CONTINUOUS MODEL THEORY AND FINITE-REPRESENTABILITY BETWEEN BANACH SPACES

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Abstract

In this thesis, we consider the problem of capturing finite-representability between Banach spaces using the tools of continuous model theory. We introduce predicates and additional sorts to capture finite-representability and show that these can be used to expand the language of Banach spaces. We then show that the class of infinite-dimensional Banach spaces expanded with this additional structure forms an elementary class $\mathcal{K}_G$, and conclude that the theory T_G of K_G is interpretable in T^{eq}, where T is the theory of infinite-dimensional Banach spaces. Finally, we show that existential equivalence in a reduct of the language implies finite-representability. Relevant background on continuous model theory and Banach space theory is provided.

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Chapter 1

Introduction

Continuous model theory, in contrast to classical model theory, is fundamentally interested in structures with an underlying metric. Whereas classical model theory is rather rigid in the sense that truth values are binary, formulas in continuous model theory take values in a bounded interval of $\mathbb{R}$, allowing for a notion of approximate satisfiability. Although it lives in a more general setting, all standard tools from classical model theory, for example the Compactness Theorem or the Lowenheim-Skolem theorems, have a continuous counterpart. These, along with a richer language, provide a setting to capture "metric" ideas, and continuous logic has found many modern applications, including in Banach space theory (for example [17]) and to operator algebras (for example [7]).

Within Banach space theory lies the study of the local properties of Banach spaces. These are the finite-dimensional, linear properties of a Banach space. The study of these properties dates back to the 1960s (see [13]) and sees many techniques employed that would be recognizable to model-theorists (see for example [10] or [19]). Considerable work was done in the 1970s and 1980s to apply logic to Banach space theory in the context of positive bounded logic (see for example [9]), but modern developments in continuous logic offer the possibility that new aspects of the theory can be effectively captured.

Closely related to these local properties is the notion of finite-representability. A Banach space $(X, \|\cdot\|_X)$ being finitely-representable in some other Banach space $(Y, \|\cdot\|_Y)$ means that an approximation of any finite-dimensional subspace of X can be found in Y. More concretely, it means that the two Banach spaces have essentially the same local properties. In the late 1960s, Ribe (see [18]) showed that local properties are essentially metric properties, and so continuous model theory seems eminently suitable to capture finite-representability.

This thesis aims to apply continuous model theory to the problem of capturing the notion of finite-representability between Banach spaces. Relevant ideas and concepts from continuous model theory are introduced in Chapter 2, and the concepts of finite-representability and local properties of Banach spaces are explained in Chapter 3. Finally, in Chapter 4, a language suitable for capturing finite-representability is introduced and proved to be captured in a reduct of the usual language of Banach spaces.

Chapter 2

The Basics of Continuous Model Theory

We first briefly outline the basics of continuous model theory. For a more detailed introduction, see [4] or [8], on which this description is based.

2.1 Languages, Formulas, and Theories

Definition 2.1.1. *A signature $\mathcal{L}$ is a triple $(\mathcal{S}, \mathcal{R}, \mathcal{F})$, where:*

1. *$\mathcal{S}$ denotes a family of sorts $(S_i : i \in \mathcal{I})$. To each sort we associate a relation symbol $d_S : S \times S \to [0, K_S]$, $K_S \in (0, \infty)$, and a modulus of uniform continuity taken to be the identity function.*

2. *$\mathcal{R}$ denotes a family of relation symbols $(R_j : j \in \mathcal{J})$. Each relation symbol has an associated domain $\prod_{i=1}^{n} S_i$, a range consisting of closed and bounded interval $B_R \subset \mathbb{R}$, and a modulus of uniform continuity δ_{R_j}.*

3. *$\mathcal{F}$ denotes a family of function symbols $(F_k : k \in \mathcal{K})$. To each function symbol we associate a domain $\prod_{i=1}^{n} S_i$ of sorts in $\mathcal{S}$, a range $S \in \mathcal{S}$, and a modulus of uniform continuity δ_{F_k}.*

By a modulus of uniform continuity above, we are referring specifically to some continuous function $\delta : [0,1] \to [0,1]$.

A structure that interprets a signature in continuous logic is called a metric structure.

Definition 2.1.2. *Let $\mathcal{L}$ be a signature as above. A metric structure $\mathcal{M}$ is a triple $(\mathcal{S}^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}}, \mathcal{F}^{\mathcal{M}})$ interpreting the signature, such that:*

1. *$\mathcal{S}^{\mathcal{M}}$ is a collection of bounded, complete metric spaces $(X_S, d_S)_{S\in\mathcal{S}}$ where d_S is taken to be the metric on X_S, under which X_S is bounded.*

2. *$\mathcal{R}^{\mathcal{M}}$ is a collection $(R^{\mathcal{M}})_{R\in\mathcal{R}}$ of uniformly continuous functions. Each $R^{\mathcal{M}} \in \mathcal{R}^{\mathcal{M}}$ has domain $\prod_{i=1}^{n} X_{S_i}$ and range B_R, where $\prod_{i=1}^{n} S_i$ and B_R are the domain and range, respectively, given to R by the signature $\mathcal{L}$, and δ_R is the modulus of continuity for $R^{\mathcal{M}}$.*

3. *$\mathcal{F}^{\mathcal{M}}$ is a collection $(F^{\mathcal{M}})_{F\in\mathcal{F}}$ of uniformly continuous functions. Each $F^{\mathcal{M}} \in \mathcal{F}^{\mathcal{M}}$ has domain $\prod_{i=1}^{n} X_{S_i}$ and range X_S, where $\prod_{i=1}^{n} S_i$ and S are the domain and range, respectively, given to F by $\mathcal{L}$, and δ_F is the modulus of continuity for $F^{\mathcal{M}}$.*

So by saying that, for example, δ_R is a modulus of uniform continuity for R, we are saying that if $d(x,y) < \delta_R(\epsilon)$, then $d(R^{\mathcal{M}}(x), R^{\mathcal{M}}(y)) < \epsilon$. Note that each function and relation in a metric structure is uniformly continuous. To be precise, we also specify that in the case where R takes inputs from multiple sorts, $d(x,y) = \max\{d_i(x_i, y_i)\}$ where i ranges over the sorts S_i appearing in x and y. The same is true for functions symbols.

It is an important fact, proven in [4], that if we have two functions $f : M \to M'$ and $g : M' \to M''$ between metric spaces, each of which has a modulus of continuity, then the composition of the moduli of continuity gives a modulus of continuity for the composition $g \circ f$. This becomes essential when we begin defining formulas, since we want the formulas we produce to also be uniformly continuous and to have a modulus of continuity.

Example 2.1.3. Let $(X, \|\cdot\|_X)$ be a Banach space. We can consider X as a metric structure in the signature of Banach spaces,

$$\begin{aligned}\mathcal{L} =& \{(B_n : n \in \mathbb{Z}_+), d_n(x,y), 0, (\mathfrak{i}_{mn} : m \leq n), (\lambda_n : n \in \mathbb{N}, \lambda \in \mathbb{C}),\\ &(+_{mn} : m \leq n), (-_n : n \in \mathbb{N})\}\end{aligned}$$

We consider X as a metric structure in this signature as follows:

- B_n is interpreted as the ball of radius n centred at the origin,

  $$B_n = \{x \in X : \|x\|_X \leq n\}$$

  These are the sorts of our structure.

- $d_n(x, y)$ is interpreted as $\|x - y\|_X$ on B_n. Note that the norm on each sort is not explicitly included in the language, but it is of course definable for each sort, i.e. if $x \in B_n$, then $\|x\|_X = d_n(x, 0)$. Due to the way we defined our sorts, each of these relations is necessarily bounded.

- 0 is defined to be the origin and additive identity in B_1.

- $\mathfrak{i}_{mn}$ is the inclusion map between B_m and B_n.

- λ_n is interpreted as the unary function of scalar multiplication by λ on B_n. The range of λ_n is B_k, where k is the least integer such that $|\lambda| \cdot n \leq k$.

- $+_{mn}$ is a binary function interpreted as addition on $X \times X$,

  $$+_{mn} : B_m \times B_n \to B_{m+n}$$

- $-_n$ is a unary function interpreted as taking the additive inverse on X,

  $$-_n : B_n \to B_n$$

A signature $\mathcal{L}$ represents the non-logical symbols of a language, with the additional logical symbols being:

1. An infinite collection of variables for each sort.
2. A symbol u for each continuous function $u : \mathbb{R}^n \to \mathbb{R}$.
3. For each sorted variable x, the quantifiers $\sup_x$ and $\inf_x$.

Just as in the discrete case, terms and formulas are built inductively from the language.

Definition 2.1.4. *Let L be a metric language. We first define $\mathcal{L}$-terms:*

1. *Variables are terms with a modulus of uniform continuity being the identity function.*
2. *If F is an n-ary function symbol from $\mathcal{L}$ and $(t_i)_{1\leq i\leq n}$ are terms with ranges matching the domain of F, then $F(t_1, ..., t_n)$ is an $\mathcal{L}$-term. The modulus of uniform continuity for it is $\delta_F(\min \delta_{t_i})$.*

The atomic $\mathcal{L}$-formulas are defined as:

1. *If R is an n-ary relation from $\mathcal{L}$ and $(t_i)_{1\leq i\leq n}$ are $\mathcal{L}$-terms with range matching the domain of R, then $R(t_1, ..., t_n)$ is an atomic formula. A modulus of uniform continuity for it is $\delta_R(\min \delta_{t_i})$.*

Finally, $\mathcal{L}$-formulas are defined as:

1. *Any atomic $\mathcal{L}$-formula is an $\mathcal{L}$-formula.*
2. *If $u : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\phi_{1\leq i\leq n}$ are $\mathcal{L}$-formulas with moduli of continuity δ_{ϕ_i}, then $u(\phi_1, ..., \phi_n)$ is an $\mathcal{L}$-formula. As ran(ϕ_i) is on a compact set, take the union of each and u is uniformly continuous on this with some modulus of continuity δ_u. The modulus of uniform continuity for the $\mathcal{L}$-formula can then be taken to be $\delta_u(\min \delta_{\phi_i})$.*

3. *If ϕ is an $\mathcal{L}$-formula and x is a sorted variable, then both $\inf_x \phi$ and $\sup_x \phi$ are $\mathcal{L}$-formulas. See [4] for a proof that if δ is a modulus of uniform continuity for ϕ, then it is also a modulus of uniform continuity for both $\inf_x \phi$ and $\sup_x \phi$.*

In all of the above cases, we determined the modulus of continuity of an $\mathcal{L}$-term or $\mathcal{L}$-formula by composition of the continuity moduli of the constituent parts.

Given any $\mathcal{L}$-formula $\phi(x)$ and an appropriately sorted tuple a from an $\mathcal{L}$-structure $\mathcal{M}$, we denote the interpretation of $\phi(x)$ at $a \in \mathcal{M}$ to be $\phi^{\mathcal{M}}(a)$, and this interpretation takes the value inductively found using the construction of $\phi(x)$.

Before continuing, we mention that it is often very convenient to assume that all formulas take values on the interval $[0, 1]$, and in many presentations this is assumed in the definitions. As all formula and relation symbols in a language $\mathcal{L}$ take values on some closed and bounded interval, this amounts to only a rescaling, and instead of taking all continuous functions $u : \mathbb{R}^n \to \mathbb{R}$, we would restrict to continuous functions $u : [0, 1]^n \to [0, 1]$ when building formulas. Making this assumption does not change any of what follows, and at times we too will assume it for convenience.

With these definitions we can recapture the notions of elementary equivalence and elementary substructures.

Definition 2.1.5. *Fix a language $\mathcal{L}$ and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures.*

1. *If dom($\mathcal{M}$) $\subseteq$ dom($\mathcal{N}$), meaning for each sort $S \in \mathcal{L}$, $S^{\mathcal{M}} \subseteq S^{\mathcal{N}}$, and if for every function $f \in \mathcal{L}$ and relation $R \in \mathcal{L}$ it is the case that,*

$$f^{\mathcal{M}} = f^{\mathcal{N}} \mid_{\mathcal{M}} \text{ and } R^{\mathcal{M}} = R^{\mathcal{N}} \mid_{\mathcal{M}}$$

   then we call $\mathcal{M}$ a substructure of $\mathcal{N}$ and denote this by writing $\mathcal{M} \subseteq \mathcal{N}$.

2. *$\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent if $\phi^{\mathcal{M}} = \phi^{\mathcal{N}}$ for any $\mathcal{L}$-sentence ϕ. We denote this by $\mathcal{M} \equiv \mathcal{N}$.*

3. *If $\mathcal{M} \subseteq \mathcal{N}$ and $\phi^{\mathcal{M}}(a) = \phi^{\mathcal{N}}(a)$ for every $\mathcal{L}$-formula $\phi(x)$ and every $a \in$ dom($\mathcal{M}$), then we call $\mathcal{M}$ an elementary substructure of $\mathcal{N}$ and write $\mathcal{M} \preceq \mathcal{N}$.*

If we fix a language $\mathcal{L}$, we use $\mathfrak{F}_{\mathcal{L}}^{\bar{x}}$ to denote the set of all $\mathcal{L}$-formulas in free variables $\bar{x}$. Note that $\mathfrak{F}_{\mathcal{L}}^{\emptyset}$ denotes the collection of $\mathcal{L}$-sentences. Using this, we define $\mathfrak{F}_{\mathcal{L}} = \cup_{\bar{x}} \mathfrak{F}_{\mathcal{L}}^{\bar{x}}$, the set of all $\mathcal{L}$-formulas.

Definition 2.1.6. *Fix a language $\mathcal{L}$. Given a metric structure $\mathcal{M}$, the theory of $\mathcal{M}$, denoted $Th(\mathcal{M})$, is the function,*

$$Th(\mathcal{M}) : \mathfrak{F}_{\mathcal{L}}^{\emptyset} \to \mathbb{R}$$

defined as, $Th(\mathcal{M})(\phi) = \phi^{\mathcal{M}}$. More generally, an $\mathcal{L}$-theory T is a partial function from $\mathfrak{F}_{\mathcal{L}}^{\emptyset}$ to $\mathbb{R}$, where for some $\mathcal{L}$-structure $\mathcal{M}$, $Th(\mathcal{M})|_{dom(T)} = T$. In this case, we say that $\mathcal{M}$ satisfies T, writing $\mathcal{M} \vDash T$. A theory T is complete if $dom(T) = \mathfrak{F}_{\mathcal{L}}^{\emptyset}$.

Consider a theory $T : \mathfrak{F}_{\mathcal{L}}^{\emptyset} \to \mathbb{R}$. We can naturally extend T to a partial linear functional on $\mathfrak{F}_{\mathcal{L}}^{\emptyset}$, and in this sense it is completely determined by its kernel. For an $\mathcal{L}$-sentence ϕ, if $T(\phi) = r$, then, since $u(x) = x - r$ is continuous, $\psi = \phi - r \in \mathfrak{F}_{\mathcal{L}}^{\emptyset}$ as well and $T(\psi) = 0$. For convenience, we often use the shorthand $\phi \in T$ to mean that $T(\phi) = 0$. Thought of this way, satisfiability in the continuous case is almost identical to the discrete one, with the exception being that we consider "truth" to be a value of 0.

Many of the key results from discrete model theory carry over to the continuous setting, though often essential to this translation is the notion of an ultraproduct of metric structures. Recall that an ultrafilter $\mathfrak{U}$ on a set I is a maximal filter on I. We say that $\mathfrak{U}$ is a principal ultrafilter if there exists some $i_0 \in I$ such that,

$$\mathfrak{U} = \{A \in \mathcal{P}(I) : i_0 \in A\}$$

Otherwise we call $\mathcal{U}$ non-principal.

Definition 2.1.7. *Let $\mathcal{U}$ be an ultrafilter on a set $\mathcal{I}$ and let $(r_i : i \in I)$ be an $\mathcal{I}$-indexed sequence of real numbers. We say the ultralimit of the sequence is $L \in \mathbb{R}$, and write*

$$L = \lim_{i \to \mathfrak{U}} r_i$$

if for every $\epsilon > 0$,

$$\{i \in \mathcal{I} : |L - r_i| < \epsilon\} \in \mathfrak{U}$$

It is not too difficult to see that for any bounded sequence of real numbers, an ultralimit exists and is in fact unique up to a change in choice of our ultrafilter on the set I.

Definition 2.1.8. *Let* $(M_i, d_i)_{i\in I}$ *be an indexed collection of bounded metric spaces with uniform bound* $B \in \mathbb{R}$ *and let* $\mathfrak{U}$ *be an ultrafilter on* I. *Consider the product* $\prod_{i\in I} \mathcal{M}_i$ *and the pseudo-metric on it given by,*

$$d(\overline{x}, \overline{y}) = \lim_{i\to\mathfrak{U}} d_i(x_i, y_i)$$

We define the ultraproduct of $(M_i, d_i)_{i\in I}$ *to be the quotient of* $\prod_{i\in I} \mathcal{M}_i$ *under the equivalence relation* $\overline{x} \sim \overline{y}$ *if* $d(\overline{x}, \overline{y}) = 0$.

Intuitively, two elements are in the same equivalence class if they are arbitrarily close on sets in the ultrafilter (we will often colloquially refer to these sets as being ultrafilter-large). Importantly, what happens on sets outside of the ultrafilter does not affect ultralimits and it does not affect the equivalence class to which an element belongs. In the above definition it was important that we had a uniform bound on the metric spaces so that we have a well-defined metric on the ultraproduct. Without this uniform bound, the ultralimit need not exist.

Now consider a setting where we have a family $(\mathcal{M}_i)_{i\in\mathcal{I}}$ of metric structures, each interpreting a language $\mathcal{L}$, and an ultrafilter $\mathfrak{U}$ on $\mathcal{I}$. We want a sensible way to take an ultraproduct of these structures that will again give us an $\mathcal{L}$-structure.

Denote the ultraproduct by,

$$\mathcal{M} = \prod_{\mathfrak{U}} \mathcal{M}_i$$

Take the sorts of $\mathcal{M}$ to be the ultraproduct of the sorts of the $\mathcal{M}_i$,

$$X_S = \prod_{\mathfrak{U}} X_{S_i}$$

This is defined as in the above definition of the ultraproduct of a collection of uniformly bounded metric spaces. We see now why we had to be so strict about having a bound associated to the sorts in our language. Without such a uniform bound, the ultraproduct of the sorts is not necessarily well-defined.

For an n-ary function symbol F, define $F^{\mathcal{M}}$ by,

$$F^{\mathcal{M}}\Big((x_{1,i})_{i\in\mathcal{I}}/_{\mathfrak{U}}, ..., (x_{n,i})_{i\in\mathcal{I}}/_{\mathfrak{U}}\Big) = F^{\mathcal{M}_i}(x_{1,i}..., x_{n,i})_{i\in\mathcal{I}}/_{\mathfrak{U}}$$

and for an n-ary relation symbol R, denote $R^{\mathcal{M}}$ by,

$$R^{\mathcal{M}}\Big((x_{1,i})_{i\in\mathcal{I}}/_{\mathfrak{U}}, ..., (x_{n,i})_{i\in\mathcal{I}}/_{\mathfrak{U}}\Big) = \lim_{i\to\mathfrak{U}} R^{\mathcal{M}_i}(x_{1,i}, ..., x_{n,i})$$

In the case where every $\mathcal{M}_i$ is the same structure $\mathcal{M}$, we denote the ultraproduct by $\mathcal{M}^{\mathfrak{U}}$ and call it the ultrapower of $\mathcal{M}$.

Proposition 2.1.9. *With the above definitions, the ultraproduct $\mathcal{M} = \prod_{\mathfrak{U}} \mathcal{M}_i$ is a metric structure.*

Recall what needs to be shown. One must show that $\mathcal{M}$ is complete and interprets the signature of $\mathcal{L}$ appropriately – functions and relations are uniformly continuous and relations are bounded as prescribed by $\mathcal{L}$. All of this can be found in [4], but we briefly note that the uniform continuity of the functions and relations is essential for the ultraproduct to be well-defined. Otherwise, there can exist tuples of elements equal in the ultrapower with different values under a continuous, but not uniformly continuous, relation or function.

With these definitions in hand, we can more generally evaluate any $\mathcal{L}$-formula in our $\mathcal{L}$-structure $\mathcal{M}$,

Theorem 2.1.10 (Łoś Theorem)**.** *Let $(\mathcal{M}_i : i \in \mathcal{I})$ be a family of $\mathcal{L}$-structures and let $\mathfrak{U}$ be an ultrafilter on $\mathcal{I}$. If $\phi(x)$ is an $\mathcal{L}$-formula and $a_k = ((a_i^k)_{i\in\mathcal{I}})_{\mathfrak{U}}$ are appropriately sorted elements of $\mathcal{M} = \prod_{\mathfrak{U}} \mathcal{M}_i$, then,*

$$\phi^{\mathcal{M}}(a) = \lim_{i\to\mathfrak{U}} \phi^{\mathcal{M}_i}(a_i)$$

Moreover, with the notion of an ultraproduct in hand, one can prove the Compactness Theorem for continuous model theory (see [4]).

Theorem 2.1.11. *Let $\mathcal{L}$ be a language and let T be an $\mathcal{L}$-theory. Then T is satisfiable if and only if every finite subset $T_0 \subseteq T$ is satisfiable.*

If a theory T has the property that every finite subset of it is satisfiable then we call the theory finitely satisfiable. The Compactness Theorem tells us that finite satisfiability implies satisfiability. We can actually further extend the theorem by defining what it means to be approximately satisfiable.

Definition 2.1.12. *Fix a language $\mathcal{L}$ and an $\mathcal{L}$-theory T. Given an $\mathcal{L}$-sentence $\phi \in T$ and some $\epsilon > 0$, we say that an $\mathcal{L}$-structure $\mathcal{M}$ satisfies the ϵ-approximation of ϕ if,*

$$|T(\phi) - \phi^{\mathcal{M}}| < \epsilon$$

We say that T is approximately satisfiable if given any finite subset $T_0 \subseteq T$ and $\epsilon > 0$, the epsilon approximation of each $\phi \in T_0$ is simultaneously satisfied by some $\mathcal{L}$-structure $\mathcal{M}$.

Theorem 2.1.13. *An $\mathcal{L}$-theory T is satisfiable if and only if it is approximately satisfiable.*

2.2 Definable Predicates and Types

We first introduce the notion of a definable predicate. What follows will largely be drawn from [4] and [8]. Recalling that $\mathfrak{F}_{\mathcal{L}}^{\overline{x}}$ denotes the collection of all $\mathcal{L}$-formulas in free variables $\overline{x}$, we can put a pseudo-metric on $\mathfrak{F}_{\mathcal{L}}^{\overline{x}}$ by fixing an $\mathcal{L}$-theory T (note this could even be the empty theory):

$$\|\phi\|_T = \sup\{\phi^{\mathcal{M}}(a) : \mathcal{M} \vDash T, a \in \mathcal{M}\}$$

We can of course turn $\mathfrak{F}_{\mathcal{L}}^{\overline{x}}$ into a metric space by quotienting it with respect to the equivalence relation $\phi \sim \psi$ if $\|\phi - \psi\|_T = 0$.

Definition 2.2.1. *Fix a language $\mathcal{L}$. A definable predicate $\phi(\overline{x})$ is a Cauchy sequence $(\phi_n(\overline{x}))$ of formulas in the metric space $\mathfrak{F}_{\mathcal{L}}^{\overline{x}}$. We interpret it by,*

$$\phi^{\mathcal{M}}(a) = \lim_{n\to\infty} \phi_n^{\mathcal{M}}(a)$$

Definable predicates can of course then be viewed as what we gain when we complete $\mathfrak{F}_{\mathcal{L}}^{\overline{x}}$ relative to the metric $\|\cdot\|_T$ induced by a given theory T. In fact, it follows then that every formula in the completion of $\mathfrak{F}_{\mathcal{L}}^{\overline{x}}$, which is often denoted by $\mathfrak{M}_{\mathcal{L}}^{\overline{x}}$, can be realized as the uniform limit of $\mathcal{L}$-formulas. Since each formula is uniformly continuous, definable predicates are thus uniformly continuous as well. In light of this, we often make no distinction between definable predicates and formulas, just choosing to work with the former.

Now having the definition of $\mathfrak{M}_{\mathcal{L}}^{\overline{x}}$, we can introduce types.

Definition 2.2.2. *Given a language $\mathcal{L}$, a (complete) type p is a linear functional from $\mathfrak{M}_{\mathcal{L}}^{\overline{x}}$ to $\mathbb{R}$ where, for some tuple $\overline{a}$ of elements in an $\mathcal{L}$-structure $\mathcal{M}$ satisfying T,*

$$p(\phi(\overline{x})) = \phi^{\mathcal{M}}(\overline{a})$$

for every $\phi(\overline{x}) \in \mathfrak{M}_{\mathcal{L}}^{\overline{x}}$. We say that $\overline{a}$ realizes p in this case. A partial type is just a partial function from $\mathfrak{M}_{\mathcal{L}}^{\overline{x}}$ to $\mathbb{R}$.

A partial type is just the restriction of a complete type to some subset of $\mathfrak{M}_{\mathcal{L}}^{\overline{x}}$. If a realizes p, then we can consider $p = Th(\mathcal{M}, a)$, where $(\mathcal{M}, a)$ is the $(\mathcal{L}, a)$-structure where the constant symbols a are just interpreted as themselves. So similarly to when we were considering theories, a type is determined by its kernel. At times, notation like $\phi(\overline{x}) \in p$ is used to denote the fact that $p(\phi(\overline{x})) = 0$, and so it can be convenient at times to consider a type as a set of conditions satisfied by some tuple a in an $\mathcal{L}$-structure $\mathcal{M}$, $p(x) = \{\phi(x) : \phi^{\mathcal{M}}(a) = 0\}$.

We denote the collection of all types in free variables $\overline{x}$ relative to some theory T by $S_{\overline{x}}(T)$. It is clear that $S_{\overline{x}}(T) \subseteq (\mathfrak{M}_{\mathcal{L}}^{\overline{x}})^*$. We consider the subspace topology on $S_{\overline{x}}(T)$ which is induced on it by the weak*-topology on $(\mathfrak{M}_{\mathcal{L}}^{\overline{x}})^*$. Recall that the weak*-topology on $(\mathfrak{M}_{\mathcal{L}}^{\overline{x}})^*$ is the weakest topology such that, identifying $\mathfrak{M}_{\mathcal{L}}^{\overline{x}}$ with its canonical image in $(\mathfrak{M}_{\mathcal{L}}^{\overline{x}})^{**}$, every $\phi \in p$ is continuous on $(\mathfrak{M}_{\mathcal{L}}^{\overline{x}})^*$. Hence the topology on $S_{\overline{x}}(T)$ is the weakest one where the linear functionals defined by $p \mapsto p(\phi)$ are continuous.

Proposition 2.2.3. *$S_{\overline{x}}(T)$ is compact.*

Proof. By the Banach-Alaoglu theorem [12], it is enough to show that $S_{\overline{x}}(T)$ is closed and bounded with respect to the norm. It is clear that $S_{\overline{x}}(T)$ is bounded. Let $p \in S_{\overline{x}}(T)$ and let a be an element realizing p. Then,

$$\|p\| = \sup_{\|\phi(x)\|=1} |p(\phi)| = \sup_{\|\phi(x)\|=1} |\phi(a)| \leq \|\phi\|$$

Now suppose (p_n) is a convergent sequence in $S_{\overline{x}}(T)$. We want to see that $p = \lim\limits_{n\to\infty} p_n \in S_{\overline{x}}(T)$. It is enough to show that there is a tuple a in some $\mathcal{L}$-structure $\mathcal{M}$ satisfying T, where for every $\phi(x) \in \mathfrak{M}_{\mathcal{L}}^{\overline{x}}$, $p(\phi) = \phi(a)$. This is a straightforward application of ultraproducts. Let a_n be a tuple in some $\mathcal{L}$-structure $\mathcal{M}_n$ satisfying T such that a_n realizes p_n. Let $\mathfrak{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Here we note the easily-proven fact that the ultralimit of a convergent sequence, with respect to a non-principal ultrafilter, agrees with the regular limit.

Consider the ultraproduct,

$$\mathcal{M} = \prod_{\mathfrak{U}} \mathcal{M}_n$$

It is easy to see by the Łoś theorem that $\mathcal{M} \models T$. Let $a \in \mathcal{M}$ be $(a_1, a_2, a_3, ...)_{\mathfrak{U}}$. Then if $\phi(x) \in \mathfrak{M}_{\mathcal{L}}^{\overline{x}}$,

$$p(\phi) = \lim_{n\to\infty} p_n(\phi) = \lim_{n\to\mathfrak{U}} \phi^{\mathcal{M}_n}(a_n) = \phi^{\mathcal{M}}(a)$$

□

We also note that the compactness of $S_{\overline{x}}(T)$ follows more directly as a corollary of the Compactness Theorem in much the same way as for classical model theory. If we have an open cover of $S_{\overline{x}}(T)$ without a finite subcover, then we can use the

Compactness Theorem to show there is a tuple a in some model $\mathcal{M}$ of T whose type is not in an open set in our cover.

We can also consider types over parameter sets. Given a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$, we can name a set B of elements in our structure as constants and add them to our language, denoting the expanded language as $(\mathcal{L}, B)$. In this case we can consider types in the space of formulas $\mathfrak{M}^{\overline{x}}_{(\mathcal{L},B)}$. In this setting, we denote the space of complete types over the parameter set B by $S_{\overline{x}}(T_B)$, where T_B is the expansion of the theory T by formulas that $\mathcal{M}$ satisfies, which contain the constant symbols in B. All of the above results still hold in this slightly more general setting.

The topology we put on $S_{\overline{x}}(T)$ necessarily made every functional of the form $p \mapsto p(\phi)$ continuous for every $\phi \in \mathfrak{M}^{\overline{x}}_{\mathcal{L}}$. The following result shows that continuous functionals on $S_{\overline{x}}(T)$ are exactly the functionals induced by formulas in $\mathfrak{M}^{\overline{x}}_{\mathcal{L}}$. This is a direct result of the Stone-Weierstrass theorem which says that the functions of the form $p \mapsto p(\phi)$ are dense in the set of all continuous functions from $S_{\overline{x}}(T)$ to a compact set in $\mathbb{R}$. In the statement of the following theorem we make use of the common tactic to treat every formula as taking values in the interval $[0, 1]$

Theorem 2.2.4. *Let $\Phi : S_{\overline{x}}(T) \to [0, 1]$ be continuous. Then $\Phi(p) = p(\psi)$ for some definable predicate ψ.*

So if $\mathcal{M}$ is an $\mathcal{L}$-structure satisfying a theory T, we would interpret $\psi^{\mathcal{M}}(a)$ as $\Phi(p)$, where p is a complete type realized by a.

We required in the definition of a type that every type p be satisfiable, in the sense that in some $\mathcal{L}$-structure $\mathcal{M}$ there is a tuple a such that $p(\phi) = \phi(a)$. We are often concerned with where this tuple a lies, and the notion of saturation comes up in response to this.

Definition 2.2.5. *Let $\mathcal{L}$ be a language and $\mathcal{M}$ an $\mathcal{L}$-structure that satisfies some theory T. We say that $\mathcal{M}$ is κ-saturated if for every parameter set $B \subseteq \mathcal{M}$, where $|B| < \kappa$, and every $p \in S_{\overline{x}}(T_B)$ there is some a in $\mathcal{M}$ realizing p.*

A nice fact proven in [4] (Proposition 7.6) is that if we have a countable language $\mathcal{L}$ and any non-principle ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, then given any family of $\mathcal{L}$-structures $(M_n)_{n\in\mathbb{N}}$, the ultraproduct,

$$\prod_{\mathfrak{U}} \mathcal{M}_n$$

is ω_1-saturated.

2.3 Elementary Classes and Definable Sets

Before discussing continuous model theory's version of definable sets, we introduce some basic categorical notions that will be useful.

Definition 2.3.1. *Given a language $\mathcal{L}$, a class $\mathcal{C}$ of $\mathcal{L}$-structures is called an elementary class if there exists some $\mathcal{L}$-theory T such that $\mathcal{C} = Mod(T)$, meaning $\mathcal{C}$ is precisely the class of all $\mathcal{L}$-structures that model T.*

We can also think of Mod(T) as a category, the objects of which are the models of T and the morphisms of which are elementary maps.

It will be important to be able to recognize when a class of structures is an elementary one. The following theorem from [8] gives us tools to do this.

Theorem 2.3.2. *Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. The following are equivalent:*

1. *$\mathcal{C}$ is an elementary class.*

2. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts, and elementary submodels.*

3. *$\mathcal{C}$ is closed under isomorphisms, ultraproducts, and ultraroots.*

Note that we say that $\mathcal{C}$ is closed under ultraroots if for any ultraproduct $\mathfrak{U}$ on any infinite set I, whenever $\mathcal{M}^{\mathfrak{U}} \in \mathcal{C}$, then $\mathcal{M} \in \mathcal{C}$ too.

One other nice way to think of elementary classes is in terms of axiomatizability.

Definition 2.3.3. *Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. We say that $\mathcal{C}$ is axiomatizable if there is some set of $\mathcal{L}$-sentences Φ such that if $\mathcal{M}$ is an $\mathcal{L}$-structure, then $\mathcal{M} \in \mathcal{C}$ if and only if $\Phi \subseteq Th(\mathcal{M})$, so $\phi^{\mathcal{M}} = 0$ for every $\phi \in \Phi$.*

Example 2.3.4. Let $\mathcal{L}$ be the language of Banach spaces that we introduced before. Then the class $\mathcal{K}$ of all Banach spaces is axiomatizable [4]. This is not too difficult to see, as the axioms for a vector space over $\mathbb{R}$ are readily converted into $\mathcal{L}$-sentences. For example, to say that there is an additive identity, take for each $n \in \mathbb{Z}_+$,

$$\sup_{x \in B_n} d_n(x, x +_{nn} i_{1n}(0))$$

Theorem 2.3.5. *Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. Then $\mathcal{C}$ is an elementary class if and only if $\mathcal{C}$ is axiomatizable.*

This can be a useful theorem in practice, for example telling us that the class of Banach spaces is an elementary class in the sense that the category of all Banach spaces and the category of models of the theory of Banach spaces are equivalent categories. In other situations though, it is not always readily apparent what the correct axioms are for the class of structures under consideration, and for that we often have to use the earlier characterization of an elementary class.

One of the main areas where continuous model theory diverges from classical model theory is in its characterization of definable sets. Unlike in the classical setting, we cannot just look at the zero set of a formula in a structure (i.e. the collection of elements in a structure satisfying the formula). We want to be more restrictive, and we will see the value in being more restrictive when we discuss imaginaries in the continuous setting.

Every definable set arises as a zero set of a definable predicate, but not every definable predicate gives us a zero set. In the following definition, Met is the category of bounded metric spaces, the morphisms of which are isometries between the spaces.

Definition 2.3.6. *Suppose $\mathcal{C}$ is an elementary class of models of some $\mathcal{L}$-theory T.*

Let $(S_i)_{1\leq i\leq m}$ be a finite collection of sorts from $\mathcal{L}$. A functor $X : \mathcal{C} \to Met$ is called a uniform assignment of closed sets relative to $\mathcal{C}$ if,

1. *For each $\mathcal{M}$ in $\mathcal{C}$, $X(\mathcal{M})$ is a closed subset of $\prod_{i=1}^{m} S_i^{\mathcal{M}}$.*
2. *For each $f : \mathcal{M} \to \mathcal{N}$, $X(f) = f|_{X(\mathcal{M})}$.*

Such a functor is called a definable set if for every $\mathcal{L}$-formula $\phi(x, y)$,

$$\inf_{x\in X(\mathcal{M})} \phi(x, y) \text{ and } \sup_{x\in X(\mathcal{M})} \phi(x, y)$$

are definable predicates.

So if we take $\phi(x, y)$ to be the formula $d(x, y)$, then if $S^{\mathcal{M}}$ is definable (meaning that for a functor X as above $X(\mathcal{M}) = S^{\mathcal{M}}$), the predicate $\inf_{x\in S^{\mathcal{M}}} d(x, y)$ gives us the distance from any element to $S^{\mathcal{M}}$, and since this is a definable predicate, $S^{\mathcal{M}}$ arises as the zero set of some definable predicate. In fact, we have an even stronger correspondence between being definable and being the zero set of some definable predicate.

Theorem 2.3.7 ([8])**.** *An assignment of the above type gives a definable set if and only if $d(x, S^{\mathcal{M}})$ is a definable predicate.*

Though this is a purely syntactic definition of definability, in practice it can be easier to work with a semantic version. What follows is stated and proven in [8].

Theorem 2.3.8 (Beth Definability)**.** *Fix a theory T in some language $\mathcal{L}$, and let T' be an extension of T in a language $\mathcal{L}'$ that in turn expands $\mathcal{L}$ without introducing new sorts. Moreover, suppose that the forgetful functor between models of T and T', $F : Mod(T') \to Mod(T)$, is an equivalence of categories. Then every predicate in $\mathcal{L}'$ is equivalent to a definable predicate in $\mathcal{L}$.*

Let us parse this. Recall that in this context, saying that F is an equivalence of categories means that if we begin with an $\mathcal{L}$-structure M, expand it to an $\mathcal{L}'$-structure and then apply the forgetful functor to "forget" the extra structure, we arrive at an

$\mathcal{L}$-structure that is isomorphic to the structure we started with. The importance of this is that the predicates (i.e. relations or functions) that we added to our language $\mathcal{L}$ could already be realized as definable predicates in our original language. We in some sense are adding nothing new. When we discuss imaginaries, we will attempt to make this precise.

An important and very useful consequence of Beth Definability is the following.

Theorem 2.3.9. *For a language $\mathcal{L}$, consider an elementary class $\mathcal{C}$ of $\mathcal{L}$-structures and an expansion $\mathcal{L}'$ of $\mathcal{L}$ by some predicate P with modulus of continuity δ_P such that for each $\mathcal{L}$-structure $\mathcal{M} \in \mathcal{C}$, $P^{\mathcal{M}}$ is a uniformly continuous, bounded function into $\mathbb{R}$. Let $\mathcal{C}'$ be the collection of all structures in $\mathcal{C}$ expanded into $\mathcal{L}'$-structures. If $\mathcal{C}'$ is an elementary class of models of some theory T', then P is T'-equivalent to a definable predicate in $\mathcal{L}$.*

A corollary of this is given in [8].

Theorem 2.3.10. *Let $\mathcal{C}$ be an elementary class and $S \mapsto S^{\mathcal{M}}$ an assignment of closed sets. This assignment is a definable set if and only if for every ultrafilter $\mathfrak{U}$ on an index set I, and family $(\mathcal{M}_i)_{i\in I}$ of structures in $\mathcal{C}$,*

$$S^{\mathcal{M}} = \prod_{\mathfrak{U}} S^{\mathcal{M}_i}$$

where $\mathcal{M}$ denotes the ultraproduct of the family $(\mathcal{M}_i)_{i\in I}$ with respect to $\mathfrak{U}$.

Colloquially we say that the assignment commutes with ultraproducts if the second half of the theorem above holds.

We will finish with an example of a set that is not definable. The result is folklore to an extent, but a reference can be found in [11].

Example 2.3.11. Recall that an ultrametric $d(x, y)$ on a metric space M is a metric with the stronger property that for every $x, y, z \in M$, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$. Consider T, the theory of pointed metric spaces of diameter 1, where a pointed metric space is a metric space with a distinguished point c. The language of pointed metric spaces of diameter 1 is just the language of pure metric spaces with one sort and a constant, $\mathcal{L} = \{M, d(x, y), c\}$. We will show that even though $B_r(c)$, the ball of radius r centred at c, is the zero set D of the formula $\phi(x) = \max\{d(x, c) - r, 0\}$, it is not a definable set.

Pick $x_n \in M$ such that $d(c, x_n) = r + \frac{1}{n}$ and let $x = (x_n)_{\mathfrak{U}}$ in $M^{\mathfrak{U}}$, where $\mathfrak{U}$ is some non-principal ultrafilter. Then clearly $d(x, c) = r$.

If $B_r(c)$ is definable, then we can find representatives $y_n \in B_r(c)$ such that $x = (y_n)_{\mathfrak{U}}$ too by theorem 2.3.10. The ultrametric inequality tells us that,

$$d(x_n, c) \leq \max\{d(x_n, y_n), d(y_n, c)\}$$

But $d(y_n, c) \leq r$ always and $d(x_n, y_n)$ is small on some $\mathfrak{U}$ big set, contradicting that $d(x_n, c) > r$.

2.4 Imaginaries in Continuous Model Theory

We recall that in classical model theory, we introduce imaginaries to our language as equivalence classes of $\emptyset$-definable equivalence relations. We have a similar notion in continuous logic, and the use of imaginaries will be essential in capturing local properties of Banach spaces.

There are three sources of imaginary sorts which we will consider in turn. Much of what follows is drawn from either [2] or [8].

1. Definable sets provide the first source of imaginary sorts. These can be extremely convenient to include, since, as was discussed earlier, these are exactly the sets over which we are able to quantify.

2. We also obtain imaginary sorts by taking countable products of sorts in our language. If $(S_i : i \in \mathbb{N})$ is a countable collection of sorts, then we consider an

imaginary sort S_P, where S_P is interpreted as the countable product,

$$\prod_{i\in\mathbb{N}} S_i$$

We also add in a metric d on S_P induced by the metrics on the S_i's and, for each S_i showing up in the product, functions $\pi_i : S_P \to S_i$ projecting S_P onto the sort S_i. Again, there is flexibility in choosing the metric on S_P, but we refer to [8] where the metric,

$$d(x, y) = \sum_{i\in\mathbb{N}} \frac{d_i(x_i, y_i)}{B_i 2^i}$$

is given as an example that one could take, where B_i is taken as a bound on S_i.

3. The last source we consider is most similar to the construction of imaginary sorts in the discrete case. Let $\phi(x, y)$ be an $\mathcal{L}$-formula, where both x and y are distinct tuples of variables from some finite product $\overline{S} := \prod_i S_i$. Moreover, suppose that $\phi(x, y)$ is interpreted as a pseudometric on $\overline{S}$ in all $\mathcal{L}$-structures. Then we add an imaginary sort S_ϕ interpreted as ${}^{\overline{S}}/_{\phi}$, the quotient of $\overline{S}$ with respect to the pseudometric $\phi(x, y)$. On this sort we define a metric d induced by $\phi(x, y)$ and a function $\pi : \overline{S} \to S_\phi$ sending a tuple in $\overline{S}$ to its equivalence class.

With all this said, suppose that we have some $\mathcal{L}$-theory T. Then we can iteratively add all sorts of the type described above to our language, and, to remain consistent with the discrete case, the resulting language is denoted $\mathcal{L}^{eq}$. We can similarly expand our theory T with $\mathcal{L}^{eq}$-formulas stating that what we have added to $\mathcal{L}$ has the properties that we intend, such as that the new metric symbols d that we added are in fact interpreted as a metric on the imaginary sorts. This expanded theory is denoted T^{eq}. Thus given any $\mathcal{L}$-structure $\mathcal{M}$, where $\mathcal{M} \models T$, there is a natural extension of $\mathcal{M}$ to an $\mathcal{L}^{eq}$-structure denoted $\mathcal{M}^{eq}$, where $\mathcal{M}^{eq} \models T^{eq}$.

An essential point is that although expanding our language with imaginary sorts and potentially any new definable predicates relative to T^{eq} gives us a richer language, it does not increase our expressive ability. The metric structures $\mathcal{M}$ and $\mathcal{M}^{eq}$ are in some sense the same. The following theorem from [8] makes this precise and echoes the discussion in the section on definable sets.

Definition 2.4.1. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages such that $\mathcal{L} \subseteq \mathcal{L}'$. If T is an $\mathcal{L}$-theory and T' is an $\mathcal{L}'$-theory, then we say that T' is a strongly conservative extension of T if the forgetful functor from Mod(T') to Mod(T) is an equivalence of categories.*

Theorem 2.4.2. *Suppose that T is a theory in a language $\mathcal{L}$. Then T^{eq} is a strongly conservative extension of T.*

It is this theorem that allows us to add imaginary sorts to our language. It is convenient to think of these sorts as already being in our language, though we generally do not need to consider a full extension of a language to $\mathcal{L}^{eq}$, instead just taking a portion of it.

We finish with a statement of Conceptual Completeness from [8].

Theorem 2.4.3 (Conceptual Completeness)**.** *Let T be an $\mathcal{L}$-theory and T' be a conservative extension of T via the forgetful functor F'. Then there exists a functor $G : Mod(T^{eq}) \to Mod(T')$ such that $F' \circ G = F$.*

Recall that T' is a conservative extension of T if every model of T can be uniquely expanded to a model of T'. As discussed in [3] and [8], suppose we have an elementary class $\mathcal{C}$ of $\mathcal{L}$-structures of some theory T. This theorem tells us that if $\mathcal{L} \subseteq \mathcal{L}'$ and we expand $\mathcal{C}$ to a class $\mathcal{C}'$ of $\mathcal{L}'$-structures, then if $\mathcal{C}'$ is also an elementary class, the forgetful functor from $\mathcal{C}'$ to $\mathcal{C}$ is an equivalence of categories and the sorts added to make $\mathcal{L}'$ are definable in $\mathcal{L}$.

Chapter 3

The Local Theory of Banach Spaces

3.1 Introduction

The notation and concepts introduced here are standard and largely drawn from [6]. Any exceptions are noted.

Recall that a Banach space over $\mathbb{R}$ is a vector space X over $\mathbb{R}$ equipped with a norm $\|\cdot\|_X$ under which X is complete. Henceforth we will assume that all Banach spaces are over $\mathbb{R}$ unless stated otherwise.

Definition 3.1.1. *A (linear) isomorphism between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is a linear bijection $T : X \longrightarrow Y$ such that*

$$K_1\|x\|_X \leq \|Tx\|_Y \leq K_2\|x\|_X$$

for some constants $K_1, K_2 \in \mathbb{R}$.

This definition is saying that both T and T^{-1} are bounded (hence continuous). Thus a linear isomorphism is a linear map $T : (X, \|\cdot\|_X) \longrightarrow (Y, \|\cdot\|_Y)$ that is a homeomorphism.

We will be primarily interested in a specific class of isomorphisms, which we will call expansive maps.

Definition 3.1.2. *Let $T : (X, \|\cdot\|_X) \longrightarrow (Y, \|\cdot\|_Y)$ be a linear operator and $\lambda \geq 1$. We say that T is a λ-expansive map if,*

$$\|x\|_X \leq \|Tx\|_Y \leq \lambda\|x\|_X$$

for all $x \in X$

Now we consider what it means for a Banach space to be (crudely) finitely representable in another.

Definition 3.1.3. *A Banach space $(X, \|\cdot\|_X)$ is crudely finitely representable in $(Y, \|\cdot\|_Y)$ if there exists a constant $\lambda \in [1, \infty)$ such that for every finite-dimensional subspace $F \subseteq X$, there exists a linear operator $T : F \longrightarrow Y$ that is λ-expansive.*

We say that $(X, \|\cdot\|_X)$ is finitely-representable in $(Y, \|\cdot\|_Y)$, if for every $\epsilon > 0$ $(X, \|\cdot\|_X)$ is crudely finitely-representable in $(Y, \|\cdot\|_Y)$ with constant $\lambda < 1 + \epsilon$.

Recall that the operator norm on the linear space $L(X, Y)$ of all bounded linear operators from X to Y is given by,

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$$

An equivalent, and sometimes useful, formulation of the definition of crude finite-representability is that $(X, \|\cdot\|_X)$ is crudely finitely-representable in $(Y, \|\cdot\|_Y)$ if there exists some $\lambda > 1$ such that for every finite-dimensional subspace $F \subset X$, there exists a linear operator $T : F \to Y$ that is isomorphic onto its image such that $\|T\| \cdot \|T^{-1}\| \leq \lambda$.

The key here is that a specific λ works for every finite-dimensional subspace of X. Up to that tolerance, we can find an approximation of any finite-dimensional subspace of X.

Example 3.1.4. [6] Any Banach space is finitely representable in c_0. Recall c_0 denotes the space of sequences of real numbers that converge to 0 with the max norm.

To see this is true, fix $\epsilon > 0$ and let X be an arbitrary Banach space and $F \subseteq X$ be a finite-dimensional subspace. Using the compactness of the unit ball in finite dimensions, let $\{x_i\}_{i=1}^n$ be a finite ϵ-net of the unit sphere S_F. For $1 \leq i \leq n$, take $l_i \in S_{X^*}$ such that $|l_i(x_i)| = 1$. Then define $T : F \to c_0$ by $x \mapsto (l_i(x))_{i=1}^n$. Note then that if $x \in X$,

$$\|Tx\| = \max_i |l_i(x)| \leq \max_i \|l_i\| \|x\| = \|x\|$$

and so $\|T\| \leq 1$. Similarly, if $x \in S_F$, fix i so that $\|x - x_i\| \leq \epsilon$, then,

$$\|Tx\| \geq |l_i(x)| = |l_i(x_i) + l_i(x - x_i)| \geq |l_i(x_i)| - \|l_i\| \|x - x_i\| \geq 1 - \epsilon = (1 - \epsilon)\|x\|$$

So for any $x \in F$, $(1-\epsilon)\|x\| \leq \|Tx\|$, hence $\|T^{-1}\| \leq \frac{1}{1-\epsilon}$. Putting the two inequalities together, we see that $\|T\|\|T^{-1}\| \leq \frac{1}{1-\epsilon}$.

We now begin applying ultraproducts to Banach spaces. Suppose $(X_i)_{i\in I}$ is a family of Banach spaces and $\mathfrak{U}$ is some ultrafilter on I. In the traditional application of ultraproducts to Banach spaces, one must restrict oneself to sequences $(x_i)_{i\in I}$ of elements that are uniformly bounded, restricting to the set of infinite tuples $(x_i)_{i\in I}$ where $\sup_{i\in I} \|x_i\|_{X_i} < \infty$. This is because, without this restriction, the ultraproduct does not necessarily give well-defined elements. For example, suppose $I = \mathbb{N}$ and we take a sequence of terms $(x_i)_{i\in\mathbb{N}}$ with $\|x_i\| = i$. Then $\lim_{i\to\mathfrak{U}} x_i \to \infty$.

In the setting of ultraproducts of metric structures the problem never arises, as ultraproducts are defined with regards to the sorts of our language and each sort is by definition uniformly bounded across all structures. It is not too difficult to see that the two notions of ultraproduct coincide. For example, one can recall the discussion in Chapter 2 that the categories of Banach spaces and the categories of models of the theory of Banach spaces are equivalent to see immediately that their respective notions of ultraproducts must coincide. Or more concretely, if $\sup_{i\in\mathbb{N}} \|x_i\|_{X_i} < M$ for some $M > 0$, then pick $n \in \mathbb{N}$ so that $n > M$ and then $x_i \in B_n^{M_i}$, where M_i is the metric structure interpreted as X_i. Similarly, if $x \in B_n = \prod_{\mathfrak{U}} B_n^{M_i}$, then $\|x_i\| \leq n$ for every i in some ultrafilter large set, and hence a representative can be taken so that $\|x_i\| \leq n$ for every $i \in \mathbb{N}$. Hence, we will not generally concern ourselves with the issue, freely taking ultraproducts when useful.

Lemma 3.1.5. *Let $(X_i : i \in I)$ be a collection of n-dimensional subspaces of a Banach space X, and let $\mathfrak{U}$ be an ultrafilter on an index set I. Then $\prod_{\mathfrak{U}} X_i$ is also n-dimensional.*

Proof. Let $(X_i : i \in I)$ be a collection of n-dimensional subspaces of a Banach space X and let $\mathfrak{U}$ be an ultrafilter on I.

Take $b_1^i, ..., b_n^i$ to be a basis for X_i, with each element of unit length and $\|b_n^i - b_m^i\| \geq \epsilon$, where $\epsilon > 0$ is some small fixed number. Take the element $b_k \in \prod_{\mathfrak{U}} X_i$ that has as a representative $(b_k^i)_{i \in \mathfrak{U}}$. We will show that $b_1, .., b_n$ is a basis for $\prod_{\mathfrak{U}} X_i$. Note first that $b_i \neq b_j$ if $i \neq j$, since we required the basis elements in each X_i be at a distance of at least ϵ from each other. Similarly, $b_i \neq 0$. Let $x \in B_1$, the unit ball of $\prod_{\mathfrak{U}} X_i$, and let $(x_i)_{\mathfrak{U}}$ be a representative for x. Then for each $i \in I$, there exists scalars c_k^i such that $x_i = \sum c_k^i b_k^i$.

Define $c_k = \lim_{i \to \mathfrak{U}} c_k^i$. Then $x = \sum c_k b_k$. □

This proof tells us, in particular, that if X is finitely-representable in a Banach space Y, then we can find an isometric copy of every finite-dimensional subspace of X in the ultrapower of Y with respect to any non-principal ultrafilter.

Lemma 3.1.6. *Let X be an n-dimensional Banach space and $\mathfrak{U}$ be an ultrafilter on an index set I. Then $X \cong X^{\mathfrak{U}}$.*

Proof. The proof of this is actually contained in the proof of lemma 3.1.5. Take the diagonal embedding of basis elements of X into $X^{\mathfrak{U}}$. □

Proposition 3.1.7. *For some fixed $n \in \mathbb{Z}_+$, let $\mathfrak{U}$ be an ultrafilter on a set I, let $X = \prod_{\mathfrak{U}} X_i$ and $Y = \prod_{\mathfrak{U}} Y_i$, where each X_i and Y_i are n-dimensional Banach spaces,* *and let* $T = \prod_{\mathfrak{U}} T_i : X \to Y$, $(x_i)_{\mathfrak{U}} \mapsto (T_i x_i)_{\mathfrak{U}}$, *where* $(T_i)_{i \in I}$ *is uniformly bounded. Then:*

1. $\|T\| = \lim_{i \to \mathfrak{U}} \|T_i\|$.
2. $\min_{\|x\|=1} \|Tx\| = \lim_{i \to \mathfrak{U}} (\min_{\|x_i\|=1} \|T_i x_i\|)$

Proof. (1) We first show that $\|T\| \geq \lim_{i \to \mathfrak{U}} \|T_i\|$. Since each X_i is finite-dimensional, then the unit sphere S_{X_i} of X_i is compact. Hence since each T_i is continuous, there exists some $x_i \in X_i$ such that $\|T_i\| = \|T_i x_i\|$. Note then that $(x_i)_{\mathfrak{U}} \in X$ and $\|x\| = \lim_{i \to \mathfrak{U}} \|x_i\| = 1$, so

$$\|T\| \geq \|Tx\| = \lim_{i \to \mathfrak{U}} \|T_i x_i\|$$

Now we want to show that $\|T\| \leq \lim_{i \to \mathfrak{U}} \|T_i\|$. Towards a contradiction, suppose that $\|T\| > \lim_{i \to \mathfrak{U}} \|T_i\|$ and let $\delta = \|T\| - \lim_{i \to \mathfrak{U}} \|T_i\|$. Take $x = (x_i)_{\mathfrak{U}}$ such that $\|T\| = \|Tx\| = \|(T_i x_i)_{\mathfrak{U}}\| = \lim_{i \to \mathfrak{U}} \|T_i x_i\|$, where $\lim_{i \to \mathfrak{U}} \|x_i\| = 1$. Thus for every $n \in \mathbb{Z}_+$,

$$A_n = \{i \in I : |1 - \|x_i\|| < 1/n\} \in \mathfrak{U}$$

and

$$B_n = \{i \in I : |\|Tx\| - \|T_i x_i\|| < 1/n\} \in \mathfrak{U}$$

Since $\mathfrak{U}$ is an ultrafilter, $C_n = A_n \cap B_n \in \mathfrak{U}$ as well. For all $i \in I$, let $y_i = \frac{1}{\|x_i\|} x_i$, so $\|y_i\| = 1$, and notice that for $n > 2$, if $i \in C_n$, then $\frac{1}{\|x_i\|} < 2$ and $\|T_i x_i\| < \|T\| + \frac{1}{2}$. Hence, when $n > 2$,

$$\begin{aligned} |\ \|Tx\| - \|T_i y_i\|\ | &\leq |\ \|Tx\| - \|T_i x_i\|\ | + |\ \|T_i x_i\| - \|T_i y_i\|\ | \\ &\leq |\ \|Tx\| - \|T_i x_i\|\ | + \|T_i y_i\| \cdot |\ 1 - \|x_i\|\ | \\ &\leq \frac{2\|T\| + 2}{n} \end{aligned}$$

So letting n grow large enough that the last inequality is smaller than δ, we see that for all $i \in C_n$, $\|T_i y_i\| > \lim_{i \to \mathfrak{U}} \|T_i\|$ on an ultrafilter-large set, which is a contradiction.

(2) is proved in much the same way.

$\square$

3.2 The Ribe Program

In [18], Ribe gave a condition for knowing when two Banach spaces are finitely-representable in each other.

Definition 3.2.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Then a bijection $f : X \to Y$ is a uniform homeomorphism if both f and f^{-1} are uniformly continuous.*

Theorem 3.2.2 (Ribe Rigidity Theorem)**.** *If two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are uniformly homeomorphic, then X is crudely finitely-representable in Y and Y is crudely finitely-representable in X.*

Ribe's theorem tells us that certain properties of the geometry of Banach spaces are essentially metric properties, in the sense that they are preserved under uniform homeomorphisms, which generally do not preserve the linear structure of the spaces. Specifically, Ribe tells us that these properties are the so-called local properties, meaning finite-dimensional linear properties. The Ribe program is concerned with using this correspondence to study general metric spaces. These local properties should be able to be viewed as metric properties.

There are many proofs of Ribe's theorem, but the one presented in [10] is in the spirit of the rest of this thesis. We will sketch it here.

Proof. Suppose that X and Y are uniformly homeomorphic Banach spaces. We will show there is some $\lambda \geq 1$ such that if $E \subseteq X$ is a finite-dimensional subspace of X, there is some $F \in Y$ and an isomorphism $T : E \to F$ such that $\|T\|\|T^{-1}\| \leq \lambda$.

Let $f : X \to Y$ be the uniform homeomorphism between the two spaces and let $\mathfrak{U}$ be a non-principal ultrafilter on $I = \mathbb{Z}_+$. Define a family of maps $f_n(x) = \frac{1}{n} f(nx)$ on X. One can check this is an equicontinuous family of maps. We now use the following clever lemma,

Lemma 3.2.3. *If $f : X \to Y$ is a uniform homeomorphism between Banach spaces X and Y, then for every $\delta > 0$ there exists some $K > 0$ such that if $\|x - y\| \geq \delta$, then $\|f(x) - f(y)\| \leq K\|x - y\|$.*

Often this is just expressed by saying that $f : X \to Y$ is Lipschitz at large distances. We use this to get a constant K for $\delta = 1$ where $\|f(x) - f(y)\| \leq K\|x - y\|$ if $\|x - y\| \geq 1$ and hence $\|f_n(x) - f_n(y)\| \leq K\|x - y\|$ if $\|x - y\| \geq \frac{1}{n}$. Hence letting $F : X^{\mathfrak{U}} \to Y^{\mathfrak{U}}$ be the map $(x_n)_{\mathfrak{U}} \mapsto (f_n(x))_{\mathfrak{U}}$ we get a Lipschitz map between the ultrapowers with constant K.

Consider $E \subseteq X$ as a subspace of $X^{\mathfrak{U}}$ and so F is a Lipschitz embedding of E into $Y^{\mathfrak{U}}$. Considering $Y^{\mathfrak{U}}$ inside of $(Y^{\mathfrak{U}})^{**}$, [10] shows there is a subspace $G \subseteq (Y^{\mathfrak{U}})^{**}$ and an isomorphism $T : E \to G$ such that $\|T\|\|T^{-1}\| \leq K + \epsilon$ for any $\epsilon > 0$. We now call upon the Principle of Local Reflexivity,

Theorem 3.2.4. *Let $X \subset X^{**}$ be a Banach space and $U \subseteq X^{**}$ be a finite-dimensional subspace. Then there exists a finite-dimensional $V \subseteq X$ and isomorphism $T : U \to V$ such that $T(x) = x$ if $x \in U$ and $\|T\|\|T^{-1}\| \leq 1 + \epsilon$.*

This is saying for any Banach space X, X^{**} is finitely-represented in X, and we can almost view it as an application of the Downward Lowenheim-Skolem theorem in the right logic, moving from X^{**} to X, the latter of which is w^*-dense in the former. Thus there exists some $G' \subseteq Y^{\mathfrak{U}}$ and an isomorphism $T : E \to G'$ with $\|T\|\|T^{-1}\| \leq (K + \epsilon)(1 + \epsilon)$. It is not too difficult to see that there must be a finite-dimensional subspace $F \subseteq Y$ and an isomorphism H from G' to F such that $\|H\|\|H^{-1}\| \leq 1 + \epsilon$. Putting all this together, we see X is finitely-representable in Y with constant $\lambda = K + \epsilon$ for any ϵ.

□

As was briefly mentioned above, uniformly homeomorphic spaces are not in general linearly isomorphic to each other. Moreover, it appears that Ribe's Theorem is the strongest general statement we can make to relate the notions of uniform homeomorphisms, finite-representability, and linearly isomorphisms among Banach spaces, as the converse is not true either.

Example 3.2.5. Let Γ be a set of the cardinality of the continuum and let $c_0(\Gamma)$

be the collection of all functions $f : \Gamma \to \mathbb{R}$ where for every $\epsilon > 0$, $|f(\gamma)| > \epsilon$ for only finitely many $\gamma \in \Gamma$. Take $\|f\| = \sup_{\gamma\in\Gamma} |f(\gamma)|$ as the norm. Note that $c_0(\mathbb{N})$ is just the familiar sequence space we usually denote c_0. In [1] it is shown that $c_0(\Gamma)$ is uniformly homeomorphic to a closed subspace of l_∞, but is not linearly isomorphic.

Example 3.2.6. In [5], it is shown that, for $p \in (1, 2)$, although $L_p(\mathbb{R})$ and l_p are finitely-representable in each other, they are not uniformly homeomorphic.

Example 3.2.7. Let us consider an example of viewing a geometric property of a Banach space as a metric property. What follows is largely drawn from [15].

We say that a Banach space X has Rademacher type p, where $p \geq 1$, if for some $T > 0$ it is the case that for any collection $\{x_i\}_{i=1}^n$ of n elements of X,

$$\frac{1}{2^n} \sum_{\epsilon_i\in\{-1,1\}} \left(\| \sum_{i=1}^{n} \epsilon_i x_i \| \right) \leq T \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

We can note immediately that this is a strengthening of the triangle property and that every Banach space has Rademacher type 1. In fact if X has Rademacher type p, then X has Rademacher type q for every $1 \leq q \leq p$. The reader can see [14] for a discussion on the usefulness of Rademacher type and for further references regarding it.

If we consider the inequality, it is clear that Rademacher type is finite-dimensional, and hence is a local property. As such, Ribe's Theorem is essentially telling us that it is a metric property in disguise and we should be able to find a formulation of it just using the metric properties of a Banach space, without reference to its linear structure.

So let (M, d) be a metric space. We say that M has Enflo type p if for some $T > 0$, it is the case that for all $n \in \mathbb{N}$ and $f : \{-1, 1\}^n \to M$,

$$\frac{1}{2^n} \sum_{\epsilon\in\{-1,1\}^n} d(f(\epsilon) - f(-\epsilon)) \leq \frac{T}{2^n} \sum_{\epsilon\in\{-1,1\}^n} \left(\sum_{i=1}^{n} d(f(\epsilon(-i)) - f(\epsilon))^p \right)^{\frac{1}{p}}$$

where $\epsilon(-i) = (\epsilon_1, ..., \epsilon_{i-1}, -\epsilon_i, \epsilon_{i+1}, ..., \epsilon_n)$.

It turns out from [16] that if a Banach space X has Rademacher type p, then it also has Enflo type $p - \epsilon$ for every $\epsilon > 0$. Note how the above definition of Enflo type is purely dependent on the metric. Notice too that the type of a metric space is expressible in continuous logic.

Chapter 4

The Model Theory of Finite-Representability

Our starting language is the language of Banach spaces we introduced earlier in Example 2.1.3. We are eventually going to add in very particular relations and sorts to our language that will help us capture the local structure of a Banach space. Before we do this though, we need to show that they satisfy the requirements of a metric language and are bounded and uniformly continuous.

For a fixed infinite-dimensional Banach space B and each $n \in \mathbb{Z}_+$, consider the set $G_n(B)$, consisting of all closed unit balls of n-dimensional subspaces of B. The inspiration here, and what G_n represents, is the n-dimensional Grassmannian, which is the space of all n-dimensional subspaces. We put the Hausdorff metric on the set, where if $X, Y \in G_n(B)$,

$$d_{G_n}(X,Y) = \max\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\}$$

Lemma 4.0.8. *$G_n(B)$ is bounded and complete with respect to the metric $d_{G_n}(X,Y)$.*

Proof. Fix n and a Banach space B and consider $G_n(B)$ and the metric $d_{G_n}(X,Y)$ on $G_n(B)$. We first note easily that $G_n(B)$ is bounded. If $X, Y \in G_n(B)$, then by definition $X, Y \subset B_1$, the ball of radius 1. Hence for any $x \in X$, since $0 \in Y$,

$d(x,Y) \leq 1$ and similarly for any $y \in Y$, $d(X,y) \leq 1$, and so $d_{G_n}(X,Y) \leq 1$ as well.

We now show that $G_n(B)$ is complete. Consider a sequence (X_n) of elements in $G_n(B)$ that is Cauchy with respect to the metric on it. Then the limit X of this sequence can be realized as the ultraproduct of the sequence with respect to any non-principal ultrafilter on $\mathbb{N}$. By lemma 3.1.5, such an ultraproduct is also n-dimensional. Moreover, if $z \in X$, then by definition of the metric, there is a sequence (z_n), with $z_n \in X_n$, in the unit ball of B converging to it. Hence $z = (z_n)_{\mathfrak{U}} \in B$, so X is an n-dimensional subspace of B.

□

We also consider a function $P_n(x,y)$ on $B_1 \times G_n(B)$, recalling that B_1 is interpreted as the unit ball, where $P_n(x,Y)$ is interpreted as the distance of $x \in B_1$ to $Y \in G_n(B)$,

$$P_n(x,Y) = \inf_{y \in Y} d(x,y)$$

Lemma 4.0.9. *For any $n \in \mathbb{Z}_+$, $P_n(x,y)$ is a bounded and uniformly continuous function.*

Proof. It is clear that $P_n(x,Y) \leq 1$ for any $x \in B_1$ and $Y \in G_n(B)$ as Y is a subspace and $0 \in Y$.

It is not much more difficult to see that $P_n(x,Y)$ is uniformly continuous. Fix $\epsilon > 0$ and suppose that $d((x_1,Y_1),(x_2,Y_2)) < \epsilon/2$, where $x_1, x_2 \in B_1$ and $Y_1, Y_2 \in G_n(B)$. Recall that $d((x_1,Y_1),(x_2,Y_2)) < \epsilon/2$ means that $\max\{d(x_1,x_2), d(Y_1,Y_2)\} < \epsilon/2$.

Since Y_1 and Y_2 are both compact, there exist $z_1 \in Y_1$ and $z_2 \in Y_2$ where $d(x_1,Y_1) = d(x_1,z_1)$ and $d(x_2,Y_2) = d(x_2,z_2)$. Noting that,

$$d(x_1,z_1) \leq d(x_2,z_2) + d(x_1,x_2) + d(z_1,z_2)$$

and

$$d(x_2,z_2) \leq d(x_1,z_1) + d(x_1,x_2) + d(z_1,z_2)$$

we see that

$$|P_n(x_1,Y_1) - P_n(x_2,Y_2)| = |d(x_1,z_1) - d(x_2,z_2)| \leq |d(x_1,x_2) + d(z_1,z_2)| < \epsilon$$

Hence $\delta(\epsilon) = \epsilon/2$ can be taken to be the modulus of uniform continuity for each $P_n(x, Y)$. $\square$

Finally, we also define relations on $G_n(B)$. For an arbitrary n-dimensional Banach space A with $\lambda \in [1, \infty)$, let $C_\lambda(A)$ denote the family of Banach spaces X for which there exists a λ-expansive map to A,

$$C_\lambda(A) = \{X : \exists T : X \to A,\ T\ \lambda\text{-expansive}\}$$

For each $\lambda \in [1, \infty)$, and for every finite-dimensional Banach space A,

$$R^\lambda_A(Z) = \inf\{\|T\| : T : Z \to X,\ T \text{ expansive},\ X \in C_\lambda(A)\}$$

As before, we want to show that these additions are bounded and uniformly continuous, and in fact we will show that they are "uniformly" uniformly continuous and bounded, in the sense that it does not matter what Banach space we evaluate $R^\lambda_A(x)$ in. To do so we will need to show they commute with ultraproducts. Note this proposition will be very important later for us when we show that the class of Banach spaces expanded by this language forms an elementary class.

Proposition 4.0.10. *Let A be an n-dimensional Banach space, $\lambda \in [1, \infty)$, $\mathfrak{U}$ be an ultrafilter on an index set I, and $(B_i)_{i \in I}$ be an I-indexed sequence from $G_n(B)$ for some Banach space B. Then,*

$$\lim_{i \to \mathfrak{U}} R^\lambda_A(B_i) = R^\lambda_A(\prod_{\mathfrak{U}} B_i)$$

Proof. It's clear that $\lim\limits_{i \to \mathfrak{U}} R^A_\lambda(B_i) \geq R^A_\lambda(\prod_{\mathfrak{U}} B_i)$. If not, let

$$\delta = R^A_\lambda(\prod_{\mathfrak{U}} B_i) - \lim_{i \to \mathfrak{U}} R^A_\lambda(B_i) > 0$$

and so there exist expansive maps $T_i : B_i \to X_i \in C_\lambda(A)$, where

$$|\ \|T_i\| - R^A_\lambda(\prod_{\mathfrak{U}} B_i)\ | > \delta/2$$

for all i in an ultrafilter-large set $I_0 \in \mathfrak{U}$. Then $T = \prod_{\mathfrak{U}} T_i : \prod_{\mathfrak{U}} B_i \to \prod_{\mathfrak{U}} X_i \in C_\lambda(A)$. But $\|T\| < R_\lambda^A(\prod_{\mathfrak{U}} B_i)$ then, which is a contradiction.

We want to also see that $R_\lambda^A(\prod_{\mathfrak{U}} B_i) \geq \lim\limits_{i \to \mathfrak{U}} R_\lambda^A(B_i)$.

Suppose instead that $\lim\limits_{i \to \mathfrak{U}} R_\lambda^A(B_i) > R_\lambda^A(\prod_{\mathfrak{U}} B_i)$. This time let

$$\delta = \frac{1}{2}(\lim_{i \to \mathfrak{U}} R_\lambda^A(B_i) - R_\lambda^A(\prod_{\mathfrak{U}} B_i))$$

By how the relations were defined, there exists $T : \prod_{\mathfrak{U}} B_i \to X$, where $X \in C_\lambda(A)$. Our goal is to eventually arrive at a contradiction by showing that if a map such as T exists, then we can find expansive maps from B_i to something in $C_\lambda(A)$ with norm less than $R_\lambda^A(\prod_{\mathfrak{U}} B_i) + \delta$ on an ultrafilter large set, which is a contradiction.

Let $\{b_1, ..., b_n\}$ be a basis for $\prod_{\mathfrak{U}} B_i$, so, defining $y_i = Tb_i$, $\{y_1, ..., y_n\}$ is a basis for X. Take representations $b_j = (b_j(i))_{i \in I}$ for each of the basis elements as well. Arguing as in Lemma 3.1.5, $\{b_1(i), ..., b_n(i)\}$ and $\{y_1, ..., y_n\}$ are bases for B_i and X respectively on an ultrafilter-large set.

Define maps $T_i : B_i \to X$ by $b_j(i) \mapsto y_j$, and note that by design $T = \prod_{\mathfrak{U}} T_i$. Moreover, since T is a bijection, T_i is a bijection on an ultrafilter large set too. However, T_i may not be expansive on an ultrafilter large set. We do know that by Proposition 3.1.7 though, for all integers $n > 1$,

$$A_n = \{i \in I : |\|T\| - \|T_i\|| < 1/n\} \in \mathfrak{U}$$

and

$$B_n = \{i \in I : |\min_{\|x\|=1} \|Tx\| - \min_{\|x_i\|=1} \|T_i x_i\|| < 1/n\} \in \mathfrak{U}$$

Note $C_n = A_n \cap B_n \in \mathfrak{U}$. If $i \in C_n$,

$$(1 - \frac{1}{n})\|x_i\| \leq \|T_i x_i\| \leq (\|T\| + \delta + \frac{1}{n})\|x_i\|$$

So for all T_i where $i \in C_2$, if T_i is expansive, then let $T_i' = T_i$, else if T_i is not expansive, define $T_i' = (\frac{n}{n-1})T_i$, where n is the greatest integer such that $i \in C_n$.

Thus, T_i' is an expansive map from B_i on an ultrafilter large set. Also,

$$\begin{aligned} \mid \|T\| - \|T_i'\| &\leq \mid \|T\| - \|T_i\| \mid + \mid \|T_i\| - \|T_i'\| \mid \\ &\leq \mid \|T\| - \|T_i\| \mid + \|T_i\| \mid 1 - \frac{n}{n-1} \mid \\ &< \frac{1}{n} + \frac{\|T_i\|}{n-1} \\ &< \frac{1}{n} + \frac{\|T\| + 1/n}{n-1} \end{aligned}$$

So for every n large enough so that the last inequality is less than δ, we get expansive maps on an ultrafilter large set of norm strictly less than $\lim_{i \to \mathfrak{U}} R_\lambda^A(B_i)$, a contradiction. □

Proposition 4.0.11. *Let A be an n-dimensional Banach space and $\lambda \in [1, \infty)$.*

1. *$R_A^\lambda(x)$ is bounded independent of the Banach space under consideration.*
2. *$R_A^\lambda(x)$ is uniformly continuous independent of the Banach space under consideration.*

Proof. For (1), suppose not and we work towards a contradiction. Assume there exists some $\lambda \in [1, \infty)$ and an n-dimensional space A such that $R_A^\lambda(x)$ is not bounded. Hence for every $k \in \mathbb{N}$, there exists some n-dimensional subspace Z_k of some Banach space X_k such that $R_A^\lambda(Z_k) > k$. From the definition of $R_A^\lambda(x)$ and $C_\lambda(A)$, for every $X \in C_\lambda(A)$, if we have any expansive map $T : Z_k \to X$, then $\|T\| > k$.

Let $T_k : Z_k \to A$ be any expansive map. Certainly $A \in C_\lambda(A)$, so $\|T_k\| > k$. Now take $\mathfrak{U}$ to be any non-principal ultrafilter on $\mathbb{N}$ so that, denoting $Z = \prod_{\mathfrak{U}} Z_k \subseteq \prod_{\mathfrak{U}} X_k$, there is some expansive map,

$$T = \prod_{\mathfrak{U}} T_k : Z \to A$$

between the finite-dimensional subspaces. Thus $\|T\| < \infty$, but we showed earlier in Proposition 3.1.7 that $\|T\| = \lim_{k \to \mathfrak{U}} \|T_k\|$, which contradicts the fact that $\|T_k\| > k$.

For (2), suppose that $R_A^\lambda(x)$ is not uniformly continuous. Fix $\epsilon > 0$. Then we can find a family of Banach spaces $(X_k)_{k \in \mathbb{N}}$ with n-dimensional subspaces $B_k, C_k \subseteq X_k$

such that $d(B_k, C_k) < 1/k$ but $|R^\lambda_A(B_k) - R^\lambda_A(C_k)| > \epsilon$. Then take $\mathfrak{U}$ to be a non-principal ultrafilter on $\mathbb{N}$, and $d(\prod_{\mathfrak{U}} B_k, \prod_{\mathfrak{U}} C_k) = 0$, but $|R^\lambda_A(\prod_{\mathfrak{U}} B_k) - R^\lambda_A(\prod_{\mathfrak{U}} C_k)| > \epsilon$, contradicting the previous proposition. $\square$

Putting all of this together, consider the expanded language,

$$\mathcal{L}_G = \mathcal{L} \cup \{G_n\}_{n\in\mathbb{Z}_+} \cup \{d_{G_n}(X,Y)\}_{n\in\mathbb{Z}_+} \cup \{P_n(x,y)\}_{n\in\mathbb{Z}_+} \cup \{R^A_\lambda(x)\}_{\lambda\in[1,\infty),\text{A f.d.}}$$

Our goal is to show that these sorts and relations are definable in $\mathcal{L}^{eq}$ already. Hence we can safely include them without changing the expressive ability of our language. To do so, we will use Conceptual Completeness and Beth definability.

Let $\mathcal{K}$ denote the class of infinite-dimensional Banach spaces. $\mathcal{K}$ is an elementary class. We noted earlier in section 1 that the class of Banach spaces is an elementary class, and it is not too difficult to see that $\mathcal{K}$ is as well. It is certainly closed under isomorphisms and ultraproducts. Also, we showed that the ultrapower of a finite-dimensional Banach space is finite-dimensional, so if $B^{\mathfrak{U}} \in \mathcal{K}$, then $B \in \mathcal{K}$. Let T_B denote the theory of infinite-dimensional Banach spaces.

We eventually want to show that the class of $\mathcal{L}_G$-structures we get by expanding the $\mathcal{L}$-structures in $\mathcal{K}$ is an elementary class, but we need one more fact first.

Proposition 4.0.12. *Let $\mathfrak{U}$ be an ultrafilter on some index set I. If $(B_i)_{i\in I}$ is a family of Banach spaces, then,*

$$G_n(\prod_{\mathfrak{U}} B_i) = \prod_{\mathfrak{U}} G_n(B_i)$$

Proof. We first just note that it is almost immediate that $\prod_{\mathfrak{U}} G_n(B_i) \subseteq G_n(\prod_{\mathfrak{U}} B_i)$. An ultraproduct of n-dimensional spaces is again n-dimensional, so if $W \in \prod_{\mathfrak{U}} G_n(B_i)$, then $W = \prod_{\mathfrak{U}} W_i$ for $W_i \in G_n(W_i)$. Hence W is n-dimensional. If $x \in W$, then $x = (x_i)_{\mathfrak{U}}$, where $x_i \in W_i$. Hence $x \in \prod_{\mathfrak{U}} B_i$ and so $W \subseteq \prod_{\mathfrak{U}} B_i$.

So now suppose that $V \in G_n(\prod_{\mathfrak{U}} B_i)$, so $V \subseteq \prod_{\mathfrak{U}} B_i$ is n-dimensional. We will show that there are $W_i \in G_n(B_i)$ such that $V = \prod_{\mathfrak{U}} W_i$.

Let $\{e_1, \cdots, e_n\}$ be a basis for V, taking a representative $e_k = (e_k(i))$ for each. For fixed i, let $W_i = \text{span}\{e_1(i), \cdots, e_n(i)\} \subseteq B_i$. It is clear that $V = \prod_{\mathfrak{U}} W_i$, but there is a concern that although the ultraproduct is n-dimensional, not all of the W_i's are n-dimensional.

So let $I_0 = \{i \in I : W_i \notin G_n(B_i)\}$. Note that $I_0 \notin \mathfrak{U}$. Suppose it were. Then for each W_i, let $d_i = \dim W_i$ and $l_i = n - d_i$. For every $1 \leq j \leq l_i$, pick $f_j(i) \in B_i$ linearly independent of each other, outside the span of W_i, and where $\|f_j(i)\|_{B_i} = 1$. Define,

$$W_i^+ = \text{span}\{e_1(i), \cdots, e_n(i), f_1(i), \cdots, f_{l_i}(i)\}$$

Note that $W_i^+ = W_i$ if $i \notin I_0$. By construction, W_i^+ is n-dimensional for every i, and hence so too is $W^+ = \prod_{\mathfrak{U}} W_i^+$. Since $e_k \in W^+$ for every k, $V \subseteq W^+$, and after comparing dimensions, it is easily seen that in fact $V = W^+$. But this leads to a contradiction. Set $y_i = 0$ if $i \notin I_0$ and $y_i = f_1(i)$ otherwise. Then $(y_i)_{\mathfrak{U}} \notin V$, but $\|y\| \neq 0$, a contradiction. So we see that $I_0 \notin \mathfrak{U}$, and hence for every $i \in I_0$, we can replace W_i with an arbitrary n-dimensional space from $G_n(B_i)$ without changing the ultraproduct. □

Theorem 4.0.13. *Given an $\mathcal{L}$-structure $X \in \mathcal{K}$, let $\hat{X}$ denote the expansion to an $\mathcal{L}_G$-structure. Let $\mathcal{K}_G$ denote the class of $\mathcal{L}_G$-structures,*

$$\mathcal{K}_G = \{\mathcal{M} : \mathcal{M} \cong \hat{X}, X \in \mathcal{K}\}$$

Then $\mathcal{K}_G$ is an elementary class.

Proof. We need to show that $\mathcal{K}_G$ is closed under isomorphisms, ultraproducts, and ultraroots.

It is clear that if $\mathcal{M} \in \mathcal{K}_G$ and $\mathcal{N} \cong \mathcal{M}$, then $\mathcal{N} \cong \hat{X}$ for some $X \in \mathcal{K}$, so $\mathcal{N} \in \mathcal{K}_G$.

Suppose that $(\mathcal{M}_i)$ is a family of structures in $\mathcal{K}_G$. We want to show that $\mathcal{M} = \prod_{\mathfrak{U}} \mathcal{M}_i \in \mathcal{K}_G$ for any ultrafilter $\mathfrak{U}$ on I. By Propositions 4.0.8 and 4.0.10, if $\mathcal{M}_i \cong \hat{X}_i$, then $\mathcal{M} \cong \widehat{\prod_{\mathfrak{U}} X_i}$, the expansion of $\prod_{\mathfrak{U}} X_i$, which is in $\mathcal{K}$, to an $\mathcal{L}_G$-structure $\widehat{\prod_{\mathfrak{U}} X_i}$.

Finally, suppose that $\mathcal{M}$ is an $\mathcal{L}_G$-structure and $\mathcal{M}^{\mathfrak{U}} \in \mathcal{K}_G$, so for some $X \in \mathcal{K}$, $\mathcal{M}^{\mathfrak{U}} \cong \hat{X}$. So $\mathcal{M}^{\mathfrak{U}}$ is isomorphic to a Banach space expanded as we described above, hence the sorts $G_n(\mathcal{M}^{\mathfrak{U}})$ capture the unit balls of the n-dimensional subspaces of $\mathcal{M}^{\mathfrak{U}}$

for each n. Since $\mathcal{M}^{\mathfrak{U}} \models T_B$ and $\mathcal{M}$ is an elementary substructure of $\mathcal{M}^{\mathfrak{U}}$, $\mathcal{M}$ also has a Banach space structure. Suppose $A \in G_n(\mathcal{M})$ for some n, so $A \in G_n(\mathcal{M}^{\mathfrak{U}})$ too. In $\mathcal{M}^{\mathfrak{U}}$, A is interpreted as an n-dimensional subspace of $\mathcal{M}^{\mathfrak{U}}$, but to see our class is elementary, we need to show A is contained inside of the Banach space structure on $\mathcal{M}$ as a substructure of $\mathcal{M}^{\mathfrak{U}}$.

It is clear that there exist $y \in \mathcal{M}$ such that $P_n^{\mathcal{M}}(y, A) = 0$. We know that $\mathcal{M}^{\mathfrak{U}} \models \inf_{x \in B_1} P_n(x, A)$, and hence $\mathcal{M}$ does as well. This means for every $\epsilon > 0$, there exists some $y_n \in \mathcal{M}$ such that $d(y, A) < \epsilon$. We take a Cauchy sequence of these to see there exists some $y \in \mathcal{M}$ where $d(y, A) = 0$. Since A is compact in $\mathcal{M}^{\mathfrak{U}}$ and $y \in \mathcal{M}^{\mathfrak{U}}$, we see $y \in A$. Moreover, we can see that the zero-set of $P_n(x, A)$ is compact in $\mathcal{M}$ as well, because we can express the fact that it has an ϵ-net covering it. Take the $\mathcal{L}_G$-sentence $\phi_{\epsilon,k}$,

$$\inf_{y_1,\dots,y_k \in B_1} \sup_{x \in B_1} \left(P_n(x, A) + \sum_{i=1}^{k} P_n(y_i, A) + \prod_{i=1}^{k} \max\{d(x, y_i) - \epsilon, 0\} \right)$$

This formula is saying that there are k-points $y_1, \dots, y_k \in B_1$, recalling B_1 is the unit ball, all of which belong to A, such that for any $x \in A$, one of the y_i are within ϵ of x. Note that for $\epsilon > 0$, there is a k such that $\mathcal{M}^{\mathfrak{U}} \models \phi_{\epsilon,k}$ and hence $\mathcal{M} \models \phi_{\epsilon,k}$ too.

It remains to show that $A \subseteq \mathcal{M}$ inside of $\mathcal{M}^{\mathfrak{U}}$. Suppose not, so there exists some $z \in \mathcal{M}^{\mathfrak{U}} \cap A$ such that $z \notin \mathcal{M}$. Let $\epsilon > 0$ be smaller than the distance from z to any point in $\mathcal{M}$. Note that we can express the fact that there are k points forming an $\frac{\epsilon}{2}$-net of the zero set of $P_n(x, A)$ in $\mathcal{M}$. We can then by assumption write a sentence saying that $\mathcal{M}^{\mathfrak{U}}$ has a point not covered by this net, but this gives a contradiction since $\mathcal{M}$ is an elementary substructure and so should satisfy this sentence too. □

Denote the theory of the $\mathcal{L}_G$-structures in $\mathcal{K}_G$ by T_G. So in fact the sorts $G_n(B)$ are definable in $\mathcal{L}^{eq}$. We actually have already shown that the relations are in $\mathcal{L}^{eq}$ as well. To show that Beth definability applies, it is enough to show that the relations commute with ultraproducts [8].

We showed in Propositions 4.0.8 and 4.0.10 that this is the case, and hence we get the following.

Theorem 4.0.14. *Let $\mathcal{L}$ be the language of Banach spaces and T_B be the theory of infinite dimensional Banach spaces. Then with $\mathcal{L}_G$ and T_G as defined above, $\mathcal{L}_G$ is definable in $\mathcal{L}^{eq}$ and T_G is interpretable in T^{eq}.*

Hence we see that both the sorts G_n and the relations R^A_λ are definable in $\mathcal{L}^{eq}$. We can thus consider the language,

$$\mathcal{L}_\lambda = \{G_n : n \in \mathbb{Z}_+\} \cup \{R^A_\lambda(x) : A \text{ a finite-dimensional Banach space}\}$$

as a reduct of $\mathcal{L}$ for any specific $\lambda \in [1, \infty)$.

We want to show that this language is effective in capturing the notion of finite-representability.

Definition 4.0.15. *Let $\mathcal{L}$ be any language and $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. We say that $\mathcal{M}$ and $\mathcal{N}$ are existentially equivalent, denoted $\mathcal{M} \equiv_\exists \mathcal{N}$ if for any $\mathcal{L}$-sentence ψ of the form $\inf_{x \in S} \phi(x)$, where ϕ is a quantifier-free formula with a single free-variable $x \in S$, $\mathcal{M} \models \psi$ if and only if $\mathcal{N} \models \psi$.*

Theorem 4.0.16. *For fixed $\lambda \in [1, \infty)$, consider $\mathcal{L}_\lambda$, a reduct of the language of Banach spaces $\mathcal{L}$, and let X and Y be two Banach spaces such that with respect to $\mathcal{L}_\lambda$, $X \equiv_\exists Y$. Then X and Y are crudely finitely-representable in each other with constant $\lambda + \epsilon$ for every $\epsilon > 0$.*

Proof. Let $F \subseteq X$ be n-dimensional and fix $\epsilon > 0$. We want to show that there exists some $(\lambda + \epsilon)$-expansive map $T : F \to Y$.

Note that $X \models \inf_{x \in G_n(X)} R^F_\lambda(x) = 1$, since $F \in C_\lambda(F)$ via the identity map, which has norm 1. Thus since X and Y are existentially equivalent, $Y \models \inf_{x \in G_n(Y)} R^F_\lambda(x) = 1$ as well. Hence for every $\epsilon > 0$, there exists an expansive map $H : Z \to A$, where $Z \subseteq Y$ is n-dimensional and $A \in C_\lambda(F)$, such that $\|H\| < 1 + \frac{\epsilon}{\lambda}$.

By definition of $C_\lambda(F)$, there exists a λ-expansive map $J : A \to F$, hence $\|J\|\|J^{-1}\| \leq \lambda$. Thus $JH : Z \to F$ is expansive since both J and H are, and $\|JH\| \leq \lambda + \epsilon$,

$$\|y\| \leq \|JHy\| \leq (\lambda + \epsilon)\|y\|$$

Take $T = (JH)^{-1}$ and note $\|T\|\|T^{-1}\| \leq \lambda + \epsilon$. The case showing X is finitely-representable in Y is the same. $\square$

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