WELL-POSEDNESS OF NONLINEAR WAVE EQUATIONS IN CHARACTERISTIC COORDINATES

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By

ANTON SAKOVICH, B.Sc.

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AUTHOR:	Anton Sakovich, B.Sc. (Belarusian State University)
SUPERVISOR:	Dr. Dmitry Pelinovsky
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Abstract

We prove global well-posedness of the short-pulse equation with small initial data in Sobolev space H^2. Our analysis relies on local well-posedness results of Schäfer and Wayne, the correspondence of the short-pulse equation to the sine-Gordon equation in characteristic coordinates, and conserved quantities of the short-pulse equation. We perform numerical computations to illustrate this result. We also prove local and global well-posedness of the sine-Gordon equation in an appropriate vector space.

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To my parents

Lyudmila and Sergei

Contents

Introduction

Fiber-optics communication lines are very efficient in rapid data transmission on long distances. In such systems a communication channel is an optical fiber with a certain balance between dispersion and nonlinearity. The data to be transmitted is encoded in pulses. If the pulses do not suffer much from dissipation and dispersion effects the data can be decoded on the other end of the communication channel. Using shorter pulses as an information carrier improves a bandwidth of the system. Modern lasers are capable of generating femtosecond laser pulses containing only a few cycles on a pulse length [23].

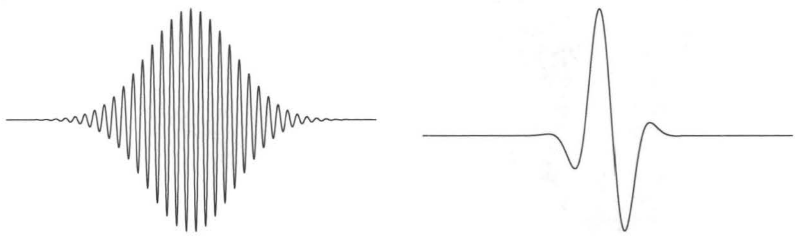


Figure 1: A wavetrain (left) and a short pulse (right).

Propagation of light pulses in an optical fiber is described by Maxwell's equations [4]. There are two alternative approaches to work with these equations. The first one relies on numerical approximations [12, 14]. This method allows us to compute evolution of any initial data provided numerical errors are sufficiently small. However, this method is computationally costly and does not say much about the analytic properties of pulse solutions. The other line of attack is based on asymptotic approximations to the Maxwell's equations. If the spectrum of a pulse is sufficiently narrow, the solution has a form of a wavetrain (see Figure 1, left) which fits into a "slowly varying envelope approximation". This pulse can be represented as a product of a slowly changing amplitude and a rapidly oscillating phase function, with the amplitude satisfying the well-known nonlinear Schrödinger equation [29]. If the spectrum of a pulse is broad, the solution has the form of a short pulse (see Figure 1, right). In the past decade, the topic of short pulses was intensively studied in the literature [5, 6, 10, 16, 27] due to rapid technological progress.

This thesis is devoted to a model equation for ultra-short pulses in silica optical fiber derived recently by Schäfer and Wayne [27]. This model is referred to as *the short-pulse equation* in literature. Chung et al. [9] justified derivation of this equation in linear case and presented numerical approximations of modulated pulse solutions. In addition to the derivation of the short-pulse equation, the pioneer paper [27] contains two important results. First, non-existence of a smooth travelling wave solution was proved in the entire range of the speed parameter. Second, the short-pulse equation was proved to be locally well-posed in a certain Sobolev space. The first result was recently extended by Costanzino, Manukian and Jones [11], who added a high-frequency dispersive term to the model of Schäfer and Wayne allowing for existence of smooth travelling solutions. To construct homoclinic solutions with slow and fast motions, the authors of [11] applied the Fenichel theory for singularly perturbed differential equations. Extension of the second result is described in the present thesis.

The short-pulse equation has a number of remarkable properties. Not only it is integrable by the Inverse Scattering Transform method [2], but also it is related to a well-studied sine-Gordon equation in characteristic coordinates through a coordinate transformation [24, 26]. This transformation endows the short-pulse equation with solitary wave solutions [19, 25] and an infinite hierarchy of conserved quantities [7]. There are two types of solitary wave solutions of the sine-Gordon equation (breathers and kinks) that generate two special solutions of the short-pulse equation (pulses and loops) [26]. The loop solutions are multi-valued, while pulse solutions are single-valued for small amplitudes.

This thesis addresses properties of the short-pulse equation and contains the following original results:

- We show that the short-pulse equation passes the Painlevé test, that is it only admits solutions "whose only movable singularities are poles" [2, 22]. Having passed the Painlevé test, the system is considered to be integrable.

- We prove *global well-posedness* of the short-pulse equation for small initial data in energy space. We rely on equivalence of the short-pulse equation and sine-Gordon equation through a coordinate transformation [24, 26] and the hierarchy of its conserved quantities [7].

- We prove that the sine-Gordon equation in characteristic coordinates is locally well-posed in a constrained Sobolev space. Our analysis is based on the Duhamel's principle, properties of the linearized problem and contraction arguments. By using the conserved quantities of the sine-Gordon equation in characteristic coordinates we prove global well-posedness for small initial data in the constrained energy space.

- We illustrate our result on global well-posedness of the short-pulse equation by numerical simulations based on the pseudospectral method [31]. The numerical approximation to the solution of the short-pulse equation remains bounded and smooth for small initial data and develops shocks and wave breaking for large initial data. We also test stability of the exact pulse solutions by numerical simulations.

The thesis is organized as follows. In Chapter 1, we show that the short-pulse equation possesses the Painléve property and review important properties of this model. Chapter 2 is devoted to global well-posedness of the short-pulse equation and well-posedness of the sine-Gordon equation in characteristic coordinates. In Chapter 3, we perform numerical computations of the short-pulse equation.

Chapter 1

Formalism of the short-pulse equation

The short-pulse equation derived by Schäfer and Wayne in [27] after rescaling of dependent and independent variables can be conveniently represented in a normalized form

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}, \tag{1.1}$$

where x and t are spatial and temporal variables correspondingly, and u is proportional to the amplitude of the pulse. This model possess a scaling invariance property

$$u(x,t) = \alpha U(\alpha^{-1}x, \alpha t), \quad \alpha \neq 0, \tag{1.2}$$

which allows to generate a one-parametric family of solutions from any known solution $U(X,T)$ to this equation.

Linearized short-pulse equation $u_{xt} = u$ restricts propagation of the wave packets to the left. Indeed, the dispersion relation $\omega(k) = k^{-1}$ for the harmonic waves

$$u(x,t) = Ae^{i(kx-\omega t)}$$

shows that the phase and group velocities are, correspondingly,

$$v_{ph} = \frac{\omega}{k} = \frac{1}{k^2} > 0, \quad v_{gr} = \frac{d\omega}{dk} = -\frac{1}{k^2} < 0,$$

and, in particular, the sign of the group velocity tells us that small-amplitude pulses of the short-pulse equation propagate leftwards.

In this chapter, we prove that the short-pulse equation (1.1) possesses a Painlevé property which is a reliable indicator of its integrability. We also review the other properties of the short-pulse equation (1.1) which stem out of its integrability. Those are Lax representation, transformation to the sine-Gordon equation in characteristic coordinates, solitary wave solutions, and conserved quantities.

1.1 Transformation to the sine-Gordon equation

Let us consider the sine-Gordon equation in characteristic coordinates in the form

$$w_{yt} = \sin w, \quad w(y,t) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}, \tag{1.3}$$

It was shown in [24] that the short-pulse equation (1.1) and the sine-Gordon equation (1.3) can be transformed to each other. This was done by transforming the generalized symmetry of the short-pulse equation to that of the sine-Gordon equation and then applying the transformations obtained directly to the equations. The derivation of this result was substantially simplified in [26] by making a change of variables that remove all but mixed second order derivatives. Using the same method as in [26] we show how the transformation can be derived.

Rewriting the short-pulse equation (1.1) in the parametric form

$$u(x,t) = v(y,t), \quad y = y(x,t), \tag{1.4}$$

we obtain

$$v_{yt}y_x + v_y y_{xt} + v_{yy}y_x y_t - v - vv_y^2 y_x^2 - \frac{1}{2}\left(v^2 v_{yy} y_x^2 + v^2 v_y y_{xx}\right) = 0. \tag{1.5}$$

To remove the terms with v_{yy}, we require

$$y_t = \frac{1}{2}v^2 y_x, \tag{1.6}$$

so that

$$y_{xt} = vv_y y_x^2 + \frac{1}{2}v^2 y_{xx},$$

and the main equation (1.5) reduces to

$$v_{yt}y_x - v = 0. \tag{1.7}$$

Let us assume that $y(x,t)$ is invertible in x and consider $x = x(y,t)$. By the chain rule,

$$x_y = \frac{1}{y_x}, \quad x_t = -x_y y_t. \tag{1.8}$$

Thus, (1.6), (1.7) and (1.8) yield

$$\begin{cases} x_y = \dfrac{1}{y_x} = \dfrac{v_{yt}}{v}, \\ x_t = -x_y y_t = -x_y \dfrac{1}{2} v^2 y_x = -\dfrac{v^2}{2}. \end{cases} \tag{1.9}$$

By compatibility condition $x_{ty} = x_{yt}$ we get

$$\left(\frac{v_{yt}}{v}\right)_t + vv_y = 0.$$

Using an integrating factor $\frac{2v_{yt}}{v}$ we arrive at

$$\frac{v_{yt}^2}{v^2} + v_y^2 = f(y), \tag{1.10}$$

where $f(y) : \mathbb{R} \mapsto \mathbb{R}_+$ is arbitrary.

The choice $f(y) \equiv 0$ gives only a trivial real solution $v(y,t) = v_0(t)$, for which $x_y \equiv 0$.

Let us consider the case $f(y) > 0$. Due to representation (1.4) and invertibility of $y(x,t)$ with respect to x, we can write the solution to the short-pulse equation in the parametric form

$$u = v(y,t), \quad x = x(y,t), \quad (y,t) \in \mathbb{R}^2.$$

This solution is invariant with respect to reparametrization $y = \psi(\tilde{y})$ with $\psi : \mathbb{R} \mapsto \mathbb{R}$. Let ψ be invertible function with the inverse $\phi = \psi^{-1} : \mathbb{R} \mapsto \mathbb{R}$. Then for $v(y,t) = \tilde{v}(\tilde{y},t)$ we have

$$v_y = \phi'(y)\tilde{v}_{\tilde{y}}.$$

One can choose the function ϕ to satisfy

$$(\phi'(y))^2 = f(y) > 0.$$

Therefore, without loss of generality we can set $f(y) = 1$ in (1.10). Now, for the equation

$$\frac{v_{yt}^2}{v^2} + v_y^2 = 1,$$

we make a potential transformation $v = w_t$, thus obtaining

$$\frac{w_{ytt}}{\sqrt{1 - w_{yt}^2}} = \pm w_t.$$

Assuming that w with its derivatives decays to zero at infinity, upon integration in t we obtain the sine-Gordon equation

$$w_{yt} = \pm \sin w,$$

where we can choose "+", due to invariance of $u(x,t)$ with respect to the transformation $y \mapsto -y$. Hence, the above computations give important formulae relating (1.1) and (1.3):

$$u(x,t) = w_t(y,t), \qquad x = x(y,t) : \begin{cases} x_t = -\frac{1}{2} w_t^2, \\ x_y = \cos w. \end{cases} \tag{1.11}$$

Formulae (1.11) make it possible to derive solutions of the short-pulse equation (1.1) from those of the sine-Gordon equation in characteristic coordinates (1.3).

We note that

$$u_x(x,t) = \tan w(y,t), \tag{1.12}$$

so that the infinite slope of $u(x,t)$ occurs in the solution of the short-pulse equation (1.1) at the same points where the solution $w(y,t)$ of the sine-Gordon equation (1.3) intersects zeros of $\cos w$. We will be interested in localized solutions $w(y,t)$ satisfying

$$\|w(\cdot,t)\|_{L^\infty} < \frac{\pi}{2}, \quad \forall t \in \mathbb{R}_+.$$

1.2 Painlevé property

To use the Inverse Scattering Transform scheme for a nonlinear partial differential equation, one can start by checking if it passes a Painlevé test [22, 30] that indicates typically integrability of the nonlinear equation. This approach, which is also known as the singularity analysis method, was adjusted to partial differential equations by Weiss, Tabor and Carnevale in [34]. The idea is to check whether the solution to the partial differential equation with independent variables $x_1, x_2, \ldots, x_n$ is a single-valued function about a movable singular manifold $\phi(x_1, x_2, \ldots, x_n) = 0$. The equation passes the Painlevé test if the expansion of a solution into Laurent series about the singular manifold has the number of arbitrary functions being equal to the order of the system, that is the highest derivative in $x_1, x_2, \ldots, x_n$.

The short-pulse equation in the form (1.1) does not allow to determine the leading order behaviour of Laurent series. Therefore, we perform the Painlevé test for the equivalent system

$$\begin{cases} v_{yt} - vx_y = 0, \\ x_t + \frac{1}{2}v^2 = 0, \end{cases} \tag{1.13}$$

which follows from (1.9). We apply the Weiss–Kruskal algorithm for Painlevé test (see [22] and references therein) to the system (1.13): the singular manifold is taken in the form

$$\phi(y,t) = y + \psi(t), \quad \psi'(t) \neq 0, \quad \forall t \in \mathbb{R},$$

and Laurent series expansions for v and x are represented as

$$\begin{cases} v(y,t) = \sum_{n=0}^{\infty} a_n(t)\phi(y,t)^{n+\alpha}, \\ x(y,t) = \sum_{n=0}^{\infty} b_n(t)\phi(y,t)^{n+\beta}. \end{cases}$$

The balance of the lowest powers of $\phi(y,t)$ in the system (1.13) occurs if

$$\begin{cases} \alpha - 2 = \alpha + \beta - 1, \\ \beta - 1 = 2\alpha, \end{cases}$$

so that $\alpha = \beta = -1$ after which we find that

$$a_0(t) = \pm 2i\psi'(t), \quad b_0(t) = -2\psi'(t).$$

We are going to consider only the branch with $a_0(t) = 2i\psi'(t)$ because the other branch $a_0(t) = -2i\psi'(t)$ will give the same expansion, but with the opposite sign due to the symmetry $v \mapsto -v$ of the system. To determine the values of r at which one or both coefficients $a_r(t)$, $b_r(t)$ can be arbitrary, we substitute the expansions

$$\begin{cases} v(y,t) \sim a_0(t)\phi^{-1}(y,t) + a_r(t)\phi^{r-1}(y,t), \\ x(y,t) \sim b_0(t)\phi^{-1}(y,t) + b_r(t)\phi^{r-1}(y,t), \end{cases}$$

into (1.13), group the terms at ϕ^{r-3} and ϕ^{r-2} for the first and second equations correspondingly, and obtain $r = -1, 1, 4$. The value $r = -1$ corresponds to the arbitrariness of the function $\psi(t)$. In order to see what actually happens at the orders $r = 1, 4$ we need to use the full series expansions. With this idea in mind, we derive the recursion relations

$$\begin{cases} (n-2)a'_{n-1} + (n-2)(n-1)a_n\psi' - \sum_{k=0}^{n}(k-1)a_{n-k}b_k = 0, \\ b'_{n-1} + (n-1)b_n\psi'(t) + \frac{1}{2}\sum_{k=0}^{n} a_k a_{n-k} = 0, \end{cases} \tag{1.14}$$

where $a_{-1} = b_{-1} = 0$ and $n = 0, 1, 2, \ldots$. The results of our computations are listed as follows:

- $n = 1$

$$a_1(t) = -i\frac{\psi''(t)}{\psi'(t)},$$

  $b_1(t)$ – arbitrary.

- $n = 2$

$$a_2(t) = -ib_2(t), \quad b_2(t) = \frac{1}{6\psi'(t)^3}\left(\psi''(t)^2 - 2b'_1(t)\psi'(t)^2\right).$$

- $n = 3$

  $a_3(t) = -ib_3(t)$,

$$b_3(t) = \frac{1}{12\psi'(t)^5}\left(b''_1(t)\psi'(t)^3 - 2b'_1(t)\psi''(t)\psi'(t)^2 - \psi''(t)\psi^3(t)\psi'(t) + 2\psi''(t)^3\right),$$

- $n = 4$

$$
\begin{aligned}
a_4(t) = \frac{i}{144\psi'(t)^7}\big(&216b_4(t)\psi'(t)^7 - 4b_1'(t)^2\psi'(t)^4 + 6{b_1}^{(3)}(t)\psi'(t)^4 \\
&- 30b_1''(t)\psi''(t)\psi'(t)^3 - 12b_1'(t)\psi^{(3)}(t)\psi'(t)^3 \\
&+ 52b_1'(t)\psi''(t)^2\psi'(t)^2 - 6\psi^{(3)}(t)^2\psi'(t)^2 \\
&- 6\psi''(t)\psi^{(4)}(t)\psi'(t)^2 + 66\psi''(t)^2\psi^{(3)}(t)\psi'(t) \\
&- 73\psi''(t)^4\big),
\end{aligned}
$$

$b_4(t)$ – arbitrary.

Higher-order corrections of the Laurent series for $v(y,t)$ and $x(y,t)$ can be obtained in the same way by recursion relation (1.14). We observe that Laurent series for the solution to system (1.13) about $\phi(y,t) = y + \psi(t)$ possess three arbitrary parameters $\psi(t)$, $b_1(t)$ and $b_4(t)$. Since system (1.13) is of the third order, this indicates that the short-pulse equation passes the Painlevé test.

1.3 Lax pair

It is common to tag a nonlinear PDE as an integrable system if it can be written as a compatibility condition of two linear operators, known as the Lax pair. In this case, the initial-value problem can be solved by means of the Inverse Scattering Transform method [2]. It has been checked for a number of examples that the existence of a Lax pair appears to be equivalent to some other tests on integrability such as the Painlevé test. There is no proof to this fact, but there is no counterexample either. A good introduction to the topic of integrable systems is given in the books [1], [2] and [30].

Following the results presented in [24] we show how to derive the Lax pair of the short-pulse equation (1.1) in the form

$$\begin{cases} \psi_x = X\psi, \\ \psi_t = T\psi, \end{cases}$$

where X and T are $n \times n$ matrices depending on x, t, function $u(x,t)$ and its derivatives, and $\psi(x,t)$ is an n-component column. The compatibility condition

$$X_t - T_x + [X, T] = 0,$$

where square brackets denote a matrix commutator, must recover the short-pulse equation (1.1). It is important to note that X, T and ψ are defined up to some gauge transformation

$$\begin{aligned} \psi &\mapsto G\psi, \\ X &\mapsto GXG^{-1} + (D_xG)G^{-1}, \\ T &\mapsto GTG^{-1} + (D_tG)G^{-1}, \end{aligned}$$

where G is a nondegenerate $n \times n$ matrix depending on x, t, $u(x,t)$ and its derivatives.

The Lax representation of the short-pulse equation can be obtained in terms of 2×2 traceless matrices X and T, upon assuming that

$$\begin{aligned} X &= Au_x + B, \\ T &= T(u, u_x), \end{aligned}$$

where A and B are 2×2 constant traceless matrices. This gives a set of commutator equations which can be solved for

$$A = i\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix},$$

with λ being an arbitrary nonzero constant, which produces

$$X = \begin{pmatrix} 0 & \lambda(1+iu_x) \\ \lambda(1-iu_x) & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} \frac{i}{2}u & \frac{i}{2}\lambda u^2 u_x + \frac{1}{2}\lambda u^2 + \frac{1}{4\lambda} \\ -\frac{i}{2}\lambda u^2 u_x + \frac{1}{2}\lambda u^2 + \frac{1}{4\lambda} & -\frac{i}{2}u \end{pmatrix}.$$

Finally, the gauge transformation induced by a matrix

$$G = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

yields another representation of the Lax pair

$$X = \begin{pmatrix} \lambda & \lambda u_x \\ \lambda u_x & -\lambda \end{pmatrix}, \quad T = \begin{pmatrix} \frac{\lambda}{2}u^2 + \frac{1}{4\lambda} & \frac{\lambda}{2}u^2 u_x - \frac{1}{2}u \\ \frac{\lambda}{2}u^2 u_x + \frac{1}{2}u & -\frac{\lambda}{2}u^2 - \frac{1}{4\lambda} \end{pmatrix}. \tag{1.15}$$

The spectral problem $\psi_x = X\psi$ produced by the operator X in (1.15) is of the Wadati–Konno–Ichikawa type [33].

1.4 Exact solutions

Using transformation (1.11), it is possible to generate solutions of the short-pulse equation (1.1) from those of the sine-Gordon equation (1.3). In [25] loop and pulse solutions were derived in this way. Later in [19], general formulae for multiloop and multibreather solutions were obtained by means of Hirota's method. In [21], some other periodic and solitary travelling-wave solutions to equation (1.1) were presented. Below we provide some explicit solutions of the short-pulse equation (1.1) following the results of [25].

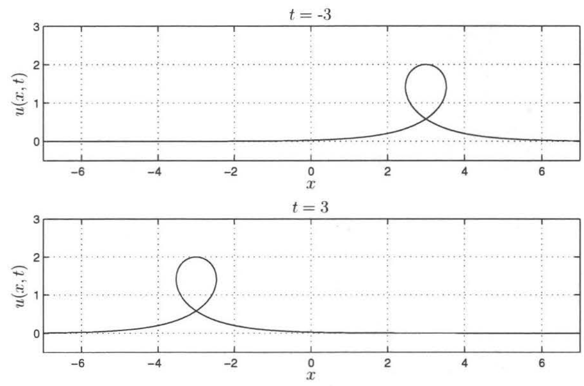


Figure 1.1: The loop solution $u(x,t)$ (1.17) to the short-pulse equation (1.1)

Consider the kink solution of the sine-Gordon equation (1.3) in the form

$$w = 4\arctan\big(\exp(y+t)\big) \tag{1.16}$$

and apply transformation (1.11) to arrive at the solution in the parametric form

$$\begin{cases} u = 2\,\mathrm{sech}(y+t), \\ x = y - 2\tanh(y+t). \end{cases} \tag{1.17}$$

This solution represents a loop soliton of the short-pulse equation (1.1) travelling with a unit speed to the left (Figure 1.1). This solution always has two singular points, because kink (1.16) passes through $w = \pi/2$ and $w = 3\pi/2$ for all $(y,t) \in \mathbb{R}^2$ (cf. (1.12)).

Consider now the breather solution to the sine-Gordon equation (1.3) in the form

$$w = -4\arctan\left(\frac{m\sin\psi}{n\cosh\phi}\right), \tag{1.18}$$

where

$$\phi = m(y+t), \quad \psi = n(y-t), \quad n = \sqrt{1-m^2}, \tag{1.19}$$

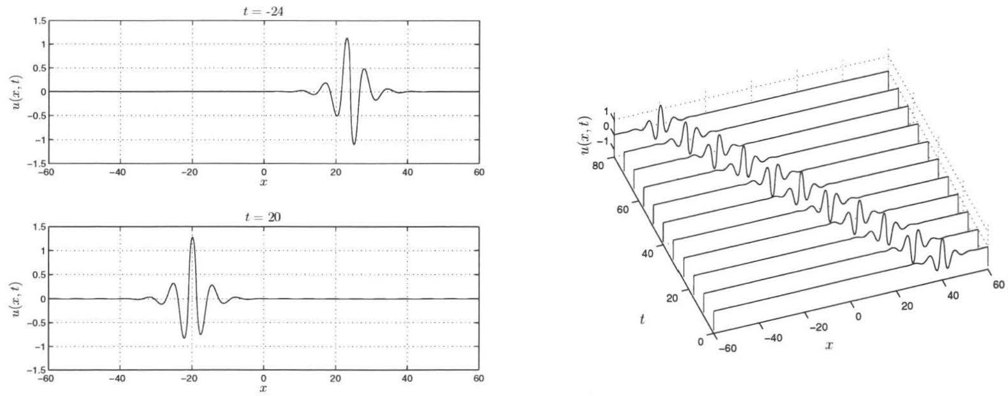


Figure 1.2: The pulse solution (1.20) to the short-pulse equation (1.1) with $m = 0.32$

and $0 < m < 1$. Since

$$|w(y,t)| < \frac{\pi}{2} \quad \text{for} \quad m < m_{cr} = \sin\frac{\pi}{8} \approx 0.383,$$

this breather solution can generate a non-singular solution of the short-pulse equation (1.1) in the parametric form

$$\begin{cases} u = 4mn\dfrac{m\sin\psi\sinh\phi + n\cos\psi\cosh\phi}{m^2\sin^2\psi + n^2\cosh^2\phi}, \\ \\ x = y + 2mn\dfrac{m\sin 2\psi - n\sinh 2\phi}{m^2\sin^2\psi + n^2\cosh^2\phi}. \end{cases} \tag{1.20}$$

This formula represents a smooth pulse solution for $0 < m < m_{cr}$ (Figure 1.2, left) and a singular pulse solution for $m > m_{cr}$ (Figure 1.3).

We note that the pulse solution (1.20) is periodic on the (x,t)-plane, according to the following property

$$\begin{cases} u(y,t) = u\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right), \\ x(y,t) = x\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right) + \frac{\pi}{m}, \end{cases} \quad \forall (y,t) \in \mathbb{R}^2.$$

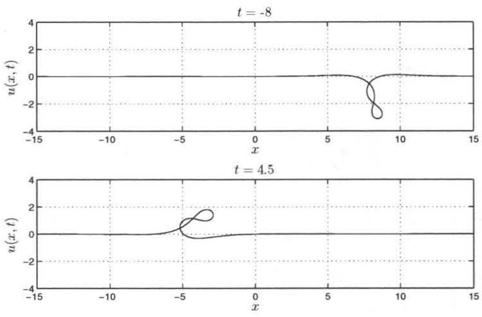


Figure 1.3: A singular solution (1.20) to the short-pulse equation (1.1) with $m = 0.80$

On the other hand, the solution is localized in any other direction on the (x, t)-plane. The solution surface of the pulse solution (1.20) with $m < m_{cr}$ is shown on Figure 1.2, right.

1.5 Conserved quantities

An infinite hierarchy of conserved quantities of the short-pulse equation (1.1) was derived by Brunelli in [7]. This was accomplished by using a bi-Hamiltonian representation of the system. Some results from [7] are reviewed below. The first few conserved quantities will be needed in Chapter 2 to prove global well-posedness of the short-pulse equation.

The short-pulse equation (1.1) can be represented in the bi-Hamiltonian form

$$u_t = \mathcal{D}_1 \frac{\delta H_0}{\delta u} = \mathcal{D}_2 \frac{\delta H_{-1}}{\delta u}, \tag{1.21}$$

where

$$\mathcal{D}_1 = \partial_x^{-1} + u_x \partial_x^{-1} u_x, \quad \mathcal{D}_2 = \partial_x,$$

and

$$H_0 = \frac{1}{2}\int_{\mathbb{R}} u^2 dx, \quad H_{-1} = \int_{\mathbb{R}} \left(\frac{1}{24}u^4 - \frac{1}{2}(\partial_x^{-1}u)^2\right) dx.$$

Here H_0, H_{-1} are conserved quantities (Hamiltonians) of the short-pulse equation and $\frac{\delta}{\delta u}$ stands for the functional derivative given by

$$\forall\, v \in L^2,\ u \in \mathrm{Dom}(f): \quad \left(v, \frac{\delta f}{\delta u}\right)_{L^2} = \frac{d}{d\epsilon} f(u+\epsilon v)\bigg|_{\epsilon=0},$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product in L^2 space. Two hierarchies of conserved quantities arise from the bi-Hamiltonian form (1.21).

The first hierarchy is generated by the recursion formula

$$\frac{\delta H_n}{\delta u} = R\frac{\delta H_{n+1}}{\delta u}, \quad n = -1, -2\ \ldots,$$

where

$$R = \mathcal{D}_2^{-1}\mathcal{D}_1 = \partial_x^{-2} + \partial^{-1}u_x\partial_x^{-1}u_x = \partial_x^{-2}\left(1 + u_x^2 + u_{xx}\partial^{-1}u_x\right).$$

The first few conserved quantities are

$$\begin{aligned}
H_0 &= \frac{1}{2}\int_{\mathbb{R}} u^2 dx,\\
H_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{24}u^4 - \tfrac{1}{2}(\partial_x^{-1}u)^2\right) dx,\\
H_{-2} &= \int_{\mathbb{R}} \left(\frac{1}{720}u^6 + \frac{1}{2}(\partial_x^{-1}u)^2 + \frac{1}{6}(\partial_x^{-2}u^3)u - \frac{1}{4}(\partial_x^{-1}u)^2u^2\right) dx,\\
&\ldots .
\end{aligned}$$

The second hierarchy of conservation laws is given recursively by

$$\frac{\delta H_{n+1}}{\delta u} = R^{-1}\frac{\delta H_n}{\delta u}, \quad n = 0, 1, 2\ldots,$$

where

$$R^{-1} = \mathcal{D}_1^{-1}\mathcal{D}_2 = \partial_x^2\left(\frac{1}{F^2} + A\partial_x^{-1}A_x\right) = \partial_x^2\frac{1}{u_{xx}}\partial_x F\partial_x^{-1}\frac{u_{xx}}{F^3},$$

and

$$F = \sqrt{1+u_x^2}, \quad A = \frac{u_x}{\sqrt{1+u_x^2}}.$$

The first few conserved quantities are

$$\begin{aligned}
H_1 &= -\int_{\mathbb{R}} F dx, \\
H_2 &= \frac{1}{2}\int_{\mathbb{R}} F A_x^2 dx, \\
H_3 &= \frac{1}{8}\int_{\mathbb{R}} \left(F A_x^4 - 4\frac{A_{xx}^2}{F} \right) dx, \\
H_4 &= \frac{1}{16}\int_{\mathbb{R}} \left(F A_x^6 + 8\frac{A A_{xx}^3}{F} - 12\frac{A_x^2 A_{xx}^2}{F} + 8\frac{A_{xxx}^2}{F^3} \right) dx, \\
&\dots .
\end{aligned}$$

Conserved quantities H_0, H_1 and H_2 will be used in Chapter 2.

Chapter 2

Well-posedness of the sine-Gordon and short-pulse equations

In this chapter, we prove local and global well-posedness of the sine-Gordon equation in characteristic coordinates (1.3) using the contraction arguments and conserved quantities. We also establish a sufficient condition on global well-posedness of the short-pulse equation (1.1). This is accomplished by using local well-posedness results of Schäfer and Wayne [27], the hierarchy of conserved quantities found by Brunelli [7] and transformation (1.11) between the sine-Gordon equation in characteristic coordinates (1.3) and the short-pulse equation (1.1) [24, 26].

2.1 Function spaces

- Let $p \geq 1$ and $f : \mathbb{R} \mapsto \mathbb{R}$. The standard L^p space is defined by the norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}.$$

  We also define L^∞ norm by

$$\|f\|_{L^\infty} = \sup_{x \in \mathbb{R}} |f(x)|.$$

- Let $s \geq 1$ be an integer. The L^2-based Sobolev space H^s is defined by the squared norm

$$\|f\|_{H^s}^2 = \sum_{j=0}^{s} \|f^{(j)}\|_{L^2}^2 = \int_{\mathbb{R}} (1 + k^2 + \cdots + k^{2s}) |\hat{f}(k)|^2 dk,$$

  where $\hat{f}$ is the Fourier transform of a function f given by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx$$

  and the equality follows from the Plancherel's identity.

  Here we list some properties of Sobolev space H^s that play an essential role in our computations (refer to e.g. [28] for details).

  - *Sobolev embedding:* If $f \in H^s$, then the functions f, f',...,$f^{(s-1)}$ are bounded uniformly continuous functions that converge to 0 at $\pm\infty$. In particular, there is a constant B_s, such that the following bound holds:

$$\forall f \in H^s : \quad \|f\|_{L^\infty} \leq B_s \|f\|_{H^s}. \tag{2.1}$$

  - *Banach Algebra Property:* There is another constant $C_s > 0$ such that

$$\forall f, g \in H^s : \quad \|fg\|_{H^s} \leq C_s \|f\|_{H^s} \|g\|_{H^s}. \tag{2.2}$$

We note that

$$\|f\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|f\|_{H^1},$$

which is obtained by using the triangle inequality for integrals and the Cauchy-Schwartz inequality as follows

$$\begin{aligned}|f(x)| &= \left|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(k)e^{ikx}dk\right| \leq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left|\hat{f}(k)\right|dk \\ &\leq \frac{1}{\sqrt{2\pi}}\sqrt{\int_{\mathbb{R}}(1+k^2)^{-1}dk}\sqrt{\int_{\mathbb{R}}(1+k^2)\left|\hat{f}(k)\right|^2 dk} = \frac{1}{\sqrt{2}}\|f\|_{H^1}.\end{aligned}$$

We also note that

$$\begin{aligned}\|fg\|_{H^1} &\leq \|f\|_{H^1}\|g\|_{H^1} \\ \|fg\|_{H^2} &\leq \|f\|_{H^2}\|g\|_{H^2},\end{aligned}$$

as follows from recent paper [20].

- Let $\int_{\mathbb{R}} f(x)dx = 0$ then $\hat{f}(0) = 0$. We define

$$\|f\|^2_{\dot{H}^{-1}} = \int_{\mathbb{R}} \frac{|\hat{f}(k)|^2}{k^2}dk = \|\partial_x^{-1} f\|^2_{L^2},$$

where

$$\partial_x^{-1} f(x) = -\int_x^\infty f(x')dx' = \int_{-\infty}^x f(x')dx' = \frac{1}{2}\left(\int_{-\infty}^x - \int_x^\infty\right) f(x')dx'.$$

Let us define the space X^s by the squared norm

$$\|f\|^2_{X^s} = \|f\|^2_{H^s} + \|f\|^2_{\dot{H}^{-1}}. \tag{2.3}$$

For a given $f_0 \in (0,1)$, a constrained version of the space X^s_c is then defined by

$$X^s_c = \{f \in X^s : \|f\|_{L^\infty} < f_0\}. \tag{2.4}$$

- *Hölder's inequality*: Let $0 < p, q \leq \infty$ such that $1 = 1/p + 1/q$, then

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}. \tag{2.5}$$

For $p = q = 2$ formula (2.5) is known as the Cauchy-Schwartz inequality.

- *Hausdorff-Young inequality*: Let $0 < p, q, r \leq \infty$ such that $1 + p^{-1} = q^{-1} + r^{-1}$, then

$$\|f \star g\|_{L^p} \leq \|f\|_{L^q}\|g\|_{L^r}, \tag{2.6}$$

where the star denotes convolution operator $(f \star g)(x) = \int_{\mathbb{R}} f(x')g(x - x')dx'$.

- To analyze the properties of a solution to the linear wave equation $Q_{xt} = Q$ we will need the following kernel functions for $x, t > 0$:

$$K_t(x) = \sqrt{\frac{t}{x}} J_0'\left(2\sqrt{tx}\right) = -\sqrt{\frac{t}{x}} J_1\left(2\sqrt{tx}\right), \tag{2.7}$$

and

$$J_t(x) = J_0\left(2\sqrt{tx}\right), \tag{2.8}$$

where J_0, J_1 stand for the Bessel functions of the first kind.

For a subsequent integral estimates it is important to determine the asymptotic behaviour of these kernel functions. Since $J_1(x) = \frac{x}{2} + \mathcal{O}(x^3)$ for small $x > 0$ and $J_1(x) \sim \frac{1}{\sqrt{x}} \cos(x - \frac{3\pi}{4})$ for large $x \gg 1$, the kernel function K_t enjoys the following properties

  - $K_t(x) = -t + \mathcal{O}(xt^2)$ for small $xt > 0$,
  - $K_t(x) = \mathcal{O}(x^{-3/4}t^{1/4})$ for large $xt \gg 1$,
  - there is a constant $C > 0$ such that $\|K_t\|_{L^\infty} \leq Ct$ and $\|K_t\|_{L^2} \leq C\sqrt{t}$ uniformly on $t \in \mathbb{R}_+$.

– $K_t \notin L^1$ for any $t > 0$.

Similarly, since $J_0(x) = 1 + O(x^2)$ for small $x > 0$ and $J_0(x) \sim \frac{1}{\sqrt{x}} \cos(x - \frac{\pi}{4})$ for large $x \gg 1$, the function J_t satisfies

– $J_t(x) = 1 + \mathcal{O}(xt) \to 1$ for small $xt > 0$,

– $J_t(x) = \mathcal{O}(x^{-1/4} t^{-1/4})$ for large $xt \gg 1$.

– $\|J_t\|_{L^\infty} \leq 1$ uniformly on $t \in \mathbb{R}_+$.

– $J_t \notin L^1 \cup L^2$ for any $t \in \mathbb{R}_+$.

The graphs of functions K_t and J_t for some values of $t > 0$ are shown on Figure 2.1. The slow decay of the functions J_t and K_t introduces a delicate problem in the well-posedness analysis of the sine-Gordon and short-pulse equations.

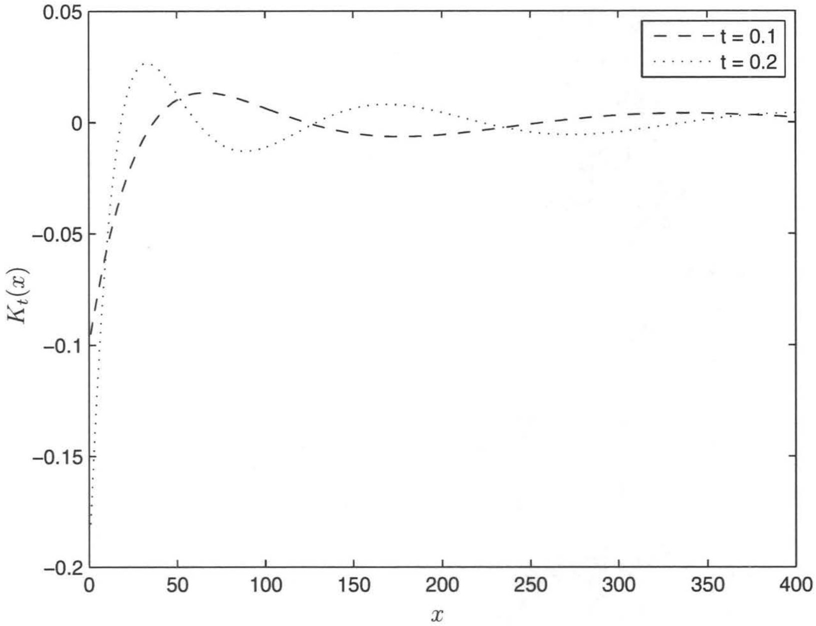
2.2 Preliminaries

The problem of global well-posedness for nonlinear equations in characteristic coordinates has been studied in a number of recent publications [13, 18, 32] in the context of Ostrovsky equation

$$u_{xt} = u + \left(u^2\right)_{xx} + \beta u_{xxxx}, \tag{2.9}$$

which has some similarity to the short-pulse equation (1.1). The Ostrovsky equation (2.9) models small-amplitude long waves in a rotating fluid. Liu and Varlamov [32] proved local well-posedness in space $H^s \cap \dot{H}^{-1}$ for $s > \frac{3}{2}$. The space $H^1 \cap \dot{H}^{-1}$ is the energy space of the Ostrovsky equation (2.9), where the mass $V(u) = \|u\|_{L^2}^2$ and the energy

$$E(u) = \int_{\mathbb{R}} \left(\beta(\partial_x u)^2 + \frac{1}{2} \left(\partial_x^{-1} u\right)^2 - \frac{1}{3} u^3 \right) dx$$



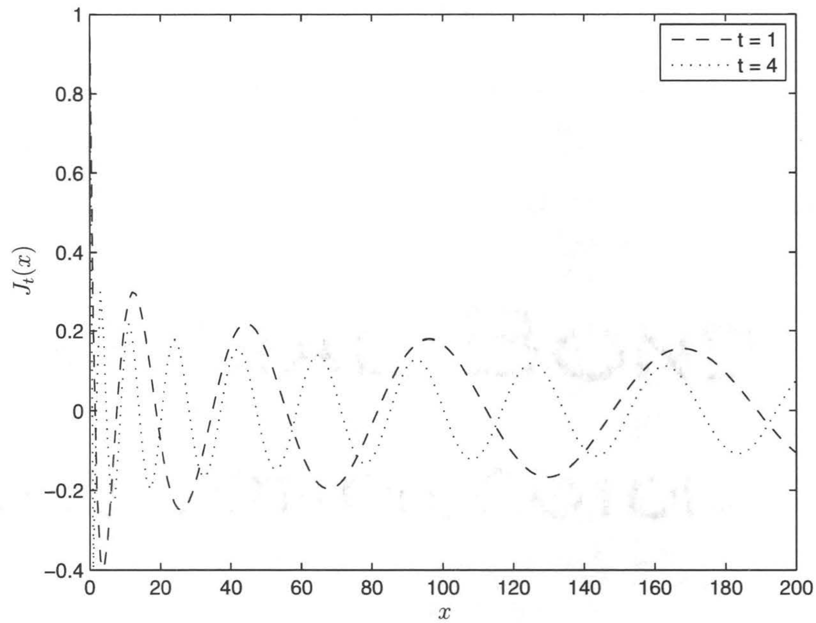


Figure 2.1: Kernel functions $J_t(x) = J_0\left(2\sqrt{tx}\right)$ and $K_t(x) = \sqrt{\frac{t}{x}}J_0'\left(2\sqrt{tx}\right)$.

conserve in time t. Using conserved quantities and local existence in $H^1 \cap \dot{H}^{-1}$, Linares and Milanes [18], and Gui and Liu [13] proved global well-posedness of the Ostrovsky equation (2.9) in the energy space. However, their proof is only valid for $\beta > 0$ and it is not applicable to the short-pulse equation (1.1).

To understand the long-term dynamics of solutions to the short-pulse equation (1.1) it is instructive to prove local and global existence of solutions to the sine-Gordon equation (1.3) in a space where the constraint

$$\|w(\cdot, t)\|_{L^\infty} < \frac{\pi}{2} \tag{2.10}$$

is kept global in time. According to the transformation (1.11) this would imply that a small initial data of the short-pulse equation (1.1) would evolve no singularities in a finite time. Together with integrability of the short-pulse equation (1.1), global well-posedness may suggest asymptotic stability of the modulated pulse solutions (1.20).

The sine-Gordon equation in characteristic coordinates was considered long ago by Kaup and Newell [15] using formal applications of the stationary phase method. Local well-posedness of this equation is a non-trivial problem due to the presence of the constraint

$$\int_{\mathbb{R}} \sin(w(y,t))dy = 0, \tag{2.11}$$

which does not guarantee that condition (2.10) is satisfied for all $t \in \mathbb{R}_+$. Our treatment of this equation is rigorous and we shall prove that the sine-Gordon equation (1.3) is locally well-posed in space $H^s \cap \dot{H}^{-1}$ for an integer $s \geq 1$ in the new variable $q = \sin w$. Global well-posedness is proved in $H^1 \cap \dot{H}^{-1}$ with the help of three conserved quantities of the sine-Gordon equation. The result can be extended in $H^s \cap \dot{H}^{-1}$ for an integer $s > 1$ if more conserved quantities are incorporated into analysis.

The sine-Gordon equation in the laboratory coordinates

$$w_{\tau\tau} - w_{\xi\xi} = \sin w$$

is known to be locally well-posed in a weaker space $L^p(\mathbb{R})$ for any $p \geq 2$, see Appendix B of Buckingham and Miller [8]. Similarly to this work, our analysis is also based on the conventional method of Picard iterations to prove local well-posedness of the sine-Gordon equation in characteristic coordinates (1.3).

2.3 Klein-Gordon equation in characteristic coordinates

To analyze the Cauchy problem for the sine-Gordon equation (1.3), we obtain information on the fundamental solution of the underlying linear problem

$$\begin{cases} Q_t = \partial_y^{-1} Q, \\ Q|_{t=0} = Q_0. \end{cases} \tag{2.12}$$

Let us denote $L = \partial_y^{-1}$ and $Q(t) = e^{tL} Q_0$. The solution operator can be represented in the Fourier transform form by

$$\mathcal{F}(e^{tL}) = e^{-\frac{it}{k}},$$

which involves a bounded oscillatory integral on $k \in \mathbb{R}$ with a singular behaviour as $k \to 0$. By the Fourier representation, the solution operator e^{tL} is a norm-preserving map from H^s to H^s for any $s \geq 0$, so that

$$\|Q(t)\|_{H^s} = \|e^{tL} Q_0\|_{H^s} = \|Q_0\|_{H^s}, \quad \forall t \in \mathbb{R}. \tag{2.13}$$

In particular, if $Q_0 \in L^2$, then $Q(\cdot, t) \in L^2$, for all $t \in \mathbb{R}$. In the following statement we shall specify an explicit representation of $Q(t)$ in the physical space.

Lemma 2.1 *Let $K_t(y)$ be defined by (2.7) with $t, y \in \mathbb{R}_+$. For any $Q_0 \in L^2 \cap L^\infty$ the linear Cauchy problem (2.12) has a solution in the form*

$$Q(y,t) = Q_0(y) + \int_y^\infty K_t(y'-y)Q_0(y')dy', \quad (y,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{2.14}$$

so that $Q(\cdot, t) \in L^\infty$ for any $t \in \mathbb{R}_+$

Proof. Applying the Laplace transform defined as

$$\tilde{f}(s) := \mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)dt,$$

with $\mathrm{Re}(s) > 0$, we reduce the linear Cauchy problem (2.12) to the form

$$s\frac{\partial \tilde{Q}}{\partial y}(y,s) - Q_0'(y) = \tilde{Q}(y,s).$$

By the method of undetermined coefficients, we find a solution

$$\tilde{Q}(y,s) = c(y,s)\exp(y/s),$$

where $c(y,s)$ solves $sc_y(y,s) = Q_0'(y)$. Since $\mathrm{Re}(s) > 0$, we integrate this equation from y to ∞ subject to the boundary condition $c(y,s) \to 0$ as $y \to \infty$ and arrive to the solution

$$c(y,s) = -\frac{1}{s}\int_y^\infty e^{-y'/s}Q_0'(y')dy'.$$

Inverting the Laplace transform, we complete computation of the solution operator as follows

$$\begin{aligned} Q(y,t) &= \mathcal{L}^{-1}\tilde{Q}(y,s) \\ &= -\int_y^\infty \mathcal{L}^{-1}\left(\frac{1}{s}e^{(y-y')/s}\right) Q_0'(y')dy' \\ &= -\int_y^\infty J_0(2\sqrt{t(y'-y)})Q_0'(y')dy', \end{aligned} \tag{2.15}$$

which gives the explicit formula (2.14) after integration by parts since $J_0(0) = 1$, $\lim_{z\to\infty} J_0(z) = 0$, and $Q_0 \in L^\infty(\mathbb{R})$. Convergence of the integral in the explicit formula (2.14) can be easily shown by the Cauchy-Schwarz inequality (2.5):

$$\begin{aligned}
|Q(y,t)| &\leq |Q_0(y)| + \int_y^\infty |K_t(y'-y)Q_0(y')|dy' \\
&\leq |Q_0(y)| + \|K_t\|_{L^2(\mathbb{R}_+)}\|Q_0\|_{L^2} < \infty,
\end{aligned}$$

so that $Q(\cdot,t) \in L^\infty$, for any $t \in \mathbb{R}_+$. □

Remark 2.2 For well-posedness analysis of the sine-Gordon equation we will need to keep track of both dependent variable and its definite integral with variable upper (lower) boundary. In particular, the variable

$$P(y,t) = -\int_y^\infty Q(y',t)dy',$$

also solves the Cauchy problem:

$$\begin{cases} P_t = \partial_y^{-1} P, \\ P|_{t=0} = P_0. \end{cases} \tag{2.16}$$

Evidently, Lemma 2.1 holds for $P(y,t)$ provided $P_0 \in L^2 \cap L^\infty$.

Remark 2.3 If $Q(\cdot,t) \in H^s$ and $P(\cdot,t) \in L^2$, then $P(\cdot,t) \in H^{s+1}$ for an integer $s \geq 0$. By Sobolev embedding, $P(\cdot,t) \in L^\infty$ and $\lim_{|y|\to\infty} P(y,t) = 0$, or equivalently,

$$\int_{\mathbb{R}} Q(y,t)dy = 0 \tag{2.17}$$

for any $t \geq 0$. Therefore, the constraint (2.17) is automatically satisfied if $Q(t) \in C(\mathbb{R}, H^s)$ and $P(t) \in C(\mathbb{R}, L^2)$ for a fixed $s \geq 0$. We recall that, if $\int_{\mathbb{R}} Q_0(y)dy \neq 0$, the solution $Q(y,t)$ still satisfies constraint (2.17) for $t > 0$ but is not smooth at $t = 0$, see Ablowitz & Villaroel [3] for analysis of a similar Kadomtsev–Petviashvili equation.

Remark 2.4 If $P_0 \in L^2 \cap L^\infty$, upon replacing $Q \mapsto P$ and $Q' \mapsto Q$ equation (2.15) gives a new representation of $P(y,t)$:

$$P(y,t) = -\int_y^\infty J_t(y'-y)Q_0(y')dy', \tag{2.18}$$

where $(t,y) \in \mathbb{R}_+ \times \mathbb{R}$, and a kernel function J_t is defined by (2.8).

2.4 Local well-posedness of the sine-Gordon equation

To simplify the constraint (2.11) for solutions of the sine-Gordon equation (1.3), we introduce a new variable

$$q = \sin w,$$

so that (2.11) becomes a linear constraint

$$\int_{\mathbb{R}} q(y,t)dy = 0. \tag{2.19}$$

The sine-Gordon equation (1.3) transforms to the evolution equation

$$q_t = \sqrt{1-q^2}\partial_y^{-1} q, \tag{2.20}$$

where the operator ∂_y^{-1} acts on an element of H^s under the constraint (2.19):

$$\partial_y^{-1} q := \int_{-\infty}^y q(y',t)dy' = -\int_y^\infty q(y',t)dy' = \frac{1}{2}\left(\int_{-\infty}^y - \int_y^\infty\right) q(y',t)dy'.$$

Let us introduce the nonlinear function

$$f(q) := 1 - \sqrt{1-q^2} = \frac{q^2}{1+\sqrt{1-q^2}} \tag{2.21}$$

and write the Cauchy problem for equation (2.20) in the equivalent form

$$\begin{cases} q_t = (1 - f(q))\partial_y^{-1} q, \\ q|_{t=0} = q_0. \end{cases} \tag{2.22}$$

The nonlinear function $f(q)$ is squeezed by the quadratic functions

$$\forall |q| < 1: \quad \frac{q^2}{2} \leq f(q) \leq q^2,$$

which allows us to interpret the term $f(q)\partial_y^{-1} q$ as a nonlinear perturbation to the linear evolution induced by $\partial_y^{-1} q$.

The local well-posedness analysis is based on the integral equation

$$q(t) = Q(t) - \int_0^t e^{(t-t')L} f(q(t'))p(t')dt', \tag{2.23}$$

which follows from Duhamel's principle for the Cauchy problem (2.22). Here

$$q(t) := q(y,t), \quad p(t) := p(y,t) = -\int_y^\infty q(y',t)dy',$$

$Q(t) = e^{tL} q_0$ is the solution of the linear problem (2.12) with $Q_0 = q_0$, and $f(q)$ is defined by (2.21). We shall work with initial data q_0 from the space X_c^s in (2.4) for an integer $s \geq 1$. Since $p_y = q$, it is clear that $p \in H^{s+1}$ if $p \in L^2$ and $q \in H^s$. We need to show that the vector field of the integral equation (2.23) is a Lipschitz map in the vector space X^s with the squared norm (2.3) rewritten as

$$\|q\|_{X^s}^2 := \|q\|_{H^s}^2 + \|p\|_{L^2}^2 \tag{2.24}$$

for any $t \in [0,T]$ and it is a contraction operator for a sufficiently small $T > 0$. If $\|q_0\|_{L^\infty} < 1$ and $q(t)$ is continuous in H^s for a fixed $s \geq 1$, then the constraint $\|q(t)\|_{L^\infty} < 1$ is satisfied on $[0,T_0] \subset [0,T]$ for some $T_0 > 0$ thanks to Sobolev's embedding $H^s \hookrightarrow L^\infty$ for any integer $s \geq 1$. Using this construction, we formulate and prove the local well-posedness theorem for the sine-Gordon equation (1.3).

Theorem 2.5 *Assume that $q_0 \in X_c^s$ for an integer $s \geq 1$. There exist a $T > 0$ such that the Cauchy problem (2.22) admits a unique local solution $q(t) \in C([0,T], X_c^s)$ satisfying $q(0) = q_0$.*

Proof. We prove this statement in two steps. First, we show that the vector field of the integral equation (2.23) is a closed map of a finite-radius ball in X_c^s that includes $q_0 \in X_c^s$ to itself on a nonempty time interval. In other words, we prove that for any $\delta \in (0, \min(B_s^{-1}, C_s^{-1}))$, and $\alpha \in (0,1)$ we can find a small $T_* = T_*(\alpha, \delta) > 0$ such that if $\|q_0\|_{X^s} \leq \alpha\delta$ and $\|q_0\|_{L^\infty} \leq \alpha$, then $\|q(t)\|_{X^s} \leq \delta$ and $\|q(t)\|_{L^\infty} < 1$ for any $t \in [0, T_*]$ (see (2.1) and (2.2) for definitions of B_s and C_s). Second, using the quadratic behaviour of $f(q)$, we prove that the map given by the integral term in (2.23) is Lipschitz with respect to the field variable q, and it is a contraction if the interval $[0,T]$ for $T \in (0, T_*)$ is sufficiently small. Existence of a unique fixed point of the integral equation (2.23) in a Banach space $C([0,T], X_c^s)$ follows by the contraction mapping principle (see e.g. [35]).

With the above scheme in mind, we start with with the bounds for $\|q(t)\|_{X^s}$ for some $t > 0$. By the triangle inequality and the norm-preserving property (2.13), we bound the H^s norm of $q(t)$ in the integral equation (2.23) for any integer $s \geq 1$ by

$$\begin{aligned} \|q(t)\|_{H^s} &\leq \|Q(t)\|_{H^s} + \int_0^t \|e^{(t-t')L} f(q(t'))p(t')\|_{H^s} dt' \\ &\leq \|q_0\|_{H^s} + C_s \int_0^t \|f(q(t'))\|_{H^s} \|p(t')\|_{H^s} dt', \end{aligned}$$

where we recall that H^s forms a Banach algebra with respect to multiplication for any integer $s \geq 1$. To deal with nonlinear function $f(q)$, we expand it in the Taylor series

$$f(q) = 1 - \sqrt{1-q^2} = \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!2^n} q^{2n},$$

which converges if $\|q\|_{L^\infty} < 1$ and involves only positive coefficients. By invoking again the Banach algebra property, we obtain

$$\begin{aligned}\|f(q)\|_{H^s} &\le \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!2^n} \|q^{2n}\|_{H^s} \le \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!2^n} C_s^{2n-1} \|q\|_{H^s}^{2n} \\ &= \frac{1}{C_s} f(C_s \|q\|_{H^s}) \le C_s \|q\|_{X^s}^2,\end{aligned}$$

if $C_s\|q\|_{H^s} < 1$, thanks to the definition (2.4) and the representation (2.21). As a result, we have

$$\|q(t)\|_{H^s} \le \|q_0\|_{H^s} + C_s \int_0^t \|q(t')\|_{X_c^s}^3 dt'$$

To estimate the L^2 norm of $p(t)$, we use the integral representation

$$p(t) = P(t) - \int_0^t L e^{(t-t')L} f(q(t'))p(t')dt',$$

where $P(t) = LQ(t)$ is defined by solution of the linear problem (2.16). Now use the triangle inequality and the norm-preservation property, to derive

$$\begin{aligned}\|p(t)\|_{L^2} &\le \|P(t)\|_{L^2} + \int_0^t \|Le^{(t-t')L} f(q(t'))p(t')\|_{L^2} dt', \\ &\le \|p_0\|_{L^2} + \int_0^t \|Le^{(t-t')L} f(q(t'))p(t')\|_{L^2} dt'.\end{aligned}$$

The norm preservation is not useful for the second term because $Lf(q(t))p(t)$ may not be in L^2. Since $J_t(y) = J_0(2\sqrt{ty})$ is bounded for any $t, y \in \mathbb{R}_+$ and

$$\|f(q)p\|_{L^1} \le \|f(q)\|_{L^1}\|p\|_{L^\infty} \le \|q\|_{L^2}^2 \|p\|_{H^1},$$

we represent the operator Le^{tL} acting on $f(q)p \in L^1$ in the convolution form (2.18), or explicitly by

$$Le^{(t-t')L} f(q(t'))p(t') = -\int_y^\infty J_{t-t'}(y'-y) f(q(y',t'))p(y',t')dy'. \qquad (2.25)$$

Using the Hausdorf-Young inequality (2.6), we obtain

$$\|Le^{(t-t')L}f(q(t'))p(t')\|_{L^2} \leq \|J_{t-t'}\|_{L^\infty}\|f(q(t'))p(t')\|_{L^{2/3}} \leq \|f(q(t'))p(t')\|_{L^{2/3}}.$$

Using the Hölder inequality (2.5), we obtain

$$\|f(q(t))p(t)\|_{L^1} \leq \|f(q(t))\|_{L^\rho}\|p(t)\|_{L^r},$$

with $\rho^{-1} + r^{-1} = 1$, so that

$$\|Le^{(t-t')L}f(q(t'))p(t')\|_{L^2} \leq \|f(q(t'))\|_{L^{2\rho/3}}\|p(t')\|_{L^{2r/3}}.$$

If we choose $r = 3$, then we have $\rho = \frac{3}{2}$ and $\|f(q)\|_{L^1} \leq \|q\|^2_{L^2}$. Thus, we conclude that

$$\|p(t)\|_{L^2} \leq \|p_0\|_{L^2} + \int_0^t \|q(t')\|^3_{X^s}dt',$$

for some $C > 0$. Altogether the above estimates give a bound on the solution norm

$$\|q(t)\|_{X^s} \leq \|q_0\|_{X^s} + C\int_0^t \|q(t')\|^3_{X^s}dt', \tag{2.26}$$

where $C > 0$. By continuity of $\|q(t)\|_{X^s}$, for any $\alpha \in (0,1)$, there is a $T > 0$ such that if $\|q_0\|_{H^s} < \alpha \min(B_s^{-1}, C_s^{-1})$, then $\|q(t)\|_{X^s} < \min(B_s^{-1}, C_s^{-1})$ for any $t \in [0,T]$.

We also need to prove that the constraint $\|q(t)\|_{L^\infty} < 1$ is satisfied for some $0 < t < \infty$. Estimating the L^∞ norm of $q(t)$ from the integral equation (2.23) we obtain

$$\|q(t)\|_{L^\infty} \leq \|Q(t)\|_{L^\infty} + \int_0^t \|e^{(t-t')L}f(q(t'))p(t')\|_{L^\infty}dt'.$$

Using the convolution formula (2.14) and the Hausdorff-Young inequality (2.6), the free term is estimated by

$$\|Q(t)\|_{L^\infty} \leq \|q_0\|_{L^\infty} + \|K_t\|_{L^2}\|q_0\|_{L^2} \leq \|q_0\|_{L^\infty} + C_1t^{1/2}\|q_0\|_{X^s},$$

for some $C_1 > 0$. The nonlinear term is estimated by

$$\begin{aligned}\|e^{(t-t')L}f(q(t'))p(t')\|_{L^\infty} &\leq \|f(q(t'))p(t')\|_{L^\infty} + \|K_{t-t'}\|_{L^\infty}\|f(q(t'))p(t')\|_{L^1} \\ &\leq C_2(1+t-t')\|q(t')\|^3_{X^s},\end{aligned}$$

for some $C_2 > 0$. Finally, we conclude that

$$\|q(t)\|_{L^\infty} \leq \|q_0\|_{L^\infty} + C_1 t^{1/2}\|q_0\|_{X^s} + C_2\int_0^t (1+t-t')\|q(t')\|^3_{X^s}dt'. \quad (2.27)$$

One can derive some explicit estimates for $T_* = T_*(\alpha, \delta)$, defined above, using the inequalities (2.26) and (2.27). That is, if we require that the solution to Cauchy problem for (2.23), with the initial norm $\|q_0\|_{X^s} \leq \alpha\delta$ and $\|q_0\|_{L^\infty} < \alpha$ for some $\alpha \in (0,1)$ and $\delta \in (0, \min(B_s^{-1}, C_s^{-1}))$, remains in the X_c^s with $\|q(t)\|_{X^s} \leq \delta$ and $\|q(t)\|_{L^\infty} < 1$ for any $t \in [0, T_*(\alpha,\delta)]$ then the bounds

$$\begin{aligned}\alpha\delta + CT_*\delta^3 &\leq \delta, \\ \alpha + C_1T_*^{1/2}\alpha\delta + C_2T_*\left(1+\frac{1}{2}T_*\right)\delta^3 &< 1\end{aligned}$$

would give the range of values for T_*.

It is left to prove that the map defined by the integral part of (2.23)

$$(Aq)(t) = \int_0^t e^{(t-t')L}f(q(t'))(\partial_x^{-1}q)(t')dt'$$

is a contraction in the X^s space on $[0,T]$ for $T \in (0, T_*)$. That is, for

$$\sup_{t\in[0,T]}\|q_1(t)\|_{X^s} \leq \delta \quad \text{and} \quad \sup_{t\in[0,T]}\|q_2(t)\|_{X^s} \leq \delta$$

there is $k \in (0,1)$ such that the inequality

$$\sup_{t\in[0,T]}\|Aq_1 - Aq_2\|_{X^s} \leq k \sup_{t\in[0,T2]}\|q_1 - q_2\|_{X^s}, \quad (2.28)$$

is satisfied. We show existence of such a constant k by a direct computation. The definition of the X^s norm (2.3) yields a convenient bound

$$\|Aq_1 - Aq_2\|_{X^s} \leq \underbrace{\|Aq_1 - Aq_2\|_{H^s}}_{(a)} + \underbrace{\|L(Aq_1 - Aq_2)\|_{L^2}}_{(b)}.$$

Let us consider the components (a) and (b) separately.

(a) Using the norm-preserving property (2.13), triangle inequality and Banach algebra property for the Sobolev space we obtain

$$\begin{aligned}
\|Aq_1 - Aq_2\|_{H^s} &\leq \int_0^t \|e^{(t-t')L}(f(q_1)p_1 - f(q_2)p_2)\|_{H^s} dt' \\
&\leq T \sup_{t\in[0,T]} \|f(q_1)p_1 - f(q_2)p_2\|_{H^s} \\
&\leq T \sup_{t\in[0,T]} \|f(q_1)(p_1 - p_2) + (f(q_1) - f(q_2))p_2\|_{H^s} \\
&\leq C_s T \sup_{t\in[0,T]} \left(\|f(q_1)\|_{H^s}\|p_1 - p_2\|_{H^s} + \|f(q_1) - f(q_2)\|_{H^s}\|p_2\|_{H^s}\right).
\end{aligned}$$

To estimate these norms we use the inequalities $\|f(q)\|_{H^s} \leq C_s\|q\|^2_{H_s}$ and $\|p\|_{H^s} \leq \|q\|_{X^s}$, where the latter follows from the identity $\|p\|_{H^{s+1}} = \|q\|_{X^s}$. By positivity of coefficients a_{nk} in Taylor series

$$\frac{1}{\sqrt{1-q_1^2} + \sqrt{1-q_1^2}} = \sum_{n,k=0}^{\infty} a_{nk} q_1^{2n} q_2^{2k}$$

we obtain a bound on the remaining term $\|f(q_1) - f(q_2)\|_{H^s}$:

$$\begin{aligned}
\|f(q_1) - f(q_2)\|_{H^s} &= \left\| \frac{q_1^2 - q_2^2}{\sqrt{1-q_1^2} + \sqrt{1-q_2^2}} \right\|_{H^s} \\
&\leq C_s \left\| \frac{1}{\sqrt{1-q_1^2} + \sqrt{1-q_2^2}} \right\|_{H^s} \|q_1^2 - q_2^2\|_{H^s} \\
&\leq C_s^2 \left(\sum_{n,k=0}^{\infty} a_{nk} C_s^{2n+2k-1} \|q_1\|^{2n}_{H^s} \|q_2\|^{2k}_{H^s} \right) \\
&\quad \times \|q_1 + q_2\|_{H^s} \|q_1 - q_2\|_{H^s} \\
&\leq C_s^2 \frac{1}{C_s} \frac{1}{\sqrt{1 - C_s^2\|q_1\|^2_{H^s}} + \sqrt{1 - C_s^2\|q_2\|^2_{H^s}}} 2\delta \|q_1 - q_2\|_{H^s} \\
&\leq \frac{\delta C_s}{\sqrt{1 - C_s^2\delta^2}} \|q_1 - q_2\|_{X^s}.
\end{aligned}$$

As a result, we arrive at the following estimate

$$\|Aq_1 - Aq_2\|_{H^s} \leq T\delta^2 C_s^2 \left(1 + \left(1 - C_s^2\delta^2\right)^{-1/2}\right) \sup_{t\in[0,T]} \|q_1 - q_2\|_{X^S}.$$

(b) Here we use the convolution representation (2.18), the Hausdorff-Young inequality and boundedness of the kernel function J_t:

$$\begin{aligned}
\|LAq_1 - LAq_2\| &\leq T \sup_{t\in[0,T]} \|J_{t-t'}\|_{L^\infty} \|f(q_1)p_1 - f(q_2)p_2\|_{L^{2/3}} \\
&\leq T \sup_{t\in[0,T]} \|f(q_1)(p_1 - p_2)\|_{L^{2/3}} + \|(f(q_1) - f(q_2))p_2\|_{L^{2/3}} \\
&\leq T \sup_{t\in[0,T]} (\|f(q_1)\|_{L^1}\|p_1 - p_2\|_{L^2} + \|f(q_1) - f(q_2)\|_{L^1}\|p_2\|_{L^2}) \\
&\leq T \sup_{t\in[0,T]} \|q_1\|_{L^2}^2 \|q_1 - q_2\|_{X^s} \\
&\quad + T \sup_{t\in[0,T]} \|q_2\|_{X^s} \left\| \frac{1}{\sqrt{1-q_1^2} + \sqrt{1-q_2^2}} \right\|_{L^\infty} \|q_1^2 - q_2^2\|_{L^1} \\
&\leq T\delta^2 \sup_{t\in[0,T]} \|q_1 - q_2\|_{X^s} \\
&\quad + T\delta \sup_{t\in[0,T]} \frac{\|q_1 + q_2\|_{L^2}\|q_1 - q_2\|_{L^2}}{\sqrt{1 - \|q_1\|_{L^\infty}^2} + \sqrt{1 - \|q_2\|_{L^\infty}^2}} \\
&\leq T\delta^2 \left(1 + \left(1 - B_s^2\delta^2\right)^{-1/2}\right) \sup_{t\in[0,T]} \|q_1 - q_2\|_{X^s}.
\end{aligned}$$

As a result, we obtain the following value for the constant k in (2.28) :

$$k = T\delta^2 \left(C_s^2 + C_s^2 \left(1 - C_s^2\delta^2\right)^{-1/2} + 1 + \left(1 - B_s^2\delta^2\right)^{-1/2}\right).$$

So that there exists $T > 0$ such that $k < 1$.

□

Corollary 2.6 *Under the conditions of Theorem 2.5, we actually have* $q(t) \in C([0,T], X_c^s) \cap C^1([0,T], H^s)$.

Proof. The assertion follows from the facts that $q_t = \sqrt{1-q^2}p$ with $q(t) \in C([0,T], X_c^s)$ and $p(t) \in C([0,T], H^{s+1})$. □

Remark 2.7 Existence of a unique solution can be proved more easily in a weaker space $C([0,T], \tilde{X}_c^s)$, where

$$\tilde{X}_c^s = \left\{ q \in \tilde{X}^s : \|q\|_{L^\infty} < 1 \right\}$$

and $\tilde{X}^s = \{q(t) \in H^s,\ \ p(t) \in L^\infty\}$, provided that $q_0 \in X_c^s$. Since $p \in H^1$ if $q, p \in L^2$, we note that Sobolev's embedding gives embedding $X_c^s \hookrightarrow \tilde{X}_c^s$. The space X_c^s turns out to be more suitable if we are to use conserved quantities of the sine-Gordon equation.

2.5 Correspondence between the short-pulse and sine-Gordon equations

We start by stating the local well-posedness theorem for the short-pulse equation from [27].

Theorem 2.8 (Schäfer & Wayne, 2004) *Let $u_0 \in H^s$ for a fixed $s \geq 2$. There exists a $T > 0$ such that the short-pulse equation (1.1) admits a unique solution*

$$u(t) \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$$

satisfying $u(0) = u_0$. Furthermore, the solution $u(t)$ depends continuously on u_0.

We can now compare the results following from Theorems 2.5 and 2.8. Using the transformation (1.11) and setting $q = \sin(w)$, we have

$$\begin{aligned} u(x,t) &= w_t(y,t) = \frac{q_t}{\sqrt{1-q^2}} = p(y,t), \\ u_x(x,t) &= \frac{w_{ty}}{\cos(w)} = \tan(w) = \frac{q}{\sqrt{1-q^2}}. \end{aligned} \tag{2.29}$$

If $q(t) \in X_c^s$, $s \geq 1$, for all $t \in [0,T]$, then there exists a uniform bound $q_0 \in (0,1)$ such that $\|q(t)\|_{L^\infty} \leq q_0$ for all $t \in [0,t]$. As a result, the H^s norm of u in x is equivalent to the H^s norm of p in y, or, since, $p_y = q$, the H^{s-1}

norm on q in y. The following lemma summarizes this correspondence between the norms.

Lemma 2.9 *Assume that* $\|q(t)\|_{L^\infty} \leq q_0 < 1$ *uniformly on* $[0, T]$ *for some* $T > 0$ *and consider the transformation (2.29). There exist* $c, C > 0$ *such that*

$$c\|p(t)\|_{H^s} \leq \|u(t)\|_{H^s} \leq C\|p(t)\|_{H^s}$$

uniformly on $[0, T]$.

Proof. The proof is given by direct computations, e.g.

$$\sqrt{1 - q_0^2}\|p\|_{L^2}^2 \leq \|u\|_{L^2}^2 \leq \|p\|_{L^2}^2,$$
$$\|\partial_y p\|_{L^2}^2 \leq \|\partial_x u\|_{L^2}^2 \leq \frac{1}{\sqrt{1 - q_0^2}}\|\partial_y p\|_{L^2}^2,$$

and so on. □

Combining Theorems 2.5 and 2.8 with Lemma 2.9, we obtain a more precise result on local well-posedness of the short-pulse and sine-Gordon equations.

Theorem 2.10 *Let* $q(t) \in C([0, T_1], X_c^{s-1}) \cap C^1([0, T_1], H^s)$ *be a solution of the sine-Gordon equation in Theorem 2.5 and Corollary 2.6 for some* $s \geq 2$ *and* $T_1 > 0$. *Let* $u(t) \in C([0, T_2], H^s) \cap C^1([0, T_2], H^{s-1})$ *be a solution of the short-pulse equation in Theorem 2.8 for the same* $s \geq 2$ *and some* $T_2 > 0$. *Let* q_0 *and* u_0 *be related by the transformations (1.11) and (2.29). Then, in fact,* $p(t) \in C^1([0, T], H^s)$ *and* $u(t) \in C^1([0, T], H^s)$ *for* $T = \min(T_1, T_2)$, *where* $p_y = q$.

Proof. If $q(t) \in X_c^{s-1}$ on $[0, T_1]$, then the bound $\|q(t)\|_{L^\infty} \leq q_0$ holds on $[0, T_1]$ for some $q_0 \in (0, 1)$. By Lemma 2.9, if $p(t) \in C([0, T_1], H^s)$ then $u(t) \in$

$C([0,T_1],H^s)$ and if $q(t) \in C^1([0,T_1],X_c^{s-1})$ then $u_x \in C^1([0,T_1],H^{s-1})$. The first constraint recovers the result of Theorem 2.8, while if $T = \min(T_1,T_2)$, the second constraint combing with $u(t) \in C^1([0,T],H^{s-1})$ implies that $u(t) \in C^1([0,T],H^s)$.

In the opposite direction, by Lemma 2.9, if $u(t) \in C([0,T_2],H^s) \cap C^1([0,T_2],H^{s-1})$, then $p(t) \in C([0,T_2],H^s) \cap C^1([0,T_2],H^{s-1})$. Combining this with the condition $q(t) \in C^1([0,T],X_c^{s-1})$ in Theorem 2.5 for $T = \min(T_1,T_2)$, we obtain that $p(t) \in C^1([0,T],H^s)$. □

Remark 2.11 Theorem 2.10 shows that the results on the sine-Gordon equation (1.3) allow us to control the C^1 property of $\|\partial_x^s u\|_{L^2}$ in the short-pulse equation (1.1), while the results on the short-pulse equation (1.1) allow us to control the C^1 property of $\|p\|_{L^2}$ in the sine-Gordon equation (1.3).

2.6 Global well-posedness of the short-pulse equation

The existence time $T > 0$ in Theorems 2.8 and 2.10 is inverse proportional to the norm $\|u_0\|_{H^s}$ of the initial data to the short-pulse equation (1.1). In order to prove the theorem on global well-posedness of the short-pulse equation (1.1) we need to show the possibility of controlling the norm $\|u(T)\|_{H^s}$ by a T-independent constant. This constant will be found from the values of conserved quantities in Section 1.5. Using Theorem 2.10, we shall make a rigorous use of the first three conserved quantities.

Lemma 2.12 *Let $u(t) \in C^1([0,T],H^2)$ be a solution of the short-pulse equa-*

tion (1.1). The following integral quantities are constant on $[0,T]$*:*

$$\begin{aligned}
H_0 &= \int_{\mathbb{R}} u^2 dx, \\
H_1 &= \int_{\mathbb{R}} \left(\sqrt{1+u_x^2} - 1\right) dx, \\
H_2 &= \int_{\mathbb{R}} \sqrt{1+u_x^2} \left[\partial_x \left(\frac{u_x}{\sqrt{1+u_x^2}}\right)\right]^2 dx.
\end{aligned}$$

Proof. We shall write the balance equations for the densities of H_0, H_1, and H_2:

$$\begin{aligned}
\partial_t \left(u^2\right) &= \partial_x \left(v^2 + \frac{1}{4}u^4\right), \\
\partial_t \left(\sqrt{1+u_x^2} - 1\right) &= \frac{1}{2}\partial_x \left(u^2\sqrt{1+u_x^2}\right), \\
\partial_t \left(\frac{u_{xx}^2}{\sqrt{(1+u_x^2)^5}}\right) &= \partial_x \left(\frac{2u_x^2}{\sqrt{1+u_x^2}} - \frac{u^2u_{xx}^2}{2\sqrt{(1+u_x^2)^5}}\right),
\end{aligned}$$

where $v = \partial_x^{-1}u = u_t - \frac{1}{2}u^2u_x$ thanks to the short-pulse equation (1.1). If $u(t) \in C^1([0,T], H^2)$, then $v(t) \in C([0,T], H^1)$. By Sobolev's embedding, we have $v(t)$, $u(t)$, $u_x(t) \in L^\infty$ and $v(t)$, $u(t)$, $u_x(t) \to 0$ as $|x| \to \infty$ for any $t \in [0,T]$. Integrating the first two balance equations on $\mathbb{R}$, we confirm conservation of H_0 and H_1. To prove conservation of H_2, we need to show that $uu_{xx} \to 0$ as $|x| \to \infty$ for any $t \in [0,T]$. Using (1.1) and (1.11), we obtain

$$\frac{1}{2}uu_{xx} - u_x^2 = \frac{u_{xt}}{u} - 1 = \tan^2(w) = \frac{q^2}{1-q^2},$$

where $u_x \to 0$ as $|x| \to \infty$ and $q = q(y,t)$. If $\|q\|_{L^\infty} \leq q_0 < 1$, then

$$\frac{\partial x}{\partial y} = \cos(w) = \sqrt{1-q^2} \geq \sqrt{1-q_0^2},$$

for any $t \in [0,T]$. As a result, the limits $y \to \pm\infty$ correspond to the limits $x \to \pm\infty$ and $uu_{xx} \to 0$ as $|x| \to \infty$ follows from $q \to 0$ as $|y| \to \infty$. $\square$

Theorem 2.13 *Assume that $u_0 \in H^2$ and the conserved quantities satisfy $2H_1 + H_2 < 1$. Then the short-pulse equation (1.1) admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.*

Proof. The values of H_0, H_1 and H_2 computed at initial data $u_0 \in H^2$ are finite thanks to the explicit estimates

$$\begin{aligned} H_0 &= \int_{\mathbb{R}} u^2 dx \le \|u_0\|_{H^2}^2, \\ H_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1+\sqrt{1+u_x^2}} dx \le \frac{1}{2}\|u_0\|_{H^2}^2, \\ H_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1+u_x^2)^{5/2}} dx \le \|u_0\|_{H^2}^2. \end{aligned}$$

By Lemma 2.12, these quantities remain constant on $[0,T]$. We will show that the quantities H_1 and H_2 give an upper bound for H^1 norm of the variable

$$\tilde{q} = \frac{u_x}{\sqrt{1+u_x^2}}. \tag{2.30}$$

Note that $\tilde{q}(x,t) = q(y,t) = \sin(w(y,t))$, where $x = x(y,t)$ is defined by the transformation (1.11). To control $\|\tilde{q}\|_{H^1}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \tilde{q}^2 dx = \int_{\mathbb{R}} \frac{u_x^2}{1+u_x^2} dx &= \int_{\mathbb{R}} \frac{u_x^2}{1+\sqrt{1+u_x^2}} \frac{1+\sqrt{1+u_x^2}}{1+u_x^2} dx \\ &\le 2\int_{\mathbb{R}} \frac{u_x^2}{1+\sqrt{1+u_x^2}} dx = 2H_1 \end{aligned}$$

and

$$\int_{\mathbb{R}} \tilde{q}_x^2 dx = \int_{\mathbb{R}} \left[\partial_x\left(\frac{u_x}{\sqrt{1+u_x^2}}\right)\right]^2 dx \le \int_{\mathbb{R}} \sqrt{1+u_x^2}\left[\partial_x\left(\frac{u_x}{\sqrt{1+u_x^2}}\right)\right]^2 dx = H_2.$$

If $u(t) \in C([0,T], H^2)$, then $\tilde{q}(t) \in C([0,T], H^1)$ and $\tilde{q}(t)$ satisfies the T-independent bound

$$\|\tilde{q}(t)\|_{H^1} \le \sqrt{2H_1 + H_2},$$

for any $t \in [0, T]$. Let us assume that $2H_1 + H_2 < 1$. Thanks to Sobolev's embedding $\|\tilde{q}\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|\tilde{q}\|_{H^1}$, we have $\|\tilde{q}(t)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\sqrt{2H_1 + H_2} < 1$. Inverting the map (2.30), we obtain

$$u_x = \frac{\tilde{q}}{\sqrt{1 - \tilde{q}^2}}.$$

Since $\|\tilde{q}(t)\|^2_{H^1} < 2H_1 + H_2 < 1$ we can derive an estimate

$$\|u_x\|_{H^1} \leq \frac{\|\tilde{q}\|_{H^1}}{\sqrt{1 - \|\tilde{q}\|^2_{H^1}}}$$

using a Taylor series expansion and Banach algebra property for Sobolev space H_1 with constant $C_1 = 1$. This results in the T-independent bound

$$\|u(T)\|_{H^2} \leq \left(H_0 + \frac{2H_1 + H_2}{1 - (2H_1 + H_2)^2}\right)^{1/2}$$

The constraint $2H_1 + H_2 < 1$ guarantees boundedness of the above expression. This allows us to choose a constant time step T_0 such that the solution $u(T_0)$ can be continued on the interval $[T_0, 2T_0]$ in space $C^1([T_0, 2T_0], H^2)$ using the same Theorems 2.8 and 2.10. Continuing the solution with a uniform time step $T_0 > 0$, we obtain global existence of solutions in space $u(t) \in C(\mathbb{R}_+, H^2)$, which completes the proof of Theorem 2.13. □

Remark 2.14 Using the whole infinite hierarchy of conserved quantitie in Section 1.5 it might be possible to extend Theorem 2.13 for $u_0 \in H^s$ for an integer $s \geq 2$. The proof would be similar for any integer $s > 2$ but more conserved quantities will be needed. If $s = 2$ is fixed, however, we need only three conserved quantities described in Lemma 2.12.

Corollary 2.15 *Assume that $u_0 \in H^2$ and the conserved quantities satisfy $2\sqrt{2H_1 H_2} < 1$. Then the short-pulse equation (1.1) admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.*

Proof. Suppose $u(x,t)$ is a solution to the short-pulse equation (1.1), then scaling invariance (1.2) gives a one-parameter family of solutions $U(X,T)$:

$$U(X,T) = \alpha u(x,t), \quad X = \alpha x, \quad T = \alpha^{-1} t,$$

where $\alpha \in \mathbb{R}_+$. This yields the following transformation for the conserved quantities

$$\tilde{H}_1 = \int_{\mathbb{R}} \frac{U_X^2}{1+\sqrt{1+U_X^2}} dX = \alpha \int_{\mathbb{R}} \frac{u_x^2}{1+\sqrt{1+u_x^2}} dx = \alpha H_1$$
$$\tilde{H}_2 = \int_{\mathbb{R}} \frac{U_{XX}^2}{(1+U_X^2)^{5/2}} dX = \alpha^{-1} \int_{\mathbb{R}} \frac{u_{xx}^2}{(1+u_x^2)^{5/2}} dx = \alpha^{-1} H_2.$$

Therefore

$$2\tilde{H}_1 + \tilde{H}_2 = 2\alpha H_1 + \alpha^{-1} H_2.$$

Function $\phi(\alpha) = 2\alpha H_1 + \alpha^{-1} H_2$ achieves its minimum of $2\sqrt{2H_1H_2}$ at $\alpha = \sqrt{\frac{H_2}{2H_1}}$, so that $2\tilde{H}_1 + \tilde{H}_2 \geq 2\sqrt{2H_1H_2}$ for all $\alpha \in \mathbb{R}_+$. If

$$2\sqrt{2H_1H_2} < 1$$

there exists some α near $\sqrt{H_2/(2H_1)}$ so that $2\tilde{H}_1 + \tilde{H}_2 < 1$ (even if $2H_1 + H_2 > 1$). By Theorem 2.13 the corresponding solution is $U(T) \in C(\mathbb{R}_+, H^2)$, so that by scaling transformation $u(t) \in C(\mathbb{R}_+, H^2)$. □

Corollary 2.16 *Assume $u_0 \in H^2$ and the solution $u(t)$ to the short-pulse equation (1.1) breaks at a finite time $t_* > 0$, so that $\lim_{t \uparrow t_*} \|u(t)\|_{H^2} = \infty$. Then, the corresponding conserved quantities satisfy $2\sqrt{2H_1H_2} \geq 1$.*

Proof. The necessary condition for the wave breaking is the negation to the sufficient condition for the global well-posedness in Corollary 2.15. □

2.7 Global well-posedness of the sine-Gordon equation

The sine-Gordon equation (1.3) has an infinite set of conserved quantities similarly to the short-pulse equation (1.1). These conserved quantities can be enumerated by the order $j \geq 0$ in the term $(\partial_y^j w)^2$ involving the highest spatial derivative. We will use only the first two conserved quantities,

$$E_0 = \int_{\mathbb{R}} (1 - \cos w) dy, \quad E_1 = \int_{\mathbb{R}} w_y^2 dy,$$

existence of which follow formally from the balance equations

$$\partial_t (1 - \cos w) = \partial_y \left(\frac{1}{2} w_t^2 \right), \quad \partial_t \left(\frac{1}{2} w_y^2 \right) = \partial_y (1 - \cos w).$$

Additionally, the sine-Gordon equation (1.3) has another infinite set of conserved quantities involving trigonometric functions of w and their integrals enumerated by $-j \leq 0$ in the term $(\partial_t^j w)^2$. Besides E_0, we need only one conserved quantity of this set,

$$E_{-1} = \int_{\mathbb{R}} w_t^2 \cos w dy,$$

existence of which follows formally from the balance equation

$$\partial_t \left(w_t^2 \cos w \right) = \partial_y \left(w_{tt}^2 - \frac{1}{4} w_t^4 \right).$$

Using the transformation $q = \sin(w)$, we rewrite the conserved quantities in the equivalent form

$$E_{-1} = \int_{\mathbb{R}} \sqrt{1 - q^2} p^2 dy, \quad E_0 = \int_{\mathbb{R}} f(q) dy, \quad E_1 = \int_{\mathbb{R}} \frac{q_y^2}{1 - q^2} dy, \qquad (2.31)$$

where $p = \partial_y^{-1} q$ and $f(q)$ is defined by (2.21). The balance equations are rewritten in the corresponding forms

$$\partial_t f(q) = \partial_y \left(\frac{1}{2} p^2 \right), \quad \partial_t \left(\frac{q_y^2}{1 - q^2} \right) = \partial_y f(q)$$

and

$$\partial_t\left(\sqrt{1-q^2p^2}\right) = \partial_y\left(p_t^2 - \frac{1}{4}p^4\right).$$

We shall check if E_1, E_0, and E_{-1} are time conserved quantities for the Cauchy problem (2.22) if $s = 2$ is fixed in Theorem 2.10. Global well-posedness in H^2 follows from analysis of the three conserved quantities. A similar analysis can be developed for any integer $s > 2$ but more conserved quantities are needed in this case.

Lemma 2.17 *Let $q(t) \in C^1([0,T], X_c^1)$ and $p(t) \in C^1([0,T], H^2)$ be the solution of the Cauchy problem (2.22) and $q(t) = \partial_y p(t)$. Then, E_1, E_0, and E_{-1} are constant for any $t \in [0,T]$.*

Proof. By Sobolev's embedding for $p(t) \in C^1([0,T], H^2)$, we have $q(t)$, $p(t)$, $p_t(t) \to 0$ as $|y| \to \infty$. Therefore, conservation of E_1, E_0, and E_{-1} follows by integrating the balance equations in y on $\mathbb{R}$. □

Theorem 2.18 *Assume that $q_0 \in X_c^1$ and $2E_0 + E_1 < 1$ for the conserved quantities (2.31). Then there exist a unique global solution $q(t) \in C(\mathbb{R}_+, X_c^1)$ of the Cauchy problem (2.22) satisfying $q(0) = q_0$.*

Proof. Let us show that the values of E_{-1}, E_0, E_1 are finite if $q_0 \in X_c^1$. Indeed, provided that $\|q_0\|_{L^\infty} < 1$, we have

$$E_{-1} \leq \|p_0\|_{L^2}^2, \quad |E_0| \leq \|q_0\|_{L^2}^2, \quad E_1 \leq \frac{1}{1-\|q_0\|_{L^\infty}^2}\|\partial_y q_0\|_{L^2}^2.$$

By Lemma 2.17, if $q(t) \in C^1([0,T], X_c^1)$ and $p(t) \in C^1([0,T], H^2)$ is a solution constructed in Theorems 2.5 and 2.10 for a fixed $T > 0$, the values of quantities E_{-1}, E_0, E_1 are constant on $t \in [0,T]$. Therefore, we only need to bound from above the norm $\|q(t)\|_{X^1}$ by a combination of values of E_{-1}, E_0, E_1. This bound is obtained from the following estimates

$$E_{-1} \geq \|p(t)\|_{L^2}^2 \sqrt{1 - \|q(t)\|_{L^\infty}^2}, \quad E_0 \geq \frac{1}{2}\|q(t)\|_{L^2}^2, \quad E_1 \geq \|\partial_y q(t)\|_{L^2}^2.$$

By Sobolev's embedding and the bounds above, we have

$$\|q(t)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|q(t)\|_{H^1} \leq \frac{1}{\sqrt{2}}\sqrt{E_1 + 2E_0} < 1,$$

since $E_1 + 2E_0 < 1$. As a result, we obtain the bound

$$\|q(t)\|_{X^1} \leq \left(E_1 + 2E_0 + \frac{E_{-1}}{\sqrt{1 - \frac{1}{2}(E_1 + 2E_0)}} \right)^{1/2}, \quad \forall t \in [0, T].$$

The time step $T > 0$ depends on $\|q(0)\|_{X^1}$. Since the above norm is bounded by the T-independent constant, one can choose a non-zero time step T_0 such that the solution can be continued on the interval $[T_0, 2T_0]$ using the same Theorems 2.5 and 2.10. Continuing the solution with a uniform time step T_0, we obtain global existence of solutions $q(t) \in C(\mathbb{R}_+, X_c^1)$. □

Remark 2.19 Theorem 2.18 is almost identical to Theorem 2.13 owing to the correspondence between the two equations in Lemma 2.9. In particular, it follows directly that $H_0 = E_{-1}$, $H_1 = E_0$ and $H_2 = E_1$.

Chapter 3

Numerical simulations of the short-pulse equation

We illustrate here our analytical results on global well-posedness of the short-pulse equation (1.1) by some numerical computations. It is worth recalling the paper by Kanattšikov & Pietrzyk [17] where the short-pulse equation was treated numerically through its multisymplectic Hamiltonian form. The authors claim their discretization allows for preservation of multisimplectic conservation laws. Our numerical approach is based on the pseudospectral method which allows us to solve the short-pulse equation in a periodic domain by means of the discrete Fourier transform. This numerical scheme is described in detail by Trefethen in [31].

3.1 Pseudospectral method

Let $u(x,t) = u(x+2\pi, t)$ be a solution of the short-pulse equation (1.1) and consider $\mathbb{T} = [-\pi, \pi]$ as the fundamental interval for u in x. We partition this interval into an *even* number N of subintervals thus obtaining a spatial discretization at the grid points

$$x_n = h\left(-\tfrac{N}{2} + n\right), \quad h = \tfrac{2\pi}{N}, \quad n \in \{1, 2, \ldots, N\},$$

where h is a mesh spacing. We denote the value of the numerical approximation to a solution $u(x,t)$ at the grid point x_n by $u_n(t)$. A discrete Fourier transform of $\{u_n(t)\}$ is defined componentwise by

$$\hat{u}_k(t) = (\mathcal{F}\{u_n\})_k := h\sum_{n=1}^{N} e^{ikx_n} u_n(t), \quad k \in \left\{-\tfrac{N}{2}+1, -\tfrac{N}{2}+2, \ldots, \tfrac{N}{2}\right\}. \tag{3.1}$$

The inverse discrete Fourier transform is written as

$$u_n(t) = \left(\mathcal{F}^{-1}\{\hat{u}\}\right)_n := \frac{1}{2\pi}\sum_{k=-N/2+1}^{N/2} e^{ikx_n}\hat{u}_k(t), \quad n \in \{1, 2, \ldots, N\}. \tag{3.2}$$

Using the discrete Fourier transform (3.1) we rewrite the short-pulse equation (1.1) in a discrete Fourier space as follows:

$$\frac{\partial}{\partial t}\hat{u}_k = -\frac{i}{k}\hat{u}_k + \frac{ik^2}{6}\mathcal{F}\left[\left(\mathcal{F}^{-1}\hat{u}\right)^3\right]_k, \quad k \neq 0, \quad t \in \mathbb{R}_+. \tag{3.3}$$

On the other hand, we have

$$\hat{u}_0(t) = 0, \quad \forall t \in \mathbb{R}_+.$$

We will use the sixth-order Runge-Kutta method to approximate solutions of the systems of ODEs (3.3). The solution in physical space is obtained by the inverse discrete Fourier transform (3.2).

3.2 Evolution of Gaussians

We numerically simulate the initial-value problem for the ODE systems (3.3) using with the initial data in the form of a Gaussian pulse:

$$u(x,0) = ae^{-bx^2} - c, \tag{3.4}$$

where $a > 0$ and $b > 0$ are arbitrary parameters and c is defined from the condition

$$\hat{u}_0(0) = h \sum_{n=1}^{N} u(x_n, 0) = 0.$$

Along with parameters a and b governing the amplitude and steepness of the Gaussian pulse, we also compute numerically parameters

$$\eta = 2H_1 + H_2 = 2\int_{-\pi}^{\pi} \frac{u_x^2}{1+\sqrt{1+u_x^2}}dx + \int_{-\pi}^{\pi} \frac{u_{xx}^2}{(1+u_x^2)^{5/2}}dx$$

and

$$\kappa = 2\sqrt{2H_1H_2}.$$

Theorem 2.13 and Corollary 2.15 on the global well-posedness guarantees that if the initial data $u_0 \in H^2(\mathbb{T})$ satisfies $\eta < 1$ or $\kappa < 1$ then the solution remains in $H^2(\mathbb{T})$ for all times $t \in \mathbb{R}_+$. However, this is just a sufficient condition on the global well-posedness which may not be sharp. Indeed, we will show that depending on the shape of $u_0 \in H^2(\mathbb{T})$ with $\eta > 1$ or $\kappa > 1$ the solution may remain in $H^2(\mathbb{T})$ or it may leave this space exhibiting wave breaking in a finite time. Table 3.1 lists some parameters of Gaussian initial data (3.4) in numerical simulations.

Figure	a	b	η	κ	H_1	H_2	$\|u_0\|^2_{H^2(\mathbb{T})}$
3.1	0.05	20	0.83	0.21	0.0070	0.82	0.86
3.2	1	3	10.72	7.80	0.84	9.04	22.27
3.3	1	4	14.09	9.49	0.92	12.24	33.09
3.4	0.05	300	36.20	2.68	0.025	36.15	48.90

Table 3.1: Parameters of initial data in the form of the Gaussian pulse (3.4)

If the amplitude a is small and the decay rate b is large with $\eta < 1$ ($\kappa < 1$), we do not observe formation of singularities (Figure 3.1). As it was expected, the Gaussian pulse breaks into a wave packet travelling leftwards. The conserved quantities guarantee that the solution does not decay to zero on the circle $\mathbb{T}$.

Even if the parameters η and κ exceed the critical value 1 the solution can remain smooth. On Figure 3.2 we show evolution of the Gaussian pulse with moderate values of a and b such that $\eta \approx 10.72$ ($\kappa \approx 7.80$). Although this initial condition has a greater amplitude and a slower decay rate than that for Figure 3.1, the numerical approximation exhibits similar behaviour. However, if we change the decay constant from $b = 3$ to $b = 4$, so that $\eta \approx 14.09$ ($\kappa \approx 9.49$) the numerical solution breaks in finite time (Figure 3.3).

The solution can remain nonsingular for some initial conditions with fairly large values of η but moderate values of κ. On Figure 3.4 we demonstrate evolution of a very narrow small-amplitude pulse with $\eta \approx 36.20$ which exhibits nonsingular behaviour. This phenomenon can be explained by the scaling transformation (1.2) since the value of $\kappa \approx 2.68$ (that gives a minimum of η along the solution family) is not large.

3.3 Evolution of perturbed pulses

Stability of the exact pulse solutions (1.20) to the short-pulse equation (1.1) has not been addressed in the literature yet. In this section we describe numerical simulations suggesting stability of pulse solutions.

First of all, it is instructive to evaluate the values of parameters $\eta = 2H_1 + H_2$ (cf. Theorem 2.13) and $\kappa = 2\sqrt{2H_1H_2}$ (cf. Corollary 2.15) for exact pulse solutions (1.20). Using breather solutions of the sine-Gordon equation

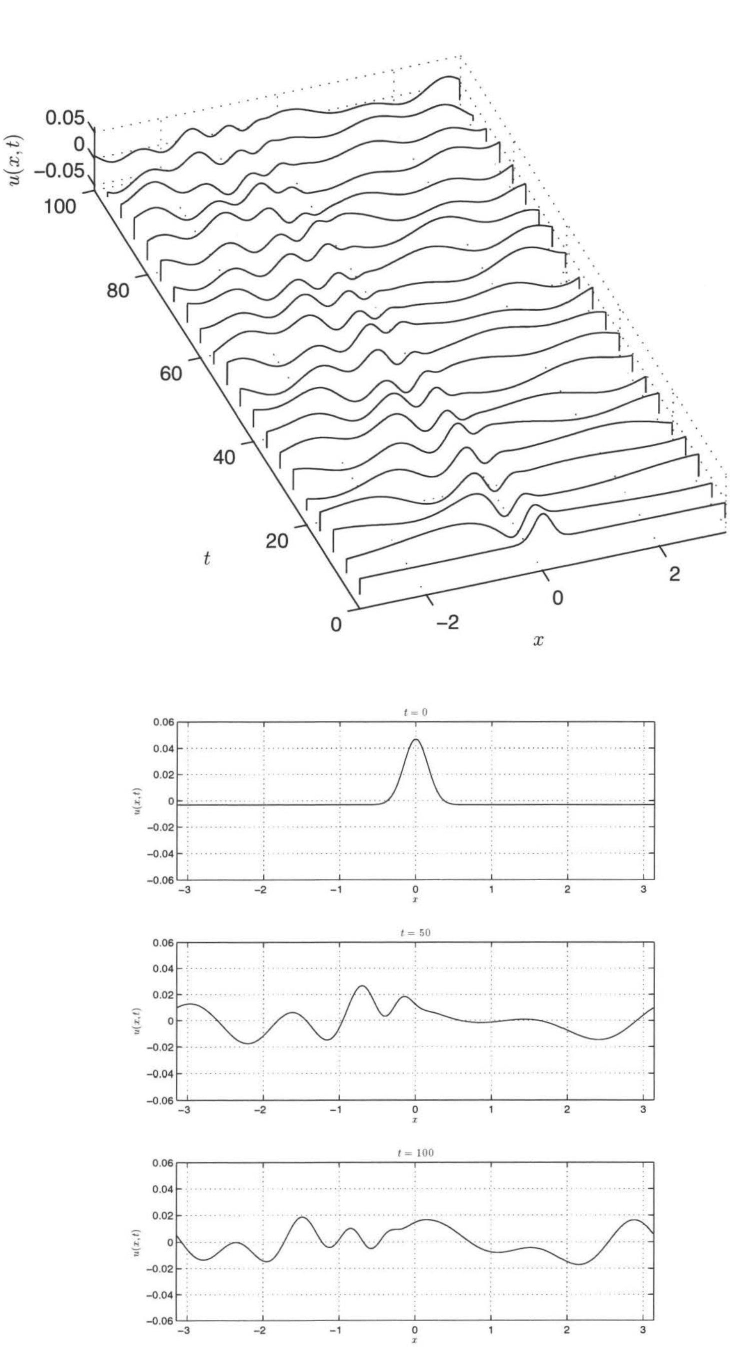


Figure 3.1: Evolution of the Gaussian pulse (3.4) with $a = 0.05$, $b = 20$, $\eta \approx 0.83 < 1$ and $\kappa \approx 0.21 < 1$.

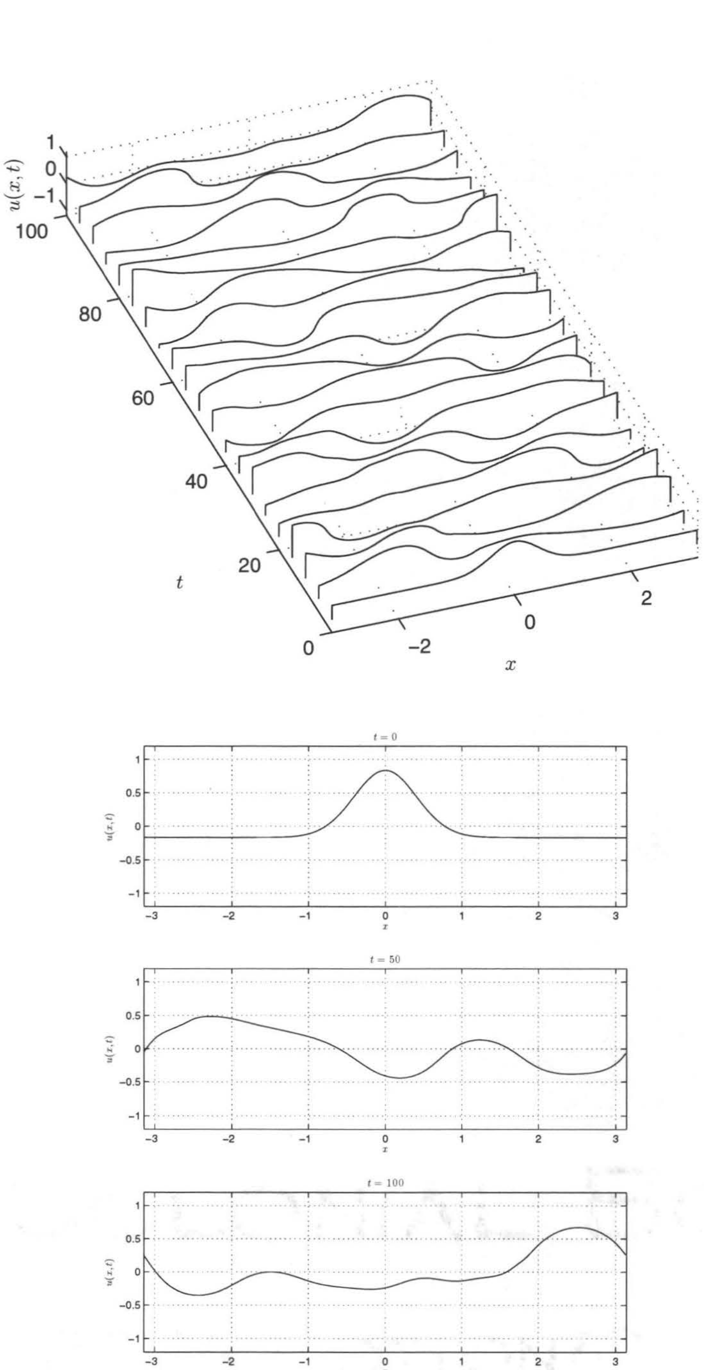


Figure 3.2: Evolution of the Gaussian pulse (3.4) with $a = 1$, $b = 3$, $\eta \approx 10.72 > 1$ and $\kappa \approx 7.79 > 1$.

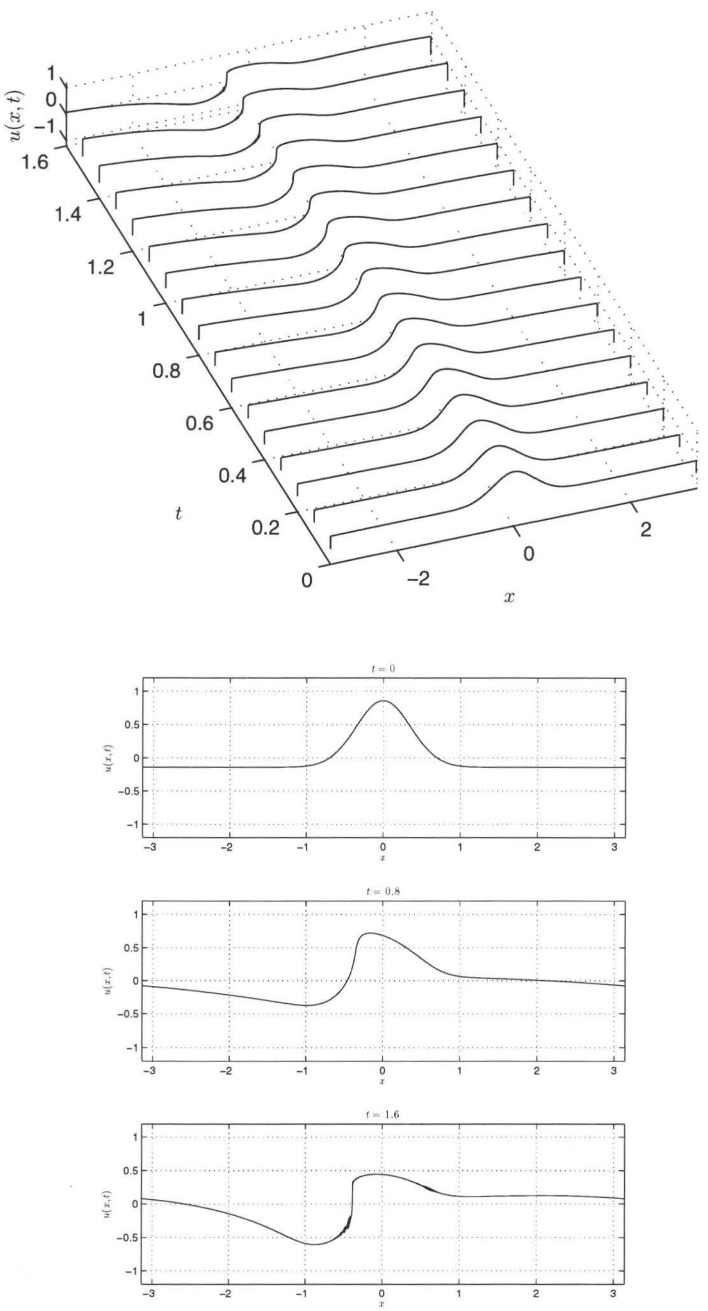


Figure 3.3: Evolution of the Gaussian pulse (3.4) with $a = 1$, $b = 4$, $\eta \approx 14.09 > 1$ and $\kappa \approx 9.49 > 1$.

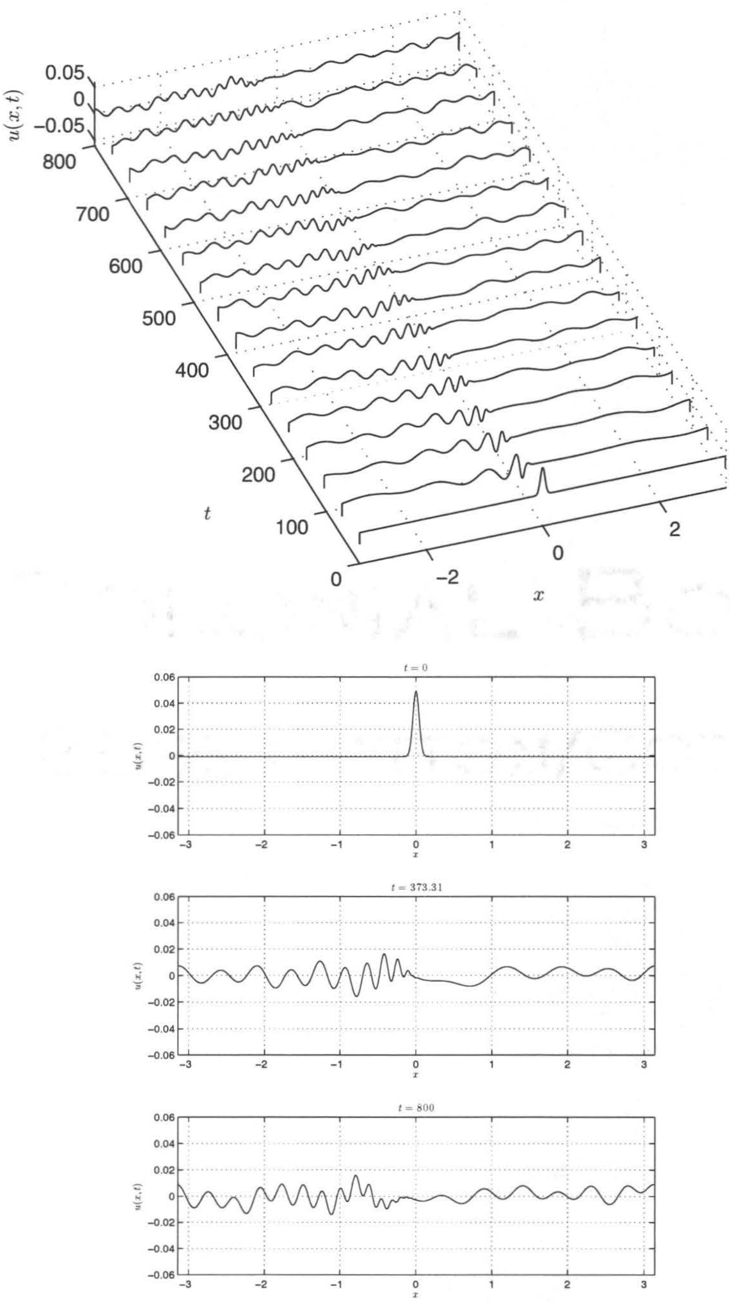


Figure 3.4: Evolution of the Gaussian pulse (3.4) with $a = 0.05$, $b = 300$, $\eta \approx 36.20 > 1$ and $\kappa \approx 2.68 > 1$.

Figure	m	ϵ	η	κ	H_1	H_2	$\|u_0\|^2_{H^2(\mathbb{T})}$
3.5	0.20	0.1	184.13	7.48	0.038	184.06	276.22
3.6	0.20	-0.1	136.17	5.35	0.026	136.12	184.91
3.7	0.30	0.05	251.18	10.21	0.052	251.07	903.87
3.8	0.30	-0.05	228.10	8.95	0.044	227.81	739.90

Table 3.2: Parameters of perturbed pulses obtained from (1.20) after rescaling (1.2) with $\alpha = 0.02$ and multiplication of u by the factor $(1+\epsilon)$.

with $2E_0 = E_1 = 16m$, we obtain

$$\begin{aligned}\eta &= 2H_1 + H_2 = 2E_0 + E_1\\ &= 2\int_{\mathbb{R}} (1-\cos w)dy + \int_{\mathbb{R}} w_y^2 dy = 32m,\end{aligned}$$

where $w(y,t)$ is given by (1.18). Since $2H_0 = H_1 = 16m$, we have $\kappa = \eta = 32m$. It is clear that $\eta = \kappa < 1$ for $m < 1/32$ and $\eta = \kappa > 1$ for $1/32 < m < m_{cr} \approx 0.38$, so that the terminal value of $\eta_{cr} = \kappa_{cr} = 32m_{cr} \approx 12.26$. This shows that the condition $\eta < 1$ of Theorem 2.13 and condition $\kappa < 1$ of Corollary 2.15 are not sharp enough, as evolution of the short-pulse equation starting with the pulse solutions does not lead to wave breaking for $t \in \mathbb{R}_+$.

Using the same numerical scheme as in Section 3.2 we consider evolution of initial data in the form of the exact pulse solution (1.20) multiplied by a factor $1+\epsilon$, with $\epsilon \in \mathbb{R}$ being a sufficiently small number. In addition, we apply scaling transformation (1.2) with $\alpha = 0.02$ to make sure that pulses fit the spatial frame of $[-\pi, \pi]$ and vanish to numerical zero on its boundary. Parameters of the initial data in the form of perturbed pulses are presented in Table 3.2. We should note that a big difference between values of η and κ is due to smallness of the scaling factor $\alpha = 0.02$. On Figure 3.5 we show what happens to a perturbed pulse solution with $m = 0.20$ and $\epsilon = 0.1$. This pulse exhibits stability under a perturbation. It moves more rapidly than the unperturbed pulse due to a higher amplitude. Some radiation propagating "out of the pulse" is also observed. If we take the same initial data with a perturbation $\epsilon = -0.1$ the pulse moves slower then the unperturbed one due to a lower amplitude and remains stable with respect to the perturbation, see Figure 3.6. On Figures 3.7–3.8 the evolution of a shorter perturbed pulse with $m = 0.30$ and $\epsilon = \pm 0.05$ is shown. The perturbation is taken smaller than in the case of Figures 3.5–3.6 to avoid wave breaking. As a result, both pulses with a higher and lower amplitude demonstrate stable propagation.

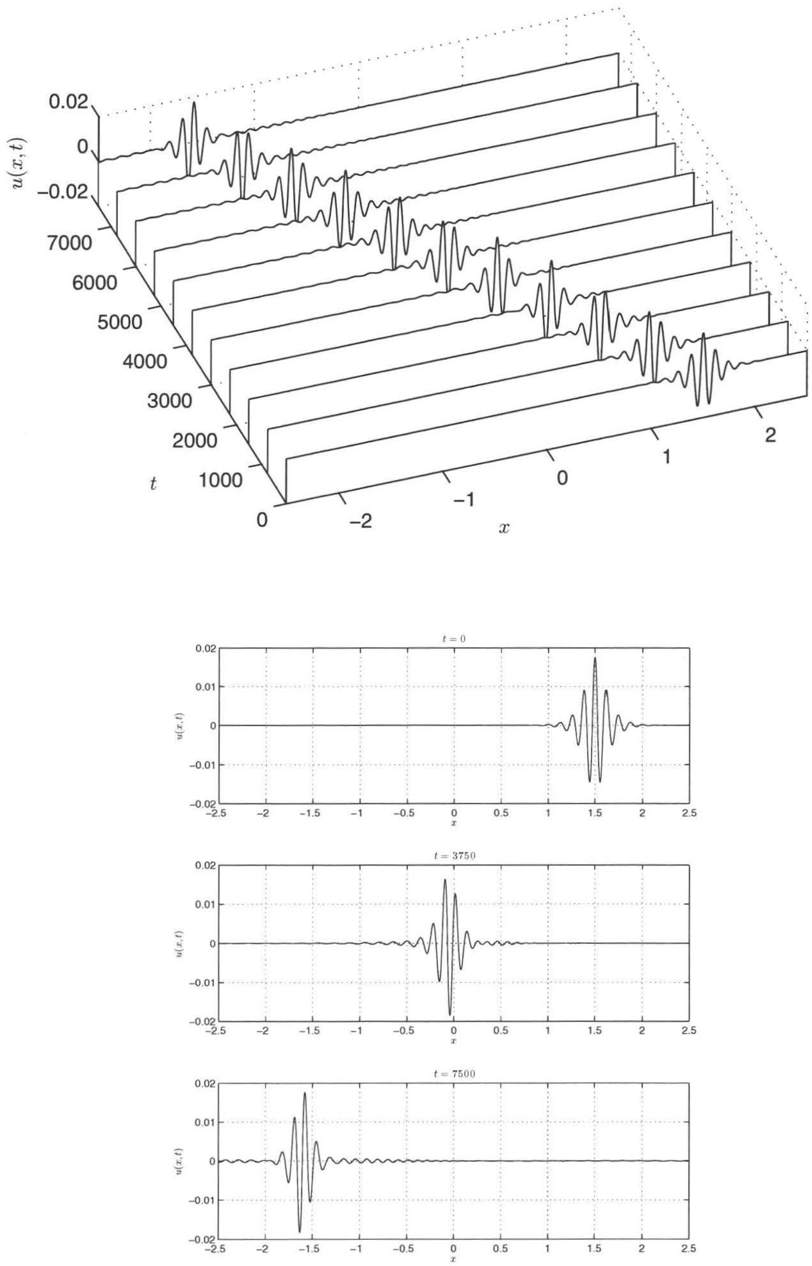


Figure 3.5: Evolution of the perturbed pulse with $m = 0.20$, $\epsilon = 0.1$

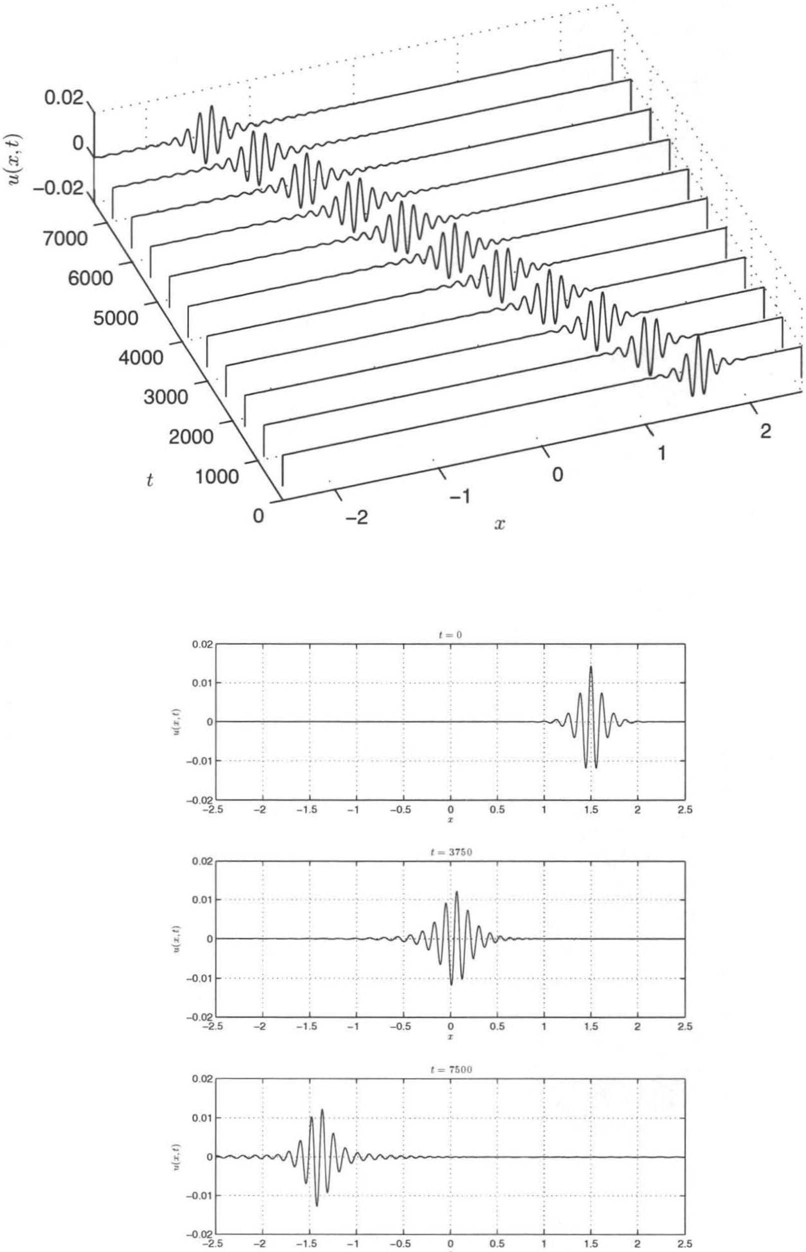


Figure 3.6: Evolution of the perturbed pulse with $m = 0.20$, $\epsilon = -0.1$

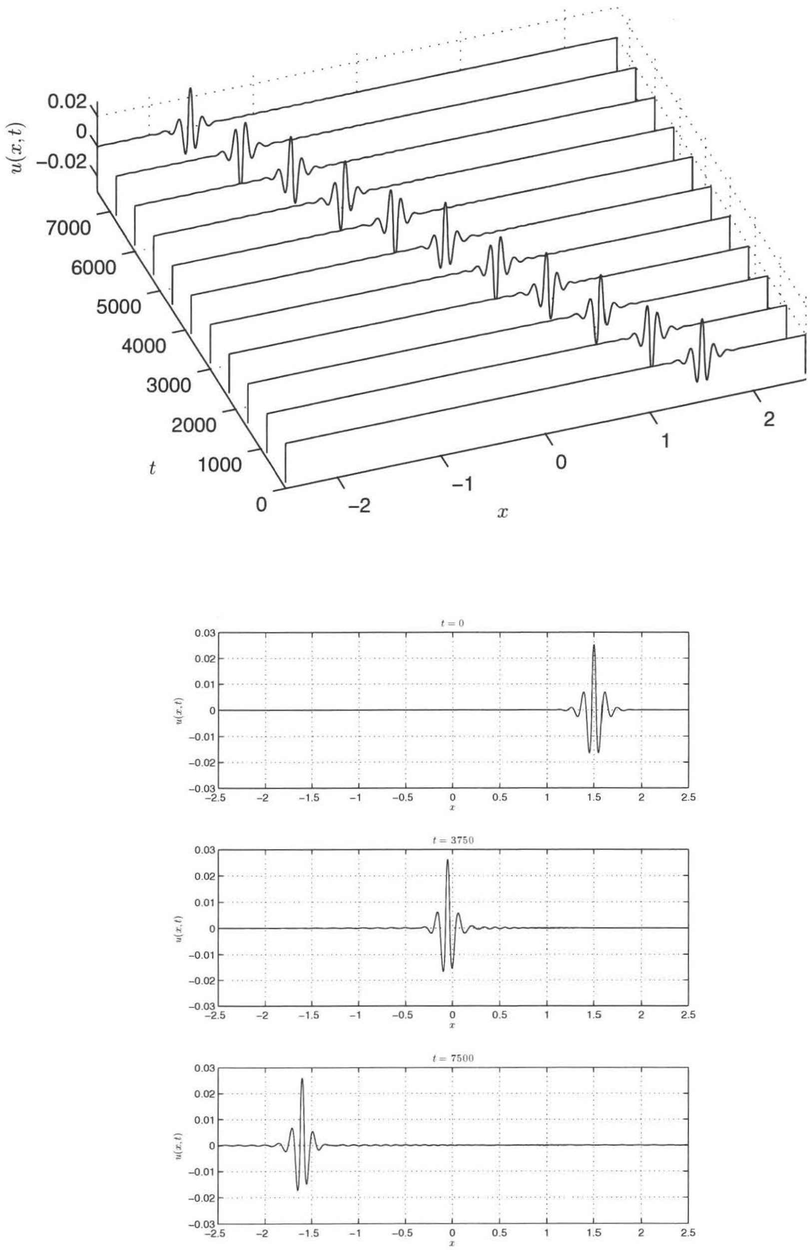


Figure 3.7: Evolution of the perturbed pulse with $m = 0.30$, $\epsilon = 0.05$

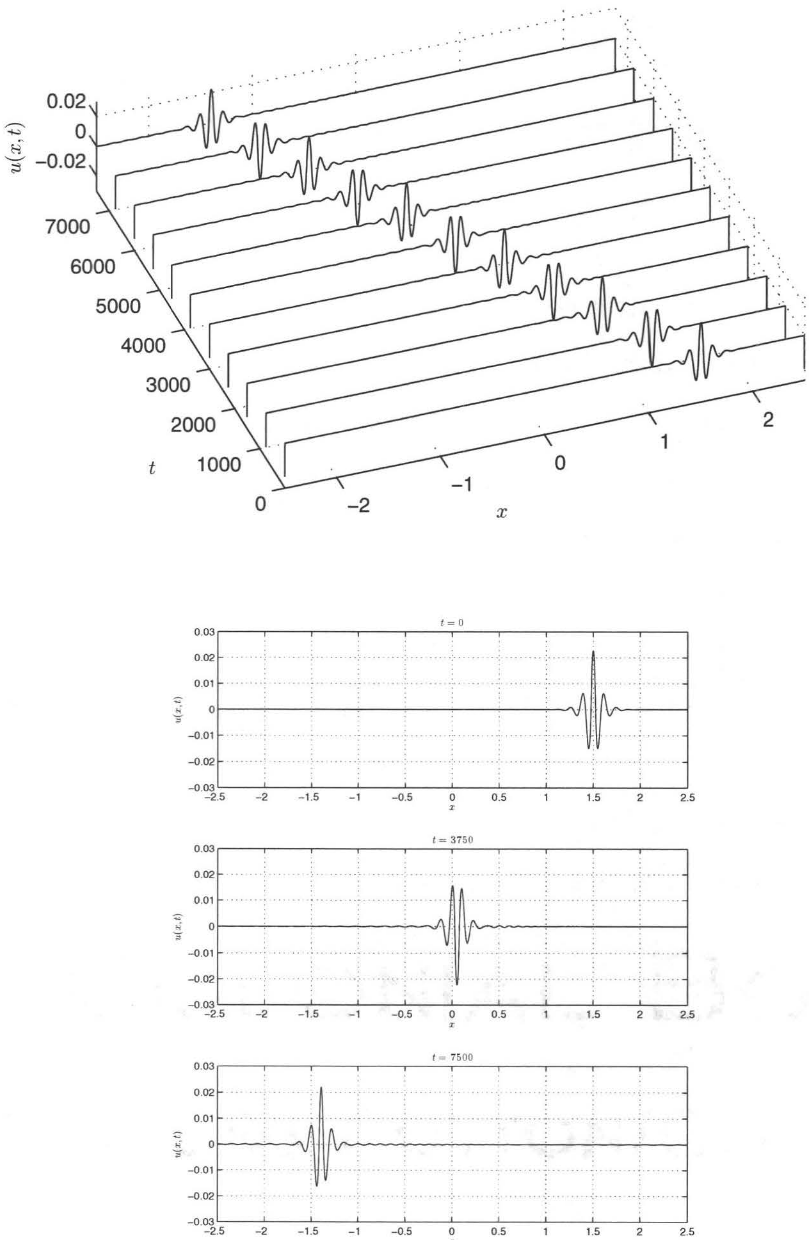


Figure 3.8: Evolution of the perturbed pulse with $m = 0.30$, $\epsilon = -0.05$

Summary and open problems

We have obtained a sufficient condition on global well-posedness of the short-pulse equation in its energy space. In accordance with this result, numerical simulations demonstrated smooth behaviour of a numerical approximation when the condition is satisfied and the possibility of wave breaking when it is violated. Some computations suggesting stability of exact pulse solutions were also presented.

We proved local and global well-posedness of the sine-Gordon equation in characteristic coordinates for small amplitudes. This gives some control on regularity of solutions to the short-pulse equation, due to the relation between these two equations.

It is worth to establish sharper conditions determining well-posedness of the short-pulse equation and the wave breaking criteria. It is also important to study regularity and wave breaking criteria for similar but more advanced models, such as the reduced Ostrovsky equation (2.9) with $\beta = 0$.

MATLAB codes for Chapter 3

This script was used to compute evolution of Gaussian pulses, generate Figures 3.1–3.4 and fill out Table 3.1 in Section 3.2.

```
%*************************************************************
% Computing evolution of Gaussians                           *
%*************************************************************

tic; % start stopwatch
clear all; close all;
a = 1; b = 3; % define amplitude and steepness of a Gaussian

% compute parameters of the Gaussian
n = 3000; x = linspace(-pi,pi,n+1);
dx = x(2)-x(1);
u = a*exp(-b*x.^2);
v = fft(u); v(1) = 0; v(n/2 + 1) = 0;
u = ifft(v);
ux = -2*a*b*x.*exp(-b*x.^2);
uxx = 2*a*b*(2*b*x.^2-1).*exp(-b*x.^2);
h22 = u.^2 + ux.^2 + uxx.^2;
```

```
h1 = ux.^2./(1 + sqrt(1 + ux.^2));
h2 = uxx.^2./(1+ux.^2).^(5/2);
% integrate by Simpson's rule
H22 = (dx/3)*(h22(1)+4*sum(h22(2:2:n))+2*sum(h22(3:2:n-1))+h22(n+1));
H1 = (dx/3)*(h1(1)+4*sum(h1(2:2:n))+2*sum(h1(3:2:n-1))+h1(n+1));
H2 = (dx/3)*(h2(1)+4*sum(h2(2:2:n))+2*sum(h2(3:2:n-1))+h2(n+1));
semilogy(x,u);
fprintf('eta = %g, kappa = %g\n', 2*H1+H2, 2*sqrt(2*H1*H2));
fprintf('H_1 = %g, H_2 = %g, H_2^2 = %g\n', H1, H2, H22);

% solve the short-pulse equation by pseudospectral method
clear n x dx u ux uxx h0 h1 H0 H1;
tmax = 10;
N = 1024; dt = .01; % number of spatial grid points and time step
x = (2*pi/N)*(-N/2:N/2-1)';
u = a*exp(-b*x.^2);
v = fft(u); v(1) = 0; v(N/2 + 1) = 0;
u = ifft(v);
H = 1.2*a;
xb = pi; L = 2*pi; xa = xb - L; % endpoints on x axis
na = 1;
nb = length(x);

figure(1); plot(x,u,'-k'); grid on;
pbaspect([3 1 1]);
axis([xa xb -H H]);
xlabel('$x$','interpreter','latex');
```

```
ylabel('$u(x,t)$','interpreter','latex');
title('$t = 0$','interpreter','latex');
saveas(gcf,'1.eps');

nplt = floor((tmax/20)/dt); % number of subplots
nmax = round(tmax/dt); % number of time steps
udata = u; tdata = 0;
h = waitbar(0,'please wait...');
k = [0.01 1:N/2-1 0.01 -N/2+1:-1]';
c1 = -i*dt./k; c2 = i*k*dt/6;
v(1) = 0; v(N/2 + 1) = 0;
for n = 1:nmax
  t = n*dt;
  a = c1.*v       + c2.*fft(real( ifft(v) ).^3);
  b = c1.*(v+a/2) + c2.*fft(real( ifft(v+a/2) ).^3);  % 4th-order
  c = c1.*(v+b/2) + c2.*fft(real( ifft(v+b/2) ).^3);  % Runge-Kutta
  d = c1.*(v+c)   + c2.*fft(real( ifft(v+c) ).^3);    % method
  v = v + (1/6)*(a + 2*(b + c) + d);
  v(1) = 0; v(N/2 + 1) = 0;
  if mod(n,nplt) == 0
    u = real(ifft(v));
    udata = [udata u]; tdata = [tdata t];
    waitbar(n/nmax);
  end
end
close(h);
toc; % stop stopwatch
```

```
figure(2); % plot u at t=T/2
k = size(udata);
nn = round(k(2)/2);
tt = tdata(nn);
plot(x(na:nb),udata(na:nb,nn),'k-'); grid on;
xlabel('$x$','interpreter','latex');
ylabel('$u(x,t)$','interpreter','latex');
axis([xa xb -H H]);
title(['$t = $ ',num2str(tt,'%g')],'interpreter','latex');
pbaspect([3 1 1]);
saveas(gcf,'2.eps');

figure(3); % plot u at t=T/2
k = size(udata);
nn = round(k(2)-1);
tt = tdata(nn);
plot(x(na:nb),udata(na:nb,nn),'k-'); grid on;
xlabel('$x$','interpreter','latex');
ylabel('$u(x,t)$','interpreter','latex');
axis([xa xb -H H]);
title(['$t = $ ',num2str(tt,'%g')],'interpreter','latex');
pbaspect([3 1 1]);
saveas(gcf,'3.eps');

figure(4); % plot u at t=T
plot(x(na:nb),udata(na:nb,k(2)),'k-'); grid on;
```

```
xlabel('$x$','interpreter','latex');
ylabel('$u(x,t)$','interpreter','latex');
axis([xa xb -H H]);
title(['$t = $ ',num2str(tmax,'%g')],'interpreter','latex');
pbaspect([3 1 1]);
saveas(gcf,'4.eps');

figure(5); % plot solution surface
waterfall(x(na:nb),tdata,udata(na:nb,:)');
colormap(1e-6*[1 1 1]);
view(-20,35);
xlabel('$x$','interpreter','latex');
ylabel('$t$','interpreter','latex');
zlabel('$u(x,t)$','interpreter','latex');
axis([xa xb 0 tmax -H H]);
pbaspect([1 2 .2]);
```

This script was used to compute the evolution of perturbed pulses and generate Figures 3.5–3.8 in Section 3.3.

```
%****************************************************************
% Computing evolution of perturbed pulses                       *
%****************************************************************

tic; % start stopwatch
clear all; close all;
r = 0.02; % scaling parameter $\alpha$
x_shift = 1.5; % initial position of center of the pulse
```

```
tmax = 10; x_shift*2/(r^2); % time of travel
N = 3072; dt = .03; % number of spatial grid point and time step
x = (2*pi/N)*(-N/2:N/2-1)';

% set up initial data
m = .30; nm = sqrt(1-m^2); % geometry of the pulse
y = linspace(-200,200,9007); % a parameter to be eliminated
ph = m*y; ps = nm*y;
A = 1 - 0.05;
denom = m^2*(sin(ps).^2) + nm^2*(cosh(ph)).^2;
uExact = A*r*4*m*nm*(m*sin(ps).*sinh(ph)+nm*cos(ps).*cosh(ph))./denom;
xExact = x_shift + r*(y + 2*m*nm*(m*sin(2*ps)-nm*sinh(2*ph))./denom);
u = spline(xExact, uExact, x); % make spline approximation
L = 5;  H = 0.03; % define a window for a graph
xb = 2.5; xa = xb - L; % endpoints on x axis
na = round( (xa+pi)*N/(2*pi) ); % margins in array
nb = round( (xb+pi)*N/(2*pi) );
figure(1); plot(x,u,'-k'); grid on;
pbaspect([3 1 1]); % aspect ratio
axis([xa xb -H H]);
xlabel('$x$','interpreter','latex');
ylabel('$u(x,t)$','interpreter','latex');
title('$t = 0$','interpreter','latex');
saveas(gcf,'1.eps');

nplt = floor((tmax/10)/dt); % number of subplots
nmax = round(tmax/dt); % number of time steps
```

```
udata = u; tdata = 0; % set up the errays for data
h = waitbar(0,'please wait...');
v = fft(u);
k = [0.01 1:N/2-1 0.01 -N/2+1:-1]';
c1 = -i*dt./k; c2 = i*k*dt/6;
v(1) = 0; v(N/2 + 1) = 0;
for n = 1:nmax
  t = n*dt;
  a = c1.*v       + c2.*fft(real( ifft(v) ).^3);
  b = c1.*(v+a/2) + c2.*fft(real( ifft(v+a/2) ).^3); % 4th-order
  c = c1.*(v+b/2) + c2.*fft(real( ifft(v+b/2) ).^3); % Runge-Kutta
  d = c1.*(v+c)   + c2.*fft(real( ifft(v+c) ).^3);   % method
  v = v + (1/6)*(a + 2*(b + c) + d);
  v(1) = 0; v(N/2 + 1) = 0;
  if mod(n,nplt) == 0
    u = real(ifft(v));
    udata = [udata u]; tdata = [tdata t];
    waitbar(n/nmax);
  end
end
close(h);

figure(2); % plot u at t=T/2
k = size(udata);
nn = round(k(2)/2);
tt = tdata(nn);
plot(x(na:nb),udata(na:nb,nn),'k-'); grid on;
```

```
xlabel('$x$','interpreter','latex');
ylabel('$u(x,t)$','interpreter','latex');
axis([xa xb -H H]);
title(['$t = $ ',num2str(tt,'%g')],'interpreter','latex');
pbaspect([3 1 1]);
saveas(gcf,'2.eps');

figure(3); % plot u at t=T
plot(x(na:nb),udata(na:nb,k(2)),'k-'); grid on;
xlabel('$x$','interpreter','latex');
ylabel('$u(x,t)$','interpreter','latex');
axis([xa xb -H H]);
title(['$t = $ ',num2str(tmax,'%g')],'interpreter','latex');
pbaspect([3 1 1]);
saveas(gcf,'3.eps');

figure(4); % a solution surface
waterfall(x(na:nb),tdata,udata(na:nb,:)');
colormap(1e-6*[1 1 1]);
view(-20,35);
xlabel('$x$','interpreter','latex');
ylabel('$t$','interpreter','latex');
zlabel('$u(x,t)$','interpreter','latex');
axis([xa xb 0 tmax -H H]);
pbaspect([1 1 .2]);
```

This code was used to compute parameters of perturbed pulses and fill out Table 3.2 in Section 3.3.

```
%*****************************************************************
% This script computes the values of parameters $\eta$ and       *
% $\kappa$ for perturbed pulses                                  *
%*****************************************************************


clear all; close all;
r = 0.02; % scaling factor \alpha
epsilon = -0.05; % perturbation
A = 1 + epsilon; % multiplier for the amplitude
N = 4000; % number of mesh points
x = linspace(-pi,pi,N)';
m = .30; nm = sqrt(1-m^2); % pulse parameters m and n
y = linspace(-400,400,9007); % a parameter to be eliminated

% generate a perturbed pulse solution
ph = m*y; ps = nm*y;
denom = m^2*(sin(ps).^2) + nm^2*(cosh(ph)).^2;
uExact = A*r*4*m*nm*(m*sin(ps).*sinh(ph)+nm*cos(ps).*cosh(ph))./denom;
xExact = r*(y + 2*m*nm*(m*sin(2*ps)-nm*sinh(2*ph))./denom);

% plot for the pulse solution
u = spline(xExact, uExact, x);
plot(x,u,'-k');  grid on;
pbaspect([3 1 1]);
```

```
H = .03;
axis([-1 1 -H H]);


ux = zeros(N,1);
uxx = zeros(N,1);
dx = x(2)-x(1);
% numerical derivatives with error ~ O(h^2)
for k = 2:N-1
    ux(k) = (u(k+1)-u(k-1))/(2*dx);
    uxx(k) = (u(k+1)-2*u(k)+u(k-1))/(dx^2);
end
h22 = u.^2 + ux.^2 + uxx.^2;
% compute integrals by Simpson's rule, error ~ O(h^3)
% Sobolev's H_2 norm squared
H22 = (dx/3)*(h22(1)+4*sum(h22(2:2:N-2))+2*sum(h22(3:2:N-1))+h22(N));
h1 = sqrt(1 + ux.^2) - 1;
h2 = uxx.^2./(1+ux.^2).^(5/2);
H1 = (dx/3)*(h1(1)+4*sum(h1(2:2:N-2))+2*sum(h1(3:2:N-1))+h1(N));
H2 = (dx/3)*(h2(1)+4*sum(h2(2:2:N-2))+2*sum(h2(3:2:N-1))+h2(N));
fprintf('eta = %g, kappa = %g\n', 2*H1+H2, 2*sqrt(2*H1*H2));
fprintf('H_1 = %g, H_2 = %g, H_2^2 = %g\n', H1, H2, H22);
% for A = 1 the below value must be approximately
% equal to \eta. test of accuracy
fprintf('16m(r + 1/r) = %g\n', 16*m*( r + 1/r ));
```

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