similarly each translation by $ww_0^{-1}Bw_0B/B$ is an affine open neighborhood of the flag wB.

By (Springer, 1981, Prop 8.5.1) these translates of the big cell give a cover for $GL_n(\mathbb{C})/B$, and thus a cover for $\mathcal{F}lags(\mathbb{C}^n)$.

Definition 1.13. Denote $\mathcal{N}_w = ww_0^{-1}Bw_0B/B$ to be the patch of $GL_n(\mathbb{C})/B$ identified with the translation of the big cell by $w \in W$. Then the set of patches $\{\mathcal{N}_w\}_{w\in W}$ cover the flag variety.

Since $ww_0^{-1}Bw_0B/B \simeq wB^-(id)B/B \simeq wU^-(id)B/B$ we can also write $\mathcal{N}_w = wU^-(id)B/B$.

We will now identify each patch of the flag variety $wU^-(id)B/B$ with a set of matrices.

Definition 1.14. Define $\mathcal{M}_w = wU^-$ to be the *matrix patch* associated to the patch $\mathcal{N}_w$. There is an isomorphism $\mathcal{M}_w \xrightarrow{\sim} \mathcal{N}_w$ given by $M \to MB/B$.

Example 1.15. $\mathcal{M}_w$ is a set of matrices we can write out, and do computations with. For n=4 and $w = w_0$ we have the matrix patch

$$\mathcal{M}_{w_0} = w_0U^- = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} * & * & * & 1 \\ * & * & 1 & 0 \\ * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}$$

Since * are arbitrary complex numbers, we will label them as variables according to their position in the matrix:

$$\mathcal{M}_{w_0} = \left\{ \begin{bmatrix} x_{11} & x_{12} & x_{13} & 1 \\ x_{21} & x_{22} & 1 & 0 \\ x_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

Chapter 2

Computing a set of generators

With a set of matrix patches $\mathcal{M}_w$ (as in Definition 1.14) that cover the flag variety in hand, we can obtain a concrete description of $Hess(X,h)$ by intersecting it with each patch. From the definition of $Hess(X,h)$ it follows that the intersection of $Hess(X,h)$ with with the patch $\mathcal{M}_w$ is given by

$$Hess(X,h) \cap \mathcal{M}_w = \{M \in \mathcal{M}_w \text{ where } XM_j \subseteq \text{span}\{M_1, M_2, ..., M_{h(j)}\} \text{ for } j,\ 1 \leq j \leq n\} \tag{2.1}$$

where M_j is the jth column of $M \in \mathcal{M}_w$.

Let's examine the condition $XM_j \subseteq \text{span}\{M_1, M_2, ..., M_{h(j)}\}$ for all j in **Equation (2.1)**. For a fixed j, $1 \leq j \leq n$ the product XM_j is in the span of the first $h(j)$

columns of $\mathcal{M}_w$ if we can find constants $\alpha_{j,\ell} \in \mathbb{C}$ which satisfy:

$$XM_j = \alpha_{j,1}M_1 + \alpha_{j,2}M_2 + ... + \alpha_{j,h(j)}M_{h(j)} \tag{2.2}$$

Fix an $M \in \mathcal{M}_w$ and fix j. Now the LHS of **Equation (2.2)** is a $n \times 1$ column vector and the RHS is the sum of $h(j)$ many $n \times 1$ column vectors. This means we may interpret **Equation (2.2)** as a system of n equations in $h(j)$ many unknowns - the $\{\alpha_{j,1}, ..., \alpha_{j,h(j)}\}$.

Definition 2.1. Let $\overline{x} = \{x_{ij}\}$ denote the set of $n(n-1)/2$ complex variables in the entries of $M \in \mathcal{M}_w$ (cf. Example 1.15).

Example 2.2. For $n = 4$ and $w = w_0$ as in Example 1.15, $\overline{x} = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{31}\}$.

Since X is a complex matrix, each of the n equations of **Equation (2.2)** is a linear polynomial in the unknowns $\{\alpha_{j,1}, ..., \alpha_{j,h(j)}\}$ with coefficients that are polynomials in the variables $\{x_{ij}\}$.

As we will see below, there is an inductive method whereby for each fixed j, we use $h(j)$ of the n equations to "solve" for the $\{\alpha_{j,1}, ..., \alpha_{j,h(j)}\}$. Here by "solve for $\alpha_{j,\ell}$" we mean that we obtain an equality of the form $\alpha_{j,\ell} =$ [an expression in the $\overline{x}$ variables]. Once $\alpha_{j,\ell}$ has been "solved" we'll write it as $\overline{\alpha_{j,\ell}}$.

Once solved we can substitute the values for $\{\overline{\alpha_{j,1}}, ..., \overline{\alpha_{j,h(j)}}\}$ into the remaining $n - h(j)$ equations. This yields a set of $n - h(j)$ many nontrivial polynomial relations among the variables $\overline{x} = \{x_{ij}\}$. These polynomials in $\mathbb{C}[\overline{x}]$ then define the intersection $Hess(X, h) \cap \mathcal{M}_w$.

Definition 2.3. For each j of **Equation (2.2)**, define $g_{ij} \in \mathbb{C}[\overline{x}]$ to be the polynomial relation obtained from substituting the solved values $\{\alpha_{j,1}, ..., \alpha_{j,h(j)}\}$ into the equation corresponding to the ith row.

These g_{ij}'s generate an ideal whose vanishing locus is $Hess(X, h) \cap \mathcal{M}_w$.

Definition 2.4. Define the *patch ideal* of $Hess(X, h) \cap \mathcal{M}_w$ to be $I(Hess(X, h) \cap \mathcal{M}_w) := \langle g_{ij} \rangle$. By construction, the vanishing locus of $\langle g_{ij} \rangle$ is indeed $Hess(X, h) \cap \mathcal{M}_w$.

Definition 2.5. Let $\overline{x}$ contain $n(n-1)/2$ many complex variables. If p is a (closed) point in the variety $Spec(\mathbb{C}[\overline{x}]) \simeq wU^-$ then we can think of p as an element of $\mathbb{C}^{n(n-1)/2}$. Define the *origin* to be the point where all variables of $\overline{x}$ equal 0, i.e. p is the point corresponding to the maximal ideal $\langle x_{ij} \rangle$.

Theorem 2.6. *(Hartshorne, 1977, pg 31) Let $V \subseteq \mathbb{C}^m$ be an affine variety, and let $f_1, \ldots, f_t \in \mathbb{C}[x_1, \ldots, x_m]$ be a set of generators for the ideal of V. We say V is nonsingular or smooth at a point $p \in V$ if the rank of the matrix $\left[\frac{\partial f_i}{\partial x_j}\right](p)$ is $m - r$, where r is the dimension of V.*

Note that the smoothness of a point $p \in Hess(X, h) \cap \mathcal{M}_w$ depends on three parameters: X, h, and w. By specializing to a particular type of linear operator, can ask the following.

Question 2.7. If we fix X to be regular nilpotent (i.e. 1's along the superdiagonal and 0's elsewhere), what conditions on h and w make $p \in Hess(X, h) \cap \mathcal{M}_w$ smooth?

This question is answered by the thesis of Erik Insko (Insko, 2012).

Theorem 2.8 (Insko 2012 Theorem 4.10)**.** *The cell $BwB/B \cap Hess(X, h)$ is singular in $Hess(X, h)$ if and only if there exists an integer k such that all of the following inequalities hold:*

$$\begin{aligned} h(w^{-1}(k)) &< w^{-1}(n) \\ h(w^{-1}(k-1)) &< w^{-1}(n-1) \\ &\vdots \\ h(w^{-1}(1)) &< w^{-1}(n-k+1) \end{aligned}$$

Motivated by the results of Insko, in this thesis we ask the following.

Question 2.9. If we fix p to be the origin, and w to be w_0, what conditions on X and h make $p \in Hess(X,h) \cap \mathcal{M}_{w_0}$ smooth?

In this thesis we will answer this question as follows.

Let X be a *regular* linear operator, i.e., an $n \times n$ complex matrix whose Jordan normal form has different eigenvalues in each separate Jordan block. Then, under the condition that $Hess(X,h) \cap \mathcal{M}_{w_0}$ is non-empty, we prove that the origin p is smooth in $Hess(X,h)$ for any h.

Since $Hess(X,h)$ is isomorphic to $Hess(gXg^{-1},h)$ for any $g \in GL_n(\mathbb{C})$, for the remainder of this thesis we assume WLOG that X is already in Jordan normal form. The following definition gives notation for writing any such X.

Definition 2.10. Let $X : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator. Define $J \subseteq \{1,2,...,n-2,n-1\}$, i.e. a subset of $[n-1]$. Let $\{\lambda_1,...,\lambda_n\}$ be a sequence of complex numbers. Define $1_i = 1$ if $i \in J$ and $1_i = 0$ otherwise. Then this data specifies a matrix X in Jordan canonical form as follows:

$$X = \begin{bmatrix} \lambda_1 & 1_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \lambda_{n-1} & 1_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}.$$

Example 2.11. We can realize any matrix in Jordan normal form using this notation. For example, the 6×6 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

is realized by setting $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = \lambda_5 = 2$, $\lambda_6 = 3$ and $J = \{1, 2, 4\}$.

Using this notation, we can compute XM_w explicitly.

Example 2.12. Let $n = 4$ and $w = w_0$.

Let X be

$$X = \begin{bmatrix} \lambda_1 & 1_1 & 0 & 0 \\ 0 & \lambda_2 & 1_2 & 0 \\ 0 & 0 & \lambda_3 & 1_3 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}.$$

We have $\mathcal{M}_{w_0}$ by Example 1.15:

$$\mathcal{M}_{w_0} = \left\{ \begin{bmatrix} x_{11} & x_{12} & x_{13} & 1 \\ x_{21} & x_{22} & 1 & 0 \\ x_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

Let $M_{w_0} \in \mathcal{M}_{w_0}$. To compute XM_{w_0} we can write

$$XM_{w_0} = \begin{bmatrix} \lambda_1 & 1_1 & 0 & 0 \\ 0 & \lambda_2 & 1_2 & 0 \\ 0 & 0 & \lambda_3 & 1_3 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & 1 \\ x_{21} & x_{22} & 1 & 0 \\ x_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which gives

$$XM_{w_0} = \begin{bmatrix} \lambda_1 x_{11} + 1_1 x_{21} & \lambda_1 x_{12} + 1_1 x_{22} & \lambda_1 x_{13} + 1_1 & \lambda_1 \\ \lambda_2 x_{21} + 1_2 x_{31} & \lambda_2 x_{22} + 1_2 & \lambda_2 & 0 \\ \lambda_3 x_{31} + 1_3 & \lambda_3 & 0 & 0 \\ \lambda_4 & 0 & 0 & 0 \end{bmatrix}.$$

We now give an explicit method for computing a set of generators g_{ij} of $I(Hess(X,h) \cap \mathcal{M}_w)$. This method does not require any assumption on X, h, or w.

Theorem 2.13. *Let $j \in \{1, 2, ...n\}$. Then there is a polynomial relation g_{ij} as defined in Definition 2.3 for each $i \in \{w(h(j)+1), w(h(j)+2), ..., w(n-1), w(n)\}$, and they are of the form:*

$$g_{w(h(j)+1),j} = [w(h(j)+1)th \text{ entry of } XM_j] - x_{w(h(j)+1),1}\overline{\alpha_{j,1}} - ... - x_{w(h(j)+1),h(j)}\overline{\alpha_{j,h(j)}}$$
$$\vdots$$
$$g_{w(n),j} = [w(n)th \text{ entry of } XM_j] - x_{w(n),1}\overline{\alpha_{j,1}} - ... - x_{w(n),h(j)}\overline{\alpha_{j,h(j)}}$$

where the $\overline{\alpha_{j,\ell}}$ denote the "solved" expression for the $\alpha_{j,\ell}$ in terms of the variables x_{ij} as explained in the paragraphs above Definition 2.3. In particular, the $\overline{\alpha_{j,\ell}}$ are polynomials in the x_{ij}.

Proof: Recall we first want to solve for the coefficients $\{\alpha_{j,1}, ..., \alpha_{j,h(j)}\}$ of

$$XM_j = \alpha_{j,1}M_1 + \alpha_{j,2}M_2 + ... + \alpha_{j,h(j)}M_{h(j)} \text{ for } 1 \leq j \leq n. \tag{2.3}$$

Fix a j. Recall M_j is defined to be the jth $n \times 1$ column of $M \in \mathcal{M}_w$ counting from the left. So the above matrix equation is a system of n equations in $h(j)$ unknowns - the $\{\alpha_{j,1}, ..., \alpha_{j,h(j)}\}$.

Henceforth we refer to the equation corresponding to the ith row of **Equation (2.3)** as the ith equation. Since X is in Jordan normal form it is not hard to see that (cf. also Example 2.12 above) the ith equation contains variables from $\{x_{i+1,j}, x_{i,1}, ..., x_{i,h(j)}\}$ and constants from $\{\lambda_i, 1_i\}$.

Notice that because $\mathcal{M}_w$ is a matrix patch, $M \in \mathcal{M}_w$ is a matrix which has a row consisting of a 1 in its left-most entry and all zeros to the right of the 1 (e.g. the 4th row in Example 1.15). This is the $w(1)th$ row.

The $w(2)th$ row has $x_{w(2),1}$ as its left-most entry, a 1 to its right, and the rest are zeros.

In general the $w(\ell)th$ row of $M \in \mathcal{M}_w$ looks like

$$\begin{bmatrix} x_{w(\ell),1} & \dots & x_{w(\ell),w(\ell)-1} & 1 & 0 & \dots & 0 \end{bmatrix}$$

So we will solve for $\alpha_{j,1}$ using the $w(1)th$ row of **Equation (2.3)** - since it's just $[w(1)th$ entry of $XM_j] = 1\alpha_{j,1}$. Note that $[w(1)th$ entry of $XM_j]$ is an expression only in the $x_{i,j}$. In this sense the equality "solves" for $\alpha_{j,1}$ in terms of $x_{i,j}$. Denote the expression of $\alpha_{j,1}$ in this form as $\overline{\alpha_{j,1}}$.

Now we can use $\overline{\alpha_{j,1}}$ to solve for $\alpha_{j,2}$ by using the $w(2)th$ row of **Equation (2.3)** which is

$[w(2)th$ entry of $XM_j] = x_{w(2),1}\overline{\alpha_{j,1}} + \alpha_{j,2}$

Thus, inductively we can solve for $\alpha_{j,3}, \alpha_{j,4}, ..., \alpha_{j,h(j)}$ by using

$[w(3)th$ entry of $XM_j] = x_{w(3),1}\overline{\alpha_{j,1}} + x_{w(3),2}\overline{\alpha_{j,2}} + \alpha_{j,3}$

$\vdots$

$[w(h(j))th$ entry of $XM_j] = x_{w(h(j)),1}\overline{\alpha_{j,1}} + ... + x_{w(h(j)),h(j)-1}\overline{\alpha_{j,h(j)-1}} + \alpha_{j,h(j)}$.

To recap, we've used $h(j)$ rows of **Equation (2.3)**, namely the $w(1)th$, $w(2)th$, ..., and $w(h(j))th$ rows, to solve for $\{\alpha_{j,1}, \alpha_{j,2}, ..., \alpha_{j,h(j)}\}$.

This leaves us with $n-h(j)$ many rows of **Equation (2.3)** for which we can substitute our solved $\{\alpha_{j,1}, \alpha_{j,2}, ..., \alpha_{j,h(j)}\}$ to get relations between the variables $\{x_{k+1,j}, x_{k,1}, ..., x_{k,h(j)}\}$.

The (polynomial) relation arising for substituting $\{\alpha_{j,1}, \alpha_{j,2}, ..., \alpha_{j,h(j)}\}$ into the ith row of **Equation (2.3)** we will label g_{ij}. For each j these i's have values $\{w(h(j)+1), w(h(j)+2), ..., w(n-1), w(n)\}$.

All together this gives:

$$\begin{aligned} g_{w(h(j)+1),j} &= [w(h(j)+1)\text{th entry of } XM_j] - x_{w(h(j)+1),1}\overline{\alpha_{j,1}} - ... - x_{w(h(j)+1),h(j)}\overline{\alpha_{j,h(j)}} \\ &\vdots \\ g_{w(n),j} &= [w(n)\text{th entry of } XM_j] - x_{w(n),1}\overline{\alpha_{j,1}} - ... - x_{w(n),h(j)}\overline{\alpha_{j,h(j)}}. \end{aligned}$$

■

The above method yields the following observations:

Observation 2.14. *The (i,j) for which there exists a corresponding generator $g_{ij} \in I(Hess(X,h) \cap \mathcal{M}_w)$ are those satisfying $h(j) < w^{-1}(i)$.*

Observation 2.15. *For each fixed j, we have a contribution of $n-h(j)$ generators to $I(Hess(X,h) \cap \mathcal{M}_w)$. We label these generators as: $\{g_{w(h(j)+1),j}, g_{w(h(j)+2),j}, ..., g_{w(n-1),j}, g_{w(n),j}\}$.*

So to understand the g_{ij}'s we need to know the form of the XM_j entries.

Recall that X is a matrix in Jordan canonical form. Since any row of X has at most two adjacent nonzero entries, λ_i and 1_i, the "most" the $w(i)th$ row/entry of XM_j can be is

$$\lambda_{w(i)}x_{w(i),j} + 1_{w(i)}x_{w(i)+1,j}. \tag{2.4}$$

The $x_{w(i),j}$ and $x_{w(i)+1,j}$ can be set to 1 or 0, depending on w. Let r be the number such that $w(r) = w(i) + 1$, then **Expression (2.4)** is subject to the constraints:

$$x_{w(i),j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } j > i \\ x_{w(i),j} & \text{if } i > j \end{cases} \text{ and } x_{w(i)+1,j} = \begin{cases} 1 & \text{if } r = j \\ 0 & \text{if } j > r \text{ or r does not exist} \\ x_{w(i)+1,j} & \text{if } r > j \end{cases} \tag{2.5}$$

Observation 2.16. *When analysing the smoothness at the origin, only the linear terms of g_{ij}'s affect the rank of the Jacobian.*

Proof: Recall our Jacobian has the form $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right]$. If g_{ij} contains a constant term, the constant vanishes when taking the partial derivative with respect to any x_{ij}. If g_{ij} contains a degree 2 or higher term, it vanishes or becomes a degree 1 or higher term when taking the partial derivative with respect to any x_{ij}.

Recall we defined evaluating the Jacobian at the origin to mean setting all $\overline{x} = \{x_{ij}\}$ equal to 0. So any degree 1 or higher monomials in the entries of the Jacobian vanish. So for the purposes of analysing the smoothness at the origin, we only need to pay attention to the degree 1 (linear) terms of g_{ij}.■

With Expression (2.4) and Observation 2.16 in mind, we will next want to write g_{ij}

as a sum organized by the degrees of its monomials.

First some notation:

Definition 2.17. Let f be a polynomial in $\mathbb{C}[\overline{x}]$. Let $(f)_k$ denote the sum of its degree k monomials.

Definition 2.18. Let $(XM)_{(i,j)}$ be the entry of the *ith* row and *jth* column of the matrix XM.

Theorem 2.19. *We can write g_{ij} as a degree-separated sum as follows:*

$g_{i,j} = (XM_{(i,j)})_0 + (XM_{(i,j)})_1 - \sum_{k=1}^{h(j)} x_{i,k}(XM_{(w(k),j)})_0 +$ *(terms of deg ≥ 2).*

Proof: Recall from the proof of Theorem 2.13 that for each fixed j the first constant we solve for is $\overline{\alpha_{j,1}}$ and that this equals $XM_{(w(1),j)}$. Also note that each entry of XM_j is a polynomial of at most degree 1 in the variables $\{x_{ij}\} = \overline{x}$ by **Expression 2.4**.

Write $\overline{\alpha_{j,1}} = (\overline{\alpha_{j,1}})_0 + (\overline{\alpha_{j,1}})_1$ a sum of its degree 0 and degree 1 terms, so $\overline{\alpha_{j,1}} = (XM_{(w(1),j)})_0 + (XM_{(w(1),j)})_1$.

Next, recall that inductively, $\alpha_{j,2}$ is solved in terms of $\overline{\alpha_{j,1}}$:

$$\overline{\alpha_{j,2}} = XM_{(w(2),1)} - \overline{\alpha_{j,1}} x_{w(2),1}$$

Then write $\overline{\alpha_{j,2}} = (XM_{(w(2),j)})_0 + (XM_{(w(2),j)})_1 - (\overline{\alpha_{j,1}} x_{w(2),1})_1 - (\overline{\alpha_{j,1}} x_{w(2),1})_2$.

In general we'll have $(\overline{\alpha_{j,i}})_0 = (XM_{(w(i),j)})_0$, since when $\overline{\alpha_{j,1}}, ..., \overline{\alpha_{j,i-1}}$ is substituted into a row of **Equation 2.3** to get a generator g_{ij}, it is always the coefficient of a variable. This implies only the degree 0 terms of the $\overline{\alpha_{j,i}}$ will affect the Jacobian at the origin, since only $(\overline{\alpha_{j,i}})_0$ can appear as a coefficient of a degree 1 monomial of a

g_{ij}.

Let us rephrase Theorem 2.13 as

$g_{i,j} = XM_{(i,j)} - \sum_{k=1}^{h(j)} x_{i,k}\overline{\alpha_{j,k}}$.

For each fixed j the index i takes values from $\{w(h(j)+1), w(h(j)+2), ..., w(n-1), w(n)\}$. And so separating g_{ij} by degrees gives

$g_{i,j} = (XM_{(i,j)})_0 + (XM_{(i,j)})_1 - \sum_{k=1}^{h(j)} x_{i,k}(\overline{\alpha_{j,k}})_0 +$ terms of deg ≥ 2.

Since we know $(\overline{\alpha_{j,i}})_0 = (XM_{(w(i),j)})_0$ this becomes

$g_{i,j} = (XM_{(i,j)})_0 + (XM_{(i,j)})_1 - \sum_{k=1}^{h(j)} x_{i,k}(XM_{(w(k),j)})_0 +$ terms of deg ≥ 2. ■

Chapter 3

$Hess(X, h) \cap \mathcal{M}_{w_0}$ is smooth at the origin

In this chapter we will restrict to the case $w = w_0$, where

$$w_0(1, 2, ..., n-1, n) = (n, n-1, ..., 2, 1)$$

and investigate the smoothness at the origin of $Hess(X, h) \cap \mathcal{M}_{w_0}$.

Let us restate the definition of a *smooth point* in a variety:

Definition 3.1. (Hartshorne 1977 pg 31) Let $V \subseteq \mathbb{C}^m$ be an affine variety, and let $f_1, \ldots, f_t \in \mathbb{C}[x_1, \ldots, x_m]$ be a set of generators for the ideal of V. We say V is *nonsingular* or *smooth at a point* $p \in V$ if the rank of the matrix $\left[\frac{\partial f_i}{\partial x_j}\right](p)$ is $m - r$, where r is the dimension of V.

Observe $m - r$ is also the codimension of V in $\mathbb{C}^m$.

This definition refers to *the* ideal of a variety V i.e. the *radical* ideal whose vanishing locus is V. The algorithm of Chapter 2 produces a set of generators of an ideal whose vanishing locus is V, but does not guarantee that this ideal is radical. We'll now argue that the radicality requirement can be relaxed when examining the smoothness of a point in $Hess(X,h) \cap \mathcal{M}_{w_0}$.

First we need to compute a dimension formula and non-emptiness criteria for $Hess(X,h) \cap \mathcal{M}_{w_0}$. To do this we use a result of (Precup, 2016).

Definition 3.2. Let Φ be the root system associated to $GL_n(\mathbb{C})$ and let Φ^+ and Φ^- be the subsets of positive and negative roots in Φ, respectively. Denote by Φ_H the root system associated with the Hessenberg space and let Φ_H^- be the negative root space in the Hessenberg space. Define the set $N^-(w) = \{\gamma \in \Phi^- | w(\gamma) \in \Phi^+\}$.

Refer to (Precup, 2016, pg 2) for a complete exposition on the Lie theoretic construction of a Hessenberg space and its associated root system. We will now state a result of (Precup 2016), translated into the notation of this paper. Recall the set $J \subseteq [n-1]$ specifies the Jordan blocks of the matrix X (cf. Definition 2.10).

Lemma 3.3 (Precup 2016 Lemma 2.5)**.** *For all $w \in W$, $Hess(X,h) \cap \mathcal{M}_w \neq \emptyset$ if and only if $w^{-1}(J) \subseteq \Phi_H$. If $Hess(X,h) \cap \mathcal{M}_w \neq \emptyset$ then $dim(Hess(X,h) \cap \mathcal{M}_w) = |N^-(w) \cap \Phi_H^-|$.*

We now set $w = w_0$ and derive the following non-emptiness criteria.

Lemma 3.4. *$Hess(X,h) \cap \mathcal{M}_{w_0} \neq \emptyset$ if and only if for $\forall i \in J$ we have $n-i+1 \leq h(n-i)$.*

Proof: Observe our permutation w_0 is its own inverse, so $w_0 = w_0^{-1}$. We also have the closed form equation $w_0(i) = n-i+1$. The root space Φ_H is in 1-1 correspondence with the set of ordered pairs:

$$\Phi_H = \{(h(i), i) | h(i) > i\}.$$

The set $w_0^{-1}(J)$ can also be associated with a set of ordered pairs:

$$w_0^{-1}(J) = \{(w_0^{-1}(i), i) | \forall i \in J\}$$

From Lemma 3.3 we know non-emptiness is equivalent to the condition $w_0^{-1}(J) \subseteq \Phi_H$. By considering the sets $w_0^{-1}(J)$ and Φ_H as sets of ordered pairs as above, and noting that $h(i) > i \iff h(i+1) \geq i$ for h which is non-decreasing, we get the following chain of equivalences:

$Hess(X, h) \cap \mathcal{M}_{w_0} \neq \emptyset \iff w_0^{-1}(J) \subseteq \Phi_H$

$\iff \forall i \in J,\, w_0^{-1}(i) \leq h((w_0{}^{-1}(i+1))$

$\iff \forall i \in J,\, n - i + 1 \leq h(n - (i+1) + 1)$

$\iff \forall i \in J,\, n - i + 1 \leq h(n - i).$ ■

Lemma 3.5. *If $\mathcal{M}_{w_0} \cap Hess(X, h) \neq \emptyset$, then $dim(\mathcal{M}_{w_0} \cap Hess(X, h)) = \sum_{i=1}^{n} h(i) - i$.*

Proof: Assuming $\mathcal{M}_{w_0} \cap Hess(X, h) \neq \emptyset$, by Lemma 3.3 we have $dim(\mathcal{M}_{w_0} \cap Hess(X, h)) = |N^-(w_0) \cap \Phi_H^-|$. Then write

$N^-(w_0) \cap \Phi_H^- = \{\gamma \in \Phi^- | w_0(\gamma) \in \Phi^+\}$

$= \{t_i - t_j$ where $i, j \in [n]$ and $i > j$ such that $t_{w(i)} - t_{w(j)} \in \Phi^+\}$

$= \{(i,j) \in [n]^2 | i > j \text{ and } w_0(i) < w_0(j)\}$

$= \{(i,j) \in [n]^2 | i > j\}$ or the number of blocks strictly below the diagonal of the Hessenberg function. So

$dim(\mathcal{M}_{w_0} \cap Hess(X,h)) = \sum_{i=1}^{n} h(i) - i.$■

From now on we will assume $\mathcal{M}_{w_0} \cap Hess(X,h) \neq \emptyset$.

Observation 3.6. *The (i,j) such that g_{ij} is a generator of $I(Hess(X,h) \cap \mathcal{M}_{w_0})$ computed by the algorithm of Chapter 2 are those that satisfy $n \geq h(j) + i$.*

Proof: By Observation 2.14 g_{ij} is a computed generator only when $h(j) < w_0^{-1}(i)$. Since $w_0 = w_0^{-1}$ and $w_0(i) = n - i + 1$ we have

g_{ij} a computed generator $\iff h(j) < n - i + 1 \iff n \geq h(j) + i.$■

Observation 3.7. *There are $\sum_{j=1}^{n} n - h(j)$ many generators g_{ij} of $I(Hess(X,h) \cap \mathcal{M}_{w_0})$ computed by the algorithm of Chapter 2.*

Proof: For a fixed j, the i's that satisfy $n \geq h(j) + i$ are $1, 2, ..., n - h(j)$. There are $n - h(j)$ many i's for each j, so there are $\sum_{j=1}^{n} n - h(j)$ many (i,j)'s that satisfy $n \geq h(j) + i$. Then by Observation 3.6 there are $\sum_{j=1}^{n} n - h(j)$ many generators g_{ij} computed by the algorithm of Chapter 2.■

Lemma 3.8. *Let $\mathcal{M}_{w_0} \cap Hess(X,h) \neq \emptyset$. Then the number of generators g_{ij} of $I(Hess(X,h) \cap \mathcal{M}_{w_0})$ computed by the algorithm of Chapter 2 equals $codim(\mathcal{M}_{w_0} \cap Hess(X,h))$.*

Proof: Recall the codimension of a variety is the dimension of the ambient space minus the dimension of the variety. Our ambient space is $\mathbb{C}^{(n(n-1)/2)}$ with dimension

$n(n-1)/2 = \sum_{i=1}^{n} n-i$, and we know the dimension of $\mathcal{M}_{w_0} \cap Hess(X,h)$ by Lemma 3.5. So we compute:

$$\begin{aligned} codim(\mathcal{M}_{w_0} \cap Hess(X,h)) &= dim(\mathbb{C}^{(n(n-1)/2)}) - dim(\mathcal{M}_{w_0} \cap Hess(X,h)) \\ &= \sum_{i=1}^{n} n-i - \sum_{i=1}^{n} h(i) - i \\ &= \sum_{i=1}^{n} n-i-h(i)+i \\ &= \sum_{i=1}^{n} n-h(i) \end{aligned}$$

By Observation 3.7 $\sum_{i=1}^{n} n-h(i)$ is the number generators g_{ij} computed. So we have shown this quantity equals $codim(\mathcal{M}_{w_0} \cap Hess(X,h))$.■

Lemma 3.9. *Let g_{ij} be a set of generators of $I(Hess(X,h) \cap \mathcal{M}_{w_0})$. Suppose the rank $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](p)$ is equal to the number of generators g_{ij}. Then the point p is smooth even if the ideal $\langle g_{ij}\rangle$ is not radical.*

Proof: If $rank(\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](p))$ = number of g_{ij}, then by Lemma 3.8 $rank(\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](p) = codim(\mathcal{M}_{w_0} \cap Hess(X,h))$.

If $\langle g_{ij}\rangle$ is radical then we conclude the point p is smooth since Definition 3.1 says the rank of $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](p)$ must equal the codimension.

Now suppose $\langle g_{ij}\rangle = I$ is not radical. Then consider the finitely generated radical of I, denoted by $\sqrt{I} = \langle p_k\rangle$. Since $I \subset \sqrt{I}$ we have $\langle g_{ij}\rangle \subset \langle p_k\rangle = \langle g_{ij}, p_k\rangle$. So the Jacobian of a set of generators which generates $\sqrt{I}$ is $\left[\frac{\partial\{g_{ij}, p_k\}}{\partial x_{ij}}\right](p)$ which must have rank of (at least) $rank\Big(\Big[\frac{\partial g_{ij}}{\partial x_{ij}}\Big](p)\Big) = codim(\mathcal{M}_{w_0} \cap Hess(X, h))$ (since we are only adding rows to the Jacobian). Thus Definition 3.1 tells us the point is smooth.■

Our strategy to prove $Hess(X, h) \cap \mathcal{M}_{w_0}$ is smooth at the origin will be to show that the rank of $\Big[\frac{\partial g_{ij}}{\partial x_{ij}}\Big](0)$ is equal to the number of generators g_{ij}. With Lemma 3.9 we don't need to be concerned whether $\langle g_{ij} \rangle = I$ is radical or not. The following example illustrates the non-emptiness criteria of Lemma 3.4.

Example 3.10. Recall the 6×6 matrix A from Example 2.14:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Here $J = \{1, 2, 4\}$ so Lemma 3.4 tells us $Hess(A, h) \cap \mathcal{M}_{w_0} \neq \emptyset$ if and only if $3 \leq h(2), 5 \leq h(4)$ and $6 \leq h(5)$.

Lemma 3.11. *For any n, the computed generators g_{ij} of $I(Hess(X, h) \cap \mathcal{M}_{w_0})$ are of the form*

$$g_{ij} = (\lambda_{w_0(j)} - \lambda_i)x_{ij} - 1_i x_{(i+1),j} + \gamma(j) 1_{w_0(j+1)} x_{i,(j+1)} \ + \textit{(terms of deg} \geq 2)$$

$$\textit{subject to } x_{ij} = \begin{cases} 1 & \textit{if } i + j = n + 1 \\ 0 & \textit{if } i + j > n + 1 \\ x_{ij} & \textit{if } i + j < n + 1 \end{cases} \textit{ and } \gamma(j) = \begin{cases} 0 & \textit{if } h(j) = j \\ 1 & \textit{if } h(j) > j \end{cases}.$$

Proof: By Theorem 2.19 we know the computed generators g_{ij} for $I(Hess(X,h) \cap \mathcal{M}_{w_0})$ are of the form

$g_{i,j} = XM_{(i,j)} - \sum_{k=1}^{h(j)} x_{i,k}(XM_{(w(k),j)})_0 +$ (terms of deg ≥ 2).

So to explicitly write our generators g_{ij} in a degree-separated form, we just need to know the form of $XM_{(i,j)}$ when $M \in \mathcal{M}_{w_0}$.

Recall from Example 2.15 that for $n = 4$, members of $X\mathcal{M}_{w_0}$ are of the form

$$XM_{w_0} = \begin{bmatrix} \lambda_1 x_{11} + 1_1 x_{21} & \lambda_1 x_{12} + 1_1 x_{22} & \lambda_1 x_{13} + 1_1 & \lambda_1 \\ \lambda_2 x_{21} + 1_2 x_{31} & \lambda_2 x_{22} + 1_2 & \lambda_2 & 0 \\ \lambda_3 x_{31} + 1_3 & \lambda_3 & 0 & 0 \\ \lambda_4 & 0 & 0 & 0 \end{bmatrix}$$

To generalize for any n, observe that the form of $XM_{(i,j)}$ is

$$XM_{(i,j)} = \lambda_i x_{ij} + 1_i x_{i+1,j} \tag{3.1}$$

subject to $\lambda_i = 0$ if $j > w_0(i)$ and $x_{ij} = \begin{cases} 1 & \text{if } i+j = n+1 \\ 0 & \text{if } i+j > n+1 \\ x_{ij} & \text{if } i+j < n+1 \end{cases}$.

We are interested in isolating the entries of XM_{w_0} which could have non-zero constant terms. Notice in the summation $\sum_{k=1}^{h(j)} x_{i,k}(XM_{(w(k),j)})_0$ we have the constant term of $XM_{(w(k),j)}$ multiplied by a variable. So if $XM_{(w(k),j)}$ is non-zero this will give a linear term of g_{ij}. Recall we are interested in linear terms since Observation 2.16 tells us only the linear terms of g_{ij} affect the rank of the Jacobian at the origin.

Notice each column of XM_{w_0} has (at most) only two entries whose constant term is potentially non-zero. In the case when $n = 4$ (cf. Example 2.15 above) column 1 of XM_{w_0} has its bottom two entries $\lambda_3 x_{31} + 1_3$ and λ_4 with potentially non-zero constants. In general, for fixed column j, the column entries of $XM_{(i,j)}$ with non-zero constants are entries $w_0(j)$ and $w_0(j+1)$.

So the two potentially linear summands of $\sum_{k=1}^{h(j)} x_{i,k}(XM_{(w(k),j)})_0$ are $x_{i,k}XM_{(w_0(j),j)}$ and $x_{i,k}XM_{(w_0(j+1),j)}$ for some k's $\in \{1, 2, ..., h(j)\}$. By inspection this requires the k's to be j and $j+1$. Hence the only two potentially linear summands of $\sum_{k=1}^{h(j)} x_{i,k}(XM_{(w(k),j)})_0$ are those corresponding to $k = j$ and $k = j+1$. The exception is when $h(j) = j$ in which case there is only one potentially non-zero summand corresponding to $k = j$.

To take care of this exceptional case define $\gamma(j) = \begin{cases} 0 & \text{if } h(j) = j \\ 1 & \text{if } h(j) > j \end{cases}$.

And now we can write Theorem 2.19

$g_{i,j} = XM_{(i,j)} - \sum_{k=1}^{h(j)} x_{i,k}(XM_{(w(k),j)})_0 +$ (terms of deg ≥ 2)

in the following way:

$$g_{i,j} = XM_{(i,j)} - x_{i,j}(XM_{(w_0(j),j)})_0 - \gamma(j)x_{i,j+1}(XM_{(w_0(j+1),j)})_0 + (\text{terms of deg} \geq 2).$$

Since $(XM_{(w_0(j),j)})_0 = \lambda_{w_0(j)}$ and $(XM_{(w_0(j+1),j)})_0 = 1_{w_0(j+1)}$ we get

$$g_{i,j} = XM_{(i,j)} - x_{i,j}\lambda_{w_0(j)} - \gamma(j)x_{i,j+1}1_{w_0(j+1)} + (\text{terms of deg} \geq 2).$$

Also by **Expression (3.1)**, $XM_{(i,j)}$ is of the form $\lambda_i x_{ij} + 1_i x_{i+1,j}$. So we get

$$g_{i,j} = \lambda_i x_{ij} + 1_i x_{i+1,j} - x_{i,j}\lambda_{w_0(j)} - \gamma(j)x_{i,j+1}1_{w_0(j+1)} + (\text{terms of deg} \geq 2).$$

Factoring and re-arranging gives the result

$$g_{ij} = (\lambda_{w_0(j)} - \lambda_i)x_{ij} - 1_i x_{(i+1),j} + \gamma(j)1_{w_0(j+1)}x_{i,(j+1)} + (\text{terms of deg} \geq 2)$$

subject to $x_{ij} = \begin{cases} 1 & \text{if } i+j = n+1 \\ 0 & \text{if } i+j > n+1 \\ x_{ij} & \text{if } i+j < n+1 \end{cases}$ and $\gamma(j) = \begin{cases} 0 & \text{if } h(j) = j \\ 1 & \text{if } h(j) > j \end{cases}$.

[Note for the sake of future convenience, in the last step we multiplied g_{ij} by -1. Since g_{ij} does not have any constant terms, this does not affect the vanishing set of $\langle g_{ij} \rangle$.]

■

Observation 3.12. *Suppose $g_{ij} \in I(Hess(X,h) \cap \mathcal{M}_{w_0})$. Then when writing g_{ij} as in Lemma 3.11, the term $(\lambda_{w_0(j)} - \lambda_i)x_{ij}$ does not simplify to 0 or $(\lambda_{w_0(j)} - \lambda_i)$.*

Proof: [This observation is saying the constraints on x_{ij} from Lemma 3.11 do not set it to 0 or 1.] If $g_{ij} \in I(Hess(X,h) \cap \mathcal{M}_{w_0})$ then by Observation 2.17 we must have $h(j) < w_0^{-1}(i)$. Since $w_0 = w_0^{-1}$ we have $h(j) < w_0(i)$. Since $j \leq h(j)$ by definition and $w_0(i) = n - i + 1$ we get $j \leq n - i + 1$ which implies $i + j < n + 1$. And so by the constraints of Lemma 3.2, we leave x_{ij} as x_{ij} and do not set it to 0 or 1. Thus $(\lambda_{w_0(j)} - \lambda_i)x_{ij}$ does not simplify to 0 or $(\lambda_{w_0(j)} - \lambda_i)$. ■

Observation 3.13. *If $Hess(X,h) \cap \mathcal{M}_{w_0} \neq \emptyset$, then the origin is in $Hess(X,h) \cap \mathcal{M}_{w_0}$.*

Proof: By Lemma 3.11 our computed generators g_{ij} do not have any constant terms. So provided $Hess(X,h) \cap \mathcal{M}_{w_0}$ is non-empty, every generator vanishes when evaluated at the origin (i.e. when each x_{ij} is set to 0). ■

Now that we have an explicit expression for any $g_{ij} \in I(Hess(X,h) \cap \mathcal{M}_{w_0})$ we can investigate its smoothness at the origin. Observation 3.3 will guide how we arrange the rows and columns of our Jacobian matrix $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right]$.

Definition 3.14. Order both g_{ij} and x_{ij} by the rules $(i_1, j_1) < (i_2, j_2)$ if $j_1 < j_2$. And if $j_1 = j_2$ then $(i_1, j_1) < (i_2, j_2)$ if $i_1 < i_2$.

Example 3.15. Suppose we have a set of ordered pairs:

$$\{(i,j)\} = \{((1,1),(1,2),(1,3),(2,3),(2,1),(2,2),(1,4),(3,1),(3,2)\}.$$

Then ordering these ordered pairs by Defintion 3.14, from least to greatest, gives: $\{(1,1),(2,1),(3,1),(1,2),(2,2),(3,2),(1,3),(2,3),(1,4)\}$.

Observation 3.16. *For each n there are $n(n-1)/2$ variables $\{x_{ij}\}$. This follows from the fact matrices in the matrix patch $\mathcal{M}_{w_0}$ only have x_{ij}'s above the anti-diagonal (see Example 1.15) of which there are $n(n-1)/2$ entries. Another way of stating this is that the (i,j) for which x_{ij} may be a variable of a generator are the (i,j) satisfying $j < w_0(i)$.*

Observation 3.17. *The algorithm of Chapter 2 produces at most $n(n-1)/2$ generators $\{g_{ij}\}$ of $I(Hess(X,h) \cap \mathcal{M}_w)$. From Observation 2.15, each j contributes $n-h(j)$ generators to $I(Hess(X,h) \cap \mathcal{M}_w)$. Since $j \leq h(j)$, the greatest number of generators each j contributes is $n-j$. And since $1 \leq j \leq n$ the greatest number of possible total generators is $(n-1)+(n-2)+...+(n-(n-1))+(n-n) = n(n-1)/2$. We know by Observation 3.7 the exact number of relations g_{ij} produced by the algorithm of Chapter 2 is $\sum_{j=1}^{n} n-h(j)$. This of course is less than or equal to $n(n-1)/2$.*

Definition 3.18. We say j contributes the *maximal number of generators* to $\{g_{ij}\}$

when it contributes $n-j$ generators. This happens exactly when $h(j)=j$. We say $I(Hess(X,h)\cap\mathcal{M}_w)$ has the *maximal number of generators* when each $j=1,...,n$ contributes maximally, and so there are $n(n-1)/2$ many total $\{g_{ij}\}$. This happens when $h(j)=j$ for all j.

If (i,j) is a pair for which we compute a corresponding g_{ij} by the algorithm of Chapter 2, then by Observation 2.14 this happens if and only if $h(j)<w^{-1}(i)$. Setting $w=w_0=w_0^{-1}$ the condition becomes $h(j)<w_0(i)$.

Definition 3.19. Define the set $S=\{(i,j)|h(j)<w_0(i)\}$. By the above paragraph this is the set of pairs (i,j) such that a corresponding g_{ij} is computed as a generator for $I(Hess(X,h)\cap\mathcal{M}_{w_0})$ by the algorithm of Chapter 2.

By Observation 3.16, the pairs (i,j) for which there is a corresponding variable x_{ij} are only those pairs satisfying $j<w_0(i)$.

Definition 3.20. Define the set $R=\{(i,j)|j<w_0(i)\}$. By the above paragraph this is the set of pairs (i,j) such that a corresponding x_{ij} is a variable.

Observation 3.21.

1. *By Observation 3.17 we have* $|S|=\sum_{j=1}^{n}n-h(j)$.
2. *By Observation 3.16, we have* $|R|=n(n-1)/2$.
3. $S\subseteq R$. *Note the defining condition of* R *is* $j<w_0(i)$ *and the defining condition of* S *is* $h(j)<w_0(i)$. *Since* $j\leq h(j)$ *by definition, it follows that any pair* (i,j) *in* S *is also in* R.

Definition 3.22. Write the Jacobian of $I(Hess(X,h)\cap\mathcal{M}_{w_0})=\langle g_{ij}\rangle$ evaluated at the origin (i.e. each x_{ij} set to 0) as $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$. Label the columns with a $\frac{\partial}{\partial x_{ij}}$ and the rows with a g_{ij}. Index the columns $\frac{\partial}{\partial x_{ij}}$ with elements of R, ordered by the rules of Definition 3.14. Index the rows g_{ij} with elements of S, ordered by the rules of

Definition 3.14.

The dimensions of the matrix $\left[\dfrac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ are $\sum_{j=1}^{n} n-h(j))\times(n(n-1)/2) = |S|\times|R|$. Note $\left[\dfrac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ will be a square $(n(n-1)/2)\times(n(n-1)/2)$ matrix exactly when $|S| = |R|$, i.e. when we've computed the maximal number of generators.

Example 3.23. Recall the ordered set of ordered pairs from Example 3.15:

$$\{(1,1),(2,1),(3,1),(1,2),(2,2),(3,2),(1,3),(2,3),(1,4)\}.$$

Suppose these are (i,j) corresponding to a set $\{g_{ij}\}$ computed by the algorithm of Chapter 2, where $n = 5$ and $h(12345) = 22345$. Then the Jacobian $\left[\dfrac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ is a 9×10 matrix of the form

$$\begin{array}{c|cccccccccc}
 & \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{21}} & \frac{\partial}{\partial x_{31}} & \frac{\partial}{\partial x_{41}} & \frac{\partial}{\partial x_{12}} & \frac{\partial}{\partial x_{22}} & \frac{\partial}{\partial x_{32}} & \frac{\partial}{\partial x_{13}} & \frac{\partial}{\partial x_{23}} & \frac{\partial}{\partial x_{14}} \\
g_{11} & \boxed{*} & * & * & * & * & * & * & * & * & * \\
g_{21} & 0 & \boxed{*} & * & * & * & * & * & * & * & * \\
g_{31} & 0 & 0 & \boxed{*} & * & * & * & * & * & * & * \\
g_{12} & 0 & 0 & 0 & 0 & \boxed{*} & * & * & * & * & * \\
g_{22} & 0 & 0 & 0 & 0 & 0 & \boxed{*} & * & * & * & * \\
g_{32} & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{*} & * & * & * \\
g_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{*} & * & * \\
g_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{*} & * \\
g_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{*}
\end{array}.$$

We have drawn boxes around the entries $\dfrac{\partial g_{ij}}{\partial x_{ij}}$ where the (i,j) of g_{ij} and x_{ij} are equal. These entries are important for the computation of the rank of $\left[\dfrac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ and so we'll give them a special name.

Definition 3.24. Define the *cross-entries* of $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ to be the entries for which the (i, j) of g_{ij} and x_{ij} are equal.

Lemma 3.25. *All entries in the Jacobian matrix* $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ *that are below, to the left, and to the south-west of a cross-entry is 0.*

Proof: By Lemma 3.11 we know that the variables that can show up in linear monomials of g_{ij} are: $x_{ij}, x_{(i+1),j}$, and $x_{i,(1+j)}$. By the ordering of Definition 3.14 x_{ij} is the first of these variables. This implies everything to the left of a cross-entry is 0. Also by Lemma 3.11, g_{ij} is the last (again by the ordering in Definition 3.14) of the generators in which x_{ij} can appear - implying everything below a cross-entry is 0. Similarly, everything to the left *and* below (or south-west) of a cross-entry is 0.

Observation 3.26.

1. *Every cross-entry of the Jacobian matrix* $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ *has the value* $\frac{\partial g_{ij}}{\partial x_{ij}}(0) = (\lambda_{w_0(j)} - \lambda_i)$. *This follows from taking the derivative of* g_{ij} *with respect to* x_{ij} *and evaluating at the origin using the expression for* g_{ij} *from Lemma 3.11.*

2. *If* $i + j < n$ *then the entry directly to the right of a cross-entry is* $\frac{\partial g_{ij}}{\partial x_{(i+1),j}}(0) = -1_i$. *This follows from taking the derivative of* g_{ij} *with respect to* $x_{(i+1),j}$ *and evaluating at the origin using the expression for* g_{ij} *from Lemma 3.11.*

Example 3.27. Suppose $n = 4$ and we compute $\{g_{ij}\} = \{g_{31}, g_{21}, g_{11}, g_{22}, g_{12}, g_{13}\}$ (so we have the maximal number of generators in this case). Ordering the set by the rules of Definition 3.5 gives: $g_{11}, g_{12}, g_{13}, g_{21}, g_{22}, g_{31}$. Then the Jacobian $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ has the form

$$
\begin{array}{ccccccc}
 & \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{21}} & \frac{\partial}{\partial x_{31}} & \frac{\partial}{\partial x_{12}} & \frac{\partial}{\partial x_{22}} & \frac{\partial}{\partial x_{13}} \\
g_{11} & \lambda_4 - \lambda_1 & -1_1 & * & * & * & * \\
g_{21} & 0 & \lambda_4 - \lambda_2 & {}_{-}1_2 & * & * & * \\
g_{31} & 0 & 0 & \lambda_4 - \lambda_3 & * & * & * \\
g_{12} & 0 & 0 & 0 & \lambda_3 - \lambda_1 & -1_1 & * \\
g_{22} & 0 & 0 & 0 & 0 & \lambda_3 - \lambda_2 & * \\
g_{31} & 0 & 0 & 0 & 0 & 0 & \lambda_2 - \lambda_1
\end{array}.
$$

Notice we can draw these blocks along the diagonal:

$$
\begin{array}{c|ccc|cc|c|}
 & \frac{\partial}{\partial x_{11}} & \frac{\partial}{\partial x_{21}} & \frac{\partial}{\partial x_{31}} & \frac{\partial}{\partial x_{12}} & \frac{\partial}{\partial x_{22}} & \frac{\partial}{\partial x_{13}} \\
\cline{2-4}
g_{11} & \lambda_4 - \lambda_1 & -1_1 & * & * & * & * \\
g_{21} & 0 & \lambda_4 - \lambda_2 & {}_{-}1_2 & * & * & * \\
g_{31} & 0 & 0 & \lambda_4 - \lambda_3 & * & * & * \\
\cline{2-6}
g_{12} & 0 & 0 & 0 & \lambda_3 - \lambda_1 & -1_1 & * \\
g_{22} & 0 & 0 & 0 & 0 & \lambda_3 - \lambda_2 & * \\
\cline{5-7}
g_{31} & 0 & 0 & 0 & 0 & 0 & \lambda_2 - \lambda_1 \\
\cline{7-7}
\end{array}.
$$

Motivated by the blocks drawn along the diagonal of Example 3.27, we define the following:

Definition 3.28. Fix a $j_0 \in \{1, ..., n-1\}$. Define the following matrix:

$$\Psi_{j_0} = \begin{bmatrix} \lambda_{w_0(j_0)} - \lambda_1 & -1_1 & 0 & 0 & 0 \\ 0 & \lambda_{w_0(j_0)} - \lambda_2 & -1_2 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & -1_{w_0(j_0)-2} \\ 0 & 0 & 0 & 0 & \lambda_{w_0(j_0)} - \lambda_{w_0(j_0)-1} \end{bmatrix}.$$

This is a square $(n-j_0)\times(n-j_0)$ matrix whose entries are $\frac{\partial g_{ij}}{\partial x_{ij}}(0)$, where j is our fixed j_0 and i are such that $j_0 < w_0(i)$. We keep the ordering of the rows and columns from $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$.

Since j_0 does not necessarily contribute the maximal number of generators (cf. Definition 3.18), not all the rows of Ψ_{j_0} are necessarily in $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$. We only want to consider the first $(n-h(j_0))$ rows of Ψ_{j_0}, leading to the following definition:

Definition 3.29. Define Λ_{j_0} as the matrix consisting to the first (or top) $(n-h(j_0))$ rows of Ψ_{j_0}. This is a $(n-h(j_0))\times(n-j_0)$ submatrix of $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$.

Observation 3.30. *Now we can write*

$$\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0) = \begin{bmatrix} \begin{pmatrix}\Lambda_1\end{pmatrix} & * & * & * & * \\ 0 & \begin{pmatrix}\Lambda_2\end{pmatrix} & * & * & * \\ 0 & 0 & \ddots & \ddots & * \\ 0 & 0 & 0 & \begin{pmatrix}\Lambda_{n-2}\end{pmatrix} & * \\ 0 & 0 & 0 & 0 & \begin{pmatrix}\Lambda_{n-1}\end{pmatrix} \end{bmatrix}.$$

Theorem 3.31. *The Jacobian matrix* $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ *has rank equal to the number of* *generators g_{ij}. Thus $Hess(X,h) \cap \mathcal{M}_{w_0}$ is smooth at the origin.*

Proof: The Jacobian matrix $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ has the same number of rows as number of generators g_{ij} computed - which is $\sum_{j=1}^{n} n - h(j)$ by Observation 3.17. We'll say a row has a *leading 1* if there is a non-zero entry which has all entries below it, to its left, and to its south-west equal to 0.

If we can show each row of $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ has a leading 1, then $\left[\frac{\partial g_{ij}}{\partial x_{ij}}\right](0)$ has rank equal to the number of generators g_{ij}, and so by Lemma 3.8 has rank equal to $codim(Hess(X,h) \cap \mathcal{M}_{w_0})$. Thus by Definition 3.1 $Hess(X,h) \cap \mathcal{M}_{w_0}$ is smooth at the origin.

We prove this by looking at two cases:

Case 1: If Λ_j has the maximal number of rows/generators, then Λ_j has a leading 1 for every row.

Proof: When Λ_j has the maximal number of generators (which is $n-j$ many) we must have $h(j) = j$. This implies Λ_j is a square matrix equal to Ψ_j and that the diagonal consists of cross-entries. We want to show the bottom right entry $\frac{\partial g_{(w_0(j)-1,j)}}{\partial x_{(w_0(j)-1,j)}}(0)$ is non-zero and is thus a leading 1 of Λ_j.

Look at the bottom right entry $\frac{\partial g_{(w_0(j)-1,j)}}{\partial x_{(w_0(j)-1,j)}}(0)$. Suppose for contradiction the bottom right entry is zero. Then by Observation 3.26.1 we have $\frac{\partial g_{(w_0(j)-1,j)}}{\partial x_{(w_0(j)-1,j)}}(0) = \lambda_{w_0(j)} - \lambda_{w_0(j)-1} = 0$ which implies $\lambda_{w_0(j)} = \lambda_{w_0(j)-1}$. This means $\lambda_{w_0(j)}$ and $\lambda_{w_0(j)-1}$ are in the same Jordan block of X and so $1_{w_0(j)-1} = 1$ and $w_0(j) - 1 \in J$.

We are assuming $Hess(X,h) \cap \mathcal{M}_{w_0}$ is non-empty, so by Lemma 3.4 since $w_0(j)-1 \in J$

we have $n - (w_0(j) - 1) + 1 \leq h(n - (w_0(j) - 1))$. Since w_0 has the closed form $w_0(j) = n - j + 1$ we can rewrite this as $n - (n - j + 1 - 1) + 1 \leq h(n - n + j)$ or $j + 1 \leq h(j)$. But this contradicts $h(j) = j$.

Thus the bottom right entry $\lambda_{w_0(j)} - \lambda_{w_0(j)-1}$ must be non-zero. This implies that every diagonal entry of Λ_j is non-zero since if $\lambda_{w_0(j)} \neq \lambda_{w_0(j)-1}$ then $\lambda_{w_0(j)} \neq \lambda_i$ for $i = 1, ..., w_0(j) - 1$.

This shows Λ_j has a leading 1 for every row in the case when Λ_j has the maximal number of generators.

Case 2: If Λ_j does not have the maximal number of rows, then the number of rows equals the number of leading 1s.

Proof: Now suppose Λ_j only has $n - h(j)$ generators where $h(j) > j$.

Look at the cross-entry of the *ith* row, where i is an integer $1 \leq i \leq n - h(j)$. By Observation 3.26.1 this cross-entry $\dfrac{\partial g_{ij}}{\partial x_{ij}}(0)$ has value $\lambda_{w_0(j)} - \lambda_i$. By Lemma 3.25 everything below it, to its left, and south-west of it is 0.

If $\lambda_{w_0(j)} \neq \lambda_i$, then the cross-entry $\dfrac{\partial g_{ij}}{\partial x_{ij}}(0) = \lambda_{w_0(j)} - \lambda_i$ is non-zero and is thus our leading 1 for the *ith* row.

Otherwise if the cross-entry $\dfrac{\partial g_{ij}}{\partial x_{ij}}(0) = \lambda_{w_0(j)} - \lambda_i = 0$ this implies $\lambda_{w_0(j)} = \lambda_i$ and so the entry to the right -1_i must equal -1 since $\lambda_{w_0(j)}$ and λ_i are in the same Jordan block.

Now we claim this -1_i is the leading 1 for the *ith* row. This is because the only potentially non-zero entry that is either below it or to its south west is the entry that is directly below it - which is $\dfrac{\partial g_{(i+1),j}}{\partial x_{(i+1),j}}(0) = \lambda_{w_0(j)} - \lambda_{i+1}$.

Suppose for contradiction that the entry $\dfrac{\partial g_{(i+1),j}}{\partial x_{(i+1),j}}(0) = \lambda_{w_0(j)} - \lambda_{i+1}$ is non-zero. Then $\lambda_{w_0(j)} \neq \lambda_{i+1}$.

Now observe that $i < i+1 < w_0(j)$. This is because we're assuming $j < h(j)$ which implies $j + n - h(j) < n$ and since $i \leq n - h(j)$ we get $j + i < n$ or $i < n - j$. Adding 1 to both sides of this inequality gives $i + 1 < n - j + 1$. But $n - j + 1 = w_0(j)$ yielding $i + 1 < w_0(j)$ as needed.

We're assuming the cross-entry is zero so $\lambda_{w_0(j)} = \lambda_i$ which means $\lambda_{w_0(j)}$ and λ_i are in the same Jordan block. The inequality $i < i+1 < w_0(j)$ requires that λ_{i+1} also belongs to the same Jordan block as $\lambda_{w_0(j)}$ and λ_i. And so $\lambda_{w_0(j)} = \lambda_{i+1}$, yielding a contradiction to the assumption that the element below -1_i can be non-zero.

What we have shown is that if the cross-entry of the ith row fails to be the leading 1, then -1_i is the leading 1 for the ith row. Since this argument holds for all i we have proven the claim of Case 2 - that Λ_j has as many leading 1s as rows when Λ_j does not have the maximal number of generators.

To recap, we have shown that in all cases each row of $\Big[\dfrac{\partial g_{ij}}{\partial x_{ij}}\Big](0)$ has a leading 1 and thus makes a contribution of 1 to the rank of $\Big[\dfrac{\partial g_{ij}}{\partial x_{ij}}\Big](0)$. Since the number of rows of $\Big[\dfrac{\partial g_{ij}}{\partial x_{ij}}\Big](0)$ equals the number of generators g_{ij}, by Lemma 3.8 it follows that $rank\Big(\Big[\dfrac{\partial g_{ij}}{\partial x_{ij}}\Big](0)\Big) = codim(Hess(X,h) \cap \mathcal{M}_{w_0})$ which is the criteria given by Definition 3.1 for the origin to be smooth in $Hess(X,h) \cap \mathcal{M}_{w_0}$. ■

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