STELLAR STRUCTURE IN SCALAR-TENSOR GRAVITY

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By

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Abstract

Stellar structure is investigated within the framework of scalar-tensor gravity. Novel perturbative analytical results are obtained for constant-density stars and for Newtonian polytropes in the quadratic model with coupling function $A(\phi) = \exp(\alpha\phi + \frac{1}{2}\beta\phi^2)$. They are compared to full numerical calculations, and possible applications to main-sequence stars, white dwarfs, and the Chandrasekhar mass are indicated.

It is found that Buchdahl's theorem is violated in Brans-Dicke theory for stars with exponentially-decaying density profiles. However, the mass-to-radius ratio M/R tends to the constant-density value in a certain limit. It is observed that for $\beta < 0$, there exists a maximum value of $\eta = P_0/\rho_0$ for constant-density stars, where P_0 and ρ_0 are the central pressure and density, respectively. It is conjectured that if such a maximum value also exists for other equations of state, and is less than the constant-density maximum value, then knowledge of P/ρ in the centre of a star can be used to constrain β.

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1 Overview

Scalar-tensor gravity is a natural generalization of general relativity. In this thesis, stellar stucture in the scalar-tensor theory of gravity with coupling function $A(\phi) = \exp(\alpha\phi + \frac{1}{2}\beta\phi^2)$ is investigated. In chapter 2, a history of the theory of gravity is presented, starting with Newtonian gravity and ending with scalar-tensor gravity. Stars and stellar structure are described in chapter 3. The discussion is limited to static spherically-symmetric stars. First, the equations of stellar structure in Newtonian gravity are derived. They are subsequently generalized to general relativity and scalar-tensor gravity.

In chapter 4, the Newtonian limit of scalar-tensor gravity is investigated. The general field equations and the geodesic equation are analyzed. This is needed for the correct identification of a star's mass. The equations of stellar structure are also analyzed, in order to describe Newtonian polytropes.

In chapter 5, analytical perturbative results are obtained for constant-density stars and Newtonian polytropes in the scalar-tensor theory with coupling function $A(\phi) = \exp(\alpha\phi + \frac{1}{2}\beta\phi^2)$. The validity of the perturbative expansion as well as astrophysical applications are discussed, and the analytical results are compared to full numerical calculations.

In the entire paper relativistic units are used, i.e. $c = 1$, and four-dimensional Lorentz metrics are taken to have the signature $(-, +, +, +)$.

2 Theories of Gravity

2.1 Newtonian Gravity

The first quantitative theory of gravity was formulated by Newton in 1687. In Newtonian gravity, any two massive particles exert an attractive force on each other. If particles A and B have gravitational masses $m_G^{(A)}, m_G^{(B)}$, and positions $\vec{x}^{(A)}, \vec{x}^{(B)}$, then the gravitational force exerted by particle A on particle B is given by

$$\vec{F}_{\rm AB} \equiv \vec{F} = G m_G^{(A)} m_G^{(B)} \cdot \frac{\vec{x}^{(A)} - \vec{x}^{(B)}}{|\vec{x}^{(A)} - \vec{x}^{(B)}|^3} , \tag{2.1}$$

where $G = 6.67 \cdot 10^{-11} \mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is Newton's gravitational constant. Equation (2.1) is called the inverse-square force law. The gravitational acceleration $\vec{a}^{(B)}$ of particle B is given by Newton's second law of motion

$$\vec{a}^{(B)} = \frac{\vec{F}}{m_I^{(B)}} = \frac{m_G^{(B)}}{m_I^{(B)}} \cdot G m_G^{(A)} \cdot \frac{\vec{x}^{(A)} - \vec{x}^{(B)}}{|\vec{x}^{(A)} - \vec{x}^{(B)}|^3}, \tag{2.2}$$

where $m_I^{(B)}$ is the inertial mass of particle B. It has been verified experimentally to a very high degree of accuracy that the gravitational mass is equal to the inertial mass, i.e. $m_G = m_I$ [48]. Consequently, the gravitational acceleration experienced by a particle is independent of its mass. This phenomenon, called the weak equivalence principle, is a surprising coincidence in Newtonian gravity.

Newtonian gravity can be formulated as a theory of a scalar field Ψ, called the Newtonian potential. The source of the scalar field is the mass density ϱ, and the gravitational field equation, called Poisson's equation is

$$\nabla^2 \Psi = 4\pi G \varrho \, . \tag{2.3}$$

The gravitational acceleration at any point in space is given by

$$\vec{a} = -\vec{\nabla}\Psi \, . \tag{2.4}$$

Setting $\varrho = m_G^{(A)} \delta^{(3)}(\vec{x} - \vec{x}^{(A)})$ for a point mass and solving Poisson's equation yields

$$\Psi = -\frac{G m_G^{(A)}}{|\vec{x} - \vec{x}^{(A)}|} + \mathrm{const} \, . \tag{2.5}$$

Upon taking the gradient of (2.5), evaluating at $\vec{x} = \vec{x}^{(B)}$, and using (2.4), equation (2.2) is recovered.

2.2 General Relativity

In Newtonian gravity, the gravitational force acts instantaneously. This is in conflict with special relativity, which states that information can not travel faster than the speed of light. In 1907-1915, Einstein reformulated gravity as a relativistic theory of the geometry of space-time in order to resolve this inconsistency [46]. In Einstein's theory, called general relativity, gravitational information propagates at the speed of light, and the weak equivalence principle is a postulate rather than an experimental observation. Space-time is modelled by a four-dimensional manifold M, and the gravitational field is described by a Lorentz metric $g_{\mu\nu}$ on M. The source of the gravitational field is relativistic energy-momentum, described by a rank-2 symmetric tensor $T_{\mu\nu}$ on M. The gravitational field equations, called Einstein's equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{2.6}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and scalar of $g_{\mu\nu}$, respectively. The Ricci tensor measures the average curvature in a 2-dimensional subspace of the tangent space, while the Ricci scalar measures the average curvature in all directions. In the special case $M = \mathbb{R}^4$ and $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ in global rectilinear coordinates, all curvature tensors vanish, gravitation is absent, and general relativity reduces to special relativity.

Einstein's equations can be derived from the variational principle $\delta(S_{\mathrm{g}} + S_{\mathrm{m}}) = 0$, where the gravitational action S_{g} is given by

$$S_{\mathrm{g}} = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} R \ . \tag{2.7}$$

In the above equation, $g = \det g_{\mu\nu}$. The energy-momentum tensor can be written in terms of the matter action S_{m} as

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} \quad \text{or} \quad T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{m}}}{\delta g_{\mu\nu}} \ . \tag{2.8}$$

The derivatives of the Ricci tensor and scalar are related by

$$\nabla_\nu R = 2 \nabla^\mu R_{\mu\nu} \ . \tag{2.9}$$

This relation follows from the symmetries of curvature, and is called the contracted form of the second Bianchi identity. The operator ∇_μ is the covariant derivative of the Levi-Civita connection of $g_{\mu\nu}$. It reduces to the partial derivative ∂_μ when acting on functions. Applying ∇^μ to Einstein's equations (2.6), and using equation (2.9) yields the statement of energy conservation

$$\nabla^\mu T_{\mu\nu} = 0 \ . \tag{2.10}$$

The trajectories of massive and massless free particles are timelike and null geodesics of the metric $g_{\mu\nu}$, respectively. They are found by solving the geodesic equation

$$\dot{x}^\mu \nabla_\mu \dot{x}^\nu = 0 \ , \tag{2.11}$$

where the world line of the particle is $x^\mu(\tau)$, and dots denote differentiation with respect to the proper time (or affine parameter, in the case of massless particles) τ. The geodesic equation can be derived from the variational principle $\delta S_{\mathrm{p}} = 0$, where the action of a free particle is given by

$$S_{\mathrm{p}} = \int \sqrt{-ds^2} = \int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\tau \ . \tag{2.12}$$

The fact that free particles travel on geodesics guarantees that the weak equivalence principle holds. In fact, more is true. All extended objects experience the same acceleration in a gravitational field, independent of their mass, composition, and internal gravitational energy. This phenomenon, called the strong equivalence principle, has been experimentally verified by laser ranging to the moon [33, 34, 32]. Its importance has been appreciated only after the formulation of alternate theories of gravity (such as scalar-tensor gravity) in which it does not hold.

General relativity predicts corrections to the Newtonian theory, which become important when gravitational fields are strong or time-dependent, or when particle velocities become relativistic. The most famous of these corrections, which led to the acceptance of general relativity upon their experimental confirmation, include the precession of the perihelion of Mercury's orbit, and the bending of starlight in the sun's gravitational field.

When applied to the entire universe, Einstein's field equations imply that it must be either contracting or expanding, and cannot be static. Troubled by this prediction, Einstein modified his field equations, adding a cosmological constant Λ, to allow a static (although unstable) universe [9]. The modified field equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{2.13}$$

However, observations of redshifts of nearby galaxies by Hubble in 1929 revealed that the universe is expanding. Upon hearing this, Einstein called the cosmological constant his "greatest blunder". In 1998, observations of type Ia supernovae showed that the expansion rate of the universe is accelerating [37, 38]. Einstein's original field equations can describe an expanding universe, but a Λ-term is needed to describe the accelerated expansion. So the cosmological constant turned out to be useful after all. It can be interpreted as a vacuum energy, with density $\rho_\Lambda = \Lambda/8\pi G$. Cosmological observations yield the constraint

$$|\rho_\Lambda| \leq (10^{-12}\mathrm{GeV})^4 \ . \tag{2.14}$$

On the other hand, if it is assumed that the standard model of particle physics can be trusted up to the Planck scale $m_{\mathrm{pl}} \sim 10^{19}\mathrm{GeV}$, then the contribution to ρ_Λ from a standard-model field is estimated to be

$$|\rho^i_\Lambda| \sim (10^{18}\mathrm{GeV})^4 \ , \tag{2.15}$$

where the index i labels the field. Since the contributions from fermion and boson fields have different signs, it is possible that they add up to a number small enough to satisfy the constraint (2.14). However, this situation is very unnatural, and there is no known mechanism to generate such large cancellations. This discrepancy, called the "cosmological constant problem", remains unsolved.

2.3 Scalar-Tensor Gravity

2.3.1 Motivation

To date, Einstein's equations with a cosmological constant are consistent with all gravitational experiments and observations [48]. However, there are many theoretical reasons for considering alternate theories of gravity, in particular, those in which the gravitational force is mediated by new massless scalar fields, in addition to the usual metric tensor.

In view of the cosmological constant problem, it is useful to consider theories of gravity in which the cosmological constant is a dynamical degree of freedom rather than a constant, and where a mechanism exists which drives it to the presently observed small value. The simplest choice for this dynamical degree of freedom is a scalar field.

It is hoped that eventually gravity will be unified with the other fundamental interactions (electroweak and strong). Superstring theory is a candidate framework for such a unification, in which space-time is ten-dimensional, and the fundamental objects are quantized one-dimensional strings [50]. Two different kinds of boundary conditions can be imposed on strings. Strings are called open or closed, depending on which boundary conditions are imposed. Different string states correspond to particles.

The massless states in the closed string spectrum can be decomposed into a scalar, a symmetric rank-2 tensor, and an antisymmetric rank-2 tensor. Since the metric tensor in general relativity is a symmetric rank-2 tensor, the symmetric rank-2 tensor string states can be interpreted as gravitational excitations, called gravitons. Graviton states should be massless, because gravity is an infinite-range force.

The scalar state is called the dilaton, and the antisymmetric rank-2 tensor states are called Kalb-Ramond states. String theory was developed in 1968 to explain strong interactions. However, when graviton states were found in the closed string spectrum, it was realized that string theory could be used to unify gravity with the electroweak and strong interactions. Since the massless closed string spectrum contains dilaton and Kalb-Ramond states in addition to graviton states, it can be conjectured that general relativity is only an approximation to a more complicated classical theory of gravitation, which contains classical dilaton and Kalb-Ramond fields, in addition to the usual metric tensor.

Space-time must be ten-dimensional for quantized superstring theory to be consistent. However, the world we live in appears to be only four-dimensional. This apparent discrepancy can be remedied by taking the six extra dimensions to be compact, and very small in size. The idea of compactification has a history much older than string theory. It was first proposed by Kaluza and Klein in 1921 [29, 31]. They tried to unify classical gravity and electrodynamics into a five-dimensional theory, with one compactified dimension. Starting with a higher-dimensional classical theory, the extra-dimensional coordinates in the action can be integrated out, to obtain an effective four-dimensional theory. The dynamics in the extra dimensions becomes encoded in a massless scalar field in the effective four-dimensional theory.

2.3.2 History

Scalar-tensor theories are a class of alternate theories of gravity, in which the gravitational force is mediated by a number of scalar fields, in addition to the usual metric tensor [13, 6, 42]. The first scalar-tensor theory of gravity was formulated by Jordan, Fierz, Brans, and Dicke in the years 1949-1961 [28, 21, 27, 5, 20]. It was motivated by Dirac's speculation that Newton's gravitational constant can change over cosmological time scales, and Mach's principle that inertia is caused by the matter distribution of the entire universe. In addition to the usual metric tensor, this theory contains one scalar field which can be related to an effective gravitational constant. A coupling function $A(\phi)$ determines the strength of interactions between the scalar field ϕ and matter fields. Its logarithmic derivative $\alpha(\phi) = A^{-1}(\phi)A'(\phi)$ can be thought of as an effective scalar-matter coupling constant. Further work on scalar-tensor gravity was done by Bergmann, Nordtvedt, and Wagoner in the years 1968-1970 [1, 35, 45].

In the simplest scalar-tensor theory of gravity, called Brans-Dicke theory, the scalar-matter coupling $\alpha(\phi)$ is constant, and $A(\phi) = \exp(\alpha\phi)$. Solar system experiments have established the bound $\alpha^2 < 10^{-5}$ [2]. Consequently, Brans-Dicke theory differs very little from general relativity, not only in the solar system, but also in the strong-field regime. The weak equivalence principle continues to hold in scalar-tensor gravity, but the strong equivalence principle does not. However, in the solar system, the violation of the strong equivalence principle is very small [34].

To construct a scalar-tensor theory which satisfies solar system constraints but differs from general relativity in the strong-field regime, a more complicated coupling function is needed. The simplest choice is $A(\phi) = \exp(\alpha\phi + \frac{1}{2}\beta\phi^2)$, where α and β are constants. This called the quadratic model. The scalar-matter coupling is linear in ϕ:

$$\alpha(\phi) = \alpha + \beta\phi \, . \tag{2.16}$$

The constant multiplying ϕ in the coupling function, as well as the logarithmic derivative of the coupling function, are both denoted by α. It should be clear from the context which is meant.

In 1993, a non-perturbative strong-field effect called spontaneous scalarization has been discovered in the stellar structure of the quadratic model by Damour and Esposito-Farèse. It occurs in massive neutron stars when $\beta \lesssim -4$ [14, 15, 39]. All quantitative descriptions of spontaneous scalarization have been carried out numerically. The focus of this thesis is on analytical perturbative results in the quadratic model, and thus spontaneous scalarization will not be discussed.

A pulsar is a neutron star which has a magnetic field, and emits radio waves at regular time intervals. A binary system consisting of at least one pulsar is a good testing ground for relativistic dynamics in strong gravitational fields, because the pulsar acts as an effective clock, allowing for a precise measurement of the orbital parameters. The first such system, called PSR 1913+16, was discovered by Hulse and Taylor in 1974 [47]. Measurements of its orbital parameters, in particular, the decay of the orbit due to emission of gravitational radiation, are in very close agreement with the predictions of general relativity. Thus, they place constraints on the coupling function of scalar-tensor gravity [19, 15, 16, 43, 12, 3]. Scalar-tensor gravity predicts the emission of both dipole scalar waves and quadrupole tensor waves from a binary pulsar, while general relativity only predicts the latter. Scalar and tensor waves are disturbances in the scalar field and metric tensor, respectively, that propagate throughout spacetime. Due to the presence of dipole scalar waves, scalar-tensor gravity predicts more radiated power than general relativity. This difference will be observable in future experiments that will directly measure gravitational waves by laser interferometry [40].

When a very massive star burns up all of its nuclear fuel, general relativity predicts that it will collapse to a space-time singularity called a black hole. The 'no-hair' theorem states that a black hole is completely described by only three parameters: mass, angular momentum (i.e. spin), and electric charge. All other information in the star is lost. Moreover, Birkhoff's theorem states that if the star is spherically symmetric, then no gravitational radiation will be emitted during the collapse.

The collapse of a spherically symmetric star can be described analytically. This solution was obtained by Oppenheimer and Snyder in 1939 [36]. Gravitational collapse in scalar-tensor gravity has a much richer structure. It was shown that the 'no-hair' theorem still holds in Brans-Dicke theory [25], while Birkhoff's theorem no longer holds. This means that when a star with non-zero scalar charge collapses, this charge must be radiated away. This process has been investigated numerically by several authors [41, 24, 30]. It would be useful to obtain an analytical description of this process. A prerequisite for this, is the analytical description of static constant-density stars. This is developed in section 5.1.

The evolution of the entire universe has been studied in scalar-tensor gravity by Damour and Nordtvedt [17, 18]. They found a mechanism which drives the theory to general relativity at late times.

2.3.3 Theoretical Framework

Consider the action of general relativity

$$S_{\mathrm{g}} = \frac{1}{16\pi\bar{G}} \int d^4x \sqrt{|g|} R \ , \tag{2.17}$$

and that of a massless scalar field ϕ ,

$$S_\phi = -\frac{1}{16\pi\bar{G}} \int d^4x \sqrt{|g|} \left(2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \ . \tag{2.18}$$

A potential $V(\phi)$ can be added to (2.18). It is usually assumed that such a potential changes significantly only over cosmological distances, and can be neglected in the study of isolated systems such as stars. $\bar{G}$ is a bare gravitational constant, which will later be related to the physical gravitational constant.

In order to introduce scalar-matter interactions, the matter fields are coupled to the metric $\tilde{g}_{\mu\nu} \equiv A^2(\phi)g_{\mu\nu}$, where $A(\phi)$ is a coupling function. The matter action is written symbolically as $S_{\mathrm{m}} = S_{\mathrm{m}}[\psi, \tilde{g}_{\mu\nu}]$, where ψ denotes the collection of all the matter fields. For example, the action for the electromagnetic field is

$$S_{\mathrm{em}}[F_{\mu\nu}, \tilde{g}_{\mu\nu}] = -\frac{1}{4\bar{\alpha}_{\mathrm{FS}}^2} \int d^4x \sqrt{|\tilde{g}|}\tilde{g}^{\mu\mu'}\tilde{g}^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'} \ , \tag{2.19}$$

where $\tilde{g} = \det \tilde{g}_{\mu\nu}$, $F_{\mu\nu}$ is the field strength tensor, and $\bar{\alpha}_{\mathrm{FS}}$ is a bare fine-structure constant. Similarly, the action for a fermion field ψ is

$$S_\psi = -\frac{1}{2} \int d^4x \sqrt{|\tilde{g}|}\bar{\psi}(e_i^{\ \mu}\gamma^i D_\mu + \bar{m})\psi \ , \tag{2.20}$$

where $\bar{m}$ is a bare mass, and

$$D_\mu = \partial_\mu + \dots \tag{2.21}$$

is the covariant derivative [4]. It is contracted with the flat-space Dirac gamma matrices γ^i by means of the tetrad $e_i^{\ \mu}$. In four space-time dimensions, (2.19) is conformally invariant, and so

$$S_{\rm em}[F_{\mu\nu}, \tilde{g}_{\mu\nu}] = S_{\rm em}[F_{\mu\nu}, g_{\mu\nu}] \ . \tag{2.22}$$

Consequently, photons do not interact with the scalar field at tree level. However, this conformal symmetry is broken upon quantization. The action of a free particle is

$$S_{\rm p} = \int \sqrt{-\tilde{ds}^2} = \int \sqrt{-\tilde{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\tau \ , \tag{2.23}$$

where $x^\mu(\tau)$ is the particle's world line parametrized by proper time (or the affine parameter, in the case of a massless particle), and dots denote $d/d\tau$. The variational principle $\delta S_{\rm p} = 0$ implies that the worldline is a geodesic of $\tilde{g}_{\mu\nu}$.

The field $g_{\mu\nu}$ is called the Einstein-frame metric, and $\tilde{g}_{\mu\nu}$ is called the Jordan-frame metric. The indices of matter-field tensors and tensors with a tilde are raised and lowered using $\tilde{g}_{\mu\nu}$. Other indices are raised and lowered using $g_{\mu\nu}$. The covariant derivative operators of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are denoted by ∇_μ and $\tilde{\nabla}_\mu$, respectively, and the d'Alembertian operators of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are denoted by $\Box$ and $\tilde{\Box}$, respectively.

Let $\tilde{R}$ be the Ricci scalar of $\tilde{g}_{\mu\nu}$. It is related to R by the formula

$$\begin{aligned} \bar{G}\Phi^2 R &= \Phi\tilde{R} - 3\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\Phi - 2(\partial_\mu\tilde{g}^{\mu\nu})(\partial_\nu\Phi) + \frac{3}{2\Phi}\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \\ &\quad +2\tilde{g}^{\mu\nu}\tilde{\Gamma}^\alpha_{\ \mu[\nu}\partial_{\alpha]}\Phi - \frac{1}{2}\tilde{g}^{\mu\nu}(\tilde{g}^{\alpha\beta}\partial_\mu\tilde{g}_{\alpha\beta})\partial_\nu\Phi \ , \end{aligned} \tag{2.24}$$

where

$$\Phi = \frac{1}{\bar{G}A^2(\phi)} \ , \tag{2.25}$$

and $\tilde{\Gamma}^\alpha_{\ \mu\nu}$ are the Christoffel symbols of $\tilde{g}_{\mu\nu}$. Using equation (2.24), the gravitational and scalar actions (2.17) and (2.18) can be re-written as

$$S_{\rm g} + S_\phi = \frac{1}{16\pi}\int d^4x\sqrt{|\tilde{g}|}\left(\Phi\tilde{R} - \frac{\omega(\Phi)}{\Phi}\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 3\tilde{\Box}\Phi\right) \ , \tag{2.26}$$

where

$$\omega(\Phi) = \frac{1}{2\alpha^2(\phi)} - \frac{3}{2} \ , \quad \alpha(\phi) = \frac{\partial \log A(\phi)}{\partial\phi} \ . \tag{2.27}$$

The last term in equation (2.26) reduces to an integral over the boundary of space-time, and is neglected:

$$S_{\rm g} + S_\phi = \frac{1}{16\pi} \int d^4x \sqrt{|\tilde{g}|} \left(\Phi \tilde{R} - \frac{\omega(\Phi)}{\Phi} \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right) . \tag{2.28}$$

The above action can be interpreted as a theory of gravity, in which the effective gravitational constant

$$G \equiv \Phi^{-1} \tag{2.29}$$

varies over space-time. Thus, in the Jordan frame, the gravitational coupling varies over space-time, while gauge couplings and fermion masses are constant. On the other hand, in the Einstein frame, the gravitational coupling is constant, while gauge couplings and fermion masses vary over space-time. To see this, the matter actions (2.19) and (2.20) need to be re-written in terms of the Einstein-frame metric.

The Jordan-frame description (2.28) of scalar-tensor gravity in terms of the variables $(\tilde{g}_{\mu\nu}, \Phi)$ is useful for understanding the physical content of the theory, since matter fields couple directly to the Jordan-frame metric $\tilde{g}_{\mu\nu}$. The Jordan-frame field equations, obtained from $\delta(S_{\rm g} + S_\phi + S_{\rm m}) = 0$ applied to (2.28), are given by

$$\begin{aligned} \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} &= 8\pi\Phi^{-1}\tilde{T}_{\mu\nu} + \Phi^{-1}(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Phi - \tilde{g}_{\mu\nu}\tilde{\Box}\Phi) \\ &\quad + \omega(\Phi)\Phi^{-2}(\partial_\mu\Phi\partial_\nu\Phi - \tfrac{1}{2}\tilde{g}^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi\tilde{g}_{\mu\nu}) , & (2.30) \\ \tilde{\Box}\Phi &= (2\omega(\Phi)+3)^{-1}(8\pi\tilde{T} - \omega'(\Phi)\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi) . & (2.31) \end{aligned}$$

They are complicated due to the mixed term $\Phi\tilde{R}$ in (2.28). It is easier to work with the Einstein-frame field equations, obtained from $\delta(S_{\rm g} + S_\phi + S_{\rm m}) = 0$ applied to (2.17) and (2.18). They are given by

$$\begin{aligned} R_{\mu\nu} &= 2\partial_\mu\phi\partial_\nu\phi + 8\pi\bar{G}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) , & (2.32) \\ \Box\phi &= -4\pi\bar{G}\alpha(\phi)T , & (2.33) \end{aligned}$$

where the Einstein-frame and Jordan-frame energy-momentum tensors, denoted by $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$, respectively, are defined by

$$\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{|g|}T_{\mu\nu} , \quad \frac{\delta S_{\rm m}}{\delta \tilde{g}^{\mu\nu}} = -\frac{1}{2}\sqrt{|\tilde{g}|}\tilde{T}_{\mu\nu} , \tag{2.34}$$

$$T = T_{\mu\nu}g^{\mu\nu} , \quad \tilde{T} = \tilde{T}_{\mu\nu}\tilde{g}^{\mu\nu} . \tag{2.35}$$

They are related by

$$T_{\mu\nu} = A^2(\phi)\tilde{T}_{\mu\nu} \ , \quad T_\mu{}^\nu = A^4(\phi)\tilde{T}_\mu{}^\nu \ , \quad T^{\mu\nu} = A^6(\phi)\tilde{T}^{\mu\nu} \ . \tag{2.36}$$

From equation (2.33) we see that the function $\alpha(\phi)$ can be interpreted as an effective scalar-matter coupling constant. In Brans-Dicke theory, this function is actually constant.

Note that the Einstein-frame energy-momentum tensor $T_{\mu\nu}$ is *not* covariantly conserved. An expression for its covariant derivative can be derived by re-writing the field equation (2.32) in the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) + 8\pi\bar{G}T_{\mu\nu} \ . \tag{2.37}$$

Applying ∇^μ to (2.37) and using the Bianchi identity (2.9) yields

$$\begin{aligned} 8\pi\bar{G}\nabla^\mu T_{\mu\nu} &= -2\nabla^\mu(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) \\ &= -2(\Box\phi)\nabla_\nu\phi \ . \end{aligned} \tag{2.38}$$

The field equation (2.33) then implies

$$\nabla^\mu T_{\mu\nu} = \alpha(\phi)T\nabla_\nu\phi \ . \tag{2.39}$$

Te Jordan-frame energy-momentum tensor *is* covariantly conserved. To demonstrate this, the Jordan-frame Christoffel symbols $\tilde{\Gamma}^\lambda{}_{\mu\nu}$ need to be related to the Einstein-frame Christoffel symbols $\Gamma^\lambda{}_{\mu\nu}$:

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + \alpha(\phi)(\delta^\lambda{}_\mu\partial_\nu\phi + \delta^\lambda{}_\nu\partial_\mu\phi - g_{\mu\nu}g^{\lambda\sigma}\partial_\sigma\phi) \ . \tag{2.40}$$

From the above equation, and the relations (2.36), it follows that

$$\tilde{\nabla}^\mu\tilde{T}_{\mu\nu} = 0 \ . \tag{2.41}$$

3 Stellar Structure

3.1 Stars

A star is a stable configuration of gaseous matter, in which the attractive gravitational force is balanced by outward pressure. Most stars spend the majority of their lifetime burning hydrogen into helium via nuclear fusion. The energy released in this process generates the pressure that balances gravity.

Once a star exhausts its nuclear fuel, there is no pressure to balance the gravitational attraction, and the star collapses. The ultimate fate depends on the mass. Stars such as the sun collapse down to a core of ionized iron and nickel. Further collapse is prohibited by the quantum degeneracy pressure of electrons, which arises due to the Pauli exclusion principle. The resulting stable configuration is called a white dwarf. Typically, white dwarfs have masses of $M \sim M_\odot$, and radii of $R \sim R_\oplus$.

Stars much heavier than the sun explode in a supernova, and then collapse to either a neutron star, or a black hole. A neutron star is a stable configuration of nuclear matter, consisting mostly of neutrons, whose quantum degeneracy pressure balances the gravitational attraction. Typically, neutron stars have masses of $M \sim M_\odot$, and radii of $R \sim 10$km. A black hole is a space-time singularity.

Stars that burn nuclear fuel, as well as white dwarfs, have relatively weak gravitational fields, and can be adequately treated with Newtonian gravity. On the other hand, general relativity is necessary for a proper description of the strong gravitational fields near neutron stars and black holes. In addition to gravity, thermodynamics plays an important role in the quantitative description of stars.

In this thesis, only static spherically symmetric stars are considered. This excludes rotating stars. Stars are modelled by a perfect fluid, with mass-energy density ρ, and isotropic pressure P.

3.2 Newtonian Gravity

In this section, the treatment of stars in Newtonian gravity is described. In the Newtonian limit, the mass-energy density ρ of the perfect fluid becomes the rest mass density, which is denoted by ϱ. Due to spherical symmetry, it is natural to work in spherical polar coordinates. Let $m(r)$ be the total mass inside the ball of radius r.

Then, conservation of mass implies that

$$m' = 4\pi r^2 \varrho \ , \tag{3.1}$$

where the prime denotes d/dr. The requirement that the star be in hydrostatic equilibrium implies that the pressure $P(r)$ must satisfy

$$P' = -\frac{G\varrho m}{r^2} \ . \tag{3.2}$$

Finally, Poisson's equation (2.3) becomes

$$\Psi' = \frac{Gm}{r^2} \ . \tag{3.3}$$

Equations (3.1)-(3.3) are called the equations of stellar structure in Newtonian gravity. These equations by themselves are not a complete system. They must be supplemented with a relationship between ϱ and P. Such a relationship can be derived from the thermodynamical properties of the star, and is called an equation of state. It is written symbolically as $P = P(\varrho)$. Alternatively, the density profile $\varrho(r)$ can be specified instead of an equation of state. Equations (3.1) and (3.2), together with an equation of state or density profile, can be integrated to obtain the profiles $m(r)$, $P(r)$, and $\varrho(r)$.

The Newtonian potential can then be calculated from equation (3.3):

$$\Psi(r) = G \int^r \frac{m}{r^2} dr \ . \tag{3.4}$$

Note that Ψ is arbitrary up to a constant. Initial conditions for the other profiles at the centre of the star ($r = 0$) are

$$m(0) = 0 \ , \quad P(0) = P_0 \ , \quad \varrho(0) = \varrho_0 \ , \tag{3.5}$$

where ϱ_0 and P_0 are related by the equation of state.

Since equation (3.2) is singular at $r = 0$, the numerical integration has to be started at some small r_0 rather than 0. Then, initial conditions at r_0 need to be related to those at $r = 0$. This can be done by means of a power series expansion of the pressure and mass functions P and m. From equations (3.1) and (3.2), it follows that

$$m(r) = \frac{4}{3}\pi\rho_0 r^3 + \mathcal{O}(r^4) \ , \tag{3.6}$$

$$P(r) = P_0 - \frac{2}{3}\pi G\rho_0^2 r^2 - \frac{7}{9}\pi G\rho_0\rho_0' r^3 + \mathcal{O}(r^4) \ , \tag{3.7}$$

where $\rho_0' = \rho'(0)$. The stellar boundary is defined to be the sphere of radius R, at which the pressure vanishes, i.e. $P(R) = 0$. The total mass of the star is then

$$M = m(R) = 4\pi \int_0^R \varrho r^2 dr \; . \tag{3.8}$$

The potential Ψ outside the star (i.e. for $r > R$) can be described analytically. Equation (3.1) implies that $m(r) = M$ for $r > R$. Then equation (3.4) implies that $\Psi = -GM/r + \text{const}$ for $r > R$.

Once an equation of state is specified, the central pressure P_0 uniquely determines the mass M and radius R. Thus,

$$R = \mathcal{F}(P_0) \; , \quad M = \mathcal{G}(P_0) \; , \tag{3.9}$$

where $\mathcal{F}$ and $\mathcal{G}$ are functions which depend on the equation of state. For a given star, M and R can be obtained from astrophysical observations, while P_0 is not easily determined. Thus, in order to compare the theoretical predictions to astrophysical data, it is useful to eliminate P_0, and obtain a relationship between M and R:

$$M = \mathcal{G}(\mathcal{F}^{-1}(R)) \; . \tag{3.10}$$

Equation (3.10) defines a one-dimensional curve in the M-R plane, which depends on the equation of state. The theory can be tested by plotting the masses and radii of stars modelled by this equation of state, and checking whether they lie on the curve (3.10).

3.3 General Relativity

In this section, the treatment of stars in general relativity is described. The derivation of the equations of relativistic stellar structure is completely different from the Newtonian derivation. However, the relativistic equations can be written in a form similar to the Newtonian equations. A static spherically-symmetric metric can be put in the form

$$g_{\mu\nu} dx^\mu dx^\nu = -f dt^2 + h dr^2 + r^2 d\Omega^2 \; , \tag{3.11}$$

where f and h depend only on r. The energy-momentum tensor of the perfect fluid that models the star is given by

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} \; , \tag{3.12}$$

where u_μ is the 4-velocity of the perfect fluid. Since the star is static, u_μ must point in the time-direction. It must also be normalized such that $u_\mu u^\mu = -1$. Thus,

$$u_\mu = \sqrt{f}(dt)_\mu \ . \tag{3.13}$$

The independent components of Einstein's field equations (2.6) are

$$\begin{aligned}
\frac{1}{2\sqrt{fh}}\left(\frac{f'}{\sqrt{fh}}\right)' + \frac{f'}{rfh} &= 4\pi G(\rho + 3P) \ , & (3.14)\\
-\frac{1}{2\sqrt{fh}}\left(\frac{f'}{\sqrt{fh}}\right)' + \frac{h'}{rh^2} &= 4\pi G(\rho - P) \ , & (3.15)\\
-\frac{f'}{2rfh} + \frac{h'}{2rh^2} + \frac{1}{r^2}\left(1 - \frac{1}{h}\right) &= 4\pi G(\rho - P) \ , & (3.16)
\end{aligned}$$

where primes denote d/dr. To re-write these equations in a form similar to the Newtonian equations (3.1)-(3.3), carry out the change of variables

$$\nu = \log f \ , \quad m = \frac{r}{2G}\left(1 - \frac{1}{h}\right) \ . \tag{3.17}$$

Equations (3.14)-(3.16) can then be re-written as

$$\begin{aligned}
m' &= 4\pi r^2 \rho \ , & (3.18)\\
P' &= -(\rho + P)\frac{4\pi G r^3 P + Gm}{r(r - 2Gm)} \ , & (3.19)\\
\nu' &= \frac{8\pi G r^3 P + 2Gm}{r(r - 2Gm)} \ . & (3.20)
\end{aligned}$$

In the non-relativistic (Newtonian) limit,

$$P \ll \rho \ , \quad Gm \ll r \ , \quad Pr^3 \ll m \ , \quad \rho \to \varrho \ . \tag{3.21}$$

In general relativity, both mass and pressure are sources of the gravitational field, while in Newtonian gravity, only mass is a source of the gravitational field. Thus, in the Newtonian limit, the contribution from the pressure to the gravitational field is negligible compared to the contribution from the mass. This justifies the first inequality. In the Newtonian limit, the metric reduces to the flat metric of special relativity, in particular, $g_{rr} \to \eta_{rr} = 1$. This justifies the second inequality. The average energy-mass density in the ball of radius r (denoted by $B_r(0)$) is

$$\begin{aligned}
\langle\rho\rangle(r) &= \frac{3}{4\pi r^3}\int_{B_r(0)} \rho d^3\vec{x}\\
&= \frac{3}{4\pi r^3}\cdot 4\pi \int_0^r \rho \frac{r^2 dr}{\sqrt{1 - 2Gm/r}} \ . & (3.22)
\end{aligned}$$

In the non-relativistic limit,

$$\begin{aligned}\langle\rho\rangle(r) &\to \frac{3}{4\pi r^3}\cdot 4\pi\int_0^r \rho r^2 dr \\ &= \frac{3m(r)}{4\pi r^3}\ .\end{aligned} \tag{3.23}$$

Thus, the third inequality is equivalent to $P \ll \langle\rho\rangle$. When (3.21) is used to simplify equations (3.18)-(3.20), the Newtonian equations (3.1)-(3.3) are recovered, provided that the identification $\nu = 2\Psi$ is made. There is, however, one subtlety. The definition of $m(r)$ in general-relativistic stellar structure (given by equations (3.17) and (3.11)), is different from that in Newtonian stellar structure. Yet, equation (3.23) shows that in the non-relativistic limit, these two definitions coincide.

The relativistic equations of stellar structure (3.18)-(3.20) are solved in the same way as the non-relativistic equations (3.1)-(3.3). Initial conditions are given by (3.5), with ϱ replaced by ρ. In the Newtonian theory, the initial condition $m(0) = 0$ immediately followed from the definition of m. On the other hand, in general relativity, this condition follows from the requirement that the metric (3.11) be well-behaved as $r \to 0$. The analogues of equations (3.6) and (3.7) are

$$m(r) = \frac{4}{3}\pi\rho_0 r^3 + \mathcal{O}(r^4)\ , \tag{3.24}$$

$$\begin{aligned}P(r) &= P_0 - \frac{2}{3}\pi G(\rho_0 + P_0)(\rho_0 + 3P_0)r^2 \\ &\quad -\frac{1}{9}\pi G\rho_0'(7\rho_0 + 15P_0)r^3 + \mathcal{O}(r^4)\ .\end{aligned} \tag{3.25}$$

If the variable ν is shifted by a constant, then the time variable t can be rescaled to remove this shift (see equations (3.11) and (3.17)). Therefore, ν is arbitrary up to a constant. It is customary to fix this constant by requiring that the metric be asymptotically flat, i.e. $\nu \to 0$ as $r \to \infty$. With this choice, ν is given by

$$\nu(r) = -\int_r^\infty \frac{8\pi G r^3 P + 2Gm}{r(r-2Gm)}dr\ . \tag{3.26}$$

The radius of the star R is defined the same way as in the Newtonian theory.

The gravitational field outside the star is found by solving equations (3.18)-(3.20) with $P = \rho = 0$. The solution is

$$m(r) = M = \text{const}\ ,\quad \nu(r) = \log(1 - 2GM/r)\ . \tag{3.27}$$

Therefore, the exterior line element is

$$ds^2 = -(1 - 2GM/r)dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\Omega^2 \ . \tag{3.28}$$

This solution is called the Schwarzschild metric. It depends on one parameter M, which is the mass of the star. This interpretation of M is justified as follows. The non-relativistic limit of the tt component of Einstein's equations (2.6) is given by

$$\nabla^2 g_{tt} = -8\pi G \varrho \ . \tag{3.29}$$

Comparing this to the Poisson equation (2.3) of Newtonian gravity shows that in the non-relativistic limit,

$$g_{tt} = -2\Psi + \text{const} \ . \tag{3.30}$$

So for the Schwarzschild metric,

$$\Psi = -\frac{GM}{r} + \text{const} \ . \tag{3.31}$$

This is the Newtonian potential of a spherically symmetric body of mass M centered at the origin (see equation (2.5)). Note that equation (3.30) is consistent with the earlier identification $\nu = 2\Psi$, to first order in ν. The value of M is found by requiring that the metric be continuous at $r = R$. The result is

$$M = m(R) = 4\pi \int_0^R \rho r^2 dr \ . \tag{3.32}$$

It follows from equation (3.22) that the non-gravitational mass-energy inside the star (called the baryonic mass of the star) is given by

$$M_{\mathrm{B}} = 4\pi \int_0^R \rho \frac{r^2 dr}{\sqrt{1 - 2Gm/r}} \ . \tag{3.33}$$

Thus, the gravitational binding energy of the star is

$$M - M_{\mathrm{B}} = 4\pi \int_0^R \rho r^2 \left(1 - \frac{1}{\sqrt{1 - 2Gm/r}}\right) dr \ . \tag{3.34}$$

For a given equation of state, a curve in the M-R plane can be calculated and used to compare the theoretical predictions to astrophysical data, as in Newtonian stellar structure.

3.4 Scalar-Tensor Gravity

In the generalization of the equations of stellar structure to scalar-tensor gravity, three complications arise. The Einstein-frame field equations are used, but the perfect fluid is described in the Jordan frame. Consequently, P and ρ are multiplied by conversion factors of $A^4(\phi)$. The exterior solution is more complicated than the Schwarzschild metric, and depends on three parameters. It can not be simply described in the coordinates (3.11). Consequently, the matching between interior and exterior solutions involves a change of coordinates, resulting in complicated formulas. In particular, the calculation of the star's total mass is more complicated. The function $m(r)$ can no longer be interpreted as the mass-energy inside the ball of radius r. The total mass is found by writing down the equation of motion of a test particle in the star's exterior, and taking the Newtonian limit. It turns out that the mass obtained in this manner is different from the mass of the Jordan-frame metric by a factor of $A(\phi_\infty)$. This calculation is presented in section 4.3.

3.4.1 Interior

Due to spherical symmetry and staticity, the Einstein-frame metric $g_{\mu\nu}$ can be put in the form (3.11), and the scalar field ϕ depends only on r. The perfect fluid is described in the Jordan frame. Its 4-velocity is

$$\tilde{u}_\mu = \sqrt{f}|A(\phi)|(dt)_\mu \ , \tag{3.35}$$

and its energy-momentum tensor is

$$\tilde{T}_{\mu\nu} = (\rho + P)\tilde{u}_\mu\tilde{u}_\nu + P\tilde{g}_{\mu\nu} \ . \tag{3.36}$$

The independent components of the Einstein-frame field equations (2.32) and (2.33) are

$$\frac{1}{2\sqrt{fh}}\left(\frac{f'}{\sqrt{fh}}\right)' + \frac{f'}{rfh} = 4\pi\bar{G}A^4(\phi)(\rho+3P) \ , \tag{3.37}$$

$$-\frac{1}{2\sqrt{fh}}\left(\frac{f'}{\sqrt{fh}}\right)' + \frac{h'}{rh^2} = \frac{2\phi'^2}{h} + 4\pi\bar{G}A^4(\phi)(\rho-P) \ , \tag{3.38}$$

$$-\frac{f'}{2rfh} + \frac{h'}{2rh^2} + \frac{1}{r^2}\left(1-\frac{1}{h}\right) = 4\pi\bar{G}A^4(\phi)(\rho-P) \ , \tag{3.39}$$

$$\frac{1}{r^2\sqrt{fh}}\left(r^2\sqrt{\frac{f}{h}}\phi'\right)' + \frac{\phi'^2}{h} = 4\pi\bar{G}\alpha(\phi)A^4(\phi)(\rho-3P) \ , \tag{3.40}$$

where primes denote d/dr. Equations (3.37)-(3.39) have the same structure as equations (3.14)-(3.16). There are, however, two differences: the rr equation has a new term involving ϕ'^2/h, and all factors of ρ and P are multiplied by $A^4(\phi)$. Upon carrying out the change of variables (3.17) (with G replaced by $\bar{G}$), the field equations (3.37)-(3.40) can be written as

$$\bar{G}m' = 4\pi\bar{G}r^2A^4(\phi)\rho + \frac{1}{2}r(r-2\bar{G}m)\phi'^2 \ , \tag{3.41}$$

$$P' = -(\rho+P)\left(\frac{4\pi\bar{G}r^3A^4(\phi)P+\bar{G}m}{r(r-2\bar{G}m)} + \frac{1}{2}r\phi'^2 + \alpha(\phi)\phi'\right) \ , \tag{3.42}$$

$$\phi'' = \frac{4\pi\bar{G}rA^4(\phi)}{r-2\bar{G}m}\big(\alpha(\phi)(\rho-3P) + r\phi'(\rho-P)\big) - \frac{2(r-\bar{G}m)}{r(r-2\bar{G}m)}\phi' \ , \tag{3.43}$$

$$\nu' = \frac{8\pi\bar{G}r^3A^4(\phi)P + 2\bar{G}m}{r(r-2\bar{G}m)} + r\phi'^2 \ . \tag{3.44}$$

These equations are solved in the same manner as the equations of stellar structure in general relativity. In addition to (3.5) (with ϱ replaced by ρ), the initial conditions $\phi(0)$ and $\phi'(0)$ need to be specified. The value $\phi_0 = \phi(0)$ can be arbitrary, but $\phi'(0)$ must vanish. To see this, evaluate (3.41) at $r = 0$ to obtain $m'(0) = 0$. Then multiply (3.43) by $r(r-2\bar{G}m)$, differentiate, and evaluate at $r = 0$ to obtain $\phi'(0) = 0$. The stellar configurations are thus a two-parameter family, which can be parametrized by the initial values (P_0, ϕ_0). In contrast, the stellar configurations of Newtonian gravity and general relativity are a one-parameter family, which can be parametrized by P_0.

The power series expansions of m, P, and ϕ, needed for connecting initial conditions at $r = 0$ to those at $r = r_0 \ll 1$, are given by

$$
\begin{aligned}
m(r) &= \frac{4\pi}{3} A_0^4 \rho_0 r^3 + \mathcal{O}(r^4) , && (3.45) \\
P(r) &= P_0 + \frac{2\pi\bar{G}}{3} A_0^4 (P_0 + \rho_0)\big(\alpha_0^2(3P_0 - \rho_0) - (3P_0 + \rho_0)\big) r^2 \\
&\quad + \frac{\pi\bar{G}}{9} A_0^4 \rho_0' \big(\alpha_0^2(9P_0 - 7\rho_0) - (15P_0 + 7\rho_0)\big) r^3 + \mathcal{O}(r^4) , && (3.46) \\
\phi(r) &= \phi_0 - \frac{2\pi\bar{G}}{3} A_0^4 \alpha_0 (3P_0 - \rho_0) r^2 + \frac{\pi\bar{G}}{3} A_0^4 \alpha_0 \rho_0' r^3 + \mathcal{O}(r^4) , && (3.47)
\end{aligned}
$$

where $A_0 = A(\phi_0)$, $\alpha_0 = \alpha(\phi_0)$, and $\rho_0' = \rho'(0)$. Note that unlike in general relativity, $m(r)$ can no longer be interpreted as the total mass in the ball of radius r. In the non-relativistic limit, equation (3.41) implies that

$$
m(r) = \int_0^r \left(4\pi r^2 A^4(\phi)\rho + \frac{1}{2\bar{G}} r^2 \phi'^2 \right) dr . \tag{3.48}
$$

On the other hand, the total mass in the ball of radius r is

$$
\int_{B_r(0)} \rho d^3\vec{x} = 4\pi \int_0^r \rho \frac{A^3(\phi) r^2 dr}{\sqrt{1 - 2\bar{G}m/r}} \rightarrow 4\pi \int_0^r \rho A^3(\phi) r^2 dr , \tag{3.49}
$$

where the arrow denotes the non-relativistic limit.

3.4.2 Exterior

The coordinates defined by (3.11) are not well-suited for the description of the stellar exterior. Instead, the coordinates

$$
g_{\mu\nu} dx^\mu dx^\nu = -e^a dt^2 + e^{-a}(d\xi^2 + e^b d\Omega^2) \tag{3.50}
$$

are used, where a and b depend only on ξ. Due to spherical symmetry and staticity, the scalar field ϕ depends only on ξ. The independent components of the Ricci tensor are then

$$
\begin{aligned}
R_{tt} &= \frac{1}{2} e^{2a-b} (a' e^b)' , && (3.51) \\
R_{\xi\xi} &= \frac{1}{2} \big(((a' e^b)' - (e^b)'') e^{-b} - (b'' + a'^2) \big) , && (3.52) \\
R_{\theta\theta} &= \frac{1}{2} (2 - (e^b)'' + (a' e^b)') , && (3.53)
\end{aligned}
$$

where primes denote $d/d\xi$, and θ denotes the polar angle. The vacuum field equations are obtained by setting $T_{\mu\nu} = 0$ in equations (2.32) and (2.33):

$$R_{tt} = 0 \ , \quad R_{\xi\xi} = 2\phi'^2 \ , \quad R_{\theta\theta} = 0 \ , \quad (\sqrt{|g|}g^{\xi\xi}\phi')' = 0 \ . \tag{3.54}$$

Substituting (3.51)-(3.53) into the above equations yields

$$\begin{aligned} (a'e^b)' &= 0 \ , &(3.55)\\ ((a'e^b)' - (e^b)'')e^{-b} - (b'' + a'^2) &= 4\phi'^2 \ , &(3.56)\\ 2 - (e^b)'' + (a'e^b)' &= 0 \ , &(3.57)\\ (e^b\phi')' &= 0 \ . &(3.58) \end{aligned}$$

The solution is

$$\begin{aligned} a &= \Gamma \log(1 - 2\bar{G}M/\Gamma\xi) \ , &(3.59)\\ b &= 2\log\xi + \log(1 - 2\bar{G}M/\Gamma\xi) \ , &(3.60)\\ \phi &= \phi_\infty - \frac{\Gamma Q}{2M}\log(1 - 2\bar{G}M/\Gamma\xi) \ , &(3.61) \end{aligned}$$

where the constants Γ, M, and Q must satisfy

$$\frac{1}{\Gamma^2} = \frac{Q^2}{M^2} + 1 \ . \tag{3.62}$$

To understand the physical significance of these constants, expand g_{tt} and ϕ in powers of $1/\xi$:

$$\begin{aligned} g_{tt} &= -1 + \frac{2\bar{G}M}{\xi} + \mathcal{O}\left(\frac{1}{\xi^2}\right) \ , &(3.63)\\ \phi &= \phi_\infty + \frac{Q}{\xi} + \mathcal{O}\left(\frac{1}{\xi^2}\right) \ . &(3.64) \end{aligned}$$

Thus, Q is the scalar charge, and ϕ_∞ is the value of the scalar field at infinity. M is called the Einstein-frame mass. It is just a formal constant in the monopole term of the multipole expansion of g_{tt}, and is *not* equal to the mass of the star. In order to find the mass of the star, the Newtonian limit of Einstein's equations and the geodesic equation need to be analyzed. This is done in section 4.

The Jordan-frame mass $\tilde{M}$ is defined by the expansion

$$\tilde{g}_{\tilde{t}\tilde{t}} = -1 + \frac{2\bar{G}\tilde{M}}{\tilde{\xi}} + \mathcal{O}\left(\frac{1}{\tilde{\xi}^2}\right) \ , \tag{3.65}$$

where $\tilde{x}^\mu = (\tilde{t}, \tilde{\xi}, \Omega)$ are rescaled coordinates in which the Jordan-frame metric is asymptotically flat. Ω denotes the angular variables, and asymptotic flatness means that

$$\tilde{g}_{\tilde{\mu}\tilde{\nu}} \longrightarrow \eta_{\tilde{\mu}\tilde{\nu}} \tag{3.66}$$

at spatial infinity. The exterior Jordan-frame metric is

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = \frac{A^2(\phi)}{A^2(\phi_\infty)}\Bigg[- \left(1-\frac{2\bar{G}M}{\Gamma\tilde{\xi}}A(\phi_\infty)\right)^\Gamma d\tilde{t}^2 + \left(1-\frac{2\bar{G}M}{\Gamma\tilde{\xi}}A(\phi_\infty)\right)^{-\Gamma} d\tilde{\xi}^2 + \left(1-\frac{2\bar{G}M}{\Gamma\tilde{\xi}}A(\phi_\infty)\right)^{1-\Gamma} \tilde{\xi}^2 d\Omega^2 \Bigg] , \tag{3.67}$$

where

$$\tilde{t} = A(\phi_\infty)t \ , \quad \tilde{\xi} = A(\phi_\infty)\xi \ . \tag{3.68}$$

Therefore,

$$\tilde{M} = (M - Q\alpha(\phi_\infty))A(\phi_\infty) \ . \tag{3.69}$$

It follows from equation (3.49) that the total baryonic mass of the star is

$$M_{\rm B} = 4\pi \int_0^R \rho \frac{A^3(\phi) r^2 dr}{\sqrt{1-2\bar{G}m/r}} \ . \tag{3.70}$$

3.4.3 Matching at the Boundary

At the stellar boundary ($r = R$), the interior solution described in section 3.4.1 is matched to the exterior solution described in section 3.4.2. In this section, the parameters of the external solution M, Q, and ϕ_∞, are written in terms of the values of m, ϕ, and ϕ' at the stellar boundary. Let Ξ be the value of the coordinate ξ that corresponds to $r = R$. Equating the metrics (3.11) and (3.50) at this point yields the relations

$$\nu = a \ , \quad \frac{1}{1-2\bar{G}m/r}\left(\frac{dr}{d\xi}\right)^2 = e^{-a} \ , \quad r^2 = e^{b-a} \ . \tag{3.71}$$

It follows from equations (3.59)-(3.61) that

$$\bar{G}M = \left.\frac{e^b}{2}\frac{da}{d\xi}\right|_{\xi=\Xi} , \tag{3.72}$$

$$Q/M = \left.-2\frac{d\phi/d\xi}{da/d\xi}\right|_{\xi=\Xi} , \tag{3.73}$$

$$\Delta\phi \equiv \phi_\infty - \phi(\Xi) = \left.-a\cdot\frac{d\phi/d\xi}{da/d\xi}\right|_{\xi=\Xi} . \tag{3.74}$$

When written in terms of m, ϕ, ϕ', and ν, the above equations become

$$\begin{aligned}
\bar{G}M &= R^2\sqrt{1-2\bar{G}m(R)/R}\frac{de^{\nu/2}}{dr}\bigg|_{r=R} , && (3.75)\\
Q/M &= -2\frac{\phi'(R)}{\nu'(R)} , && (3.76)\\
\Delta\phi &= -\frac{\phi'(R)}{\nu'(R)}\nu(R) . && (3.77)
\end{aligned}$$

To calculate the external parameters, $\nu(R)$ and $\nu'(R)$ are needed. It turns out that these quantities can be written in terms of $m(R)$, $\phi(R)$, and $\phi'(R)$. This is useful, because then the external parameters can be calculated without solving equation (3.44). Evaluating equation (3.44) at $r = R$ yields

$$\nu'(R) = R\phi'^2(R) + \frac{2\bar{G}m(R)}{R(R-2\bar{G}m(R))} . \tag{3.78}$$

Combinining the last two relations in (3.71), and using equations (3.59)-(3.60) yields

$$\sqrt{1-2\bar{G}m(R)/R} = \frac{1-(1+1/\Gamma)\bar{G}M/\Xi}{\sqrt{1-2\bar{G}M/\Gamma\Xi}} . \tag{3.79}$$

From equations (3.78), (3.79), (3.71), and (3.59)-(3.61), it follows that

$$\nu(R) = \frac{-2\nu'(R)}{\sqrt{\nu'^2(R)+4\phi'^2(R)}}\text{arctanh}\left(\frac{\sqrt{\nu'^2(R)+4\phi'^2(R)}}{\nu'(R)+2/R}\right) . \tag{3.80}$$

The final expressions for the external parameters can be written in the form

$$\begin{aligned}
\frac{\bar{G}M}{R} &= \frac{\mathcal{K}(x,y)}{2\sqrt{1-2x}}\exp\left[-\frac{\mathcal{K}(x,y)}{\mathcal{H}(x,y)}\text{arctanh}\left(\frac{\mathcal{H}(x,y)}{2(1-x)+y(1-2x)}\right)\right] , && (3.81)\\
\frac{Q}{M} &= -\frac{2\sqrt{y}(1-2x)}{\mathcal{K}(x,y)} , && (3.82)\\
\Delta\phi &= \frac{2\sqrt{y}(1-2x)}{\mathcal{H}(x,y)}\text{arctanh}\left(\frac{\mathcal{H}(x,y)}{2(1-x)+y(1-2x)}\right) , && (3.83)
\end{aligned}$$

where $x = \bar{G}m(R)/R$, $y = R^2\phi'^2(R)$, and

$$\begin{aligned}
\mathcal{H}(x,y) &= \sqrt{\mathcal{K}^2(x,y)+4y(1-2x)^2} , && (3.84)\\
\mathcal{K}(x,y) &= 2x(1-y)+y . && (3.85)
\end{aligned}$$

For a given equation of state, specification of P_0 and ϕ_0 uniquely determines the external parameters R, M, Q, and ϕ_∞. Thus, the point (R, M, Q, ϕ_∞) is constrained to lie in a 2-dimensional hypersurface. This constraint is much stronger than what would be naively expected. If the equations of stellar structure admitted an arbitrary $\phi'(0)$, then this hypersurface would be 3-dimensional.

4 Newtonian Limit of Scalar-Tensor Gravity

In the limit of weak and slowly varying gravitational fields, and non-relativistic particle velocities, general relativity reduces to Newtonian gravity. This limit is applied to scalar-tensor gravity in the Einstein frame.

4.1 Field Equations

The Einstein-frame metric is taken to be

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \ , \tag{4.1}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, and $\gamma_{\mu\nu}$ is a small perturbation. It is assumed that there exists a global rectilinear coordinate system in which $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, and $|\gamma_{\mu\nu}| \ll 1$. In this section, terms quadratic (or higher) in $\gamma_{\mu\nu}$ are always dropped, and indices are raised and lowered with $\eta_{\mu\nu}$. The trace-reverse of $\gamma_{\mu\nu}$ is defined by

$$\bar{\gamma}_{\mu\nu} \equiv \gamma_{\mu\nu} - \frac{1}{2}\gamma\eta_{\mu\nu} \ , \tag{4.2}$$

where $\gamma = \gamma_\mu{}^\mu = \eta^{\mu\nu}\gamma_{\mu\nu}$ is the trace of $\gamma_{\mu\nu}$. The gauge $\partial^\nu\bar{\gamma}_{\mu\nu} = 0$ is chosen. Then the trace-reverse of the Ricci tensor is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{1}{2}\Box\bar{\gamma}_{\mu\nu} \ , \tag{4.3}$$

where $\Box$ is the d'Alembertian operator of $\eta_{\mu\nu}$. The gravitational fields due to energy flux, momentum density, and momentum flux are assumed to be negligible. The energy-momentum tensor can then be taken to be

$$T_{\mu\nu} = \varrho u_\mu u_\nu \ , \tag{4.4}$$

where ϱ is the mass density, $u_\mu = (dt)_\mu$, and t is the global time coordinate. If time derivatives of ϕ and $\gamma_{\mu\nu}$ are neglected, the Einstein-frame field equations (2.32) and (2.33) become

$$\begin{aligned}
\nabla^2\bar{\gamma}_{00} &= -16\pi\bar{G}\varrho + 2(\gamma_{00}-1)|\vec{\nabla}\phi|^2 \ , && (4.5)\\
\nabla^2\bar{\gamma}_{0i} &= 2\gamma_{0i}|\vec{\nabla}\phi|^2 \ , && (4.6)\\
\nabla^2\bar{\gamma}_{ij} &= -4\partial_i\phi\partial_j\phi + 2(\gamma_{ij}+\delta_{ij})|\vec{\nabla}\phi|^2 \ , && (4.7)\\
\nabla^2\phi &= 4\pi\bar{G}\alpha(\phi)\varrho \ . && (4.8)
\end{aligned}$$

Latin indicies run over the spatial coordinates $(1 \ldots 3)$, and 0 denotes the time coordinate. When written in terms of $\gamma_{\mu\nu}$, the field equations are

$$\begin{aligned}
\nabla^2 \gamma_{00} &= -8\pi \bar{G} \varrho + (2\gamma_{00} + \gamma)|\vec{\nabla}\phi|^2 , && (4.9) \\
\nabla^2 \gamma &= -16\pi \bar{G} \varrho - 2(\gamma + 2)|\vec{\nabla}\phi|^2 , && (4.10) \\
\nabla^2 \gamma_{0i} &= 2\gamma_{0i}|\vec{\nabla}\phi|^2 , && (4.11) \\
\nabla^2 \gamma_{ij} &= -4\partial_i\phi\partial_j\phi + 2(\gamma_{ij} + \delta_{ij})|\vec{\nabla}\phi|^2 && (4.12) \\
\nabla^2 \phi &= 4\pi \bar{G} \alpha(\phi) \varrho . && (4.13)
\end{aligned}$$

Note that in the Newtonian limit of general relativity, γ_{00} is the only non-vanishing component of $\gamma_{\mu\nu}$. It can be related to the Newtonian potential. However, in the Newtonian limit of scalar-tensor gravity, none of the components of $\gamma_{\mu\nu}$ vanish.

4.2 Stellar Structure

The Newtonian limit (3.21) of the equations of stellar structure (3.41)-(3.44) is

$$\begin{aligned}
\bar{G}m' &= 4\pi \bar{G} r^2 A^4(\phi)\varrho + \frac{1}{2} r^2 \phi'^2 , && (4.14) \\
P' &= -\varrho \left(\frac{\bar{G}m}{r^2} + \frac{1}{2} r\phi'^2 + \alpha(\phi)\phi' \right) , && (4.15) \\
\phi'' &= 4\pi \bar{G} A^4(\phi)\varrho(\alpha(\phi) + r\phi') - \frac{2}{r}\phi' , && (4.16) \\
\nu' &= \frac{2\bar{G}m}{r^2} + r\phi'^2 . && (4.17)
\end{aligned}$$

In the absence of sources $(P = \varrho = 0)$, the solution to these equations is

$$\begin{aligned}
\nu &= -2\bar{G}M/r , && (4.18) \\
\bar{G}m &= \bar{G}M - \bar{G}^2 Q^2/2r , && (4.19) \\
\phi &= \phi_\infty + \bar{G}Q/r . && (4.20)
\end{aligned}$$

The parameters M, Q, and ϕ_∞ can be expressed in terms of the fields at the stellar boundary:

$$\begin{aligned}
\bar{G}M &= \bar{G}m(R) + \frac{1}{2} R^3 \phi'^2(R) , && (4.21) \\
\bar{G}Q &= -R^2 \phi'(R) , && (4.22) \\
\phi_\infty &= \phi(R) + R\phi'(R) . && (4.23)
\end{aligned}$$

4.3 Equations of Motion

In the Newtonian limit, the equations of motion of a test particle become

$$\frac{d^2x^i}{dt^2} = -\tilde{\Gamma}^i{}_{00} \,. \tag{4.24}$$

It follows from equation (2.40) that

$$\tilde{\Gamma}^\mu{}_{00} = \Gamma^\mu{}_{00} - \alpha(\phi)(\partial^\mu\phi(\gamma_{00}-1) + \gamma^{\mu i}\partial_i\phi) \,. \tag{4.25}$$

The relevant Einstein-frame Christoffel symbols are given by

$$\Gamma^\mu{}_{00} = -\frac{1}{2}\partial^\mu\gamma_{00} \,. \tag{4.26}$$

Note that the fields γ_{0i} do not enter into the equations of motion.

Now specialize to the exterior field of a spherical body. The gravitational and scalar fields are given by equations (4.18)-(4.20). Thus, the spherical components of $\gamma_{\mu\nu}$ are given by

$$\gamma_{00} = 1 - e^{-2\bar{G}M/r} \,, \quad \gamma_{rr} = \frac{1}{1 - 2\bar{G}M/r + \bar{G}^2Q^2/r^2} - 1 \,. \tag{4.27}$$

The cartesian components of $\gamma_{\mu\nu}$ can be calculated using the formulae

$$\gamma_{ii} = \frac{1}{r^2}\left(\gamma_{rr}(x^i)^2 + \sum_{j\neq i}(x^j)^2\right) \quad \text{(no sum on i)} \,, \tag{4.28}$$

$$\gamma_{ij} = (\gamma_{rr}-1)\frac{x^ix^j}{r^2} \quad (i \neq j) \,. \tag{4.29}$$

In vector notation, equation (4.24) becomes

$$\begin{aligned}\frac{d^2\vec{x}}{dt^2} &= -\frac{\bar{G}\vec{x}}{r^3}\Bigg[e^{-2\bar{G}M/r}(M - Q\alpha(\phi)) \\ &\qquad + Q\alpha(\phi)\left(\frac{1}{1 - 2\bar{G}M/r + \bar{G}^2Q^2/r^2} - 1\right)\Bigg] \\ &= -\frac{\bar{G}\tilde{M}\vec{x}}{r^3A(\phi_\infty)}\left[1 - \frac{2\bar{G}M}{r} + \frac{\bar{G}QA(\phi_\infty)}{r\tilde{M}}(2M\alpha(\phi_\infty) - Q\alpha'(\phi_\infty))\right] \\ &\quad + \mathcal{O}(\bar{G}^3) \,. \end{aligned} \tag{4.30}$$

When written in terms of the coordinates

$$\hat{t} = tA(\phi_\infty) \,, \quad \hat{\vec{x}} = \vec{x}A(\phi_\infty) \,, \tag{4.31}$$

in which the Jordan-frame metric is asymptotically flat, the equations of motion become

$$\begin{aligned}\frac{d^2\hat{\vec{x}}}{d\hat{t}^2} &= -\frac{\bar{G}\tilde{M}\hat{\vec{x}}}{\hat{r}^3}\bigg[1 - \frac{2\bar{G}MA(\phi_\infty)}{\hat{r}} \\ &\quad + \frac{\bar{G}QA^2(\phi_\infty)}{\hat{r}\tilde{M}}(2M\alpha(\phi_\infty) - Q\alpha'(\phi_\infty))\bigg] + \mathcal{O}(\bar{G}^3)\ . \qquad (4.32)\end{aligned}$$

Re-expressed in terms of the effective gravitational constant (defined in equation (2.29)), the above equation becomes

$$\frac{d^2\hat{\vec{x}}}{d\hat{t}^2} = -\frac{G\tilde{M}\hat{\vec{x}}}{\hat{r}^3 A^2(\phi_\infty)} + \mathcal{O}(G^2)\ . \qquad (4.33)$$

Therefore, the gravitational mass is

$$\begin{aligned}M_{\rm G} &= \tilde{M}/A^2(\phi_\infty) \\ &= \frac{M - Q\alpha(\phi_\infty)}{A(\phi_\infty)}\ . \qquad (4.34)\end{aligned}$$

5 The Quadratic Model

A framework has been developed to systematically add relativistic corrections to the Newtonian limit described in the previous section. It is called the parametrized post-Newtonian formalism (PPN) [48]. For the scalar-tensor theory defined in section 2.3.3, the first post-Newtonian corrections depend only on $\alpha(\phi_\infty)$ and $\alpha'(\phi_\infty)$. The Jordan-frame 1PN parameters $\tilde{\gamma}$, $\tilde{\beta}$ are given by

$$\tilde{\gamma} = 1 - \left.\frac{2\alpha^2}{1+\alpha^2}\right|_{\phi=\phi(\infty)}, \tag{5.1}$$

$$\tilde{\beta} = 1 + \left.\frac{\alpha^2\alpha'}{2(1+\alpha^2)^2}\right|_{\phi=\phi(\infty)}. \tag{5.2}$$

Thus, if attention is restricted to first post-Newtonian corrections, the most general coupling function is $A(\phi) = \exp(\alpha\phi + \frac{1}{2}\beta\phi^2)$, called the quadratic model.

It is difficult to obtain exact analytical solutions for stellar interiors in scalar-tensor gravity. One such solution has been found by Yazadjiev [49], for the rather complicated coupling function

$$A(\phi) = e^{-b\phi/(2-a)}\left[(3-a)e^{\phi/b} - (2-a)\right]^{b^2/(2-a)(3-a)}, \tag{5.3}$$

$$\alpha(\phi) = \frac{b(1-e^{\phi/b})}{(3-a)e^{\phi/b} - (2-a)}, \tag{5.4}$$

where a and b are constants satisfying $a^2 + b^2 = 1$. The density profile and equation of state of Yazadjiev's solution are involved, and it is not clear whether they describe realistic stars.

In this section, stellar structure in the quadratic model is investigated. Perturbation theory is used to find analytic stellar interior solutions to zeroth order in the couplings $\chi \equiv \alpha^2(\phi_0)$ and β. The results are compared to full numerical calculations, obtained using the Runge-Kutta-Fehlberg algorithm available in Maple.

5.1 Constant-Density Stars

It is hard to describe realistic stars analytically, particularily in the strong-field regime. However, many properties of stellar structure can be understood from a simplified model, where the energy-mass density is taken to be constant. No equation of state is specified, and the pressure profile is found by solving the equation of hydrostatic equilibrium.

In general relativity, the constant-density solution was obtained by Schwarzschild in 1916. Moreover, a theorem of Buchdahl establishes an inequality between the mass-radius ratio of a star with any equation of state, and that of a constant-density star [7].

5.1.1 Equations

Consider a star of constant density ρ_0 in the quadratic model. Replace the coupling constant α with $\chi = \alpha^2(\phi_0) = (\alpha + \beta\phi_0)^2$, and carry out the change of variables

$$\mu = \bar{G}m/r \ , \ \Pi = P/\rho_0 \ , \ \varphi = (\phi - \phi_0)/\alpha(\phi_0) \ , \ u = 8\pi\bar{G}\rho_0 A^4(\phi_0)r^2 \ . \tag{5.5}$$

The equations of stellar structure (3.41)-(3.44) then become

$$\begin{aligned}
\dot{\mu} &= -\frac{\mu}{2u} + \frac{e^{4\chi\varphi(1+\beta\varphi/2)}}{4} + \chi u(1-2\mu)\dot{\varphi}^2 \ , && (5.6)\\
\dot{\Pi} &= -(1+\Pi)\left(\frac{\mu}{2u(1-2\mu)} + \frac{\Pi e^{4\chi\varphi(1+\beta\varphi/2)}}{4(1-2\mu)} + \chi\dot{\varphi}(1+u\dot{\varphi}+\beta\varphi)\right) \ , && (5.7)\\
\ddot{\varphi} &= -\frac{(3-4\mu)\dot{\varphi}}{2u(1-2\mu)} + \frac{e^{4\chi\varphi(1+\beta\varphi/2)}}{8u(1-2\mu)}\big((1+\beta\varphi)(1-3\Pi) + 2(1-\Pi)u\dot{\varphi}\big) \ , && (5.8)\\
\dot{\nu} &= \frac{\Pi e^{4\chi\varphi(1+\beta\varphi/2)}}{2(1-2\mu)} + \frac{\mu}{u(1-2\mu)} + 2\chi u\dot{\varphi}^2, && (5.9)
\end{aligned}$$

where dots denote d/du. The initial conditions at $u = 0$ are

$$\mu(0) = 0 \ , \quad \Pi(0) = P_0/\rho_0 \equiv \eta \ , \quad \varphi(0) = 0 \ , \quad \dot{\varphi}(0) = (1-3\eta)/12 \ . \tag{5.10}$$

The condition for μ is required for the metric (3.11) to be regular at $r = 0$. The condition for $\dot{\varphi}$ is found by multiplying equation (5.8) by $u(1-2\mu)$, and evaluating at $u = 0$. Note that $\dot{\varphi}(0)$ vanishes when $\eta = \frac{1}{3}$. This happens when the energy-momentum tensor is traceless at the centre of the star, i.e. $\tilde{T}(0) = 3P_0 - \rho_0 = 0$. In this case, φ is constant, and the field equations reduce to those of general relativity.

The variable ν can be written in terms of the other variables, by combining equations (5.9) and (5.7):

$$-\frac{\dot{\Pi}}{1+\Pi} = \frac{\dot{\nu}}{2} + \chi(1+\beta\varphi)\dot{\varphi} \ . \tag{5.11}$$

Integration of the above equation yields

$$\nu(u) = \nu_0 + 2\log\left(\frac{1+\eta}{1+\Pi(u)}\right) - 2\chi\varphi(u)(1+\beta\varphi(u)/2) , \tag{5.12}$$

where $\nu_0 = \nu(0)$. Now expand the remaining dependent variables μ, Π, and φ in a series in the couplings χ and β:

$$\mu = \sum_{i,j\geq 0} \mu^{(i,j)}\chi^i\beta^j , \quad \Pi = \sum_{i,j\geq 0} \Pi^{(i,j)}\chi^i\beta^j , \quad \varphi = \sum_{i,j\geq 0} \varphi^{(i,j)}\chi^i\beta^j . \tag{5.13}$$

5.1.2 Interior Profiles

In the limit $\chi = \beta = 0$, equations (5.6)-(5.8) can be solved analytically:

$$\mu^{(0,0)} = \frac{1}{2}\sin^2\frac{v}{2} , \tag{5.14}$$

$$\Pi^{(0,0)} = \frac{(1+\eta)-(1+3\eta)\cos\frac{v}{2}}{(1+3\eta)\cos\frac{v}{2}-3(1+\eta)} , \tag{5.15}$$

$$\dot{\varphi}^{(0,0)} = \frac{(\frac{1}{3}+\eta)\sec\frac{v}{2}+\frac{3}{4}(1+\eta)(1-v\csc v)\csc^2\frac{v}{2}}{(1+3\eta)\cos\frac{v}{2}-3(1+\eta)} , \tag{5.16}$$

$$U^{(0,0)} = \frac{12\eta(1+2\eta)}{(1+3\eta)^2} , \tag{5.17}$$

where $u = 3\sin^2\frac{v}{2}$, and U is the value of the independent variable corresponding to $r = R$. Integration of equation (5.16) yields

$$\begin{aligned}\varphi^{(0,0)} = {} & \frac{81}{32(2+3\eta)^{3/2}}\Bigg\{(\eta+\tfrac{1}{3})^2(\eta+1)\left[2\mathrm{dilog}\Omega_- - 2\mathrm{dilog}\Omega_+ + iv\log\frac{\Omega_-}{\Omega_+}\right] \\ & + \frac{4}{3}\sqrt{2+3\eta}\Bigg[(\eta+\tfrac{5}{9})(\eta-\tfrac{1}{3})\log(6(1+\eta)e^{iv/2} \\ & \qquad - (1+3\eta)(e^{iv}+1)) \\ & - \frac{iv}{e^{iv}-1}\Bigg(\tfrac{8}{9}(\eta+\tfrac{2}{3})+\tfrac{1}{3}(1+\eta)(1+3\eta)e^{iv/2} \\ & \qquad + (\eta^2+\tfrac{10}{9}\eta+\tfrac{11}{27})e^{iv}\Bigg)\Bigg]\Bigg\} \\ & + C(\eta) ,\end{aligned} \tag{5.18}$$

where

$$\Omega_\pm = 1 - \frac{(1+3\eta)e^{iv/2}}{3(1+\eta)\pm 2\sqrt{2+3\eta}} , \tag{5.19}$$

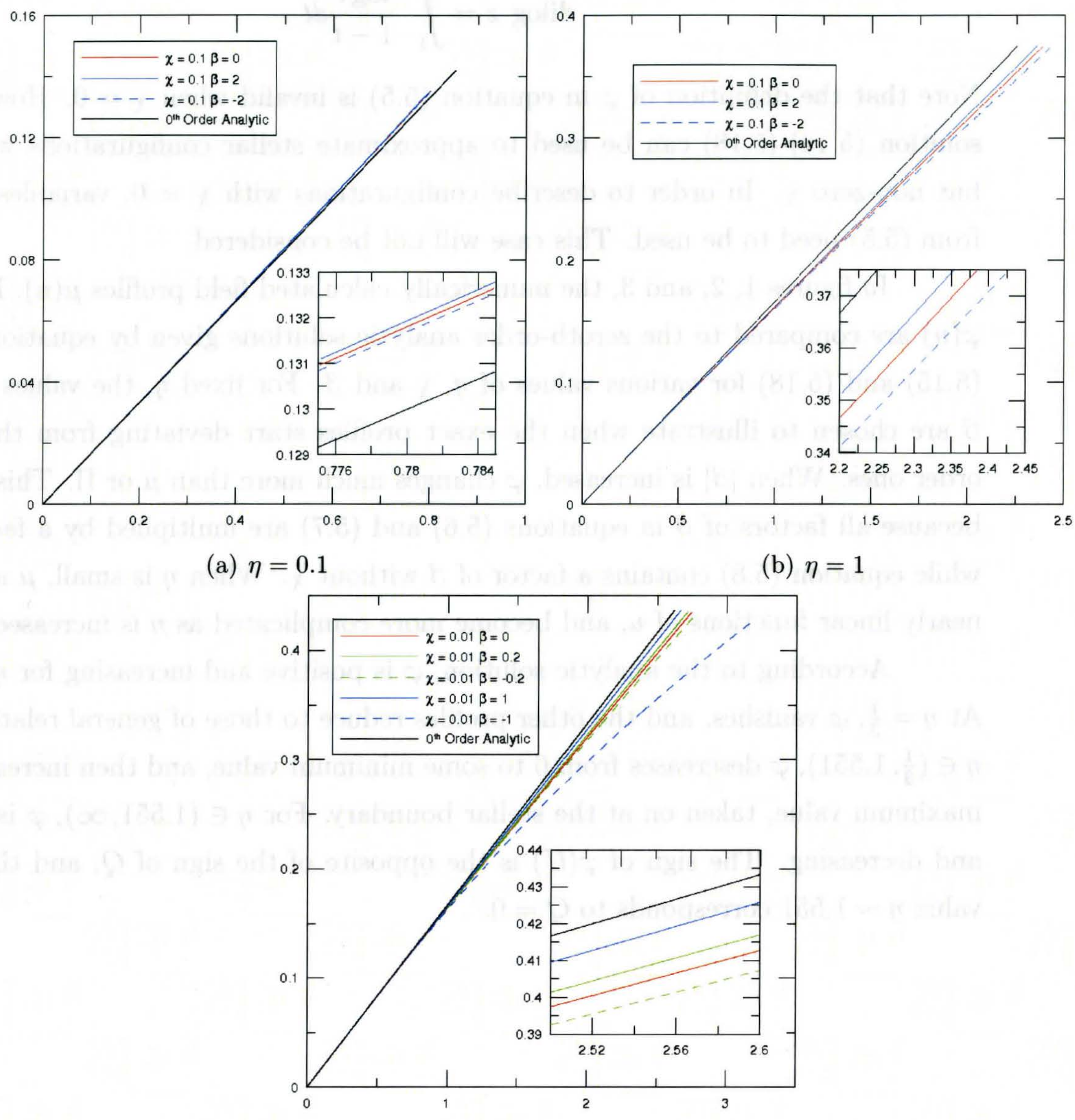
$C(\eta)$ is chosen such that $\varphi^{(0,0)}(0) = 0$, and the dilogarithm is defined by

$$\text{dilog } z = \int_1^z \frac{\log t}{1-t} dt \ . \tag{5.20}$$

Note that the definition of φ in equation (5.5) is invalid when $\chi = 0$. However, the solution (5.14)-(5.18) can be used to approximate stellar configurations with small but non-zero χ. In order to describe configurations with $\chi = 0$, variables different from (5.5) need to be used. This case will not be considered.

In figures 1, 2, and 3, the numerically calculated field profiles $\mu(u)$, $\Pi(u)$, and $\varphi(u)$ are compared to the zeroth-order analytic solutions given by equations (5.14), (5.15) and (5.18) for various values of η, χ and β. For fixed η, the values of χ and β are chosen to illustrate when the exact profiles start deviating from the zeroth-order ones. When $|\beta|$ is increased, φ changes much more than μ or Π. This happens because all factors of β in equations (5.6) and (5.7) are multiplied by a factor of χ, while equation (5.8) contains a factor of β without χ. When η is small, μ and Π are nearly linear functions of u, and become more complicated as η is increased.

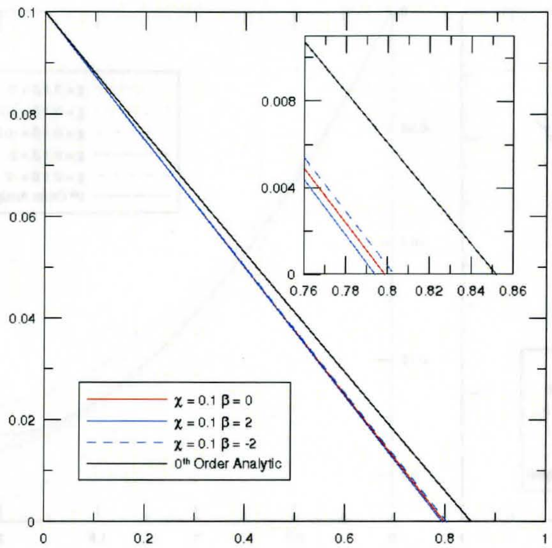
According to the analytic solution, φ is positive and increasing for $\eta \in (0, \frac{1}{3})$. At $\eta = \frac{1}{3}$, φ vanishes, and the other profiles reduce to those of general relativity. For $\eta \in (\frac{1}{3}, 1.551)$, φ descreases from 0 to some minimum value, and then increases to its maximum value, taken on at the stellar boundary. For $\eta \in (1.551, \infty)$, φ is negative and decreasing. The sign of $\dot{\varphi}(U)$ is the opposite of the sign of Q, and the critical value $\eta \sim 1.551$ corresponds to $Q = 0$.



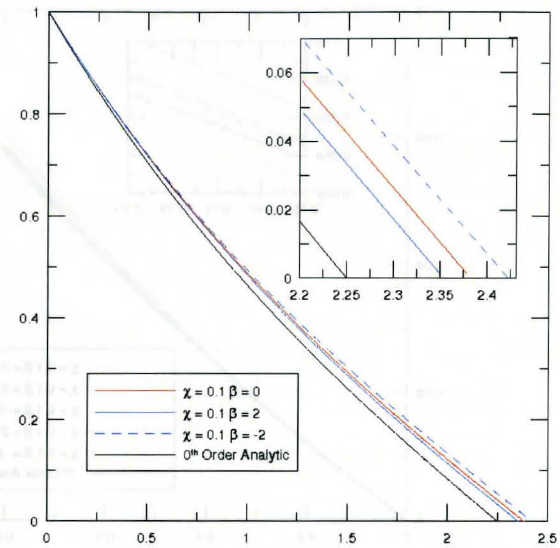
(a) $\eta = 0.1$ (b) $\eta = 1$

(c) $\eta = 10$

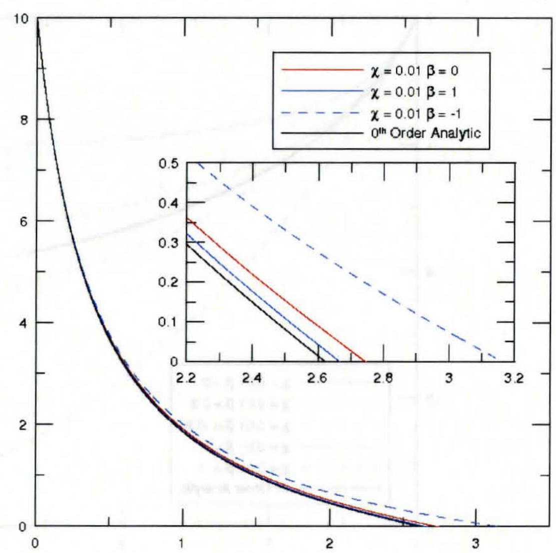
Figure 1: The function $\mu = (1 - g^{rr})/2$ plotted versus the dimensionless variable $u = 8\pi\bar{G}\rho_0 A^4(\phi_0) r^2$ for a star of constant density ρ_0 and central pressure $P_0 = \eta\rho_0$, for various values of the couplings $\chi = \alpha^2(\phi_0)$ and β.



(a) $\eta = 0.1$

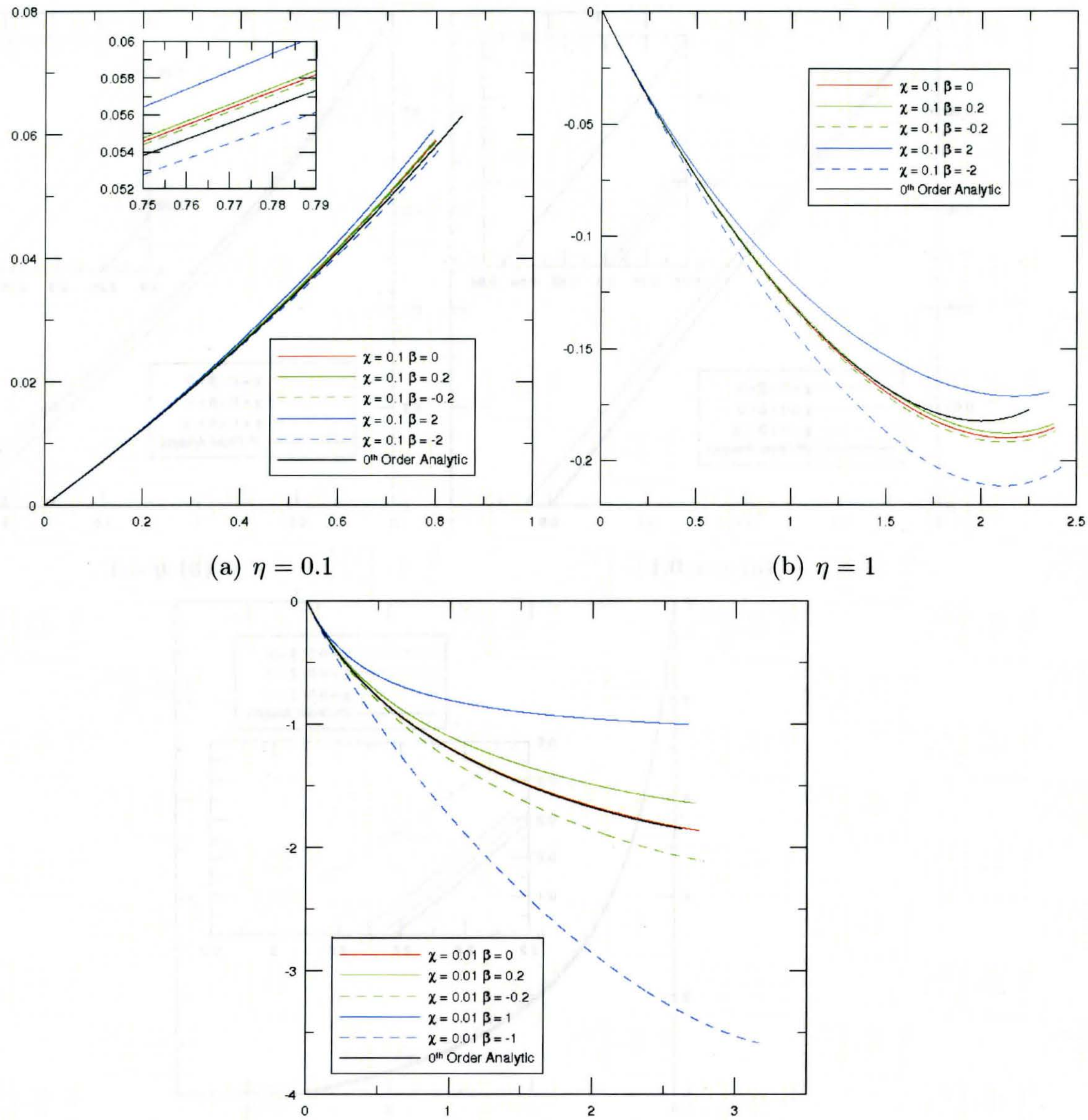


(b) $\eta = 1$



(c) $\eta = 10$

Figure 2: The rescaled pressure $\Pi = P/\rho_0$ plotted versus u for a star of constant density.



(a) $\eta = 0.1$

(b) $\eta = 1$

(c) $\eta = 10$

Figure 3: The normalized and rescaled scalar field $\varphi = (\phi - \phi_0)/\alpha(\phi_0)$ plotted versus u for a star of constant density.

By means of perturbation theory, the first corrections to the zeroth-order profiles can be expressed as the following integrals:

$$\mu^{(1,0)} = \frac{1}{\sqrt{u}} \int du \sqrt{u} [\varphi^{(0,0)} + u(1-u/3)\dot{\varphi^{(0,0)}}^2] \,, \tag{5.21}$$

$$U^{(1,0)} = 12 \left(\frac{1+\eta}{1+3\eta} \right)^2 \Pi^{(1,0)}(U^{(0,0)}) \,, \tag{5.22}$$

$$\begin{aligned}
\Pi^{(1,0)} &= -\frac{4(1+\eta)\mu^{(1,0)}}{\sqrt{1-u/3}} \cdot \frac{(1+\eta)(u-3/2)\sqrt{1-u/3} + \frac{1}{2}(1+3\eta)}{((1+3\eta)\sqrt{1-u/3} - 3(1+\eta))^2} \\
&+ \frac{2(1+\eta)\varphi^{(0,0)}}{(1+3\eta)\sqrt{1-u/3} - 3(1+\eta)} \\
&+ \frac{4(1+\eta)\sqrt{1-u/3}}{((1+3\eta)\sqrt{1-u/3} - 3(1+\eta))^2} \times \\
& \int du \left\{ \frac{u(\dot{\varphi}^{(0,0)})^2}{\sqrt{1-u/3}} \left[(1+3\eta)\sqrt{1-u/3} - \tfrac{1}{3}(1+\eta)(u^2 - \tfrac{9}{2}u + 9) \right] \right. \\
& \qquad \left. - \frac{(1+\eta)\varphi^{(0,0)}(u^2 - \frac{9}{2}u + \frac{9}{4})}{3(1-u/3)^{3/2}} \right\} ,
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
\dot{\varphi^{(0,1)}} &= \frac{1}{4u^{3/2}\sqrt{1-u/3}[(1+3\eta)\sqrt{1-u/3} - 3(1+\eta)]} \times \\
& \int du \varphi^{(0,0)} \sqrt{\frac{u}{1-u/3}} [2(1+3\eta)\sqrt{1-u/3} - 3(1+\eta)] \,,
\end{aligned} \tag{5.24}$$

$$\mu^{(0,1)} = \Pi^{(0,1)} = U^{(0,1)} = 0 \,. \tag{5.25}$$

The full expressions for $\mu^{(1,0)}$ and $\Pi^{(1,0)}$ are complicated. They can be approximated by an expansion in $1/\eta$. The power series expansion of $\mu^{(1,0)}$ about $u = 0$ has the form

$$\begin{aligned}
\mu^{(1,0)}(u) &= (a_{22}\eta^2 + a_{21}\eta + a_{20})u^2 \\
&+ (a_{33}\eta^3 + a_{32}\eta^2 + a_{31}\eta + a_{30})u^3 \\
&+ (a_{44}\eta^4 + a_{43}\eta^3 + a_{42}\eta^2 + a_{41}\eta + a_{40})u^4 \\
&+ \dots
\end{aligned} \tag{5.26}$$

$$\equiv g^{(0)}(\eta u) + \frac{g^{(1)}(\eta u)}{\eta} + \frac{g^{(2)}(\eta u)}{\eta^2} + \frac{g^{(3)}(\eta u)}{\eta^3} + \dots \,, \tag{5.27}$$

where

$$\begin{aligned} g^{(0)}(t) &= a_{22}t^2 + a_{33}t^3 + a_{44}t^4 + \dots , \\ g^{(1)}(t) &= a_{21}t^2 + a_{32}t^3 + a_{43}t^4 + \dots , \\ g^{(2)}(t) &= a_{20}t^2 + a_{31}t^3 + a_{42}t^4 + \dots . \end{aligned} \tag{5.28}$$

The coefficients a_{ij} can be calculated symbolically using a computer algebra program such as Maple, by repeatedly differentiating the equations of stellar structure, and evaluating them at $u = 0$. The results of such a calculation suggest that the functions $g^{(k)}$ have the general form

$$g^{(k)}(t) = a_k t^k + \sum_{n=k+1}^{\infty} (-t/4)^n \frac{p_k(n)}{(2n+1)(n-1)} , \tag{5.29}$$

where a_k are rational numbers, and p_k are polynomials of degree $2 + \lfloor k/2 \rfloor$ with rational coefficients. If p_k is given, then the infinite sum in (5.29) can be calculated analytically. The first few polynomials p_k and functions $g^{(k)}$ are tabulated in Appendix A.

In principle, it should be possible to prove that (5.29) holds for all k, and derive an analytic expression for $p_k(n)$, by expanding equations (5.16) and (5.21) in power series. However, carrying out this calculation results in very long formulas that involve multiple sums and hypergeometric functions.

The sum in equation (5.29) converges only for $|t| < 4$, i.e. $u < 4/\eta$. However, if p_k is given, then $g^{(k)}$ can be evaluated analytically, and the resulting function is well-defined for all u.

A similar expansion can be carried out for $\Pi^{(1,0)}$. Rather than working directly with Π, it is more convenient to define

$$\xi = \frac{3(1+\Pi)\sqrt{1-2\mu} - (1+3\Pi)}{(1+3\Pi) - (1+\Pi)\sqrt{1-2\mu}} . \tag{5.30}$$

Expand ξ in powers of χ and β:

$$\xi = \sum_{i,j \geq 0} \xi^{(i,j)} \chi^i \beta^j . \tag{5.31}$$

Then $\xi^{(0,0)} = 1/\eta$, and $\xi^{(1,0)}$ has a power series expansion of the form

$$\begin{aligned}
\xi^{(1,0)}(u) &= (b_{1,1}\eta + b_{1,0} + b_{1,-1}\eta^{-1} + b_{1,-2}\eta^{-2})u \\
&\quad +(b_{2,2}\eta^2 + b_{2,1}\eta + b_{2,0} + b_{2,-1}\eta^{-1} + b_{2,-2}\eta^{-2})u^2 \\
&\quad +(b_{3,3}\eta^3 + b_{3,2}\eta^2 + b_{3,1}\eta + b_{3,0} + b_{3,-1}\eta^{-1} + b_{3,-2}\eta^{-2})u^3 \\
&\quad +\dots \qquad (5.32) \\
&\equiv h^{(0)}(u\eta) + \frac{h^{(1)}(u\eta)}{\eta} + \frac{h^{(2)}(u\eta)}{\eta^2} + \dots , \qquad (5.33)
\end{aligned}$$

where

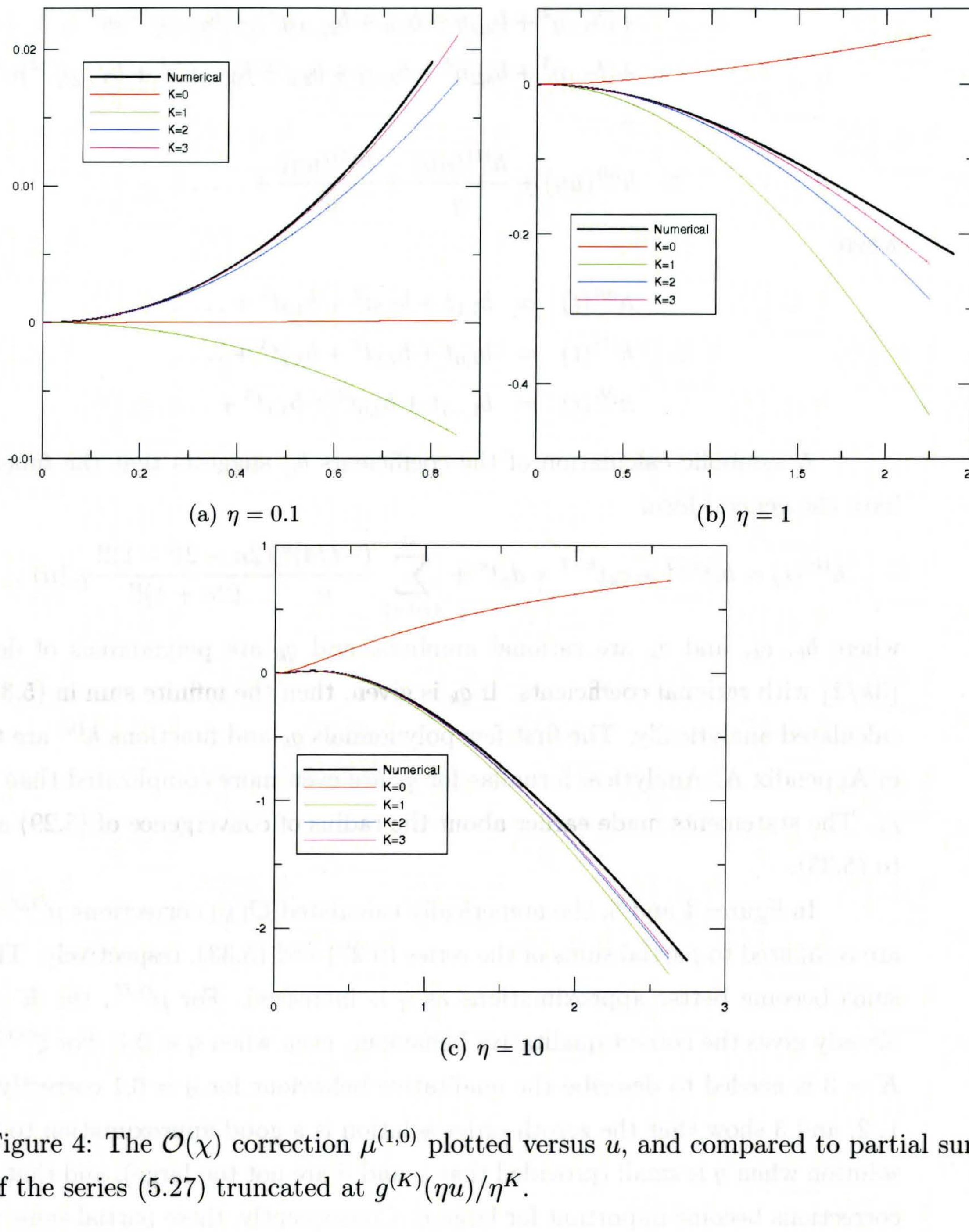
$$\begin{aligned}
h^{(0)}(t) &= b_{1,1}t + b_{2,2}t^2 + b_{3,3}t^3 + \dots , \\
h^{(1)}(t) &= b_{1,0}t + b_{2,1}t^2 + b_{3,2}t^3 + \dots , \\
h^{(2)}(t) &= b_{1,-1}t + b_{2,0}t^2 + b_{3,1}t^3 + \dots . \qquad (5.34)
\end{aligned}$$

A symbolic calculation of the coefficients b_{ij} suggests that the functions $h^{(k)}$ have the general form

$$h^{(k)}(t) = b_k t^{k-2} + c_k t^{k-1} + d_k t^k + \sum_{n=k+1}^{\infty} \frac{(-t/4)^n}{n} \frac{(2n-2k-1)!!}{(2n+1)!!} q_k(n) , \qquad (5.35)$$

where b_k, c_k, and d_k are rational numbers, and q_k are polynomials of degree $2 + \lfloor 3k/2 \rfloor$ with rational coefficients. If q_k is given, then the infinite sum in (5.35) can be calculated analytically. The first few polynomials q_k and functions $h^{(k)}$ are tabulated in Appendix A. Analytical formulas for q_k are even more complicated than those for p_k. The statements made earlier about the radius of convergence of (5.29) apply also to (5.35).

In figures 4 and 5, the numerically calculated $\mathcal{O}(\chi)$ corrections $\mu^{(1,0)}$ and $\xi^{(1,0)}$ are compared to partial sums of the series (5.27) and (5.33), respectively. The partial sums become better approximations as η is increased. For $\mu^{(1,0)}$, the $K = 2$ term already gives the correct qualitative behaviour, even when $\eta = 0.1$. For $\xi^{(1,0)}$, at least $K = 3$ is needed to describe the qualitative behaviour for $\eta = 0.1$ correctly. Figures 1, 2, and 3 show that the zeroth-order solution is a good approximation to the exact solution when η is small (provided that χ and β are not too large), and that the $\mathcal{O}(\chi)$ corrections become important for large η. Consequently, these partial sums work well in the regime where the corrections are important, and don't work well in the regime where they are not important.



(a) $\eta = 0.1$ (b) $\eta = 1$

(c) $\eta = 10$

Figure 4: The $\mathcal{O}(\chi)$ correction $\mu^{(1,0)}$ plotted versus u, and compared to partial sums of the series (5.27) truncated at $g^{(K)}(\eta u)/\eta^K$.

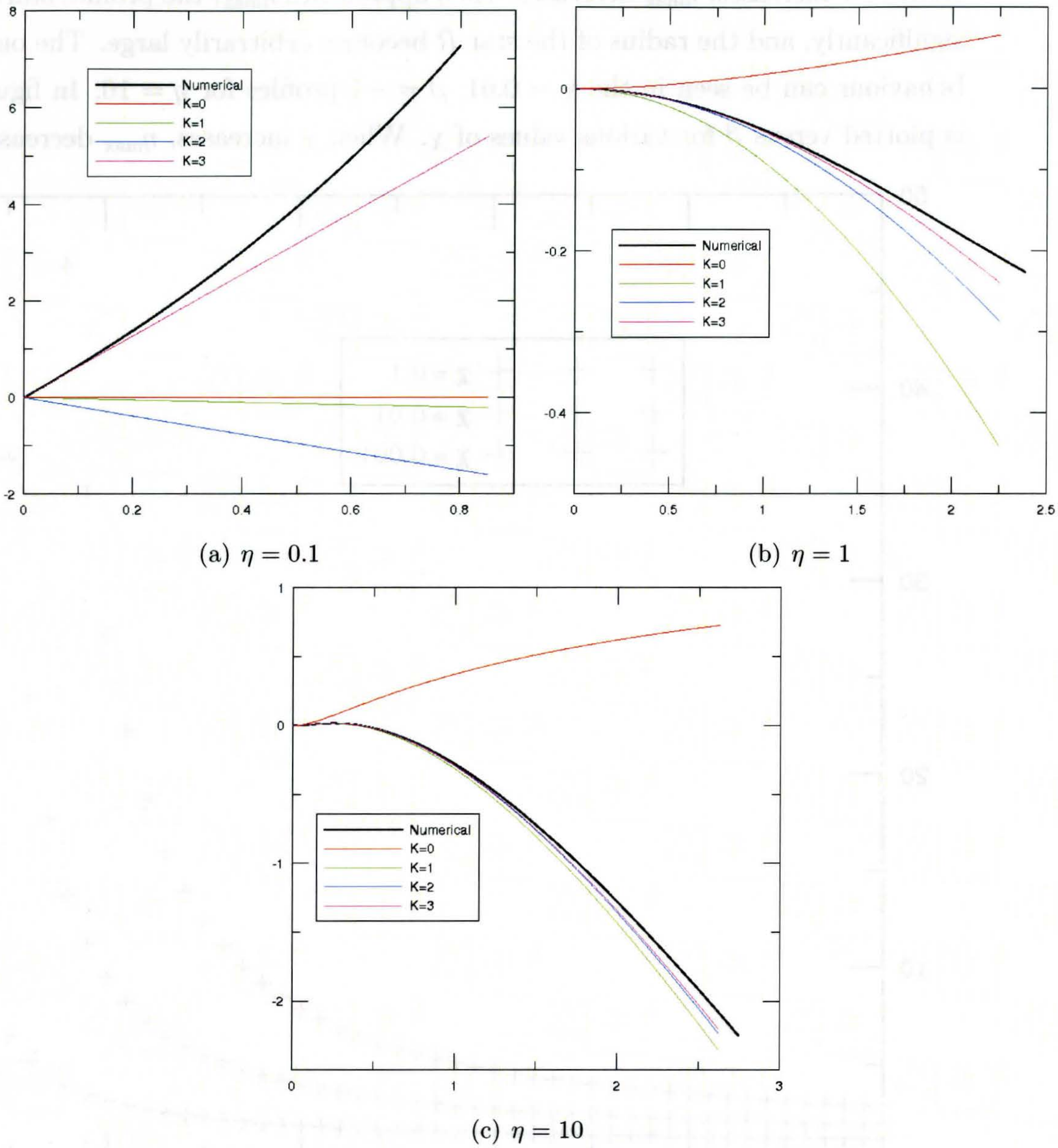


Figure 5: The $\mathcal{O}(\chi)$ correction $\xi^{(1,0)}$ plotted versus u, and compared to partial sums of the series (5.33) truncated at $h^{(K)}(\eta u)/\eta^K$.

It was found numerically that if β is negative, there is a maximum value $\eta_{\mathrm{max}}(\chi, \beta)$, above which no constant-density stellar solutions exist. As the absolute value of β increases, η_{max} decreases. As η approaches η_{max}, the profiles start changing significantly, and the radius of the star R becomes arbitrarily large. The onset of this behaviour can be seen in the $\chi = 0.01$, $\beta = -1$ profiles for $\eta = 10$. In figure 6, η_{max} is plotted versus β for various values of χ. When χ increases, η_{max} decreases.

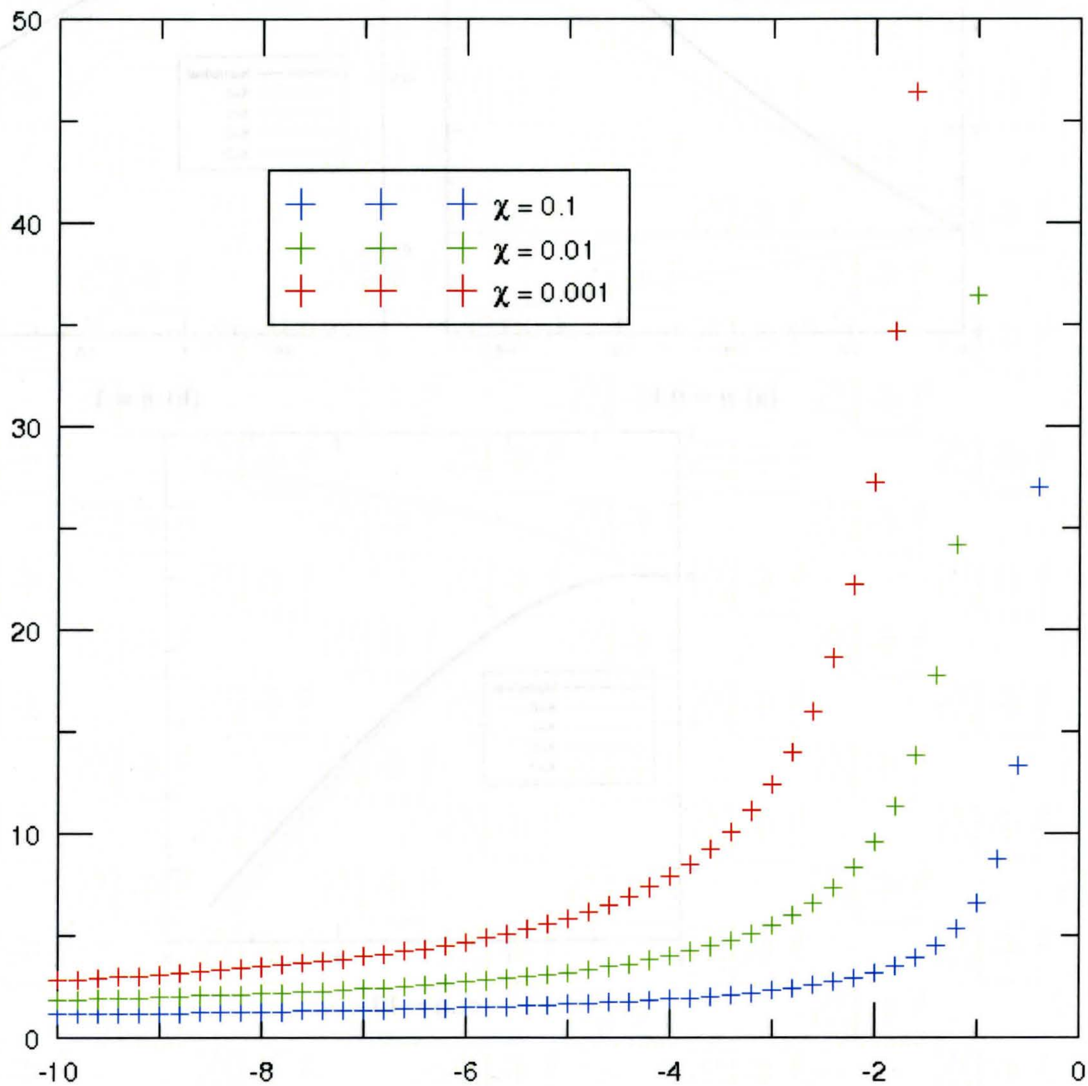


Figure 6: The maximum value of η for constant-density stars, plotted versus β, for various values of χ.

5.1.3 Effective Couplings

In figures 7 and 8, the effective coupling constants G and $\alpha(\phi)$, which were defined in equations (2.29) and (2.27), are plotted in the vincinity of a constant-density star. The vertical lines denote the stellar boundary. In the $\eta = 0.1$ and $\eta = 1$ figures, only the stellar boundary of the $\beta = 0$ profile is shown. The boundaries of the other profiles are very close to this one. For the case of $\eta = 10$, the boundary of the $\beta = -1$ profile is shown, because it differs substantially from the $\beta = 0$ boundary.

In terms of the variables (5.5), G and α are given by

$$\frac{G}{G_0} = \exp(2\chi\varphi(1 + \beta\varphi/2)) \; , \tag{5.36}$$

$$\frac{\alpha}{\alpha_0} = 1 + \beta\varphi \; , \tag{5.37}$$

where the subscript 0 denotes evaluation at $u = 0$. These ratios are independent of ϕ_0.

For $\eta = 0.1$, the effective gravitational constant G increases as one moves radially outward. For $\eta = 1$, G decreases to a minimum value near the stellar boundary, and then increases in the exterior. For $\eta = 10$, G decreases in the stellar interior, and continues decreasing at a slower rate in the exterior. A positive value of β causes G to increase faster or decrease slower, while a negative value of β causes G to increase slower or decrease faster.

For $\eta = 0.1$, a positive value of β causes the scalar-matter coupling α to increase as one moves radially outward, while a negative value of β causes α to decrease. One observes an approximate symmetry:

$$\beta \to -\beta \; , \; \frac{\alpha}{\alpha_0} \to 2 - \frac{\alpha}{\alpha_0} \; . \tag{5.38}$$

This symmetric behaviour is violated as η is increased. For $\eta = 1$ and $\eta = 10$, a positive value of β causes α to decrease, while a negative value of β causes α to increase.

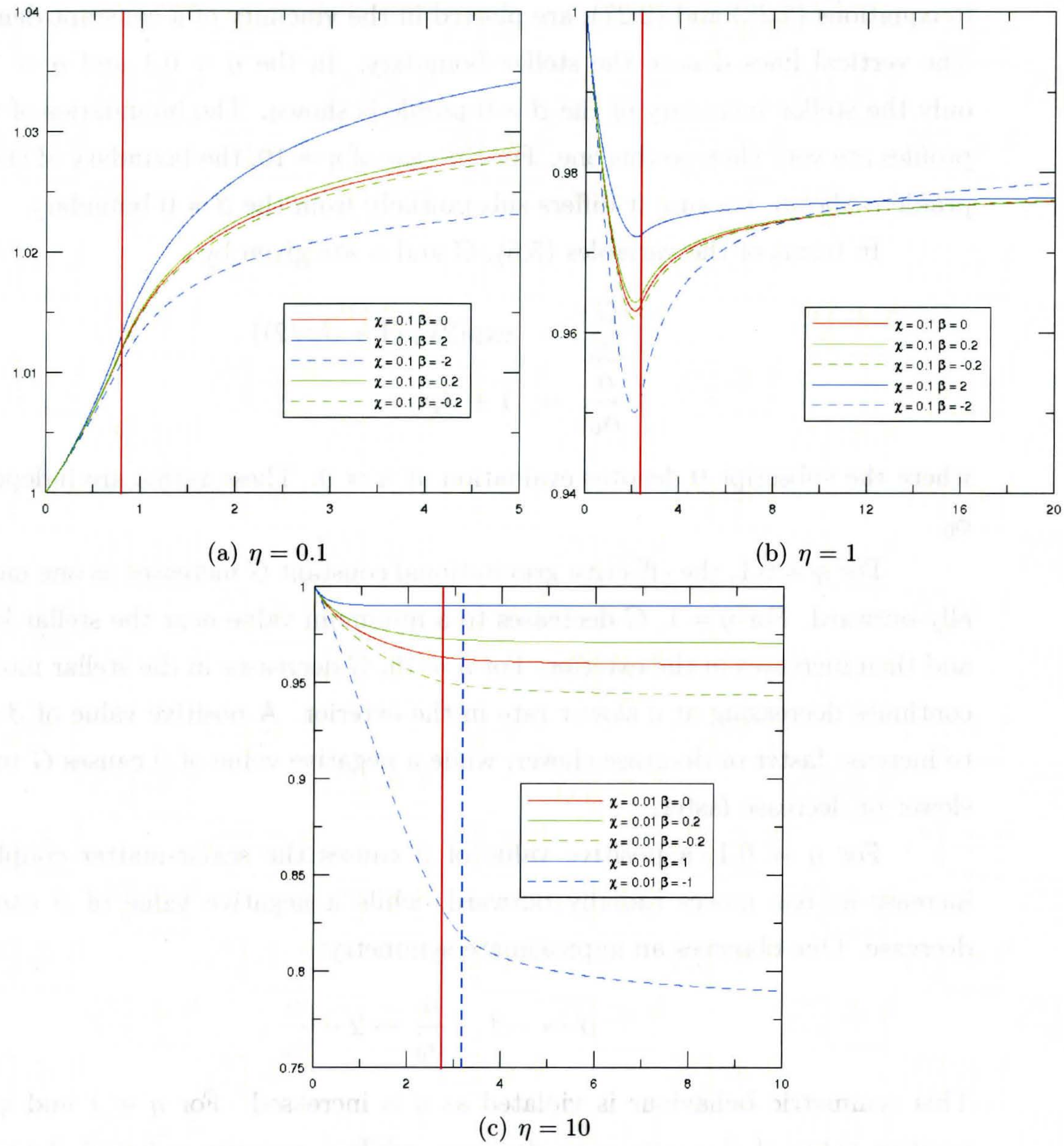
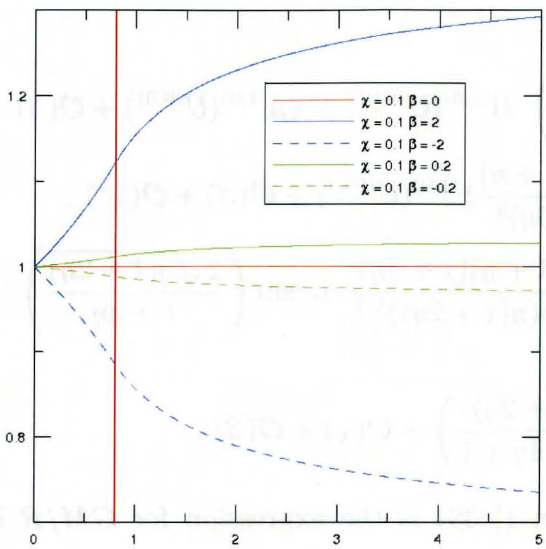
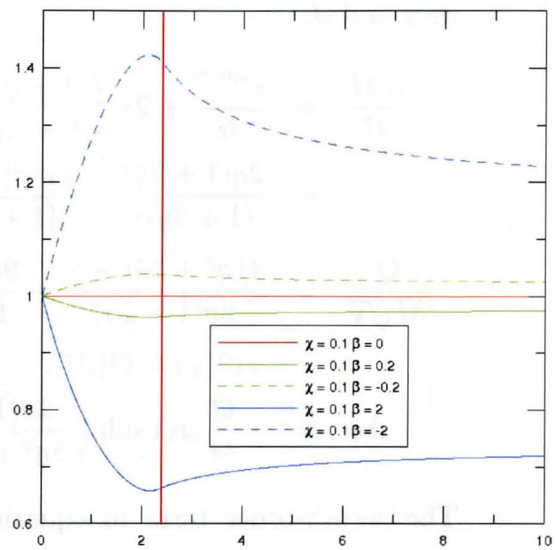


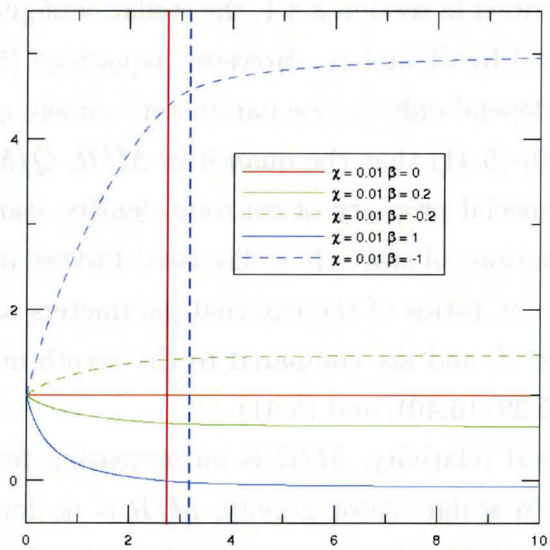
Figure 7: The normalized effective gravitational constant G/G_0 plotted versus u for a star of constant density. The vertical line denotes the stellar boundary.



(a) $\eta = 0.1$



(b) $\eta = 1$



(c) $\eta = 10$

Figure 8: The normalized scalar-matter coupling α/α_0 plotted versus u for a star of constant density. The vertical line denotes the stellar boundary.

5.1.4 External Parameters

The external parameters calculated in section 3.4.3 can be expanded perturbatively in χ and β:

$$\begin{aligned}
\frac{\bar{G}M}{R} &= \frac{U^{(0,0)}}{6} + 2\chi\left(\frac{1+\eta}{1+3\eta}\right)^2 \Pi^{(1,0)}(U^{(0,0)}) + \chi\mu^{(1,0)}(U^{(0,0)}) + \mathcal{O}(\beta) + \mathcal{O}(\chi^2) \\
&= \frac{2\eta(1+2\eta)}{(1+3\eta)^2} - \frac{2\chi\eta^2(1+\eta)}{(1+3\eta)^3}\xi^{(1,0)}(U^{(0,0)}) + \mathcal{O}(\beta) + \mathcal{O}(\chi^2)\ , \qquad (5.39)\\
\frac{Q}{M\sqrt{\chi}} &= \frac{41\eta^2+34\eta+9}{8\eta(1+2\eta)} - \frac{9(1+\eta)(1+3\eta)^2}{16(\eta(1+2\eta))^{3/2}}\arcsin\left(\frac{2\sqrt{\eta(1+2\eta)}}{1+3\eta}\right) \\
&\quad +\mathcal{O}(\chi) + \mathcal{O}(\beta)\ , \qquad (5.40)\\
\Delta\phi &= -\frac{Q}{M}\text{arctanh}\left(\frac{2\eta(1+2\eta)}{5\eta^2+4\eta+1}\right) + \mathcal{O}(\chi) + \mathcal{O}(\beta)\ . \qquad (5.41)
\end{aligned}$$

The zeroth-order term in equation (5.39) is the expression for GM/R in general relativity. The $\mathcal{O}(\chi)$ term depends only on the function ξ.

As described in section 3.4.1, the stellar configurations in scalar-tensor gravity are parametrized by P_0 and ϕ_0. However, equations (5.6)-(5.9) and their initial conditions (5.10) depend only on one parameter, namely $\eta = P_0/\rho_0$. It then follows from equations (5.39)-(5.41) that the quantities M/R, Q/M, and $\Delta\phi$ are independent of ϕ_0. This is a special property of constant-density stars, and does not hold for stars with other equations of state. It is discussed further in section 5.3.

In figure 9, ratios of the external parameters are plotted versus η for various values of χ and β, and are compared to the zeroth-order analytic expressions given in equations (5.39),(5.40), and (5.41).

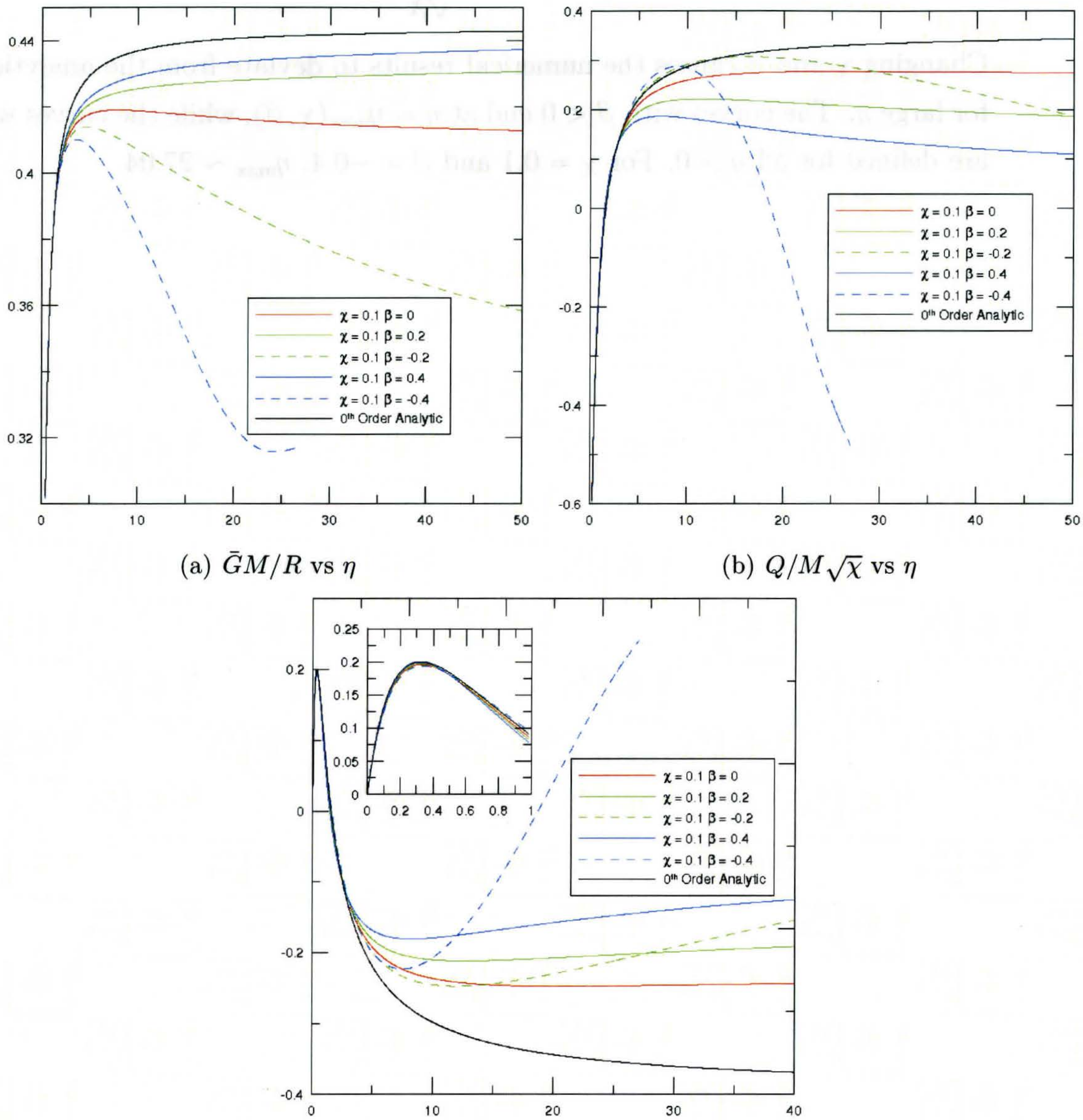
In general relativity, M/R is an increasing function of η, with a maximum value of 4/9. In scalar-tensor gravity, M/R is no longer monotonic. According to equation (5.40), $Q/M\sqrt{\chi}$ is an increasing function of η, which starts with a minimum value of -1 at $\eta = 0$, vanishes at $\eta \sim 1.551$, and approaches a maximum value of ~ 0.359 as $\eta \to \infty$. Thus, to zeroth order in χ and β,

$$-1 \leq \frac{Q}{M\sqrt{\chi}} \leq 0.359\ . \qquad (5.42)$$

According to equation (5.41), $\Delta\phi/\sqrt{\chi}$ increases from 0 to ~ 0.200 for $\eta \in (0, \sim 0.321)$, and then decreases from ~ 0.200 to ~ -0.395 for $\eta \in (0.321, \infty)$. It goes through zero at $\eta \sim 1.551$. Thus, to zeroth order in χ and β,

$$-0.395 \leq \frac{\Delta\phi}{\sqrt{\chi}} \leq 0.200 \,. \tag{5.43}$$

Changing χ and β causes the numerical results to deviate from the analytical curves for large η. The curves with $\beta < 0$ end at $\eta = \eta_{\max}(\chi, \beta)$, while the curves with $\beta \geq 0$ are defined for all $\eta > 0$. For $\chi = 0.1$ and $\beta = -0.4$, $\eta_{\max} \sim 27.04$.



(a) $\bar{G}M/R$ vs η

(b) $Q/M\sqrt{\chi}$ vs η

(c) $\Delta\phi/\sqrt{\chi}$ vs η

Figure 9: Ratios of the external parameters of constant-density stars plotted versus η.

5.1.5 The Limit $\eta \to \infty$

The parameter η measures how relativistic the star is. For non-relativistic stars (such as main-sequence stars or white dwarfs), $P \ll \rho$, so $\eta \ll 1$. For neutron stars, $\eta \sim 10^{-2} \ldots 10$.

Consider the limit $\eta \to \infty$. This limit might be useful when considering an idealized model of gravitational collapse. In this limit, the zeroth-order solution given by equations (5.14)-(5.18) becomes

$$\mu^{(0,0)} = \frac{u}{6} \tag{5.44}$$

$$\Pi^{(0,0)} = \frac{1 - 3\cos\frac{v}{2}}{3(\cos\frac{v}{2} - 1)} , \tag{5.45}$$

$$\dot{\varphi}^{(0,0)} = -\frac{\left(1+\cos\frac{v}{2}\right)\left(\frac{1}{3}\sec\frac{v}{2} + \frac{1}{4}(1 - v\csc v)\csc^2\frac{v}{2}\right)}{\sin^2\frac{v}{2}} , \tag{5.46}$$

$$\begin{aligned}\varphi^{(0,0)} + \log\eta &= \frac{19}{24} - \log 3 - 2\log\sin\frac{v}{4} - \frac{3}{8}v\cot\frac{v}{2} \\ &\quad + \frac{7 - 3\cos\frac{v}{2}}{16\sin^2\frac{v}{4}} - \frac{v(3\cos\frac{v}{4} - \cos\frac{3v}{4})}{32\sin^3\frac{v}{4}} ,\end{aligned} \tag{5.47}$$

$$U^{(0,0)} = \frac{8}{3} . \tag{5.48}$$

Equation (5.47) implies that φ diverges as $-\log\eta$ as $\eta \to \infty$. Also,

$$\mu^{(1,0)} + \frac{2}{3}u\log\eta = \mu_\infty^{(1,0)}(u) , \tag{5.49}$$

where the right-hand side is a finite function of u. Thus, the $\mathcal{O}(\chi)$ correction to μ also diverges. These divergences indicate that the variables (5.5) are not well-suited for describing this limit. They disappear when the variables m,P,ϕ, and r are used, and ϕ_0 is chosen to be

$$\phi_0 = \frac{\phi_0^0 + \alpha\log\eta}{1 - \beta\log\eta} , \tag{5.50}$$

where ϕ_0^0 is independent of η. For example, a perturbative expansion of $m(r)$ yields

$$m(r) = \frac{1}{6}\zeta\rho_0(1 + 4\alpha\phi_0^0)r^3 + \alpha r\mu_\infty^{(1,0)}(\zeta\rho_0 r^2) + \mathcal{O}(\alpha^2) + \mathcal{O}(\beta) . \tag{5.51}$$

5.2 Newtonian Polytropes

Many stars can be modelled by a polytropic equation of state

$$P = K\varrho^{1+1/n} , \tag{5.52}$$

where ϱ is the rest mass density, n is the polytropic index (a constant that need not be an integer), and K is a constant.

5.2.1 Equations

Specialize equations (4.14)-(4.17) to the equation of state (5.52), and change variables to

$$r = aw \ , \quad \varrho = \varrho_0 \theta^n \ , \tag{5.53}$$

where $\rho_0 = \rho(0)$, and a is a length scale that will be specified later. Then, equations (4.14)-(4.17) become

$$\begin{aligned} \ddot{\phi} &= -2\dot{\phi}/w + CA^4(\phi)\theta^n(\alpha(\phi) + w\dot{\phi}) \ , & (5.54) \\ \frac{d}{dw}\left(\zeta w^2\dot{\theta} + \frac{1}{2}w^3\dot{\phi}^2 + w^2\alpha(\phi)\dot{\phi}\right) &= -CA^4(\phi)w^2\theta^n - \frac{1}{2}w^2\dot{\phi}^2 \ , & (5.55) \\ \bar{G}m &= -aw^2\left(\zeta\dot{\theta} + \frac{1}{2}w\dot{\phi}^2 + \alpha(\phi)\dot{\phi}\right) \ , & (5.56) \end{aligned}$$

where dots denote d/dw, and

$$\zeta = K(n+1)\varrho_0^{1/n} \ , \quad C = 4\pi\bar{G}a^2\varrho_0 \ . \tag{5.57}$$

The initial conditions are

$$\begin{aligned} \theta(0) = 1 \quad &, \quad \dot{\theta}(0) = 0 \ , & (5.58) \\ \phi(0) = \phi_0 \quad &, \quad \dot{\phi}(0) = 0 \ . & (5.59) \end{aligned}$$

Now specialize to the model $A(\phi) = \exp(\alpha\phi + \frac{1}{2}\beta\phi^2)$. Define

$$\chi = \alpha^2(\phi_0) = (\alpha + \beta\phi_0)^2 \ , \quad \varphi = (\phi - \phi_0)/\alpha(\phi_0) \ . \tag{5.60}$$

Choose

$$a = \sqrt{\frac{K(n+1)}{4\pi\bar{G}}}\varrho_0^{(1-n)/2n}e^{-2(\alpha\phi_0 + \frac{1}{2}\beta\phi_0^2)} \ , \tag{5.61}$$

so that

$$\zeta = Ce^{4\alpha\phi_0 + 2\beta\phi_0^2} \ . \tag{5.62}$$

Then equations (5.54)-(5.56) become

$$\begin{aligned} -(w^2\dot{\theta})^{\cdot} - \frac{\chi\beta}{\zeta}w^2\dot{\varphi}^2 &= w^2e^{4\chi\varphi(1+\beta\varphi/2)}\theta^n(1 + \chi(1 + w\dot{\varphi} + \beta\varphi)^2) \ , & (5.63) \\ \ddot{\varphi} &= -2\dot{\varphi}/w + \zeta e^{4\chi\varphi(1+\beta\varphi/2)}\theta^n(1 + w\dot{\varphi} + \beta\varphi) \ , & (5.64) \\ \bar{G}m &= -aw^2\left(\zeta\dot{\theta} + \frac{1}{2}\chi w\dot{\varphi}^2 + \chi(1 + \beta\varphi)\dot{\varphi}\right) \ , & (5.65) \end{aligned}$$

with the initial conditions for φ:

$$\varphi(0) = 0 \; , \quad \dot{\varphi}(0) = 0 \; . \tag{5.66}$$

Since the system of differential equations (5.63)-(5.64) is singular at $w = 0$, power series expansions are used to evaluate initial conditions at some small w_0:

$$\theta(w) = 1 - \frac{1}{6}(1+\chi)w^2 + \mathcal{O}(w^4) \; , \tag{5.67}$$

$$\varphi(w) = \frac{1}{6}\zeta w^2 + \mathcal{O}(w^4) \; . \tag{5.68}$$

Expand θ and φ in a power series in the couplings χ and β, as well as the parameter ζ:

$$\theta = \sum_{i,j} \theta^{(i,j)} \chi^i \beta^j = \sum_{i,j,k} \theta^{(i,j,k)} \chi^i \beta^j \zeta^k \; , \tag{5.69}$$

$$\varphi = \sum_{i,j} \varphi^{(i,j)} \chi^i \beta^j = \sum_{i,j,k} \varphi^{(i,j,k)} \chi^i \beta^j \zeta^k \; . \tag{5.70}$$

5.2.2 Interior Profiles

In the limit $\chi = \beta = 0$, equations (5.63)-(5.64) become

$$\ddot{\theta} = -2\dot{\theta}/w - \theta^n \; , \tag{5.71}$$

$$\ddot{\varphi} = -2\dot{\varphi}/w + \zeta\theta^n(1 + w\dot{\varphi}) \; . \tag{5.72}$$

Equation (5.71) is called the Lane-Emden equation. It plays an important role in the theory of stellar structure, and the properties of its solutions are well-known. It can be solved analytically when $n = 0, 1, 5$ [11]. The solutions of the Lane-Emden equation with initial conditions (5.58) are called the Lane-Emden functions, and are denoted by $\Theta_n(w)$. Thus,

$$\theta^{(0,0)}(w) = \Theta_n(w) \; . \tag{5.73}$$

Equation (5.72) can be solved to write φ in terms of θ:

$$\varphi^{(\dot{0},0)} = -\frac{\zeta}{w^2} \exp(-\zeta(w\dot{\Theta}_n + \Theta_n)) \int (w^2\dot{\Theta}_n)^{\cdot} \exp(\zeta(w\dot{\Theta}_n + \Theta_n)) dw \; . \tag{5.74}$$

Expansion of the above equation in powers of ζ yields

$$\varphi^{(0,0)} = \zeta(1 - \Theta_n) + \mathcal{O}(\zeta^2) \; . \tag{5.75}$$

Thus,

$$\varphi^{(0,0,0)} = 0 \; , \quad \varphi^{(0,0,1)} = 1 - \Theta_n \; . \tag{5.76}$$

5.2.3 External Parameters

When written in terms of the variables θ, φ, and w, equations (4.21)-(4.23) become

$$\begin{aligned}
R &= aW \ , && (5.77)\\
\bar{G}M &= -aW^2(\zeta\dot{\theta}(W) + \chi(1+\beta\varphi(W))\dot{\varphi}(W)) \ , && (5.78)\\
\frac{\bar{G}Q}{\sqrt{\chi}} &= -aW^2\dot{\varphi}(W) \ , && (5.79)\\
\frac{\Delta\phi}{\sqrt{\chi}} \equiv \frac{\phi_\infty - \phi(W)}{\sqrt{\chi}} &= W\dot{\varphi}(W) \ , && (5.80)
\end{aligned}$$

where W denotes the value of the independent variable corresponding to $r = R$. The polytropes can be parametrized by K, n, P_0, and ϕ_0. However, equations (5.63)-(5.64) and their initial conditions depend only on n and ζ. It follows that the quantities M/R, Q/M, and $\Delta\phi$ depend only on n and ζ, and are independent of ϕ_0. Moreover, Q, $\Delta\phi$, and R are related by

$$\frac{\bar{G}Q}{R} = -\Delta\phi \ . \qquad (5.81)$$

The external parameters then have the perturbative expansion

$$\begin{aligned}
\frac{\bar{G}M}{R} &= -\zeta W\dot{\theta}(W) + \mathcal{O}(\chi) + \mathcal{O}(\beta) + \mathcal{O}(\zeta^2) \ , && (5.82)\\
\frac{Q}{M\sqrt{\chi}} &= -1 + \mathcal{O}(\chi) + \mathcal{O}(\beta) + \mathcal{O}(\zeta^2) \ , && (5.83)\\
\frac{\Delta\phi}{\sqrt{\chi}} &= -\zeta W\dot{\theta}(W) + \mathcal{O}(\chi) + \mathcal{O}(\beta) + \mathcal{O}(\zeta^2) \ . && (5.84)
\end{aligned}$$

5.2.4 The Chandrasekhar Limit

A white dwarf can be modelled by a degenerate fermion gas. In the ultra-relativistic limit, this gas satisfies a polytropic equation of state (5.52) with

$$K = \frac{3^{1/3}\pi^{2/3}}{4}\hbar c(Y_e/m_B)^{4/3} \ , \quad n = 3 \ , \qquad (5.85)$$

where Y_e is the number of electrons per nucleon, and m_B is the average nucleon mass [8]. Although the fermion gas is highly relativistic, the gravitational field generated by it is non-relativistic, and the results developed above can be applied. The typical central densities of white dwarfs are $\varrho_0 \sim (10^7 \ldots 10^{14})\mathrm{kg\,m^{-3}}$ [8]. These values correspond to $\zeta \sim 10^{-5} \ldots 10^{-2}$ for $Y_e = 1/2$. Therefore, a perturbative expansion in ζ is a good approximation for white dwarfs.

Since the ultra-relativistic limit of the fermion gas is used, the mass calculated using equation (4.34) is greater than the actual mass of the white dwarf. In general relativity, this upper bound on the mass of white dwarfs is independent of ζ, and is called the Chandrasekhar limit. For $Y_e = \frac{1}{2}$, its value is approximately $1.4 M_\odot$ [10].

In the quadratic model, the maximum white dwarf mass is given by an expression of the form

$$M_{\rm max} = \mathcal{F}(\zeta) A^{-2}(\phi_0) \ . \tag{5.86}$$

For small scalar-matter couplings, $\mathcal{F}(\zeta)$ is nearly constant over the range of white-dwarf densities. However, $M_{\rm max}$ depends exponentially on ϕ_0, and can thus become arbitrarily large.

5.2.5 Main-Sequence Stars

In the Eddington standard stellar model, a star is modelled by an ideal gas, and energy is transported only by radiation. Moreover, it is assumed that

$$\beta \equiv \frac{P_{\rm gas}}{P} = \text{const} \ , \tag{5.87}$$

where $P_{\rm gas}$ is the pressure of the ideal gas, $P_{\rm rad}$ is the radiation pressure, and $P = P_{\rm gas} + P_{\rm rad}$ is the total pressure. The Eddington model leads to a polytropic equation of state with index $n = 3$, and

$$K = \left(\left(\frac{R}{\mu} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right)^{1/3} \ , \tag{5.88}$$

where R is the universal gas constant, μ is the molar mass of the ideal gas,

$$a = \frac{\pi^2 k^4}{15 c^3 \hbar^3} \tag{5.89}$$

is a constant related to blackbody radiation, and k is Boltzmann's constant [23]. Main-sequence stars can be described approximately by the Eddington standard model.

For a more accurate single-polytrope model of the sun, $n = 3.35$ and $\zeta \sim 10^{-5}$ [26], so the perturbative expansion in ζ is a good approximation.

Consider the following problem. A star has gravitational mass $M_{\rm G}$ and radius R, and is modelled by a polytrope of index n. Using the theory of gravity, calculate the central pressure and density of the star.

In general relativity, this problem has a unique solution [26]. In Brans-Dicke theory, the situation is more complicated. In order to solve the equations of stellar structure (5.63)-(5.64), a value of ζ needs to be specified. However, the central pressure and density are needed to calculate ζ. One way to proceed, is to guess a value of ζ. Using equations (5.82)-(5.84), (5.77), (4.34), and (2.25), the external parameters M_{G}, R, ϕ_0, and ϕ_∞ can be related to the boundary values of θ and φ:

$$\frac{G_\infty M_{\mathrm{G}}}{R}(1-2\alpha\phi_\infty) = -\zeta W\dot{\theta}(W) , \tag{5.90}$$

$$\phi_\infty - \phi_0 = \alpha\cdot(\varphi(W) - \zeta W\dot{\theta}(W)) , \tag{5.91}$$

$$\frac{R}{W} = \sqrt{\frac{\zeta}{4\pi G_\infty \varrho_0}}(1+\alpha\cdot(\phi_\infty - 2\phi_0)) , \tag{5.92}$$

where $G_\infty = G(r=\infty)$ is the gravitational constant far away from the star. Use equation (5.90) to find ϕ_∞, then use equation (5.91) to find ϕ_0. Finally, use equation (5.92) to find ϱ_0. P_0 can then be found from equations (5.57) and (5.52). In summary, *any* (positive) value of ζ is permissible, and there is a one-parameter family of polytrope profiles with prescribed values of M_{G}, R, and n. In order to obtain a unique profile, either ϕ_0 or ϕ_∞ must be chosen. Then equation (5.90) or (5.91) constrain the permissible values of ζ. Note that prescription of Q does not constrain ζ, because

$$Q = -\alpha M_{\mathrm{G}} . \tag{5.93}$$

5.3 Other Equations of State and Buchdahl's Theorem

Consider a star with initial values $\rho(0) = \rho_0$, $P(0) = P_0$, $\phi(0) = \phi_0$, radius R, external parameters M, Q, $\Delta\phi$, a density profile that decreases with r, and an arbitrary equation of state. Now compare it to a star with constant density ρ_0, and initial values $P(0) = P_0$ and $\phi(0) = \phi_0$. Denote the radius and external parameters of this constant-density star by $\hat{R}$ and $\hat{M}$, $\hat{Q}$, $\hat{\Delta\phi}$, respectively. In general relativity, Buchdahl's theorem [7] states that

$$\frac{M}{R} \leq \frac{\hat{M}}{\hat{R}} \, . \tag{5.94}$$

It turns out that this theorem no longer holds in scalar-tensor gravity. This will be demonstrated for a density profile that falls off exponentially:

$$\rho(r) = \rho_0 \exp(-r/\xi\hat{R}) \, , \tag{5.95}$$

where ξ is a constant. Stars with this density profile can be parametrized by $\eta \equiv P_0/\rho_0$ and ϕ_0, where $\eta \in (0, \infty)$ and $\phi_0 \in (\phi_0^\star, \infty)$. As ϕ_0 approaches the critical value $\phi_0^\star$, the radius of the star becomes arbitrarily large. When $\xi \to \infty$, $\phi_0^\star$ tends to $-\infty$, and equation (5.95) reduces to a constant-density profile.

In figures 10, 11, and 12, ratios of the external parameters M, Q, R, and $\Delta\phi$ are plotted versus ϕ_0 for various values of η. These figures show that Buchdahl's theorem is violated in Brans-Dicke theory, even when the scalar-matter coupling is very weak ($\chi = 10^{-4}$). They also suggest that as $\phi_0 \to \infty$, the quantities M/R, Q/M, and $\Delta\phi$ tend to the corresponding 'hatted' quantities.

Buchdahl's theorem in scalar-tensor gravity has been investigated by Tsuchida, et al. [44]. They made the additional assumption that

$$\rho_{\text{eff}} \equiv A^4(\phi)\rho + \frac{\phi'^2}{8\pi\bar{G}}(1 - 2\bar{G}m/r) \tag{5.96}$$

decreases with r, and derived the inequalities

$$\frac{GM}{\Xi} \leq \frac{4}{9} \, , \quad \left|\frac{GQ}{\Xi}\right| \leq \frac{2\sqrt{3}}{9} \, . \tag{5.97}$$

The radius of the star R is related to Ξ by

$$R^2 = \Xi^2 \left(1 - \frac{2\bar{G}M}{\Gamma\Xi}\right)^{1-\Gamma} \, . \tag{5.98}$$

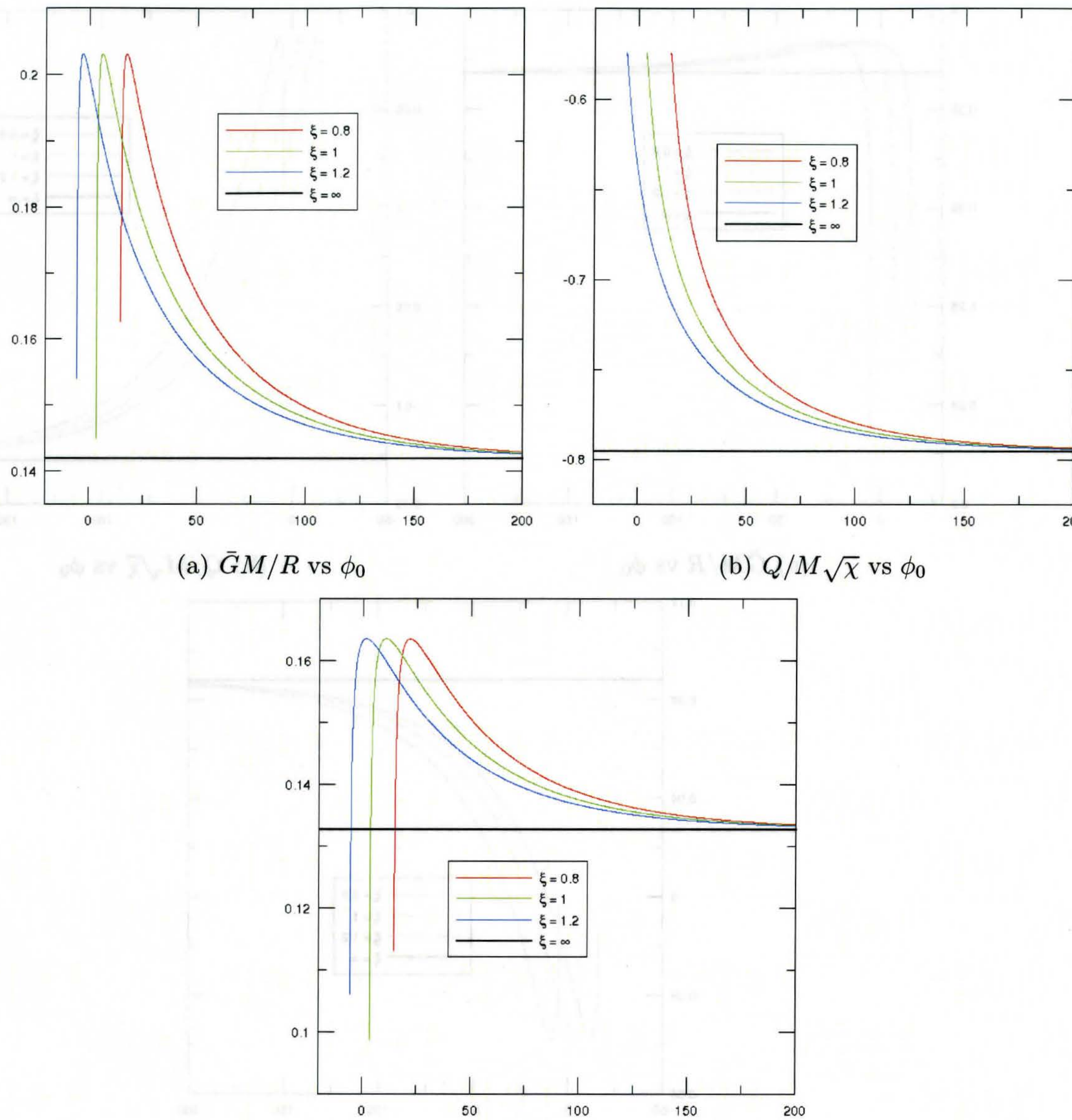
Therefore, there is no simple way to write the inequalities (5.97) in terms of M/R or Q/R. The definition of ρ_{eff} allows equation (3.41) to be written in the simple form

$$m' = 4\pi r^2 \rho_{\text{eff}} \, , \tag{5.99}$$

analogous to equation (3.18). In the variables (5.5), ρ_{eff} is given by

$$\rho_{\text{eff}} = \rho_0 A^4(\phi_0) \left(e^{4\chi\varphi(1+\beta\varphi)} + 4\chi u(1-2\mu)\dot{\varphi}^2 \right) \, . \tag{5.100}$$

For constant-density stars and couplings $\chi = 10^{-3}$ and $\beta = 0$, ρ_{eff} exhibits the following behaviour: For $\eta = 0.1$, ρ_{eff} is an increasing function of r. For $\eta = 1$, ρ_{eff} is increasing near the surface of the star, and decreasing everywhere else. For $\eta = 10$, ρ_{eff} is increasing near the centre of the star, and decreasing everywhere else.

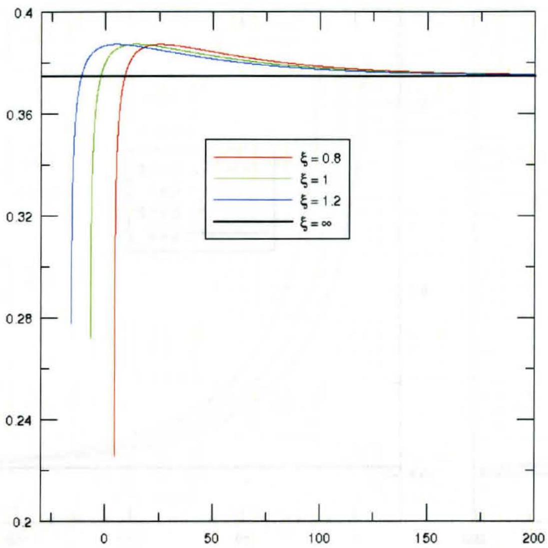


(a) $\bar{G}M/R$ vs ϕ_0

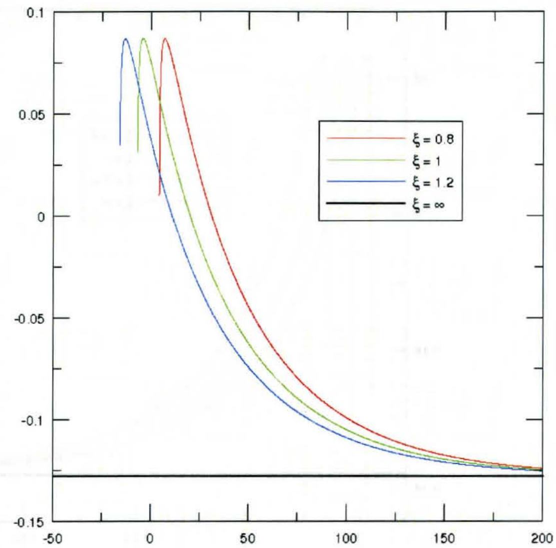
(b) $Q/M\sqrt{\chi}$ vs ϕ_0

(c) $\Delta\phi/\sqrt{\chi}$ vs ϕ_0

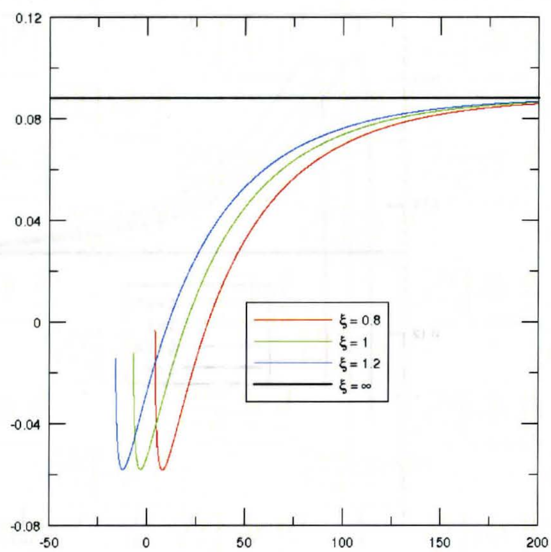
Figure 10: Ratios of the external parameters of stars with density profile (5.95) plotted versus ϕ_0, for $\chi = 10^{-4}$, $\beta = 0$, and $\eta = 0.1$.



(a) $\bar{G}M/R$ vs ϕ_0

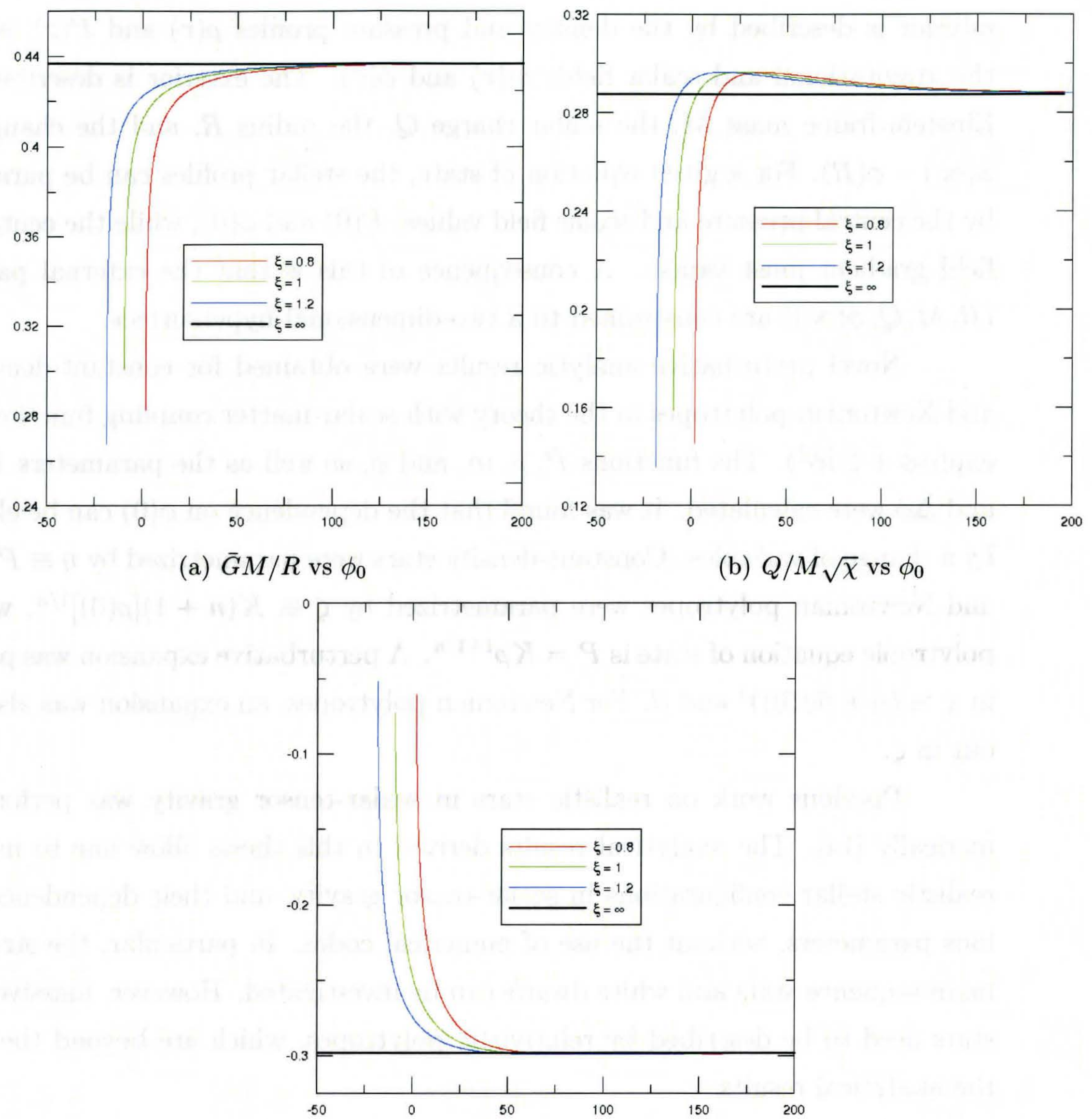


(b) $Q/M\sqrt{\chi}$ vs ϕ_0



(c) $\Delta\phi/\sqrt{\chi}$ vs ϕ_0

Figure 11: Ratios of the external parameters of stars with density profile (5.95) plotted versus ϕ_0, for $\chi = 10^{-4}$, $\beta = 0$, and $\eta = 1$.



(a) GM/R vs ϕ_0

(b) $Q/M\sqrt{\chi}$ vs ϕ_0

(c) $\Delta\phi/\sqrt{\chi}$ vs ϕ_0

Figure 12: Ratios of the external parameters of stars with density profile (5.95) plotted versus ϕ_0, for $\chi = 10^{-4}$, $\beta = 0$, and $\eta = 10$.

6 Conclusions

Static, spherically-symmetric stars were investigated in scalar-tensor gravity. A star's interior is described by the density and pressure profiles $\rho(r)$ and $P(r)$, as well as the gravitational and scalar fields $m(r)$ and $\phi(r)$. The exterior is described by the Einstein-frame mass M, the scalar charge Q, the radius R, and the change $\Delta\phi \equiv \phi(\infty) - \phi(R)$. For a given equation of state, the stellar profiles can be parametrized by the central pressure and scalar field values, $P(0)$ and $\phi(0)$, while the central scalar field gradient must vanish. A consequence of this is that the external parameters $(R, M, Q, \phi(\infty))$ are constrained to a two-dimensional hypersurface.

Novel perturbative analytic results were obtained for constant-density stars and Newtonian polytropes in the theory with scalar-matter coupling function $A(\phi) = \exp(\alpha\phi + \frac{1}{2}\beta\phi^2)$. The functions P, ρ, m, and ϕ, as well as the parameters M, Q, R, and $\Delta\phi$ were calculated. It was found that the dependence on $\phi(0)$ can be eliminated by a change of variables. Constant-density stars were parametrized by $\eta \equiv P(0)/\rho(0)$, and Newtonian polytropes were parametrized by $\zeta \equiv K(n+1)[\rho(0)]^{1/n}$, where the polytropic equation of state is $P = K\rho^{1+1/n}$. A perturbative expansion was performed in $\chi \equiv (\alpha + \beta\phi(0))^2$ and β. For Newtonian polytropes, an expansion was also carried out in ζ.

Previous work on realistic stars in scalar-tensor gravity was performed numerically [14]. The analytical results derived in this thesis allow one to investigate realistic stellar configurations in scalar-tensor gravity, and their dependence on various parameters, without the use of numerical codes. In particular, the structure of main-sequence stars and white dwarfs can be investigated. However, massive neutron stars need to be described by relativistic polytropes, which are beyond the scope of the analytical results.

For negative coupling $\beta < 0$, it was found that there exists a maximum value $\eta_{\rm max}(\chi, \beta)$, such that no constant-density stars exist with $\eta > \eta_{\rm max}$. On the other hand, if $\beta \geq 0$, constant-density solutions could be found for arbitrarily large η.

It was verified that a similar phenomenon occurs for several other equations of state. There exists a maximum value of $P(0)/\rho(0)$, which depends on the equation of state and $\phi(0)$, as well as the couplings χ and β. Numerical calculations suggest that this maximum value is less than the constant-density maximum $\eta_{\rm max}(\chi, \beta)$. If such an inequality holds for all reasonable equations of state, then knowledge of P/ρ in the centre of any star can be used to constrain β.

When β is negative and $P(0)/\rho(0)$ is sufficiently large, the effective gravitational constant G decreases, and the effective scalar-matter coupling α increases, as one moves radially outward (see figures 7(c) and 8(c)). Also, $\phi' < 0$ (see figure 3(c)). In equation (3.42), all terms in the large brackets are positive, except for $\alpha(\phi)\phi'$, which is negative. If $P(0)/\rho(0)$ exceeds the maximum value described above, then at some critical radius $r = r_\star$, $\alpha(\phi)\phi'$ cancels the other terms, creating a minimum in the pressure profile $P(r)$:

$$P'(r_\star) = 0 \ , \quad P(r_\star) > 0 \ . \tag{6.1}$$

For very large r, $\alpha(\phi)\phi'$ dominates over the other terms, and drives the pressure to infinity:

$$\lim_{r\to\infty} P = \infty \ , \quad \lim_{r\to\infty} G = 0 \ , \quad \lim_{r\to\infty} \alpha = \infty \ . \tag{6.2}$$

Such a pressure profile can not describe a star.

Stars with exponentially-decaying density profiles were investigated numerically. It was found that the dependence on ϕ_0 could not be eliminated, as was possible for constant-density stars and Newtonian polytropes. The parameters M, Q, R, and $\Delta\phi$ were compared to those of a star with constant density $\rho(0)$, and it was found that Buchdahl's theorem is violated, namely,

$$\frac{M}{R} \not\leq \frac{\hat{M}}{\hat{R}} \ , \tag{6.3}$$

even when $\beta = 0$ and α is very small. The 'hatted' external parameters are those of the constant-density star.

In general relativity, Buchdahl's theorem is important because it gives information about the external parameters that is independent of the equation of state, which is not always well-known. Even if Buchdahl's theorem fails in scalar-tensor gravity, more subtle relationships may hold between general stars and constant-density stars. For example, it was found from the numerical work that

$$\lim_{\phi(0)\to\infty} \frac{M}{R} = \frac{\hat{M}}{\hat{R}} \ , \quad \lim_{\phi(0)\to\infty} \frac{Q}{M} = \frac{\hat{Q}}{\hat{M}} \ , \quad \lim_{\phi(0)\to\infty} \Delta\phi = \hat{\Delta\phi} \ , \tag{6.4}$$

for several different equations of state. A prospect for future work would be an analytical proof of (6.4) for 'reasonable' equations of state.

In order to determine whether the obtained solutions are physically realistic, their stability against perturbations needs to be analyzed. This is a possible prospect for future work.

Another prospect for future work, is the calculation of the dependence of the Einstein-frame gauge couplings and fermion masses on ϕ. Constraints on the variations of these constants [22] can then be used to constrain the scalar-matter coupling function $A(\phi)$.

A The functions $g^{(k)}$ and $h^{(k)}$

In tables 1 and 2, the coefficients and polynomials needed to calculate $g^{(k)}$ and $h^{(k)}$ are tabulated for $k = 0 \ldots 3$. The infinite sums in equations (5.29) and (5.35) can be evaluated analytically:

$$g^{(0)}(t) = 2 \cdot \frac{6+t}{4+t} - 6\frac{\arctan(\sqrt{t}/2)}{\sqrt{t}} , \tag{A.1}$$

$$g^{(1)}(t) = \frac{t^2}{5(4+t)} - \frac{2}{3}t\log(1+t/4) , \tag{A.2}$$

$$g^{(2)}(t) = \frac{1311t^4 - 8392t^3 + 8176t^2 + 417200t + 1001280}{189000(4+t)^2} - \frac{149}{225}\frac{\arctan(\sqrt{t}/2)}{\sqrt{t}} + \frac{8}{45}t\log(1+t/4) , \tag{A.3}$$

$$g^{(3)}(t) = \frac{346t^5 + 16663t^4 + 117312t^3 + 152544t^2 - 781200t - 1874880}{396900(4+t)^2} + \frac{62}{105}\frac{\arctan(\sqrt{t}/2)}{\sqrt{t}} - \frac{128}{945}t\log(1+t/4) . \tag{A.4}$$

$$h^{(0)}(t) = -3\frac{6+t}{4+t} + 9\frac{\arctan(\sqrt{t}/2)}{\sqrt{t}} , \tag{A.5}$$

$$h^{(1)}(t) = \frac{19t^2 - 80t - 720}{20(4+t)} + 18(1-t/4)\frac{\arctan(\sqrt{t}/2)}{\sqrt{t}} + 6\log(1+t/4) , \tag{A.6}$$

$$h^{(2)}(t) = \frac{24939t^4 + 527092t^3 + 2377424t^2 + 238000t - 10073280}{126000(4+t)^2} \tag{A.7}$$

$$+\frac{1499 - 75t(18+5t/4)}{150}\frac{\arctan(\sqrt{t}/2)}{\sqrt{t}} + \frac{128}{15}\log(1+t/4) , \tag{A.8}$$

$$h^{(3)}(t) = \frac{269671t^5 + 4656214t^4 + 35009064t^3}{5292000(4+t)^2} + \frac{122200176t^2 + 146435520t - 46609920}{5292000(4+t)^2} - \frac{1225t^3 + 10500t^2 + 41972t - 9248}{8400}\frac{\arctan(\sqrt{t}/2)}{\sqrt{t}} + \frac{436}{175}\log(1+t/4) . \tag{A.9}$$

Table 1: Coefficients and polynomials for the calculation of $g^{(k)}$

k	a_k	$p_k(n)$
0	0	$2(n-1)^2$
1	0	$(3n-13)(2n+1)$
2	$\frac{13}{360}$	$\frac{2}{1575}(175n^3 - 718n^2 + 1087n + 1136)$
3	$\frac{41}{7560}$	$\frac{4}{14175}(105n^3 - 811n^2 - 2467n - 2587)$

Table 2: Coefficients and polynomials for the calculation for $h^{(k)}$

k	b_k	c_k	d_k	$q_k(n)$
0	0	0	0	$-3n(n-1)$
1	0	0	$-\frac{1}{4}$	$-\frac{6}{5}(4n^3 - 10n^2 - n - 5)$
2	0	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{525}(700n^5 - 1472n^4 - 1495n^3 + 5394n^2 + 8737n + 13440)$
3	$\frac{1}{12}$	$-\frac{1}{80}$	$\frac{103}{10080}$	$-\frac{2}{4725}(840n^6 - 6908n^5 + 12830n^4 - 20605n^3 - 90272n^2 - 152079n - 88290)$

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