SCALING SOLUTIONS IN SUPERGRAVITY

TIME-DEPENDENT SCALING SOLUTIONS IN D DIMENSIONAL SUPERGRAVITY

By

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree
Master of Science

McMaster University

MASTER OF SCIENCE (2008)
(Physics)

McMaster University
Hamilton, Ontario

TITLE: Time-dependent Scaling Solutions in D Dimensional Supergravity

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NUMBER OF PAGES: vii, 46

Abstract

We look for time-dependent solutions to a general class of supergravity models in an arbitrary amount of dimensions. Previously, many static solutions of these models have been found and studied, of which a subclass of these solutions support membrane-like configurations. While many properties of these solutions are known, their dynamics - and therefore stability - are not. We follow this motivation, and investigate the possibility of time dependent solutions that will also support this membrane configuration. Under various conditions, it turns out this is the case, bringing a better understanding to the stability of these branes. In addition, the form of the time dependence found suggest possible applications of supergravity to cosmological models.

Acknowledgements

I'd like to thank my supervisor Cliff Burgess for suggesting this problem and giving valuable insight into the physics of these systems, as well as providing beer at group meetings. Much thanks are deserved for Andrew Tolley as well, for if it wasn't for his extensive knowledge in this area, I may have never found these solutions. Thank you to both Sung-Sik Lee and Walter Craig for reading my thesis and being on my defence committee (while asking me useful and interesting questions). I must also thank many professors I've had throughout my career in physics, as they have given me great insight, inspiration, and excitement towards this fascinating subject.

My peers have done much to help me understand the subject, as well as provided an incredible atmosphere towards both research and socializing. Phillip Ashby and Prasanna Balasubramanian have taught me much and given me goals to aim for, while Eric Mills made me want to learn more about the world in general. Patrick Rogers, Josh McGraw, Mike Young, Clare Armstrong, and Marc-Antoni Goulet have provided much-needed excuses to have fun, while showing me how one can be in physics, and still be "cool." The list goes on, and it is these people that have made the graduate experience an enjoyable one. I'd also like to thank my parents. Even though they had no direct effect on me becoming a scientist (they're both strongly interested and talented in music,) their incredible intelligence and reason has no doubt affected me in my life and career.

Finally, my wife deserves the most thanks. If it wasn't for her endless encouragement and sacrifice (as well as temporarily waiving my household chores!) this thesis would be much harder to produce. She deserves much praise for arranging our wedding almost entirely by herself while I worked on this project, and I am in debt to her sacrifice.

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Chapter 1

Introduction

Supergravity was originally developed as an extension to a supersymmetric theory that incorporated gravity. While it is beyond the scope of this thesis to fully motivate supergravity (and supersymmetry), as well as the field content, reviews of supergravity (as well as motivation) can be found here [1, 2, 3], as well as original research developing supergravity here [4, 5]. As a brief summary, the original motivation was that if supersymmetry was to exist (a special symmetry of flat space-time involving particles of different spin), then some form of supergravity must exist as well (supersymmetry in a curved space-time, a natural generalization to flat space-time).

Also, since string theory itself requires supersymmetry to be a consistent theory, supergravity again is an important theory (in fact, it describes the low energy behavior of string theory). Since string theory requires ten dimensions to be a consistent theory [5], the supergravity from string theory exists in more dimensions than was originally investigated. Since string theory, at this point in time, is a very promising candidate as a theory of nature, the study of supergravity and its ramifications is thus well motivated.

1.1 Branes in Supergravity

The class of supergravities considered in this thesis are those which are derived from string theory. This class can be found here [2]. Since string theory itself has a very constrained field content [2], so must the corresponding class of supergravities. One of these fields,

$$F_{M_1 \dots M_n}(x^M), \tag{1.1}$$

is an n-form. A n-form is simply an antisymmetric rank-n tensor, with some convenient mathematical properties. One of these is that it naturally describes a n-hypersurface in a D dimensional manifold (therefore $n < D$). A good example of this is a parallelogram, which has area described by the antisymmetric product (cross product) of the two vectors corresponding to its two sides. It is thus this sort of term which can naturally source the presence of a special hypersurface (henceforth referred to as a brane) in our manifold. The brane can naturally couple to the gauge potential $n-1$-form of the field strength n-form itself (which is referred to as the electric case in the literature), or the gauge potential of its dual $D-n$-form (the magnetic case). The dual of a form can simply be thought of as the dimensions the n-form doesn't span. It turns out that this general class of supergravity theories has been solved with some specific symmetry in a static configuration in both the electric and magnetic case.

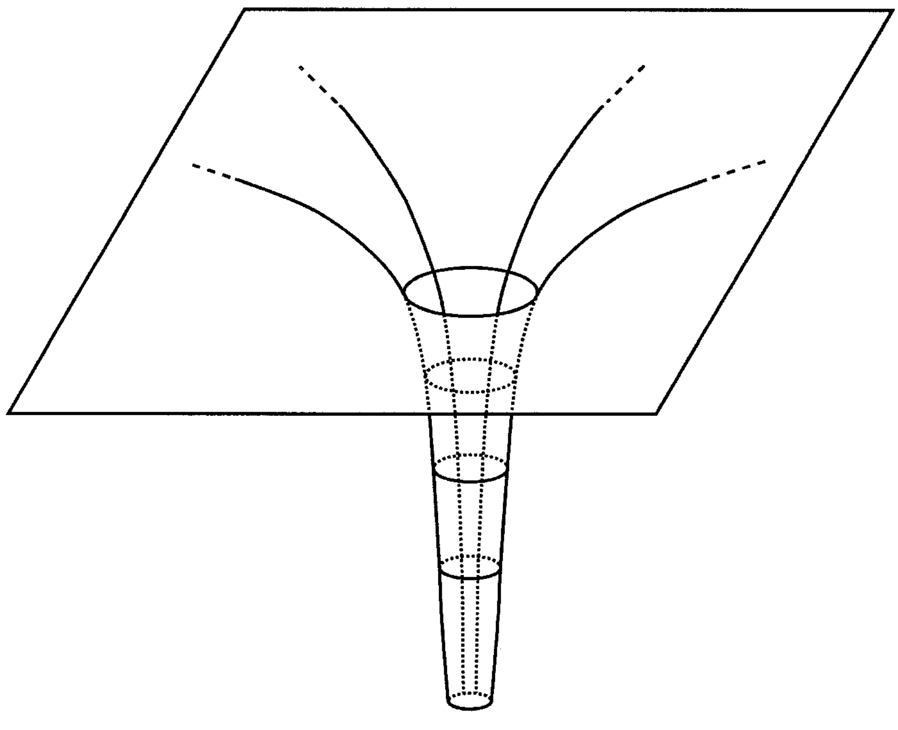

Figure 1.1: Spatial mock-up of both electric and magnetic static solutions. The radial distance from the "throat" represents radial distance from brane. The angular direction about the "throat" represents all angular directions about brane. The directions on the brane are maximally (Poincaré) symmetric. Image taken from K.S. Stelle review on supergravity p-branes.

These turn out to have many of the same properties of black holes, and are thus dubbed "black branes"[1]. In both cases, there exist a horizon of no return, making a striking connection to Schwarzschild-like solutions in canonical gravity. However, the dynamics of these branes are unknown, and embedding them in a cosmological scenario is impossible without including time dependence (since observations have made it fairly clear that the universe is not static.) This is interesting as it may lead to possibilities of our visible universe being confined to one of these "black branes".

[1]Aside from Stelle's review, see [6, 7, 8, 9, 10, 11, 12, 13] for an explicit treatment of these branes, and their relation to D-branes in string theory.

1.2 Time Dependence - Motivation and Methods

As was mentioned, time dependence opens up interesting possibilities due to investigation of stability as well as cosmological ramifications of black branes. However, including time dependence undoubtedly makes the problem of solving the field equations more difficult. The inclusion of a second variable dependence changes the problem from a set of Ordinary Differential Equations (ODE's) to a set of Partial Differential Equations (PDE's), as well as a reduction in the overall symmetry involved in the problem. These two factors can combine to create quite an intractable problem in general.

However, some work has already been done to include time dependence to already-known static solutions in specific cases of supergravity.[14] It is with this ansatz of a power-law time dependence that the time dependence remarkably decouples from the field equations, leaving a slightly more complicated set of ODE's than in the static case. Remarkably, this type of solution is possible for not just the aforementioned supergravity (six dimensional chiral flavour), since it exploits a scaling symmetry which is common to all of these supergravities.

Finally, this same research has led to useful methods in solving these equations. The methods employ essentially using the Hamiltonian formalism of general relativity (for a thorough treatment of the Lagrangian/Hamiltonian formalism, as well as the ADM decomposition, see appendix E in Wald [15]). We have the understanding that the reduced field equations (after applying an ansatz) must also be able to be formulated from a Lagrangian standpoint (which can be thought of as substituting the ansatz in the full action) with a constraint equation in the form of a Lagrange multiplier (to preserve the Bianchi identity). From this reduced action, many properties become apparent about the symmetries, as well as methods for simplifying the equations, which could not originally be easily seen from the equations of motion themselves. In this, we use both of these methods for finding time-dependent brane solutions in a general supergravity.

1.3 Conventions

We use the mostly plus metric, diag(-1,1,...,1). To denote all dimensions, we use $M, N...$. To denote the compact dimensions, we use $m, n...$. To denote the usual extended dimensions, we use $\mu, \nu...$. To denote the extended spatial dimensions, we use $i, j...$. Finally, in chapter 2, denoting all spatial dimensions (both compact and non-compact) is necessary. For this chapter, and this chapter alone, we use $\alpha, \beta...$ to denote these conventions. Curvature conventions are those which are used in Weinberg [16]. Finally, the conventions for field normalization and coupling constants are different from those in [2], with appropriate redefinitions connecting the two conventions given in Appendix A.

Chapter 2

Equations of Motion

Looking for solutions in any gravity theory requires careful exploitation of the symmetries at hand. The focus of this chapter is first to find the equations of motion that govern a general supergravity, and then look for symmetries in these equations that may be exploited to look for time dependence. While the results of this chapter will be more general than necessary when investigating the properties of branes, it will give us a better idea of what ansatz to use (in terms of time dependence) in later chapters when more symmetries are available to us.

2.1 Action and the Full Field Equations

The action for a general D-Dimensional supergravity, parametrized by a scalar field (called the dilaton) coupling a, and the rank of the form field n is

$$S = -\int d^D x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + \nabla_M\phi\nabla^M\phi\right) + \frac{1}{4}e^{a\phi}F^2_{[n]}\right] \tag{2.1}$$

$$= \int d^D x \mathscr{L}, \tag{2.2}$$

where F^2 means $F_{MN\ldots}F^{MN\ldots}$. The Einstein equation is obtained by the condition that

$$\frac{\delta S}{\delta g^{MN}} = 0. \tag{2.3}$$

We trace-reverse the Einstein equations by employing the following trick. Using the fact that the functional derivative is Leibniz, and (2.3), we get

$$\frac{\delta}{\delta g^{MN}}\mathscr{L} = \frac{\mathscr{L}}{\sqrt{-g}}\frac{\delta}{\delta g^{MN}}\sqrt{-g} + \sqrt{-g}\frac{\delta}{\delta g^{MN}}\frac{\mathscr{L}}{\sqrt{-g}}$$

$$0 = \left[-\frac{1}{2}g_{MN}\frac{\mathscr{L}}{\sqrt{-g}} + \frac{\delta}{\delta g^{MN}}\frac{\mathscr{L}}{\sqrt{-g}}\right], \tag{2.4}$$

where $g = \det(g_{MN})$ - making (2.4) our Einstein equation. Taking the trace of (2.4), we obtain

$$\frac{D}{2}\frac{\mathscr{L}}{\sqrt{-g}} = g^{MN}\frac{\delta}{\delta g^{MN}}\frac{\mathscr{L}}{\sqrt{-g}}. \tag{2.5}$$

We now take the functional derivative of $\frac{\mathscr{L}}{\sqrt{-g}}$ with respect to the metric. We use the fact that the Ricci tensor transforms as a total derivative, and thus is effectively constant with respect to the variation, as is shown in Chapter eleven of [17]. This gives

$$-\frac{\delta}{\delta g^{MN}}\frac{\mathscr{L}}{\sqrt{-g}} = \frac{1}{2\kappa^2}\left(\frac{\delta}{\delta g^{MN}}g^{PQ}R_{PQ} + \frac{\delta}{\delta g^{MN}}g^{PQ}\nabla_P\phi\nabla_Q\phi\right) \tag{2.6}$$

$$+\frac{1}{4}e^{a\phi}\frac{\delta}{\delta g^{MN}}g^{AP}g^{BQ}...F_{AB...}F_{PQ...}$$

$$= \frac{1}{2\kappa^2}\left(R_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{n}{4}e^{a\phi}F_{M...}F_N{}^{...}. \tag{2.7}$$

We now trace-reverse the Einstein equations. The meaning and purpose of trace-reversing is to use the relationship of the Ricci scalar in terms of the other scalars (that is, F^2 and $(\partial\phi)^2$) to eliminate it from the Einstein equations. This in turn makes the resulting field equations have a different form than the canonical Einstein equations. Using the trace condition, (2.5), we obtain

$$\frac{1}{2\kappa^2}\left(R + (\partial\phi)^2\right) + \frac{n}{4}e^{a\phi}F^2 = \frac{D}{2}\left[\frac{1}{2\kappa^2}\left(R + (\partial\phi)^2\right) + \frac{1}{4}e^{a\phi}F^2\right]$$

$$\frac{1}{2\kappa^2}\left(R + (\partial\phi)^2\right) = \frac{D-2n}{8-4D}e^{a\phi}F^2. \tag{2.8}$$

Substituting (2.8) into (2.4) gives us

$$\frac{1}{2\kappa^2}\left(R_{MN} + \partial_M\phi\partial_N\phi\right) + e^{a\phi}\left[\frac{n}{4}F_{M...}F_N{}^{...} + g_{MN}\frac{1}{8}\left(\frac{D-2n}{D-2} - 1\right)F^2\right] = 0$$

$$\frac{1}{2\kappa^2}\left(R_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{n}{4}e^{a\phi}\left[F_{M...}F_N{}^{...} - g_{MN}\left(\frac{n-1}{n(D-2)}\right)F^2\right] = 0, \tag{2.9}$$

which are our Einstein equations. The dilaton equation comes from taking the functional derivative with respect to ϕ. This is simply

$$\frac{\delta S}{\delta\phi} = 0 \tag{2.10}$$

$$\frac{\delta}{\delta\phi}\frac{\mathscr{L}}{\sqrt{-g}} = \frac{1}{\kappa^2}\partial_M\phi\partial^M\frac{\delta}{\delta\phi}\phi + \frac{1}{4}\frac{\delta}{\delta\phi}e^{a\phi}F^2 \tag{2.11}$$

$$= -\Box\phi + \frac{a}{4}e^{a\phi}F^2$$

$$\Box\phi = \frac{\kappa^2 a}{4}e^{a\phi}F^2, \tag{2.12}$$

where $\Box = \frac{1}{\sqrt{-g}}\partial_M\sqrt{-g}\partial^M$. Finally, to obtain the equations for the Maxwell field, we take the functional derivative with respect to the $n-1$ form field $A_{M...}$. Remembering the antisymmetry of the form field, we take

$$\frac{\delta S}{\delta A_{MN...}} = 0 \tag{2.13}$$

$$\begin{aligned}
\frac{\delta}{\delta A_{MN...}}\frac{\mathscr{L}}{\sqrt{-g}} &= \frac{1}{2}e^{a\phi}F^{PQ...}\frac{\delta}{\delta A_{MN...}}\left[\nabla_P A_{QR...} - \nabla_Q A_{PR...} - \nabla_R A_{QP} - ...\right] \\
&= -\frac{1}{2}\nabla_L\left(e^{a\phi}F^{PQ...}\right)\left[\delta^L_P\delta^M_Q\delta^N_R ... - \delta^L_Q\delta^M_P\delta^N_R ... - \delta^L_R\delta^M_Q\delta^N_P ... - ...\right] \\
&= -\frac{n}{2}\nabla_M\left(e^{a\phi}F^{MN...}\right)
\end{aligned}$$

$$\nabla_M\left(e^{a\phi}F^{MN...}\right) = 0, \tag{2.14}$$

which gives us our final set of field equations - the Maxwell equation. We're in a position to fully state the full field equations relevant to the problem at hand. These are

$$\frac{1}{2\kappa^2}\left(R_{MN} + \partial_M\phi\partial_N\phi\right) + \frac{n}{4}e^{a\phi}\left[F_{M...}F_N^{\cdots} - g_{MN}\left(\frac{n-1}{n(D-2)}\right)F^2\right] = 0 \tag{2.15}$$

$$\Box\phi - \frac{\kappa^2 a}{4}e^{a\phi}F^2 = 0 \tag{2.16}$$

$$\nabla_M\left(e^{a\phi}F^{MN...}\right) = 0. \tag{2.17}$$

These field equations posses a scaling symmetry. To see this we take

$$g_{MN} \longrightarrow \lambda^{2h}g_{MN} \tag{2.18}$$

$$e^{a\phi} \longrightarrow \lambda^p e^{a\phi}, \tag{2.19}$$

giving us a rescaled action

$$S = -\int d^Dx\sqrt{-g}\lambda^{-2h}g^{MN}\left[\frac{1}{2\kappa^2}R_{MN} + \nabla_M\phi\nabla_N\phi + \frac{\lambda^{p-(n-1)2h}}{4}e^{a\phi}F_{M..}F_N^{\cdot\cdot}\right]. \tag{2.20}$$

Clearly, the condition

$$p = (n-1)2h \tag{2.21}$$

scales the action by an overall constant, which doesn't affect the dynamics of the fields. To exploit this symmetry, we will look for an ansatz which makes this scaling symmetry explicit. To find this ansatz, we look for a homothetic Killing vector. A homothetic killing vector is a vector which satisfies the condition,

$$\mathcal{L}_X g_{MN} = hg_{MN}, \tag{2.22}$$

where $\mathcal{L}_X$ is the Lie derivative along the "flow" of the vector X^M. This is the statement that the metric simply scales as it flows along the the vector X^M, hence giving us the motivation associated with the action possessing the scaling symmetry. We note that this condition on the metric, along with the action of the Lie derivative on the dilaton,

$$\mathcal{L}_X\phi = \frac{p}{a}, \tag{2.23}$$

with the condition (2.21) will give the action when the Lie derivative acts on it,

$$\mathcal{L}_X S = (D-2)hS, \tag{2.24}$$

leading us to the ansatz of the time dependence in the next section.

2.2 Metric

Since we are looking for a metric which has time dependence manifest in this scaling symmetry, we adopt a power law time dependence ansatz which gives us the required relations of (2.18) and (2.19). The fields are of the form

$$ds^2 = -(Ht)^{2h-2}K^2(x^\alpha)dt^2 + 2(Ht)^{2h-1}V_\beta(x^\alpha)dtdx^\beta + (Ht)^{2h}h(x^\alpha)_{\beta\gamma}dx^\beta dx^\gamma \tag{2.25}$$

$$\phi = \tilde{\phi}(x^\alpha) + \frac{p}{a}\ln(t) \tag{2.26}$$

$$F_{M\ldots} = F_{\alpha\ldots}(x^\beta) \quad \text{others zero.} \tag{2.27}$$

This undergoes (2.18) and (2.19) when $t \to \lambda t$, as required. The homothetic Killing vector associated with this symmetry is

$$X^M = t\delta_t^M \tag{2.28}$$

This scaling symmetry can be modified into translational one by the transformation $Ht \to e^\tau$, which gives the metric and dilaton

$$ds^2 = e^{2h\tau}(-K^2(x^\alpha)d\tau^2 + 2V_\beta(x^\alpha)dtdx^\beta + h_{\beta\gamma}(x^\alpha)dx^\beta dx^\gamma) \tag{2.29}$$

$$= e^{2h\tau}d\tilde{s}^2 \tag{2.30}$$

$$\phi = \tilde{\phi} + \frac{p}{a}\tau, \tag{2.31}$$

as well as the associated homothetic Killing vector $X^M = \delta_t^M$. It is convenient to express the field equations in terms of the conformally related metric, $\tilde{g}_{MN}$, where

$$g_{MN} = \Omega^2 \tilde{g}_{MN} \tag{2.32}$$

$$\Omega = e^{h\tau}. \tag{2.33}$$

Now we can explicitly remove the time dependence by using an identity for conformally related metrics. For further reference, this is proved in appendix B. As a reminder, the Ricci tensor is

$$R_{MN} = \partial_M \Gamma^P_{NP} - \partial_P \Gamma^P_{MN} + \Gamma^P_{MQ}\Gamma^Q_{NP} - \Gamma^P_{MN}\Gamma^Q_{PQ}, \tag{2.34}$$

with Γ being the connections defined through the metric (Christoffel symbols). Using the identity for conformally related metrics, we relate the Ricci tensors of the respective metrics. That is,

$$\begin{aligned} R_{MN} = {} & \tilde{R}_{MN} + (D-2)\tilde{\nabla}_M\tilde{\nabla}_N \ln\Omega + \tilde{g}_{MN}\tilde{g}^{OP}\tilde{\nabla}_O\tilde{\nabla}_P \ln\Omega \\ & - (D-2)(\tilde{\nabla}_M \ln\Omega)(\tilde{\nabla}_N \ln\Omega) + (D-2)\tilde{g}_{MN}\tilde{g}^{OP}(\tilde{\nabla}_O \ln\Omega)(\tilde{\nabla}_P \ln\Omega), \end{aligned} \tag{2.35}$$

where $\tilde{R}_{MN}$ is the metric for $\tilde{g}_{MN}$, and covariant derivatives, $\tilde{\nabla}_M$ are with respect to the metric $\tilde{g}_{MN}$. Plugging in (2.33), the components of the Ricci tensor are

$$R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - (D-2)h\tilde{\Gamma}^\tau_{\alpha\beta} - h\tilde{g}_{\alpha\beta}\tilde{g}^{OP}\tilde{\Gamma}^\tau_{OP} + (D-2)h^2\tilde{g}_{\alpha\beta}\tilde{g}^{\tau\tau} \tag{2.36}$$

$$R_{\alpha\tau} = \tilde{R}_{\alpha\tau} - (D-2)h\tilde{\Gamma}^\tau_{\alpha\tau} - h\tilde{g}_{\alpha\tau}\tilde{g}^{OP}\tilde{\Gamma}^\tau_{OP} + (D-2)h^2\tilde{g}_{\alpha\tau}\tilde{g}^{\tau\tau} \tag{2.37}$$

$$R_{\tau\tau} = \tilde{R}_{\tau\tau} - (D-2)h\tilde{\Gamma}^\tau_{\tau\tau} - h\tilde{g}_{\tau\tau}\tilde{g}^{OP}\tilde{\Gamma}^\tau_{OP} - (D-2)h^2 + (D-2)h^2\tilde{g}_{\tau\tau}\tilde{g}^{\tau\tau}. \tag{2.38}$$

We are now in a position to express the full field equations in terms of this conformal scaling. Using (2.15)-(2.17), the new field equations are

$$\frac{1}{2\kappa^2}\Big(R_{\tau\tau} - (D-2)h\tilde{\Gamma}^{\tau}_{\tau\tau} - h\tilde{g}_{\tau\tau}\tilde{g}^{OP}\tilde{\Gamma}^{\tau}_{OP} - (D-2)h^2 + (D-2)h^2\tilde{g}_{\tau\tau}\tilde{g}^{\tau\tau} + \frac{p^2}{a^2}\Big) + \frac{n}{4}e^{a\tilde{\phi}}\left(F_{\tau\ldots}F_{\tau}{}^{\cdots} - \tilde{g}_{\tau\tau}\frac{n-1}{n(D-2)}F^2\right) = 0 \quad (2.39)$$

$$\frac{1}{2\kappa^2}\Big(R_{\alpha\tau} - (D-2)h\tilde{\Gamma}^{\tau}_{\alpha\tau} - h\tilde{g}_{\alpha\tau}\tilde{g}^{OP}\tilde{\Gamma}^{\tau}_{OP} + (D-2)h^2\tilde{g}_{\alpha\tau}\tilde{g}^{\tau\tau} + \frac{p}{a}\partial_\alpha\tilde{\phi}\Big) + \frac{n}{4}e^{a\tilde{\phi}}\left(F_{\alpha\ldots}F_{\tau}{}^{\cdots} - \tilde{g}_{\alpha\tau}\frac{n-1}{n(D-2)}F^2\right) = 0 \quad (2.40)$$

$$\frac{1}{2\kappa^2}\Big(R_{\alpha\beta} - (D-2)h\tilde{\Gamma}^{\tau}_{\alpha\beta} - h\tilde{g}_{\alpha\beta}\tilde{g}^{OP}\tilde{\Gamma}^{\tau}_{OP} + (D-2)h^2\tilde{g}_{\alpha\beta}\tilde{g}^{\tau\tau} + \partial_\alpha\tilde{\phi}\partial_\beta\tilde{\phi}\Big) + \frac{n}{4}e^{a\tilde{\phi}}\left(F_{\alpha\ldots}F_{\beta}{}^{\cdots} - \tilde{g}_{\alpha\beta}\frac{n-1}{n(D-2)}F^2\right) = 0 \quad (2.41)$$

$$(D-2)h\tilde{g}^{\tau N}\partial_N\left(\tilde{\phi} + \frac{p}{a}\tau\right) + \tilde{\Box}\tilde{\phi} - \frac{\kappa^2 a}{4}e^{a\tilde{\phi}}F^2 = 0 \quad (2.42)$$

$$\tilde{\nabla}_M\left(e^{a\tilde{\phi}}F^{MN\ldots}\right) = 0. \quad (2.43)$$

All factors of Ω from the metric in F^2 are canceled with the $e^{a\tilde{\phi}}$ term due to condition (2.21).

2.3 Covariant form

While the form of the field equations are apparently non-overconstraining, they were not manifestly covariant. That is to say that connections are appearing in our field equations (which are obviously not-covariant). The reason that these connections are appearing is from making a co-ordinate system choice, i.e. we've chosen our time dependence to be (2.33). However, we still have made no assumptions about the spatial dependence of the metric, suggesting that we may be able to write the field equations in a covariant form in terms of the spatial components of the metric. This is made explicit given that our stationary metric is general and has the form

$$\tilde{ds}^2 = -K^2(x^\alpha)d\tau^2 + 2V_\beta(x^\alpha)d\tau dx^\beta + h_{\beta\gamma}(x^\alpha)dx^\beta dx^\gamma. \quad (2.44)$$

This is similar to the Arnowitt-Desner-Misner (ADM) decomposition. The ADM decomposition simply decomposes the metric into a scalar, vector, and smaller tensor (see [18] for a thorough treatment on this decomposition.) From here we can find an inverse metric in terms of these functions.

We employ the definitions

$$h_{\alpha\beta}h^{\beta\gamma} = \delta^{\gamma}_{\alpha} \tag{2.45}$$

$$V^{\alpha} = h^{\alpha\beta}v_{\beta} \tag{2.46}$$

$$V^2 = V^{\alpha}V_{\alpha}. \tag{2.47}$$

Using these definitions, the inverse metric turns out to be

$$\tilde{g}^{\tau\tau} = \frac{-1}{K^2+V^2} \tag{2.48}$$

$$\tilde{g}^{\tau\alpha} = \frac{V^{\alpha}}{K^2+V^2} \tag{2.49}$$

$$\tilde{g}^{\alpha\beta} = h^{\alpha\beta} - \frac{V^{\alpha}V^{\beta}}{K^2+V^2}. \tag{2.50}$$

This allows us to express the Christoffel connection symbols in terms of our newly defined scalar, tensor, and vector. These turn out to be

$$\tilde{\Gamma}^{\tau}_{\tau\tau} = \frac{KV^{\alpha}\partial_{\alpha}K}{K^2+V^2} \tag{2.51}$$

$$\tilde{\Gamma}^{\tau}_{\alpha\tau} = \frac{1}{K^2+V^2}\left(K\partial_{\alpha}K + \frac{1}{2}V^{\beta}\left(D_{\alpha}V_{\beta} - D_{\beta}V_{\alpha}\right)\right) \tag{2.52}$$

$$\tilde{\Gamma}^{\alpha}_{\tau\tau} = \left(h^{\alpha\beta} - \frac{V^{\alpha}V^{\beta}}{K^2+V^2}\right)K\partial_{\beta}K \tag{2.53}$$

$$\tilde{\Gamma}^{\tau}_{\alpha\beta} = \frac{-1}{2(K^2+V^2)}\left(D_{\alpha}V_{\beta} + D_{\beta}V_{\alpha}\right) \tag{2.54}$$

$$\tilde{\Gamma}^{\alpha}_{\tau\beta} = \frac{1}{2}\left(h^{\alpha\gamma} - \frac{V^{\alpha}V^{\gamma}}{K^2+V^2}\right)\left(D_{\beta}V_{\gamma} - D_{\gamma}V_{\beta}\right) - \frac{V^{\alpha}K\partial_{\beta}K}{K^2+V^2} \tag{2.55}$$

$$\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \tilde{\Gamma}^{\gamma}_{\alpha\beta}(h_{\alpha\beta}) + \frac{V^{\gamma}}{2(K^2+V^2)}\left(D_{\alpha}V_{\beta} + D_{\beta}V_{\alpha}\right)), \tag{2.56}$$

where $\tilde{\Gamma}(h_{\alpha\beta})$ is the connection corresponding to the hypersurface metric $h_{\alpha\beta}$, and D_{α} is the corresponding covariant derivative. Before we go ahead and use these relationships for the connection, there is one useful contraction that comes up a lot in the Ricci tensor. This is

$$\begin{aligned}
\tilde{g}^{MN}\tilde{\Gamma}^{\tau}_{MN} =& \tilde{g}^{\tau\tau}\tilde{\Gamma}^{\tau}_{\tau\tau} + 2\tilde{g}^{\alpha\tau}\tilde{\Gamma}^{\tau}_{\tau\alpha} + \tilde{g}^{\alpha\beta}\tilde{\Gamma}^{\tau}_{\alpha\beta} \\
=& -\frac{KV^{\alpha}\partial_{\alpha}K}{(K^2+V^2)^2} + 2V^{\alpha}\frac{1}{(K^2+V^2)^2}\left(K\partial_{\alpha}K + \frac{1}{2}V^{\beta}\left(D_{\alpha}V_{\beta} - D_{\beta}V_{\alpha}\right)\right) \\
&+ \left(h^{\alpha\beta} - \frac{V^{\alpha}V^{\beta}}{K^2+V^2}\right)\frac{-1}{2(K^2+V^2)}\left(D_{\alpha}V_{\beta} + D_{\beta}V_{\alpha}\right) \\
=& \frac{V^{\alpha}\partial_{\alpha}(K^2+V^2)}{(K^2+V^2)^2} - \frac{D_{\alpha}V^{\alpha}}{K^2+V^2}.
\end{aligned} \tag{2.57}$$

Using all these definitions, we calculate the Ricci tensor, and the full field equations for a stationary supergravity that is scaled by the conformal factor $e^{h\tau}$. These are

$$h(D-2)\frac{1}{K^2+V^2}\left[V^\alpha\partial_\alpha\tilde{\phi}-\frac{p}{a}\right]+\tilde{\Box}\tilde{\phi}-\frac{1}{K^2+V^2}\Big(V^\alpha V^\beta D_\alpha\partial_\beta\tilde{\phi}- \left(D_\alpha V^\alpha+\frac{V^\alpha V^\beta}{K^2+V^2}\right)V^\gamma\partial_\gamma\tilde{\phi}\Big)-\frac{\kappa^2 a}{4}e^{a\tilde{\phi}}\mathcal{F}^2=0\ \ (\tilde{\phi}) \tag{2.58}$$

$$\frac{1}{2\kappa^2}\Bigg[\frac{1}{2}\left[D_\alpha\left(\frac{V^\alpha V^\beta}{K^2+V^2}\partial_\beta K^2\right)-\tilde{\Box}K^2\right]-\frac{1}{4}\Bigg[\frac{\partial^\alpha K^2 D_\alpha V^2-(V^\alpha\partial_\alpha K^2)^2}{K^2+V^2} -\frac{\partial^\alpha K^2\partial_\alpha K^2}{K^2+V^2}-h^{\alpha\delta}D_{[\beta}V_{\delta]}D_{[\alpha}V_{\gamma]}\left(h^{\beta\gamma}+\frac{2V^\beta V^\gamma}{K^2+V^2}\right)\Bigg]-(D-2)h\frac{V^\alpha\partial_\alpha K^2}{2(K^2+V^2)} +hK^2\left(\frac{V^\alpha\partial_\alpha(K^2+V^2)}{2(K^2+V^2)^2}-\frac{D_\alpha V^\alpha}{K^2+V^2}\right)-(D-2)h^2\frac{V^2}{K^2+V^2}+\frac{p^2}{a^2}\Bigg] +\frac{n}{4}e^{a\tilde{\phi}}\left(K^2\frac{n-1}{n(D-2)}\mathcal{F}^2\right)=0\ \ (\tau\tau) \tag{2.59}$$

$$\frac{1}{2\kappa^2}\Bigg[\frac{1}{2}\left[D_\alpha\left(V^\alpha\frac{\partial_\rho K^2}{K^2+V^2}\right)-D_\alpha\left(\left(h^{\alpha\beta}+\frac{V^\alpha V^\beta}{K^2+V^2}\right)D_{[\rho}V_{\beta]}\right)\right] -\frac{D_{\{\rho}V_{\alpha\}}\partial^\alpha K^2}{4(K^2+V^2)}+V^\alpha\partial_\alpha(K^2+V^2)\frac{\partial_\rho K^2+D_{[\rho}V_{\beta]}V^\beta}{4(K^2+V^2)^2} -(D-2)h\frac{\partial_\alpha K^2-V^\beta D_{[\alpha}V_{\beta]}}{2(K^2+V^2)}+hV_\rho\left(\frac{V^\alpha\partial_\alpha(K^2+V^2)}{(K^2+V^2)^2}-\frac{D_\alpha V^\alpha}{K^2+V^2}\right) -(D-2)h^2\frac{V_\rho}{K^2+V^2}+\frac{p}{a}\partial_\rho\tilde{\phi}\Bigg]-\frac{n}{4}e^{a\tilde{\phi}}\left(V_\rho\frac{n-1}{n(D-2)}\mathcal{F}^2\right)=0\ \ (\rho\tau) \tag{2.60}$$

$$\frac{1}{2\kappa^2}\Bigg[\mathcal{R}_{\rho\lambda}-\frac{1}{2}\left[D_\gamma\left(\frac{V^\gamma D_{\{\rho}V_{\lambda\}}}{K^2+V^2}\right)-D_\rho\partial_\lambda\ln(K^2+V^2)\right] -\frac{1}{4(K^2+V^2)^2}\left[D_{\{\rho}V_{\lambda\}}V^\gamma\partial_\gamma(V^2+K^2)-\partial_\rho(K^2+V^2)\partial_\lambda(K^2+V^2)\right] +(D-2)h\frac{D_{\{\rho}V_{\lambda\}}}{2(K^2+V^2)}-hh_{\rho\lambda}\left(\frac{V^\alpha\partial_\alpha(K^2+V^2)}{(K^2+V^2)^2}-\frac{D_\alpha V^\alpha}{K^2+V^2}\right) -(D-2)h^2\frac{h_{\rho\lambda}}{K^2+V^2}+\partial_\rho\tilde{\phi}\partial_\lambda\tilde{\phi}\Bigg]+\frac{n}{4}e^{a\tilde{\phi}}\left(\mathcal{F}_{\rho\ldots}\mathcal{F}_\lambda{}^{\cdots}-h_{\rho\lambda}\frac{n-1}{n(D-2)}\mathcal{F}^2\right)=0\ \ (\rho\lambda) \tag{2.61}$$

$$D_M\mathcal{F}^{M\ldots}+\frac{V^\gamma D_{\{\alpha}V_{\gamma\}}}{2(K^2+V^2)}\mathcal{F}^{\alpha\ldots}=0. \tag{2.62}$$

As a reminder, h is the exponent of time, $h_{\alpha\beta}$ is the spatial hypersurface metric, covariant derivatives are with respect to $h_{\alpha\beta}$, and all Greek indices correspond to only the spatial directions. The Ricci tensor, $\mathcal{R}$, and the d'Alembertian, $\tilde{\Box}$ are also in terms of the hypersurface metric $h_{\alpha\beta}$. $\mathcal{F}_{M\ldots}$ is in place of $F_{M\ldots}$ to make it clear that the field strength tensor is contracted with the original metric g_{MN}.

While these equations aren't particularly useful in themselves in terms of finding solutions (they're horrendous), they do provide an excellent consistency check. This is because these equations should transform like tensors, scalars, or vectors under spatial co-ordinate transformations (if they're zero in one set of co-ordinates, they should be in all co-ordinates.) While it wasn't surprising that $\tilde{R}_{MN}$ properly decomposed to a scalar, vector, and tensor (since the ADM decomposition does this for the Ricci tensor,) it was uncertain whether the terms proportional to h and h^2 would be covariant under this decomposition (the parameter that controls the time dependence). Since these equations are covariant, we have found a form of time dependence which is covariantly consistent, while not over-constraining the field equations. This is the message to take away - we have found a form of time dependence that is perfectly consistent, and will be using the idea of power law time dependence in the remaining chapters as our ansatz.

Chapter 3

Symmetry Specialization Ansatz

We have now seen how adding a power law time dependence in a very special way allows us to have fully consistent field equations. This will facilitate us in finding brane solutions with a similar power law dependence. Due to the symmetries associated with a brane, we're not concerned about grouping all spatial dimensions together. As mentioned in the introduction, we can now take Greek characters α, β, and γ to be constants, as opposed to indices.

See Stelle's review [2] for the symmetries of the metric in the static case. The symmetries associated with a brane are such that translations along the brane leave the metric invariant (ie a homogeneous brane). In addition, we wish for the brane to be isotropic. In terms of the bulk, we wish for the brane to "source" the bulk curvature, and nothing else to source the bulk curvature. It is therefore natural to assume that the only dependence of the metric should be the distance from the brane in the transverse directions. That is to say the bulk is isotropic about the location of the brane. In the static case, this would mean a Poincaré$_{(d)}$ $\times$ $SO(D-d)$ symmetry (Poincaré symmetry along the d dimensional brane and spherical symmetry in $D-d$ dimensions about the brane,) however, due to time dependence this is broken down to a $E(d-1) \times SO(D-d)$ group since time translations and boosts no longer leave the metric invariant, where $E(n)$ is the Euclidean group. The most general metric associated with this symmetry is

$$ds^2 = -(Ht)^{2\gamma} e^{2\mathcal{C}(r)} dt^2 + (Ht)^{2\alpha} \delta_{ij} e^{2\mathcal{A}(r)} dx^i dx^j + (Ht)^{2\beta} \delta_{mn} e^{2\mathcal{B}(r)} dy^m dy^n, \tag{3.1}$$

with the dilaton and relation

$$\tilde{\phi} = \varphi(r) + \frac{p}{a} \ln(t), \tag{3.2}$$

$$p = 2\beta(n-1) \tag{3.3}$$

where $r = \sqrt{y^m y^m}$. Notice how this metric doesn't posses the homothetic symmetry imposed by

(2.28). Currently the motivation for the powers of time is the fact that there will be more options for solving the equations of motion (in addition to the homothetic symmetric case). The relation (3.3) is imposed for the same reason that (2.21) is. This will be seen when we take the magnetic brane ansatz (4.1), since only the bulk components of the form field will be contracted with the metric, allowing us only the need to relate the bulk exponent of time and the dilaton exponent. Under this ansatz, the Ricci tensor is

$$
\begin{aligned}
R_{tt} =& - e^{2(\mathcal{C}-\mathcal{B})}(Ht)^{2(\gamma-\beta)}\left(\mathcal{C}'' + (\mathcal{C}')^2 + (d-1)\mathcal{C}'\mathcal{A}' + \tilde{d}\mathcal{C}'\mathcal{B}' + \frac{\tilde{d}+1}{r}\mathcal{C}'\right) \\
& - \frac{H^2}{(Ht)^2}\Big((d-1)\alpha(1+\gamma-\alpha) + (\tilde{d}+2)\beta(1+\gamma-\beta)\Big) \qquad (3.4) \\
R_{ij} =& \delta_{ij}e^{2(\mathcal{A}-\mathcal{B})}(Ht)^{2(\alpha-\beta)}\left(\mathcal{A}'' + (d-1)(\mathcal{A}')^2 + \mathcal{A}'\mathcal{C}' + \tilde{d}\mathcal{A}'\mathcal{B}' + \frac{\tilde{d}+1}{r}\mathcal{A}'\right) \\
& + \delta_{ij}e^{2(\mathcal{A}-\mathcal{C})}(Ht)^{2(\alpha-\gamma-2)}H^2\alpha\Big(1+\gamma-(d-1)\alpha-(\tilde{d}+2)\beta\Big) \qquad (3.5) \\
R_{mn} =& \delta_{mn}\Bigg(\mathcal{B}'' + \tilde{d}(\mathcal{B}')^2 + (d-1)\mathcal{A}'\mathcal{B}' + \mathcal{C}'\mathcal{B}' + \frac{2\tilde{d}+1}{r}\mathcal{B}' + \frac{d-1}{r}\mathcal{A}' + \frac{1}{r}\mathcal{C}' \\
& + e^{2(\mathcal{B}-\mathcal{C})}(Ht)^{2(\beta-\gamma-1)}H^2\beta\Big(1+\gamma-(d-1)\alpha-(\tilde{d}+2)\beta\Big)\Bigg) \\
& + \frac{y^m y^n}{r^2}\Big(\tilde{d}\mathcal{B}'' + (d-1)\mathcal{A}'' + \mathcal{C}'' - 2(d-1)\mathcal{A}'\mathcal{B}' - 2\mathcal{C}'\mathcal{B}' \\
& + (d-1)(\mathcal{A}')^2 + (\mathcal{C}')^2 - \tilde{d}(\mathcal{B}')^2 - \frac{\tilde{d}}{r}\mathcal{B}' - \frac{d-1}{r}\mathcal{A}' - \frac{1}{r}\mathcal{C}'\Big) \qquad (3.6) \\
R_{tn} =& \frac{y^n}{tr}\Big((d-1)\alpha\left(\mathcal{A}'-\mathcal{C}'\right) - \beta\Big((\tilde{d}+1)\mathcal{C}' + (d-1)\mathcal{A}'\Big)\Big), \qquad (3.7)
\end{aligned}
$$

where $\tilde{d} = D - d - 2$. There are two options at this point to keep the equations from being overconstrained. We can either ensure that the field equations all scale as the same power of time (in essence this is equivalent to adding a "source" term to the field equations), or the time dependence is can celled an appropriate choice of exponents (in this case, the "sourcing" is can celling itself out, while still leaving non-trivial powers of time.) This thesis will be mainly devoted to finding solutions to the former case. However, there is an immediate solution available to the latter case, and will be treated in section 3.2.

3.1 Metric With Homothetic Symmetry

The first option is to ensure that each component of the Ricci tensor scales as only one power of time. This amounts to an imposition of conditions (and redefinitions.) These conditions are

$$\begin{aligned} \beta = \gamma + 1 = h \\ \alpha = h - l, \end{aligned} \tag{3.8}$$

where both h and l are arbitrary constants. In this case, the metric is

$$ds^2 = -(Ht)^{2(h-1)}e^{2\mathcal{C}(r)}dt^2 + (Ht)^{2(h-l)}\delta_{ij}e^{2\mathcal{A}(r)}dx^i dx^j + (Ht)^{2h}\delta_{mn}e^{2\mathcal{B}(r)}dy^m dy^n, \tag{3.9}$$

The definition of l has been made for several reasons. It seems that the homothetic symmetry seen in (2.25) is not here in this case. However, we realize that if we make the transformation $x^i \to t^l x^i$ that this is, in fact, a special case of (2.25). The homothetic symmetry is seen by taking $t \to \lambda t, x^i \to \lambda^l x^i$, changing the original isometric of (2.22) to

$$X^M = t\delta_t^M + lx^i\delta_i^M \quad \text{(no sum)}. \tag{3.10}$$

This allows us to scan through a larger class of solutions than we'd naively expect. Normally, we would expect $l = 0$ to be the only possibility (as seen from (2.25)), with a flat space in the brane directions, and warped in the bulk. However, due to the Euclidean symmetry, the homothetic Killing vector can have a spatial component (in the brane directions), making a vector pointing radially away from the origin in a d-dimensional space-time, parametrized by its distance from the t axis with l. This enlarged parameter space allows for a more general class of solutions to a time-dependent brane. The Ricci tensor associated with (3.9) is

$$\begin{aligned} R_{tt} =& -\frac{e^{2(\mathcal{C}-\mathcal{B})}}{(Ht)^2}\Bigg(\mathcal{C}'' + (\mathcal{C}')^2 + (d-1)\mathcal{C}'\mathcal{A}' + \tilde{d}\mathcal{C}'\mathcal{B}' + \frac{\tilde{d}+1}{r}\mathcal{C}' \\ & + e^{2(\mathcal{B}-\mathcal{C})}H^2 l(h-l)(d-1)\Bigg) \end{aligned} \tag{3.11}$$

$$\begin{aligned} R_{ij} =& \delta_{ij}\frac{e^{2(\mathcal{A}-\mathcal{B})}}{(Ht)^{-2l}}\Bigg(\mathcal{A}'' + (d-1)(\mathcal{A}')^2 + \mathcal{A}'\mathcal{C}' + \tilde{d}\mathcal{A}'\mathcal{B}' + \frac{\tilde{d}+1}{r}\mathcal{A}' \\ & - e^{2(\mathcal{B}-\mathcal{C})}H^2(h-l)\left(h(D-2) - l(d-1)\right)\Bigg) \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
R_{mn} =& \delta_{mn}\Bigg(\mathcal{B}'' + \tilde{d}(\mathcal{B}')^2 + (d-1)\mathcal{A}'\mathcal{B}' + \mathcal{C}'\mathcal{B}' + \frac{2\tilde{d}+1}{r}\mathcal{B}' + \frac{d-1}{r}\mathcal{A}' + \frac{1}{r}\mathcal{C}' \\
& - e^{2(\mathcal{B}-\mathcal{C})}H^2 h\left(h(D-2) - l(d-1)\right)\Bigg) \\
& + \frac{y^m y^n}{r^2}\Bigg(\tilde{d}\mathcal{B}'' + (d-1)\mathcal{A}'' + \mathcal{C}'' - 2(d-1)\mathcal{A}'\mathcal{B}' - 2\mathcal{C}'\mathcal{B}' \\
& + (d-1)(\mathcal{A}')^2 + (\mathcal{C}')^2 - \tilde{d}(\mathcal{B}')^2 - \frac{\tilde{d}}{r}\mathcal{B}' - \frac{d-1}{r}\mathcal{A}' - \frac{1}{r}\mathcal{C}'\Bigg) \qquad (3.13) \\
R_{tn} =& -\frac{y^n}{tr}\left(h(D-2)\mathcal{C}' + l(d-1)(\mathcal{A}' - \mathcal{C}')\right), \qquad (3.14)
\end{aligned}
$$

which reduces to the static brane case in [2] when $\mathcal{C} = \mathcal{A}$ and $h = l = 0$. The field equations under this metric will be explored in the next chapter.

3.2 Cancellation of Powers

We notice in (3.4)-(3.6) that the derivative terms scale as a different power of t than the terms proportional to the metric functions. We can instead make these metric function terms vanish entirely (as opposed to scaling with the same power of t). The motivation for this is that the exponents cancel and leave entirely different field equations. While the following conditions may seem to be pulled out of a hat, it will soon be clear why they were imposed. Using the conditions

$$\alpha = -\left[\left(\tilde{d}+2\right) \pm \sqrt{\frac{1}{d-1}\left(\tfrac{4\tilde{d}^2}{a^2}(d-2) + \left(\tilde{d}+2\right)(D-2)\right)}\right]\frac{\beta}{(d-2)} \qquad (3.15)$$

$$\gamma = -\left[\left(\tilde{d}+2\right) \pm \sqrt{(d-1)\left(\tfrac{4\tilde{d}^2}{a^2}(d-2) + \left(\tilde{d}+2\right)(D-2)\right)}\right]\frac{\beta}{(d-2)} - 1, \qquad (3.16)$$

as well as the source ansatz (4.1), and the condition $\mathcal{C} = \mathcal{A}$, we produce the field equations

$$\mathcal{A}'' + (d)(\mathcal{A}')^2 + \tilde{d}\mathcal{A}'\mathcal{B}' + \frac{\tilde{d}+1}{r}\mathcal{A}' = \frac{\kappa^2\tilde{d}}{2(D-2)}S^2 \qquad (3.17)$$

$$\mathcal{B}'' + \tilde{d}(\mathcal{B}')^2 + (d)\mathcal{A}'\mathcal{B}' + \frac{2\tilde{d}+1}{r}\mathcal{B}' + \frac{d}{r}\mathcal{A}' = -\frac{\kappa^2 d}{2(D-2)}S^2 \qquad (3.18)$$

$$\tilde{d}\mathcal{B}'' + (d)\mathcal{A}'' - 2d\mathcal{A}'\mathcal{B}' + d(\mathcal{A}')^2 - \tilde{d}(\mathcal{B}')^2 - \frac{\tilde{d}}{r}\mathcal{B}' - \frac{d}{r}\mathcal{A}' + (\varphi')^2 = \frac{\kappa^2}{2}S^2 \qquad (3.19)$$

$$(D-2)\mathcal{A}' = \frac{2\tilde{d}}{a}\varphi' \qquad (3.20)$$

$$\varphi'' + d\mathcal{A}'\varphi' + \tilde{d}\mathcal{B}'\varphi' + \frac{\tilde{d}+1}{r}\varphi' = \frac{\kappa^2 a}{4}S^2. \qquad (3.21)$$

These are identical to the field equations for a magnetic brane ansatz in [2], with the addition of (3.20). However, (3.20) is automatically satisfied by the supersymmetric p-brane solution - automatically giving us a solution! The solutions to these equations are

$$ds^2 = H^{\frac{-4\tilde{d}}{\sigma(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d}{\sigma(D-2)}} dy^m dy^n \delta_{mn} \tag{3.22}$$

$$e^\phi = H^{-\frac{a}{\sigma}} \tag{3.23}$$

$$H = 1 + \frac{k}{r^{\tilde{d}}} \tag{3.24}$$

$$\sigma = \frac{a^2}{2} + \frac{2\tilde{d}d}{(D-2)}, \tag{3.25}$$

with k being an integration constant and interpreted as the mass of the brane. The reason why these conditions have generated a solution that is identical to the static one, is that the static solution has supersymmetry imposed. This has the property that $\Box = \partial_r^2 + \frac{\tilde{d}+1}{r}$. However, this condition is no longer satisfied with time dependence, unless we make $\sqrt{-g}g^{tt}$ independent of time. This is precisely what (3.15) and (3.16) do.

This is probably not the most general solution that cancel the powers of time (for the same reason that a supersymmetric brane is not the most general solution with the ansatz (4.1), see chapter 4,) but it is an immediately visible solution. Notice how β is still a free parameter, allowing us to have a class of scaling solutions. While there is the possibility of generating more general solutions from this option of generating consistent equations, the remainder of this thesis will concentrate on the homothetic case (3.9).

The lesson here is that multiple powers can be imposed (instead of the one in section 2.2) when greater symmetries are available, and thus allowing two entirely different types of solutions. One of these will be explored in greater detail in the following chapters.

Chapter 4

Magnetic Brane Ansatz

We use the metric (3.1), dilaton(3.2), and relation (3.8) of the previous section with the ansätz

$$F_{m_1 \dots m_n} = \lambda \epsilon_{m_1 \dots m_n q} \frac{y^q}{r^{n+1}},$$
$$\tilde{d} = n - 1 \tag{4.1}$$

to obtain the field equations. Here, λ is considered the charge coupled to the form field, and thus interpreted as the charge of the brane. This is a magnetic source which naturally couples the n-form to the transverse directions, as well as satisfies (2.17)[1]. It may strike the reader as odd to choose an ansätz for the form field strength, as opposed to the Gage field. Doing this of course requires a careful check that in fact $F_{m_1 \dots m_n}$ is exact (which is the same statement that its closed, since our manifold is simply connected.) To check this, we must take the exterior derivative of our field strength F. The partial derivative is

$$\partial_q F_{m_1 .. m_n} = r^{-(n+1)} \left(\epsilon_{m_1 .. m_n q} - (n+1) \epsilon_{m_1 .. m_n p} \frac{y^p y_q}{r^2} \right). \tag{4.2}$$

To find the exterior derivative, we find the completely antisymmetric component of the partial derivative. To this end, we contract with the $n+1$ dimensional completely anti-symmetric tensor. Doing this gives

$$\epsilon^{m_1 .. m_n q} \partial_q F_{m_1 .. m_n} = r^{-(n+1)} \epsilon^{m_1 .. m_n q} \left(\epsilon_{m_1 .. m_n q} - (n+1) \epsilon_{m_1 .. m_n p} \frac{y^p y_q}{r^2} \right) \tag{4.3}$$

$$= r^{-(n+1)} \left((n+1)! - (n+1) n! \delta^q_p \frac{y^p y_q}{r^2} \right) \tag{4.4}$$

$$= 0 \tag{4.5}$$

[1]This is because $g^{n m_1} \nabla_n f(r) \epsilon_{m_1 \dots m_n q} y^q$ ends up contracting two indices in the epsilon tensor with either the metric or $y^m y^n$, yielding zero

using the well known contraction properties of the ϵ tensor. Indeed, this is a consistent ansätz to make for the field strength tensor.

The opposite ansätz to make instead of the magnetic is the electric one. This would have the Gage form field be

$$A_{mu_1..mu_{n-1}} = \epsilon_{mu_1..mu_{n-1}} e^{f(r)}, \tag{4.6}$$

with the field strength being the exterior derivative of this (and $n-1=d$). However, we realize that this means that $F_{\mu...}$ is nonzero, which violates (2.27), forcing us to concentrate on the magnetic case. Using (2.15) and (2.16), along with the Ricci tensor from (3.9), the field equations are

$$\begin{aligned}
&\mathcal{C}'' + (\mathcal{C}')^2 + (d-1)\mathcal{C}'\mathcal{A}' + \tilde{d}\mathcal{C}'\mathcal{B}' + \frac{\tilde{d}+1}{r}\mathcal{C}' \\
&\quad -e^{2(\mathcal{B}-\mathcal{C})}H^2\left(\frac{4\tilde{d}^2h^2}{a^2} - l(h-l)(d-1)\right) = \frac{\kappa^2\tilde{d}}{2(D-2)}S^2 \quad (tt) && (4.7)\\
&\mathcal{A}'' + (d-1)(\mathcal{A}')^2 + \mathcal{A}'\mathcal{C}' + \tilde{d}\mathcal{A}'\mathcal{B}' + \frac{\tilde{d}+1}{r}\mathcal{A}' \\
&\quad -e^{2(\mathcal{B}-\mathcal{C})}H^2(h-l)\left(h(D-2) - l(d-1)\right) = \frac{\kappa^2\tilde{d}}{2(D-2)}S^2 \quad (\delta_{ij}) && (4.8)\\
&\mathcal{B}'' + \tilde{d}(\mathcal{B}')^2 + (d-1)\mathcal{A}'\mathcal{B}' + \mathcal{C}'\mathcal{B}' + \frac{2\tilde{d}+1}{r}\mathcal{B}' + \frac{d-1}{r}\mathcal{A}' \\
&\quad +\frac{1}{r}\mathcal{C}' - e^{2(\mathcal{B}-\mathcal{C})}H^2h(h(D-2) - l(d-1)) = -\frac{\kappa^2 d}{2(D-2)}S^2 \quad (\delta_{mn}) && (4.9)\\
&\tilde{d}\mathcal{B}'' + (d-1)\mathcal{A}'' + \mathcal{C}'' - 2(d-1)\mathcal{A}'\mathcal{B}' - 2\mathcal{C}'\mathcal{B}' + (d-1)(\mathcal{A}')^2 \\
&\quad +(\mathcal{C}')^2 - \tilde{d}(\mathcal{B}')^2 - \frac{\tilde{d}}{r}\mathcal{B}' - \frac{d-1}{r}\mathcal{A}' - \frac{1}{r}\mathcal{C}' + (\varphi')^2 = \frac{\kappa^2}{2}S^2 \quad (y^m y^n) && (4.10)\\
&\quad (h(D-2) - l(d-1))\,\mathcal{C}' + l(d-1)\mathcal{A}' = \frac{2\tilde{d}h}{a}\varphi' \quad (tn) && (4.11)\\
&\varphi'' + \mathcal{C}'\varphi' + (d-1)\mathcal{A}'\varphi' + \tilde{d}\mathcal{B}'\varphi' + \frac{\tilde{d}+1}{r}\varphi' \\
&\quad -e^{2(\mathcal{B}-\mathcal{C})}H^2\frac{2\tilde{d}h}{a}((D-2)h - l(d-1)) = \frac{\kappa^2 a}{4}S^2. \quad (\varphi) && (4.12)
\end{aligned}$$

We have used (2.21) to get (4.7) and (4.12). Both (4.9) and (4.10) are obtained from the mn piece of the Einstein equation by switching to the radial variable. Finally, the derivative of (4.11) is a consequence of (4.7), (4.8), and (4.12) (this is relatively easy to see) thanks to a Bianchi identity. The source (right-hand side, RHS) of these equations is

$$S = \lambda\sqrt{n!}e^{\frac{1}{2}a\varphi - \tilde{d}\mathcal{B}}r^{-\tilde{d}-1}. \tag{4.13}$$

The constraint equation obtained by taking (4.10)$-\tilde{d}$(4.9)$-(d-1)$(4.8)$-$(4.7) is

$$\begin{aligned}
&((d-1)-(d-1)^2)(\mathcal{A}')^2-\tilde{d}(\tilde{d}+1)(\mathcal{B}')^2-2(d-1)(\tilde{d}+1)\mathcal{A}'\mathcal{B}'-2(d-1)\mathcal{A}'\mathcal{C}'\\
&\quad-2(\tilde{d}+1)\mathcal{B}'\mathcal{C}'-2\frac{\tilde{d}(\tilde{d}+1)}{r}\mathcal{B}'-2\frac{(d-1)(\tilde{d}+1)}{r}\mathcal{A}'-2\frac{(\tilde{d}+1)}{r}\mathcal{C}'+(\varphi')^2\\
&\quad+e^{2(\mathcal{B}-\mathcal{C})}H^2\left((h(D-2)-l(d-1))^2+\frac{4\tilde{d}^2h^2}{a^2}+l^2(d-1)-h^2(D-2)\right)=\frac{\kappa^2}{2}S^2. \qquad (4.14)
\end{aligned}$$

Once again a more difficult Bianchi identity is obtained here. It can be seen as

$$\begin{aligned}
(4.14)' =&2\mathcal{B}'(4.14)+2\varphi'(4.12)-2\mathcal{C}'\left((4.10)+(4.9)-(4.7)\right)\\
&-2(\tilde{d}+1)\mathcal{B}'(4.10)-2(d-1)\mathcal{A}'\left((4.10)+(4.9)-(4.8)\right)-2\frac{(\tilde{d}+1)}{r}(4.10). \qquad (4.15)
\end{aligned}$$

We would like to search for a redefinition of variables that simplifies this system of equations. To aid this we look at the above constraint equation since this is the equation that most directly resembles the action governing the system. In particular the kinetic term in the constraint should be the same as the kinetic term in the action. The first thing to do is to solve the *tn* constraint for $\mathcal{C}$ in terms of the other variables. Doing this will easily connect us to the static case since we know in that case $\mathcal{C}=\mathcal{A}$. The relation from (4.11),

$$\mathcal{C}=\frac{1}{\Delta}\left(\frac{2\tilde{d}h}{a}\varphi-l(d-1)\mathcal{A}\right) \qquad (4.16)$$

$$\Delta=(D-2)h-(d-1)l, \qquad (4.17)$$

leaves us with equations for $\mathcal{A}$, $\mathcal{B}$ and φ. We now have a simple connection to the static case which can be seen by taking $h\to 0$, then $l\to 0$ (in that order). The remaining functions satisfy the nontrivial constraint

$$\begin{aligned}
&-(d-1)(d-2)(\mathcal{A}')^2-\tilde{d}(\tilde{d}+1)(\mathcal{B}')^2-2(d-1)(\tilde{d}+1)\mathcal{A}'\mathcal{B}'\\
&-2(d-1)\mathcal{A}'\mathcal{C}'-2(\tilde{d}+1)\mathcal{B}'\mathcal{C}'-2\frac{\tilde{d}(\tilde{d}+1)}{r}\mathcal{B}'-2\frac{(d-1)(\tilde{d}+1)}{r}\mathcal{A}'\\
&\qquad-2\frac{(\tilde{d}+1)}{r}\mathcal{C}'+(\varphi')^2+e^{2(\mathcal{B}-\mathcal{C})}H^2\Sigma^2=\frac{\kappa^2}{2}S^2. \qquad (4.18)
\end{aligned}$$

where

$$\Sigma^2=\Delta^2+\frac{4\tilde{d}^2h^2}{a^2}+l^2(d-1)-h^2(D-2) \qquad (4.19)$$

with $\mathcal{C}$ being defined through (4.16). The trick is to look for a field redenition that is linear in three new variables X, Y, Z, and $\ln(r)$. The latter is necessary to remove the terms in the constraint which are linear in r derivatives since any properly diagonalized action will only have quadratic derivatives. The following redefinition greatly simplifies things

$$\varphi(r) = \frac{a}{2|u|}Y(r) + \frac{\Delta - l}{a\Sigma|u|}\sqrt{\frac{\tilde{d}(d-1)}{D-2}}Z(r) \tag{4.20}$$

$$\mathcal{A}(r) = \frac{\Delta}{(d-1)(\Delta - l)}\left(|u|Y(r) - \left(\frac{a}{2} + \frac{2\tilde{d}h}{a\Delta}\right)\varphi(r)\right) \tag{4.21}$$

$$\mathcal{B}(r) = \frac{1}{\tilde{d}}\left(|w|X(r) - (d-1)\left(1 - \frac{l}{\Delta}\right)\mathcal{A}(r) - \frac{2\tilde{d}h}{a\Delta}\varphi(r)\right) - \ln(r) \tag{4.22}$$

which is designed so that

$$\frac{a}{2}\varphi + (d-1)\mathcal{A} + \mathcal{C} = |u|Y \tag{4.23}$$

$$\mathcal{C} + (d-1)\mathcal{A} + \tilde{d}\mathcal{B} + \tilde{d}\ln(r) = |w|X. \tag{4.24}$$

While it may not be obvious in this co-ordinate system why this combination of functions is important, it will be clear later that this definition greatly simplifies the field equations. This redefinition also implies that both source terms in the constraints happen to scale as $1/r^2$. The coefficients $|u|$ and $|w|$, in terms of the original parameters, is

$$|u| = \frac{a}{2}\sqrt{1 + \frac{4\tilde{d}d}{a^2(D-2)}} \tag{4.25}$$

$$|w| = \sqrt{\frac{\tilde{d}}{\tilde{d}+1}}. \tag{4.26}$$

This reduces the constraint to

$$-(X')^2 + (Y')^2 + (Z')^2 + \frac{1}{r^2}\left(e^{2(\mathcal{B}-\mathcal{C})}H^2\Sigma^2 + \tilde{d}\left(\tilde{d}+1\right) - \frac{\kappa^2}{2}S^2\right) = 0. \tag{4.27}$$

Field redefinitions have not been substituted in the "source" (non-derivative) terms yet since it will make the equations more opaque, and considerable simplifications will be made when co-ordinate transformations are performed. Given this, all the equations of motion and the above constraint can be obtained by varying the action,

$$S = \int dr e^{\tilde{\Omega}(r)}\left[N^{-1}\left(-(X')^2 + (Y')^2 + (Z')^2\right) - \frac{1}{r^2}N\left(e^{2(\mathcal{B}-\mathcal{C})}H^2\Sigma^2 + \tilde{d}\left(\tilde{d}+1\right) - \frac{\kappa^2}{2}S^2\right)\right], \tag{4.28}$$

with $\tilde{\Omega}(r) = \mathcal{C} + (d-1)\mathcal{A} + \tilde{d}\mathcal{B} + (\tilde{d}+1)\ln(r)$. This is unsurprising, since it is the measure of integration, $\sqrt{-g}g^{rr}$. The function N is nothing other than the lapse function in the ADM decomposition in the Hamiltonian formulation of general relativity, and can be set to unity after variation. This is because it is a non-dynamical gauge degree of freedom, and thus is unphysical. We notice that our action has an overall factor in front of the kinetic term, making it an unusual action. However, we can turn this into a canonical action for 3 scalar fields by performing the co-ordinate transformation to eliminate the explicit r dependence in the action,

$$r = e^{\rho} \tag{4.29}$$

$$\Omega = \tilde{\Omega} - \ln(r), \tag{4.30}$$

giving us the action

$$\begin{aligned} S = \int d\rho e^{\Omega} \Big[& N^{-1} \left(-(X')^2 + (Y')^2 + (Z')^2\right) \\ & - N \left(e^{2(\mathcal{B}-\mathcal{C})} H^2 \Sigma^2 + \tilde{d}\left(\tilde{d}+1\right) - \frac{\kappa^2}{2} S^2 \right) \Big], \end{aligned} \tag{4.31}$$

remembering that derivatives are now with respect to ρ. We now can eliminate the factor multiplying the kinetic terms by performing the co-ordinate transformation,

$$d\rho = e^{\Omega} d\mu \tag{4.32}$$

Finally we get a canonical action

$$\begin{aligned} S = \int d\mu N^{-1} & \left(-(X')^2 + (Y')^2 + (Z')^2\right) \\ & - N \left(H^2 \Sigma^2 e^{2v\cdot X} + \tilde{d}\left(\tilde{d}+1\right) e^{2w\cdot X} - \frac{\kappa^2 \lambda^2 n!}{2} e^{2u\cdot X} \right), \end{aligned} \tag{4.33}$$

with it being understood that primes are now with respect to μ. The vectors u, v, and w are defined through the target space metric that the kinetic terms in (4.33) induce. That is the metric $\text{diag}(-1, 1, 1)$. The target space vector X (contravariant) is

$$X = \begin{pmatrix} X(\mu) \\ Y(\mu) \\ Z(\mu) \end{pmatrix}, \tag{4.34}$$

while the 'coeffecient' vectors (covariant) are

$$u = \begin{pmatrix} 0 \\ |u| \\ 0 \end{pmatrix} \tag{4.35}$$

$$v = \begin{pmatrix} \frac{1}{|w|} \\ -\frac{1}{|u|} \\ v_3 \end{pmatrix} \tag{4.36}$$

$$w = \begin{pmatrix} |w| \\ 0 \\ 0 \end{pmatrix}, \tag{4.37}$$

with squared terms the length of these vectors under the target space metric. The length of v is

$$v^2 = -\frac{\Delta^2}{\Sigma^2}, \tag{4.38}$$

with the explicit form of v_3 (which determines if the vectors are linearly dependant) being

$$(v_3) = \frac{1}{\Sigma|u|}\left(\frac{4\tilde{d}^2 h}{a^2} + (l-h)(D-2)\right)\sqrt{\frac{d-1}{\tilde{d}(D-2)}}. \tag{4.39}$$

The relations between these vectors are quite remarkable, and are

$$u \cdot w = 0 \tag{4.40}$$

$$u \cdot v = -1 \tag{4.41}$$

$$w \cdot v = -1, \tag{4.42}$$

which shows that their relative orientation is independent of our choice of parameters. In fact, the relative orientation (and magnitude) of these vectors is why we were able to redefine the original fields in such a clean way to get a diagonal target space metric. In general, the EOMs are impossible to fully solve analytically. In the current form, however, it is made obvious when they are linearly independent, and thus we can decouple at least one of the fields (namely Z). This is when $v_3 = 0$, which is true for some choice of l.

We have now nailed down the field equations (or, rather, action) in a fairly elegant and simple form without imposing any additional assumptions. This form will also allow us to analyze the behavior of the solutions associated with the equations much more transparently (since this is now an action for a relativistic particle in a potential in 2+1 dimensions.) as well as quickly identify cases in which it is easier to solve.

Chapter 5

Special Cases

This chapter will be looking at the various cases in which we can solve the equations (at least in part) analytically. In order to not lose sight of the problem at hand, we remind the reader of the original metric we were solving in terms of the r co-ordinate (in spherical co-ordinates for the transverse dimensions). For convenience, we initially eliminate all the linear derivative terms in the EOMs that go as $\frac{1}{r}$ by letting $\mathcal{B} \to \mathcal{B} - \ln(r)$ so the inverse transformation doesn't involve the radial co-ordinate explicitly.

$$ds^2 = -(Ht)^{2(h-1)} e^{2\mathcal{C}(r)} dt^2 + (Ht)^{2(h-l)} \delta_{ij} e^{2\mathcal{A}(r)} dx^i dx^j + (Ht)^{2h} e^{2\mathcal{B}(r)} \left(\frac{dr^2}{r^2} + d\Omega^2 \right) \tag{5.1}$$

is the metric in this co-ordinate system. When we switch to the μ radial co-ordinate (under a gauge transformation), the metric becomes

$$\begin{aligned} ds^2 = & -(Ht)^{2(h-1)} e^{2\mathcal{C}(\mu)} dt^2 + (Ht)^{2(h-l)} \delta_{ij} e^{2\mathcal{A}(\mu)} dx^i dx^j \\ & + (Ht)^{2h} \left(e^{2(\mathcal{C}(\mu) + (d-1)\mathcal{A}(\mu) + (\tilde{d}+1)\mathcal{B}(\mu))} d\mu^2 + e^{2\mathcal{B}(\mu)} d\Omega^2 \right). \end{aligned} \tag{5.2}$$

With the various metric functions related to the functions in the diagonalized action by (4.20)-(4.22) except without the logarithm in the redefinition of $\mathcal{B}$ since this redefinition was already performed on the metric (5.1).

5.1 Special case - $\Sigma = 0$

The first case is when $\Sigma = 0$. Solving for h, we obtain the condition

$$h = l \frac{\left(\sqrt{-(d-1)\,\tilde{d}\left((4\,d + a^2)\,\tilde{d} + a^2 d \right)} + a\,(d-1)\left(d + \tilde{d} \right) \right) a}{\left(d + \tilde{d} \right)\left(d + \tilde{d} - 1 \right) a^2 + 4\,\tilde{d}^2}, \tag{5.3}$$

which, for non-zero l, is an imaginary number assuming a is real, and d and $\tilde{d}$ are positive. This tells us that Σ is a strictly non-negative number, meaning that it only vanishes when $h = l = 0$. Therefore, Σ can be thought of a paraboloid in l-h space with its minima at the origin for all physical values of the other parameters $(a, d, \tilde{d})$. Since Σ only vanishes when h and l do, this leads us to the conclusion that this is the static case (taking h to zero first, then l as stated in chapter 4). The simplified action then becomes

$$S = \int d\mu N^{-1}\left(-(X')^2 + (Y')^2 + (Z')^2\right) - N\left(\tilde{d}\left(\tilde{d}+1\right)e^{2|w|X} - \frac{\kappa^2\lambda^2 n!}{2}e^{2|u|Y}\right). \tag{5.4}$$

The solutions to this action are

$$X = -\frac{1}{|w|}\ln\left(\frac{1}{X_1}\sin(X_1|w|(\mu - X_0))\sqrt{\tilde{d}\left(\tilde{d}+1\right)}\right) \tag{5.5}$$

$$Y = -\frac{1}{|u|}\ln\left(\frac{1}{\sqrt{2}Y_1}\sin(Y_1|u|(\mu - Y_0))\kappa\lambda\sqrt{n!}\right) \tag{5.6}$$

$$Z = Z_0 + Z_1\mu \tag{5.7}$$

$$0 = (Y_1)^2 - (X_1)^2 - (Z_1)^2, \tag{5.8}$$

with (5.8) coming from the constraint equation. This implies that $(Y_1)^2 > (X_1)^2$ and $(Y_1)^2 > (Z_1)^2$ to ensure the reality of the metric.

These solutions are more general than those of Stelle [2], since linearity conditions were imposed to preserve supersymmetry, as well as to simplify the equations. Here we have solved all three independent equations of motion without imposing this linearity.

Analyzing these solutions, it is suggestive where the brane is located (as well as $r = \infty$.) For the X (and similarly Y) field, we find that the brane is located at $\mu = X_0$ (since the metric functions are exponentials of these fields.) Infinity corresponds to the point in which the sine is unity. If we were to impose the condition that the brane and spatial infinity is located in the same place for $\mathcal{A}, \mathcal{B}, \varphi$ (and thus X, Y, Z), this implies either

$$X_0 = Y_0 \tag{5.9}$$

$$Y_1 = X_1 \tag{5.10}$$

$$Z_1 = 0, \tag{5.11}$$

or

$$Y_1 = Z_1 \tag{5.12}$$

$$X_1 = 0, \tag{5.13}$$

where we have used the constraint condition (5.8). This in turn leaves us with only one free parameter, the mass of the brane (which would go as $\frac{1}{Y_1}$). Finally, the case of Stelle is a special case of these solutions. If we let $X_1, Y_1, X_0 \to 0$ and $Y_0 = \frac{\sqrt{2}}{\kappa\lambda\sqrt{n!}}$, we obtain the solution (3.24) in "our" μ co-ordinates,

$$X = -\frac{1}{|w|}\ln(-\tilde{d}\mu) \tag{5.14}$$

$$Y = -\frac{1}{|u|}\ln\left(1 - \frac{|u|}{\tilde{d}}\lambda\kappa\sqrt{n!}\mu\right) \tag{5.15}$$

$$Z = 0, \tag{5.16}$$

with the coeffecient of μ in the Y field identified as the mass of the brane (relating the charge and the mass).

5.2 Special case - $\tilde{d} = 0$

In this case, one of the terms in the potential of the action (4.33) is removed when we set $\tilde{d} = 0$ (which is the statement that there are only 2 transverse directions since $\tilde{d} = D - d - 2$). This leaves us with only two coeffecient vectors on the target space. We can use the $O(2,1)$ symmetry on the target space to rotate these vectors, decoupling one of the functions. With the knowledge that now $|u| = \frac{a}{2}$, the vectors then become

$$u = \begin{pmatrix} 0 \\ \frac{a}{2} \\ 0 \end{pmatrix} \tag{5.17}$$

$$v = \begin{pmatrix} \sqrt{-v^2 + \frac{4}{a^2}} \\ -\frac{2}{a} \\ 0 \end{pmatrix}, \tag{5.18}$$

with the magnitudes unchanged applying. Under the assumption that $\tilde{d} =$, the length of v is

$$v^2 = -\frac{\Delta^2}{\Sigma^2} \tag{5.19}$$

$$\Delta = (dh - (d-1)l) \tag{5.20}$$

$$\Sigma^2 = (h - l)^2 d(d-1). \tag{5.21}$$

This gives us the action

$$S = \int d\mu N^{-1}\left(-(X')^2 + (Y')^2 + (Z')^2\right) - N\left(H^2\Sigma^2 e^{2\left(\sqrt{-v^2+\frac{1}{u^2}}X - \frac{1}{|u|}Y\right)} - \frac{\kappa^2\lambda^2 n!}{2}e^{2|u|Y}\right). \tag{5.22}$$

Here Z has decoupled from the other field equations, and can once again be solved with solution $Z(\mu) = Z_0 + Z_1\mu$. This leaves us with the reduced action

$$S = \int d\mu N^{-1}\left(-(X')^2 + (Y')^2 + (Z_1)^2\right) - N\left(H^2\Sigma^2 e^{2\left(\sqrt{-v^2+\frac{1}{u^2}}X - \frac{1}{|u|}Y\right)} - \frac{\kappa^2\lambda^2 n!}{2}e^{2|u|Y}\right). \tag{5.23}$$

Once again, the remaining functions cannot be solved analytically in general. However, we notice in (5.3) that once $\tilde{d} = 0$, both h and l can be real and non-zero for Σ to vanish. This becomes the condition that $h = l$, with the action

$$S = \int d\mu N^{-1}\left(-(X')^2 + (Y')^2 + (Z_1)^2\right) - N\left(-\frac{\kappa^2\lambda^2 n!}{2}e^{2|u|Y}\right), \tag{5.24}$$

and solution

$$Y = -\frac{1}{|u|}\ln\left(\frac{1}{\sqrt{2}Y_1}\sin(Y_1|u|(\mu - Y_0))\kappa\lambda\sqrt{n!}\right) \tag{5.25}$$

$$X = X_0 + X_1\mu \tag{5.26}$$

$$Z = Z_0 + Z_1\mu \tag{5.27}$$

$$0 = (Z_1)^2 - (X_1)^2 - (Y_1)^2. \tag{5.28}$$

This is a different class of solutions from the static one, since the constraint equation has changed. In fact, this case cannot reduce to the static case, since the ratio h/l is set, and thus we cannot take them to zero seperately anymore. We do notice that there is in fact a brane still present, since Y still diverges. Once again, we can allow $Y_1 \to 0$, and $Y_0 = \frac{\sqrt{2}}{\kappa\lambda\sqrt{n!}}$, giving us the function and condition

$$Y = -\frac{1}{|u|}\ln\left(1 - \frac{|u|}{\tilde{d}}\lambda\kappa\sqrt{n!}\mu\right) \tag{5.29}$$

$$X_1 = Z_1. \tag{5.30}$$

These also solve the reduced action with the condition $h = l$. Since we performed an $O(2,1)$ rotation on the fields, the original relations (4.20)-(4.22) do not apply. The new field redefinitions are

$$\varphi = -\frac{2}{a\sqrt{-v^2 + \frac{4}{a^2}}} X + Y + \frac{2}{a}\sqrt{-v^2 + \frac{4}{a^2}} Z \tag{5.31}$$

$$\mathcal{A} = \frac{a\Delta}{2(d-1)(\Delta - l)} (Y - \varphi) \tag{5.32}$$

$$\mathcal{B} = \sqrt{-v^2 + \frac{4}{a^2}} X - \frac{2}{a} Y - (d-1)\mathcal{A}, \tag{5.33}$$

which finalizes what can be solved analytically in this case.

5.3 Special case - linearly dependent vectors ($v_3 = 0$)

This section will deal with the case in which the vectors lie in a plane, (as opposed to spanning the full target space). In this case, $v_3 = 0$, which is satisfied by taking (4.39) to be zero. This gives us the condition

$$l = h\left(1 - \frac{4\tilde{d}^2}{(D-2)a^2}\right). \tag{5.34}$$

Again, as in the previous case, we automatically obtain a solution $Z = Z_0 + Z_1\mu$ with the reduced action

$$S = \int d\mu N^{-1} \left(-(X')^2 + (Y')^2 + (Z_1)^2\right) \\ - N\left(H^2\Sigma^2 e^{2\left(\frac{1}{|w|}X - \frac{1}{|u|}Y\right)} + \tilde{d}\left(\tilde{d}+1\right) e^{2|w|X} - \frac{\kappa^2\lambda^2 n!}{2} e^{2|u|Y}\right). \tag{5.35}$$

This system is more complicated than the case of taking $\tilde{d} = 0$, since there's an added term in the potential. This concludes the various special cases in which the action can be simplified. The upshot is that various special cases can help show us the behavior of these solutions in the dynamical case, as well as provide an exact solution in one case. The next chapter will use these special cases to get an idea of how the general case can behave under various conditions.

Chapter 6

Analysis

We now take the results of the previous chapter to analyze the behavior of the solutions. This will allow us to connect the dynamic solutions with the static ones, determining if qualitatively the solutions are different. We will also analyze the asymptotics of the full (unspecialized) field equations to determine if asymptotically flat solutions are viable in the time dependent case.

6.1 Static Case

While we have full solutions for the static case (5.5)-(5.6), analysis is still required in order to connect it with the dynamic cases. For instance, we know that branes are located when the functions are singular, as well as the asymptotic region is located where the function becomes flat (constant). This is the point where the functions X, Y, Z vanish, since the exponentials of this would be unity. For the convenience of the reader, we graph the solution for Y.

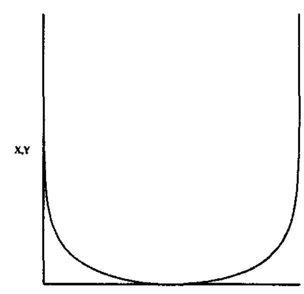

Figure 6.1: Configuration of the field $Y(\mu)$ and $X(\mu)$ for $X_0 = Y_0 = 0$, in the static limit. The horizontal scale is different for the different fields. Both are singular at $\mu = 0$, and at the point where the argument of the sine is π.

We can see explicitly the location of the brane(s) where the solution diverges, as well as the asymptotic region where $Y = 0$. The behavior for these fields can be seen from the form of their field equations. The form

$$Y'' \propto e^{\alpha Y} \tag{6.1}$$

leads to a constant "force" in the positive Y direction. The following plot shows what such a potential would look like. We see that any initial position would have the field "roll" down the potential well

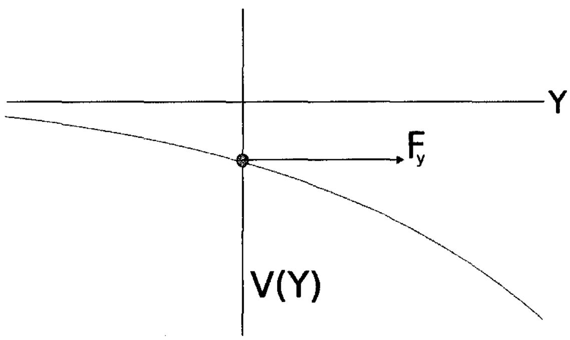

Figure 6.2: Potential of field Y, and force, F_y, it experiences in such a potential

into positive infinity. As we can see from the plot of the solution, this happens at a finite point μ. We identify this as the brane, and subsequent analysis will identify the brane as the point where X, Y and Z diverge.

6.2 General Case Analysis

Now that we have the tools to identify branes, we look at the general equations of motion from the action (4.33). These are

$$X'' = \frac{H^2\Sigma^2}{|w|}e^{2v\cdot X} + \tilde{d}\left(\tilde{d}+1\right)e^{2|w|X} \tag{6.2}$$

$$Y'' = \frac{H^2\Sigma^2}{|u|}e^{2v\cdot X} + \frac{\kappa^2\lambda^2 n!}{2}e^{2|u|Y} \tag{6.3}$$

$$Z'' = -H^2\Sigma^2 v_3 e^{2v\cdot X} \tag{6.4}$$

$$0 = \left(-(X')^2 + (Y')^2 + (Z')^2\right) + \left(H^2\Sigma^2 e^{2v\cdot X} + \tilde{d}\left(\tilde{d}+1\right)e^{2w\cdot X} - \frac{\kappa^2\lambda^2 n!}{2}e^{2u\cdot X}\right), \tag{6.5}$$

immediately showing us that two of these fields are being "accelerated" into the positive direction indefininately. Furthermore, one may worry that the constraint may limit the possibility of having these fields fall to positive infinity (that is, the fields have enough "escape" energy not to be bound by the potential term.) However, the constraint equation isn't affected by the direction (sign) of the initial velocity, and the signature of the target space metric allows for the fields to have various initial conditions, as long as they are compensated to preserve the constraint equation.

The last worry is that the Z field may be growing negatively faster than X, and Y are, and thus destroying a brane solution (since a metric function that goes as $e^{aX+bY+cZ}$ may go to zero instead of infinity). We do note that the "force" terms in (6.2) and (6.3) are larger than those in (6.4), notwithstanding a large v_3. However, the form of v_3 prevents it from being made arbitrarily large, so we are justified in our claim that these very general equations permit brane solutions. Furthermore, the special cases have $Z'' = 0$, and thus similarly have brane solutions. Unfortunately, due to the negative signature of the target space metric, it is impossible to provide a visualization of the potential in terms of a particle "rolling down" a potential well. Instead, it is understood that in the X direction, the particle "rolls up" the potential well (due to the different sign in kinetic energy).

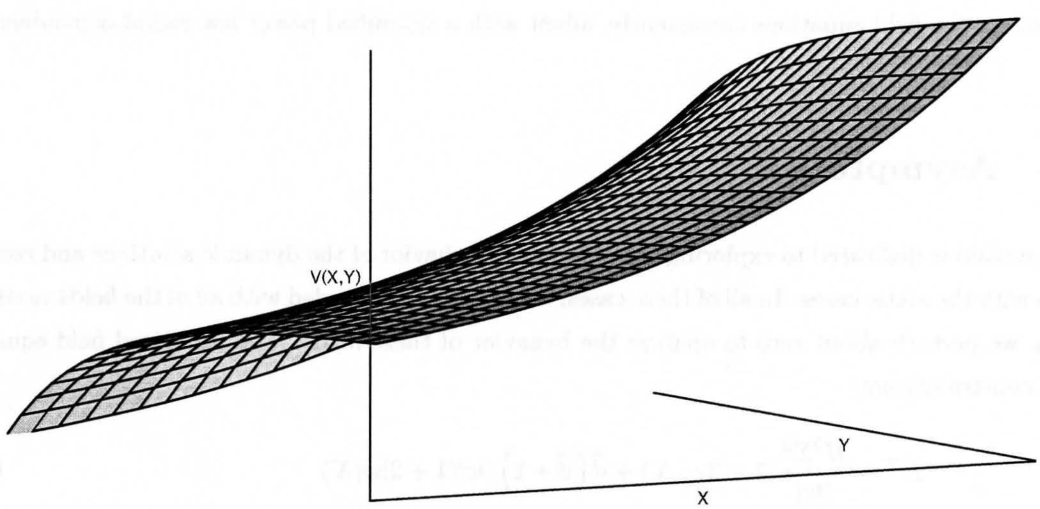

Figure 6.3: Surface plot of "potential" for X and Y in the case(s) where Z decouples. We see that it bends upwards in the positive X direction, and downwards in the positive Y direction. This will cause any static (vanishing derivative) initial configuration to eventually roll in the positive X and Y direction.

We see in the case where Z decouples, that the fields X and Y "roll" into positive infinity due to the curvature of the potential.

Finally, following the approach of Stelle, we look for solutions that go as a power law in these diagonalized set of co-ordinates. It turns out that

$$X(\mu) = -\frac{1}{|w|}\ln(k_x\mu) \tag{6.6}$$

$$Y(\mu) = -\frac{1}{|u|}\ln(k_y\mu) \tag{6.7}$$

$$Z(\mu) = \alpha\ln(k_z\mu) \tag{6.8}$$

$$\alpha = \frac{1}{v_3}\left(-1 - \frac{1}{|u|^2} + \frac{1}{|w|^2}\right) \tag{6.9}$$

$$k_x = \frac{\tilde{d}v_3}{\sqrt{1+v^2}} \tag{6.10}$$

$$k_y = \frac{\sqrt{n!}\kappa\lambda u v_3}{\sqrt{2}\sqrt{1+v^2}} \tag{6.11}$$

$$(k_z)^\alpha = \frac{(k_x)^{\frac{1}{|w|^2}}\sqrt{\alpha}}{(k_y)^{\frac{1}{|u|^2}}H\Sigma\sqrt{v_3}} \tag{6.12}$$

also solves the field equations consistently, albeit with a simplified power law radial dependence.

6.3 Asymptotics

This section is dedicated to exploring the asymptotic behavior of the dynamic solutions and compare them with the static cases. In all of these cases, flat space corresponded with all of the fields vanishing. Thus, we perturb about zero to analyze the behavior of these fields. The linearized field equations (and constraint) are

$$X'' = \frac{H^2\Sigma^2}{|w|}(1+2v\cdot X) + \tilde{d}\left(\tilde{d}+1\right)|w|(1+2|w|X) \tag{6.13}$$

$$Y'' = \frac{H^2\Sigma^2}{|u|}(1+2v\cdot X) + \frac{\lambda^2\kappa^2 n!}{2}|u|(1+2|u|Y) \tag{6.14}$$

$$Z'' = -H^2\Sigma^2 v_3(1+2v\cdot X) \tag{6.15}$$

$$0 = H^2\Sigma^2(1+2v\cdot X) + \tilde{d}\left(\tilde{d}+1\right)(1+2|w|X) - \frac{\lambda^2\kappa^2 n!}{2}(1+2|u|Y), \tag{6.16}$$

where we only trust the solutions to these equations for small values of the fields and their derivatives. Now that the equations are linearlized, we can eliminate Z from the field equations, as well as perform the redefinition

$$X = \tilde{X} - \frac{1}{2|w|} \tag{6.17}$$

$$Y = \tilde{Y} - \frac{1}{2|u|} \tag{6.18}$$

to eliminate the inhomogeneous pieces of the field equations (6.13)-(6.16). The resulting field equations,

$$X'' = -2\tilde{d}X + \frac{\lambda^2\kappa^2 n!|u|}{|w|}Y \tag{6.19}$$

$$Y'' = \lambda^2\kappa^2 n!(1+u^2)Y - 2\tilde{d}\left(\tilde{d}+1\right)\frac{|w|}{|u|}X, \tag{6.20}$$

are a pair of coupled harmonic oscillators (although we want the exponential behavior, and not the oscillation behavior). The normal modes (eigenvalues), are

$$k_\pm = \frac{1}{2}\lambda^2\kappa^2 n!\left(1+u^2\right) - \tilde{d} \pm \sqrt{(1+u^2)^2\left(\frac{1}{2}\lambda^2\kappa^2 n!\right)^2 + 2\,\tilde{d}\left(u^2 - 1 - 2\,\tilde{d}\right)\frac{1}{2}\lambda^2\kappa^2 n! + \tilde{d}^2}, \tag{6.21}$$

with eigenvectors

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.22}$$

$$V_2 = \begin{pmatrix} \frac{\lambda^2\kappa^2 n!|u|}{|w|k_+} \\ \frac{\lambda^2\kappa^2 n!|u|}{|w|k_-} \end{pmatrix}. \tag{6.23}$$

The condition that these eigenvalues are real and positive is such that

$$\frac{1}{2}\lambda^2\kappa^2 n! > \frac{\left(1 + 2\,\tilde{d} - u^2 + 2\sqrt{\left(\tilde{d}+1\right)\left(\tilde{d}-u^2\right)}\right)\tilde{d}}{\left(1+u^2\right)^2} \tag{6.24}$$

and $u^2 < \tilde{d}$. After analyzing the static case and its asymptotics, we take our initial conditions for asymptotic solutions to be $X = 0, Y = 0, X' = 0, Y' = 0$. This corresponds to $\tilde{X} = \frac{1}{2|w|}, \tilde{Y} = \frac{1}{2|u|}, \tilde{X}' = 0, \tilde{Y}' = 0$. We plot a force field diagram with these initial conditions.

We immediately see that given certain conditions, the system (at infinity) will simply blow up in the positive direction for X and Y, which is the behavior we want to correspond to a brane forming at $X, Y = \infty$. While we know from the full field equations that a brane can form regardless of the parameters (with appropriate initial conditions), we find that these parameters affect how the fields behave asymptotically. That is, if the charge of the brane is too small, we find that oscillating behavior can ensue as the fields evolve in parameter μ. While this doesn't ruin the possibility of a brane, it does suggest that there will be fluctuations at infinity which doesn't have a clear effect. Finally, in the case of decoupling Z from the field equations (see special cases chapter), the "force" on X and Y are definite-positive when linearized, allowing for asymtotic behavior regardless of the strength of the charge of the brane.

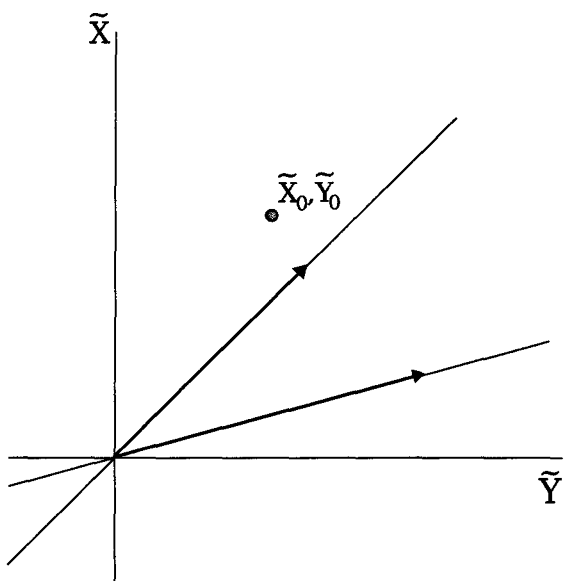

Figure 6.4: Force field plot in $\tilde{X}$-$\tilde{Y}$ phase-space. Eigenvectors are shown as well as initial configuration.

Chapter 7

Conclusion and Discussion

Many solutions (albeit some partial) have been found in a general class of supergravity theories in D dimensions. In fact, we have found static solutions which, although perhaps not supersymmetric, are more general than those of [6]. This has allowed us to see another class of solutions, which may be transformed into a set of co-ordinates making the structure of the brane more explicit.

These static solutions have helped immensely in the finding of other solutions. In the easiest case, we have time-dependent solutions with the static solution as the radial profile for appropriate choices of exponents of time. This is in addition to a free parameter as an exponent, which is good news, as we now have twice the amount of parameters describing the brane - the charge, and the exponent of time at which it expands (or contracts). However, this is probably not the most general case, as enforcing $\mathcal{C} = \mathcal{A}$ (which was required to find an immediate solution since it reproduced the static field equations) reduces our degrees of freedom (without justification from symmetry principles). Even if it may not be the most general case, we are encouraged in potentially finding more solutions in the case that exponents of time cancel in the field equations.

As for the other class of solutions (where field equations scale as one power of time), we have found the power law solution that has a similar form to that of Stelle's solution in the static case. In (6.6)-(6.12) we find that these may support a brane at $\mu = 0$. Furthermore, we have found partial solutions in the cases in which Z decouples from the field equations. In the general case, after analyzing the field equations, we see that with sufficient initial conditions on the fields, we can always generate a divergence, telling us that the whole class of these power law solutions may support a brane given appropriate initial field configurations.

The asymptotics of these solutions are a little different, since they may require a specific set of initial conditions on the field configuration (like starting at infinity and integrating inwards). Requiring asymptotic flatness seems to have the requirement that the charge, λ, of the brane be sufficiently large to "pull" the fields from infinity without fluctuation. A small charge seems to have a fluctuating effect on the fields at infinity, without it being entirely clear what the field configurations would end up at infinity. In all cases, asymtotic flatness means the space-time is flat at some finite time t_0.

It is clear that branes may be supported in our time-dependence ansätz for this general class of supergravities (of which needs to be analyzed further in future work), thus confirming what we set out to do. Many more possible interesting properties about these branes have been found, as well as how one can generalize them to having cosmological-like time dependence. At this point, the hope is that these solutions can be combined with cosmological arguments to help strengthen solutions to string theory and their connection to the visible universe. While it is unclear whether these solutions may help string cosmology, the techniques used here are of some import in solving similar problems in supergravity in the future.

Appendix A

Conversion to Stelle's Conventions

In Stelle's review of supergravity p-branes, the conventions used were different than the ones here. From a differential geometry standpoint, while using the same metric as ours (mostly plus, diag(-1, 1,1,...)), the curvature conventions were different. This can be seen in full from Stelle's action versus our action. The action,

$$S = \int d^D x \sqrt{-g} \left[R - \frac{1}{2} \nabla_M \phi \nabla^M \phi - \frac{1}{2n!} e^{a\phi} F_{[n]}^2 \right], \tag{A.1}$$

does in fact look different from our action (as a reminder)

$$S = -\int d^D x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(R + \nabla_M \phi \nabla^M \phi \right) + \frac{1}{4} e^{a\phi} F_{[n]}^2 \right]. \tag{A.2}$$

Stelle's action has the advantage that the field equations involving the form field as a source has no factors of $n!$, while our action has the advantage that the string tension is explicitly in our field equations (which is useful for finding limits). One does note that Stelle's action has a Ricci scalar that is the negative of ours. However, this is just due to a different set of curvature conventions (MTW[19] vs Weinberg[16]). In order to recover one action from the other, fields and couplings must be scaled accordingly. Denoting $[\ ]_s$ and $[\ ]_o$ for field and/or coupling corresponding to Stelle's and ours, respectively. Doing this rigorously, we find

$$[g_{MN}]_o = [g_{MN}]_s \left(2\kappa^2\right)^{\frac{2}{D-2}} \tag{A.3}$$

$$[\phi]_o = \frac{1}{\sqrt{2}}[\phi]_s + \frac{1}{\sqrt{2}a_s}\left(n \ln\left(2\kappa^2\right) - \ln\left(\frac{n!}{2}\right)\right) \tag{A.4}$$

$$[a]_o = \sqrt{2}[a]_s \tag{A.5}$$

connects the two actions together appropriately, showing us the proper correspondence between conventions. Alternatively to see the correspondence between the solutions, one can simply make the redefinitions in the solutions we found. The value and redefinitions,

$$\kappa^2 = \frac{1}{n!} \tag{A.6}$$

$$[\phi]_o = \frac{1}{\sqrt{2}}[\phi]_s \tag{A.7}$$

$$[a]_o = \sqrt{2}[a]_s, \tag{A.8}$$

properly connect Stelle's solutions to ours.

Appendix B

Conformal Transformation of Ricci Tensor

Here we prove a result used in chapter 2. That is, how the Ricci tensor transforms when the metric undergoes a conformal transformation. Given

$$g_{MN}(x^P) = \Omega^2(x^P)\tilde{g}_{MN}(x^P), \tag{B.1}$$

we start by calculating the effect of the transormation on the Christoffel symbols of the second kind (covariant connections). This is

$$\begin{aligned}
\Gamma^M_{NP} &= \frac{1}{2} g^{MQ}\left(\partial_N g_{PQ} + \partial_P g_{NQ} - \partial_Q g_{NP}\right) && \text{(B.2)}\\
&= \frac{1}{2}\tilde{g}^{MQ}\Omega^{-2}\left[\partial_N \tilde{g}_{PQ} + \partial_P \tilde{g}_{NQ} - \partial_Q \tilde{g}_{NP} + 2\Omega\left(g_{PQ}\partial_N\Omega + g_{NQ}\partial_P\Omega - g_{PN}\partial_Q\Omega\right)\right]\\
&= \tilde{\Gamma}^M_{NP} + \delta^M_P \partial_N \ln\Omega + \delta^M_N \partial_P \ln\Omega - \tilde{g}_{PN}\tilde{g}^{MQ}\partial_Q \ln\Omega, && \text{(B.3)}
\end{aligned}$$

allowing us to now calculate the Ricci Tensor. Plugging this into the definition of the Ricci tensor gives,

$$\begin{aligned}
R_{MN} =& \partial_M \Gamma^P_{NP} - \partial_P \Gamma^P_{MN} + \Gamma^Q_{MP}\Gamma^P_{NQ} - \Gamma^Q_{MN}\Gamma^P_{PQ} && \text{(B.4)}\\
=& \tilde{R}_{MN} - 2\tilde{\nabla}_M\tilde{\nabla}_N \ln\Omega + \tilde{g}_{MN}\tilde{g}^{PQ}\tilde{\nabla}_P\tilde{\nabla}_Q \ln\Omega + D\tilde{\nabla}_M\tilde{\nabla}_N \ln\Omega\\
& (D+2)\tilde{\nabla}_M \ln\Omega\tilde{\nabla}_N \ln\Omega - 2\tilde{g}_{MN}\tilde{g}^{PQ}\tilde{\nabla}_P \ln\Omega\tilde{\nabla}_Q \ln\Omega\\
& - D\left(2\tilde{\nabla}_M \ln\Omega\tilde{\nabla}_N \ln\Omega + \tilde{g}_{MN}\tilde{g}^{PQ}\tilde{\nabla}_P \ln\Omega\tilde{\nabla}_Q \ln\Omega\right)\\
=& \tilde{R}_{MN} + (D-2)\tilde{\nabla}_M\tilde{\nabla}_N \ln\Omega + \tilde{g}_{MN}\tilde{g}^{PQ}\tilde{\nabla}_P\tilde{\nabla}_Q \ln\Omega\\
& - (D-2)\tilde{\nabla}_M \ln\Omega\tilde{\nabla}_N \ln\Omega + (D-2)\tilde{g}_{MN}\tilde{g}^{PQ}\tilde{\nabla}_P \ln\Omega\tilde{\nabla}_Q \ln\Omega, && \text{(B.5)}
\end{aligned}$$

with tilde's on the Ricci tensor and the covariant derivative belonging to the conformally related metric ($\tilde{g}_{MN}$), thus completing the proof.

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