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THE BETHE-ANSATZ FOR GAUDIN SPIN CHAINS

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By

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ABSTRACT

We investigate a special case of the quantum integrable Heisenberg spin chain known as Gaudin model. The Gaudin model is an important example of quantum integrable systems. We study the Gaudin model for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The key problem is to find the spectrum and the corresponding eigenvectors of the commuting Hamiltonians. The standard method to solve this type of classical problem was introduced by H. Bethe and is known as the Bethe-Ansatz. Bethe's technique has proven to be very powerful in various areas of modern many-body theory and statistical mechanics. [19], [14], [4]

Following Sklyanin's ideas in [19] , we derive the Bethe-Ansatz equations for $\mathfrak{sl}_2(\mathbb{C})$. Solving the Bethe-Ansatz equations is equivalent to finding polynomial solutions of the Lamé differential equation, which has a meaning in electrostatics. We derive this equation for $\mathfrak{sl}_2(\mathbb{C})$, and investigate its special cases. We discuss classical and more recent results on the Gaudin spin chain for $\mathfrak{sl}_2(\mathbb{C})$ and provide numerical evidence for new observations in the real case of the Lamé equation. Using roots of classical polynomials known as Jacobi polynomials, which are solutions to a special case of the Lamé equation, we numerically approximate solutions to the Lamé equation in more complicated settings.

We discuss the Gaudin model associated to the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Using the Bethe-Ansatz equations for $\mathfrak{sl}_3(\mathbb{C})$, we provide solutions in special cases.

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Chapter 1

Introduction

The Gaudin spin chain model corresponding to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has been extensively studied. The Bethe-Asantz was introduced in the context of spin chains in quantum physics. The problem of diagonalizing the Gaudin Hamiltonians is equivalent to solving the Bethe-Ansatz equations [19]. An equivalent to the Bethe-Ansatz equations, which relates to infinite dimensional irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$, is a classical differential equation called the Lamé equation, which has an electrostatic interpretation. The Lamé equation was developed by Heine [7] in the 19th century in the context of orthogonal polynomials. In special cases, solutions to the Lamé equation are classical orthogonal polynomials called Jacobi polynomials.

For cases of the Lamé equation corresponding to infinite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$, the exact number of solutions and their distribution on the real line are known. This result was first proven by Stieltjes in 1885 [21], and is known as the Heine-Stieltjes theorem. In the finite dimensional case, still relatively little is known. Recent work in this area, [12], shows interesting patterns in the complex case.

We derive the equations and prove some of these results. For each of the discussed cases, we provide numerical algorithms. We observe and provide numerical evidence that in certain cases, a solution to a simpler problem, to which solutions are completely described, can be used to approximate solutions in more complicated settings. We numerically obtain a single solution to our problem from iterating two sets of distinct solutions. Finally, we provide numerical evidence that there exist solutions to equations that are similar to the Bethe-Ansatz equations corresponding to the Gaudin spin chain model associated to the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. The case of the Gaudin Model to $\mathfrak{sl}_3(\mathbb{C})$ has recently sparked the interest of researchers, [13], [14], however, much is still unknown. We provide a method of finding solutions in a special setting and explain why the Heine-Stieltjes theorem does not hold in the $\mathfrak{sl}_3(\mathbb{C})$ case.

We begin the discussion with a basic introduction to quantum mechanics, explaining the terminology and mathematical formulation of a quantum mechanical system.

In chapter 2, we describe the Gaudin spin chain model, introduce the Gaudin Hamiltonians, and explain the correspondence with the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The commutativity of the generating function corresponding to the $\mathfrak{sl}_2(\mathbb{C})$ Gaudin model is then proven, which implies commutativity of the Gaudin Hamiltonians.

The Bethe-Ansatz is an approach to find joint eigenvectors and eigenvalues of Gaudin Hamiltonians. In chapter 3, we prove that solving Bethe-Ansatz equations is equivalent to diagonalizing the generating function.

Chapter 4 discusses two equivalent methods to describe solutions to the Bethe-Ansatz equations associated to the $\mathfrak{sl}_2(\mathbb{C})$ Gaudin spin chain model. The Bethe vectors can be thought of as a collection of numbers satisfying critical point equations of the Master function. The other way to solve the Bethe-Ansatz equations is by looking at polynomial solutions of a differential equation called the Lamé equation.

In Chapter 5, we discuss solutions to the Lamé equation in the case of two real sites. These solutions correspond to classical orthogonal polynomials, known as Jacobi polynomials. This Chapter begins with an overview of the theory of orthogonal polynomials and discusses a subclass of Jacobi polynomials, known as Chebyshev polynomials, in more details. The last section of this chapter shows how the theory of orthogonal polynomials is used to obtain numerical solutions of the Lamé equation in this special case.

Chapter 6 contains experimental numerical results. We provide numerical evidence for obtaining approximations of solutions to the Lamé equation of more than two real sites using numerical approximations of roots of Jacobi polynomials. We also make observations and provide numerical evidence of the existence of solutions to other equations that are similar to the Bethe-Ansatz equations for $\mathfrak{sl}_2(\mathbb{C})$.

In chapter 7, we consider the Lamé equation with complex parameters and having three complex sites. This special instance of the Lamé equations is then translated into an eigenvalue problem. Finally, we provide expository discussion on the recent progress in this direction and deliver numerical algorithms for each considered variant.

The Gaudin spin chain model can be associated to any semi-simple Lie algebra. In chapter 8, we derive a solution set of the Bethe-Ansatz equations corresponding to the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. We end the thesis with concluding remarks and outlook for future research in this area.

Chapter 2

Quantum Mechanical Systems

This chapter is a basic introduction to quantum mechanics. We explain the terminology used in quantum mechanics and mathematical formulation.

2.1 Mathematical Formulation of Quantum Mechanics

The foundations of quantum mechanics were established during the first half of the 20^{th} century by Werner Heisenberg and Max Planck. The mathematical formulation of quantum mechanics was developed by Paul Dirac and John von Neumann, [16].

The state of a system at a given time is described by a complex wave function, and more generally, by non-zero vectors of a complex Hilbert space called the state space. An element of this space is called ket and is denoted by $| >$.

An observable is a physical quantity that can be measured by an experiment and whose result is a real number (for example: energy, position, momentum of a particle). Each observable is represented by a Hermitian operator acting on the state space.

Quantum mechanics does not assign definite values to observables; it makes predictions about probability distributions (i.e. the probability of obtaining each of the possible outcomes from measuring an observable). These probabilites depend on the quantum state at the instant of the measurement. The states that correspond to a definite value of a particular observable are known as eigenstates. Each eigenstate of an observable corresponds to an eigenvector of the operator and the associated eigenvalue corresponds to the value of the observable in that eigenstate. The possible results of a measurement are the eigenvalues of the operator - which explains the choice of Hermitian operators for which all eigenvalues are real.

In quantum mechanics, we are interested in finding the spectrum of commuting operators.

2.2 Quantum Integrable Systems

Although there is no formal formal definition of a quantum integrable system, there is a working definition that is analogous to a definition of an integrable system in classical mechanics. The notion of Poisson commuting functions, which occurs in classical setting is replaced with self-adjoint, commuting operators on a Hilbert space in the quantum setting. Since there is no clear definition of independence of operators, except for special classes, the working definition of a quantum integrable system requires the existence of a maximal set of commuting operators including the Hamiltonians.

Chapter 3

Gaudin spin chain for $\mathfrak{sl}_2(\mathbb{C})$

The first section of this chapter introduces a quantum integrable system called the Gaudin spin chain model and explains its association to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We explicitly write the Hermitian operators of the system; called the Gaudin Hamiltonians. Next, we define the generating function $t(u)$ and prove its commutativity property, which in turn implies commutativity of the Gaudin Hamiltonians. Next, we introduce the Bethe-Ansatz method and prove that solving the Bethe-Ansatz equations is equivalent to diagonalizing the operator $t(u)$.

3.1 The Gaudin spin chain model for $\mathfrak{sl}_2(\mathbb{C})$

We study an integrable quantum system associated to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ called Gaudin model of statistical mechanics.

Let e, f, h be the generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ satisfying the commutation relations:

$$[h,e] = 2e$$

$$[h,f] = -2f$$

$$[e,f]=h.$$

Denote by V_λ, an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with the highest weight λ and the highest weight vector $|0>$ (also called vacuum) with

$$e|0>=0$$

and

$$h|0>=\lambda|0>.$$

The Casimir operator t for $\mathfrak{sl}_2(\mathbb{C})$, which commutes with all elements of $\mathfrak{sl}_2(\mathbb{C})$ is given by:

$$2t=ef+fe+\frac{1}{2}h^2 \tag{3.1}$$

Please refer to Appendix A for a discussion on $\mathfrak{sl}_2(\mathbb{C})$ representation theory.

Proposition 3.1.1.

$$2t=\frac{\lambda(\lambda+2)}{2}$$

Proof:

$$\begin{aligned}
2t|0> &= ef|0>+fe|0>+\frac{1}{2}h^2|0> \\
&= fe|0>+[e,f]|0>+fe|0>+\frac{1}{2}h^2|0> \\
&= h|0>+\frac{1}{2}hh|0> \\
&= \lambda|0>+\lambda^2|0> \\
&= \frac{\lambda(\lambda+2)}{2}|0>
\end{aligned}$$

□

A spin chain can be visualized as a string of particles with magnetic spin. The

Gaudin model is a quantum spin chain where the space of states of the model is the tensor product of irreducible $\mathfrak{sl}_2(\mathbb{C})$ representations. The Gaudin Model can be associated to any semi-simple complex Lie algebra, [19], [14], [4], [18].

Let $(\lambda) := (\lambda_0, ..., \lambda_N)$ be a set of the highest weights of $\mathfrak{sl}_2(\mathbb{C})$. Consider the tensor product:

$$V_\lambda := V_{\lambda_0} \otimes \cdots \otimes V_{\lambda_N}$$

Associate with each factor V_{λ_n}, a distinct complex number α_n, for $n = 0 \cdots N$, also called sites.

Denote by $|0>$ the tensor product of the highest weight vectors:

$$|0> := |0>_0 \otimes \cdots \otimes |0>_N$$

Each V_{λ_n} is a spin space on n particles located at distinct points $\alpha_0, \alpha_1, ..., \alpha_N$. Denote the action of f, h, e on the n^{th} factor of the tensor product by f_n, h_n, e_n respectively. In particular, f_n acts as f on the n^{th} factor of the tensor product and as the identity on the rest. The mutually commuting linear operators in the Gaudin model, called Gaudin Hamiltonians $\mathbb{H}_n$, are given by

$$\mathbb{H}_n = \sum_{m \neq n} \frac{e_n f_m + f_n e_m + \frac{1}{2} h_n h_m}{\alpha_n - \alpha_m}, n = 0, \cdots, N \tag{3.2}$$

Let $u \neq \alpha_n \forall n$.

Consider rational functions with coefficients in the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ that form the following one parameter operator families:

$$h(u) = \sum_{n=0}^{N} \frac{h_n}{u - \alpha_n}$$

$$e(u) = \sum_{n=0}^{N} \frac{e_n}{u - \alpha_n}$$

$$f(u) = \sum_{n=0}^{N} \frac{f_n}{u - \alpha_n.}$$

The operators $e(u), f(u), h(u)$ form the highest weight module over the infinite dimensional loop algebra derived from the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ where

$$e(u)|0>= 0$$

$$h(u)|0>= \lambda(u)|0>$$

and a scalar function called the highest weight

$$\lambda(u) = \sum_{n=1}^{N} \frac{\lambda_n}{u - \alpha_n}.$$

The bracket relations of the loop algebra are given by:

$$\begin{aligned}[h(u), e(v)] &:= \sum_{n=0}^{N} \frac{[h_n, e_n]}{(u-\alpha_n)(v-\alpha_n)} \\ &:= \sum_{n=0}^{N} \frac{2e_n}{(u-\alpha_n)(v-\alpha_n)} \\ &:= \frac{2}{v-u} \sum_{n=0}^{N} (\frac{e_n}{u-\alpha_n} - \frac{e_n}{v-\alpha_n}) \\ &:= \frac{2(e(u)-e(v))}{v-u} \end{aligned} \tag{3.3}$$

$$[h(u), f(v)] = \frac{-2(f(u)-f(v))}{v-u} \tag{3.4}$$

$$[e(u), f(v)] = \frac{(h(u) - h(v))}{v - u} \tag{3.5}$$

$$[e(u), e(v)] = [f(u), f(v)] = [h(u), h(v)] = 0 \tag{3.6}$$

We differentiate $h(u), e(u), f(u)$.

$$h'(u) = -\sum_{n=0}^{N} \frac{h_n}{(u - \alpha_n)^2}$$

$$f'(u) = -\sum_{n=0}^{N} \frac{f_n}{(u - \alpha_n)^2}$$

$$e'(u) = -\sum_{n=0}^{N} \frac{e_n}{(u - \alpha_n)^2}$$

Using $h'(u), f'(u)$ and $e'(u)$ the following brackets are computed:

$$[h(u), e(u)] = \sum_{n=0}^{N} \frac{2e_n}{(u - \alpha_n)^2} = -2e'(u)$$

$$[h(u), f(u)] = -\sum_{n=0}^{N} \frac{2f_n}{(u - \alpha_n)^2} = 2f'(u)$$

$$[e(u), f(u)] = \sum_{n=0}^{N} \frac{h_n}{(u - \alpha_n)^2} = -h'(u)$$

Replace e, f, h with $e(u), f(u), h(u)$ in the Casimir 3.1 and define the generating function $t(u)$:

$$2t(u) = e(u)f(u) + f(u)e(u) + \frac{1}{2}h^2(u). \tag{3.7}$$

Proposition 3.1.2. $[t(u), t(v)] = 0, \forall u, v.$

Proof:

$$
\begin{aligned}
2[t(u),t(v)] &= 2((u)t(v) - t(v)t(u)) \\
&= \left(e(u)f(u) + f(u)e(u) + \frac{1}{2}h^2(u)\right)\left(e(v)f(v) + f(v)e(v) + \frac{1}{2}h^2(v)\right) \\
&- \left(e(v)f(v) + f(v)e(v) + \frac{1}{2}h^2(v)\right)\left(e(u)f(u) + f(u)e(u) + \frac{1}{2}h^2(u)\right) \\
&= e(u)f(u)e(v)f(v) + e(u)(f(u)f(v)e(v) + e(u)f(u)\frac{1}{2}h^2(v) \\
&+ f(u)e(u)e(v)f(v) + f(u)e(u)f(v)e(v) + f(u)e(u)\frac{1}{2}h^2(v) \\
&+ \frac{1}{2}h^2(u)e(v)f(v) + \frac{1}{2}h^2(u)f(v)e(v) + \frac{1}{4}h^2(u)h^2(v) \\
&- e(v)f(v)e(u)f(u) - e(v)f(v)f(u)e(u) - e(v)f(v)\frac{1}{2}h^2(v) \\
&- f(v)e(v)e(u)f(u) - f(v)e(v)f(u)e(u) - f(v)e(v)\frac{1}{2}h^2(u) \\
&- \frac{1}{2}h^2(v)e(u)f(u) - \frac{1}{2}h^2(v)f(u)e(u) - \frac{1}{4}h^2(v)h^2(u)
\end{aligned}
$$

The above expression is easily rewritten such that it only involves brackets:

$$
\begin{aligned}
&[e(u)f(u), e(v)f(v)] + \qquad [e(u)f(u), f(v)e(v)] && (3.8) \\
&+[f(u)e(u), e(v)f(v)] + \qquad [f(u)e(u), f(v)e(v)] && \\
&+\left[e(u)f(u), \frac{1}{2}h^2(v)\right] + \qquad \left[f(u)e(u), \frac{1}{2}h^2(v)\right] && (3.9) \\
&+\left[\frac{1}{2}h^2(u), e(v)f(v)\right] + \qquad \left[\frac{1}{2}h^2)u), f(v)e(v)\right] && (3.10)
\end{aligned}
$$

Using the Power Rule (ref. Appendix B) for Lie brackets and the bracket relations 3.3, 3.4, 3.5, and 3.6, we further expand the above expressions. We first expand each bracket individually:

$$
\begin{aligned}
&[e(u)f(u),e(v)f(v)] = [e(u)f(u),e(v)]\,f(v)+e(v)\,[e(u)f(v),f(v)] \\
= \;& [e(u),e(v)]f(u)f(v)+e(u)\,[f(u),e(v)]\,f(v) \\
+ \;& e(v)\,[e(u),f(v)]\,f(u)+e(v)e(u)\,[f(u),f(v)] \\
= \;& e(u)\left(\frac{h(u)-h(v)}{u-v}\right)f(v)+e(v)\left(\frac{h(u)-h(v)}{v-u}\right)f(u)
\end{aligned}
$$

Similarly, we expand the remaining brackets and obtain:

$$
[e(u)f(u),f(v)e(v)] = \left(\frac{h(u)-h(v)}{v-u}\right)+f(v)e(u)\left(\frac{h(u)-h(v)}{u-v}\right)
$$

$$
[f(u)e(u),e(v)f(v)] = \left(\frac{h(u)-h(v)}{u-v}\right)e(u)f(v)+e(v)f(u)\left(\frac{h(u)-h(v)}{v-u}\right)
$$

$$
[f(u)e(u),f(v)e(v)] = f(u)\left(\frac{h(u)-h(v)}{v-u}\right)e(v)+f(v)\left(\frac{h(u)-h(v)}{u-v}\right)e(u)
$$

$$
\begin{aligned}
\left[e(u)f(u),\frac{1}{2}h^2(v)\right] \;=\;& \left(\frac{e(u)-e(v)}{u-v}\right)f(u)h(v)+e(u)\left(\frac{f(v)-f(u)}{u-v}\right)h(v) \\
+\;& h(v)\left(\frac{e(u)-e(v)}{u-v}\right)f(u)+f(v)e(u)\left(\frac{f(v)-f(u)}{u-v}\right)
\end{aligned}
$$

$$\begin{aligned}\left[f(u)e(u),\frac{1}{2}h^2(v)\right] &= \left(\frac{f(v)-f(u)}{u-v}\right)e(u)h(v)+f(u)\left(\frac{e(u)-e(v)}{u-v}\right)h(v)\\ &+ h(v)\left(\frac{f(v)-f(u)}{u-v}\right)e(u)+h(v)f(u)\left(\frac{e(u)-e(v)}{u-v}\right)\end{aligned}$$

$$\begin{aligned}\left[\frac{1}{2}h^2(u),e(v)f(v)\right] &= \left(\frac{e(u)-e(v)}{v-u}\right)h(u)f(v)+h(u)\left(\frac{e(u)-e(v)}{v-u}\right)f(v)\\ &+ e(v)\left(\frac{f(u)-f(v)}{u-v}\right)h(u)+e(v)h(u)\left(\frac{f(u)-f(v)}{u-v}\right)\end{aligned}$$

After grouping the expanded brackets and making appropriate cancellations, expression 3.8 becomes:

$$\begin{aligned}\frac{1}{u-v}(&- e(u)h(v)f(v)+e(v)h(v)f(u)+f(u)h(v)e(v)-f(v)h(v)e(u)\\ &+ e(u)f(v)h(v)-h(v)e(v)f(u)-f(u)e(v)h(v)+h(v)f(v)e(u)\\ &+ h(u)e(v)f(v)-e(v)f(v)h(u)-h(u)f(v)e(v)+f(v)e(v)h(u))\end{aligned}$$

We group the terms to obtain new brackets:

$$
\begin{aligned}
&\frac{1}{u-v}(e(v)h(v)f(u) - f(u)e(v)h(v) + h(v)f(v)e(u) - e(u)h(v)f(v) \\
&\quad +f(u)h(v)e(v) - h(v)e(v)f(u) + e(u)f(v)h(v) - f(v)h(v)e(u) \\
&\quad +h(u)e(v)f(v) - e(v)f(v)h(u) + f(v)e(v)h(u) - h(u)f(v)e(v)) \\
&= \frac{1}{u-v}([e(v)h(v), f(u)] + [h(v)f(v), e(u)] \\
&\quad + [f(u), h(v)e(v)] + [e(u), f(v)h(v)] \\
&\quad + [h(u), e(v)f(v)] + [f(v)e(v), h(u)])
\end{aligned} \tag{3.11}
$$

We then merge these brackets in the following way:

$$
\begin{aligned}
&[e(v)h(v), f(u)] + [f(u), h(v)e(v)] \\
&= [e(v), f(u)]\, h(v) + e(v)\, [h(v), f(u)] + [f(u), h(v)]\, e(v) + h(v)\, [f(u), e(v)] \\
&= \left(\frac{h(v) - h(u)}{u-v}\right) h(v) - 2e(v) \left(\frac{f(v) - f(u)}{u-v}\right) \\
&- 2\left(\frac{f(u) - f(v)}{u-v}\right) e(v) + h(v) \left(\frac{h(u) - h(v)}{u-v}\right) \\
&= \frac{1}{u-v}\,(h(v)h(v) - h(u)h(v) - 2e(v)f(v) + 2e(v)f(u)) \\
&- \frac{1}{u-v}\,(2f(u)e(v) + 2f(v)e(v) + h(v)h(u) - h(v)h(v)) \\
&= \frac{2}{u-v}\,([f(v), e(v)] + [e(v), f(u)])
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
&[h(v)f(v),e(u)]+[e(u),f(v)h(v)]\\
&=[h(v),e(u)]f(v)+h(v)[f(v),e(u)]+[e(u),f(v)]h(v)+f(v)[e(u),h(v)]\\
&=\left(2\frac{e(v)-e(u)}{u-v}\right)f(v)+h(v)\left(\frac{h(u)-h(v)}{u-v}\right)\\
&-\left(\frac{h(u)-h(v)}{u-v}\right)h(v)-2f(v)\left(\frac{e(v)-e(u)}{u-v}\right)\\
&=\frac{1}{u-v}(2e(v)f(v)-2e(u)f(v)+h(v)h(u)-h(v)h(v))\\
&-\frac{1}{u-v}(h(u)h(v)-h(v)h(v)-2f(v)e(u)+2f(v)e(u))\\
&=\frac{2}{u-v}([e(v),f(v)]+[f(v),e(u)]) \qquad (3.13)
\end{aligned}
$$

We observe that brackets are anti-symmetric in variables u and v. Thus, grouping expressions obtained in 3.12 and 3.13 gives:

$$
\begin{aligned}
&\frac{2}{u-v}([f(v),e(v)]+[e(v),f(u)]+[e(v),f(v)]+[f(v),e(u)])\\
=&\frac{2}{u-v}\left(-\frac{h(u)-h(v)}{u-v}\right)+\left(\frac{h(u)-h(v)}{u-v}\right)\\
=&\ 0
\end{aligned}
$$

Finally, we expand the remaining brackets in equation 3.11 and obtain the desired cancellations.

$$
\begin{aligned}
& [h(u),e(v)f(v)]+[f(v)e(v),h(u)] \\
= \quad & [h(u),e(v)]\,f(v)+e(v)\,[h(u),f(v)] \\
+ \quad & [f(v),h(u)]\,e(v)+f(v)\,[e(v),h(u)] \\
= \quad & -2\left(\tfrac{e(u)-e(v)}{u-v}\right)f(v)+2e(v)\left(\tfrac{f(u)-f(v)}{u-v}\right) \\
- \quad & 2\left(\tfrac{f(u)-f(v)}{u-v}\right)e(v)+2f(v)\left(\tfrac{e(u)-e(v)}{u-v}\right) \\
= \quad & \tfrac{2}{u-v}\left(-e(u)f(v)+e(v)f(v)+e(v)f(u)-e(v)f(v)\right) \\
+ \quad & \tfrac{2}{u-v}\left(f(u)e(v)+f(v)e(v)+f(v)e(u)-f(v)e(v)\right) \\
= \quad & \tfrac{2}{u-v}\left([f(v),e(u)]+[e(v),f(u)]\right) \\
= \quad & \frac{2}{u-v}\left(\frac{h(u)-h(v)}{u-v}-\frac{h(u)-h(v)}{u-v}\right) \\
= \quad & 0
\end{aligned}
$$

□

Proposition 3.1.3. *The vacuum vector $|0>$ is an eigenvector of the operator $t(u)$.*

Proof:

$$
\begin{aligned}
t(u)|0> &= \frac{1}{2}\left(e(u)f(u)+f(u)e(u)+\frac{1}{2}h^2(u)\right)|0> \\
&= \frac{1}{2}\,[e(u),f(u)]\,|0> + f(u)e(u)|0> + \frac{1}{4}h^2(u)|0> \\
&= -\frac{1}{2}h'(u)|0> + \frac{1}{4}h^2(u)|0> \\
&= \frac{1}{4}\lambda(u)|0> - \frac{1}{2}\lambda'(u)|0>
\end{aligned}
\tag{3.14}
$$

□

We also observe that:

Proposition 3.1.4.

$$t(u)=\sum_{n=0}^{N}\frac{\mathbb{H}_n}{u-\alpha_n}+\frac{\frac{1}{4}\lambda_n(\lambda_n+2)}{(u-\alpha_n)^2}$$

Proof:

$$\begin{aligned}
t(u) = & \quad e(u)f(u)+f(u)e(u)+\frac{1}{2}h^2(u) \\
= & \quad \sum_{n=0}^{N}\sum_{m\neq n}\frac{e_nf_m+f_ne_m+\frac{1}{2}h_nh_m}{(u-\alpha_n)(u-\alpha_m)}+\frac{e_nf_n+f_ne_n+\frac{1}{2}h_n^2}{(u-\alpha_n)^2} \\
= & \quad \sum_{n=0}^{N}\sum_{m\neq n}\frac{e_nf_m+f_ne_m+\frac{1}{2}h_nh_m}{\alpha_m-\alpha_n}\left(\frac{1}{\alpha_m-u}-\frac{1}{\alpha_n-u}\right)+\frac{t}{(u-\alpha_n)^2} \\
= & \quad \sum_{n=0}^{N}\frac{\mathbb{H}_n}{u-\alpha_n}+\frac{\frac{1}{4}\lambda_n(\lambda_n+2)}{(u-\alpha_n)^2} \qquad (3.15)
\end{aligned}$$

□

Proposition 3.1.5. *Gaudin Hamiltonians are diagonalizable.*

Proof: We need to show that:

$$[\mathbb{H}_n,\mathbb{H}_m]=0, n\neq m.$$

By Proposition 3.1.4, we deduce that

$$\left[\sum_{n=0}^{N}\frac{\mathbb{H}_n}{u-\alpha_n},\sum_{n=0}^{N}\frac{\mathbb{H}_n}{v-\alpha_n}\right]=0, u\neq v$$

Choose $N = 1$. Then,

$$\left[\frac{\mathbb{H}_0}{u-\alpha_n}+\frac{\mathbb{H}_1}{u-\alpha_n},\frac{\mathbb{H}_0}{v-\alpha_n}+\frac{\mathbb{H}_1}{v-\alpha_n}\right]=0.$$

$$\frac{\mathbb{H}_0}{u-\alpha_0}\frac{\mathbb{H}_1}{u-\alpha_1}+\frac{\mathbb{H}_0}{v-\alpha_0}\frac{\mathbb{H}_1}{v-\alpha_1}=0$$

$$\frac{1}{(u-\alpha_0)(v-\alpha_1)}[\mathbb{H}_0,\mathbb{H}_1]+\frac{1}{(v-\alpha_0)(u-\alpha_1)}[\mathbb{H}_1,\mathbb{H}_0]=0$$

$$\implies [\mathbb{H}_0,\mathbb{H}_1]=0.$$

□

3.2 Bethe-Ansatz equations for $\mathfrak{sl}_2(\mathbb{C})$

The Bethe-Ansatz is an approach to find joint eigenvectors and eigenvalues of Gaudin Hamiltonians 3.2. The equations which determine these special vectors are called the Bethe-Ansatz equations and the joint eigenvector that corresponds to a solution of the Bethe-Ansatz equations is called the Bethe vector.

We recall a well known theorem from linear algebra:

Theorem 3.2.1. *Hermitian operators commute if and only if there exists a basis of eigenvectors that is common to both.*

Definition 3.2.1. *The Bethe vectors are defined for any finite set of complex numbers $\mathcal{V}$ as $|\mathcal{V}>:=\prod_{v\in\mathcal{V}}f(v)|0>$*

Theorem 3.2.2. *[19] The vector $|\mathcal{V}>$ is a joint eigenvector of the commuting Hamiltonians, or equivalently of the operators $t(u)$ if and only if the spectral parameters $v\in\mathcal{V}$ satisfy the Bethe-ansatz equations:*

$$\lambda(v) = \sum_{v' \neq v} \frac{2}{v - v'}, \forall v \in \mathcal{V} \tag{3.16}$$

The corresponding eigenvalue $\tau(u)$ *of* $t(u)$,

$$t(u)|\mathcal{V}> = \tau(u)|\mathcal{V}>$$

is then

$$\tau(u) = \frac{1}{4}\tilde{\lambda}^2(u) - \frac{1}{2}\partial_u \tilde{\lambda}(u) \tag{3.17}$$

where

$$\tilde{\lambda}(u) := \lambda(u) - \sum_{v \in |\mathcal{V}>} \frac{2}{u - v} \tag{3.18}$$

.

3.2.1 Proof of theorem 3.2.2

Let $\mathcal{V} = \{v_1, v_2, ..., v_k, ..., v_K\}, v_k \in \mathbb{C}$.

$$f(u)|v_i> = f(u)f(v_i)|0> = |v_i \cup \{u\}> = |v_i u>$$

$$f(u)|\mathcal{V}> = |\mathcal{V} \cup \{u\}> \tag{3.19}$$

Equation 3.19 follows from the definition of Bethe vectors.

Lemma 3.2.1.

$$h(u)|\mathcal{V}> = \left(\lambda(u) - \sum_{v\in\mathcal{V}} \frac{2}{u-v}\right)|\mathcal{V}> + \sum_{v\in\mathcal{V}} \frac{2}{u-v}|\mathcal{V}\setminus\{v\}\cup\{u\}>$$

Proof: We use induction with base case given by:

$$\begin{aligned}
h(u)|v_i> &= f(v_i)h(u)|0> + \frac{2}{v_i-u}(f(u)-f(v_i)) \\
&= f(v_1)\lambda(u)|0> - \frac{2}{v_i-u}(f(u)-f(v_i)) \\
&= \lambda(u)|v_i> - \frac{2}{v_i-u}(|u> - |v_i>) \\
&= (\lambda(u) + \frac{2}{v_i-u})|v_i> - \frac{2}{v_i-u}|u> \\
&= (\lambda(u) - \frac{2}{u-v_i})|v_i> + \frac{2}{u-v_i}|u>
\end{aligned}$$

Inductive Hypothesis: Let $\mathcal{W} = \{v_1 ... v_{n-1}\}$ Assume the statement is true for W, i.e.

$$h(u)|W> = \left(\lambda(u) - \sum_{v\in\mathcal{W}} \frac{2}{u-v}\right)\mathcal{W} + \sum_{v\in\mathcal{W}} \frac{2}{u-v}|\mathcal{W}\setminus\{v\}\cup\{u\}>$$

$$
\begin{aligned}
h(u)|\mathcal{V}> &= h(u)|v_n v_1 ... v_n - 1> \\
&= f(v_n)h(u)|\mathcal{W}> + \frac{2}{u-v_n}(f(u)-f(v_n))|\mathcal{W}> \\
&= f(v_n)(\lambda(u) - \sum_{v\in\mathcal{W}} \frac{2}{u-v})|\mathcal{W}> + \sum_{v\in\mathcal{W}} \frac{2}{u-v}|\mathcal{W}\backslash\{v\}\cup\{u\}>) \\
&+ \frac{2}{u-v_n}|uv_1\cdots v_{n-1}> - \frac{2}{u-v_n}|v_n v_1\cdots v_{n-1}> \\
&= \lambda(u)|v_n v_1\cdots v_{n-1}> - \sum_{v\in\mathcal{W}} \frac{2}{u-v}|v_n v_1\cdots v_n - 1> \\
&+ \sum_{v\in\mathcal{W}} \frac{2}{u-v}|\mathcal{W}\backslash\{v\}\cup\{u\}> \\
&+ \frac{2}{u-v_n}|v_1\cdots v_{n-1}u> - \frac{2}{u-v_n}|v_1\cdots v_n> \\
&= \lambda(u)|v_1\cdots v_n> - \sum_{v\in\mathcal{W}} \frac{2}{u-v}|v_1\cdots v_n> \\
&- \frac{2}{u-v_n}|v_1\cdots v_n> + \sum_{v\in\mathcal{W}} \frac{2}{u-v}|\mathcal{W}\backslash\{v\}\cup\{u\}> + \frac{2}{u-v_n}|v_1\cdots v_{n-1}u> \\
&= \left(\lambda(u) - \sum_{v\in\mathcal{V}} \frac{2}{u-v}\right)|v_1\cdots v_n> + \sum_{v\in\mathcal{V}} \frac{2}{u-v}|\mathcal{V}\backslash\{v\}\cup\{u\}>
\end{aligned}
$$

□

Lemma 3.2.2.

$$
\begin{aligned}
e(u)|\mathcal{V}> &= \sum_{v\in\mathcal{V}} -\frac{1}{u-v}\left(\lambda(u) - \lambda(v) - \sum_{v'\neq v} 2\left(\frac{1}{u-v'} - \frac{1}{v-v'}\right)\right) \\
&+ \sum_{v,v'\in\mathcal{V},v\neq v'} -\frac{2}{(u-v)(u-v')}|\mathcal{V}\backslash\{v,v'\}\cup\{u\}>
\end{aligned}
$$

Proof: To prove Lemma 3.2.2, we also use induction. Base case:

$$\begin{aligned} e(u)|v_i> &= f(v_i)e(u)|0> + \frac{h(u)-h(v_i)}{v_i-u}|0> \\ &= \frac{1}{v_i-u}(h(u)|0> - h(v_i|0>) \\ &= \frac{1}{v_i-u}(\lambda(u)|0> - \lambda(v_i)|0>) \\ &= -\frac{1}{u-v_i}(\lambda(u)|0> - \lambda(v_i)|0>) \end{aligned}$$

Inductive Hypothesis. Let $\mathcal{W} = \{v_1 \ldots v_{n-1}\}$ Assume that the statement is true for W, i.e.

$$\begin{aligned} e(u)|\mathcal{W}> &= \sum_{v\in\mathcal{W}} -\frac{1}{u-v}\left(\lambda(u)-\lambda(v)-\sum_{v'\neq v}2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right)\right)|\mathcal{W}\backslash\{v\}> \\ &+ \sum_{v,v'\in\mathcal{W},v\neq v'} -\frac{2}{(u-v)(u-v')}|\mathcal{W}\backslash\{v,v'\}\cup\{u\}> \end{aligned}$$

$$
\begin{aligned}
e(u)|\mathcal{V}> &= e(u)|v_n v_1 \cdots v_{n-1}> \\
&= f(v_n)e(u)|v_1 \cdots v_{n-1}> + \frac{h(v_n)-h(u)}{u-v_n}|v_1 \cdots v_{n-1}> \\
&= f(v_n)\sum_{v\in\mathcal{W}'} -\frac{1}{u-v}(\lambda(u)-\lambda(v_n)-\sum_{v'\neq v} 2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right))|\mathcal{V}\backslash\{v\}> \\
&+ f(v_n)\sum_{v,v'\in\mathcal{W},v\neq v'} -\frac{2}{(u-v)(u-v')}|\mathcal{W}\backslash\{v,v'\}\cup\{u\}> \\
&+ \frac{1}{u-v_n}\left(h(v_n)-h(u)\right)|\mathcal{W}> \\
&= \sum_{v\in\mathcal{W}} -\frac{1}{u-v}\lambda(u)f(v_n)|\mathcal{W}\backslash\{v\}> + \sum_{v\in\mathcal{W}}\frac{1}{u-v}\lambda(v_n)f(v_n)|\mathcal{W}\backslash\{v\}> \\
&+ \sum_{v\in\mathcal{W}}\frac{1}{u-v}\sum_{v'\neq v} 2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right)f(v_n)|\mathcal{W}\backslash\{v\}> \\
&+ f(v_n)\sum_{v,v'\in\mathcal{W},v\neq v'} 2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right)|\mathcal{W}\backslash\{v,v'\}\cup\{u\}> \\
&+ \frac{1}{u-v_n}\left(\left(\lambda(v_n)-\sum_{v\in\mathcal{W}}\frac{2}{v_n-v}|\mathcal{W}>\right)+\sum_{v\in\mathcal{W}}\frac{2}{v_n-v}|\mathcal{W}\backslash\{v\}\cup\{v_n\}>\right) \\
&- \frac{1}{u-v_n}\left(\left(\lambda(u)-\sum_{v\in\mathcal{W}}\frac{2}{u-v}|\mathcal{W}>+\sum_{v\in\mathcal{W}}\frac{2}{u-v}\right)|\mathcal{W}\backslash\{v\}\cup\{u\}\right)> \\
&= \sum_{v\in\mathcal{W}} -\frac{1}{u-v}\lambda(u)|\mathcal{W}\backslash\{v\}\cup\{v_n\}> - \frac{1}{u-v_n}\lambda(u)|\mathcal{W}> \\
&+ \sum_{v\in\mathcal{W}}\frac{1}{u-v}\lambda(v_n)|\mathcal{W}\backslash\{v\}\cup\{v_n\}> + \frac{1}{u-v_n}\lambda(v_n)|\mathcal{W}> \\
&+ \sum_{v\in\mathcal{W}}\frac{1}{u-v}\sum_{v'\neq v} 2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right)|\mathcal{W}\backslash\{v\}\cup\{v_n\}> \\
&+ \sum_{v,v'\in\mathcal{W},v\neq v'} 2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right)|\mathcal{W}\backslash\{v,v'\}\cup\{u\}\cup\{v_n\}> \\
&- \frac{1}{u-v_n}\sum_{v\in\mathcal{W}}\frac{2}{v_n-v}|\mathcal{W}> + \frac{1}{u-v_n}\sum_{v\in\mathcal{W}}\frac{2}{v_n-v}|\mathcal{W}\backslash\{v\}\cup\{v_n\}> \\
&+ \frac{1}{u-v_n}\sum_{v\in\mathcal{W}}\frac{2}{u-v}|\mathcal{W}> - \frac{1}{u-v_n}\sum_{v\in\mathcal{W}}\frac{2}{u-v}|\mathcal{W}\backslash\{v\}\cup\{u\}>
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{v\in\mathcal{V}} -\frac{1}{u-v}(\lambda(u)-\lambda(v))|\mathcal{V}\backslash\{v\}> \\
&+ \sum_{v\in\mathcal{W}} \frac{1}{u-v}\sum_{v'\neq v} 2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right)|\mathcal{W}\backslash\{v\}\cup\{v_n\}> \\
&+ \sum_{v,v'\in\mathcal{W}} -\frac{2}{(u-v)(u-v')}|\mathcal{W}\backslash\{v,v'\}\cup\{u\}> \\
&+ \sum_{v\in\mathcal{W}} \frac{2}{(u-v_n)(v-v_n)}|\mathcal{W}> - \sum_{v\in\mathcal{W}} \frac{2}{(u-v_n)(v-v_n)}|\mathcal{W}\backslash\{v\}\cup\{v_n\}> \\
&+ \sum_{v\in\mathcal{W}} \frac{2}{(u-v_n)(u-v)}|\mathcal{W}> - \sum_{v\in\mathcal{W}} \frac{2}{(u-v_n)(u-v)}|\mathcal{W}\backslash\{v\}\cup\{u\}> \\
&= \sum_{v\in\mathcal{V}} -\frac{1}{u-v}(\lambda(u)-\lambda(v))|\mathcal{V}\backslash\{v\}> \\
&+ \sum_{v\in\mathcal{W}} \frac{1}{u-v}\sum_{v'\neq v} 2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right)|\mathcal{W}\backslash\{v\}\cup\{v_n\}> \\
&+ \frac{1}{u-v_n}\sum_{v'\neq v} 2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right)|\mathcal{W}\backslash\{v\}\cup\{v_n\}> \\
&+ \sum_{v,v'\in\mathcal{W}} -\frac{2}{(u-v)(u-v')}|\mathcal{W}\backslash\{v,v'\}\cup\{u\}> \\
&- \frac{2}{(u-v_n)(u-v)}|\mathcal{W}\backslash\{v\}\cup\{u\}> \\
&= \sum_{v\in\mathcal{V}} -\frac{1}{u-v}\left(\lambda(u)-\lambda(v)-\sum_{v'\neq v} 2\left(\frac{1}{u-v'}-\frac{1}{v-v'}\right)\right)|\mathcal{V}\backslash\{v\}> \\
&+ \sum_{v,v'\in\mathcal{V},v\neq v'} -\frac{2}{(u-v)(u-v')}|\mathcal{V}\backslash\{v,v'\}\cup\{u\}>
\end{aligned}
$$

□

We also compute the following brackets:

Lemma 3.2.3.

$$[t(u),f(v_i)] = \frac{1}{u-v_i}(f(u)h(v_i)-f(v_i)h(u))$$

Proof:

$$
\begin{aligned}
[2t(u), f(v_i)] &= \left[\frac{1}{2}h^2(u) + e(u)f(u) + f(u)e(u), f(v_i)\right] \\
&= \left[\frac{1}{2}h^2(u), f(v_i)\right] + [e(u)f(u), f(v_i)] + [f(u)e(u), f(v_i)] \\
&= \left[\frac{1}{2}h(u), f(v_i)\right] h(u) + \frac{1}{2}h(u)\,[h(u), f(v_i)] \\
&+ [e(u), f(v_i)]\, f(u) + e(u)\,[f(u), f(v_i)] \\
&+ [f(u), f(v_i)]\, e(u) + f(u)\,[(e(u), f(v_i)] \\
&= \frac{f(u) - f(v_i)}{u - v_i} h(u) + h(u)\frac{f(u) - f(v_i)}{u - v_i} \\
&- \frac{h(u) - h(v_i)}{u - v_i} f(u) - f(u)\frac{h(u) - h(v_i)}{u - v_i} \\
&= \frac{1}{u - v_i}\left(f(u)h(u) - f(v_i)h(v_i) + h(u)f(u) - h(u)f(v_i)\right) \\
&- \frac{1}{u - v_i}\left(h(u)f(u) + h(v_i)f(u) - f(u)h(u) + f(u)h(v_i)\right) \\
&= \frac{1}{u - v_i}\left(h(v_i)f(u) - h(u)f(v_i) + f(u)h(v_i) - f(v_i)h(u)\right) \\
&= \frac{1}{u - v_i}(f(u)h(v_i) + [h(v_i), f(u)] \\
&+ f(u)h(v_i) - f(v_i)h(u) - [f(v_i), h(u)] - f(v_i)h(u)) \\
&= \frac{1}{u - v_i}\left(2f(u)h(v_i) - 2f(v_i)h(u)\right)
\end{aligned}
$$

$$
\Longrightarrow [t(u), f(v_i)] = \frac{1}{u - v_i}\left(f(u)h(v_i) - f(v_i)h(u)\right) \qquad (3.20)
$$

□

Using the above as the base case we formulate the following recursive procedure:

$$
\begin{aligned}
&[t(u), f(v_1)f(v_2)...f(v_n)] \\
&= [t(u), f(v_1)f(v_2)...f(v_{n-1})]\, f(v_n) + f(v_1)f(v_2)...f(v_{n-1})\, [t(u), f(v_n)]
\end{aligned}
$$

Lemma 3.2.4.

$$[t(u), e(v_i)] = \frac{1}{u - v_i}\,(e(v_i)h(u) - e(u)h(v_i))$$

Proof:

$$\begin{aligned}
[2t(u),e(v_i)] &= \left[\frac{1}{2}h^2(u)+e(u)f(u)+f(u)e(u),e(v_i)\right] \\
&= \left[\frac{1}{2}h^2(u),e(v_i)\right]+[e(u)f(u),e(v_i)]+[f(u)e(u),e(v_i)] \\
&= \left[\frac{1}{2}h(u),e(v_i)\right]h(u)+\frac{1}{2}h(u)\,[h(u),e(v_i)] \\
&+ [e(u),e(v_i)]\,f(u)+e(u)\,[f(u),e(v_i)] \\
&+ [f(u),e(v_i)]\,e(u)+f(u)\,[(e(u),e(v_i)] \\
&= -\frac{e(u)-e(v_i)}{u-v_i}h(u)-h(u)\frac{e(u)-e(v_i)}{u-v_i} \\
&+ e(u)\frac{h(u)-h(v_i)}{u-v_i}f(u)+\frac{h(u)-h(v_i)}{u-v_i}e(u) \\
&= \frac{1}{u-v_i}\left(-e(u)h(u)+e(v_i)h(u)-h(u)e(u)+h(u)e(v_i)\right) \\
&+ \frac{1}{u-v_i}\left(e(u)h(u)-e(u)h(v_i)+h(u)e(u)-h(v_i)e(u)\right) \\
&= \frac{1}{u-v_i}\left(e(v_i)h(u)+h(u)e(v_i)-e(u)h(v_i)-h(v_i)e(u)\right) \\
&= \frac{1}{u-v_i}(e(v_i)h(u)+e(v_i)h(u)+[h(u),e(v_i)] \\
&- e(u)h(v_i)-e(u)h(v_i)-[h(v_i),e(u)]) \\
&= \frac{1}{u-v_i}\left(2e(v_i)h(u)-2e(u)h(v_i)\right)
\end{aligned}$$

$$\Longrightarrow [t(u),e(v_i)] = \frac{1}{u-v_i}\left(e(v_i)h(u)-e(u)h(v_i)\right) \tag{3.21}$$

□

Lemma 3.2.5.

$$[t(u),h(v_i)] = \frac{1}{u-v_i}\left(f(v_i)e(u)-f(u)e(v_i)\right)$$

Proof:

$$
\begin{aligned}
[2t(u),h(v_i)] &= \left[\frac{1}{2}h^2(u)+e(u)f(u)+f(u)e(u),h(v_i)\right] \\
&= \left[\frac{1}{2}h^2(u),h(v_i)\right]+[e(u)f(u),h(v_i)]+[f(u)e(u),h(v_i)] \\
&= [e(u),h(v_i)]\,f(u)+e(u)\,[f(u),h(v_i)] \\
&+ [f(u),h(v_i)]\,e(u)+f(u)\,[e(u),h(v_i)] \\
&= 2\frac{e(u)-e(v_i)}{u-v_i}f(u)-2e(u)\frac{e(u)-e(v_i)}{u-v_i} \\
&- 2\frac{f(u)-f(v_i)}{u-v_i}e(u)+2f(u)\frac{e(u)-e(v_i)}{u-v_i} \\
&= \frac{2}{u-v_i}\left(e(u)f(u)+e(v_i)f(u)-e(u)f(u)+e(u)f(v_i)\right) \\
&- \frac{2}{u-v_i}\left(f(u)e(u)+f(v_i)e(u)+f(u)e(u)-f(u)e(v_i)\right) \\
&= \frac{2}{u-v_i}\left(f(v_i)e(u)+e(u)f(v_i)-e(v_i)f(u)-f(u)e(v_i)\right) \\
&= \frac{2}{u-v_i}(f(v_i)e(u)+f(v_i)e(u)+[e(u),f(v_i)] \\
&- f(u)e(v_i)-[e(v_i),f(u)]-f(u)e(v_i)) \\
&= \frac{2}{u-v_i}\left(2f(v_i)e(u)-2f(u)e(v_i)\right)
\end{aligned}
$$

$$
\Longrightarrow [t(u),h(v_i)] = \frac{1}{u-v_i}\left(f(v_i)e(u)-f(u)e(v_i)\right) \tag{3.22}
$$

□

Proof of theorem 3.2.2

$$
\begin{aligned}
t(u) \quad & |v_1 v_2 ... v_K> = [t(u), f(v_1)]\,|v_2 ... v_K> \\
+ \quad & f(v_1)\,[t(u), f(v_2)]\,|v_3 ... v_K> \\
+ \quad & f(v_1) f(v_2)\,[t(u), f(v_3)]\,|v_4 ... v_K> \\
+ \quad & \cdots \\
+ \quad & f(v_1) f(v_2) \cdots f(v_{K-1})\,[t(u), f(v_K)]\,|0> \\
+ \quad & f(v_1) f(v_2) ... f(v_K) t(u) |0> \\
= \quad & \frac{1}{u-v_1}\left(f(u)h(v_1) - f(v_1)h(u)\right)|v_2 \cdots v_K> \\
+ \quad & \frac{1}{u-v_2} f(v_1)\left(f(u)h(v_2) - f(v_2)h(u)\right)|v_3 \cdots v_K> \\
+ \quad & \cdots \\
+ \quad & \frac{1}{u-v_{K-1}} f(v_1) f(v_2) \cdots f(v_{K-2})\left(f(u)h(v_{K-1}) - f(v_{K-1})h(u)\right)|v_K> \\
+ \quad & \frac{1}{u-v_K} f(v_1) \cdots f(v_{K-1})\left(f(u)h(v_K) - f(v_K)h(u)\right)|0> \\
+ \quad & f(v_1) f(v_2) ... f(v_K) t(u) |0> \\
= \quad & \frac{1}{u-v_1}(f(u)\left(\lambda(v_1) - \sum_{k=2}^{K} \frac{2}{v_1 - v_k}\right)|v_2 \cdots v_K> \\
+ \quad & \sum_{k=2}^{K} \frac{2}{v_1 - v_k} |u v_2 \cdots v_K \backslash \{v_k\} \cup \{v_1\}>) \\
- \quad & \frac{1}{u-v_1}(f(v_1)\left(\lambda(u) - \sum_{k=2}^{K} \frac{2}{u - v_k}\right)|v_2 \cdots v_K> \\
+ \quad & \sum_{k=2}^{K} \frac{2}{u - v_k} |v_1 v_2 \cdots v_K \backslash \{v_k\} \cup \{u\}>) \\
+ \quad & \frac{1}{u-v_2}(f(v_1) f(u)\left(\lambda(v_2) - \sum_{k=3}^{K} \frac{2}{v_2 - v_k}\right)|v_3 \cdots v_K> \\
+ \quad & \sum_{k=3}^{K} \frac{2}{v_2 - v_k} |u v_1 v_3 \cdots v_K \backslash \{v_k\} \cup \{v_2\}>)
\end{aligned}
$$

$$
\begin{aligned}
- & \frac{1}{u-v_2}(f(v_1)f(v_2)\left(\lambda(u)-\sum_{k=3}^{K}\frac{2}{u-v_k}\right)|v_3\cdots v_K> \\
+ & \sum_{k=3}^{K}\frac{2}{u-v_k}|v_1v_2\cdots v_K\backslash\{v_k\}\cup\{u\}>) \\
+ & \cdots \\
+ & \frac{1}{u-v_{K-1}}f(v_1)f(v_2)\cdots f(v_{K-2})f(u)(\lambda(v_{K-1}) \\
- & \frac{2}{v_{K-1}-v_K})|v_K> + \frac{2}{v_{K-1}-v_K}|uv_1v_2\cdots v_{K-1}> \\
- & \frac{1}{u-v_{K-1}}f(v_1)f(v_2)\cdots f(v_{K-2})f(v_{K-1})(\lambda(u) \\
- & \frac{2}{u-v_K})|v_K> + \frac{2}{u-v_K}|uv_1v_2\cdots v_{K-1}> \\
+ & \frac{1}{u-v_K}f(v_1)\cdots f(v_{K-1})f(u)\lambda(v_K)|0> - \frac{1}{u-v_K}f(v_1)\cdots f(v_K)\lambda(u)|0> \\
+ & \frac{1}{4}\lambda^2(u)-\frac{1}{2}\lambda'(u)
\end{aligned}
$$

We group all terms that are not the eigenvector, i.e. the terms that involve the spectral parameter u.

$$
\begin{aligned}
& \frac{1}{u-v_1}\left(\lambda(v_1)-\sum_{k=2}^{K}\frac{2}{v_1-v_k}\right)|uv_2\cdots v_K> \\
+ & \frac{1}{u-v_2}\left(\lambda(v_2)-\sum_{k=3}^{K}\frac{2}{v_2-v_k}\right)|uv_1v_3\cdots v_K> \\
+ & \cdots \\
+ & \frac{1}{u-v_{K-2}}\left(\lambda(v_{K-2})-\sum_{k=K-1}^{K}\frac{2}{v_{K-2}-v_k}\right)|uv_1\cdots v_{K-3}v_{K-1}v_K> \\
+ & \frac{1}{u-v_{K-1}}\left(\lambda(v_{K-1})-\frac{2}{v_{K-1}-v_K}\right)|uv_1\cdots v_{K-2}v_K> \\
+ & \frac{1}{u-v_K}\lambda(v_K)|uv_1\cdots v_{K-1}> \qquad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{u-v_1}\sum_{k=2}^{K}\frac{2}{v_1-v_k}|uv_2\cdots v_K\setminus\{v_k\}\cup\{v_1\}> \\
- & \frac{1}{u-v_1}\sum_{k=2}^{K}\frac{2}{u-v_k}|v_1v_2\cdots v_K\setminus\{v_k\}\cup\{u\}> \\
+ & \frac{1}{u-v_2}\sum_{k=3}^{K}\frac{2}{v_2-v_k}|uv_1v_3\cdots v_K\setminus\{v_k\}\cup\{v_2\}> \\
- & \frac{1}{u-v_2}\sum_{k=3}^{K}\frac{2}{u-v_k}|v_1v_2\cdots v_K\setminus\{v_k\}\cup\{u\}> \\
+ & \cdots \\
+ & \frac{1}{u-v_{K-1}}\frac{2}{v_{K-1}-v_K}|uv_1\cdots v_{K-1}> \\
- & \frac{1}{u-v_{K-1}}\frac{2}{u-v_K}|uv_1v_2\cdots v_{K-1}> \qquad (3.24)
\end{aligned}
$$

We observe that the vectors in 3.24 have the following coefficients (we use partial fraction decomposition):

$$|uv_1v_2\cdots v_{K-1}>:$$

$$\begin{aligned}
& \frac{1}{u-v_1}\frac{2}{v_1-v_K}-\frac{1}{u-v_1}\frac{2}{u-v_K} \\
+ & \frac{1}{u-v_2}\frac{2}{v_2-v_K}-\frac{1}{u-v_2}\frac{2}{u-v_K}+\cdots \\
+ & \frac{1}{u-v_{K-1}}\frac{2}{v_{K-1}-v_K}-\frac{1}{u-v_{K-1}}\frac{2}{u-v_K} \\
= & \frac{1}{u-v_K}(-\frac{2}{u-v_1}-\frac{2}{u-v_2}-\cdots-\frac{2}{u-v_{K-1}} \\
+ & \frac{2}{u-v_1}-\frac{2}{v_K-v_1}+\frac{2}{u-v_2}-\frac{2}{v_K-v_2}+\cdots \\
+ & \frac{2}{u-v_{K-1}}-\frac{2}{v_K-v_{K-1}}) \\
= & \frac{1}{u-v_K}(-\frac{2}{v_K-v_1}-\frac{2}{v_K-v_2}-\cdots-\frac{2}{v_K-v_{K-1}} \\
= & -\frac{1}{u-v_K}\left(\sum_{k=1}^{K-1}\frac{2}{v_K-v_k}\right)
\end{aligned}$$

$$|uv_1v_2\cdots v_{K-2}v_K>:$$

$$
\begin{aligned}
& \frac{1}{u-v_1}\frac{2}{v_1-v_{K-1}} - \frac{1}{u-v_1}\frac{2}{u-v_{K-1}} \\
+ & \frac{1}{u-v_2}\frac{2}{v_2-v_{K-1}} - \frac{1}{u-v_2}\frac{2}{u-v_{K-1}} + \cdots \\
+ & \frac{1}{u-v_{K-2}}\frac{2}{v_{K-2}-v_{K-1}} - \frac{1}{u-v_{K-2}}\frac{2}{u-v_{K-1}} \\
= & \frac{1}{u-v_{K-1}}(-\frac{2}{u-v_1} - \frac{2}{u-v_2} - \cdots - \frac{2}{u-v_{K-2}} + \frac{2}{u-v_1} \\
- & \frac{2}{v_{K-1}-v_1} + \frac{2}{u-v_2} - \frac{2}{v_{K-1}-v_2} + \cdots \\
+ & \frac{2}{u-v_{K-2}} - \frac{2}{v_{K-1}-v_{K-2}}) \\
= & \frac{1}{u-v_{K-1}}(-\frac{2}{v_{K-1}-v_1} - \frac{2}{v_{K-1}-v_2} - \cdots - \frac{2}{v_{K-1}-v_{K-2}} \\
= & -\frac{1}{u-v_{K-1}}\left(\sum_{k=1}^{K-2}\frac{2}{v_{K-1}-v_k}\right)
\end{aligned}
$$

$$|uv_1v_2\cdots v_{K-3}v_{K-1}v_K>$$

$$\cdots$$

$$|uv_1v_3\cdots v_K>:$$

$$
\begin{aligned}
& \frac{1}{u-v_1}\frac{2}{v_1-v_2} - \frac{1}{u-v_1}\frac{2}{u-v_2} \\
= & \frac{1}{u-v_2}\left(\frac{2}{u-v_1} + \frac{2}{v_2-v_1}\right) = \frac{1}{u-v_2}\left(-\frac{2}{v_2-v_1}\right)
\end{aligned}
$$

Thus the terms which involve the spectral parameter u (Equations 3.23 and 3.24) take the following form:

$$
\begin{aligned}
&\frac{1}{u-v_1}\left(\lambda(v_1)-\sum_{k=2}^{K}\frac{2}{v_1-v_k}\right)|uv_2\cdots v_K>\\
&+\frac{1}{u-v_2}\left(\lambda(v_2)-\sum_{k=3}^{K}\frac{2}{v_2-v_k}-\frac{2}{v_2-v_1}\right)|uv_1v_3\cdots v_K>\\
&+\cdots\\
&+\frac{1}{u-v_{K-2}}\left(\lambda(v_{K-2})-\sum_{k=K-1}^{K}\frac{2}{v_{K-2}-v_k}-\sum_{k=1}^{K-3}\frac{2}{v_{K-2}-v_k}\right)|uv_1\cdots v_{K-3}v_{K-1}v_K>\\
&+\frac{1}{u-v_{K-1}}\left(\lambda(v_{K-1})-\frac{2}{v_{K-1}-v_K}-\sum_{k=1}^{K-2}\frac{2}{v_{K-1}-v_k}\right)|uv_1\cdots v_{K-2}v_K>\\
&+\frac{1}{u-v_K}\left(\lambda(v_K)-\sum_{k=1}^{K-1}\frac{2}{v_K-v_k}\right)|uv_1\cdots v_{K-1}> \qquad (3.25)
\end{aligned}
$$

We now consider the remaining terms in equation 3.23 (i.e. the terms that contribute to the coefficient of vector $|v_1\cdots v_K>$).

$$
\begin{aligned}
&-\frac{1}{u-v_1}\left(\lambda(u)-\sum_{k=2}^{K}\frac{2}{u-v_k}\right)\\
&-\frac{1}{u-v_2}\left(\lambda(u)-\sum_{k=3}^{K}\frac{2}{u-v_k}\right)\\
&-\cdots\\
&-\frac{1}{u-v_{K-1}}\left(\lambda(u)-\frac{2}{u-v_k}\right)\\
&-\frac{1}{u-v_K}\lambda(u)\\
&+\frac{1}{4}\lambda^2(u)-\frac{1}{2}\lambda'(u)\\
&=\frac{1}{4}\lambda^2(u)-\frac{1}{2}\lambda'(u)-\lambda(u)\sum_{k=1}^{K}\frac{1}{u-v_k}\\
&+\frac{1}{u-v_1}\sum_{k=2}^{K}\frac{2}{u-v_k}+\frac{1}{u-v_2}\sum_{k=3}^{K}\frac{2}{u-v_k}+\cdots+\frac{1}{u-v_{K-1}}\frac{2}{u-v_K}\\
&+\sum_{k=1}^{K}\frac{1}{(u-v_k)^2}-\sum_{k=1}^{K}(\frac{1}{u-v_k})^2
\end{aligned}
\tag{3.26}
$$

Remark: In equation 3.26, we added and subtracted the term:

$$
\sum_{k=1}^{K}(\frac{1}{u-v_k})^2
$$

.

Consider the terms:

$$\frac{1}{u-v_1}\sum_{k=2}^{K}\frac{2}{u-v_k}+\frac{1}{u-v_2}\sum_{k=3}^{K}\frac{2}{u-v_k}$$
$$+\cdots+$$
$$\frac{1}{u-v_{K-1}}\frac{2}{u-v_K}+\sum_{k=1}^{K}\frac{1}{(u-v_k)^2}$$

in equation 3.26 .

Lemma 3.2.6.

$$\frac{1}{u-v_1}\sum_{k=2}^{K}\frac{2}{u-v_k}+\frac{1}{u-v_2}\sum_{k=3}^{K}\frac{2}{u-v_k}+\cdots+\frac{1}{u-v_{K-1}}\frac{2}{u-v_K}+\sum_{k=1}^{K}\frac{1}{(u-v_k)^2}$$
$$= \left(\sum_{k=1}^{K}\frac{1}{u-v_k}\right)^2$$

Proof of Lemma 3.2.6:

$$
\begin{aligned}
& \frac{1}{u-v_1}\sum_{k=2}^{K}\frac{2}{u-v_k}+\frac{1}{u-v_2}\sum_{k=3}^{K}\frac{2}{u-v_k}+\cdots+\frac{1}{u-v_{K-1}}\frac{2}{u-v_K}+\sum_{k=1}^{K}\frac{1}{(u-v_k)^2} \\
= & \frac{1}{u-v_1}\frac{2}{u-v_2}+\frac{1}{u-v_1}\frac{2}{u-v_3}+\cdots+\frac{1}{u-v_1}\frac{2}{u-v_{K-1}}+\frac{1}{u-v_1}\frac{2}{u-v_K} \\
+ & \frac{1}{u-v_2}\frac{2}{u-v_3}+\frac{1}{u-v_2}\frac{2}{u-v_4}+\cdots+\frac{1}{u-v_2}\frac{2}{u-v_{K-1}}+\frac{1}{u-v_2}\frac{2}{u-v_K} \\
+ & \frac{1}{u-v_3}\frac{2}{u-v_4}+\frac{1}{u-v_3}\frac{2}{u-v_5}+\cdots+\frac{1}{u-v_3}\frac{2}{u-v_{K-1}}+\frac{1}{u-v_3}\frac{2}{u-v_K} \\
+ & \cdots \\
+ & \frac{1}{u-v_{K-2}}\frac{2}{u-v_{K-1}}+\frac{1}{u-v_{K-2}}\frac{2}{u-v_K} \\
+ & \frac{1}{u-v_{K-1}}\frac{2}{u-v_K}+\sum_{k=1}^{K}\left(\frac{1}{u-v_k}\right)^2 \\
= & 2(\frac{1}{u-v_2}\frac{1}{u-v_1}+\frac{1}{u-v_3}\sum_{k=1}^{2}\frac{1}{u-v_k}+\frac{1}{u-v_4}\sum_{k=1}^{3}\frac{1}{u-v_k} \\
+ & \cdots \\
+ & \frac{1}{u-v_{K-2}}\sum_{k=1}^{K-2}\frac{1}{u-v_k}+\frac{1}{u-v_{K-1}}\sum_{k=1}^{K-1}\frac{1}{u-v_k})+\sum_{k=1}^{K}\left(\frac{1}{u-v_k}\right)^2 \\
= & \left(\frac{1}{u-v_1}\right)^2+2\frac{1}{u-v_1}\frac{1}{u-v_2}+\left(\frac{1}{u-v_2}\right)^2+2\left(\sum_{k=1}^{2}\frac{1}{u-v_k}\right)\frac{1}{u-v_3} \\
+ & \left(\frac{1}{u-v_3}\right)^2+2\left(\sum_{k=1}^{3}\frac{1}{u-v_k}\right)\frac{1}{u-v_4} \\
+ & \cdots \\
+ & 2\left(\sum_{k=1}^{K-1}\frac{1}{u-v_k}\right)\frac{1}{u-v_K}+\left(\frac{1}{u-v_k}\right)^2 \\
= & \left(\sum_{k=1}^{K}\frac{1}{u-v_k}\right)^2
\end{aligned}
$$

□

Thus the coefficient of vector $|v_1\cdots v_K>$ (equation 3.26) becomes:

$$\frac{1}{4}\lambda^2(u) - \frac{1}{2}\lambda'(u) - \lambda(u)\sum_{k=1}^{K}\frac{1}{u-v_k} + \left(\sum_{k=1}^{K}\frac{1}{u-v_k}\right)^2 \tag{3.27}$$

Finally, we group equations 3.25 and 3.27 and obtain:

$$\begin{aligned}
& t(u)|v_1 \ldots v_K> \\
= & \left(\frac{1}{4}\lambda^2(u) - \frac{1}{2}\lambda'(u) - \lambda(u)\sum_{k=1}^{K}\frac{1}{u-v_k} + \left(\sum_{k=1}^{K}\frac{1}{u-v_k}\right)^2\right)|v_1 \cdots v_K> \\
+ & \frac{1}{u-v_1}\left(\lambda(v_1) - \sum_{k=2}^{K}\frac{2}{v_1-v_k}\right)|uv_2 \cdots v_K> \\
+ & \frac{1}{u-v_2}\left(\lambda(v_2) - \sum_{k=3}^{K}\frac{2}{v_2-v_k} - \frac{2}{v_2-v_1}\right)|uv_1v_3 \cdots v_K> \\
+ & \cdots \\
+ & \frac{1}{u-v_{K-2}}(\lambda(v_{K-2}) - \sum_{k=K-1}^{K}\frac{2}{v_{K-2}-v_k} \\
- & \sum_{k=1}^{K-3}\frac{2}{v_{K-2}-v_k})|uv_1 \cdots v_{K-3}v_{K-1}v_K> \\
+ & \frac{1}{u-v_{K-1}}\left(\lambda(v_{K-1}) - \frac{2}{v_{K-1}-v_K} - \sum_{k=1}^{K-2}\frac{2}{v_{K-1}-v_k}\right)|uv_1 \cdots v_{K-2}v_K> \\
+ & \frac{1}{u-v_K}\left(\lambda(v_K) - \sum_{k=1}^{K-1}\frac{2}{v_K-v_k}\right)|uv_1 \cdots v_{K-1}>
\end{aligned}$$

We thus conclude that vector $|\mathcal{V}> = |v_1 \cdots v_K>$ is a joint eigenvector of the commuting operator $t(u)$ if and only if the spectral parameters $v \in \mathcal{V}$ satisfy the Bethe-Ansatz equations.

□

Remark: Another proof of theorem 3.2.2 is based on Sklyanin's, [19], [14], separation of variables method. The eigenfunction equations are transformed into a differential equation for which the differential operator is the same as the differential operator in the Bethe-Ansatz equations.

Chapter 4

Lamé differential equation

There are two ways to describe the solutions to the Bethe-Ansatz equations associated with the $\mathfrak{sl}_2(\mathbb{C})$ Gaudin spin chain model. One of the them is to regard the Bethe vectors as a collection of numbers satisfying the critical point equations of the Master function. The problem of solving the Bethe-Ansatz equations can also be approached by looking at polynomial solutions of a differential equation, called the Lamé equation, [14], [4], [15], [13]. In what follows, we discuss these two equivalent methods.

4.1 The Lamé equation

In 1878, H.E. Heine [7] motivated by his work in the area of orthogonal polynomials, formulated the following problem:

Given $a(z), b(z) \in \mathbb{C}[z]$ of degrees $N+1$ and N respectively. Heine was interested in the polynomials $c(z) \in \mathbb{C}[z]$ of degree $N-1$ such that the equation

$$a(z)\psi''(z) + b(z)\psi'(z) + c(z)\psi(z) = 0 \tag{4.1}$$

has non-trivial, monic polynomial solution $\psi(z)$ of a given degree K. Equation 4.1 is called the Lamé equation. The solution to the Lamé equation is a pair $c(z), \psi(z)$. Heine proved that there are at most

$$\sigma(N,K) = \frac{(N+K-1)}{K!(N-1)!}$$

polynomial solutions of degree K to equation 4.1. These polynomial solutions $\psi(z)$ are called the Heine-Stieltjes polynomials and the corresponding polynomials $c(z)$ are known as Van Vleck polynomials.

4.2 Heine-Stieltjes Theorem

In 1885, T.J. Stieltjes [20] considered a special case of the Lamé equation 4.1. Let

$$a(z) = \prod_{n=0}^{N} (z - \alpha_n),$$

$$b(z) = -a(z) \sum_{n=0}^{N} \frac{\lambda_n}{z - \alpha_n}, \lambda_n < 0 \tag{4.2}$$

where all $\alpha_0 < \alpha_1 < ... < \alpha_N$ are distinct and real roots of $a(z)$ and $\lambda_1, \lambda_2, ..., \lambda_K$ are negative. These assumptions force the roots of $b(z)$ to also be real. Note: $\lambda_n < 0$ corresponds to infinite dimensional irreducible representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ in the Gaudin spin chain model. Please refer to Chapter 2 and to Appendix A for more details.

Under these new assumptions, the Lamé equation 4.1 takes the following form:

$$\prod_{n=0}^{N}(z-\alpha_n)\psi''(z)-a(z)\sum_{n=0}^{N}\frac{\lambda_n}{z-\alpha_n}\psi'(z)+c(z)\psi(z)=0 \tag{4.3}$$

Stieltjes proved the following theorem:

Theorem 4.2.1. *Heine-Stieltjes Theorem. There are exactly $\sigma(N,K)$ polynomial solutions to equation 4.3. The location of zeros of the Heine-Stieltjes polynomials is completely characterized by their distribution in subintervals*

$$(\alpha_0,\alpha_1),\ldots,(\alpha_{N-1},\alpha_N).$$

The Lamé equation 4.3 has a natural interpretation in electrostatics, which had lead Stieltjes to further investigation. Consider an electrostatic field with the logarithmic potential energy acting upon the system. The zeros of the Heine-Stieltjes polynomials $\psi(z)$ can be interpreted as equilibrium positions of an electrostatic system with logarithmic potential. Suppose there are magnets with charges λ_n, located at the points $\alpha_n \in \mathbb{R}$. Place K identical electrons (unit charges) at the points v_k allowed to move freely on the real line. The charges repel each other according to the logarithmic potential (particles repel each other with a force proportional to their masses and inversely proportional to their distance). Stieltjes, [20], proved that the energy of the field has a local minimum. Szegö [21] then proved that this minimum is unique and thus results in stability of the equilibrium.

Let

$$\Xi(\alpha,v)=\prod_{k=1}^{L}\prod_{n=0}^{N}(v_k-\alpha_n)^{-\lambda}\prod_{k\neq l}^{K}(v_k-v_l)^2 \tag{4.4}$$

$\Xi(v)$ is called the Master Function.

The logarithmic electrostatic potential S of the system is given by the logarithm of the Master function:

$$S(v) = -\log(\prod_{k=1}^{L}\prod_{n=0}^{N}(v_k - \alpha_n)^{-\lambda}\prod_{k\neq l}^{K}(v_k - v_l)^2)$$

Proposition 4.2.1. *The Bethe-Ansatz equations are critical points of the Master function.*

Proof: Consider the function

$$\begin{aligned} -\log\Xi = S(v) &= -log(\prod_{k=1}^{K}\prod_{n=0}^{N}(v_k - \alpha_n)^{-\lambda}\prod_{k\neq l}^{K}(v_k - v_l)^2) \\ &= -\sum_{k=1}^{K}\sum_{n=0}^{N}\lambda_n log(\frac{1}{v_k - \alpha_n}) + \sum_{k\neq l}^{K} 2log(\frac{1}{v_k - v_l}) \end{aligned} \quad (4.5)$$

Since $S' = \frac{\Xi'}{\Xi}$, both functions $\Xi(v)$ and $S(v)$ have the same critical points. Differentiating $S(v)$ gives

$$\begin{aligned} \frac{d(S(v))}{dv} &= \frac{\Xi'(v)}{\Xi(v)} \\ &= \sum_{k=1}^{K}\sum_{n=0}^{N}\frac{\lambda_n}{v_k - \alpha_n} - \sum_{k\neq l}^{K}\frac{2}{v_k - v_l} \end{aligned}$$

which proves the proposition. □

At an equilibrium position, the sum of all forces acting on each electron is zero. It occurs when $\nabla S = 0$.

In what follows, we shall prove that there exists a one-to-one correspondence between the set of Heine-Stieltjes polynomials and the points at an equilibrium of the electrostatic system under discussion.

Lemma 4.2.1. *The roots of ψ are simple.*

Proof: Let v_k be any root of ψ. Assume for a contradiction that $\psi(v_k) = \psi'(v_k) = 0$. After repeated differentiation of equation 4.1, we observe that all derivatives of ψ are equal to zero when evaluated at v_k, and thus ψ is a zero polynomial. Therefore, we conclude that $\psi'(v_k) \neq 0$ and the roots of ψ are simple. □

Proposition 4.2.2. *The roots of ψ form a solution to the Bethe-Ansatz equations 3.16 if and only if there exists a polynomial $c(z)$ of degree not greater than $N-1$ such that ψ is a solution to the Lamé equation 4.3.*

Let $v_1 \cdots v_K$ be distinct roots of $\psi(z)$.

Proof:

$$\psi(z) = (z - v_1)...(z - v_K)$$

Let $\psi_k(z) = \frac{\psi(z)}{z - v_k}$

$$\psi(z) = (z - v_k)\psi_k(z)$$

$$\psi'(z) = \psi_k(z) + (z - v_k)\psi_k'(z)$$

$$\psi'(v_k) = \psi_k(v_k)$$

$$\psi''(z) = \psi_k'(z) + \psi_k'(z) + (z - v_k)\psi_k''(z)$$

$$\psi''(v_k) = 2\psi_k'(v_k)$$

$$\implies \frac{\psi''(v_k)}{\psi'(v_k)} = 2\frac{\psi_k'(v_k)}{\psi_k(v_k)}$$

Note: $\psi'(v_k) \neq 0$ since roots of ψ are simple by lemma 4.2.1.

We also use the following known fact: $\frac{\psi_k'(v_k)}{\psi_k(v_k)} = \sum_{v_1}^{K} \frac{1}{z - v_k}$

We prove one direction first by assuming the Bethe-Ansatz equations.

$$\frac{\psi''(v_k)}{\psi'(v_k)} = \frac{2\psi'(v_k)}{\psi(v_k)} = \sum_{k \neq l} \frac{2}{v_k - v_l} = \sum_{n=0}^{N} \frac{\lambda_n}{v_k - \alpha_n} = \frac{-b(v_k)}{a(v_k)}$$

$$(a\psi'' + b\psi')(v_k) = 0$$

$$\implies \psi(z) \text{ divides } a\psi''(z) + b\psi'(z)$$

$$\implies \exists c \text{ such that } a(z)\psi''(z) + b(z)\psi'(z) = -c(z)\psi(z)$$

$$\text{Thus, } a(z)\psi''(z) + b(z)\psi'(z) + c(z)\psi(z) = 0$$

To prove the converse of the above statement, we assume that $\psi(z)$ is a solution to the Lamé equation 4.1. By evaluating 4.1 at any of the roots of $\psi(z)$, the expression $c(v_k)\psi(v_k)$ vanishes.
Equation 4.1 becomes:

$$a(v_k)\psi''(v_k) + b(v_k)\psi'(v_k) = 0$$

$$\frac{\psi''(v_k)}{\psi'(v_k)} = -\frac{b(v_k)}{a(v_k)}$$

Thus,

$$\sum_{n=0}^{N} \frac{\lambda_n}{v_k - \alpha_n} = \frac{-b(v_k)}{a(v_k)} = \frac{\psi''(v_k)}{\psi'(v_k)} = 2\frac{\psi_k(v_k)}{\psi_k(v_k)} = \sum_{k \neq l} \frac{2}{v_k - v_l}$$

□

Definition 4.2.1. *Fix $\alpha_0 < \alpha_1 < ... < \alpha_N$, and $K > 0$, a positive integer. A configuration m is a multindex $d = (d_1, ..., d_N)$ such that $|d| = \sum_{w=1}^{N} d_w = K$. For a given configuration m denote by D the set of all $z \in \mathbb{R}^k$ with the arrangement that there is d_1 of z_k in (α_0, α_1), d_2 of z_k in (α_1, α_2),etc... This is equivalent to saying that*

$$\alpha_0 < z_1 < ... < z_{d_1} < \alpha_1 < z_{1+d_1} < ... < z_{d_2+d_1} < \alpha_2 < ...$$

$$< \alpha_{N-1} < z_{1+m_{N-1}+...+m_1} < ... < z_{m_N+m_{N-1}+...+m_1} < \alpha_N$$

Definition 4.2.2. *An equilibrium configuration is a configuration m where $\{z_k\}_{k=1}^{K}$ are solutions to the Bethe-Ansatz equations 3.16.*

A simple counting argument shows that the number of possible configurations is $\sigma(N,K)$. D is a connected, convex and open subset of $\mathbb{R}^k$.

Theorem 4.2.2. *The electrostatic potential function S has a unique minimum.*

Proof: Let H be the Hessian of the electrostatic potential S. Then H is a symmetric KxK matrix with entries

$$H_{kl} = \frac{\partial^2(S)}{\partial(v_k)\partial(v_l)}$$

On the diagonal of H we have:

$$H_{kk} = \sum_{l \neq k} \frac{2}{(v_k - v_l)^2} - \sum_{n=0}^{K} \frac{\lambda_n}{(v_k - \alpha_n)^2}$$

and the off diagonal entries are:

$$H_{kl} = -\frac{2}{(v_k - v_l)^2}$$

or in the matrix format:

$$H = \begin{pmatrix} \sum_{l \neq 1}^{K} \frac{2}{(v_1 - v_l)^2} - \sum_{n=0}^{N} \frac{\lambda_n}{(v_1 - \alpha_n)^2} & -\frac{2}{(v_1 - v_2)^2} & \cdot\cdot & -\frac{2}{(v_1 - v_K)^2} \\ -\frac{2}{(v_2 - v_1)^2} & \sum_{l \neq 2}^{K} \frac{2}{(v_2 - v_l)^2} - \sum_{n=0}^{N} \frac{\lambda_n}{(v_2 - \alpha_n)^2} & \cdot\cdot & -\frac{2}{(v_2 - v_K)^2} \\ \vdots & \cdot\cdot & \ddots & \vdots \\ -\frac{2}{(v_K - v_1)^2} & \cdots & \cdots & \sum_{l \neq K}^{K} \frac{2}{(v_K - v_l)^2} - \sum_{n=0}^{N} \frac{\lambda_n}{(v_K - \alpha_n)^2} \end{pmatrix}$$

and we have the following equation

$$H_{kk} - \sum_{l \neq k} |H_{kl}| = -\sum_{n=0}^{N} \frac{\lambda_n}{(v_k - \alpha_n)^2}, k = 1, \cdots, K$$

By the Gershgorin circle theorem (ref. to Appendix B), all eigenvalues of H lie in one of the disks D_k where D_k is a disk centered at H_{kk} with radius $\sum_{l \neq k} |H_{kl}|$.

Since $\{v_k\}_{k=1}^K$, $\{\alpha_n\}_{n=0}^N$ are real and $\{\lambda_n\}_{n=0}^N$ are negative, every eigenvalue η of H satisfies

$$\eta >= \min_k \left(-\sum_{n=0}^{N} \frac{\lambda_n}{(v_k - \alpha_n)^2} \right)$$

Thus all eigenvalues of H are strictly positive.

For a fixed configuration m let $S = S(v)$ and restrict $H = H(v)$ to $D(\alpha, m)$. Then S is convex since D is connected, open, bounded, and H is a positive definite KxK matrix. Also, $\lim_{x->\partial D} S(v) = +\infty$. Thus S has a unique minimum and no maxima. This unique minimum occurs at the equilibrium configuration.

It remains to show that for a given polynomial $c(z)$, there cannot be two linearly independent polynomials that are solutions to the Lamé equation 4.3. Suppose that this is possible and call these two solutions $\psi_1(z)$ and $\psi_2(z)$. Let $w(z)$ be the Wrońskian of $\psi_1(z)$ and $\psi_2(z)$. Then the $w(z) = \psi_1(z)\psi_2'(z) + \psi_1'(z)\psi_2(z)$. By Abel's theorem (ref. to Appendix C), $w(z)$ satisfies the following differential equation:

$$a(z)w'(z) + b(z)w(z) = 0$$

and

$$w(z) = w_0 e^{-\int \frac{b(z)}{a(z)} dz}$$

where w_0 is any constant.

By equation 4.2,

$$\frac{b(z)}{a(z)} \rightarrow \infty \text{ as } z \rightarrow \alpha_n$$

Therefore

$$w(z) \rightarrow 0 \text{ as } z \rightarrow \alpha_n,$$

which cannot be true for Wrońskian $w(z)$ of two independent functions.

Thus, if energy of the electrostatic potential function has a unique minimum for a given configuration m, then there exists a unique pair (c,ψ) for the Lamé equation 4.1. Conversely, if there is a unique pair (C,ψ) of a Van Vleck and a Heine-Stieltjes polynomial for the Lamé equation 4.1, such that the zeros of the polynomial ψ form a configuration, then the energy of the electrostatic field has a unique minimum. □

The Heine-Steiltjes Theorem does not apply when the roots of $a(z)$ are complex. The first author who has obtained a result in the complex case was Pólya, [17], who proved that the zeros of the Heine-Stieltjes polynomial lie inside the convex hull $CH(\alpha_1,\cdots,\alpha_N)$ of the roots of $a(z)$.

Proposition 4.2.3. *The Bethe-Ansatz equations are invariant under complex affine transformations of* $\{v_k\}_{k=1}^K$ *and* $\{\alpha_n\}_{n=0}^N$.

Proof: Let $r,s\in\mathbb{C}$. Applying the change of variables rv_k+s and $r\alpha_n+s$ to the Bethe-Ansatz equations, we obtain:

$$\sum_{n=0}^{N}\frac{\lambda_n}{(rv_k+s)-(r\alpha_n+s)}=\sum_{l\neq k}^{K}\frac{2}{(rv_k+s)-(rv_l+s)},$$

which also has the form of the Bethe-Ansatz equation.

□

Thus, proposition 4.2.3 allows us to fix two sites α_0 and α_1 on the real line, which is very useful for numerical and symbolic computations of solutions.

Proposition 4.2.4. *The zeros of the Heine-Stieltjes polynomials lie inside the convex hull of the zeros of* $a(z)$.

Proof: Let v_k be any root of ψ. From equation 4.1, we see that

$$\frac{\psi''(v_k)}{2\psi'(v_k)} + \frac{B(v_k)}{A(v_k)} = 0$$

By proposition 4.2.2, the Lamé equation 4.1 is equivalent to the Bethe-Ansatz 3.16 equations. After evaluating equation 3.16 at v_k, we obtain:

$$\sum_{n=0}^{N} \frac{\lambda_n}{v_k - \alpha_n} = \sum_{k \neq l} \frac{2}{v_k - v_l}$$

Thus, by the Gauss-Lucas theorem (ref. to Appendix E), v_k must lie inside the convex hull $CH(\alpha_1, \cdots, \alpha_N)$.

□

Marden, [11], used a similar argument to that of Pólya to show that zeros of the Van Vleck polynomial also lie inside the convex hull of the zeros of $a(z)$.

Chapter 5

Classical Jacobi Polynomials - special case of the Lamé equation

In this chapter we investigate a well known case of the Lamé equation 4.3, which occurs when $a(z) = z^2 - 1$. In this case the solutions to equation 4.3 are classical othogonal polynomials, known as Jacobi polynomials. We begin the discussion with basic introduction to orthogonal polynomials. The last section of this chapter shows plots of solutions obtained numerically.

5.1 Orthogonal Polynomials

Definition 5.1.1. *Let $h : [a,b] \to \mathbb{R}$ such that h is strictly positive on the interior (a,b) but h may go to infinity at the endpoints. h is called a weight function if for any polynomial f the integral $\int_a^b f(x)h(x)dx$ is finite. Orthogonal polynomials with respect to h are defined as a sequence of $\{P_n\}$ polynomials satisfying the* *orthogonality condition:*

$$<P_n, P_m> = \int_a^b P_n(z)P_m(z)h(z)dz = 0, n \neq m$$

where $\{P_n\}$ *represents the* n^{th} *degree polynomial, and* $< \cdot, \cdot >$ *is an inner product on the vector space of all polynomials. This means that a sequence of orthogonal polynomials forms a basis of the infinite dimensional vector space of all polynomials with the condition that* P_n *has degree n.*

5.2 Jacobi Polynomials

Jacobi polynomials are orthogonal polynomials $\{P_n^{\alpha,\beta}\}$ on the interval $[-1,1]$ with respect to the weight function $h(z) = (1-z)^{\alpha}(1+z)^{\beta}$, where the parameters α and β are required to be greater than -1, and n is the degree of the polynomial.

Jacobi polynomials $\{P_K^{\alpha,\beta}\}$ are solutions of

$$(1-z^2)\psi''(z) + (\beta - \alpha - (\alpha+\beta+2)z)\psi'(z) + K(K+\alpha+\beta)\psi(z) = 0 \quad (5.1)$$

[21]

Equation 5.1 is a special case of the Lamé equation 4.3 with sites -1 and 1. In what follows, we relate parameters α and β in equation 5.1 to charges λ_0 and λ_1 that occur in equation 4.3 . Let $a(z) = z^2 - 1$. Then $b(z) = -(z^2 - 1)\left(\frac{\lambda_0}{z-1} - \frac{\lambda_1}{z+1}\right) = -(\lambda_0+\lambda_1)z + \lambda_0 - \lambda_1$ and equation 4.3 takes the form

$$(z^2-1)\psi''(z) - ((\lambda_0+\lambda_1)z + \lambda_0 - \lambda_1)\psi'(z) + c\psi(z) = 0 \quad (5.2)$$

According to the Heine-Stieltjes theorem, equation 5.2 has K solutions (where

K is the degree of ψ) corresponding to a unique degree zero polynomial (c is a constant polynomial in the case of two sites). This problem can be approached by solving an eigenvalue problem of the form: let D be the differential operator $D(\psi) = (z^2-1)\psi''(z) - ((\lambda_0+\lambda_1)z+\lambda_0-\lambda_1)\psi'(z)$. The problem is to find eigenvectors ψ and the corresponding eigenvalues such that $D(\psi) = -c\psi$.

Remark: To compute the relations between α, β and λ_0, λ_1, we use the negative of equation 5.1.

Proposition 5.2.1. *In equations 5.1 and 5.2, the parameters* α,β *and* λ_0,λ_1 *are such that* $\lambda_0 = -\beta-1$ *and* $\lambda_0 = -\alpha-1$

Proof: Equating the coefficients of $\psi'(z)$ from both equations gives the following system of equations:

$$-\beta+\alpha = \lambda_0-\lambda_1$$

$$\alpha+\beta+2 = \lambda_0-\lambda_1$$

$$\implies \lambda_0 = -\beta-1, \lambda_1 = -\alpha-1$$

□

Equation 5.1 can be further classified in the theory of orthogonal polynomials. Let $\alpha = \beta = -\frac{1}{2}$:

$$(z^2-1)\psi''(z)+z\psi'(z)+K^2\psi(z) = 0 \tag{5.3}$$

Solutions to equation 5.3 form a subclass of Jacobi polynomials and are known as Chebyshev polynomials, [1].

Chebyshev polynomials are obtained from the following recurrence relations, [9]:

$$T_{k+1} = 2zT_k(z) - T_{k-1}(z) \tag{5.4}$$

and

$$T_k(z) = cos(kcos^{-1}(z)),$$

which puts roots of the Chebyshev polynomial inside the interval of orthogonality, [9]. This fact can be proven by using recurrence formula 5.4.

5.3 Numerical solutions

A Matlab code that computes roots of the Chebyshev polynomial using recurrence relation 5.4 is attached in Appendix D (ChebyRoots.m). Figure 5.1 shows distribution of roots of the Chebyshev polynomial of degree $K = 40$, which is a solution to equation 5.3.

We use roots of the Chebyshev polynomials for numerical computation of roots of general Jacobi polynomials. Roots of general Jacobi polynomials for different values of parameters α and β can be obtained by iterating roots of the Chebyshev polynomial using Newton's method. It turns out that roots of the Chebyshev polynomial are close to roots of the Jacobi polynomial for α and β relatively close to $-\frac{1}{2}$. A Matlab algorithm to compute zeros of Jacobi polynomials in the interval $[-1, 1]$ can be found in Appendix D.

The Heine-Stieltjes theorem is demonstrated in figure 5.4. The Matlab algorithm that generates figure 5.4 can be found in Appendix D.

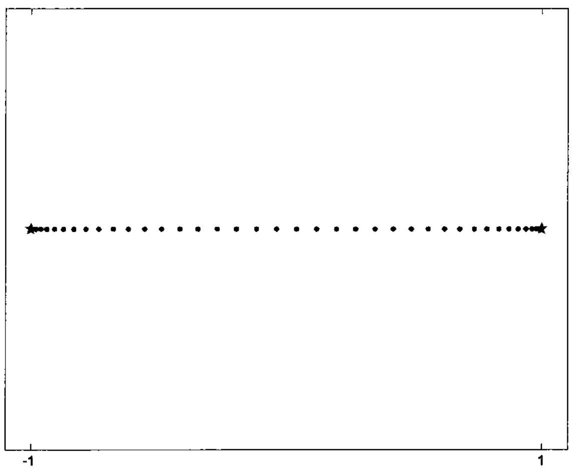

Figure 5.1: An $\mathfrak{sl}_2(\mathbb{C})$ equillibrium configuration with two sites $\alpha_0 = -1, \alpha_1 = 1$. The sites are represented by stars. Roots of the Chebyshev polynomial of degree K=40 are represented by dots.

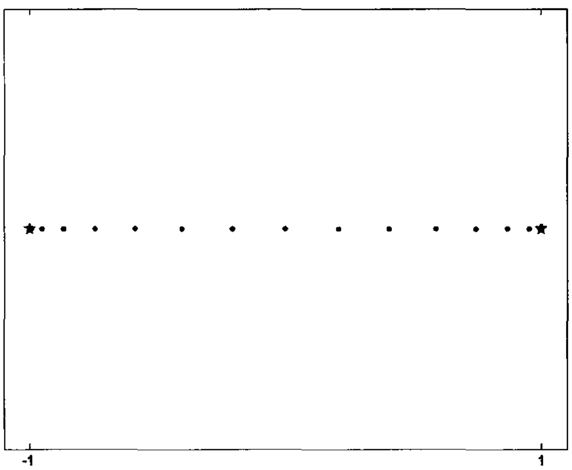

Figure 5.2: An equillibrium configuration for $\mathfrak{sl}_2(\mathbb{C})$ with two sites $\alpha_0 = -1, \alpha_1 = 1$. The sites are represented by stars. Roots of the Chebyshev polynomial of degree K=15 are represented by dots.

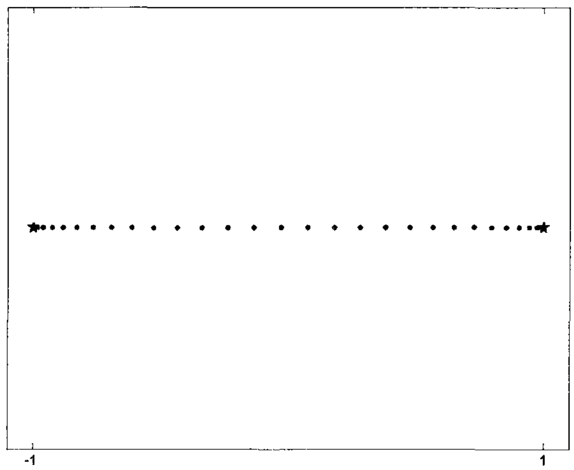

Figure 5.3: An $\mathfrak{sl}_2(\mathbb{C})$ equillibrium configuration with two sites $\alpha_0 = -1, \alpha_1 = 1$. The sites are represented by stars. Roots of the Jacobi polynomial of degree $K = 29$ for $\alpha = -.2$ and $\beta = 1$ are represented by dots.

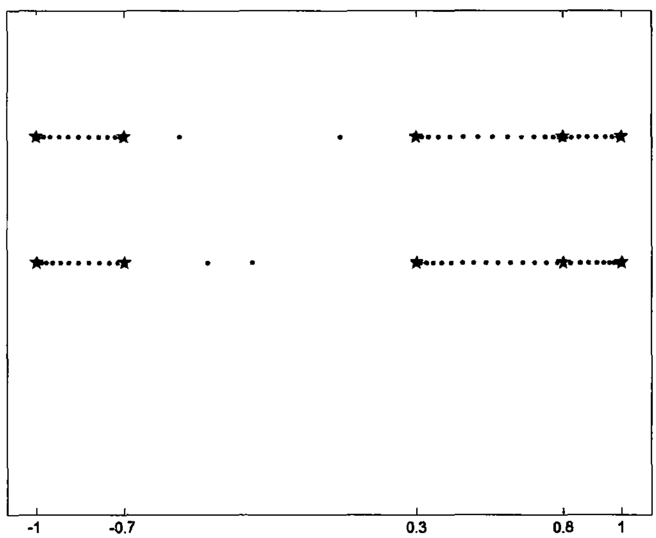

Figure 5.4: An equilibrium configuration for $\mathfrak{sl}_2(\mathbb{C})$ with sites $\alpha_0 = -1, \alpha_1 = -.7, \alpha_2 = .3, \alpha_3 = .8, \alpha_4 = 1$ with weights $\lambda_0 = -.5, \lambda_1 = -1, \lambda_2 = -1.3, \lambda_3 = -.5, \lambda_4 = -.8$. The sites are represented by stars. The $K = 40$ Bethe parameters are represented by dots.

Chapter 6

Experimental numerical results

In this chater we provide numerical evidence for obtaining approximations of solutions to the Lamé equation of more than two real sites using numerical approximations of roots of Jacobi polynomials. We also make observations and provide numerical evidence for obtaining solutions to other equations, which are similar to the Bethe-Ansatz equations for $\mathfrak{sl}_2(\mathbb{C})$.

6.1 Two site solution of the Bethe-Ansatz equations converging to a multisite $\mathfrak{sl}_2(\mathbb{C})$ equilibrium configuration

Using numerical approximations to roots of the Jacobi polynomial $\{P_K^{(\alpha,\beta)}\}$ as initial values for Newton's method, we approximate solutions to the Lamé equation with more than two real sites $(N > 1)$. We write the Bethe-Ansatz equations 3.16 as:

$$f_k(v) = \sum_{n=0}^{N} \frac{\lambda_n}{v_k - \alpha_n} - \sum_{k \neq l} \frac{2}{v_k - v_l} = 0, k = 1, \cdots K$$

where K is the degree of $\psi(z)$ and $v \in (-1,1)$.

$f(v) = (f_1(v), f_2(v), \cdots, f_K(v))^T = \nabla S(v) : \mathbb{R}^k \to \mathbb{R}^k$ where S is the logarithmic potential function, (ref. to 4.5). We are interested in the unique equilibrium position $v^* \in (-1,1)$ of the given particles $v \in (-1,1)$. We use Newton's iteration $v_{m+1} = g(v_m)$ where $g(v) = v - (f'(v))^{-1} f(v)$. $f'(v) = H(v)$ is the Hessian of S. In the proof of the Heine-Stieltjes Theorem, we see that $f'(v)$ is a positive definite KxK matrix and thus, it is invertible at any $v \in (-1,1)$. At the equilibrium $g'(v^*) = 0$. Therefore g is a contraction in some neighborhood of v^*. The convergence rate for Newton's method is quadratic if the initial position of the points is close to their equilibrium position. It turns out that roots of the Jacobi polynomial are very good initial values for Newton's method. In the case of the degree of ψ equal to 40, our algorithm (Demosl2J.m, ref. Appendix D) produces figure 6.1.

We observe that after applying Newton's iteration to roots of the Jacobi Polynomial, they will stay in the same configuration at their equilibrium position (provided that roots of the Jacobi polynomial do not coincide with the sites α_n). Numerical evidence suggests that an $\mathfrak{sl}_2(\mathbb{C})$ equilibrium configuration with $N = 1$ (two sites) is a good approximation to the electrostatic equilibrium configuration of particles where $N > 1$. We observe, that any $\mathfrak{sl}_2(\mathbb{C})$ configuration with two sites will converge to a unique $\mathfrak{sl}_2(\mathbb{C})$ equilibrium configuration of more than two sites. Below is a table of absolute differences between roots of the Jacobi polynomial in the interval [-1,1] having weights $\lambda_0 = -2, \lambda_1 = -.5$ and roots of the polynomial that is a solution to the Lamé equation 4.1 in the multi-site setting with $\alpha_0 = -1, \alpha_1 = -0.2, \alpha_2 = 0.1 \alpha_3 = 0.7, \alpha_4 = 1$ having weights $\lambda_0 = -2, \lambda_1 = -.7, \lambda_2 = -.9, \lambda_3 = -.4, \lambda_4 = -.5$. Table 6.2 corresponds to figure 6.1.

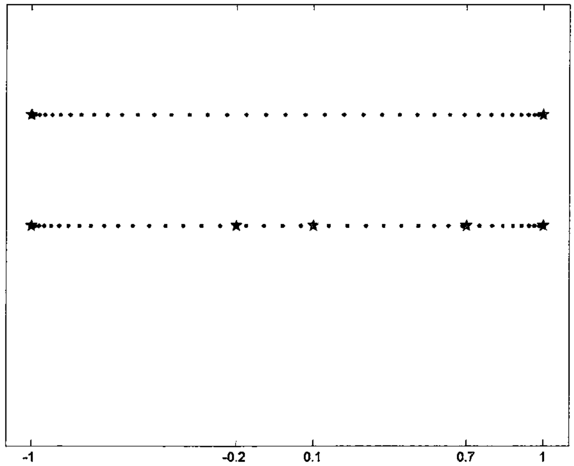

Figure 6.1: The upper configuration is an equillibrium configuration with two sites $\alpha_0 = -1, \alpha_1 = 1$ with weights $\lambda_0 = -2, \lambda_1 = -.5$ computed using roots of the Jacobi polynomial in the interval $[-1, 1]$. The sites are represented by stars. Roots of the Jacobi polynomial of degree $K = 40$ are represented by dots. The lower configuration is an equillibrium configuration with five sites: $\alpha_0 = -1, \alpha_1 = -0.2, \alpha_2 = 0.1\alpha_3 = 0.7, \alpha_4 = 1$ with weights $\lambda_0 = -2, \lambda_1 = -.7, \lambda_2 = -.9, \lambda_3 = -.4, \lambda_4 = -.5$. The sites are represented by stars and roots are represented by dots.

Abs(initial-new)	Abs(initial-new)	Abs(initial-new)
0.0002	0.0235	0.0030
0.0007	0.0251	0.0082
0.0014	0.0260	0.0049
0.0024	0.0007	0.0049
0.0036	0.0061	0.0043
0.0051	0.0110	0.0035
0.0067	0.0164	0.0026
0.0085	0.0163	0.0018
0.0105	0.0138	0.0011
0.0126	0.0112	0.0006
0.0148	0.0086	0.0002
0.0171	0.0059	0.0000
0.0193	0.0031	
0.0215	0.0003	

Figure 6.2: Absolute differences between roots of the Jacobi polynomial in the interval [-1,1] with weights $\lambda_0 = -2, \lambda_1 = -.5$ and roots of the polynomial that is a solution to the Lamé equation 4.1 in the multisite setting with $\alpha_0 = -1, \alpha_1 = -0.2, \alpha_2 = 0.1\alpha_3 = 0.7, \alpha_4 = 1$ having weights $\lambda_0 = -2, \lambda_1 = -.7, \lambda_2 = -.9, \lambda_3 = -.4, \lambda_4 = -.5$

6.2 Roots of Jacobi Polynomials converging to solutions of other equations

Numerical evidence suggests that for a positive number r, equations

$$\sum_{n=0}^{1} \frac{\lambda_0}{v_k^{(1)} - \alpha_n} - \sum_{l \neq k}^{K_1} \frac{2}{v_k^{(1)} - v_l^{(1)}} - \sum_{l=1}^{K_2} \frac{r}{v_k^{(1)} - v_l^{(2)}} = 0, \tag{6.1}$$

$k = 1, \cdots, K_1$

$$\sum_{n=0}^{1} \frac{\lambda_0}{v_k^{(2)} - \alpha_n} - \sum_{l \neq k}^{K_2} \frac{2}{v_k^{(2)} - v_l^{(2)}} - \sum_{l=1}^{K_1} \frac{r}{v_k^{(2)} - v_l^{(1)}} = 0, \tag{6.2}$$

$k = 1, \cdots, K_2$,
with $\lambda_n < 0$, have a solution in the interval (α_0, α_1).

Given two distinct $\mathfrak{sl}_2(\mathbb{C})$ equilibrium configurations $\{v^{*(1)}\}, \{v^{*(2)}\}$, we use Newton's method to iterate equations 6.1 and 6.1.

We iterate these equations such that for each equation we use points obtained from iterating the other equation. We iterate equations 6.1 with respect to the first kind of points and equations 6.2 with respect to the second kind of points while keeping the other kind of points fixed. This is the same as treating one kind of points as sites with weight r. This procedure is repeated until desired accuracy is obtained. We are able to obtain solutions to equations 6.1 and 6.1 for $K \leq 10$. A Matlab algorithm using this procedure is attached in Appendix D (DemoRsl2.m).

Figures 6.3 and 6.4 show distributions of points before and after applying Newton's method where $r = 10$ and $r = 3$, with sites $\alpha_1 = -1, \alpha_2 = 1$. $v_k^{(1)}$ is represented by dots, and $v_k^{(2)}$ is represented by crosses.

Solutions to equations 6.1 and 6.2 can be thought of as points at an equilib-

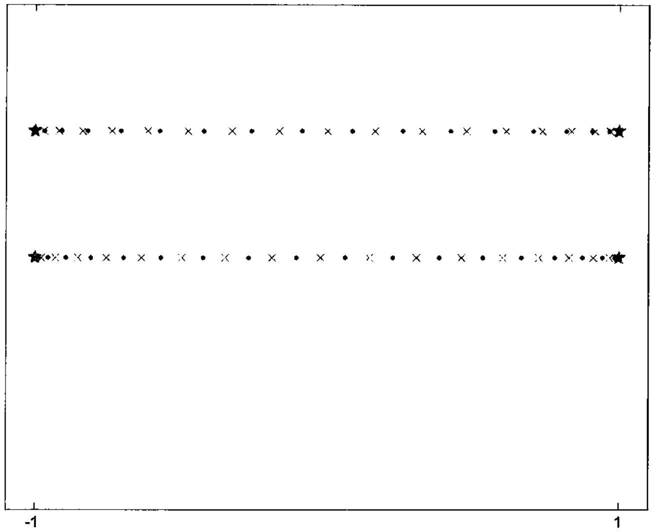

Figure 6.3: Distribution of points before and after applying Newton's method to equations 6.1 and 6.1 where $r = 3$ and $\alpha_1 = -1, \alpha_2 = 1, \lambda_0 = -.7, \lambda_1 = -2$. $v_k^{(1)}$ is represented by dots, and $v_k^{(2)}$ is represented by crosses. $K_1 = 18$ and $K_2 = 19$.

rium of the electrostatic potential as explained in Chapter 4 with the additional requirement that each two particles of different kind repel each other with force proportional to r.

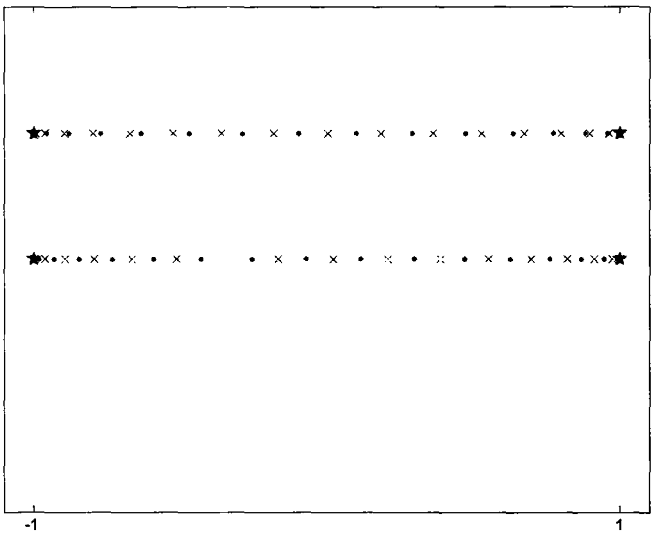

Figure 6.4: Distribution of points before and after applying Newton's method to equations 6.1 and 6.1 where $r = 10$ and $\alpha_1 = -1, \alpha_2 = 1, \lambda_0 = -3.5, \lambda_1 = -.8$. $v_k^{(1)}$ is represented by dots, and $v_k^{(2)}$ is represented by crosses. $K_1 = 16$ and $K_2 = 17$.

6.3 Two $\mathfrak{sl}_2(\mathbb{C})$ equilibrium configurations converging to a single $\mathfrak{sl}_2(\mathbb{C})$ configuration

Let $r = 2$ in equations 6.1 and 6.1. Iterating these equations separately, with the initial values as explained in the previous section, gives a solution to the Bethe-Ansatz equations:

$$\sum_{n=0}^{N} \frac{\lambda_n}{v_k - \alpha_n} - \sum_{l \neq k}^{K} \frac{2}{v_k - v_l} = 0$$

where $K = K_1 + K_2$

We conclude that given two distinct $\mathfrak{sl}_2(\mathbb{C})$ equilibrium configurations we obtain a single $\mathfrak{sl}_2(\mathbb{C})$ equilibrium configuration by slightly shifting the points. That is, given two Lamé equations:

$$a(z)\Psi_1''(z)+b(z)\Psi_1'(z)+c(z)\Psi_1(z)=0 \tag{6.3}$$

$$a(z)\Psi_2''(z)+b(z)\Psi_2'(z)+c(z)\Psi_2(z)=0 \tag{6.4}$$

with $deg(\Psi_1)\neq deg(\Psi_2)$, the product of polynomials $\Psi_1\Psi_2$ is a good approximation to a solution of the Lamé equation

$$a(z)\Psi''(z)+b(z)\Psi'(z)+c(z)\Psi(z)=0 \tag{6.5}$$

where

$$deg(\Psi)=deg(\Psi_1)+deg(\Psi_2).$$

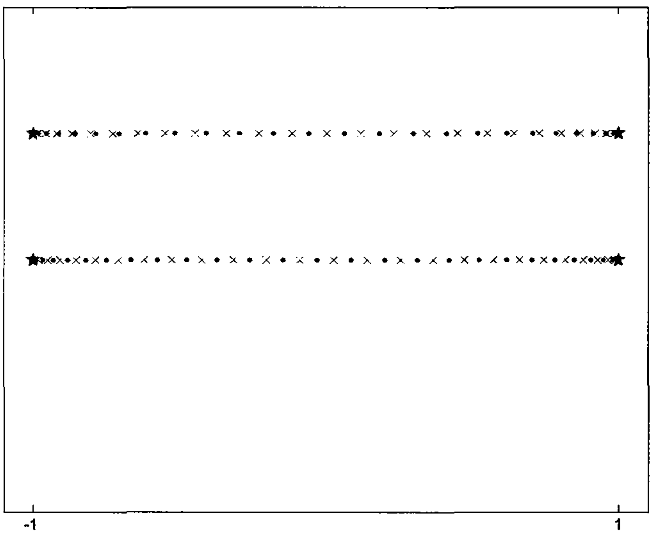

Figure 6.5: The first configuration illustrates two distinct $\mathfrak{sl}_2(\mathbb{C})$ configurations; the first one is represented by dots, the second one is represented by crosses. $K_1 = 26$ and $K_2 = 27$, $\alpha_0 = -1, \alpha_1 = 1$, and $\lambda_0 = -2.5, \lambda_1 = -.7$. The second configuration is a solution to equations 6.1 and 6.2 with the parameter $r = 2$. In this case, solutions to equations 6.1 and 6.2 converge to a single $\mathfrak{sl}_2(\mathbb{C})$ equilibrium configuration.

Chapter 7

The Lamé equation with Complex parameters

Substantial amounts of research have considered the case of real number parameters of the Lamé equation 4.1. In this chapter, we consider the Lamé equation with complex parameters having three complex sites. The Lamé equation is invariant under affine transformations, hence two of its sites can be assumed to be fixed on the real line. This allows us to convert the Lamé equation into an eigenvalue problem, for which, we derive a linear operator matrix. Finally, we discuss recent research development in this direction and provide numerical evidence to each considered variant.

7.1 Solutions to the Lamé equation with three complex sites

Let $a \in \mathbb{C}$. In the Lamé equation, 4.1, let

$$a(z) = z(z-1)(z-a) = z^3 - (a+1)z^2 + az.$$

Then

$$b(z) = -(z-1)(z-a)\lambda_0 - z(z-a)\lambda_1 - z(z-1)\lambda_2$$

and

$$c(z) = c_1 z + c_0$$

for some $c_1, c_0 \in \mathbb{C}$.

For a given K, a solution to the Lamé equation is a polynomial of the form:

$$\Psi(z) = z^K + \cdots p_1 z + p_0.$$

Then,

$$\begin{aligned} \Psi'(z) &= Kz^{K-1} + p_{K-1}(K-1)z^{K-2} + \cdots + 2p_2 z + p_1 \\ \Psi''(z) &= K(K-1)z^{K-2} + (K-1)(K-2)p_{K-1}z^{K-3} + \cdots + 2p_2. \end{aligned}$$

Recall from Chapter 4 that zeros of polynomials Ψ and $c(z)$ lie inside the convex hull of zeros of $a(z)$.

Compute:

$$\begin{aligned} a(z)\Psi''(z) &= (z^3-(a+1)z^2+az)(K(K-1)z^{K-2}+\cdots+2p_2) \\ &= K(K-1)z^{K+1}-(a+1)K(K-1)z^K+aK(K-1)z^{K-1} \\ &+ \ldots+2p_2\left(z^3-(a+1)z^2+az\right) \end{aligned}$$

$$\begin{aligned} b(z)\Psi'(z) &= -((z-1)(z-a)\lambda_0+z(z-a)\lambda_1 \\ &+ z(z-1)\lambda_2)(Kz^{K-1}+p_{K-1}(K-1)z^{K-1}+\cdots+2p_2z+p_1) \\ &= -K(\lambda_0+\lambda_1+\lambda_2)z^{K+1}+\ldots+-p_1(\lambda_0+\lambda_1+\lambda_2)z^2 \end{aligned}$$

$$\begin{aligned} c(z)\Psi(z) &= (c_1z+c_0)\left(z^K+p_{K-1}z^{K-1}+\cdots+p_1z+p_0\right) \\ &= c_1z^{K+1}+c_0z^K+\cdots+c_1p_0z+c_0p_0 \end{aligned}$$

We rewrite the Lamé equation:

$$a(z)\Psi''(z)+b(z)\Psi'(z)+(c_1z+c_0)\Psi(z) \tag{7.1}$$

as

$$a(z)\Psi''(z)+b(z)\Psi'(z)+c_1z\Psi(z)=-c_0\Psi(z), \tag{7.2}$$

which is an eigenvalue problem.

Since the z^{K+1} term does not occur on the R.H.S of this equation, we make the highest power coefficients equal to zero and obtain an expression for c_1:

$$c_1 = -K^2 + K(1+\lambda_0+\lambda_1+\lambda_2)$$

Thus,

$$a(z)\Psi''(z) + b(z)\Psi'(z) + (-K^2 + K(1+\lambda_0+\lambda_1+\lambda_2)z\Psi(z) = -c_0\Psi(z). \quad (7.3)$$

Let

$$L\Psi = -c_0\Psi.$$

We have:

$$z^k \mapsto a(z)(z^k)'' = (z^3-(1+a)z^2+az)k(k-1)z^{k-2}$$

$$= k(k-1)z^{k+1} - (1+a)k(k-1)z^k + ak(k-1)z^{k-1}$$

$$z^k \mapsto b(z)(z^k)' = -k(\lambda_0+\lambda_1\lambda_2)z^{k+1} + ((a+1)\lambda_0 + a\lambda_1+\lambda_2)kz^k - a\lambda_0 kz^{k-1}$$

$$z^k \mapsto c_1(z)z^{k+1}$$

Collecting coefficients of the same power of z gives:

$$\left(k^2 - K^2 - (k-K)(1+\lambda_0+\lambda_1+\lambda_2)\right)z^{k+1}$$

$$\left(-(1+a)k(k-1) + k((a+1)\lambda_0 + a\lambda_1+\lambda_2)\right)z^k$$

$$\left(ak(k-1) - a\lambda_0 k\right)z^{k-1}$$

L is a linear operator acting on polynomials of degree K. We now express L as a $K+1$x$K+1$ matrix with respect to basis $\{z^K, z^{K-1}, \ldots, 1\}$:

$$L = \begin{pmatrix} D_K & D_K^+ & \dots & 0 \\ D_K^- & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 \\ \vdots & \vdots & 0 & 0 \end{pmatrix}$$

where

$$D_K^+ = k^2 - K^2 - (k-K)(1+\lambda_0+\lambda_1+\lambda_2)$$

$$D_K = -(1+a)k(k-1) + k((a+1)\lambda_0 + a\lambda_1 + \lambda_2)$$

$$D_K^- = ak(k-1) - a\lambda_0 k$$

$$k = 1, \dots, K$$

and Ψ is a Kx1 vector:

$$\Psi = \begin{pmatrix} w_K \\ w_{K-1} \\ \vdots \\ w_1 \\ w_0 \end{pmatrix}$$

For each solution pair $(\Psi, c(z))$, the column vector Ψ with entries w_k is an eigenvector of L with eigenvalue $-c_0$ and corresponds to coefficients of the Heine-Stieltjes polynomial.

The Matlab algorithm which was used to solve the eigenvalue problem of the linear operator L can be found in Appendix D (Demosl2Complex.m). Figures 7.1 are 7.2 are outputs of the program with different parameters. Figure 7.3 shows one pair of solutions to the Bethe-Ansatz equations and figure 7.7 shows zeros of the Van Vleck polynomials.

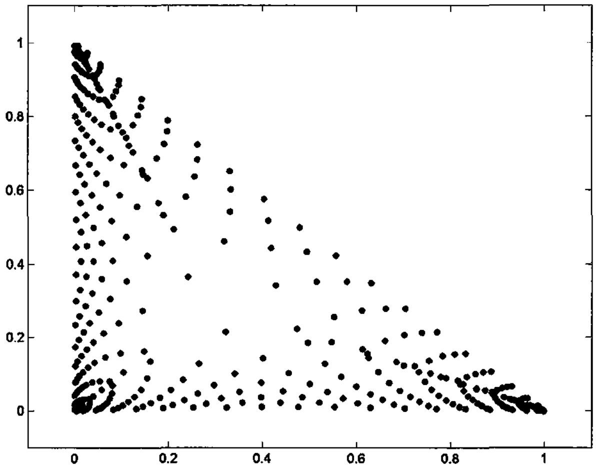

Figure 7.1: Solutions of the Bethe-Ansatz equations where $\alpha_0 = 0, \alpha_1 = i, \alpha_2 = 1$ with $\lambda_0 = -1, \lambda_1 = -0.5, \lambda_2 = -2$. The dots are roots of the Heine-Stieltjes polynomials of degree 20.

An electrostatic interpretation still applies in the complex case, however the complex case significantly differs from the real case, [10]. If the matrix L has $K+1$ distinct eigenvalues then there are $K+1$ distinct degree K polynomial solutions. In fact, distinct eigenvalues occur for generic choices of the parameter a in the linear operator L. The proof of the Heine-Stieltjes theorem is not valid in the complex case since the convexity argument does not hold.

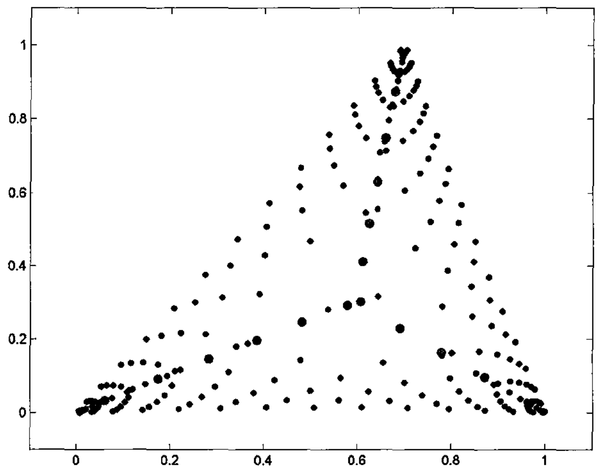

Figure 7.2: Solutions of the Bethe-Ansatz equations where $\alpha_0 = 0, \alpha_1 = .7 + i, \alpha_2 = 1$ with weights $\lambda_0 = -1.1, \lambda_1 = -.7, \lambda_2 = -2$. Smaller dots are roots of the Heine-Stieltjes polynomial of degree 15 and larger dots are roots of the corresponding Van Vleck polynomial.

7.2 Location of zeros of complex Van Vleck polynomials

In their recent paper, A. McMillen, A. Bourget and A. Agnew [12] investigated the complex case of the Lamé equation. They observed that roots of the Van Vleck polynomials have a deeper structure in the case where $\{\alpha_n\}_{n=0}^{N}$ are vertices of an equilateral triangle. Since the Lamé equation is invariant under complex affine transformations, we may assume that $\{\alpha_n\}_{n=0}^{N}$are the third roots of unity:

$$\alpha_{n+1} = e^{\frac{2\pi i}{3}n}$$

where $n = 0, 1, 2$.

Let the charges $\lambda = \lambda_1 = \lambda_2 = \lambda_3$ be negative and equal for all three sites. Un-

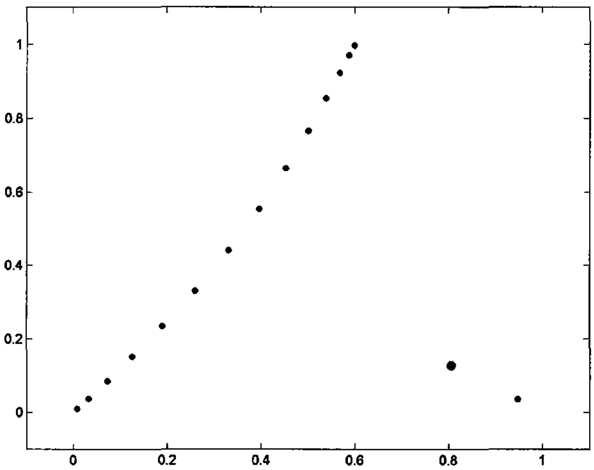

Figure 7.3: One pair of solutions of the Bethe-Ansatz equations where $\alpha_0 = 0, \alpha_1 = 0.6 + i, \alpha_2 = 1$ with weights $\lambda_0 = -1.5, \lambda_1 = -2, \lambda_2 = -.7$. Smaller dots are roots of the Heine-Stieltjes polynomials of degree 15 and larger dots correspond to a root of the Van Vleck polynomials.

der these assumptions, we obtain the following special case of the Lamé equation:

$$(z^3 - 1)\Psi''(z) + 6\lambda z^2\Psi'(z) = \mu(z - c)\Psi(z) \tag{7.4}$$

For each $k \in \mathbb{N}$, denote by $N(k)$ the number of roots of all distinct Van Vleck polynomials $c(z) = \mu(z - c)$ corresponding to the Heine-Stieltjes polynomials of degree k.

Theorem 7.2.1. *[12] Let $\alpha_1, \alpha_2, \alpha_3$ and $\lambda_1, \lambda_2, \lambda_3$ be as above. For any $k \in \mathbb{N}$, let $k+1 = 3p+q, q = 0, 1$ or 2. Then there exists p real, distinct $r \in (0,1)$, such that $3p$ of the zeros of the accociated Van Vleck polynomials are nonzero and have the form:*

$$re^{\frac{2\pi i}{3}} for j = 0, 1, 2$$

and the remaining zeros of the Van Vleck polynomials are zero.

Thus:

(i) If $q=0$ or $q=1$, then $N(k)=k+1$.

(ii) if $q=2$, then $N(k)=k$.

Therefore, Van Vleck zeros lie on the lines that are angle bisectors of the equilateral triangle with vertices $\alpha_1, \alpha_2, \alpha_3$. Figures 7.4, 7.5, 7.6 illustrate cases (i) and (ii) (for $q=0,1,2$ respectively) of theorem 7.2.1, where the equilateral triangle has been moved by an affine transformation to vertices 0 and 1 and $0.5+\frac{\sqrt{3}}{2}i$.

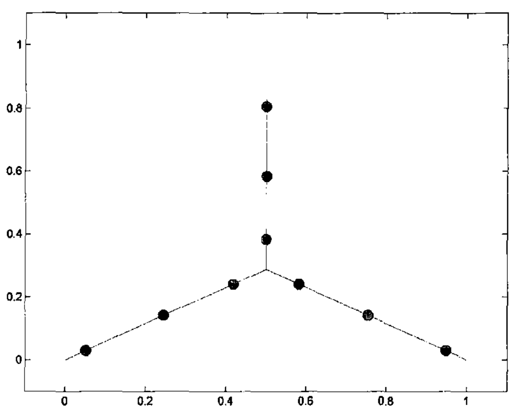

Figure 7.4: Illustration of theorem 7.2.1 where $q=0$, $K=8$. The zeros of Van Vleck polynomials distribute over the bisectrices of the equilateral triangle with vertices $\alpha_1=0, \alpha_2=0.5+\frac{\sqrt{3}}{2}i, \alpha_3=1$.

As a consequence of Theorem 7.2.1, we have the following corollary:

Corollary 7.2.1. *With $\alpha_1, \alpha_2, \alpha_3$ and $\lambda_1, \lambda_2, \lambda_3$ as above, the Lamé equation 7.4 has exactly $k+1$ polynomial solutions of degree k if $k \equiv 0(mod3)$ or if $k \equiv 2(mod3)$, but has only k polynomial solutions if $k \equiv 1(mod3)$.*

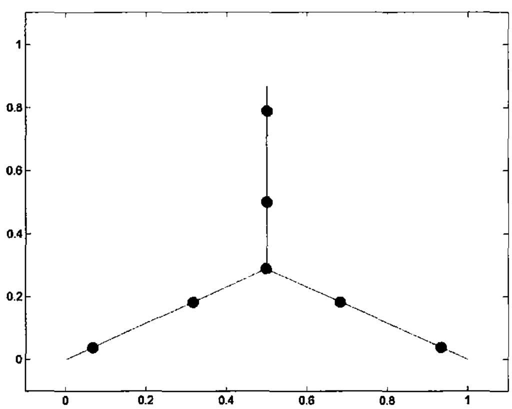

Figure 7.5: Illustration of theorem 7.2.1 where $q = 1$, $K = 6$. The zeros of Van Vleck polynomials distribute over the bisectrices of the equilateral triangle with vertices $\alpha_1 = 0, \alpha_2 = 0.5 + \frac{\sqrt{3}}{2}i, \alpha_3 = 1$.

Theorem 7.2.1 does not hold in the case when the charges λ_n are not equal (figure 7.7) and when the triangle is not equilateral (figures 7.8 and 7.9).

A. McMillen, A. Bourget and A. Agnew further investigated this case of the Lamé equation and considered to vary λ_n from 0 to ∞. The $\lambda_n = 0$ case does not correspond the the Lamé equation but can be understood as the limiting case $\lambda_n \longrightarrow 0$ when the charges are reduced to zero. Even in this case, there exists an equilibrium configuration of free charges that are located inside the triangle formed by fixed charges $\lambda = 0$. This observation has led to the following corollary:

Corollary 7.2.2. *Let the hypotheses of Theorem 7.2.1 hold. Then, if* $\lambda = 0$ *and* $k \geq 2$*, three Van Vleck zeros lie on the vertices of the triangle and the remaining zeros lie inside the triangle. As* $\lambda \longrightarrow \infty$*, the zeros concentrate at the center of the triangle.*

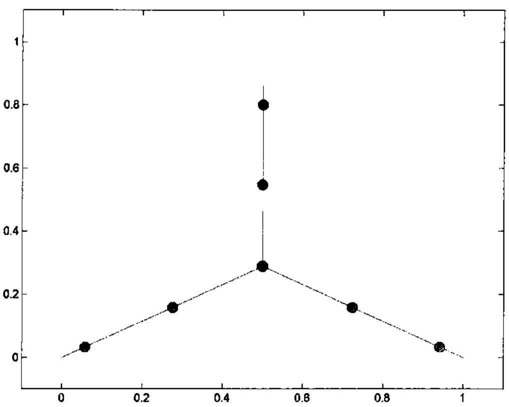

Figure 7.6: Illustration of theorem 7.2.1 where $q = 2$, $K = 7$. Zeros of the Van Vleck polynomials distribute over the bisectrices of the equilateral triangle with vertices $\alpha_1 = 0, \alpha_2 = 0.5 + \frac{\sqrt{3}}{2}i, \alpha_3 = 1$.

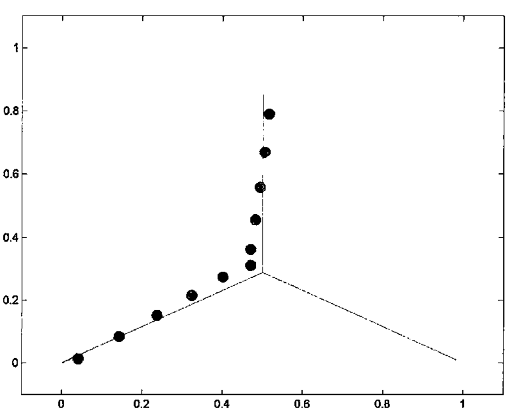

Figure 7.7: Illustrates the Van Vleck zeros where the sites are vertices of an equilateral triangle ($\alpha_1 = 0, \alpha_2 = .5 + \frac{\sqrt{3}}{2}i, \alpha_3 = 1$) with charges $\lambda_1 = -1.1, \lambda_2 = -15, \lambda_3 = -2$.

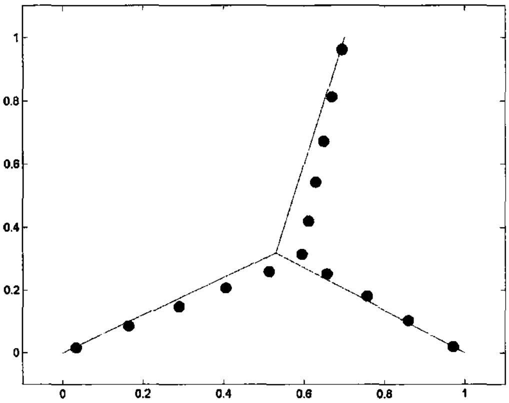

Figure 7.8: Illustrates the Van Vleck zeros where the sites have vertices $\alpha_1 = 0, \alpha_2 = .7+i, \alpha_3 = 1$) with charges $\lambda_1 = \lambda_2 = \lambda_3 = -.5$.

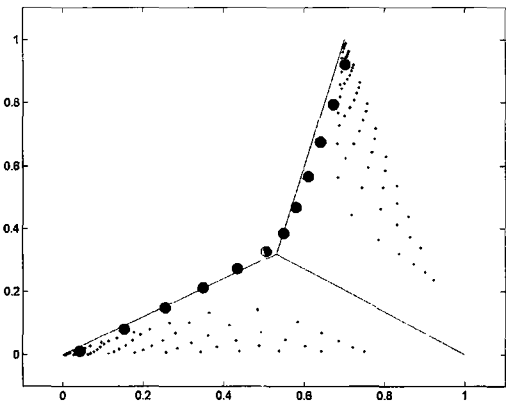

Figure 7.9: Illustrates the Van Vleck zeros where the sites have vertices $\alpha_1 = 0, \alpha_2 = .7+i, \alpha_3 = 1$) with charges $\lambda_0 = -1.1, \lambda_2 = -15, \lambda_3 = -2$

Chapter 8

Conclusions and Outlook

The Gaudin spin chain model can be associated to any semi-simple Lie algebra. In the first section of this chapter, we derive a solution set of the Bethe-Ansatz equations corresponding to the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. The last section contains final remarks and outlook.

8.1 Gaudin spin chain associated to $\mathfrak{sl}_3(\mathbb{C})$

Letting $r = -1$ in equations 6.1 and 6.2 gives:

$$\sum_{n=0}^{N} \frac{\lambda_n}{v_k^{(1)} - \alpha_n} - \sum_{l \neq k}^{K_1} \frac{2}{v_k^{(1)} - v_l^{(1)}} + \sum_{l=1}^{K_2} \frac{1}{v_k^{(1)} - v_l^{(2)}} = 0 \tag{8.1}$$

$k = 1, \cdots, K_1$

$$\sum_{n=0}^{N} \frac{\lambda_n}{v_k^{(2)} - \alpha_n} - \sum_{l \neq k}^{K_2} \frac{2}{v_k^{(2)} - v_l^{(2)}} + \sum_{l=1}^{K_1} \frac{1}{v_k^{(2)} - v_l^{(1)}} = 0 \tag{8.2}$$

$k = 1, \cdots, K_2$

It turns out that equations 8.1 and 8.1 correspond to the Gaudin spin chain associated with the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. [13], [14], [15], [4]. Very little is known about solutions to equations 8.1 and 8.2. We provide a method which produces infinitely many solutions in the case when $N = 1$, $\alpha_0 = -1, \alpha_1 = 1$ with charges $\lambda_0 = \lambda_1 = \frac{1}{2}$. The points v_k are located symmetrically around zero with two of the points inside the interval $(-1, 1)$, and two of them outside the interval $(-1, 1)$. We denote by $x, -x \in (-1, 1)$, one kind of points, and by $y, -y$ the points outside the interval $(-1, 1)$. That is:

$$-y < -1 < -x < 0 < x < 1 < y.$$

Under these assumptions, equations 8.1 and 8.2 take the following form:

$$\begin{aligned} \frac{x}{x^2-1} + \frac{1}{x} - \frac{2x}{x^2-y^2} &= 0 \\ \frac{y}{y^2-1} + \frac{1}{y} - \frac{2y}{y^2-x^2} &= 0 \end{aligned} \tag{8.3}$$

Let

$$\varepsilon = 1 - x^2 > 0$$

and

$$\delta = y^2 - 1 > 0.$$

We conclude that if

$$\delta = \frac{\varepsilon}{1-2\varepsilon},$$

or equivalently

$$\varepsilon = \frac{\delta}{1+2\delta},$$

we get infinitely many solutions to equations 8.1 and 8.2.

For example: Let $\delta = \frac{1}{4}$, $\varepsilon = \frac{1}{6}$. Then

$$x \pm \sqrt{\frac{5}{6}}$$

and

$$y \pm \frac{\sqrt{5}}{2}. \tag{8.4}$$

Recall that in the $\mathfrak{sl}_2(\mathbb{C})$ case, solutions of the Bethe-Ansatz equations lie inside the convex hull of the $\{\alpha_n\}_{n=0}^{N}$. In the $\mathfrak{sl}_3(\mathbb{C})$ case, this assertion does not hold and 8.4 gives a counterexample.

Let

$$F = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

where

$$f_1 = \frac{x}{x^2-1} + \frac{1}{x} - \frac{2x}{x^2-y^2}$$

$$f_2 = \frac{y}{y^2-1} + \frac{1}{y} - \frac{2y}{y^2-x^2}$$

Since $F^{-1}(0,0)$ is a one parameter curve, dF is not invertible and the map F cannot be a diffeomorphism by the Inverse Function theorem. In particular, Newton's method cannot be used to solve the $\mathfrak{sl}_3(\mathbb{C})$ Bethe-Ansatz equations.

8.2 Conclusions and Outlook

The Gaudin spin chain model can be associated to any semi-simple complex Lie algebra; hence it presents many possibilities for future considerations. Even the case of finite irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$, associated to the Gaudin spin chain, is still unknown. Furthermore, asymptotic properties of the model can be studied, [2], [3]. Moreover, the problem of diagonalizing the Gaudin's Hamiltonians has an interpretation in algebraic geometry, [6]. As a further matter, similar techniques used for diagonalizing Hamiltonians of the Gaudin spin chain model, can be applied to other Heisenberg spin chains.

Appendix A

A.1 Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

Intuitively, a representation of a Lie algebra is a way of representing a Lie algebra in terms of matrices such that the Lie bracket is the commutator. These matrices are endomorphisms of some vector space.

Let $\mathfrak{g}$ be a Lie algebra.

Definition A.1.1. *A linear map $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a homomorphism of Lie algebras if $\phi([X,Y]) = [\phi(X), \phi(Y)]$, i.e. a linear map that preserves the bracket operation.*

Definition A.1.2. *A vector space V over a field F is a representation (or an $\mathfrak{g}$-module) if there exists a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.*

Definition A.1.3. *$W \subset V$ is a submodule of a $\mathfrak{g}$-module V if W a linear subspace of V such that $x \cdot w \in W$ for all $x \in \mathfrak{g}$.*

A g-module V is irreducible if there exists no proper submodule, that is no submodule other than 0 and $\mathfrak{g}$ itself.

We also state the well known theorem by Weyl:

Theorem A.1.1. *Every finite-dimensional representation of a complex semisimple Lie algebra is completely reducible.*

[5]

The vector space,

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} | a,b,c \in \mathbb{C} \right\}$$

together with the bracket operation $[A,B] = AB - BA$, is a semisimple Lie algebra.

We choose a basis of $\mathfrak{sl}_2(\mathbb{C})$ as follows:

$$\left\{ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

The basis elements satisfy the commutation relations:

$$[h,e] = 2e$$

$$[h,f] = -2f$$

$$[e,f] = h.$$

Let V is an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module. Then we have a decomposition of V as follows:

$$V = \oplus_{\mu \in \mathbb{C}} V_\mu$$

where $V_\mu = \{v \in V | hv = \mu v\}$. If $V_\mu \neq 0$ we call μ a weight and V_μ a weight space. Let $v \in V_\mu$. The action of elements e and f on the weight spaces is given by:

$$ev \in V_{\mu+2}$$

$$fv \in V_{\mu-2}.$$

If $V_{\lambda+2} = 0$ then λ is called the highest weight and any non-zero vector in V_λ is called vacuum and is denoted by $|0>$ with

$$e|0>=0$$

and

$$h|0>=\lambda|0>$$

V_λ is called the highest weight module.

The highest weights λ of $\mathfrak{sl}_2(\mathbb{C})$ can be identified with complex numbers, however the following theorem gives further classification of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

Theorem A.1.2. *[5], [8]*
(a) V_λ is isomorphic to V.
(b) If λ has a positive integral value then it corresponds to an finite dimensional irreducible representation and other values of λ give rise to infinite dimensional irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

A.2 Loop algebra

Definition A.2.1. *[6] Given a Lie algebra $\mathfrak{g}$. The loop algebra of $\mathfrak{g}$ is defined as the tensor product of $\mathfrak{g}$ with $\mathbb{C}[u,u^{-1}]$, the Laurent polynomials in the variable u over $\mathbb{C}$ is an infinite-dimensional Lie algebra with the Lie bracket given by*

$$[g_1 \otimes f_1, g_2 \otimes f_2] = [g_1, g_2] \otimes f_1 f_2$$

where g_1 and g_2 belong to $\mathfrak{g}$ and f_1 and f_2 are elements of $\mathbb{C}[u,u^{-1}]$. It is called a loop algebra because it can be thought of as a smooth parameterized loop in $\mathfrak{g}$.

A.3 The Casimir operator for semisimple Lie algebras

Let $\mathfrak{g}$ be an n-dimensional, semisimple Lie algebra. Let $x, y \in \mathfrak{g}$.

Definition A.3.1. *The Killing form on $\mathfrak{g}$ is an invariant symmetric bilinear form given by*

$$\kappa(x,y) := tr(adx \circ ady)$$

The bilinearity of the Killing form follows from the the fact that ad is linear, the composition is bilinear and tr is linear. The Killing form is symmetric since $tr(ab) = tr(ba)$ for any linear maps a, b. The Killing form is invariant in the sense that it is associative:

$$\kappa([x,y],y) = \kappa(x,[y,z])$$

Let V be a faithful $\mathfrak{g}$-module. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Definition A.3.2. *[5] The Casimir element of the Lie algebra $\mathfrak{g}$ is the linear map $c : V \to V$ defined by*

$$t(v) = \sum_{i=1}^{m} \rho(x_i)\rho(y_i)$$

where x_i is any basis of $\mathfrak{g}$ and y_i is any dual basis of $\mathfrak{g}$ with respect to a fixed invariant bilinear form (e.g. the Killing form on $\mathfrak{g}$ satisfying $\kappa(x_i, y_j) = \delta_{ij}$).

A.4 The Casimir operator for $\mathfrak{sl}_2$

The basis for $\mathfrak{sl}_2$ that has been used throughout the thesis is:

$$\left\{ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

Using the given basis we observe that

$$\begin{aligned}\kappa(e,f) &= 0\\ \kappa(f,e) &= 0\\ \kappa(h,\frac{1}{2}h) &= 0\end{aligned}$$

Thus the Casimir operator for $\mathfrak{sl}_2$ is: $t_{\mathfrak{sl}_2} = ef + fe + \frac{1}{2}h^2$.

A.5 The Casimir operator for $\mathfrak{sl}_3$

The basis for $\mathfrak{sl}_3$ that has been used throughout the thesis is:

$$\begin{aligned}
h_1 &= \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, h_2 = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},\\
e_{12} &= \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, e_{13} = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, e_{23} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix},\\
e_{21} &= \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, e_{31} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, e_{32} = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}
\end{aligned}$$

We make the following observations:

$$
\begin{aligned}
\kappa(e_{12}, e_{21}) &= 0 \\
\kappa(e_{13}, e_{31}) &= 0 \\
\kappa(e_{23}, e_{32}) &= 0 \\
\kappa(e_{21}, e_{12}) &= 0 \\
\kappa(e_{31}, e_{13}) &= 0 \\
\kappa(e_{32}, e_{23}) &= 0 \\
\kappa(h_1, \frac{1}{3}h_2 + \frac{1}{3}h_1h_2) &= 0 \\
\kappa(h_2, \frac{1}{3}h_1 + \frac{1}{3}h_1h_2) &= 0
\end{aligned}
$$

and obtain the Casimir operator for $\mathfrak{sl}_3$:

$$
\begin{aligned}
t_{\mathfrak{sl}_3} &= e_{12}e_{21} + e_{13}e_{31} + e_{23}e_{32} \\
&+ e_{21}e_{12} + e_{31}e_{13} + e_{32}e_{23} \\
&+ \frac{2}{3}h_1^2 + \frac{2}{3}h_2^2 + \frac{2}{3}h_1h_2
\end{aligned}
$$

Appendix B

B.1 Product Rule for Lie brackets

Proposition B.1.1.

$$[a,bc] = [a,b]c + b[a,c]$$

Proof:

$$\begin{aligned} [a,b]c + b[a,c] &= abc - bac + bac - bca \\ &= [a,bc] \end{aligned}$$

□

Proposition B.1.2.

$$[ab,c] = [a,c]b + a[b,c]$$

Proof:

$$\begin{aligned} [a,c]b + a[b,c] &= acb - cab + abc - acb \\ &= [ab,c] \end{aligned}$$

□

Appendix C

C.1 Gershgorin circle theorem

Theorem C.1.1. *(Gershgorin's circle theorem) Let A be an nxn matrix, with complex entries a_{ij}. For $i \in \{1,n\}$ let $R_i = \sum_{j \neq i} |a_{ij}|$ where $|a_{ij}|$ is the complex norm of a_{ij}. The closed disc $D(a_{ii}, R_i)$ centered at a_{ii} with radius R_i is called a Gershgorin disc. Every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{ii}, R_i)$.*

Proof: Let η be an eigenvalue of A and let $x = [x_j]$ the corresponding eigenvector. Choose $i \in \{1,n\}$ such that $|x_i| = max_j |x_j|$. Then $|x_i| > 0$, otherwise $x = 0$. Since x is an eigenvector, $Ax = \eta x$ or equivalently:

$$\sum_j a_{ij} x_j = \eta x_i \quad \forall i \in 1 \ldots n$$

$$\sum_{j \neq i} a_{ij} x_j = \eta x_i - a_{ii} x_i$$

Choose i as above, we have $x_i \neq 0$. Divide both sides by x_i, and take the norm to obtain:

$$|\eta - a_{ii}| = \left| \frac{\sum_{j \neq i} a_{ij} x_j}{x_i} \right| \leq \sum_{j \neq i} |a_{ij}| = R_i.$$

The last inequality is valid because

$$\frac{x_j}{x_i} \leq 1 \quad \forall j \neq i$$

. ☐

C.2 Abel's Theorem

Theorem C.2.1. *(Abel's theorem) Given a homogeneous second-order ordinary differential equation:*

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \tag{C.1}$$

The Wrońskian of two linearly independent solutions to C.1 is given by

$$w(x) = ce^{-\int p(x)dx}$$

where c is any constant.

Proof: let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions to C.1 Then,

$$y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x) = 0$$

and

$$y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x) = 0.$$

We have,

$$0 = y_1(x)(y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x)) - y_2(x)(y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x))$$

$$0 = (y_1(x)y_2''(x) - y_2(x)y_1''(x)) + P(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x))$$
$$+Q(x)(y_1(x)y_2(x) - y_1(x)y_2(x))$$

$$0 = (y_1(x)y_2''(x) - y_2(x)y_1''(x)) + P(x)(y_1(x)y_2'(x) - y_1'(x)y_2(x)) \tag{C.2}$$

We compute the Wronskian:

$$w(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

$$w'(x) = (y_1'(x)y_2'(x) + y_1(x)y_2''(x)) - (y_1'(x)y_2'(x) + y_1''(x)y_2(x))$$
$$= y_1(x)y_2''(x) - y_1''(x)y_2(x)$$

Combining C.2 with w and w' gives:

$$w'(x) + P(x)w(x) = 0.$$

Integrate both sides:

$$\int \frac{dw(x)}{w(x)} = \int -P(x)dx$$

$$\ln\left|\frac{w(x)}{c}\right| = -\int P(x)d(x)$$

$$w(x) = ce^{-\int P(x)dx}$$

where c is a constant of integration. □

C.3 Gauss-Lucas Theorem

Lemma C.3.1. *Let $z \in \mathbb{C}$ and let $r_1, \cdots, r_M$ be positive real numbers. Suppose $w_1 \cdots w_M \in \mathbb{C}$ and $w_m \neq z$. The equality*

$$\frac{r_1}{z-w_1} + \cdots + \frac{r_M}{z-w_M} = 0 \tag{C.3}$$

only holds when z lies inside the convex hull $CH(w_1, \cdots, w_M)$ of $\{w_m\}, m = 1 \cdots M$

Proof: Suppose that z lies outside of $CH(w_1, \cdots, w_M)$. Let L be the line that separates z and $CH(w_1, \cdots, w_M)$. Since equation C.3 is invariant under affine transformations, we can choose a and b such that $aL + b$ is the x-axis, the points $aw_m + b$ have positive imaginary part and $az + b$ has negative imaginary part. Then the left hand side of equation C.3 has strictly negative imaginary part and therefore cannot be zero. □

Theorem C.3.1. *(Gauss-Lucas) Let $p(z) \in \mathbb{C}$. The zeros of $p'(z)$ lie inside the convex hull of the zeros of $p(z)$.*

Proof: Suppose $z_1, \cdots, z_M$ are zeros of $p(z)$. Assume that there exists z for which $p'(z) = 0$ and that lies outside $CH(z_1, \cdots, z_M)$. We have $p(z) \neq 0$ and

$$\frac{p'(z)}{p(z)} = \sum_{m=1}^{M} \frac{1}{z - z_m} = 0$$

This cannot be true by lemma C.3.1. □

Appendix D

Matlab code

```
\small{
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%ChebyRoots.m
%Computes the roots of the Chebychev polynomial
%in the interval [-1,1] for a give K
%%
%%a - vector of sites
%%w - vector of weights
%%K - number of points
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
K = 15;
a = [-1 1];
w = [-0.5, -0.5];
%find the roots of the Chebyshev polynomial in the interval [-1,1]
v = roots(ChebyshevPoly(K));

%improve with Newton iteration
```

```
v = newtonf(v, 1E-100, 20, a, w);

%plot(real(a), imag(a)-1,'rs',real(v), imag(v)-1,'b.');

plot(real(a), imag(a)-1,'rp','MarkerEdgeColor','r',
'MarkerFaceColor','r','MarkerSize',10);

set(gca, 'YTick', [])
xlim([-1.1 1.1])
ylim([-3 1])
set(gca, 'XTick', a)
set(gca, 'XTickLabel',{'-1','1'})
hold on

plot(real(v), imag(v)-1,'b.','MarkerSize',13);

%end of ChebyRoots.m


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%JacobiRoots.m
%Computes the roots of the Jacobi polynomial
%in the interval [-1,1] for a given K
%%
%%a - vector of sites
%%w - vector of weights
%%K - number of points
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

K = 29;
a = [-1 1];
%w = [0.6, 3];
w = [-0.8, -2];
%find the roots of the Chebyshev polynomial in the interval [-1,1]
v = roots(ChebyshevPoly(K));

%Find roots of the Jacobi Polynomial with Newton iteration
%using roots of the Chebyshev polynomial as initial values
v = newtonf(v, 1E-100, 20, a, w)

plot(real(a), imag(a)-1,'rp','MarkerEdgeColor','r',
'MarkerFaceColor','r','MarkerSize',10);

set(gca, 'YTick', [])
xlim([-1.1 1.1])
ylim([-3 1])
set(gca, 'XTick', a)
set(gca, 'XTickLabel',{'-1','1'})
hold on

plot(real(v), imag(v)-1,'b.','MarkerSize',13);

%end of JacobiRoots.m
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%DemoMultisites12.m
%Demonstrates the Heine-Stieltjes theorem.
%Fix \{ \alpha_n\}_{n=0}^{N} the charges \lambda_ns.
%Compute the equilibrium
%distribution at each interval
%(i.e. approx. to roots of the Jacobi
%polynomial after scaling each two sites to [-1,1]).
%Apply the Newtons
%method to all points v with all sites \{ \alpha_n\}_{n=0}^{N}.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%number of points
K = 40;

%number of sites
N = 5;
%number of intervals = N-1
%sites
a = [-1 -.7 .3 .8 1];

%number of points in each interval
k = [13 2 15 10];

w=[-.5 -1 -1.3 -.5 -.8];
%create the weight vector
% for n=2:N
```

```
%      w = [w,-0.5];
% end
% w;
%scale points and compute roots of the Chebyshev in each interval

for m=1:N-1
tempv = roots(ChebyshevPoly(k(m)));
sort(tempv);
    tempv = [-1 tempv' 1];
    tempv = sort(tempv);
    tempv = scaledata(tempv, a(m), a(m+1));
    for r=1:k(m)
        if r==1
            v2 = tempv(2);
        else
            v2 = [v2 tempv(r+1)];
        end
        end
     atemp = [a(m), a(m+1)]
      wtemp = [w(m), w(m+1)]
      v2=v2';
      %Get Jacobi roots in each interval
      v2 = newtonf(v2, 1E-100, 20, atemp, wtemp);
      if m==1
        v=v2';
      else
        v  = [v v2'];
```

```
      end

end

figure(1);
plot(real(a), imag(a),'rp','MarkerEdgeColor','r',
'MarkerFaceColor','r','MarkerSize',10);
set(gca, 'YTick', [])
xlim([-1.1 1.1])
ylim([-3 1])
set(gca, 'XTick', a)
hold on

 figure(1);
 plot(real(a), imag(a),'rp','MarkerEdgeColor','r',
 'MarkerFaceColor','r','MarkerSize',10);
 set(gca, 'YTick', [])
 xlim([-1.1 1.1])
 ylim([-3 1])
 set(gca, 'XTick', a)
 hold on
plot(real(v), imag(v),'b.','MarkerEdgeColor','b',
'MarkerFaceColor','b','MarkerSize',10);

v1 = v';
v1 = newtonf(v1, 1E-100, 100, a, w);
plot(real(a), imag(a)-1,'rp','MarkerEdgeColor','r',
```

```
'MarkerFaceColor','r','MarkerSize',10);
set(gca, 'YTick', [])
xlim([-1.1 1.1])
ylim([-3 1])
set(gca, 'XTick', a)
hold on
plot(real(v1), imag(v1)-1,'b.','MarkerEdgeColor','b',
'MarkerFaceColor','b','MarkerSize',10);

%evaluate the function at v1
v1 = baf(v1, a, w)

%end of DemoMultisitesl2.m

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%5
%%Demosl2J.m
%%Illustrates an sl2 equilibrium confuguration
%of points of two sites
%%converging to an sl2 multisite equillibrium configuration;
%%(solutions of the Bethe-Ansatz equations for multiple sites).
%%The differences between the
%initial and final position of points is also
%%computed.
%a - vector of sites
%%w - vector of weights
%%K - number of points
%%N -number of sites
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%

K = 40;
N = 5;
a=[-1 -0.2 .1 .7 1];
w=[-2 -.7 -.9 -.4 -.5];

v = roots(ChebyshevPoly(K));
v = sort(v');

a1 = [-1,1];
w1 = [-2, -.5];
v = v';
v = newtonf(v, 1E-100, 100, a1, w1);

figure(1);
plot(real(a1), imag(a1),'rp','MarkerEdgeColor','r',
'MarkerFaceColor','r','MarkerSize',10);
set(gca, 'YTick', [])
xlim([-1.1 1.1])
ylim([-3 1])
set(gca, 'XTick', a)
hold on

plot(real(v), imag(v),'b.','MarkerSize',10);
```

```
v1 = newtonf(v, 1E-100, 100, a, w);

plot(real(a), imag(a)-1,'rp','MarkerEdgeColor','r',
'MarkerFaceColor','r','MarkerSize',10);
plot(real(a), imag(a)-1,'rp',real(v1), imag(v1)-1,'b.',
'MarkerSize',10);

va = baf(v, a1, w1);
v1a = baf(v1, a, w);
abs(v-v1)

%end of Demosl2J.m



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%DemoRsl2.m
%%Finds solutions to the 'Bethe-Ansatz like' equations
%%a - vector of sites
%%w - vector of weights
%%K1 - number of points of the first kind
%%K2 - number of points of the second kind

%
% K1 = 16;
% K2 = 17;
% a = [-1 1];
% w = [-.5, -.5];
```

```
% nw = -10;
% r = -10;
K1 = 18;
K2 = 19;
a = [-1 1];
w = [-.5, -.5];
nw = -3;
r = -3;
w1 = [nw];
w2 = [nw];
for k=2:K1
    w1 = [w1,nw];
end
for k=2:K2
    w2 = [w2, nw];
end

v1 = roots(ChebyshevPoly(K1));
v2 = roots(ChebyshevPoly(K2));
v1 = sort(v1);
v2 = sort(v2);

figure(1);
plot(real(a), imag(a),'rp','MarkerEdgeColor','r',
'MarkerFaceColor','r','MarkerSize',10);
set(gca, 'YTick', [])
xlim([-1.1 1.1])
```

```
ylim([-3 1])
set(gca, 'XTick', a)
hold on
plot(real(v1), imag(v1),'b.','MarkerEdgeColor',
'b','MarkerFaceColor','b','MarkerSize',10);
plot(real(v2), imag(v2),'bx', 'MarkerSize',7);


v1new = v1;
v1_sites = append_sites(a,v1new);
v1_weights = append_weights(w,w1);
v2_weights = append_weights(w,w2);
v2new = v2;

maxiterations = 40;

 for k=1:maxiterations
     v2new = newtonf(v2new, 1E-100, 20, v1_sites, v1_weights);
     v2new = sort(v2new);
     v2_sites = append_sites(a,v2new);
     v1new = newtonf(v1new, 1E-100, 20, v2_sites, v2_weights);
     v1new = sort(v1new);
     v1_sites = append_sites(a,v1new);
 end

plot(real(a), imag(a)-1,'rp','MarkerEdgeColor','r'
,'MarkerFaceColor','r','MarkerSize',10);
```

```
hold on
plot(real(v1new), imag(v1new)-1,'b.'
,'MarkerEdgeColor','b','MarkerFaceColor','b','MarkerSize',10);
hold on
plot(real(v2new), imag(v2new)-1,'bx', 'MarkerSize',7);

v1r = rf(v1new,v2new, a, w,r)
v2r = rf(v2new,v1new, a, w,r)

%end of DemoRsl2.m

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%newtonf.m
%%function: [v,st]=newtonf(v_init, tol, maxiter,sites,weights)
%%Computes the next iterate of v_init using Newton's method
%%Input parameters:
%%v_init: vector with initial points
% for the Newton Iteration to be
%%performed
%%tol: tolerance value
%%maxiter: maximum number of iterations
%%sites: vector of sites
%%weights: vector of weights
%%Output:
%%v: vector of points after applying the Newton's Iteration
%%st: status of acuarracy
```

```
function [v,st]=newtonf(v_init, tol, maxiter,sites,weights)

if nargin <2
    tol = 1E-100;
    maxiter=20;
end

v=v_init;
v = v-Jbaf(v,sites,weights)\baf(v,sites,weights);
k=1;
while abs(v-v_init) >tol *abs(v_init) & k<maxiter
    v_init=v;
    v=v-Jbaf(v,sites,weights)\baf(v,sites,weights);
    k=k+1;
end
if abs(v-v_init) > tol*abs(v_init)
    st =0;
    st=1;
end

%end of function newtonf

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%Jbaf.m
%%function Jac=Jbaf(v_k,sites,weights) returns the Jacobian
%%of the Bethe-Ansatz equations at v_k
%%Input parameters:
```

```
%%v_k: vector of values at which the Jacobian is evaluated
%%sites: vector of sites
%%weights: vector of weights
%%Output:
%%Jac: Jacobian of the Bethe-Ansatz equations at v_k

function Jac=Jbaf(v_k,sites,weights)

L = length(v_k);
[X,Y]=meshgrid(v_k,v_k);

%off diagonal entries
Jac = -(X-Y).^2;
Jac = Jac+eye(L);
Jac = 1./Jac;
%diagonal entries
for k=1:L
     Jac(k,k) = sum(2./(v_k([1:k-1 k+1:L])
     -v_k(k)).^2)-(1./(sites-v_k(k)).^2)*weights';
end

%end of Jbaf



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%baf.m
%%function ba=baf(v_k,sites,weights)
```

```
%%Evaluates the Bethe-Ansatz equations at v_k
%%Input parameters:
%%v_k: vector of values at which the function is evaluated
%%sites: vector of sites
%%weights: vector of weights
%%Output:
%%ba: value of the function at v_k

function ba=baf(v_k,sites,weights)
L = length(v_k);
ba = zeros(size(v_k));
for k=1:L
    ba(k)=(1./(v_k(k) - sites))*weights'
    - sum(2./(v_k(k) - v_k([1:k-1 k+1:L])));
end

%end of function baf

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%ChebyshevPoly.m
%%function coef = ChebyshevPoly(n)
%%Given nonnegative integer n, the function computes the
%%Chebyshev polynomial T_n.
%%Output:
%%Returns the result as a column vector whose mth
%%element is the coefficient of x^(n+1-m).
```

```
function coef = ChebyshevPoly(n)

if n==0
    coef = 1;
elseif n==1
    coef = [1 0]';
else
 v  coef2 = zeros(n+1,1);
   coef2(n+1) = 1;
   coef1 = zeros(n+1,1);
   coef1(n) = 1;

    for k=2:1:n

        coef = zeros(n+1,1);

        for ind=n-k+1:2:n
            coef(ind) = 2*coefm1(ind+1) -coef2(ind);
        end

        if k<n
           coef2 =coef1;
           coef1 = coef;
        end

        if mod(k,2)==0
            coef(n+1) = (-1)^(k/2);
```

```
        end

    end

end

%end of function ChebyshevPoly.m

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%Demosl2Complex.m
%%function Demosl2Complex
%%Computes and plots solutions of the
%BA equations with three sites: 0,1,a
%%This algorithm plots roots of the
%H-S polynomials and roots of the Van Vleck
%%polynomials

function Demosl2Complex

%third site
%a = input('a: ');
a = 0.7 + i;
% a=i;
%a = 0.5 + (sqrt(3)/2)*i;
%degree of the Heine-Stieltjes polynomial
%n = input('degree of q: ');
n = 14;
```

```
lambda_0 = -.5;
lambda_1 = -.5;
lambda_2 = -.5;


% lambda_0 = -1.1;
% lambda_1 = -15   ;
% lambda_2 = -2;
A = zeros(n+1, n+1);
k = n;
 for iter=1:n+1
      A(iter,iter) = -(1+a)*k*(k-1) + k*((a+1)*lambda_0
      + a*lambda_1 + lambda_2);
    if iter==n+1;
        else A(iter+1, iter) = a*k*(k-1) - a*lambda_0*k;
        end
if iter ~= 1
    A(iter-1, iter)= k^2 - n^2 - (k-n)*(1+lambda_0
    + lambda_1 + lambda_2);
end
     k = k-1;
end



[X D] = eig(A);
Cmatr=-1/(n^2 - n*(1 + lambda_0 + lambda_1 + lambda_2))*D;
for j=1:1:n+1
C(j) = Cmatr(j,j);
```

```
end

for iter=1:n+1
    p= X(:,iter);
    y(:, iter) = roots(p);

end

%Compute the coordinates of the incentre of the triangle
%with vertices 0, 1, a
distance0 = sqrt(sum(([0,0]-[real(a), imag(a)]).^2));
distance1 = sqrt(sum(([1,0]-[real(a), imag(a)]).^2));
incenter = (distance0/(distance0+1+distance1))*[0,0]
+ (1/(distance0 + 1 + distance1))*[1,0] +
 (distance1/(distance0 + 1 + distance1))*[real(a),imag(a)];
b1 = [[0,0]; [incenter(1), incenter(2)]];
b2 = [[1,0]; [incenter(1), incenter(2)]];
b3 = [[real(a),imag(a)]; [incenter(1), incenter(2)]];

bind = 1:2;
k=1:n;
j=1:n+1;
j1=1:n+1;
b1
figure(2);
plot(y(k, j),'b.');
hold on
```

```
plot(b1(bind,1), b1(bind, 2), 'b-');
hold on
plot(b2(bind,1), b2(bind, 2), 'b-');
hold on
plot(b3(bind,1), b3(bind, 2), 'b-');
hold on
plot(C(j),'ro','MarkerEdgeColor','r',
'MarkerFaceColor','r','MarkerSize',10);


xlim([-.1 1.1])
ylim([-0.1 1.1])
hold off

%end of Demosl2Complex.m
}
```

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