ASPECTS OF MESH THEORY

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By

Daniel Hambleton

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AUTHOR:	Daniel Hambleton, B.A. (McGill University)
SUPERVISOR:	Andrew Nicas
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To My Parents

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Abstract

The main purpose of this paper is to give a thorough account of planar quadrilateral and offset meshes. To this end, abstract meshes, the linear spaces $C(M)$, $P(M)$, and the distance functions $d_*(M_1, M_2)$ are defined and applied to planar quadrilateral meshes. We then study the discrete analogue of the Gauss map as well as the defining properties of circular and conical meshes. The proof of the angle condition for conical meshes is given and used as motivation for the study of Lie sphere geometry. We apply the theory of Laguerre and Moebius transformations to conical and circular meshes respectively. All of this theory is then applied to the creation of a planar quadrilateral mesh with the face offset property in the context of a study project.

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Chapter 1

Introduction

It is fairly unusual for a problem in the construction industry to lead to the development of a rich mathematical theory, but in the case of architectural geometry, this is exactly what happened. Originally, the problem concerned the design and panelization of freeform glass surfaces. Architects, structural engineers, and mathematicians alike sought answers to questions such as: Is it possible to panel an arbitrary shape with planar quadrilateral panels? Why, when the surface is designed as a two dimensional object, do so many errors occur when structural elements are implemented? Since many innovative and abstract shapes have been designed and succesfully realized as buildings, it is clear that these questions are not insurmountable on an individual basis. However, as projects became increasingly complex, a need arose to deal with these design problems in a more efficient and systematic way. It turned out that the key to solving these problems lay in combining the fields of architecture, discrete differential geometry, and optimization. Although mathematics has always been an integral part of the design and construction process, it was not until the study of panelization began that architectural geometry became known as a new and emerging field. Since 2005, there have been major breakthroughs concerning panelization of freeform surfaces, the latest of which was published by Pottman et al; [7], [8], [6]. These papers are primarily summaries of main results. It is my belief that in order to make the next step in architectural geometry, the theory and methods behind these results should be more transparent. In this thesis, I will attempt to give a thorough account of the theory behind the panelization problem, concrete methods for its implementation, and demonstrate how these methods can be applied to a study project.

In general, discrete differential geometry deals with multidimensional discrete nets, called meshes, that display certain geometric properties. From this point of view, discrete surfaces are simply two dimensional layers of multidimensional nets, and the classical transformations, such as the Backlund, Darboux, and Bianchi transformations, can then be viewed as shifts in the transverse net directions. This realization led to a more fundamental understanding of some important geometrical constructions. Moreover, the continuous theory can be recovered by refining the mesh in two directions to obtain the continuous surfaces and leaving the transverse directions discrete, as shown in Figure 1.1 [3]. Since the heart of the panelization problem lies in understanding how to attach layers to a freeform surface that perserve certain properties, Figure 1.1 also suggests that discrete differential geometry will play an important part in this understanding.

In the world of architecture and building science, the panelization problem

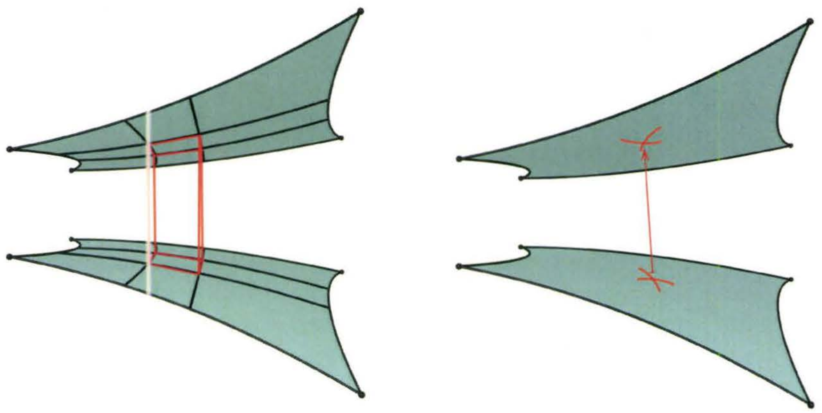

Figure 1.1: Discrete And Continuous Theories

manifests itself in a very clear way: since glass does not bend, at least not economically, is it possible to panel a freeform surface so that the design intent is preserved and waste is minimized? The standard technique has been to triangulate the surface, and since three points define a plane, such triangulations are well suited for glass panels. The advantages of this method are that the meshing lines can be chosen to suit the taste of the designer while still approximating key geometrical features. However the disadvantages are fairly severe. Most of the problems associated with triangular meshes stem from the fact that they typically have six edges intersecting at a vertex, which means that when structural support is added, there are six beams at every vertex. This will lead to visibility issues and also structural issues, since the more beams that meet at a node, the harder it is to fabricate the structural element for each node. Secondly, the industry standard for cutting glass panels is such that exactly one panel can be cut out of a given rectangular sheet of glass, suggesting that finding a panel that better fits the rectangular sheet will help minimize waste.

It is then natural then to turn our attention to planar quadrilateral meshes (PQ meshes). Such a mesh, if it exists for an arbitrary surface, would have only four beams meeting at a typical node, thus facilitating construction and increasing transparency. We say "typical" node, since many surfaces will force the panelization scheme to have nodes with an exceptional number of edges. These nodes exist due to topological considerations and are dealt with individually. Nonetheless, quadrilaterals, even irregular ones, fit a bounding rectangle much more efficiently than triangles. Moreover, PQ meshes offer the additional advantage of sometimes having what are called exact offset meshes of the original surface. These offset meshes incorporate the geometry of the support structure in the mathematical description of the mesh, meaning the surface becomes a three dimensional structure, which more closely resembles the actual architectural situation. Unfortunately, panelling a freeform surface with a PQ mesh is a significantly more difficult problem for the simple reason that if four points are chosen at random in space, they will almost certainly not lie in a plane. We will see that if a planar quadrilateral mesh is given, then, in certain cases, an application of projective geometry will produce associated offset meshes, solving the second part of the panelization problem. However, in order to produce a planar quadrilateral mesh in the first place, we need to use optimization techniques, a key idea that is credited to H.Pottman et al [7].

This thesis is divided into three parts. The first part deals with the basic definitions and concepts of mesh theory, starting with the definition of a mesh as both a combinatorial and concrete object, and then moving on to a discussion of mesh parallelism and the definition of the discrete Gauss map. The second part briefly summarizes the main concepts of Lie sphere geometry, with the goal of understanding the theoretical basis for the offset construction and mesh manipulation. In the third section, step by step instructions of how to construct and manipulate meshes are given and some of the main results of mesh theory are applied to a study project.

Part I

Preliminaries And Basic Theorems

Chapter 2

What Is A Mesh?

Intuitively, a mesh is simply the discrete analogue of a surface. While this intuitive picture is helpful, it misses some of the fundamental differences between surfaces and meshes. For example, meshes depend not only on the position of their vertices, but also on the relationships between the vertices. Preserving these relationships while moving the vertices around can create families of meshes that are visually very different although the underlying structure remains unchanged. On the other hand, it is also possible to represent the same surface by a number of meshes that have very different underlying structures. Indeed, a given surface can usually be meshed using triangular, quadrilateral, or even hexagonal panels. In this thesis, we are motivated by architectural applications, and so the focus will be on quadrilateral meshes. This, in part, is due to the fact that strips of quadrilaterals, obtained by connecting quadrilaterals along opposite edges, can be viewed as coordinate lines on the quadrilateral surface [3], a property that helps the transition from continuous geometry to discrete geometry.

2.1 Abstract And Quadrilateral Meshes

Consider the following definition,

Definition 2.1.1. An abstract mesh $\mathcal{M}$ is a triple $\mathcal{M} = (V, E, F)$, with,

1. $V = \{v_1, v_2, ..., v_n\}$, a finite set of points, referred to as vertices.
2. $E = \{e_1, e_2, ..., e_m\}$, a set of two element subsets of V, referred to as edges.
3. $F = \{f_1, f_2, ..., f_l\}$, a set of cyclically ordered four element subsets of V, referred to as faces. Moreover, F has the property that for any $f = (v_i, v_j, v_k, v_t) \in F$, only $(v_i, v_j), (v_j, v_k), (v_k, v_t), (v_t, v_i)$ are contained in E, and any edge is contained in at most two faces.

This somewhat technical definition serves to keep track of what is called the combinatorics of a mesh, which is no more than the information of which vertex is beside which and which vertices define the corner points of a face.

Definition 2.1.2. A mesh $M = (\mathcal{M}, \mathcal{X}, \mathcal{F})$ is a triple consisting of,

1. An abstract mesh $\mathcal{M} = (V, E, F)$
2. A vector space $\mathcal{X}$
3. A map $\mathcal{F}$ that assigns a point in $\mathcal{X}$ to each point in V, i.e. a function $\mathcal{F} : V \to \mathcal{X}$

It is also important to note that two meshes that look vastly different might have the same underlying abstract mesh. For instance, it is possible that the map $\mathcal{F}$ will be such that the resulting mesh will have self intersections, or edges that cross and perhaps degenerate. These meshes, although they satisfy the definitions, are not the focus of this thesis and so we will restrict our attention to "nice" meshes. In what follows, we use $\mathcal{X} = \mathbb{R}^3$ as the basic vector space. Also, if the function $\mathcal{F}$ is clear from the context, we will usually write $M = (V, E, F)$ where the elements in V are given points in $\mathbb{R}^3$.

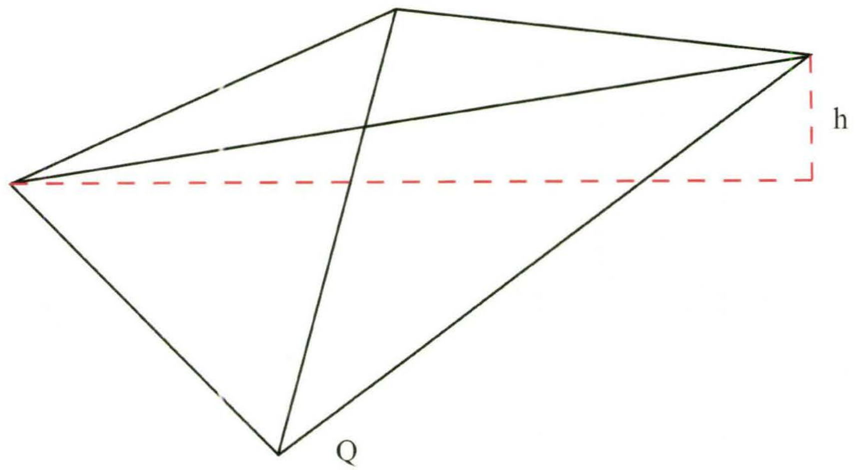

Figure 2.1: The Volume Of A Tetrahedron

Definition 2.1.3. Two meshes M_1 and M_2 are combinatorially equivalent if and only if the abstract meshes $\mathcal{M}_1$, $\mathcal{M}_2$ corresponding to M_1, M_2 are isomorphic.

The true power of quad meshes becomes apparent by restricting our attention to the distinguished subset of quad meshes whose faces are all planar. More formally,

Definition 2.1.4. A planar quadrilateral mesh (PQ mesh) is a mesh M where the volume of the tetrahedron formed by the four vertices of each face is zero.

PQ meshes were first suggested by Sauer [9] as discrete analogs of surfaces parametrized by conjugate curves. To see the connection between smooth surfaces and PQ meshes, we need some definitions from classical differential geometry.

Definition 2.1.5. [4] A developable surface is a surface with zero Gaussian curvature.

Developable surfaces have already been the subject of intertest in the construction industry, as such surfaces are well suited for constructions using sheet metal.

Definition 2.1.6. [4] A ruled surface S is a surface such that at every point on S, there exists a straight line that lies entirely on S.

In $\mathbb{R}^3$, all developable surfaces are ruled surfaces.

Definition 2.1.7. [6, p.3] A conjugate curve network consists of two families of curves A, and B, such that the envelope of tangent planes along a curve in A is a developable surface whose rulings are tangent to curves in the family B.

Definition 2.1.7, while required for a complete understanding of the discretization process, is not the most useful for our purpose. For further information concerning the concepts used in the above definition, we refer the reader to [4]. Instead, it is more convenient to work with the following,

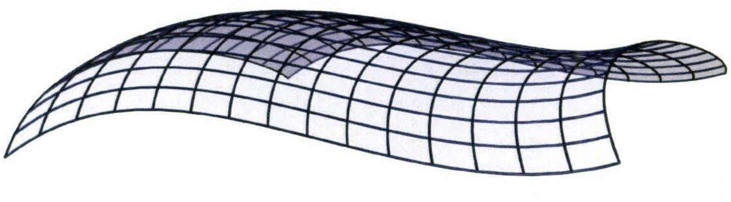

Figure 2.2: A Conjugate Curve Network

Definition 2.1.8. [3] A surface $X(u, v)$ is said to be parametrized by conjugate curves if,

$$X_{uv}(u, v) = c_1(u, v)X_u(u, v) + c_2(u, v)X_v(u, v)$$

where X_u, X_v are the first partial derivatives and X_{uv} is the mixed second partial derivative.

The condition in the above definition can be regard as an "infinitesimally planarity"' condition. In most cases, the principal curvature lines of a surface, which are a conjugate curve network, will be used to create the associated PQ mesh, but it is also possible to model with different families of conjugate curve networks. One technique for doing this is discussed in Appendix A. To see that the limit surface of an arbitrary PQ mesh is a conjugate curve network note that the row and column polylines of a PQ mesh display a discrete analog of the infinitesimally planar condition: each row of faces f_i is a PQ strip which represents a discrete developable surface tangent to the mesh [6, p.3].

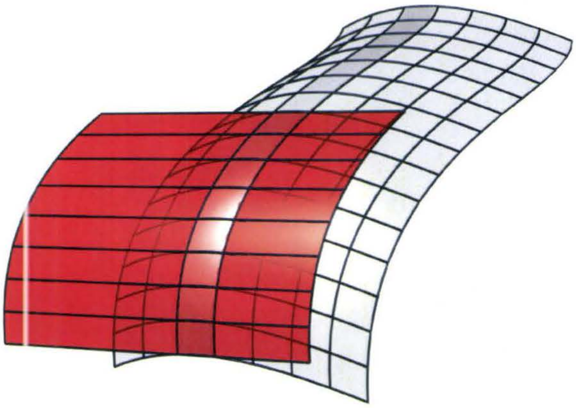

Figure 2.3: PQ Strip With Developable Surface

The row of vertices $v_{0,k}, ..., v_{n,k}$ can be seen as the polyline of tangency between the mesh and the developable surface represented by the f_i's. Moreover, the rulings of the developable surface are spanned by the edges $v_{i,k}, v_{i,k+1}$ for $i = 1, ..., n$. The same lines are also the tangent lines of the column polylines

$v_{i,0}, \ldots, v_{i,m}$. It then follows that the system of row and column polylines are a discrete conjugate network of polylines. Indeed, a discrete developable surface tangent to a PQ mesh along a polyline is given by a row (or a column) of quad faces. We then get the following theorem:

Proposition 2.1.1. *[6, p.3] If a subdivision process preserves the PQ property, refines a PQ mesh, and produces a curve network in the limit, then the limit is a conjugate curve network on a surface.*

Figure 2.4: A Subdivision Process

The following definitions deal with incorporating a support structure into the mesh.

Definition 2.1.9. A system of lines passing through the vertices V of a mesh $M = (V, E, F)$ are *node axes* if and only if neighbouring lines are coplanar.

The existence of node axes is particularly important when applying the theory of meshes to architecture. This is because the existence of node axes implies a node that has support beams aligned with the neighbouring node axes would be torsion free [7, p.3].

Definition 2.1.10. [7, p.3] A geometric support structure for a mesh M is a collection of planar quadrilaterals, defined by the node axes, which are transverse to M and share a common edge on the node axis.

2.2 The Linear Spaces $C(M)$ and $P(M)$

Definition 2.2.1. [7, p.4] Two meshes M_1 and M_2 are parallel if and only if they are combinatorially equivalent and corresponding edges, viewed as vectors, are parallel.

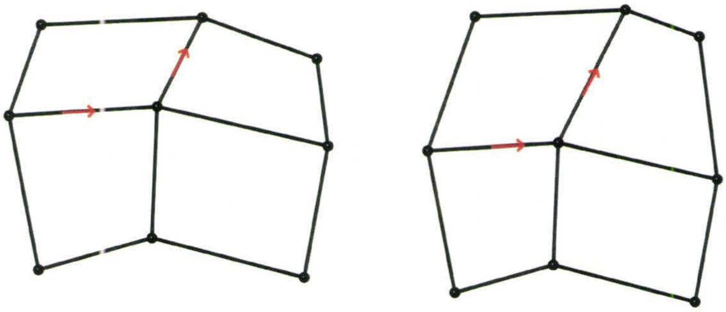

Figure 2.5: Two parallel meshes

Identifying M with a list of its vertex vectors $v_1, v_2, ..., v_n$ while remembering the mesh combinatorics allows us to represent M as a vector in $\mathbb{R}^{3n}$. We can then define the following:

Definition 2.2.2. [7, p.4] $C(M)$ denotes the set $\mathbb{R}^{3n}$ and can be seen as consisting of all combinatorially equivalent meshes to a given mesh M. $P(M)$ is the subset of $C(M)$ consisting of all meshes parallel to a given mesh M.

Since $\mathbb{R}^{3n}$ is a vector space, $C(M)$ is also a vector space. Moreover, we find that

Proposition 2.2.1. *[7, p.4] $P(M)$ is a linear subspace of $C(M)$.*

Proof: Consider two meshes M_1, M_2 in $P(M)$. Then, the vectors $v'_i - v'_j$ corresponding to the edges $v'_i v'_j$ of M_2 must be multiples of the corresponding edges of M_1, namely,

$$v_i' - v_j' = c_i(v_i - v_j)$$

Therefore, the edges in the mesh defined by $d_1M_1+d_2M_2$ must also be multiples of edges in M_1. So $P(M)$ is a linear subspace of $C(M)$. □

In addition, we get that $P(M)$ is the solution space of the system of equations:

$$(m_i' - m_j') \times (m_i - m_j) = 0, \forall m_i, m_j$$

2.3 The Distance Functions $d_*(M_1, M_2)$ and $O_*(M)$

Given any two combinatorially equivalent meshes, that is, meshes that have the same abstract mesh, it is reasonable to wonder what the distance between them is. There are in fact three different ways to measure distance:

Definition 2.3.1. [7, p.5] Let M_1, M_2, be two parallel PQ meshes. Then the distance functions, $d_*(M_1, M_2)$, where $* = v, e, f$ which stand for vertex, edge, and face respectively, are defined by:

1. $d_v(M_1, M_2)$ = distance between corresponding vertices, i.e. $||v_i - v_i'||$
2. $d_e(M_1, M_2)$ = distance between corresponding edges, i.e. $||p_i||$, where $||p_i||$ is the length of a line perpendicular to both edges.
3. $d_f(M_1, M_2)$ = distance between corresponding faces, i.e. $||n_i||$, where $||n_i||$ is the length of a line normal to both planes.

It is important to note that the distance functions defined above depend on the particular vertex, edge, or face at which it is being evaluated, that is, it is a local measure of distance. Indeed, the issue at the heart of the panelization problem is understanding for what kinds of meshes is this distance function a constant function.

Definition 2.3.2. [7, p.5] Two meshes M_1, M_2, are said to be offsets of each other if and only if they are parallel and $d_*(M_1, M_2) = d$, where d is a constant. Note that $d_*(M_1, M_2) = d$ means that the distance function is measured in one of the three admissible ways. Moreover, we call $O_*(M)$ the set of all offset meshes to given mesh.

From here on, the focus will be on understanding offset meshes.

Chapter 3

Mesh Properties

3.1 The Discrete Gauss Map

In classical differential geometry, the Gauss map assigns a point on the unit sphere to each point on a two dimensional surface by placing the unit normal at the point on the surface at the origin. By looking at the resulting points on the sphere, it is possible to gain some understanding of the original surface. In the discrete theory, there is no way to uniquely define a normal vector at a vertex, and so an exact analogue of the Gauss map is not possible. However, in the case that a mesh M_1 has an offset mesh M_2, the line connecting corresponding vertices will be a node axis, and can be used as a normal vector. We can now define the discrete Gauss map.

Definition 3.1.1. [7, p.5] Let M_1 be a mesh such that there exits a mesh M_2 where $d_*(M_1, M_2) = d$. Then the discrete Gauss map is a mapping:

$$N : O_*(M) \to P(M), \ \ S = \frac{(M_1 - M_2)}{d}$$

This means that if we are given a mesh $M_1 = (V, E, F)$, we can construct the discrete Gauss map of M_1 by considering another mesh $M_2 = (V', E, F)$ such that $d_*(M_1, M_2) = d$ and constructing the mesh S, whose vertices are given by the linear combination of vectors,

$$v_i^s = \frac{(v_i^1 - v_i^2)}{d}$$

It is clear that S is a mesh parallel to M. In the case that $d_v(M_1, M_2) = d$, the discrete Gauss map sends the vertices of M_1 to points on the unit sphere, but this is not true in the other two cases. The following theorem is therefore somewhat surprising:

Theorem 3.1.1. *[7, p.5] The following holds true for a PQ mesh M_1 and the unit sphere in $\mathbb{R}^3$:*

1. *A mesh M_1 has a vertex offset M_2 at distance d $\Longleftrightarrow$ vertices v_i of S lie on S^2.*
2. *A mesh M_1 has a edge offset M_2 at distance d $\Longleftrightarrow$ edges e_i of S are tangent to S^2.*
3. *A mesh M_1 has a face offset M_2 at distance d $\Longleftrightarrow$ faces f_i of S are tangent to S^2.*

Proof:

(1) ($\Rightarrow$) Assume M_1 has a vertex offset M_2. Since $d_v(M_1, M_2) = d$,

$$|s_i| = \left|\frac{(v_i^1 - v_i^2)}{d}\right| = 1$$

and therefore lie on S^2.

($\Leftarrow$) Assume that S has vertices the lie on S^2 and that $S \in P(M)$. Clearly,

$$v_i^2 = dv_i^s + v_i^1$$

defines the vertices of a mesh M_2 such that $d_v(M_1, M_2) = d$ (Figure 3.1).

(2) ($\Rightarrow$) Assume M_1 has an edge offset M_2. Construct a line segment through v_i^2 and perpendicular to e_i^1 and denote the intersection point of that

line with e_i^1, A. Since $d_e(M_1, M_2) = d$, $|(v_i^2 - A)| = d$. Now, construct the discrete Gauss map S. Then, the line passing through the origin and perpendicular to e_i^s will be parallel to the vector $(v_i^2 - A)$, and therefore,

$$\left|(v_i^2 - A)\right| = 1$$

($\Leftarrow$) Assume that S has edges tangent to S^2 and that $S \in P(M)$. Construct a line segment between the point of tangency A' between e_i^s and S^2 and the origin. As before, construct the mesh M_2 with vertices $v_i^2 = dv_i^s + v_i^1$. Then, the line perpendicular to e_i^1 and passing through v_i^2 will be parallel to n, and therefore,

$$|A'| = 1$$

Refer to Figure 3.2.

(3) The face offset case is similar to the edge offset case (Figure 3.3). □

3.2 Circular And Conical Meshes

Having defined the Gauss map for PQ meshes, we can now focus on two important classes of offset meshes, namely circular and conical meshes. It is desirable to know, for instance, what conditions an arbitrary mesh M_1 must satisfy in order for it to admit one of the three offset constructions. These results will follow from Theorem 3.1.1. Consider the following definitions,

Definition 3.2.1. [3]A circular mesh is a PQ mesh such that for all $f_i = \{v_{i1}, v_{i2}, v_{i3}, v_{i4}\} \in F$, the v_{ij}'s lie on the circumference of a circle.

Definition 3.2.2. [6, p.5] A conical mesh is PQ mesh such that for every vertex $v_i \in V$ the four faces f_i that contain v_i are tangent to a cone whose axis passes through v_i.

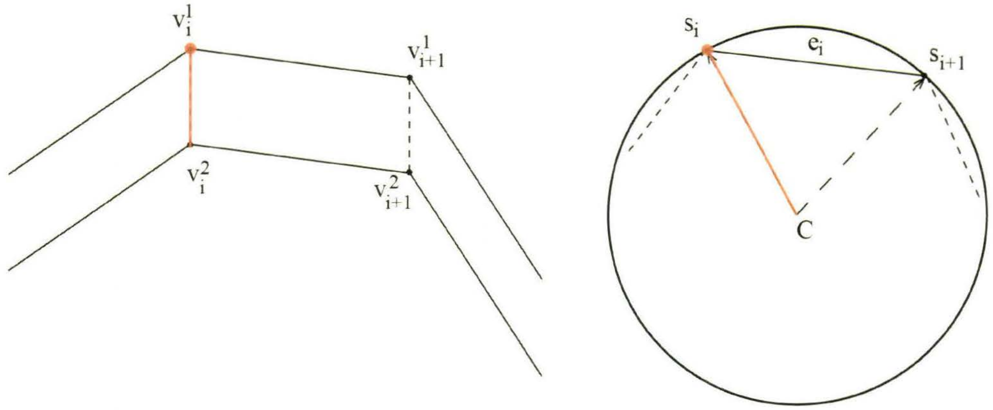

Figure 3.1: The Gauss Map Of A Vertex Offset Mesh

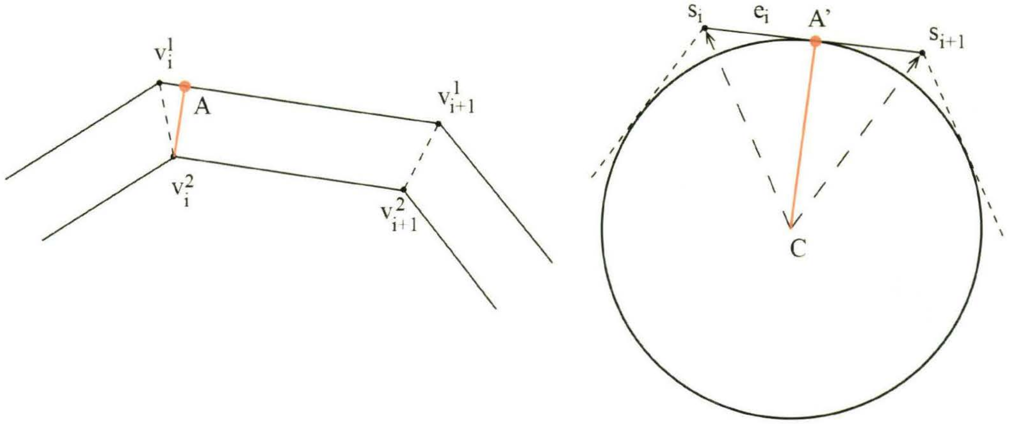

Figure 3.2: The Gauss Map Of A Edge Offset Mesh

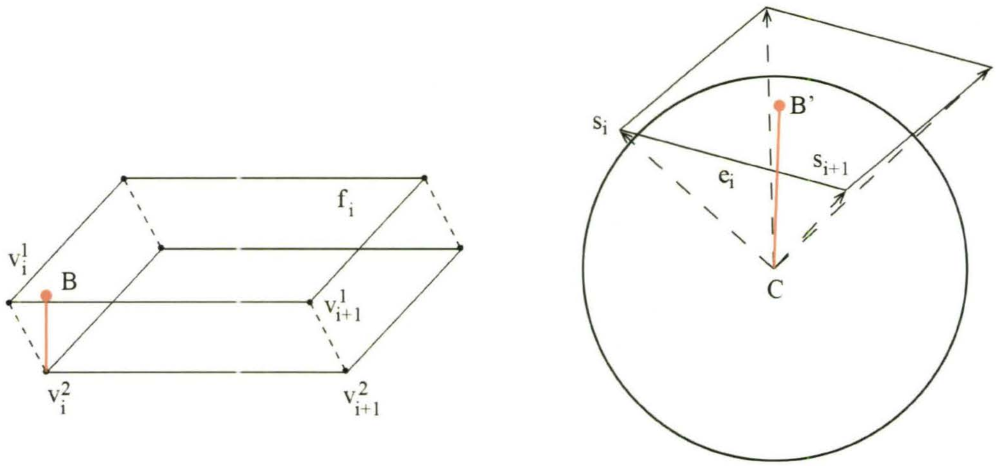

Figure 3.3: The Gauss Map Of A Face Offset Mesh

It is a well known fact that a planar quadrilateral is inscribed in a circle if and only if the angles at the vertices of the quadrilateral satisfy $\alpha+\gamma=\beta+\delta$.

It is shown in [7] that circular meshes have a discrete Gauss map with vertices that lie on the unit sphere. So, by Theorem 2.15, all circular meshes admit the vertex offset construction. Thus, in order to decide if a given mesh admits a vertex offset, it suffices to check whether or not the above angle balance is satisfied at every vertex. Moreover, a given mesh can be transformed into a circular mesh by perturbing the vertices so that the angle balance is achieved. This last statement is actually a statement about minimizing a non-convex penalty function, so there are some initialization conditions on the initial mesh that need to be satisfied in order for the optimization algorithm

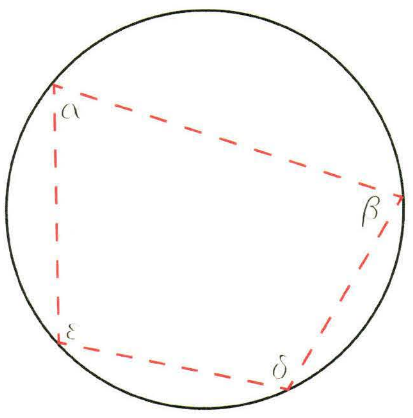

Figure 3.4: An inscribed quadrilateral

to be successful. It turns out that conical meshes also satisfy an angle balance,namely, $\alpha + \gamma = \beta + \delta$ where these angles are measured at a vertex. The proof of this is due to [10] and an outline of it will be given in the next section.

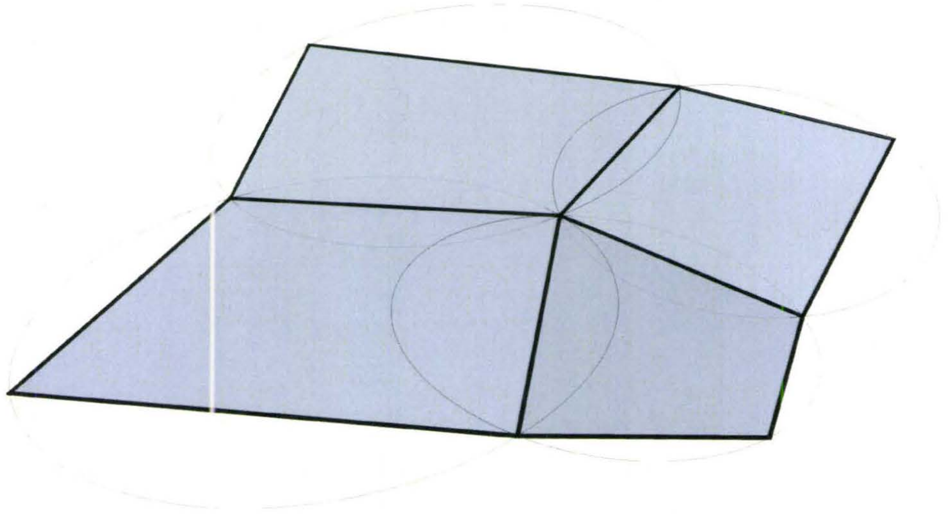

Figure 3.5: A Circular Mesh

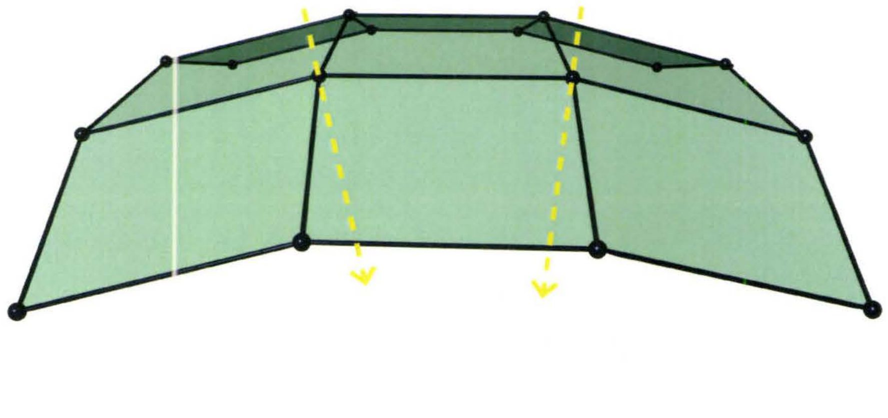

Figure 3.6: A Conical Mesh

Part II

The Theoretical Framework

Chapter 4

The Angle Criteria For Conical Meshes

4.1 Proof Of Geometry Fact 1

Theorem 4.1.1. *[10, p.2] A vertex v of a planar, quadrilateral, mesh is conical if and only if the sums of the opposite angles are equal, that is, if* $\alpha+\gamma=\beta+\delta$.

Proof: [10, p.2] We will only consider a special case of the proof, and for further details refer the reader to [10]. We first need a lemma:

Lemma 4.1.2. *[10, p.2] Suppose a spherical convex quadrilateral with consecutive sides* $e_1, ..., e_4$ *has an inscribed circle, referred to as an "incircle"'. Let* α_i *be the length of the side* e_i*. Then* $\alpha_1+\alpha_3=\alpha_2+\alpha_4$*. Conversely, a convex spherical quadrilateral with the property* $\alpha_1+\alpha_3=\alpha_2+\alpha_4$ *has an incircle.*

It is interesting to note that the above lemma does not use any properties particular to spherical geometry, and so we can use the same argument that is used to prove the well known Euclidean version of this statement.

Proof: [10, p.3] Assume that a convex quad has an incircle. Suppose also that the incircle touches the four sides e_i at the points $p_i \in e_i$ as shown in Figure 4.1. Let u_i denote the vertex which is at the intersection point of the

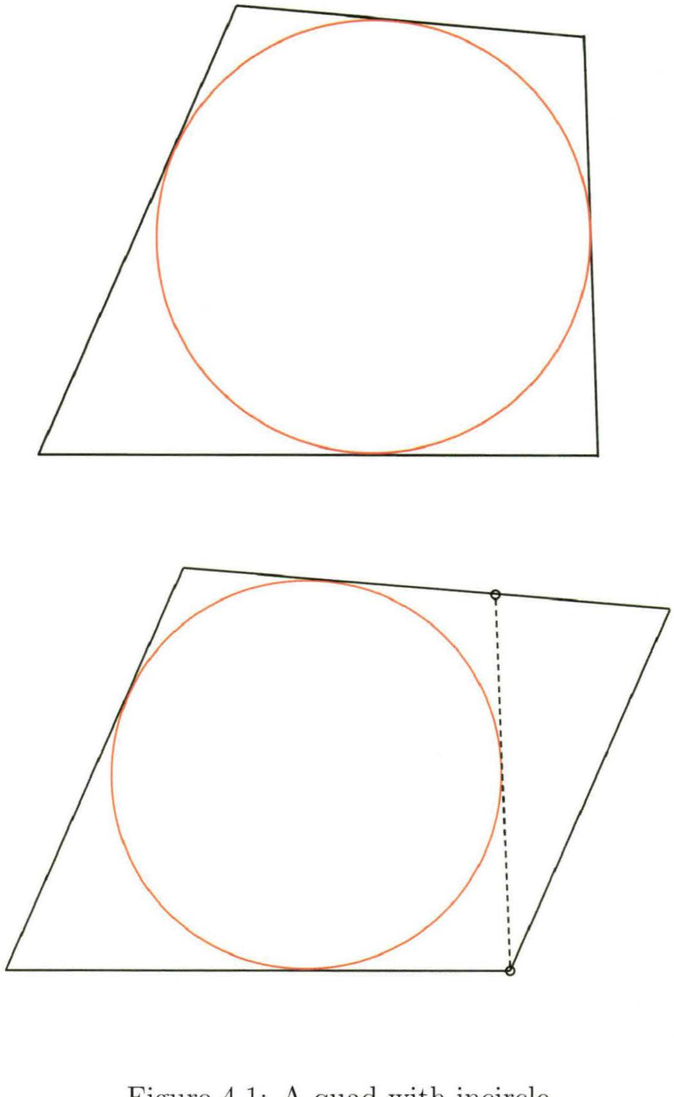

Figure 4.1: A quad with incircle

sides e_i and e_{i+1}. Also, let $d(a,b)$ denote the distance between the points a and b, and is defined to be the angle of the smallest arc of a great circle on S^2 connecting a and b. From the fact that the two sides incident with a vertex are tangents of the same incircle, we get that,

$$d(u_1,p_1) = d(u_1,p_1)$$

$$d(u_2,p_2) = d(u_2,p_3)$$

$$d(u_3,p_3) = d(u_3,p_4)$$

$$d(u_4,p_4) = d(u_4,p_1)$$

It then follows that,

$$\alpha_1 + \alpha_3 = d(u_1,p_1) + d(u_4,p_1) + d(u_2,p_3) + d(u_3,p_3)$$

$$= d(u_1,p_2) + d(u_4,p_4) + d(u_2,p_2) + d(u_3,p_4) = \alpha_2 + \alpha_4$$

To show the other direction, assume that.

$$\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 \qquad (1)$$

Also, suppose that the quad $Q = u_1u_2u_3u_4$ does not have an incircle. To derive the contradiction, consider the family of circles that are contained in the convex quad Q and tangent to e_2 and e_3. This family either contains a circle, which we will call C, which is tangent to e_1 but not to e_4, or a circle which is tangent to e_4 but not to e_1. Without loss of generality, suppose that the first case is true. Now, let u_4' be the unique point on e_1 that lies between p_1 and u_4, such that the side $e_4' = u_3u_4'$ is tangent to the circle C at p_4. Then, by the same reasoning as above, the convex quad $Q' = u_1u_2u_3u_4'$ satisfies the angle criterion,

$$\alpha_1' + \alpha_3 = \alpha_2 + \alpha_4' \qquad (2)$$

where $\alpha_1' = d(u_1, u_4')$ and $\alpha_4' = d(u_3, u_4')$. Subtracting (1) from (2) gives,

$$\alpha_1 - \alpha_1' = \alpha_4 + \alpha_4'$$

It then follows that $\alpha_4 = \alpha_4' + \alpha_1 - \alpha_1' = \alpha_4 + d(u_4, u_4')$. In addition, the triangle inequality gives that,

$$\alpha_4 < \alpha_4' + d(u_4, u_4')$$

which is a contradiction. Therefore, Q has an incircle. □

Although Theorem 4.1.1 will prove to be essential for creating conical meshes by optimization, it will also be necessary to know how to construct the tangent cone of a vertex v_i, knowing only that the faces meeting at that vertex are planar and that the angle criterion is satisfied. To do this, construct a unit sphere centered at the vertex v_i, and intersect it with the planes meeting at that vertex to produce a spherical quadrilateral, as above. We know now that this spherical quadrilateral must have an inscribed circle, which is constructed in exactly the same way as in Euclidean geometry: one simply bisects the vertex angles. In this case, will do this with planes. The resulting inscribed circle then serves as the base of the vertex cone. Figure 4.2 depicts these steps.

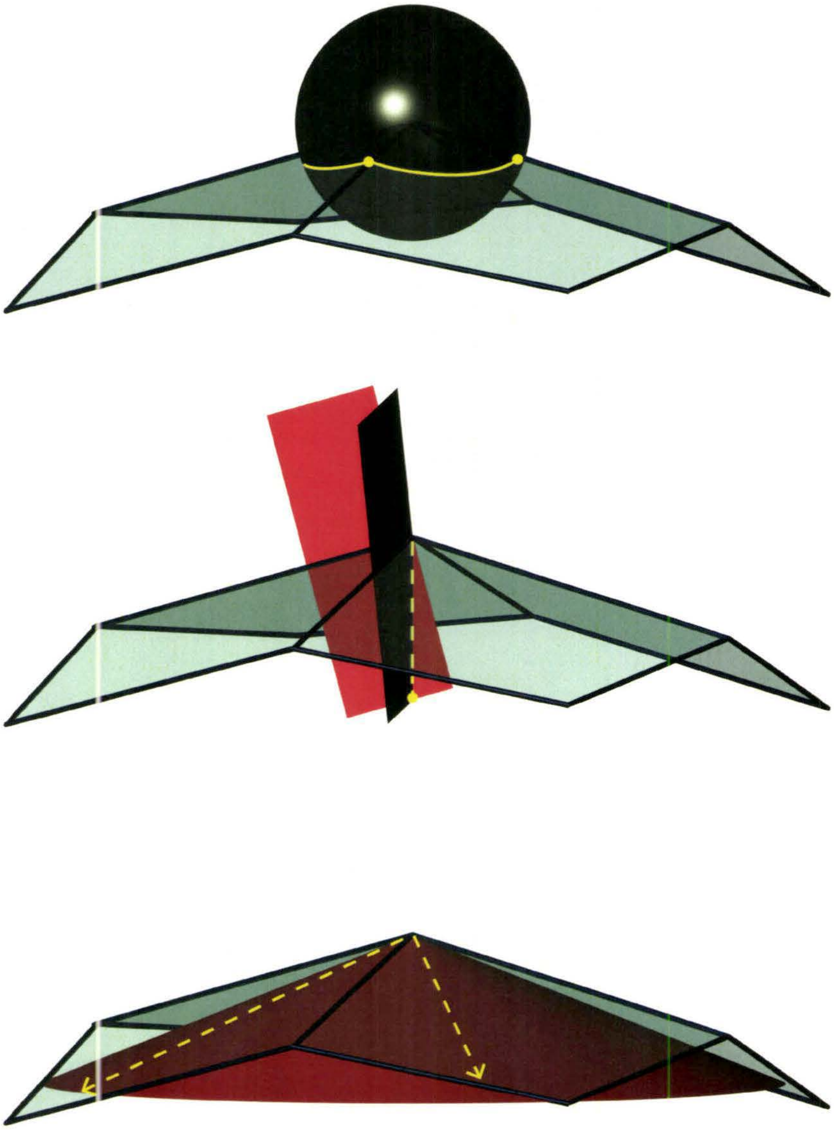

Figure 4.2: Constructing A Vertex Cone

4.2 Edge Offset Meshes

Edge offset meshes are theoretically the most difficult type offset mesh to deal with in the sense that so far, it is not completely understood what kind of shapes admit an edge offset mesh. What is known, though, is that if a mesh M has an edge offset mesh M', then the discrete Gauss map of M will create what is known as a *Koebe polyhedron* [8]. Moreover, A. Bobenko has shown in [3] that it is possible to create Koebe polyhedra by minimizing a nonlinear functional. So, at the moment, it is possible to create any number of edge offset meshes, but constructing one that closely approximates a given surface is still an open problem.

Chapter 5

Lie Sphere Geometry

5.1 The Lie Quadric

The theory of Lie sphere geometry was originally introduced by Sophus Lie in [5] and has recently been revived in the context of circular and conical meshes by H. Pottman ([7], [8]), and by A. Bobenko ([3]). The basis of the Lie sphere geometry is the bijective correspondence between the set of all Lie spheres, that is oriented hyperspheres, oriented hyperplanes viewed as spheres with infinite radii, and point spheres viewed as spheres with zero radii, and the set of all points on the quadric hypersurface Q^{n+1} in real projective space $\mathbb{P}^{n+2}$ as described by the equation $\langle x, x\rangle = 0$, where $\langle \cdot, \cdot \rangle$ is an indefinite scalar product with signature $(n+1, 2)$ on $\mathbb{R}^{n+3}$. The hypersurface Q^{n+1} is called the Lie quadric and contains projective lines but no higher dimensional subspaces [2].

Having constructed the space of Lie spheres, it is natural to ask about the kind of transformations that the space admits. Indeed, a Lie sphere transformation is a projective transformation of $\mathbb{P}^{n+2}$ which maps the Lie quadric Q^{n+1} to itself. This amounts to sending Lie spheres to Lie spheres. Lie proved the "fundamental theorem of Lie sphere geometry" for the case $n = 3$ which U. Pinkall then generalized to higher dimensions. The theorem states that a diffeomorphism of the Lie quadric Q^{n+1} that preserves lines is the restriction to Q^{n+1} of a projective transformation of $\mathbb{P}^{n+2}$. More succinctly, an oriented contact preserving transformation of the space of Lie spheres is a Lie sphere transformation [2].

In this context, we will be using Lie sphere geometry to manipulate meshes with the offset property. This is natural, since the conditions for having an offset of one of the three admissible kinds can be reformulated in the language of spherical geometry, via the discrete Gauss map. First, however, we must construct the framework in which Lie sphere geometry takes place.

Our notation will be that $\mathbb{R}_k^{n+2}$ stands for the $n+2$-dimensional Euclidean space equipped with a scalar inner product of the form,

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 - x_2y_2 - \cdots - x_ky_k + x_{k+1}y_{k+1} + \cdots + x_{n+2}y_{n+2}$$

If there is only one minus sign, that is, if we are working in $\mathbb{R}_1^{n+2}$, we get the following definitions.

Definition 5.1.1. [2, p.10] The set of vectors, called lightlike vectors, given by,

$$x_1^2 = x_2^2 + \cdots + x_{n+2}^2$$

is called a light cone. Those vectors that satisfy $\langle v, v \rangle < 0$ are inside the cone and are called timelike vector, and those that satisfy $\langle v, v \rangle > 0$ are outside the cone and are called spacelike vectors.

We can then let W^{n+1} be the set of vectors in $\mathbb{R}_1^{n+2}$ that satisfy $\langle \zeta, \zeta \rangle = 1$. Geometrically, these are the position vectors of points lying on a hyperboloid of revolution of one sheet in $\mathbb{R}_1^{n+2}$. If α is a spacelike point in $\mathbb{P}^{n+1}$, or equivalently a line in $\mathbb{R}_1^{n+2}$ with a spacelike direction vector, then there are exactly two vectors, $\pm\zeta \in W^{n+1}$ such that $\alpha = [\zeta]$. Later on, these two vectors will be taken to correspond to the orientations of the oriented sphere or hyperplane represented by α. In order to establish the correspondence between α and the two oriented spheres, we first embed $\mathbb{R}_1^{n+2}$ into $\mathbb{P}^{n+2}$ by sending $z \longrightarrow [(z, 1)]$. Notice that if $\zeta \in W^{n+1}$, then

$$1 = -\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_{n+1}^2$$

This means that the point $[(\zeta, 1)] \in \mathbb{P}^{n+2}$ must lie on the quadric $Q^{n+1} \in \mathbb{P}^{n+2}$ given, in homogeneous coordinates, by the equation,

$$\langle \mathbf{x}, \mathbf{x} \rangle = -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - x_{n+3}^2 = 0$$

The quadric Q^{n+1} is called the Lie quadric equipped with the Lie metric defined above [2, p.15]. Now, suppose that $x \in Q^{n+1}$ with homogeneous coordinate x_{n+3} not equal to zero. It is then possible to represent x by a vector of the form $(\zeta, 1)$, with Lorentz scalar product $(\zeta, \zeta) = 1$. Moreover, suppose that the first two coordinates of ζ satisfy $\zeta_1 + \zeta_2$ not equal to zero, then $[\zeta]$ can be represented by a vector of the form,

$$\chi = \left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p \right)$$

where $p \in \mathbb{R}^n$ and $r \in \mathbb{R}$ [2, p. 12]. This particular way of writing ζ comes from the formulae for stereographic projection. In addition,

$$(\chi, \chi) = r^2$$

which implies that ζ must be one of $\pm\chi/r$. So, in $\mathbb{P}^{n+2}$, we get that,

$$[(\zeta, 1)] = [(\chi/r, 1)] = [(\chi, r)]$$

At this point we establish the convention that a positive signed radius means an inward pointing field of unit normals on the sphere, whereas a negatively signed radius signals an outward pointing field of unit normals. Thus, given a sphere in $\mathbb{R}^n$ with center p and unsigned radius $r > 0$, we can represent the two orientations of the sphere by the two projective points [2, p.15],

$$\left[\left(\frac{1+p\cdot p-r^2}{2}, \frac{1-p\cdot p+r^2}{2}, p, \pm r\right)\right] \in Q^{n+1}$$

Moreover, χ is the pole of the hyperplane that cuts out the original sphere in $\mathbb{R}^n$ on S^n.

5.2 Lie Sphere Transformations

Having constructed the Lie quadric Q^{n+1}, we want to study those transformations that preserve Q^{n+1}. Indeed, we get the following definition:

Definition 5.2.1. [2, p.25] A Lie sphere transformation is a projective transformation that sends Q^n to itself.

A linear transformation A induces a projective transformation by acting on a representative of the equivalence class denoted by homogeneous coordinates. That is, if A is a linear transformation, then we can define the induced projective transformation by letting $P(A)[x] = [Ax]$. The map P sends $GL(n+1) \to PGL(n)$, and it's kernel consists of all non-zero scalar multiples of I_{n+1}.

In order to determine the group of Lie Sphere transformations, we need the following theorem:

Theorem 5.2.1. *[2, p.26] Let A be an non-singular, linear transformation on the indefinite scalar product space $\mathbb{R}^n_k$, $1 \leq k \leq n-1$, such that A takes lightlike vectors to lightlike vectors.*

1. *Then there is a nonzero constant* λ *such that* $\langle Av, Aw\rangle = \lambda\langle v, w\rangle$ *for all* v, w *in* $\mathbb{R}^n_k$. *Here,* $\langle , \rangle$ *denotes the scalar product on* $\mathbb{R}^n_k$.

2. *Furthermore, if* $k \neq n - k$, *then* $\lambda > 0$.

Proof: [2, p.26] (1) The inequality $1 \leq k \leq n-1$ implies that there exist both timelike and spacelike vectors in $\mathbb{R}^n_k$. Supposing that v is a unit timelike vector and w is a unit spacelike vector such that $\langle v, w\rangle = 0$, we get that

$$\langle v + w, v + w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle = 0$$

$$\langle v - w, v - w\rangle = \langle v, v\rangle - 2\langle v, w\rangle + \langle w, w\rangle = 0$$

So, both $v + w$ and $v - w$ are lightlike. However, the theorem states that $A(v + w)$ and $A(v - w)$ are both lightlike. So,

$$\langle A(v + w), A(v + w)\rangle = \langle Av, Av\rangle + 2\langle Av, Aw\rangle + \langle Aw, Aw\rangle$$

$$\langle A(v - w), A(v - w)\rangle = \langle Av, Av\rangle - 2\langle Av, Aw\rangle + \langle Aw, Aw\rangle$$

Subtracting the second from the first equation, we get $\langle Av, Aw\rangle = 0$, and substituting this into either equation gives,

$$-\langle Av, Av\rangle = \langle Aw, Aw\rangle = \lambda \qquad (1)$$

for some $\lambda \in \mathbb{R}$. Now, it is a basic property of the scalar product space $\mathbb{R}^n_k$ that we can construct an orthonormal basis $v_1, ..., v_k, w_1, ..., w_{n-k}$, where the v's are timelike and the w's are spacelike. From the reasoning above, we know that $\langle Av_i, Aw_j\rangle = 0$ for all i and j. In addition, if we hold v constant and vary w and vice versa, we get that $-\langle Av_i, Av_i)\rangle = \langle Aw_j, Aw_j\rangle$ for all i and j. To show that $\langle Av_i, Av_j\rangle = 0$ and $\langle Aw_i, Aw_j\rangle = 0$ for $i \neq j$, consider the vector

$w = \frac{(w_i+w_j)}{\sqrt{2}}$. Clearly, w is a unit spacelike vector orthogonal to v_1 and so we can apply (1) implying, $\langle Aw, Aw\rangle = \lambda$. We can rewrite this as,

$$2\lambda = \langle A(w_i + w_j), A(w_i + w_j)\rangle = \langle Aw_i, Aw_i\rangle + 2\langle Aw_i, Aw_j\rangle + \langle Aw_j, Aw_j\rangle$$

But since $\langle Aw_i, Aw_i\rangle = \langle Aw_j, Aw_j\rangle = \lambda$, we get that $\langle Aw_i, Aw_j\rangle = 0$ for $i \neq j$. Substituting v_i for w_j in the above reasoning, we get that $\langle Av_i, Av_j\rangle = 0$ for $i \neq j$. Since the relation holds on all orthonormal basis vectors,

$$\langle Ax, Ay\rangle = \lambda\langle x, y\rangle$$

holds for all vectors $x, y \in \mathbb{R}^n_k$.

(2) Note that on $\mathbb{R}^n_k$, $\langle , \rangle$ has signature $(k, n-k)$. So $k \neq n-k$ must mean that Av_i is timelike and so Aw_i must be spacelike. This implies $\lambda > 0$ [2]. □

It is now clear which what the group of Lie sphere transformations actually is, [2, p. 28]:

Corollary 5.2.2. *[2, p.27] following are true,*

1. *The group G of Lie sphere transformations is isomorphic to $O(n+1, 2)/\pm I$*
2. *The group H of Moebius transformations is isomorphic to $O(n+1, 1)/\pm I$*

Proof: [2, p.27] (1) Let $P(A)$ be a Lie sphere transformation. Theorem 2 states that $\langle Av, Aw\rangle = \lambda\langle v, w\rangle$ for all v, w in $\mathbb{R}^n_k$, with $\lambda > 0$. Define a new transformation $B = A/\sqrt{\lambda}$. We then get that $B \in O(n+1, 2)$ and $P(A) = P(B)$, meaning that every Lie sphere transformation can be written as an element in $O(n+1, 2)$. Conversely, for any $B \in O(n+1, 2)$, $P(B)$ is a Lie sphere transformation. It remains to show that the map $\phi : O(n+1, 2) \to G$, which is the restriction of the homomorphism $P : GL(n+1) \to G$ to $O(n+1, 2)$, is surjective. To do this, simply note that the kernel of ϕ is equal to the intersection of $Ker(P)$ with $O(n+1, 2)$, that is, $Ker(\phi) = \pm I$. Thus, ϕ is surjective.

(2) The second statement of the corollary follows in exactly the same way except that instead of using the Lie metric, we use the Lorentz metric.

5.3 Laguerre Transformations

We now shift our attention to a specific subgroup of Lie sphere transformations, namely the Laguerre transformations. After establishing the construction of the group of Laguerre transformations, we will use them to manipulate conical meshes.

Using the correspondence between spheres in $\mathbb{R}^n$ and points in Q^{n+1}, planes in $\mathbb{R}^n$ are mapped to points in the Lie quadric Q^{n+1} that satisfy $x_1 + x_2 = 0$. Geometrically this can be seen as the intersection of the plane $x_1 + x_2 = 0$ with the Lie quadric. The remaining points satisfy the condition $x_1 + x_2 \neq 0$ and correspond to spheres in $\mathbb{R}^n$ with radius $r \geq 0$ [2, p.37].

Lemma 5.3.1. *[2, p.38] A Lie sphere transformation is determined by its restriction to the set of points $[x] \in Q^{n+1}$ with $x_1 + x_2 \neq 0$.*

Proof: [2, p.38] All that is needed to prove this lemma is to construct a basis for $\mathbb{R}^{n+3}_2$ consisting of lightlike vectors that satisfy $x_1 + x_2 \neq 0$. Indeed,

$$v_1 = e_2 + e_{n+3}$$

$$v_i = e_1 + e_i, 2 \leq i \leq n+2$$

$$v_{n+3} = \frac{1}{\sqrt{2}}(e_1 - e_{n+3}) + e_2$$

is such a basis. □

Moreover, if we multiply by the appropriate scalar multiple, we can assume that the homogeneous coordinates of $[x]$ satisfy $x_1 + x_2 = 1$. Denote,

$$X = (x_3, ..., x_{n+3})$$

and let

$$(X, X) = x_3^2 + \cdots + x_{n+2}^2 - x_{n+3}^2$$

be the restriction of the scalar product $\langle , \rangle$ to $\mathbb{R}_1^{n+1}$. The condition $x_1 + x_2 = 1$ implies,

$$0 = \langle x, x \rangle = -x_1^2 + x_2^2 + (X, X) = -x_1 + x_2 + (X, X)$$

Thus, $x_1 - x_2 = (X, X)$, and so,

$$x_1 = \frac{1 + (X, X)}{2}, \qquad x_2 = \frac{1 - (X, X)}{2}$$

We now have a diffeomorphism $[x] \to X$ that maps the open set U of points in Q^{n+1} such that $x_1 + x_2 \neq 0$ to points $X \in \mathbb{R}_1^{n+1}$ [2, p.38].

A fundamental concept of Laguerre geometry is the tangential distance between two spheres in $\mathbb{R}^n$. Indeed, Laguerre transformations preserve the tangential distance between two spheres. However, some spheres, such as concentric spheres, don't admit a tangent plane, and so the tangential distance is undefined on these pairs. The following lemma characterizes those pairs of spheres for which the measure of tangential distance is applicable.

Lemma 5.3.2. *[2, p.40] The oriented sphere in $\mathbb{R}^n$ corresponding to the points X and Y in $\mathbb{R}_1^{n+1}$ have a common tangent plane if and only if $X - Y$ is lightlike or spacelike.*

By definition a Laguerre transformations maps the improper point to itself. However, Laguerre transformations also preserve oriented contact, since oriented contact occurs when the tangential distance is zero. This is implies that Laguerre transformations also send planes to planes. This last statement combined with the reasoning above shows that Laguerre transformations map the open set U consisting of points that satisfy $x_1 + x_2 \neq 0$ to itself. Moreover, the diffeomorphism between U and $\mathbb{R}_1^{n+1}$ implies that Laguerre transformations induce a transformation sending $\mathbb{R}_1^{n+1}$ onto itself. In fact, this transformation is affine and so Laguerre transformations must have the form,

$$Y = TX + B$$

where $T \in O(n, 1)$ and B is some translation vector [2, p.42].

As an example consider the sphere S in $\mathbb{R}^3$ with center $p = (1, 2, 3)$ and radius $r = 5$. We find the appropriate point on the Lie quadric and the use the diffeomorphism from $Q^{n+1} \to \mathbb{R}_1^{n+1}$,

$$S \to \left[\left(\frac{1 + p \cdot p - r^2}{2}, \frac{1 - p \cdot p + r^2}{2}, p, \pm r\right)\right] \to (1, 2, 3, 5)$$

Now, act on the point $X = (1, 2, 3, 5)$ by the transformation $T \in O(3, 1)$, where,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(12) & -\sin(12) \\ 0 & 0 & \sin(12) & \cos(12) \end{pmatrix}$$

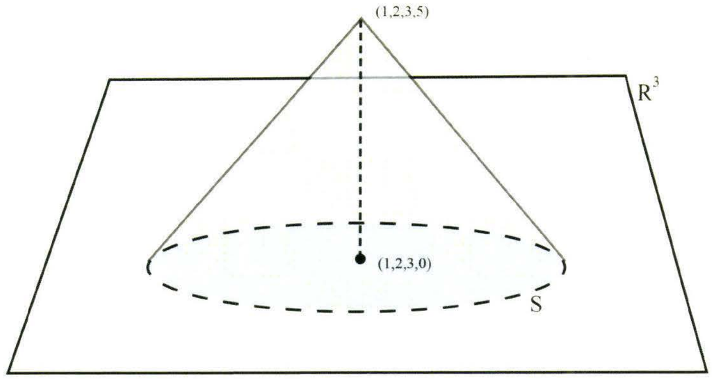

Figure 5.1: Recovery of Sphere in $\mathbb{R}^3$

So,

$$TX = \begin{pmatrix} 1 \\ 2 \\ 2.486 \\ 5.275 \end{pmatrix} = Y$$

Thus, S is mapped to S', where S' has center $(1, 2, 2.486)$ and radius (5.275). To recover the sphere S' from the its corresponding point on the quadric geometrically, we construct the lightlike cone with vertex at ζ' and a plane through the point $(1, 2, 3, 0)$ with normal vector $(0, 0, 0, 5)$. Note that the transformation T, is not an isometry of $\mathbb{R}^3$, since it does not keep the radius of S constant. The intersection of the light cone and the plane will produce the circle S' in $\mathbb{R}^3$, as shown in Figure 5.1.

Laguerre transformations have a very nice connection to conical meshes. Recall that a conical mesh is a mesh M such that at every vertex v_i of M it is

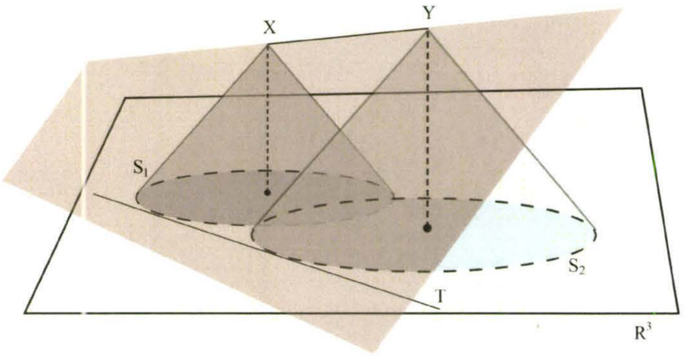

Figure 5.2: Recovery of Tangent Plane in $\mathbb{R}^3$

possible to construct a cone that is tangent to the four faces that meet at v_i. However, a cone is uniquely determined by two spheres that satisfy Lemma 5.3.2, and of course, given a cone, it is possible to construct two spheres whose centers lie on the cone axis and that are tangent to the cone, and so it is possible to map conical meshes to conical meshes using Laguerre transformations. This relationship has been briefly stated in the literature ([6], [7]), but not formally shown to be true.

Theorem 5.3.3. *Let M be a conical mesh, and $A = TX + B$ a Laguerre transformation. Let $\Gamma = \{\Gamma(v_1), \Gamma(v_2), ..., \Gamma(v_n)\}$ be the set of cones with vertices corresponding to vertices of M. If A is applied to all elements in Γ, then the vertices of the resulting set of cones Γ' is again a conical mesh.*

Proof: The basic idea of this proof is to construct a bijective map L using Laguerre transformations that sends the vertices of M to new points in $\mathbb{R}^3$ such that when the combinatorial structure of M is imposed on this new set of points, the resulting mesh is again a conical mesh. First, we construct the

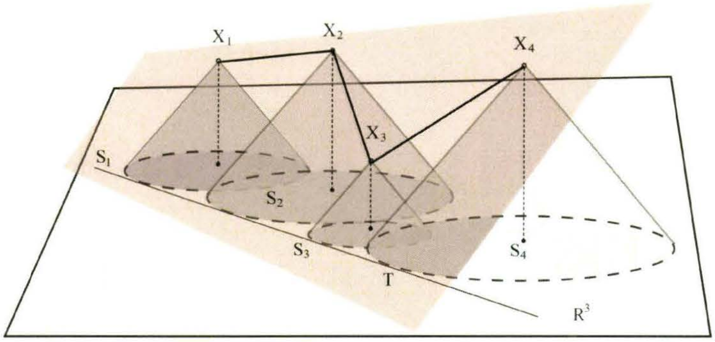

Figure 5.3: Spheres With A Common Tangent Plane in $\mathbb{R}^3$

map L. As noted above, Γ is the set of cones with vertices coinciding with the vertices of M. Define Γ' to be the the set of cones obtained by acting on the two spheres associated with each $\Gamma(v_i) \in \Gamma$ by a Laguerre transformation A. Define V' to be the set of vertices of the cones in Γ', that is $V' = \{v'_1, v'_2, ..., v'_n\}$. The map L is then defined by

$$L : v_i \rightarrow v'_i$$

To show that the mesh $M' = (V', E, F)$ is conical we need to show that every face of M' is planar and that for each v'_i there exits a cone with with vertex v'_i that is tangent to the four faces meeting at v'_i. First, consider two non-concentric spheres in $\mathbb{R}^3$, S_1 and S_2 and their corresponding points in $\mathbb{R}^4$, X and Y. To construct a common tangent plane, first choose any lightlike vector n in the space perpendicular to the line $[(X - Y)]$, denoted $(X - Y)^{\perp}$. The hyperplane π with pole n will contain $[(X - Y)]$ and so will be tangent to

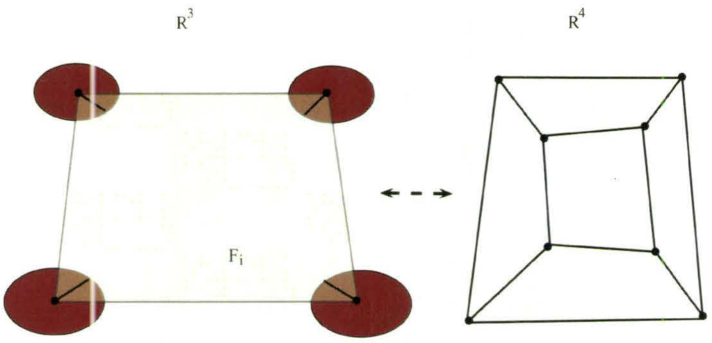

Figure 5.4: Application To A Single Face

both light cones. Intersection of π with the copy of $\mathbb{R}^3$ spanned by the x,y, and z axes will then give a plane in $\mathbb{R}^3$ that is tangent to both spheres. This is shown in Figure 5.2. Of course, there are many choices for n, and indeed the envelope of all tangent planes defined by the different choices of n is the unique cone tangent to S_1 and S_2. Moreover, if a series of spheres, $S_1, ..., S_4$ with associated points $X_1, ..., X_4$, share a common tangent plane T, then T must be the intersection of $\mathbb{R}^3$ with the plane π containing the line segments $[(X_1 - X_2)], [(X_2 - X_3)], [(X_3 - X_4)]$. Thus, the points $X_1, ..., X_4$ must be coplanar, in the sense that they are contained in a single hyperplane π, as shown in Figure 5.3.

Applying this reasoning to a single face f_i of M, together with the four vertex cones and their pairs of inscribed spheres, $S_1, ..., S_8$, we get a picture similar to the one shown in Figure 5.4. Since the plane P defined by f_i is tangent to all eight spheres, the corresponding points $X_1, ...X_8$ in $\mathbb{R}^4$ must be coplanar. Now, we act on these points by the Laguerre transformation $A = TX + B$. Since $T \in O(3, 1)$, A is a linear transformation, and so it does not affect the linear dependence of the vectors $[(X_i - X_j)]$. Thus, the resulting points $X_1', ...X_8'$ must also be coplanar. Intersecting the plane containing the X_i's with $\mathbb{R}^3$ yields a plane P' tangent to the eight spheres $S_1', ..., S_2'$. Following the definition of the map L, we now construct the cones defined by the appropriate pairs of spheres and call their vertices v_i, the vertices of the new mesh M'. To see that M' a planar quad mesh, notice that images of the vertices contained in any face f_i must lie on the plane P', and so must be coplanar. Moreover, applying L to a square of nine vertices with center vertex v_i, the planes $P_i, ...P_{i+3}$ must pass through the vertex of the cone $\Gamma(v_i')$. Since $P_i, ...P_{i+4}$ also contain the images of the other eight vertices, it is possible to construct a cone at each vertex of M' that is tangent to the faces of M' meeting at that vertex. Thus, M' is conical □

This technique is particularly effective when a basic starting conical mesh is chosen, such as a piece of a sphere, or cylinder. Since the curvature lines of these shapes are well known, their associated conical meshes are easily created. Since a well chosen Laguerre transformation will change the radii of the spheres it is acting on, the angle of the new vertex cone will change, and as a result the curvature and geometry of the resulting mesh will not be as trivial as the original surface.

A specific example of this is the offset operation for face offset meshes and is shown in Figure 5.5. In that transformation, $A = I_{4\times 4}$ and $C = (0, 0, 0, d)$, where d is the offset distance.

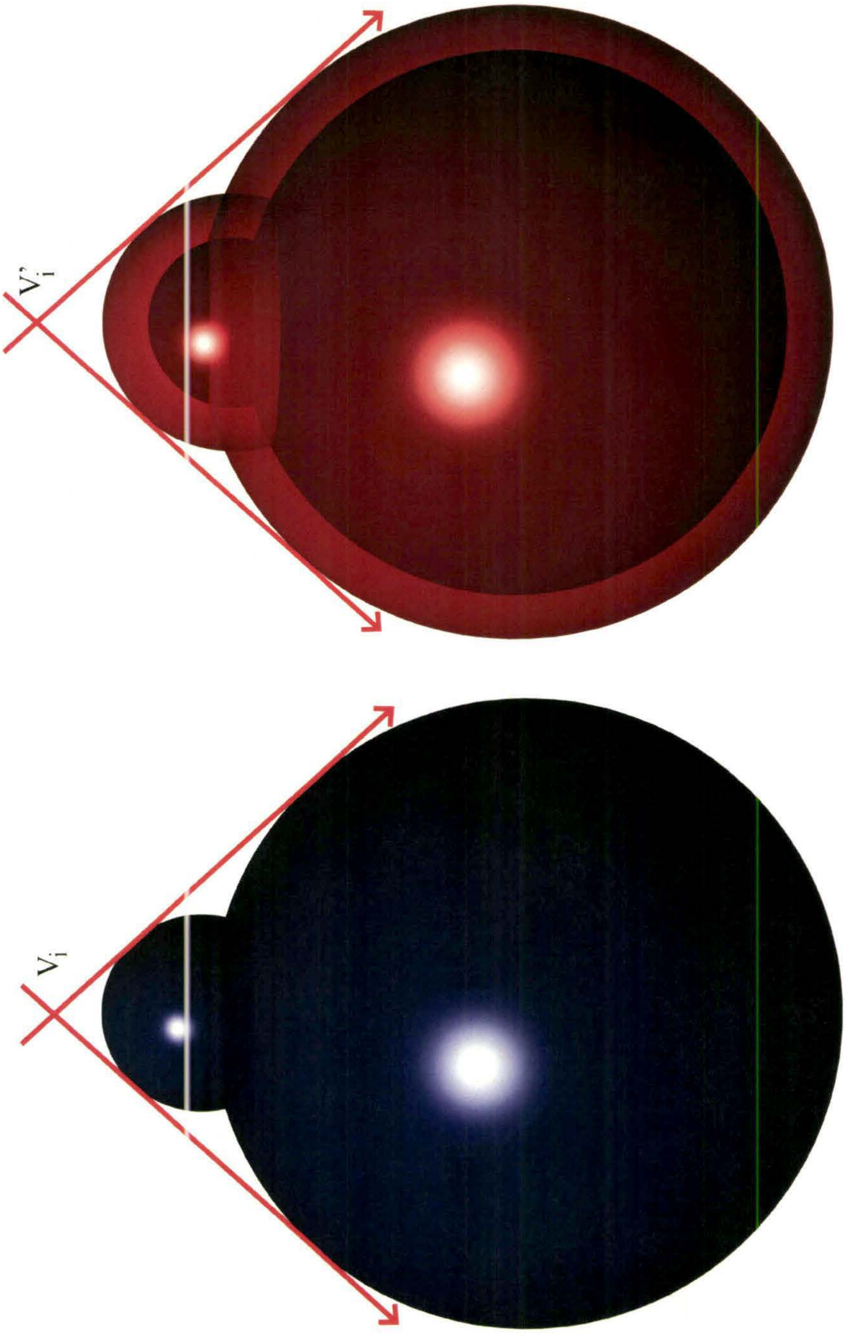

Figure 5.5: The Offset Transformation

5.4 Moebius Transformations

In the same way that Laguerre transformations can be used to manipulate conical meshes, Moebius transformations can be used to manipulate circular meshes. In Corollary 1, it was shown that the group of Moebius transformations is isomorphic to the group $O(n+1,1)/\pm I$. Geometrically, this means that points in the Lie Quadric Q^{n+1} that have zero in the last coordinate, that is, they correspond to spheres of zero radius, get mapped to other such points when acted upon by a Moebius transformation. Moreover, although tangential distance between spheres in $\mathbb{R}^3$ is not preserved, the angle between their centres is. Indeed, it this property that ensures that the circular property, which is that every face of a mesh M can be inscribed in a circle, is preserved under Moebius transformations. Since the architectural applications of circular meshes are limited, we will not pursue them further.

Part III

Applications And Numerical Experiments

Chapter 6

Mesh Creation

In practice, it is necessary to know how to approximate a given surface with a discrete one, a process known as discretization. In architecture, this is particularly important since most often the defining geometrical characteristics of a surface are contained in sketches, which are inherently smooth. As always, the focus here is on planar quadrilateral meshes, and so we will develop techniques that will produce discretizations of smooth surfaces that are not only accurate representations of that surface, but that are also conducive for the construction of planar quadrilateral meshes.

6.1 PQ Meshes By Optimization

In order to construct a PQ mesh based on a given smooth surface, a discretization of that surface needs to be chosen. In general, there are many different discretizations with different characteristics and different uses. However, a basic property that is fundamental to all of them is that if a discretization is refined enough, it will approximate the original surface arbitrarily well. This just means that the defining characteristics of the smooth surface, such as genus or curvature, are encoded in the discrete version. For different examples of discretizations of the same smooth surface, refer to Figure 6.1.

To create a mesh M from a discretization, we simply apply the appropriate combinatorial structure, which could be anything in general, but in this case, it is that of quadrilateral mesh. This can be done by taking the intersection points of the curve network that defines the discretization as the vertices of the mesh. The edges are defined as straight line segments connecting adjacent vertices. At this point, it is not possible to realize the faces as actual geometrical objects, as the region that is enclosed by the appropriate four vertices might not be planar, and therefore not unique. However, as a combinatorial entity, the face is perfectly well defined: it is the ordered four element subset of the vertex set. So, given a smooth surface, we can construct a quad mesh M whose approximation of the surface depends on how fine the mesh is.

The question is, how can the vertices of M be rearranged so that the the set of points in each face are coplanar, but that no edges degenerate. The current solution to this problem was introduced in [6] and involves a fairly complex non linear optimization. It is important to note that with the introduction of numerical techniques, the term "planar" will mean "planar to within a very small error" and not "geometrically planar".

The general idea of the optimization is this, given a quadrilateral mesh M, define a penalty function P that measures the aggregate amount of "non planarity" of the mesh. More formally,

$$P = \sum_{i=0}^{n} [V_{f_i}]^2$$

That is, P measures the sum of the volumes of the verex tetrahedra. Indeed, P

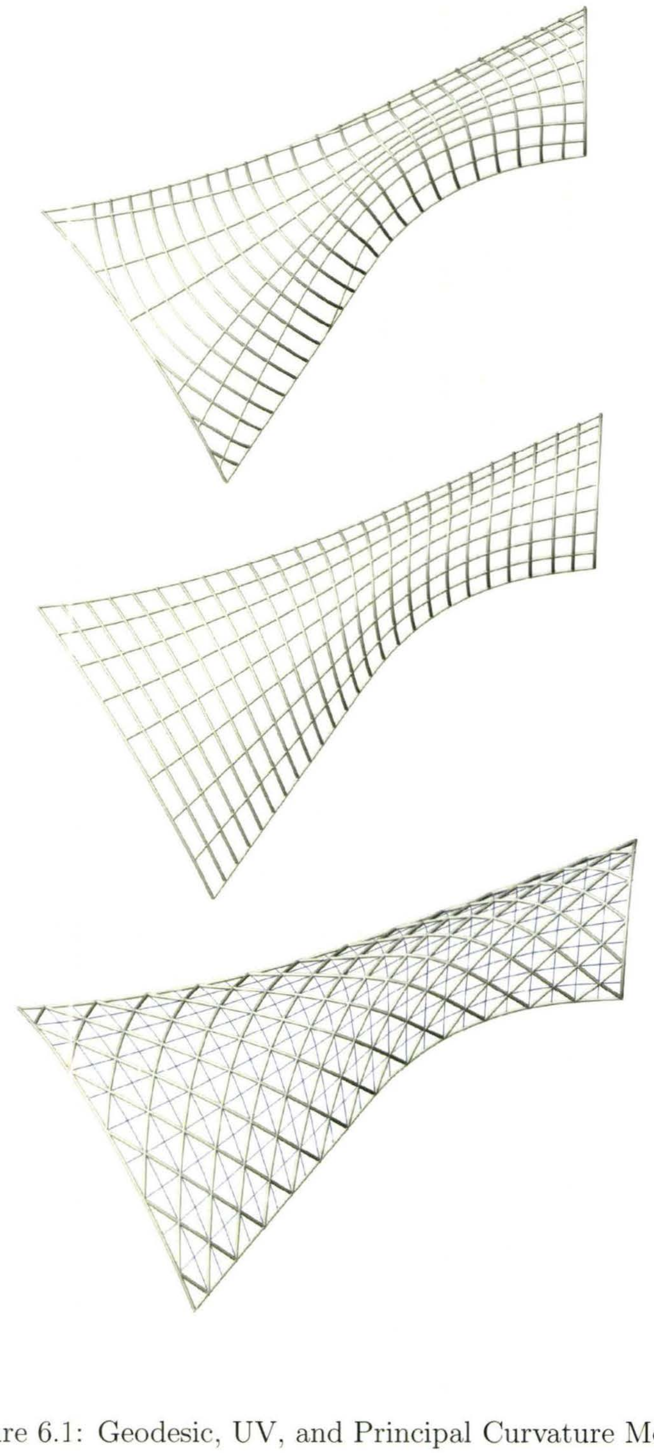

Figure 6.1: Geodesic, UV, and Principal Curvature Meshes

will be zero if and only if each quadrilateral of the mesh is planar. This is a non-linear, non-convex, optimization problem, and therefore it is not guaranteed that a global minimum will be found. However, recall that Proposition 2.1.1 states that if a PQ mesh is refined, it becomes a conjugate curve network. This implies that if the original surface is discretized by such a curve network, the optimization process will in fact arrive at a minimum, and indeed numerical experiments support this. Moreover, they show that if the optimization process is started on a mesh that is not derived from a conjugate curve network then the vertices will be moved so far that the mesh will not approximate the original surface. For more details of the numerical methods used to minimize this penalty function, see [6].

In summary, if the original surface can be parametrized by a conjugate curve network, then the associated mesh can be optimized into a PQ mesh without perturbing the vertices very much. If the original surface is discretized in some other way, then the associated mesh can still be optimized, but it is unclear whether the result will be desirable.

6.2 Conical Meshes From Principal Curvature Networks

It is also desirable to know, given a surface, if it is possible to approximate an arbitrary surface with a PQ mesh that admits the constant offset construction. Of course, since there are three different types offsets, a surface could admit, for example, a vertex offset and not an edge offset. In its most general form, this question is still unanswered, and this is due to the fact that the theory

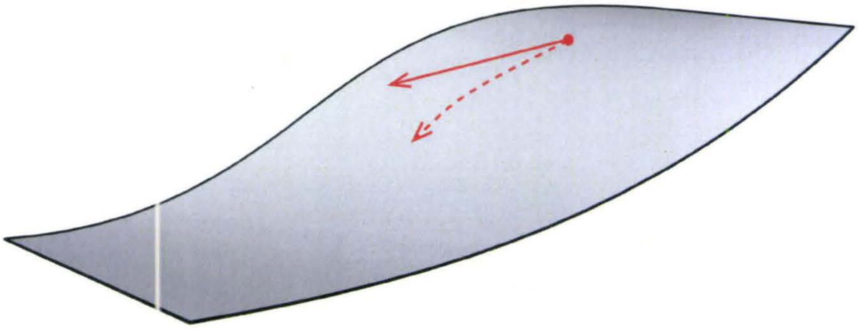

Figure 6.2: Computation of Principal Curvature Lines

of edge offsets is significantly more complicated than that of both vertex and face offsets.

Let us now see how it is possible to go from an arbitrary, initial surface to a discrete mesh that allows for the face offset property. This method was introduced and described in [6].

In the previous section, it was shown how to construct a PQ mesh that approximates an arbitrary smooth surface, with the key step being the discretization by a conjugate curve network. In [6] it is shown that face offset meshes are the discrete analogs of networks of Principal curvature lines. So, in order to initialize the optimization process, the surface needs to be discretized by its maximum and minimum Principal curvature lines. There are many techniques to for doing this, in particular the one in [1] is very efficient. For the purposes of this paper, we will use a simplified integration technique that simply starts at a point on the surface, computes the unit principal direction vectors, and then projects down onto the surface to get the next point of the iteration, as displayed in Figure 6.2. Although very sensitive to noise and not efficient, this technique, if refined enough, will produce a line of principal curvature. Since in addition to being the continuous analog to a face offset mesh, the network of principal curvature lines is a conjugate curve network, the mesh constructed from the intersection points of the principal curvature lines will be a good initialization for a discrete offset mesh. As before, we define a penalty function P_F to be

$$P_F = \lambda_1 P + \lambda_2 \left(\sum_{i=0}^{n} [(a_i + b_i + c_i + d_i) - 2\pi]^2 \right)$$

where P is the planarity penalty function defined above, the second term is difference between the sum of the angles at a vertex and 2π summed over all the vertices, and λ_1, λ_2 are scalar multiples. Since a PQ mesh is conical if and only if at each vertex it is possible to construct a cone tangent to all faces meeting at that vertex, if the above function is zero, the resulting mesh will be a conical mesh.

Chapter 7

Case Study - A Freeform Glass Roof

Throughout this paper, we have been discussing the theoretical framework behind mesh creation and mesh manipulation. Indeed, at the very beginning of the paper, we stated that the main application of the theory of meshes is in the architecture and construction industry, particularly concerning the problem of paneling freeform glass surfaces. In order to demonstrate the true power of discrete differential geometry, we will apply all of the techniques discussed in this paper to a fictional, but plausible, project. The general scope of the project is to design a surface that can be paneled with glass elements, that has no inherent symmetry (i.e a freeform surface), and that spans the open area bordered by the existing buildings at College St. and University Avenue, in Toronto, Ontario. The purpose of the project is to cover that open area in order to promote pedestrian traffic. In Figure 7.1, the site plan along with the initial sketch of the surface is depicted. To further mimic the process that a real project might go through, we will assume that an architect proposed a surface, but not a panelization scheme. Moreover, although the architect is

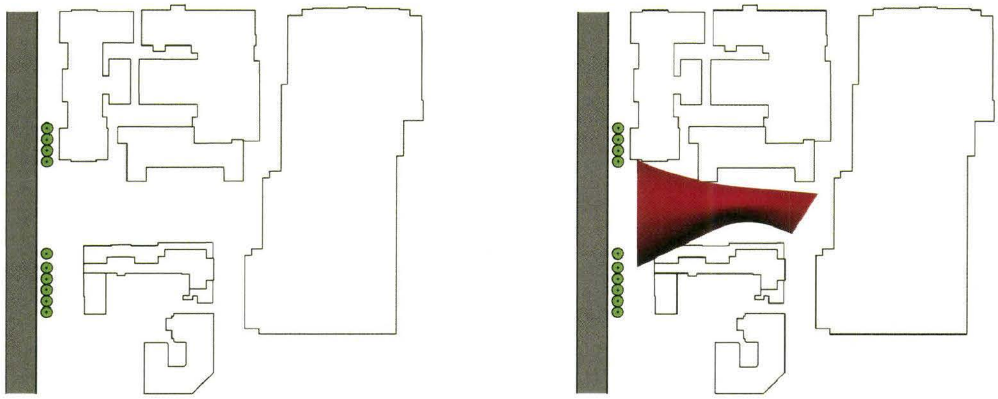

Figure 7.1: The Site Plan

in principle willing to modify the shape slightly, the goal is to stay as close as possible to the existing sketch. The first step is to construct a parametrization of the surface based on a conjugate curve network. In addition, since we will want to take advantage of the offset construction, we will need to parametrize this surface by principal curvature lines. To do this, we use the integration technique outlined in the Chapter 6 to construct the nodes of the mesh.

A quick visual inspection of the principal curvature mesh of the original surface, Figure 7.3, shows that the geometry of the surface will have to be modified slightly in order for such a mesh to be viable. By decreasing the curvature in the centre region of the surface and rerunning the principal curvture line computation, we arrive at a surface and its discretization that is well suited for panelization. Figure 7.4 shows this modified surface.

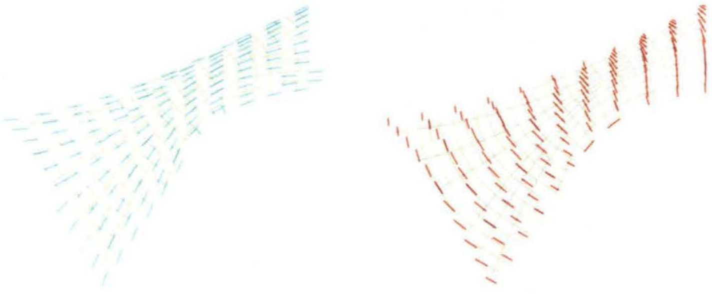

Figure 7.2: Computation of Principal Curvature Directions

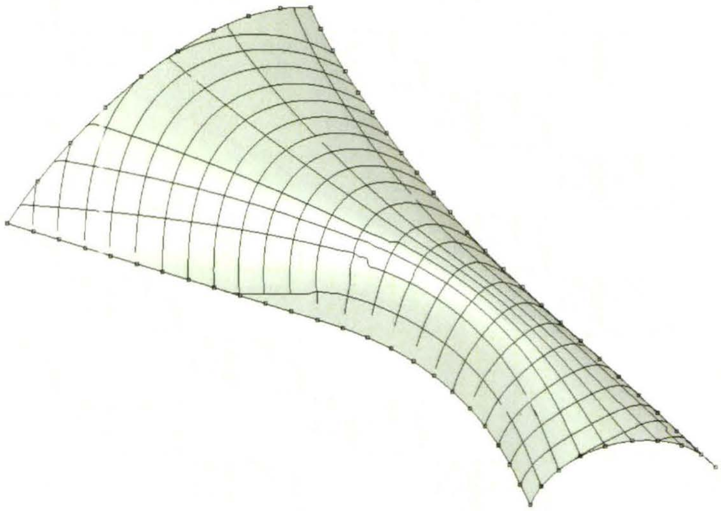

Figure 7.3: Computation of Principal Curvature Lines

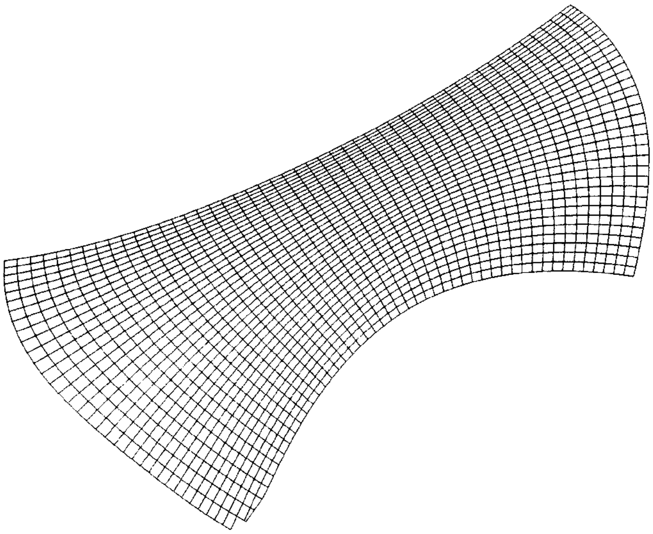

Figure 7.4: The Modified Surface

Using the discretization by principal curvature lines, we have a quadrilateral mesh M that approximates the original surface. The panels, however, are not yet planar. Not only do we want planar panels, but we want the mesh M to satisfy the conical angle condition at every vertex. We then run the modified optimization algorithm that optimizes both the planarity and the conical penalty functions. Figure 7.5 shows a comparison between two mesh M and M' before and after the optimization. Notice that although the planarity function stays more or less constant, implying that the original mesh was already very planar, the conical penalty function is extremely different.

Once the mesh has satisfied the planarity and conical conditions up to an acceptable error, the offset construction can be used to create a series of parallel meshes where the distance between corresponding faces is constant over the whole mesh. This allows for the design of an extremely "clean" node,

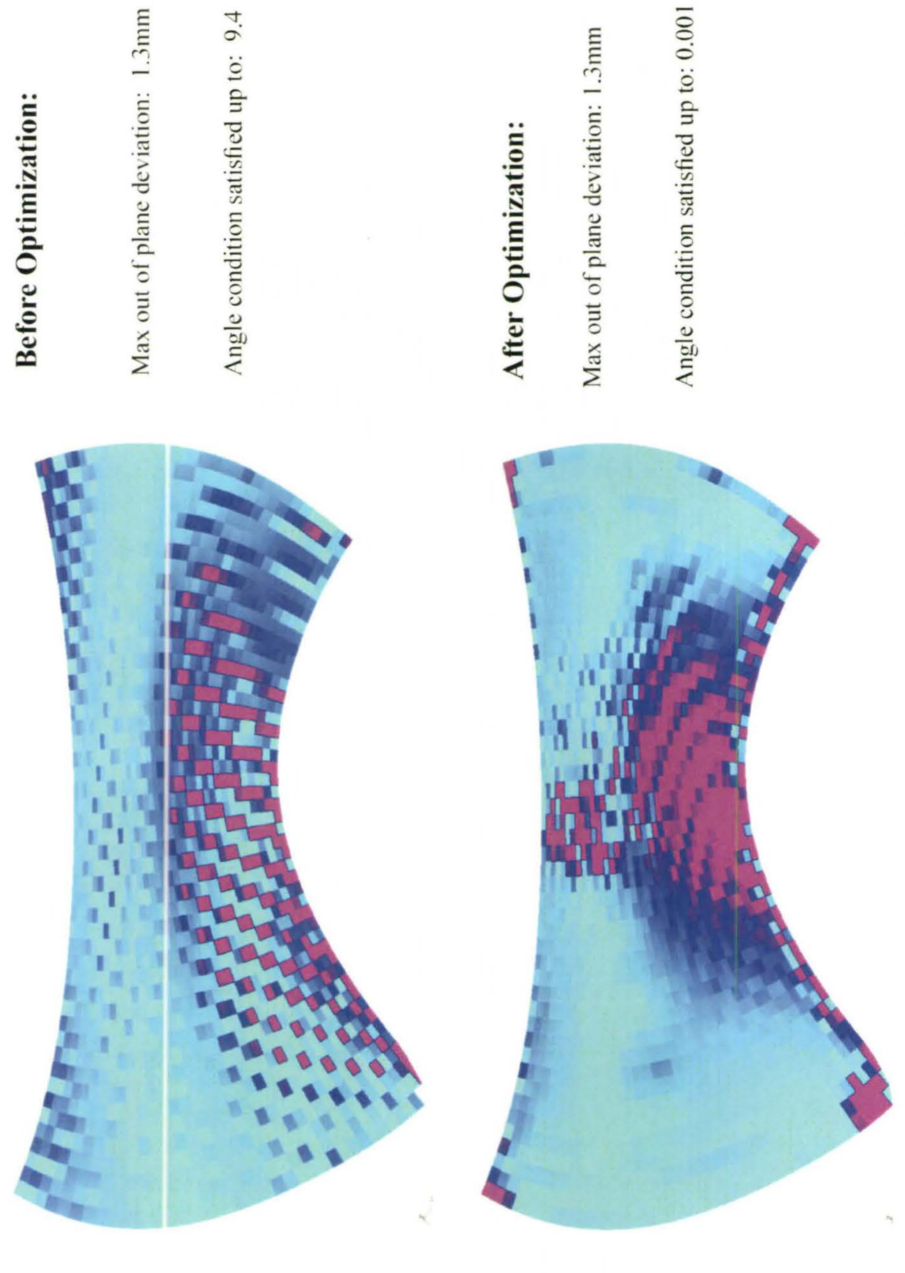

Figure 7.5: Planarity And Angle Condition Optimization

since it is not necessary to design elements that deal with eccentricities present in other discretizations of the same surface. Figure 7.6 depicts such a node. In addition, know that the distance between faces is constant means that is possible to design standardized spacing elements between the layers of glass in a typical panel.

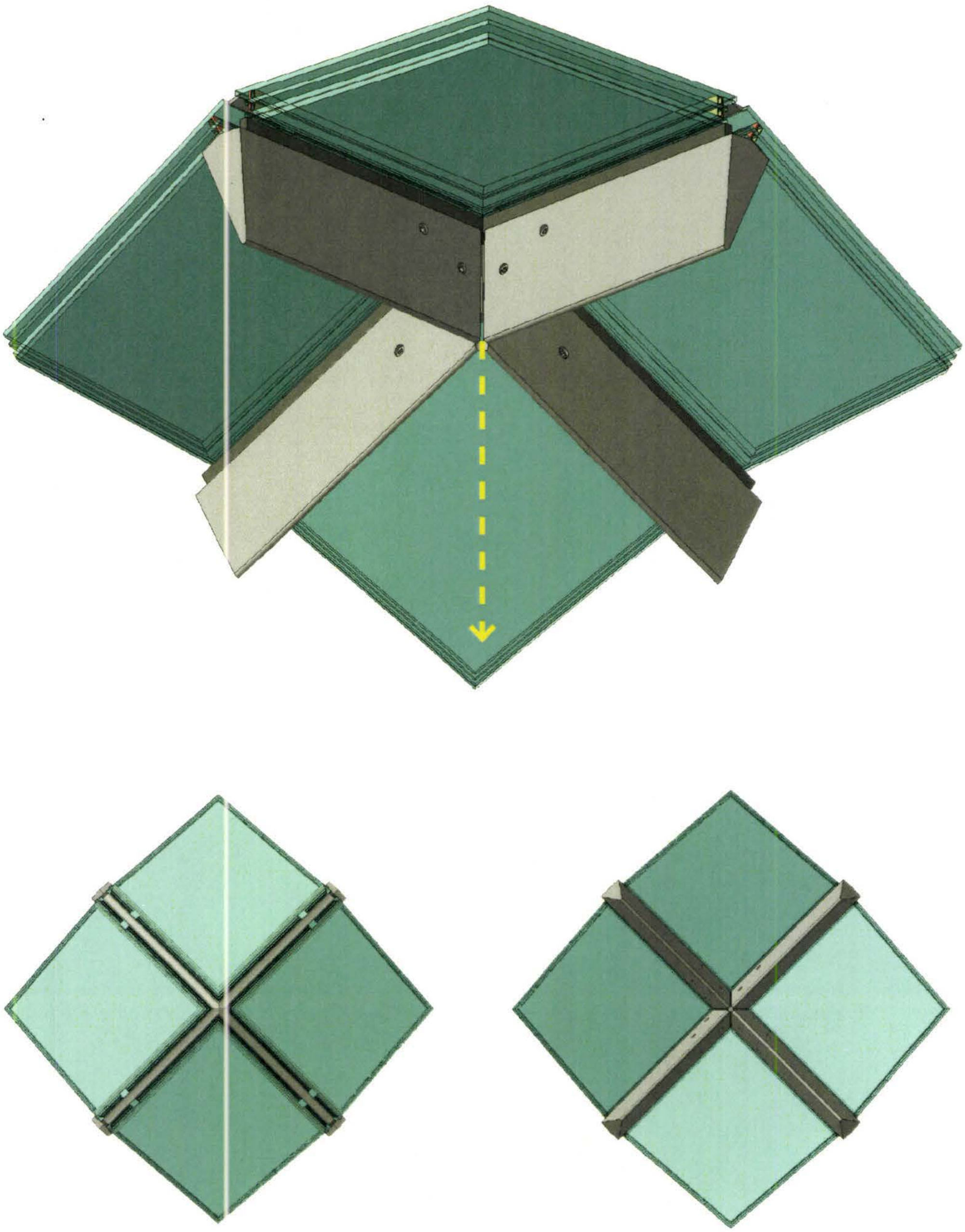

Figure 7.6: A Conical Node

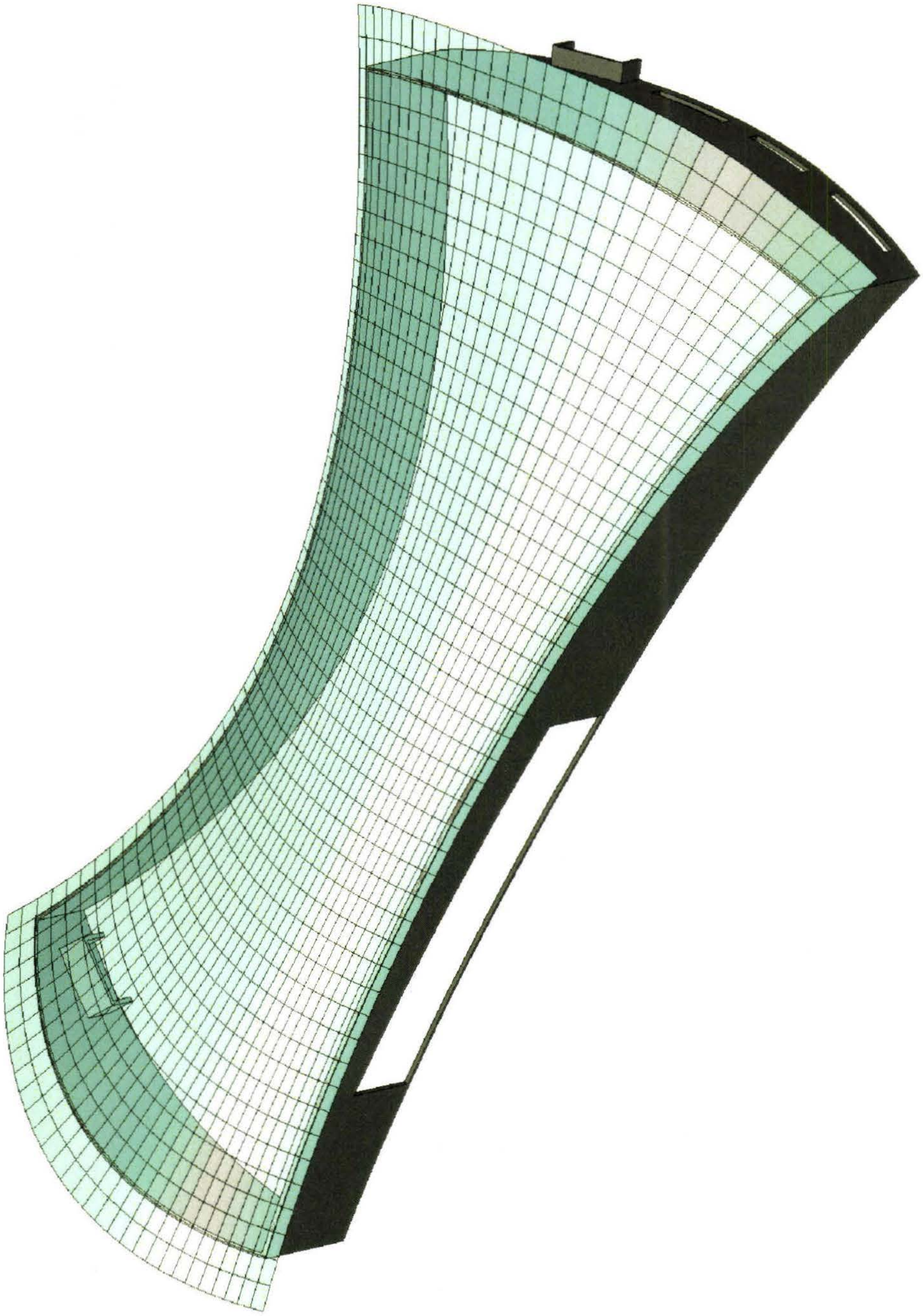

Figure 7.7: The Final Panel Layout

Chapter 8

Summary

In thesis, we have seen that by aligning an initial mesh with a conjugate curve network on a given surface, it is possible to transform it into a PQ mesh. Since conjugate curve networks possess the continuous version of the planarity condition, the resulting PQ mesh will still approximate the given surface. Moreover, if the principal curves are chosen as the conjugate curve network, the resulting PQ mesh will also possess a one parameter family of face offset meshes [7]. These meshes are called conical meshes and are extremely useful in the design of freeform glass surfaces. Given a conical mesh, we can transform it by using Laguerre transformations, which preserve the conical property. These transformations can be used to generate meshes with interesting geometries from very simple ones.

Thus, pure mathematics has been used to solve an extremely pragmatic problem, namely that of attaching structure to an otherwise two dimensional surface. Fortunately, there are many more unsolved problems. For instance, edge offset meshes seem to lend themselves naturally to beam standardization, since such a mesh would allow beams to have a constant width. However, they have proved to be theoretically the most difficult to deal with, and the kinds of shapes that can be discretized by edge offset meshes are not yet fully understood. Furthermore, the dependency of face offset meshes on the principal curvature lines often leads to unusable or visually bizarre meshes. While there have been some results that address this issue, it has by no means been fully dealt with.

It is interesting to note that without the theory of conjugate curve networks and projective geometry, it would not be possible to arrive at the arrangement of vertices that are found in an offset mesh "by hand". This implies that the study of offset meshes has given designers and engineers a tool that truly expands the scope of freeform geometry. However, inasmuch as projective geometry is an unlikely topic of conversation in an architecture office, beam and node layout for glass panels is an unusual problem for pure mathematicians to tackle. Yet, thinking about this problem is what led researchers to develop a whole new field of thought, coined "architectural geometry". It is perhaps due to this latest symbiotic relationship between application and theory that architectural geometry is shaping up to be an intellectually rich and exciting field.

Appendix A

Appendix

A.1 Modeling with Conjugate Curve Networks

In the section above, we outlined a process that would take an arbitrary surface as input, and produce a PQ mesh with the face offset property as output. What about the other way? It is, in many cases, desirable to be able to explore different surfaces that still have the PQ property. As an example of such a technique, we will restrict our attention to surfaces that have polynomials in two variables of degree three as coordinate functions. That is,

$$X(u,v) = (F^1(u,v), F^2(u,v), F^3(u,v))$$

where,

$$F^i(u,v) = a^i_{00}+a^i_{10}u+a^i_{01}v+a^i_{11}uv+a^i_{20}u^2+a^i_{02}v^2+a^i_{21}u^2v+a^i_{12}uv^2+a^i_{30}u^3a^i_{03}v^3$$

and the first few partial derivatives can be written as,

$$F^i_u(u,v) = a^i_{10} + a^i_{11}v + 2a^i_{20}u + 2a^i_{21}uv + a^i_{12}v^2 + 3a^i_{30}u^2$$

$$F^i_v(u,v) = a^i_{01} + a^i_{02}v + 2a^i_{11}u + 2a^i_{12}uv + a^i_{21}u^2 + 3a^i_{03}v^2$$

$$F^i_{uv}(u, v) = a^i_{11} + 2a^i_{21}u + 2a^i_{12}v$$

The idea is that we will construct a surface $X(u, v)$ whose parameter curves are in fact a family of conjugate curves. To do this, recall that the definition of a surface to be parametrized by conjugate curves is,

$$X_{uv}(u, v) = c_1 X_u(u, v) + c_2 X_v(u, v)$$

Combining the conjugacy condition with the partial derivatives of the coordinate functions and comparing coefficients, we get the following relations:

1. $a^i_{11} = c_1 a^i_{10} + c_2 a^i_{01}$
2. $2a^i_{21} = 2c_1 a^i_{20} + c_2 a^i_{11}$
3. $2a^i_{12} = 2c_2 a^i_{02} + c_1 a^i_{11}$
4. $0 = 2c_1 a^i_{21} + 2c_2 a^i_{12}$
5. $0 = c_1 a^i_{12} + 3c_2 a^i_{03}$
6. $0 = 3c_1 a^i_{30} + c_2 a^i_{21}$

These relations can be used to write all the coefficients as linear combinations of $\mathbf{a_{10}}, \mathbf{a_{01}}, \mathbf{a_{20}} \in \mathbf{R^3}$, namely,

1. $\mathbf{a_{11}} = \mathbf{c_1 a_{10}} + \mathbf{c_2 a_{01}}$
2. $\mathbf{a_{21}} = \mathbf{c_1 a_{20}} + \frac{\mathbf{c_2}}{\mathbf{2}}\mathbf{a_{11}}$
3. $\mathbf{a_{12}} = -\frac{\mathbf{c_1}}{\mathbf{c_2}}\mathbf{a_{21}}$
4. $\mathbf{a_{02}} = \frac{\mathbf{2a_{12} - c_1 a_{11}}}{\mathbf{2c_2}}$

5. $\mathbf{a_{03}} = -\frac{\mathbf{c_1}}{\mathbf{3c_2}}\mathbf{a_{12}}$

6. $\mathbf{a_{30}} = -\frac{\mathbf{c_2}}{\mathbf{3c_1}}\mathbf{a_{21}}$

It is also necessary to choose $c_1, c_2 \in \mathbf{R}$ and $\mathbf{a_{00}} \in \mathbf{R^3}$, which is simply the constant term in the polynomial functions. To illustrate this procedure, let,

$$\mathbf{a_{10}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{a_{01}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{a_{20}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and let, $c_1 = c_2 = 1$. This gives the following surface,

$$X(u, v) = (F^1(u, v), F^2(u, v), F^3(u, v))$$

where,

$$F^1(u, v) = u + uv - v^2 + \frac{1}{2}u^2v - \frac{1}{2}uv^2 - \frac{1}{6}u^3 + \frac{1}{6}v^3$$
$$F^2(u, v) = v + uv - v^2 + \frac{1}{2}u^2v - \frac{1}{2}uv^2 - \frac{1}{6}u^3 + \frac{1}{6}v^3$$
$$F^3(u, v) = u^2 - v^2 + u^2v - uv^2 - \frac{1}{3}u^3 + \frac{1}{3}v^3$$

Figures 7 and 8 show the continuous and discrete versions of the surface.

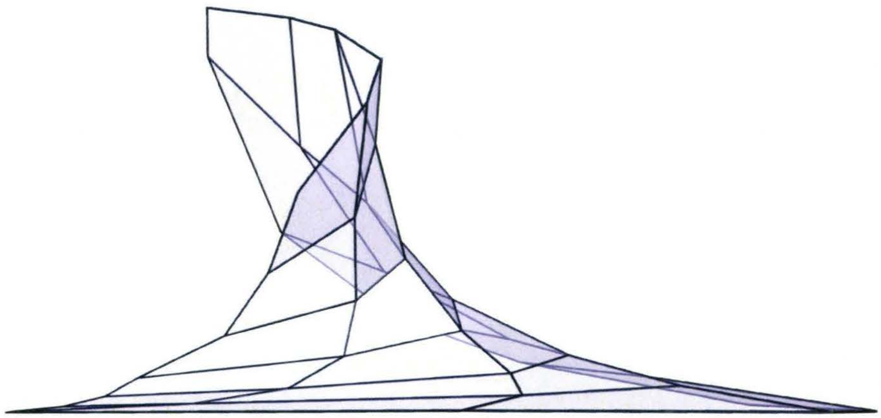

Figure A.1: The Associated PQ Mesh To A Conjugate Curve Network

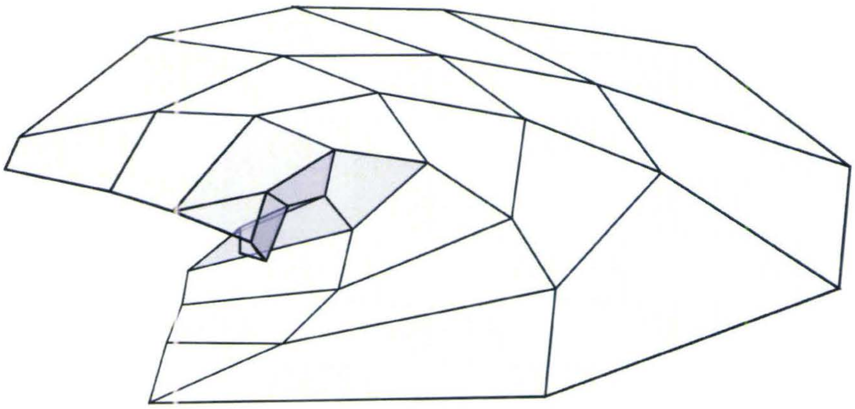

Figure A.2: The Associated PQ Mesh To A Conjugate Curve Network

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