k-FOLD SYSTEMS OF PROJECTIONS AND CONGRUENCE MODULARITY

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Abstract

Bergman showed that systems of projections of algebras in a variety will satisfy a certain property if the variety has a near-unanimity term. The converse of this theorem was left open. This paper investigates this open question, and shows that in a locally finite variety, Bergman's Condition implies congruence modularity.

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1 Introduction

The motivation behind this thesis is an open question posed by George Bergman in a 1977 paper (see [3]). In this paper, he took the celebrated theorem of Baker and Pixley on the existence of near-unanimity terms, and considered a related condition in terms of systems of projections. He showed that if a variety satisfies the equivalent conditions of the Baker-Pixley theorem, any system of projections of algebras in the variety will satisfy a certain condition. His speculation as to whether the converse was true lead to the work in this thesis.

While investigating possible counterexamples to the converse of Bergman's Theorem, we found that Bergman's condition concerning systems of projections ties into the property of congruence modularity. We present one construction that shows that if a variety satisfies Bergman's condition for systems of 2-fold projections over 4 coordinates, the variety will be congruence modular. Then, if we further suppose that our variety is locally finite, we are able to present a different construction that shows the same result for systems of k-fold projections. This gives partial verification of Bergman's original problem, because the existence of a near-unanimity term implies congruence modularity. For both of these results, we assume that our variety is idempotent. In fact, Bergman's condition is a feature of the idempotent reduct of a variety, and so this restriction is not essential.

The necessary background material in universal algebra can be found in [5] or [14], but we will include several basic definitions here.

Definition 1. A *type* of algebras is a set $\mathscr{F}$ of function symbols such that each $f \in \mathscr{F}$ is assigned a nonnegative integer n. We call n the *arity* of f.

Definition 2. An *algebra* $\mathbf{A}$ of type $\mathscr{F}$ is an ordered pair $\langle A, F\rangle$ where A is a nonempty set and F is a family of finitary operations on A (called the *basic operations* of $\mathbf{A}$) indexed by $\mathscr{F}$ such that corresponding to each n-ary function symbol $f \in \mathscr{F}$, there is an n-ary operation $f^{\mathbf{A}}$ on A. The *term operations* of $\mathbf{A}$ are those operations on A that can be obtained via composition from the basic operations of $\mathbf{A}$ and the projection operations on $\mathbf{A}$. A nonempty class of algebras of the same type that is closed under subalgebras, homomorphic images, and direct products is called a *variety.*

Definition 3. Let θ be an equivalence relation on an algebra $\mathbf{A}$. We call θ a *congruence* if, for each n-ary function $f \in F$, $a_i, b_i \in A$, the following holds: If $a_i \theta b_i$ for all $1 \leq i \leq n$, then $f(a_1, \ldots, a_n) \theta f(b_1, \ldots, b_n)$. The set of congruences of an algebra forms a lattice under inclusion and is called the *congruence lattice* of the algebra.

The *relational product* of two congruences θ_1 and θ_2 is the relation

$$\theta_1 \circ \theta_2 = \{(x, y) \mid \exists z((x, z) \in \theta_1 \text{ and } (z, y) \in \theta_2)\}.$$

Definition 4. An algebra $\mathbf{A}$ is *congruence modular* if the congruence lattice of $\mathbf{A}$ is modular, i.e., for any congruences θ_1, θ_2, and θ_3 of $\mathbf{A}$,

$$\theta_1 \subseteq \theta_2 \Rightarrow \theta_1 \vee (\theta_2 \wedge \theta_3) = \theta_2 \wedge (\theta_1 \vee \theta_3).$$

Definition 5. An *identity* of type $\mathscr{F}$ over X, for X a set of variables, is an expression of the form

$$p \approx q$$

for terms p, q over X. An algebra $\mathbf{A}$ of type $\mathscr{F}$ satisfies an identity

$$p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$$

for $x_i \in X$ if for any $a_1, \ldots, a_n \in A$ we have

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = q^{\mathbf{A}}(a_1, \ldots, a_n).$$

Definition 6. (This was first described in [15], along with other intersection properties.) Let $r > 0$ and $\mathbf{A}_i$ be algebras of the same type for $1 \leq i \leq r$. For $k > 0$ and $\mathbf{B}, \mathbf{C} \leq \prod_{i=1}^{r} \mathbf{A}_i$, we say that $\mathbf{B}$ and $\mathbf{C}$ are *k-equal*, and write $\mathbf{B} =_k \mathbf{C}$, if for every $I \subseteq \{1, 2, \ldots, r\}, |I| \leq k$, the projections of $\mathbf{B}$ and $\mathbf{C}$ onto the coordinates I are equal. If $\mathbf{B} =_k \prod_{i=1}^{r} \mathbf{A}_i$, then we say that $\mathbf{B}$ is *k-complete* with respect to $\prod_{i=1}^{r} \mathbf{A}_i$.

We will write $proj_I \mathbf{B}$ to indicate the projection of $\mathbf{B}$ onto the coordinates I.

Definition 7. An operation t on a set A is *idempotent* if, for all $x \in A$, we have $t(x, x, \ldots, x) = x$. An algebra is *idempotent* if all of its operations are. The *idempotent reduct* of an algebra $\mathbf{A}$ is the algebra with universe A whose basic operations consist of all of the idempotent term operations of $\mathbf{A}$. We define the idempotent reduct of a variety $\mathcal{V}$ to be the variety generated by the idempotent reduct of the $\mathcal{V}$-free algebra on countably many generators.

2 The Baker-Pixley Theorem and Bergman's Condition

The work in this thesis was motivated by a result of Bergman, which, in turn, was motivated by the Baker-Pixley theorem mentioned earlier. This result, Theorem 2.1 in [1], is as follows:

Baker-Pixley Theorem. *For a variety V and integer $k \geq 2$, the following conditions are equivalent:*

(i) V has a (k+1)-variable term operation $m(x_0, \ldots, x_k)$ satisfying the "near-unanimity" identities; i.e. $m(x, \ldots, x, y, x, \ldots, x) = x$ for all positions of y.

(ii) In V, if A is a subalgebra of a direct product $A_1 \times \cdots \times A_r$, $k \leq r < \infty$, then A is uniquely determined by its images under the projections of $A_1 \times \cdots \times A_r$ on all products $\prod_I A_i$ with $I \subseteq \{1, \ldots, r\}, |I| = k$.

(iii) In any algebra $A \in V$, if r congruences $x \equiv a_i \mod \theta_i, 1 \leq i \leq r(k \leq r)$, are solvable k at a time, then they are solvable simultaneously.

(iv) For any algebra $A \in V$, integer $n \geq 1$, and finite partial function $f : A^n \to A$, if the restriction of f to each subset of its domain with k or fewer elements has an interpolating term operation, then so does f itself.

(v) f, as given in (iv), has an interpolating term operation if and only if all subalgebras of A^k are closed under f (where defined).

Let $\mathcal{V}$ be a variety, $k \leq r$ positive integers, and $\mathbf{A}_1, \ldots, \mathbf{A}_r$ be algebras in $\mathcal{V}$. For each $I \subseteq \{1, \ldots, r\}$ with $|I| = k$, suppose we are given a k-fold projection $S_I \leq \prod_I \mathbf{A}_i$. For every J with $|J| \geq k$, let $S_J \leq \prod_J \mathbf{A}_j$ be the intersection, over all $I \subseteq J, |I| = k$, of the inverse image of S_I under the natural map $\prod_J \mathbf{A}_j \to \prod_I \mathbf{A}_i$.

Definition 8. We call the given system of subalgebras $(S_I)_{|I|=k}$ *consistent* on J if for every k-element subset $I \subseteq J$, the projection of S_J in $\prod_I \mathbf{A}_i$ is all of S_I. This means that for every $I \subseteq J$, each k-tuple in S_I can be extended to a $|J|$-tuple in $\prod_J \mathbf{A}_j$, each sub-k-tuple of which belongs to the appropriate subalgebra $S_{I'}, I' \subseteq J$.

We write $C(e, f)$ if $(S_J)_{|J|=e}$ is consistent on all K of cardinality f, i.e., if each e-tuple can be extended to a consistent f-tuple.

Note that consistency in this sense is transitive: if $C(d, e)$ and $C(e, f)$ hold, then $C(d, f)$ holds.

Bergman's Theorem (Theorem 1 in [3]). *Let V be a variety, and k a positive integer, satisfying the equivalent conditions of the Baker-Pixley Theorem. For $r \geq k$, let $A_1, \ldots, A_r \in V$, and for every subset $I \subseteq \{1, \ldots, r\}, |I| = k$, let S_I be a subalgebra of $\prod_I S_i$. Then there exists a subalgebra $S \subseteq A_1 \times \cdots \times A_r$ whose projection in each $\prod_I S_i, |I| = k$ is S_I (i.e., the given system is consistent on $\{1, \ldots, r\}$) if and only if the given system $(S_I)_{|I|=k}$ is consistent on every J with $|J| = k + 1$.*

Bergman's Condition. In other words, this theorem states that if a variety $\mathcal{V}$ satisfies the equivalent conditions of the Baker-Pixley Theorem, then for any r and any system of k-fold projections over r members of $\mathcal{V}$, the system will satisfy $C(k, r)$ for all $r \geq k$ if and only if it satisfies $C(k, k+1)$, i.e.,

$$C(k, r) \Leftrightarrow C(k, k+1).$$

Now, if $C(k, r)$ holds, then certainly $C(k, k+1)$ will hold as well. To prove the other direction, that $C(k, k+1) \Rightarrow C(k, r)$, Bergman showed that in a variety satisfying the right conditions, $C(e-1, e) \Rightarrow C(e, e+1)$ for all $e > k$. Using the transitivity property from above, this gives $C(k, k+1) \Rightarrow C(k, r)$. Accordingly, we will call

$$C(k, k+1) \Rightarrow C(k, r) \text{ for all } r > k$$

Bergman's Condition for k. Now Bergman's Theorem above states that any variety satisfying the equivalent conditions of the Baker-Pixley Theorem will also satisfy Bergman's Condition for k. Bergman poses the question of whether the converse of the theorem above is true, i.e., whether Bergman's Condition for k implies the existence of a $(k+1)$-ary near-unanimity term.

3 Congruence Modular Varieties

Our key result relates Bergman's Condition to congruence modularity. In order to prove it, we will need several known results about congruence modular varieties. First, we need the following theorem, 2.2 in [9]. This result first appeared in Day's McMaster Masters thesis and was also published in [7].

Theorem 1. *A variety V is congruence modular if and only if for some n there are terms $m_0(x,y,z,u),\ldots,m_n(x,y,z,u)$ such that $\mathcal{V}$ satisfies*

(i) $m_0(x,y,z,u) \approx x, m_n(x,y,z,u) \approx u$

(ii) $m_i(x,y,y,x) \approx x, i \leq n$

(iii) $m_i(x,x,y,y) \approx m_{i+1}(x,x,y,y),$ *for all even* $i < n$

(iv) $m_i(x,y,y,z) \approx m_{i+1}(x,y,y,z),$ *for all odd* $i < n$.

Terms satisfying these requirements are called ***Day terms.***

These equations imply that Day terms are idempotent. Hence, congruence modularity is a feature of the idempotent reduct of a variety.

Consider the following condition on an algebra $\mathbf{A}$ with $\alpha, \beta, \gamma \in Con(\mathbf{A})$:

$$(*) \qquad \begin{array}{c} \text{Let } a,b,c,d \in \mathbf{A}, (a,b),(c,d) \in \beta, \\ (a,c),(b,d) \in \gamma, \text{ and } \gamma \wedge \beta \subseteq \alpha. \\ \text{Then } (a,b) \in \alpha \Rightarrow (c,d) \in \alpha. \text{ , i.e.,} \end{array}$$

$$\begin{tikzcd} a \arrow[r, dash, "\gamma"] \arrow[d, dash, "\beta"'] \arrow[d, bend left, dash, "\alpha"] & c \arrow[d, dash] \\ b \arrow[r, dash] & d \end{tikzcd}$$

implies

$$\begin{tikzcd} a \arrow[r, dash, "\gamma"] \arrow[d, dash, "\beta"'] \arrow[d, bend left, dash, "\alpha"] & c \arrow[d, dash] \arrow[d, bend left, dash] \\ b \arrow[r, dash] & d \end{tikzcd}$$

(where parallel lines are assumed to have the same label).

The Shifting Lemma (2.4 in [9]). *For a variety, condition (*) is equivalent to congruence modularity.*

Proof. Sufficiency. Suppose **A** is congruence modular. Then

$$\beta \wedge (\gamma \vee (\beta \wedge \alpha)) = (\beta \wedge \gamma) \vee (\beta \wedge \alpha).$$

Since $c\gamma a(\beta \wedge \alpha)b\gamma d$ and $\beta \wedge \alpha \subseteq \alpha$,

$$\begin{aligned} &(c, d) \in \beta \wedge (\gamma \vee (\beta \wedge \alpha)) \\ \Rightarrow &(c, d) \in (\beta \wedge \gamma) \vee (\beta \wedge \alpha) \subseteq \alpha. \end{aligned}$$

Necessity. Suppose (*) holds in a variety $\mathcal{V}$. Let $\mathbf{F}_{\mathcal{V}}(x, y, z, u)$ be the free $\mathcal{V}$-algebra generated by $\{x, y, z, u\}$ and let

$$\begin{aligned} \beta' &= Cg(x, u) \vee Cg(y, z), \\ \gamma' &= Cg(x, y) \vee Cg(z, u), \\ \alpha' &= Cg(y, z). \end{aligned}$$

By (*), $(x, u) \in \alpha'$, so $(x, u) \in \alpha' \vee (\beta' \wedge \gamma')$. Now we show that our variety has Day terms $m_0(x, y, z, u), \ldots, m_n(x, y, z, u)$.

The fact that (x, u) is in $\alpha' \vee (\beta' \wedge \gamma')$ implies that, for some n, there are elements

$$w_o = x, w_1, \ldots, w_n = u \in \mathbf{F}_{\mathcal{V}}(x, y, z, u)$$

such that $w_i(\beta' \wedge \gamma')w_{i+1}$ if i is even and $w_i(\alpha')w_{i+1}$ if i is odd. Let $x = m_0(x, y, z, u), m_1(x, y, z, u), \ldots, m_n(x, y, z, u) = u$ be the terms representing $w_0, w_1, \ldots, w_n$, i.e. $w_i = m_i^{\mathbf{F}}(x, y, z, u)$. Clearly, condition (i) above is satisfied. Now, since $\alpha' \subseteq \beta'$, all the w_i's are in the same β' class. So we have $x\beta' m_i^{\mathbf{F}}(x, y, z, u)\beta' m_i^{\mathbf{F}}(x, y, y, x)$, and since β' restricted to the subalgebra generated by x and y is trivial, $x = m_i^{\mathbf{F}}(x, y, y, x)$. Hence, $x \approx m_i(x, y, y, x)$ holds in $\mathcal{V}$, and so (ii) is satisfied. Similarly, (iii) and (iv) hold in $\mathcal{V}$. □

4 Bergman's Condition and The Idempotent Reduct of a Variety

We will establish that Bergman's Condition for k is a feature of the idempotent reduct of a variety. The following theorem is an extension of

Bergman's observation in the final section of [3], which involved the case where $k = 2$.

Theorem 2. *Let $\hat{\mathcal{V}}$ be the idempotent reduct of a variety $\mathcal{V}$. Then $\hat{\mathcal{V}}$ satisfies Bergman's Condition for k (i.e. $C(k, k+1) \Rightarrow C(k, r)$ for all $r > k$) if and only if $\mathcal{V}$ does.*

Proof. One direction is immediate: if $\hat{\mathcal{V}}$ satisfies Bergman's Condition for k, then $\mathcal{V}$ will as well, because any system of k-fold projections in $\mathcal{V}$ can be realized by one in $\hat{\mathcal{V}}$.

Suppose $\mathcal{V}$ satisfies Bergman's Condition for k and let $\hat{A}_i \in \hat{\mathcal{V}}$ for $i \in \{1, \ldots, r\}$. Let $\hat{S}_I, I \subset \{1, \ldots, r\}, |I| = k$ be a system of projections over the $\hat{A}_i$'s that satisfies $C(k, k+1)$. We will show that it satisfies $C(k, r)$.

Let $F_i = F_{\mathcal{V}}(\hat{A}_i)$, the free algebra in $\mathcal{V}$ generated by $\hat{A}_i$, and let $S_I = Sg_{\prod_{i \in I} F_i}(\hat{S}_I)$, the subalgebra of $\prod_{i \in I} F_i$ generated by $\hat{S}_I$.

First, we will show that S_I satisfies $C(k, k+1)$:

Let $\vec{v} \in S_I$ for $I = \{1, \ldots, k\}$. Then there exist a term t and vectors $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_m \in \hat{S}_I$ such that $\vec{v} = t(\vec{s}_1, \ldots, \vec{s}_m)$. Since $\hat{S}_I$ satisfies $C(k, k+1)$, each of these $\vec{s}_j$'s can be extended consistently to any other coordinate, in particular, to $k+1$. For each j, let $\vec{s}_j{}' \in \hat{S}_{\{1, \ldots, k+1\}}$ be an extension of $\vec{s}_j$ that is consistent with our system and let $\vec{v}\,' = t(\vec{s}_1{}', \ldots, \vec{s}_m{}')$. Then $\vec{v}\,'$ extends $\vec{v}$ appropriately, i.e., $\vec{v}\,' \in S_J$, for $J = \{1, 2, \ldots, k+1\}$. By symmetry, for any $I \subset J \subset \{1, \ldots, r\}, |I| = k, |J| = k+1$, we can extend $\vec{v} \in S_I$ to a $\vec{v}\,' \in S_J$. Hence, S_I satisfies $C(k, k+1)$. By assumption, S_I also satisfies $C(k, r)$.

Let $\vec{x} \in \hat{S}_I, I = \{1, \ldots, k\}$, and let $\vec{x}\,' \in S_{\{1, \ldots, r\}}$ extend $\vec{x}$ to an r-tuple. For all $J \subset \{1, \ldots, r\}, |J| = k$, let $\vec{x}_J{}'$ be the projection of $\vec{x}\,'$ onto the coordinates J. For each such J, there exists, for some $m = m_J$, $\vec{s}_J{}^1, \ldots, \vec{s}_J{}^m \in \hat{S}_J$ and a term t_J such that

$$\vec{x}_J{}' = t_J(\vec{s}_J{}^1, \ldots, \vec{s}_J{}^m).$$

Note that for $J = I, \vec{x}_J{}' = \vec{x}$, and we can take t_J to be the term $\vec{x}$.

For all $J, K \subset \{1, \ldots, r\}, |J|, |K| = k$, if $j \in J \cap K$, we have

$$\begin{aligned} t_J(\vec{s}_J{}^1, \ldots, \vec{s}_J{}^m)(j) &= \vec{x}\,'(j) \\ &= t_K(\vec{s}_K{}^1, \ldots, \vec{s}_K{}^n)(j), \end{aligned}$$

where $m = m_J$ and $n = m_K$. We will call this condition $*_{j,J,K}$.

If t_J and t_K are idempotent, then this condition will also hold in $\hat{\mathcal{V}}$, the idempotent reduct of $\mathcal{V}$. Then, we can use these terms to extend any k-tuple in $\hat{S}_I$ to a consistent r-tuple, and we'll be done. Now, if $j \in J$ and $j \in \{1, \ldots, k\}$, then $*_{j,J,\{1,\ldots,k\}}$ gives

$$t_J(\vec{s}_J{}^1, \ldots, \vec{s}_J{}^m)(j) = \vec{x}(j).$$

Since the $\vec{s}_J{}^i(j)$'s are free variables, this gives that

$$t_J(x, x, \ldots, x) = x$$

in $\mathcal{V}$, and so t_J is idempotent. Now, for an arbitrary J, we can find a K with $K \cap \{1, 2, \ldots, k\}$ nonempty and some $j \in J \cap K$. Since t_K is idempotent and $*_{j,J,K}$ gives

$$\begin{aligned} t_J(\vec{s}_J{}^1, \ldots, \vec{s}_J{}^m)(j) &= t_K(\vec{s}_K{}^1, \ldots, \vec{s}_K{}^n)(j) \\ \Rightarrow t_J(x, \ldots, x) &= t_K(x, \ldots, x) \\ &= x \end{aligned}$$

then t_J is idempotent as well. □

5 Results

We will first describe a result concerning Bergman's Condition for the case $k = 2$, and then use a different construction to obtain a more general result.

Lemma 1. *In any idempotent variety $\mathcal{V}$ that is not congruence modular, there is an algebra $\mathbf{A} = \mathbf{B} \times \mathbf{C}$, where $\mathbf{B}$ is generated by elements 0 and $1 \in \mathbf{B}$, and $\mathbf{C}$ is generated by elements 0 and $1 \in \mathbf{C}$, with congruences α, β, and γ satisfying*

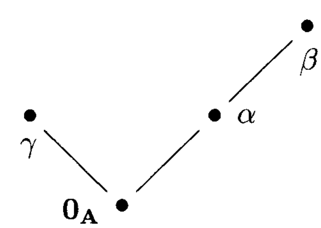
(i) β and γ are the projection kernels of $\mathbf{A}$ onto $\mathbf{B}$ and $\mathbf{C}$, respectively

(ii) $\alpha = Cg_{\mathbf{A}}((a, b)), \beta = Cg_{\mathbf{A}}((a, b), (c, d))$, and $\gamma = Cg_{\mathbf{A}}((a, c), (b, d))$, where $a = (0, 0), b = (0, 1), c = (1, 0)$, and $d = (1, 1)$

(iii) $\gamma \circ \beta = \mathbf{1_A}$

(iv) $((1,0),(1,1)) \notin \alpha$ *and* $((0,i),(0,j)) \in \alpha$ *for all* $i,j \in C$.

Proof. Suppose that $\mathcal{V}$ is not congruence modular. By the Shifting Lemma, for some $\mathbf{A} \in \mathcal{V}$ we can find $a,b,c,d \in \mathbf{A}$ and $\alpha,\beta,\gamma \in Con(\mathbf{A})$ such that $(a,b),(c,d) \in \beta, (a,c),(b,d) \in \gamma$, $\gamma \wedge \beta \subseteq \alpha, (a,b) \in \alpha$ and $(c,d) \notin \alpha$. Since $(a,b) \in \alpha \wedge \beta$ and $(c,d) \notin \alpha \wedge \beta$, we can assume $\alpha \subset \beta$. Furthermore, we can assume that $\gamma \wedge \beta = \mathbf{0_A}$ by taking a suitable quotient of $\mathbf{A}$. So we have:



We may replace $\mathbf{A}$ by $Sg_{\mathbf{A}}(\{a,b,c,d\})$, and α,β, and γ by $Cg_{\mathbf{A}}((a,b))$, $Cg_{\mathbf{A}}((a,b),(c,d))$, and $Cg_{\mathbf{A}}((a,c),(b,d))$, respectively, since we will have the same configuration. Because $(c,d) \in \gamma \vee \alpha$, we must have $\beta \subseteq \gamma \vee \alpha$. Hence, $\gamma \vee \beta = \gamma \vee \alpha$. Now, since $a/(\gamma \vee \beta)$ contains $\{a,b,c,d\}$ (and, in fact, a,b,c, and d are all $\gamma \circ \beta$-related to each other), $a/(\gamma \vee \beta) = \mathbf{A}$ (since $\mathbf{A}$ is idempotent). Therefore, $\beta \vee \gamma = \mathbf{1_A}$. So in an idempotent variety, we have an algebra $\mathbf{A}$ generated by a,b,c,d with congruences α,β, and γ as pictured in Figure 1.

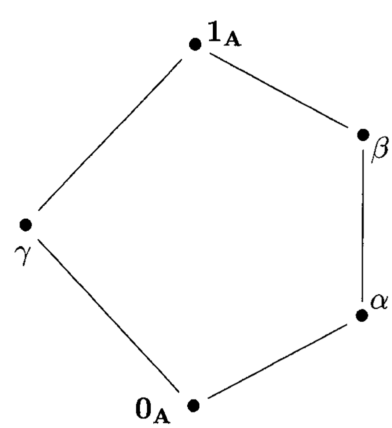


Figure 1: α,β, and γ

Claim 1. $\gamma \circ \beta = \mathbf{1_A}$.

Proof. Let $x, y \in \mathbf{A}$. Then $x = t_1(a,b,c,d)$ and $y = t_2(a,b,c,d)$ for some terms t_1, t_2 of $\mathbf{A}$. It was noted that a, b, c, and d are all $(\gamma \circ \beta)$-related to a, and so, $x = t_1(a,b,c,d)(\gamma \circ \beta)t_1(a,a,a,a) = a$. By symmetry, x is also $(\gamma \circ \beta)$-related to b, c, and d. Hence, $x = t_2(x,x,x,x)(\gamma \circ \beta)t_2(a,b,c,d) = y$. Hence, $\gamma \circ \beta = \mathbf{1_A}$. □

It follows that $\beta \circ \gamma = \mathbf{1_A}$, and so β and γ are a pair of factor congruences. Hence, by Theorem 7.5 in [5], $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$, for $\mathbf{B} = \mathbf{A}/\beta, \mathbf{C} = \mathbf{A}/\gamma$. We can assume that $\mathbf{A} = \mathbf{B} \times \mathbf{C}$.

If we consider the projection of $\mathbf{B} \times \mathbf{C}$ onto $\mathbf{B}$ or $\mathbf{C}$, then, viewing β and γ as kernels of the corresponding projection maps, we obtain

$$\begin{aligned} \beta &= \{(a,b) \in A^2 \mid \pi_1(a) = \pi_1(b)\} \\ &= \{((q,r),(s,t)) \in (B \times C)^2 \mid q = s\} \\ &= \{((u,v),(u,w)) \mid u \in B \text{ and } v, w \in C\}. \end{aligned}$$

Similarly, $\gamma = \{((u,v),(x,v)) \mid u, x \in B \text{ and } v \in C\}$. We have assumed that $A = B \times C$ for sets B and C, and so we can represent a, b, c, and d as ordered pairs.

Taking $B = \mu$ and $C = \nu$ for (possibly infinite) cardinals μ, ν, we may assume that $a = (0,0), b = (0,1), c = (1,0)$, and $d = (1,1)$. Since $\mathbf{A}$ is generated by a, b, c, d, if we consider its projection onto its first or second coordinates, we must have that $\mathbf{B}$ is generated by $\{0,1\}$ and $\mathbf{C}$ is generated by $\{0,1\}$. It follows that, since $(a,b) = ((0,0),(0,1)) \in \alpha$, $(0,0)/\alpha = (0,0)/\beta = \{(0,j) \mid j \in C\}$.

We also have that $\gamma \circ \alpha \circ \gamma = \mathbf{1_A}$: For any $(x_1,x_2),(y_1,y_2) \in \mathbf{A}$,

$$(x_1,x_2)\gamma(0,x_2)\alpha(0,y_2)\gamma(y_1,y_2).$$

Also note that $((1,0),(1,1)) \notin \alpha \circ \gamma \circ \alpha$: If $((1,0),(1,1)) \in \alpha \circ \gamma \circ \alpha$, then there exist $(x,y),(z,u) \in A$ such that

$$(1,0)\alpha(x,y)\gamma(z,u)\alpha(1,1).$$

This gives $y = u$ and $x = z = 1$. Hence, we have

$$(1,0)\alpha(1,u)\gamma(1,u)\alpha(1,1),$$

which implies $(1,1)$ and $(1,0)$ are in the same α-class as $(1,u)$, and, thus, α-related to each other. This is a contradiction. $\square$

To obtain the following result, we used java programs developed by Barry Dewitt during his summer research term with Dr. Matthew Valeriote at McMaster University. His software enabled us to find various counterexamples for Bergman's Condition for k in specific algebras that are not congruence modular, which we were then able to extend to a more general setting.

Theorem 3. *Le $\mathcal{V}$ be a variety. If Bergman's Condition holds for $k = 2$ (in particular, if $C(2,3) \Rightarrow C(2,4)$), then $\mathcal{V}$ is congruence modular.*

Proof. We will prove the contrapositive. Suppose $\mathcal{V}$ is not congruence modular. By Theorems 1 and 2, we may assume that $\mathcal{V}$ is idempotent. Then, we have the situation presented in the previous lemma. Since sets defined by primitive positive formulas using α, β, and γ are subalgebras of subpowers of $\mathbf{A}$, we can use them to build a system of projections $(S_I)_{|I|=2}$ over $\mathbf{A}^4$ that is consistent on every 3-element subset of $\{1,2,3,4\}$, but not consistent over all four elements.

Let $S_{\{1,2\}} = S_{\{3,4\}} = \beta, S_{\{1,3\}} = \alpha \circ \gamma, S_{\{1,4\}} = \gamma \circ \alpha, S_{\{2,3\}} = \mathbf{1_A}$, and $S_{\{2,4\}} = \gamma$.

First we will show that if $((a_1,b_1),(a_2,b_2)) \in S_{\{1,2\}} = \beta$, then there exist (a_3,b_3) and (a_4,b_4) such that all of the 2-fold projections of $((a_1,b_1),(a_2,b_2),(a_3,b_3))$ and $((a_1,b_1),(a_2,b_2),(a_4,b_4))$ belong to the appropriate subsystem. Because elements that are β-related must have the same first coordinate, $a_1 = a_2$.

We can take $(a_3,b_3) = (a_1,b_1)$ and $(a_4,b_4) = (0,b_2)$. Then $((a_1,b_1),(a_1,b_1)) \in S_{\{1,3\}} = \alpha \circ \gamma$ and $((a_2,b_2),(a_1,b_1)) \in S_{\{2,3\}} = \mathbf{1_A}$. $(a_1,b_1)\gamma(0,b_1)\alpha(0,b_2)$, so $((a_1,b_1),(0,b_2)) \in \gamma \circ \alpha = S_{\{1,4\}}$.

Also, $((a_2,b_2),(a_4,b_4)) = ((a_1,b_2),(0,b_2)) \in \gamma = S_{\{2,4\}}$ because elements with the same second coordinate are γ-related.

Next, if we are given $((a_2,b_2),(a_3,b_3)) \in S_{\{2,3\}} = \mathbf{1_A}$, we can extend to elements (a_1,b_1) and (a_4,b_4). Take $(a_1,b_1) = (a_2,b_3)$ and $(a_4,b_4) = (a_3,b_2)$. Then

$$\begin{aligned}
((a_2,b_3),(a_2,b_2)) &\in \beta = S_{\{1,2\}}, \\
((a_2,b_3),(a_3,b_3)) &\in \gamma \subseteq \alpha \circ \gamma = S_{\{1,3\}}, \\
((a_2,b_2),(a_3,b_2)) &\in \gamma = S_{\{2,4\}}, \text{ and} \\
((a_3,b_3),(a_3,b_2)) &\in \beta = S_{\{3,4\}}.
\end{aligned}$$

Continuing in this fashion, given

- $((a_1,b_1),(a_3,b_3)) \in S_{\{1,3\}}$, take $(a_2,b_2)=(a_1,b_1)$ and $(a_4,b_4)=(a_3,b_1)$;
- $((a_1,b_1),(a_4,b_4)) \in S_{\{1,4\}}$, take $(a_2,b_2)=(a_1,b_4)$ and $(a_3,b_3)=(a_4,b_1)$;
- $((a_2,b_2),(a_4,b_4)) \in S_{\{2,4\}}$, take $(a_1,b_1)=(a_2,b_4)$ and $(a_3,b_3)=(a_4,b_4)$;
- $((a_3,b_3),(a_4,b_4)) \in S_{\{3,4\}}$, take $(a_1,b_1)=(0,b_4)$ and $(a_2,b_2)=(a_4,b_4)$.

Next, we show that $(S_I)_{|I|=2}$ is not consistent on $\{1,2,3,4\}$. Consider the element $((1,0),(1,1)) \in S_{\{2,3\}} = \mathbf{1_A}$. We'll show that there aren't elements $(x_1,x_2),(y_1,y_2) \in \mathbf{A}$ such that $((x_1,x_2),(1,0),(1,1),(y_1,y_2))$ projects correctly. If such elements exist, then

$$
\begin{aligned}
((x_1,x_2),(1,0)) &\in S_{\{1,2\}} = \beta \\
&\Rightarrow x_1 = 1 \text{ and} \\
((1,x_2),(1,1)) &\in S_{\{1,3\}} = \alpha \circ \gamma \\
&\Rightarrow (1,x_2)\alpha(1,z_2)\gamma(1,1) \text{ for some } z_2 \in C \\
&\Rightarrow (1,x_2)\alpha(1,1).
\end{aligned}
$$

Next,

$$
\begin{aligned}
((1,0),(y_1,y_2)) &\in S_{\{2,4\}} = \gamma \\
&\Rightarrow y_2 = 0 \text{ and} \\
((1,1),(y_1,0)) &\in S_{\{3,4\}} = \beta \\
&\Rightarrow y_1 = 1.
\end{aligned}
$$

So we must have

$$
\begin{aligned}
((1,x_2),(1,0)) &\in S_{\{1,4\}} = \gamma \circ \alpha, \text{ where } x_2 \neq 0 \\
&\Rightarrow (1,x_2)\gamma(z_1,x_2)\alpha(1,0) \text{ for some } z_1 \in B \\
&\Rightarrow z_1 = 1, \text{ so } (1,x_2)\alpha(1,0) \\
&\Rightarrow (1,1)\alpha(1,0),
\end{aligned}
$$

a contradiction. Hence, $(S_I)_{|I|=2}$ satisfies $C(2,3)$ but not $C(2,4)$. □

Using a different construction, we are able to generalize this result for any natural number $k > 1$, but only under a stronger hypothesis. In an idempotent variety, we know we can find an algebra $\mathbf{A}$ and congruences α, β, and γ satisfying the properties listed in Lemma 1. For $\mathcal{V}$ locally finite, we can take this one step further and assume that the α-classes are "uniform," in the following sense. Recalling that $A = B \times C$, we can partition B into sets P and Q such that for all $p \in P, (p,c)\alpha(p,d)$ for all $c, d \in C$, and for all $q \in Q$, there are c and d in C with $((q,c),(q,d)) \notin \alpha$. By the properties listed in Lemma 1, we know that both P and Q are nonempty (since $0 \in P$ and $1 \in Q$).

Lemma 2. *Assume $\mathcal{V}$ is locally finite. Then we may assume that for all $q_1, q_2 \in Q$, and $c_1, c_2 \in C$,*

$$(q_1, c_1)\alpha(q_1, c_2) \Leftrightarrow (q_2, c_1)\alpha(q_2, c_2). \tag{1}$$

Pictorially, we have a situation as in Figure 2.

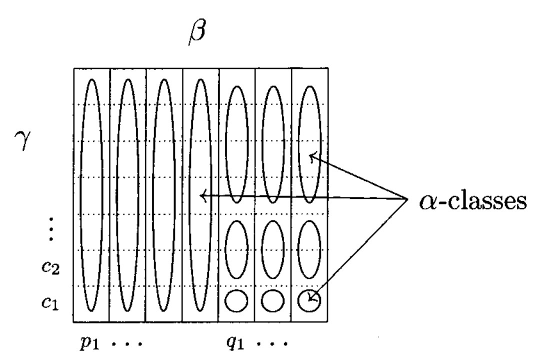


Figure 2: $(P \cup Q) \times C$

Here, each box represents a different pair in $B \times C$, where $B = P \cup Q$, and $p_i \in P, q_i \in Q$, and $c_i \in C$ for all i. The vertical partitions are β-classes, the horizontal partitions are γ-classes, and the α-classes are subsets of the β-classes.

Proof. Now, we obviously have at least one element, $0 \in B$, in our set P. We need to show that we can reduce to a situation in which Q satisfies (1), such that $B = P \cup Q$.

Suppose we have elements $q_1, q_2 \in Q$ and $c_1, c_2 \in C$ such that $(q_1, c_1)\alpha(q_1, c_2)$ and $((q_2, c_1), (q_2, c_2)) \notin \alpha$. Consider $A' = \{(a,b) \mid (a,b)(\alpha \circ \gamma)(q_1, c_1)\}$.

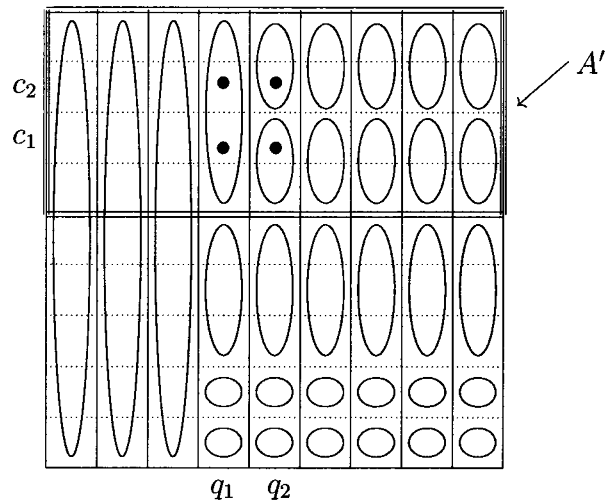


Figure 3: Reducing to a uniform α

Since **A** is idempotent, A' is a subuniverse of **A**, and the restrictions of α, β, and γ to $\mathbf{A}'$ satisfy the same properties that were assumed to hold in **A**.

Furthermore, our new algebra is the direct product of two algebras **B** and $\mathbf{C}'$, and if we consider P and Q defined relative to $\mathbf{A}'$, we now have $q_1 \in P$, and at least one element (namely, q_2) in Q. We can repeat this process as necessary until we have an algebra **A** with underlying set $A = (P \cup Q) \times C$ and with the desired uniformity for α. □

Note that this is the only point in our argument where we use the local finiteness of $\mathcal{V}$. It is likely that in the infinite case, we will still be able to establish this sort of uniformity for α. If so, our result would extend to non-locally finite varieties, and Theorems 3 and 4 would apply to all varieties.

Since $(q_1, c)\alpha(q_1, d)$ iff $(q_2, c)\alpha(q_2, d)$ for all $q_1, q_2 \in Q$, we will say that $c(\alpha_Q)d$, or $(c, d) \in \alpha_Q$, if $(q, c)\alpha(q, d)$ for any $q \in Q$.

In obtaining the following result, the Universal Algebra Calculator computer software ([8]) was used in order to test certain hypotheses for small values of k.

Theorem 4. *Let $\mathcal{V}$ be a locally finite variety. If Bergman's Condition holds for some $k > 1$, then $\mathcal{V}$ is congruence modular.*

Proof. By Theorems 1 and 2, we may assume that $\mathcal{V}$ is idempotent. Suppose that $\mathcal{V}$ is not congruence modular. As established before, we have an algebra $\mathbf{A}$, with $A = (P \cup Q) \times C$, satisfying (1) above.

Define $T \subseteq \mathbf{A}^{k+1}$ to be the following union:

$$\begin{array}{rll} & \{((x_1, y_1), \dots, (x_{k+1}, y_{k+1})) \mid & x_i \in P \text{ for some } i, 1 < i < k+1\} \\ \cup & \{((x_1, y_1), \dots, (x_{k+1}, y_{k+1})) \mid & x_i \in Q \text{ for all } i, 1 < i < k+1, \\ & & (y_1, y_{k+1}) \in \alpha_Q, \\ & & \text{and } x_1 \text{ or } x_{k+1} \in P\} \\ \cup & \{((x_1, y_1), \dots, (x_{k+1}, y_{k+1})) \mid & x_i \in Q \text{ for all } i \\ & & \text{and } (y_i, y_j) \in \alpha_Q \text{ for all } i, j\} \end{array}$$

We will refer to the first set above as X, the second as Y, and the third as Z.

Claim 2. *T is a subuniverse of* $\mathbf{A}^{k+1}$.

Proof. It will suffice to show that T is closed under any operation that preserves α, β, and γ, since α, β, and γ are congruences of $\mathbf{A}$. Let f be an n-ary operation that preserves α, β, and γ. Since $\mathbf{A} = \mathbf{B} \times \mathbf{C}$, we can consider f as a pair (f_1, f_2), where

$$f((a_1, b_1), \dots, (a_n, b_n)) = (f_1((a_1, b_1), \dots, (a_n, b_n)), f_2((a_1, b_1), \dots, (a_n, b_n)))$$

for $(a_i, b_i) \in \mathbf{A}$.

We can translate the preservation of α, β, and γ into conditions placed on f_1 and f_2. Because $(a_i, b_i)\beta(a_i, c_i)$ for all $a_i \in B, b_i, c_i \in C$, f preserves β

$$\begin{array}{rl} \Leftrightarrow & f((a_1, b_1), \dots, (a_n, b_n))\beta f((a_1, c_1), \dots (a_n, c_n)) \\ \Leftrightarrow & (f_1((a_1, b_1), \dots, (a_n, b_n)), f_2((a_1, b_1), \dots, (a_n, b_n))) \\ & \quad \beta(f_1((a_1, c_1), \dots, (a_n, c_n)), f_2((a_1, c_1), \dots, (a_n, c_n))) \\ \Leftrightarrow & f_1((a_1, b_1), \dots, (a_n, b_n)) = f_1((a_1, c_1), \dots, (a_n, c_n)). \end{array}$$

Hence, f_1 only depends on the first coordinates of a given tuple, so we can simply write $f_1(a_1, \dots, a_n)$ for $a_i \in B$. Similarly, preserving γ implies that f_2 only depends on the second coordinates. We can conclude that

$$f((a_1, b_1), \ldots, (a_n, b_n)) = (f_1(a_1, \ldots, a_n), f_2(b_1, \ldots, b_n)).$$

Next, we see what it means for f to preserve α. Now, if $(a_i, b_i)\alpha(a_i, b_i')$, then we have

$$f((a_1, b_1), \ldots, (a_n, b_n))\alpha f((a_1, b_1'), \ldots, (a_n, b_n')).$$

If $f_1(a_1, \ldots, a_n) = q$ for some $q \in Q$, then, if we have $(b_i, b_i') \in \alpha_Q$ for all $a_i \in Q$,

$$f_2(b_1, \ldots, b_n)(\alpha_Q)f_2(b_1', \ldots, b_n')$$

(i.e., if $a_i \in P, b_i$ and b_i' can be anything). Hence, if $f_1(a_1, \ldots, a_n) \in Q$, f_2 does not depend (modulo α_Q) on its variables y_i with $a_i \in P$.

For $1 \leq i \leq n$, let t_i be a member of T. We would like to show that $f(t_1, \ldots, t_n) \in T$. Without loss of generality, we may assume that there are $l < m \leq n$ such that $t_i \in X$ if $1 \leq i \leq l$, $t_i \in Y$ if $l < i \leq m$ and $t_i \in Z$ if $m < i$.

For each i, we write t_i as $((x_1^i, y_1^i), \ldots, (x_{k+1}^i, y_{k+1}^i))$. If $f_1(x_i^1, x_i^2, \ldots, x_i^n) \in P$ for some i with $1 < i < k+1$, then

$$f(t_1, t_2, \ldots, t_n) \in X \subseteq T.$$

Suppose $f_1(x_i^1, x_i^2, \ldots, x_i^n) \in Q$ for all i with $1 < i < k+1$. Since $t_i \in X$ for $1 \leq i \leq l$, we have that, for each i, there exists a $j_i, 1 < j_i < k+1$ such that $x_{j_i}^i \in P$. So, for some $p_1, p_2, \ldots, p_l \in P$, we have

$$\begin{aligned}
f_1(p_1, x_{j_1}^2, x_{j_1}^3, \ldots, x_{j_1}^n) &\in Q, \\
f_1(x_{j_2}^1, p_2, x_{j_2}^2, \ldots, x_{j_2}^n) &\in Q, \\
f_1(x_{j_3}^1, x_{j_3}^2, p_3, \ldots, x_{j_3}^n) &\in Q, \\
&\vdots \\
f_1(\ldots, x_{j_l}^{l-1}, p_l, x_{j_l}^{l+1}, \ldots) &\in Q.
\end{aligned}$$

Hence, f_2 does not depend on its first l coordinates, modulo α_Q, and so

$(f_2(y_1^1, y_1^2, \ldots, y_1^n), f_2(y_{k+1}^1, y_{k+1}^2, \ldots, y_{k+1}^n)) \in \alpha_Q$.

If $f_1(x_1^1, \ldots, x_1^n)$ or $f_1(x_{k+1}^1, \ldots, x_{k+1}^n) \in P$, then our resulting $k+1$-tuple is in Y and we are done. If both of these elements are in Q, then we will show our resulting $k+1$-tuple must be in Z. To do this, we only need to show that $(f_2(y_i^1, \ldots, y_i^n), f_2(y_j^1, \ldots, y_j^n)) \in \alpha_Q$ for all i, j.

¿From above, we know that, modulo α_Q, f_2 does not depend on its first l coordinates. Next, we will show that f_2 also does not depend on any of its coordinates, modulo α_Q, between $l+1$ and m, inclusive:

Since for each $i, l+1 \leq i \leq m$, $t_i \in Y$, we have x_1^i or $x_{k+1}^i \in P$. Also, both $f_1(x_1^1, \ldots, x_1^n)$ and $f_1(x_{k+1}^1, \ldots, x_{k+1}^n) \in Q$. Hence, because f preserves α, f_2 does not depend on its first m coordinates, modulo α_Q.

Now, for $m+1 \leq i \leq n$, $(y_{j_1}^i, y_{j_2}^i) \in \alpha_Q$ for all j_1, j_2. Hence, we have that $(f_2(y_i^1, \ldots, y_i^n), f_2(y_j^1, \ldots, y_j^n)) \in \alpha_Q$ for all i, j, and so our element must lie in Z. This proves our first claim.

Claim 3. $\mathbf{T}$ *is k-complete with respect to* $\mathbf{A}^{k+1}$.

Proof. We need to show that $\mathbf{T} =_k \mathbf{A}^{k+1}$. Since $\mathbf{T} \leq \mathbf{A}^{k+1}$, for every $I \subseteq \{1, 2, \ldots, k+1\}$ with $|I| \leq k$, the projection of any $t \in \mathbf{T}$ onto the coordinates I will belong to $\prod_{i \in I} A_i$ (i.e. $proj_I \mathbf{T} \subseteq proj_I \mathbf{A}^{k+1}$). Suppose $I = \{1, 2, \ldots, k\}$. Then, for any $a = ((a_1, b_1), \ldots, (a_k, b_k)) \in \mathbf{A}^k$, consider the element $t = ((a_1, b_1), \ldots, (a_k, b_k), (p, b_1))$, where p is any element in P. Then $t \in \mathbf{T}$, and so $a \in proj_I \mathbf{T}$.

If $I = \{1, \ldots, j-1, j+1, \ldots, k+1\}$ for some j between 1 and $k+1$, and we're given

$$a = ((a_1, b_1), \ldots, (a_{j-1}, b_{j-1}), (a_{j+1}, b_{j+1}), \ldots, (a_{k+1}, b_{k+1})) \in \mathbf{A}^k,$$

we can consider

$$t = ((a_1, b_1), \ldots, (a_{j-1}, b_{j-1}), (p, b_1), (a_{j+1}, b_{j+1}), \ldots, (a_{k+1}, b_{k+1})) \in \mathbf{T},$$

where, again, $p \in P$. Finally, for $I = \{2, \ldots, k+1\}$ and

$$a = ((a_2, b_2), \ldots, (a_{k+1}, b_{k+1})) \in \mathbf{A}^k,$$

we can consider

$$t = ((p, b_{k+1}), (a_2, b_2), \ldots, (a_{k+1}, b_{k+1})) \in \mathbf{T}.$$

This shows that $proj_I \mathbf{A}^{k+1} \subseteq proj_I \mathbf{T}$, for all $I \subseteq \{1, 2, \ldots, k+1\}$ with $|I| \leq k$ and so $\mathbf{T} =_k \mathbf{A}^{k+1}$.

We will now define a system of projections over $\mathbf{T}^{k+2}$ that satisfies $C(k, k+1)$ but not $C(k, k+2)$. This construction was first used by Valeriote in [16] and is based on Bergman's counterexample for the variety of abelian groups, found in section 2 of [3].

Let (S_I) be the system of k-fold projections of $\prod_{i=1}^{k+2} \mathbf{T}_i$, where $\mathbf{T}_i = \mathbf{T}$ for all i, defined as follows: for $I \subseteq \{1, 2, \ldots, k+2\}, |I| = k$, take S_I to be the set of k-tuples $(\vec{v}_i \mid i \in I)$ from $\prod_{i \in I} \mathbf{T}_i$ such that

$$\begin{aligned} \vec{v}_i(j) &= \vec{v}_j(i) \text{ if } i, j < k+2 \text{ and } i, j \in I \text{ and} \\ \vec{v}_i(i) &= \vec{v}_{k+2}(i) \text{ if } i < k+2 \text{ and } i, k+2 \in I. \end{aligned}$$

For any I, the set S_I is a nonempty subuniverse of $\prod_{i \in I} \mathbf{T}_i$.

Claim 4. *The system $(S_I)_{|I|=k}$ of k-fold projections is consistent on every $(k+1)$-element subset J of $\{1, 2, \ldots, k+2\}$, i.e., it satisfies $C(k, k+1)$.*

Proof. To show this, we need to show that any k-tuple $(\vec{v}_i \mid i \in I) \in S_I$ can be extended to a $(k+1)$-tuple that projects as necessary.

Suppose $J = \{1, 2, \ldots, k+1\}$ and $I = \{1, 2, \ldots, k\}$. Let $(\vec{v}_1, \ldots, \vec{v}_k) \in S_I$. We need to find a $\vec{v}_{k+1} \in \mathbf{T}$ such that the projection of $(\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1})$ onto any k-element set of coordinates $I' \subset \{1, 2, \ldots, k+1\}$ belongs to $S_{I'}$.

Now, for some $a \in \mathbf{A}, (\vec{v}_1(k+1), \ldots, \vec{v}_k(k+1), a) \in \mathbf{T}$, since $\mathbf{T}$ is k-complete with respect to $\mathbf{A}^{k+1}$. Taking this vector as $\vec{v}_{k+1}$ gives us what we need, as for any $I' \subset \{1, 2, \ldots, k+1\}, |I'| = k$, the projection of $(\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1})$ onto I' is in $S_{I'}$. This procedure will work for any $I \subset J \subset \{1, 2, \ldots, k+2\}, |J| = k+1$ and $|I| = k$, and so the system satisfies $C(k, k+1)$.

Claim 5. *The system $(S_I)_{|I|=k}$ of k-fold projections of $\prod_{i=1}^{k+2} \mathbf{T}_i$ is not consistent on $\{1, 2, \ldots, k+2\}$, i.e., it fails $C(k, k+2)$.*

Proof. Choose elements $c, d \in C$ such that $(c, d) \notin \alpha_Q$. Suppose $q \in Q$ and $p \in P$. For $1 \leq i \leq k$, let $\vec{v}_i$ be the $(k+1)$-tuple with $\vec{v}_i(i) = (q, d), \vec{v}_i(k+1) = (q, c)$ and $\vec{v}_i(j) = (p, c)$ otherwise. So, we have

$$\begin{aligned} \vec{v}_1 &= ((q, d), (p, c), (p, c), \ldots, (p, c), (q, c)), \\ \vec{v}_2 &= ((p, c), (q, d), (p, c), \ldots, (p, c), (q, c)), \end{aligned}$$

$$
\begin{aligned}
\vec{v}_3 &= ((p,c),(p,c),(q,d),\ldots,(p,c),(q,c)),\\
&\vdots\\
\vec{v}_k &= ((p,c),(p,c),(p,c),\ldots,(q,d),(q,c)).
\end{aligned}
$$

We have $\vec{v}_i \in \mathbf{T}$ for all i, and $(\vec{v}_1,\ldots,\vec{v}_k) \in S_I$, for $I = \{1,2,\ldots,k\}$. We will show that this k-tuple cannot be extended to a $(k+2)$-tuple that projects appropriately onto each sub-k-tuple.

Any extension must have $\vec{v}_{k+1}(i) = (q,c)$ and $\vec{v}_{k+2}(i) = (q,d)$ for all $i, 1 \leq i \leq k$. We also must have $\vec{v}_{k+1}(k+1) = \vec{v}_{k+2}(k+1)$. Hence, for some $x \in B, y \in C$, we have

$$
\begin{aligned}
\vec{v}_{k+1} &= ((q,c),(q,c),\ldots,(q,c),(x,y)),\\
\vec{v}_{k+2} &= ((q,d),(q,d),\ldots,(q,d),(x,y)).
\end{aligned}
$$

But in order for these vectors to be in $\mathbf{T}$, we must have $(c,y) \in \alpha_Q$ and $(d,y) \in \alpha_Q$, which contradicts $(c,d) \notin \alpha_Q$. Hence, this system is not consistent over $k+2$ coordinates.

We can now put all this together to prove the theorem. We have shown that in any locally finite, idempotent variety that is not congruence modular, we can find a system of k-fold projections that satisfies $C(k,k+1)$ but not $C(k,k+2)$, for $k > 1$. Hence, the variety will fail Bergman's Condition for any $k > 1$. Therefore, if a locally finite variety satisfies Bergman's Condition for some $k > 1$, it must be congruence modular. $\square$

6 Connections with the Constraint Satisfaction Problem

Definition 9. An *instance of the Constraint Satisfaction Problem (CSP)* is of the form $P = (\mathcal{A}, \mathcal{C})$, where

- $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ is a sequence of finite, nonempty sets, called the *domains* of P, and
- $\mathcal{C}$ is a set of constraints $\{C_1, \ldots, C_q\}$ where each C_i is a pair (S_i, R_i) with

- S_i a nonempty subset of $\{1, 2, \ldots, n\}$ called the *scope* of C_i, and
- R_i an $|S_i|$-ary relation over $(A_j \mid j \in S_i)$, called the *constraint relation* of C_i.

A *solution* to P is an n-tuple $\vec{x}$ over the sequence $(A_i \mid 1 \leq i \leq n)$ such that $proj_{S_j}(\vec{x}) \in R_j$ for each $1 \leq j \leq q$.

If $\mathcal{V}$ is a variety and the A_j's are universes of algebras from $\mathcal{V}$ such that each of the constraint relations R_i is the universe of a subalgebra of the corresponding product of the A_j's, then we say that P is an instance of the CSP from $\mathcal{V}$.

Note that this is a generalization of the typical one-sorted definition of the Constraint Satisfaction Problem (see, for example, definition 2.2 in [4]) . Each standard instance of the CSP can equivalently be expressed in the way we've defined it here. The entire collection of Constraint Satisfaction Problems forms an **NP**-complete class of problems, but there are many tractable subclasses. Finding such subclasses is an area of active research. For examples of this, see [4, 11, 12, 13].

Definition 10. For $I \subset \{1, \ldots, n\}$, a *partial solution* of P over I is a tuple $\vec{a} = (a_i \mid i \in I, a_i \in A_i)$ such that for all $C_i = (S_i, R_i) \in \mathcal{C}, proj_{I \cap S_i}(\vec{a}) \in proj_{I \cap S_i}(R_i)$.

Definition 11. An instance P satisfies the *k-extendability property* if: every partial solution of P over k variables can be extended to a solution of P if and only if every partial solution of P over k variables can be extended to a partial solution over every other variable of P.

We can consider CSPs in algebraic terms, and translate Bergman's Condition accordingly. Given a variety $\mathcal{V}$ and $A_1, \ldots, A_n \in \mathcal{V}$, all finite, and given a system S of k-fold projections over the A_i's, S determines the following instance of the CSP from $\mathcal{V}$:

$$P_S = ((A_1, \ldots, A_n), \{C_I \mid I \subseteq \{1, \ldots, n\}, |I| = k\}),$$

where $C_I = (I, S_I)$. If $\mathcal{V}$ satisfies Bergman's Condition for k, then, for the above set-up, if for every $I \subseteq \{1, \ldots, n\}, |I| = k, j \notin I$, and $\vec{a}$ a partial solution of P_S over I, $\vec{a}$ can be extended to a partial solution of P_S over $I \cup \{j\}$, then every partial solution of P_S over a k-element set of coordinates can be extended to a solution of P_S. (Note that solutions of P_S are exactly those n-tuples that are consistent with S, and partial solutions of P_S over some subset J correspond to members of S_J.) So, if $\mathcal{V}$ satisfies Bergman's Condition for k, then an instance P of the CSP from $\mathcal{V}$ will satisfy the k-extendability property.

In fact, if we consider constraint relations with some bound k, there is no loss of generality by considering instances of the form P_S. Given $P = (\mathcal{A}, \mathcal{C})$ such that the size of the scopes of the constraints in $\mathcal{C}$ are bounded by k, for each I with $|I| = k$, let S_I be the set of partial solutions of P over I. For fixed k, finding S_I is a polynomial-time problem, and so the instance P_S can be constructed from P in polynomial time. Note that if P is an instance from a variety $\mathcal{V}$, then P_S will be as well.

Claim 6. *P and P_S have the same set of solutions.*

Proof. Let $\vec{x}$ be a solution of P. Then, since $proj_I\vec{x} \in S_I$, $\vec{x}$ is consistent over S, and is a solution of P_S. Next, suppose $\vec{x}$ is a solution of P_S. Then, $proj_I\vec{x}$ is a partial solution of P over I, for all $I, |I| = k$, and so x will satisfy all of the constraints of P, since each constraint has scope contained in some I with $|I| = k$. Hence, $\vec{x}$ is a solution of P. $\square$

So now, our earlier result becomes:

Theorem 5. *Let $\mathcal{V}$ be a locally finite variety and let $k > 1$. If all instances of the CSP from $\mathcal{V}$ whose constraint scopes all have size at most k satisfy the k-extendability property, then $\mathcal{V}$ is congruence modular.*

7 Conclusion

The converse of Bergman's Theorem was left open in [3]. Bergman's question was, essentially, whether Bergman's Condition for k implies the existence of a $(k+1)$-ary near-unanimity term in a variety $\mathcal{V}$. This work does not settle that question, but it does show that Bergman's Condition for k implies congruence modularity. This would be consistent with a positive answer to Bergman's question.

In fact, our result has some further implications. In terms of tame congruence theory (see [6] for an overview), Valeriote ([17]) showed that if a locally finite variety satisfies Bergman's Condition, it omits types 1,2, and 5. Combined with the result of this thesis, it then follows that if $\mathcal{V}$ is locally finite, then Bergman's Condition for k implies congruence distributivity (see [10]).

In work related to the Constraint Satisfaction Problem, Barto and Kozik ([2]) have claimed that in a locally finite variety with a 4-ary near-unanimity term, $C(2,3) \Rightarrow C(2,r)$ always holds for $r > 2$. If correct, this would give a negative answer to Bergman's question, as it implies that the two examples posed in the final section of [3] by Bergman are, indeed, counterexamples to the converse of Bergman's Theorem.

It is not known whether the congruence distributivity of a variety $\mathcal{V}$ implies that Bergman's Condition holds in $\mathcal{V}$ for some k, which would indicate that Bergman's Condition characterizes congruence distributivity. It may be possible to find examples of congruence distributive varieties with no near-unanimity terms in which Bergman's Condition for k is satisfied.

In terms of the Constraint Satisfaction Problem, and the extendability of partial solutions, we can pose the following open question: in a congruence distributive variety, is there some k for which this extendability property holds?

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