SOME NICE RESULTS ABOUT ANISOTROPIC MEAN CURVATURE FLOW

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Introduction

Imagine stretching out a rubber band on a flat surface and letting go suddenly. Picture the way the rubber band contracts in slow motion and that should give you a good idea of how mean curvature flow dictates the evolution of plane curves. The more stretched out the rubber band, the faster it snaps back. Just like the rubber band returns to it's original round shape no matter how it is stretched, any smooth plane curve will evolve under mean curvature flow to a circle. Suppose that you try to kink the rubber band, try to force a sharp corner into it. As soon as you let go those kinks disappear. Similarly a piecewise smooth curve will smooth out instantaneously under mean curvature flow. Now suppose that you stretch out the rubber band and put kinks in it, but instead of letting go completely, you hold those kinks in place. The rest of the rubber band will still try to shrink back to it's original circular shape. This is the major topic of this paper-how do piecewise smooth curves behave under mean curvature flow if their kinks are held fast? It turns out that the initial evolution of a curve in such a situation depends completely on the number and precise angles of those kinks.

One of the earliest references on mean curvature flow is a 1956 paper [15] which explored a specific case of piecewise smooth curves evolving by mean curvature and found that by counting the number of sides one could determine how the enclosed area would change (Theorem 4.1). This was a suprising result because in the smooth case, the area enclosed is *always* shrinking, but by adding some sharp corners it became possible that the area would increase initially. Little attention seems to have been paid to piecewise smooth curves and mean curvature flow since then, with one notable exception being a paper by L. Bronsard and F. Reitich [5] which proved that the curves analyzed in the 1956 paper could really exist!

The main result of this paper is Theorem 4.4 which is a generalization of the aforementioned Theorem 4.1. The new result generalizes the original in two ways: first it is non-specific with respect to the angles at the corners, and second, it allows for the the flow to be *anisotropic*; the evolution of the curve may depend on it's orientation in the plane. Two proofs of this result are presented. One uses ideas from the 1956 paper and is fairly intuitive. The other proof follows the strategy of a more recent paper [10] and proves the result as an intrinsic property of the curve. The final section of the paper mentions some other questions and topics related to mean curvature flow and includes a new result about the behavior of curves evolving on the unit sphere according to a generalized version of mean curvature flow.

1 Mean Curvature Flow

The closed plane curves we will consider are embedded in $\mathbf{R}^2$, that is they do not cross themselves. At first our curves will be smooth, but later when we are working with piecewise smooth curves we require that they have only finitely many vertices. By smooth we mean C^∞. We will adopt the counterclockwise orientation, so that the curvature of the unit circle is $+1$. Finally we will assume that all smooth portions of the curves have bounded curvature. Let us now define mean curvature flow,

Definition 1.1. *If $\boldsymbol{F}(\cdot,t)$ is a one parameter family of curves such that*

$$\langle \frac{\partial \boldsymbol{F}}{\partial t}, \boldsymbol{N} \rangle = \kappa \tag{1}$$

then the curves $\boldsymbol{F}(\cdot,t)$ are said to be evolving by Mean Curvature, where κ is the plane curvature, $\boldsymbol{N}$ is the inward pointing unit normal and $\langle \cdot,\cdot \rangle$ is the usual scalar product.

Example 1.2. *The circle provides a fairly intuitive example for mean curvature flow. The curvature is constant for a circle so the normal vectors all have equal length and point inward towards the center.*

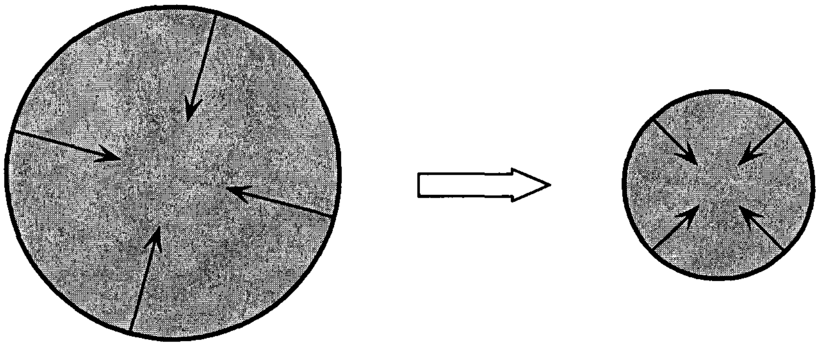


Figure 1: A circle evolving by mean curvature flow.

Straight line segments have zero curvature and hence have no velocity in the normal direction. With this in mind we will not consider curves that have any interval of constant curvature equal to zero.

If we parameterize by arclength s, then the length of a curve $\mathbf{F}(s,t)$ is

$$L = \int_0^L ds. \tag{2}$$

We wish first to calculate a general formula for the time derivative of the length. Then we will see that Equation (1) reflects the choice of the fastest length minimizing flow, that is, the flow which shrinks the initial curve $\mathbf{F}(\cdot, 0)$ most efficiently. The mathematical obstacle here is the fact that the arclength parameter s (and hence the length L) is time dependent. To overcome this, we reparameterize the curve in the form

$$\mathbf{F}(u,t) = (x(u,t), y(u,t))$$

where $u \epsilon [0,1]$ is a time independent parameter. As we will discuss shortly, there are a number of existence type results which ensure that such a parameterization can be made at each time so that $\mathbf{F}(u,t)$ is differentiable in time. Then the length becomes

$$L = \int_0^1 \sqrt{(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2} du, \tag{3}$$

and, if we denote

$$v = \sqrt{(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2}$$

then

$$ds = vdu. \tag{4}$$

This also gives us a useful relation between the associated partial derivative operators:

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}. \tag{5}$$

Now the time derivative can be interchanged with the integral so that

$$\frac{\partial L}{\partial t} = \frac{\partial}{\partial t}\int_0^1 vdu = \int_0^1 \frac{\partial v}{\partial t} du.$$

So what we need is a more meaningful expression of $\frac{\partial v}{\partial t}$.

Lemma 1.3. *The time derivative of v is*

$$\frac{\partial v}{\partial t} = v\langle \boldsymbol{T}, \frac{\partial}{\partial s}(\frac{\partial \boldsymbol{F}}{\partial t})\rangle \tag{6}$$

where $\boldsymbol{T}$ is the unit tangent vector to the curve.

Proof. First, recall that

$$v = |\frac{\partial \mathbf{F}}{\partial u}|$$

so that

$$v^2 = |\frac{\partial \mathbf{F}}{\partial u}|^2 = \langle \frac{\partial \mathbf{F}}{\partial u}, \frac{\partial \mathbf{F}}{\partial u}\rangle.$$

Then

$$\frac{\partial}{\partial t}(v^2) = 2v(\frac{\partial v}{\partial t})$$
$$= \frac{\partial}{\partial t}\langle\frac{\partial \mathbf{F}}{\partial u}, \frac{\partial \mathbf{F}}{\partial u}\rangle$$
$$= 2\langle\frac{\partial \mathbf{F}}{\partial u}, \frac{\partial^2 \mathbf{F}}{\partial t \partial u}\rangle.$$

Since u is a time independent parameter the operators

$$\frac{\partial}{\partial t} \text{ and } \frac{\partial}{\partial u}$$

commute and so

$$2v(\frac{\partial v}{\partial t}) = 2\langle\frac{\partial \mathbf{F}}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial \mathbf{F}}{\partial t})\rangle.$$

The vector $\frac{\partial \mathbf{F}}{\partial u}$ points in the tangential direction and has magnitude v, and using the relation (5) to rewrite the operator $\frac{\partial}{\partial u}$ we have

$$2v(\frac{\partial v}{\partial t}) = 2v^2\langle\mathbf{T}, \frac{\partial}{\partial s}(\frac{\partial \mathbf{F}}{\partial t})\rangle. \tag{7}$$

Note that because our curve is smooth, it has a nonzero tangent vector at every point, so

$$v = |\frac{\partial \mathbf{F}}{\partial u}| = \sqrt{(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2} \neq 0.$$

Thus $2v$ can be cancelled from both sides of Equation (7) which proves the lemma.

Now we have that

$$\frac{\partial L}{\partial t} = \int_0^1 v\langle\mathbf{T}, \frac{\partial}{\partial s}(\frac{\partial \mathbf{F}}{\partial t})\rangle du$$

or

$$\frac{\partial L}{\partial t} = \int_0^L \langle\mathbf{T}, \frac{\partial}{\partial s}(\frac{\partial \mathbf{F}}{\partial t})\rangle ds \tag{8}$$

since

$$vdu = ds.$$

If we express $\frac{\partial \mathbf{F}}{\partial t}$ in the orthonormal basis $\{\mathbf{T}, \mathbf{N}\}$ as

$$\frac{\partial \mathbf{F}}{\partial t} = a\mathbf{T} + b\mathbf{N} \tag{9}$$

for some C^1 real valued functions a and b, then

$$\begin{aligned}
\frac{\partial L}{\partial t} &= \int_0^L \langle \mathbf{T}, \frac{\partial}{\partial s}(a\mathbf{T} + b\mathbf{N}) \rangle ds \\
&= \int_0^L \langle \mathbf{T}, \frac{\partial}{\partial s}(a\mathbf{T}) \rangle ds + \int_0^L \langle \mathbf{T}, \frac{\partial}{\partial s}(b\mathbf{N}) \rangle ds \\
&= \int_0^L \langle \mathbf{T}, \frac{\partial a}{\partial s}\mathbf{T} \rangle ds + \int_0^L \langle \mathbf{T}, a\frac{\partial \mathbf{T}}{\partial s} \rangle ds \\
&+ \int_0^L \langle \mathbf{T}, \frac{\partial b}{\partial s}\mathbf{N} \rangle ds + \int_0^L \langle \mathbf{T}, b\frac{\partial \mathbf{N}}{\partial s} \rangle ds.
\end{aligned}$$

There is a quick, and helpful digression to be made here. For plane curves, the Frenet Equations are

$$\begin{aligned}
\frac{\partial \mathbf{F}}{\partial s} &= \mathbf{T} \\
\frac{\partial \mathbf{T}}{\partial s} &= \kappa \mathbf{N} \\
\frac{\partial \mathbf{N}}{\partial s} &= -\kappa \mathbf{T}.
\end{aligned}$$

With the reparameterization by u the Frenet Equations become

$$\frac{\partial \mathbf{F}}{\partial u} = v\mathbf{T} \tag{10}$$

$$\frac{\partial \mathbf{T}}{\partial u} = v\kappa\mathbf{N} \tag{11}$$

$$\frac{\partial \mathbf{N}}{\partial u} = -v\kappa\mathbf{T}, \tag{12}$$

and both versions will be used repeatedly throughout this paper. With Frenet in mind

$$\frac{\partial L}{\partial t} = \int_0^L \langle \mathbf{T}, \frac{\partial a}{\partial s}\mathbf{T}\rangle ds + \int_0^L \langle \mathbf{T}, a\kappa\mathbf{N}\rangle ds$$
$$+ \int_0^L \langle \mathbf{T}, \frac{\partial b}{\partial s}\mathbf{N}\rangle ds + \int_0^L \langle \mathbf{T}, -b\kappa\mathbf{T}\rangle ds,$$

but because

$$\mathbf{T} \perp \mathbf{N}$$

and both $\mathbf{T}$ and $\mathbf{N}$ have unit length the second and third terms are zero and we are left with

$$\frac{\partial L}{\partial t} = \int_0^L \frac{\partial a}{\partial s} ds - \int_0^L b\kappa ds$$
$$= (a(L) - a(0)) - \int_0^L b\kappa ds.$$

Because we are considering smooth, closed curves,

$$a(L) = a(0)$$

and so

$$\frac{\partial L}{\partial t} = -\int_0^L b\kappa ds. \tag{13}$$

Using Holder's Inequality

$$\int_0^L (\kappa b) ds \leq \int_0^L |\kappa b| ds$$
$$\leq (\int_0^L |\kappa|^2 ds)^{\frac{1}{2}} (\int_0^L |b|^2 ds)^{\frac{1}{2}}$$

with equality if and only if $|b|$ is a multiple of $|\kappa|$, that is,

$$|b| = c|\kappa|$$

for some positive constant c. So the fastest length minimizing flow can be written as

$$\frac{\partial L}{\partial t} = -c\int_0^L \kappa^2 ds. \tag{14}$$

Notice that this formula is independent of the tangential component a. If we choose $a = 0$ then

$$\frac{\partial \mathbf{F}}{\partial t} = c\kappa\mathbf{N} \tag{15}$$

and we would still have the fastest length minimizing flow given by Equation (14). Moreover, if the evolution of a curve by mean curvature flow is viewed like a movie, the constant c acts like the fast-forward/slow-motion button. Changing c affects the speed at which a curve changes but not the nature of its evolution and so we will agree to set $c \equiv 1$. All of this means that we could consider the evolution equation

$$\frac{\partial \mathbf{F}}{\partial t} = \kappa\mathbf{N} \tag{16}$$

instead of the flow given by Equation (1), and the formula for

$$\frac{\partial L}{\partial t}$$

would be unchanged. In fact, many of our references prefer this flow, but the more general nature of Equation (1) will be helpful when we examine piecewise smooth curves later on.

2 Smooth Curves

Because mean curvature flow is the maximal length shrinking flow much analysis has been done. Accordingly there are standard results which will be discussed here. The "behavioral" results, those which illuminate *how* a curve evolves under mean curvature will be proved in this section, or discussed later on. The *existence* type results are arguably more fundamental but will be cited without the proofs which are beyond the scope of this paper. The first theorem, which we have already proved, is the defining attribute of mean curvature flow.

Theorem 2.1. *Under mean curvature flow*

$$\frac{\partial L}{\partial t} = -\int_0^L \kappa^2 ds. \tag{17}$$

The next theorem is not a suprising result in the sense that it is implied (at least in the long run) by Theorem 2.1, but it's generalization to piecewise smooth curves does hold some suprises and will play a central role in this paper.

Theorem 2.2. *The time derivative of the area enclosed by a curve which is evolving by mean curvature is*

$$\frac{\partial A}{\partial t} = -2\pi. \tag{18}$$

A proof of this theorem can be found in [10], but the next section will be devoted to proving the more general result Theorem 3.8. In Section 4 two proofs of a still more general result, Theorem 4.4, will be presented. One implication of Theorem 2.2 is that we can calculate how long it will take for a curve to disappear.

Corollary 2.3. *A simple closed curve shrinks to a point in time $\frac{A}{2\pi}$, where A is the area enclosed by the initial curve.*

Example 2.4. *Determining the time $\frac{A}{2\pi}$ for familiar shapes is straightforward. For example, the ellipse given by*

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

encloses an area of πab and so would disappear under mean curvature flow in time $\frac{ab}{2}$.

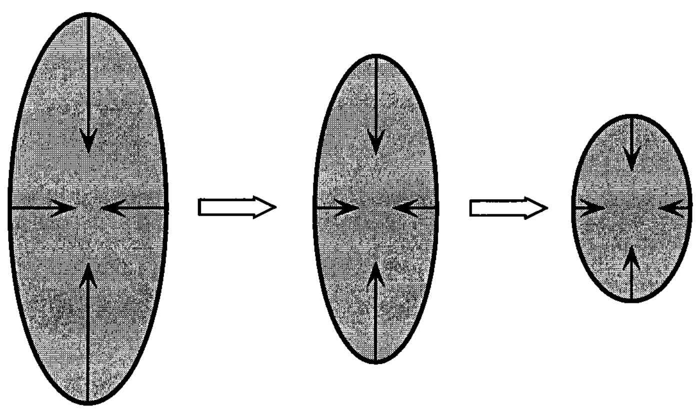


Figure 2: An ellipse evolving by isotropic mean curvature.

Thanks to George Green, we have a very useful formula for the area enclosed by an arbitrary plane curve. Let R be the region whose area we are interested in, with boundary curve C given by $\mathbf{F}(u) = (x(u), y(u))$ where $u\epsilon[0,1]$. We can define the area of R as

$$A = \int\int_R 1dA.$$

The divergence of the vector field

$$(x,y) \mapsto \frac{1}{2}(x,y) = \frac{1}{2}\mathbf{F}$$

is 1 so by The Divergence Theorem, and using the equality $ds = vdu$,

$$\begin{aligned} A = \int\int_R 1dA &= -\frac{1}{2}\int_C \langle \mathbf{F}, \mathbf{N}\rangle ds \\ &= -\frac{1}{2}\int_0^1 \langle \mathbf{F}, v\mathbf{N}\rangle du. \end{aligned} \tag{19}$$

Note that the minus sign appears because $\mathbf{N}$ is the *inward* pointing unit normal vector. We note that this formula generalizes directly to the case where C is piecewise smooth, the integral around the boundary is replaced by the sum of the integrals along each side. In coordinates,

$$\mathbf{N} = \frac{1}{v}(-\frac{dy}{du}, \frac{dx}{du})$$

so we can compute

$$\begin{aligned} \langle \mathbf{F}, v\mathbf{N}\rangle &= \langle (x,y), v\frac{1}{v}(-\frac{dy}{du}, \frac{dx}{du})\rangle \\ &= -x\frac{dy}{du} + y\frac{dx}{du} \end{aligned}$$

whence (19) becomes

$$A = \frac{1}{2}\int_0^1 (xdy - ydx)\frac{1}{\partial u}du = \frac{1}{2}\int_C xdy - ydx$$

which is the area formula familiar to many. This second expression is particularly useful when we want to consider polar coordinates,

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

in which case

$$xdy - ydx = (x\frac{dy}{d\theta} - y\frac{dx}{d\theta})d\theta = r^2 d\theta,$$

and

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta. \tag{20}$$

Theorems 2.1 and 2.2 and Corollary 2.3 are fairly comprehensive. Together they tell us that a smooth, closed, plane curve will shrink under mean curvature flow, how quickly it will shrink and, with Green's Formula in hand, how long it will take to disappear completely. We have mentioned existence theorems, and this is where we must rely on those results since otherwise we wouldn't know if there are actually any curves to analyze at all! With some animated intuition, or perhaps from the figures in Examples 1.2 and 2.4 the reader could conjecture the following result.

Theorem 2.5 (Fundamental Theorem for Plane Curves). *A smooth simple closed curve in $\boldsymbol{R}^2$, evolving by mean curvature flow shrinks to a point, and becomes asymptotic to a shrinking circle.*

Example 2.6. *The curve on the left of figure 3 is "unrolling" from both ends, but at the same time it's circular shape causes it to shrink. The Fundamental Theorem ensures that the spiral will unroll before the whole shape collapses on itself. Similarly, the theorem guarantees that the curve on the right will not "pinch off" into two separately evolving curves.*

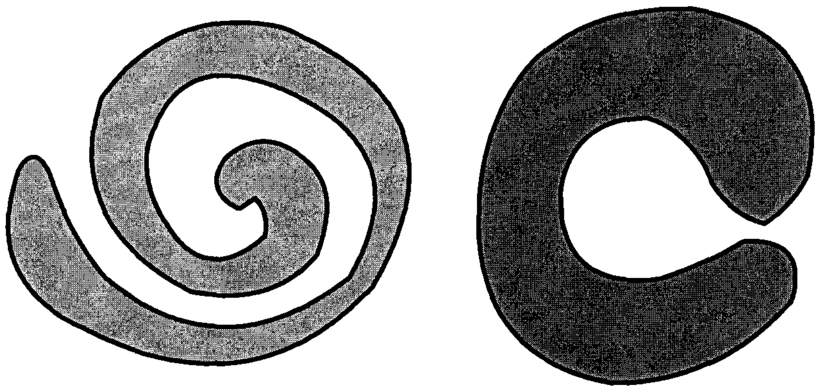


Figure 3: Potentially Troublesome Shapes

The Fundamental Theorem was proved in two parts. In 1986 M. Gage and R. Hamilton [10] proved that convex plane curves shrink to circular points and one year later, M. Grayson published a paper [12] in which he proved that simple closed curves become convex under mean curvature flow. Notice that the mean curvature evolution law Equation (1) really defines a system of PDE's so the Fundamental Theorem is really a result about the existence and nature of solutions to this system. It turns out that finding curves which evolve "nicely" by mean curvature is equivalent to solving a system of parabolic PDE's, which is the approach used by Gage and Hamilton, and also by D. DeTurck whose work on a more general problem [6] implies the following theorem:

Theorem 2.7. *A piecewise smooth initial curve allowed to evolve by mean curvature will become instantaneously smooth.*

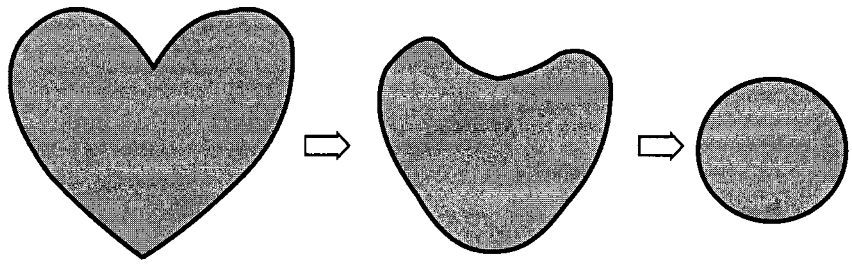


Figure 4: Piecewise smooth curves are smooth for all positive time.

With these existence theorems, the evolution of plane curves by mean curvature really is completely characterized, but in true mathematical fashion we will not be satisfied until these results have been generalized.

3 Anisotropic Mean Curvature Flow

As promised, this section will be devoted to a generalization of Theorem 2.2, and the proof will closely follow Gage and Hamilton's proof in [10]. The evolution of a curve under Equation (1) is *isotropic*; the curve will evolve independently of it's orientation in the plane. Now we turn to a similar flow, but one which includes a dependence on orientation.

Definition 3.1. *If $\boldsymbol{F}(\cdot, t)$ is a one parameter family of curves such that*

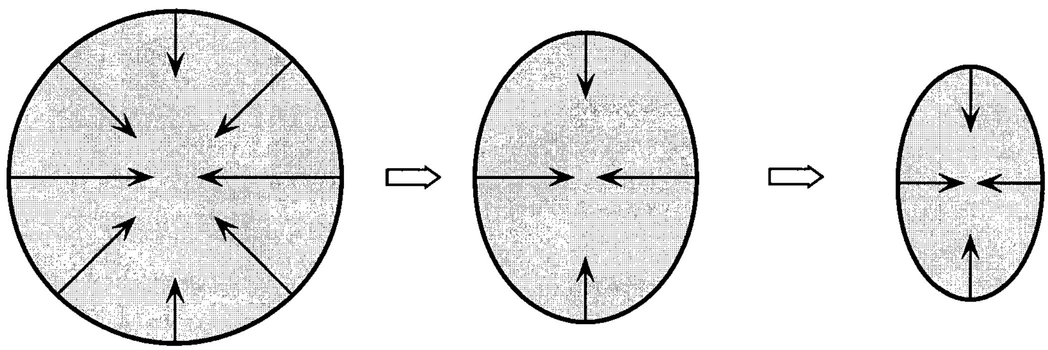
$$\langle \frac{\partial \boldsymbol{F}}{\partial t}, \boldsymbol{N} \rangle = \gamma \kappa \tag{21}$$

where

$$\gamma : [0, 2\pi) \to \boldsymbol{R}_+ \tag{22}$$

is a C^∞, 2π-periodic function of the angle α between the tangent vector $\boldsymbol{T}$ to the curve and the horizontal axis, then the curves $\boldsymbol{F}(\cdot, t)$ are said to be evolving by Anisotropic Mean Curvature.

As mentioned, anisotropic evolution depends on the orientation of a curve in the plane. This is because rotating C also rotates its tangent vectors, changing the value of γ at each point. The requirement that γ be positive is conventional and comes from the physical applications of anisotropic mean curvature flow. To the boundary of any material can be associated its free energy, which depends on both the microscopic structure of the material as well as its macroscopic shape. The function γ reflects this physical property and would depend on the specific material under investigation.



Example 3.2. *This circle is evolving by anisotropic mean curvature with*

$$\gamma(\alpha) = \cos^2 \alpha + 1,$$

which is a positive, 2π-periodic function. In addition

$$\gamma(\alpha + \pi) = \gamma(\alpha),$$

and so Gage and Li's results about anisotropic evolution in [11] imply that the circle will evolve into an elliptical point in finite time.

Lemma 3.3. *The time derivative of v is*

$$\frac{\partial v}{\partial t} = -\gamma \kappa^2 v. \tag{23}$$

Proof. Recall that because u and t are independent variables so the operators

$$\frac{\partial}{\partial t} \text{ and } \frac{\partial}{\partial u}$$

commute. Following the strategy of Lemma 1.3 we have, on one hand,

$$\frac{\partial}{\partial t}(v^2) = 2v\frac{\partial v}{\partial t}$$

and on the other hand

$$\frac{\partial}{\partial t}(v^2) = \frac{\partial}{\partial t}\langle\frac{\partial \mathbf{F}}{\partial u}, \frac{\partial \mathbf{F}}{\partial u}\rangle = 2\langle\frac{\partial \mathbf{F}}{\partial u}, \frac{\partial^2 \mathbf{F}}{\partial t \partial u}\rangle = 2\langle\frac{\partial \mathbf{F}}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial \mathbf{F}}{\partial t})\rangle.$$

Recall that

$$\frac{\partial \mathbf{F}}{\partial u} = v\mathbf{T}$$

and so using the Equation (21),

$$2\langle\frac{\partial \mathbf{F}}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial \mathbf{F}}{\partial t})\rangle = 2\langle v\mathbf{T}, \frac{\partial}{\partial u}(\gamma\kappa\mathbf{N})\rangle.$$

Applying the chain rule to the second term, we try to collect the terms in a sensible way,

$$2\langle v\mathbf{T}, \frac{\partial}{\partial u}(\gamma\kappa\mathbf{N})\rangle = 2\langle v\mathbf{T}, \frac{\partial \gamma}{\partial u}\kappa\mathbf{N} + \gamma\frac{\partial \kappa}{\partial u}\mathbf{N} + \gamma\kappa\frac{\partial \mathbf{N}}{\partial u}\rangle =$$

$$2\langle v\mathbf{T}, (\frac{\partial \gamma}{\partial u}\kappa + \gamma\frac{\partial \kappa}{\partial u})\mathbf{N} + \gamma\kappa\frac{\partial \mathbf{N}}{\partial u}\rangle.$$

The Frenet Equation (12) together with the fact that $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors give us

$$2\langle v\mathbf{T}, (\frac{\partial \gamma}{\partial u}\kappa + \gamma\frac{\partial \kappa}{\partial u})\mathbf{N} + \gamma\kappa\frac{\partial \mathbf{N}}{\partial u}\rangle =$$

$$2\langle v\mathbf{T}, (\frac{\partial \gamma}{\partial u}\kappa + \gamma\frac{\partial \kappa}{\partial u})\mathbf{N}\rangle + 2\langle v\mathbf{T}, -v\gamma\kappa^2\mathbf{T}\rangle =$$

$$0 - 2v^2\gamma\kappa^2\langle \mathbf{T}, \mathbf{T}\rangle = -2v^2\gamma\kappa^2.$$

Remember, we can cancel $2v$ from both sides which proves the claim.

Unfortunately, the operators

$$\frac{\partial}{\partial s} \text{ and } \frac{\partial}{\partial t}$$

do not commute because s depends on t, but we can calculate in some sense how closely they are to commuting.

Lemma 3.4.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + \gamma\kappa^2\frac{\partial}{\partial s}. \tag{24}$$

Proof. To begin, we will rewrite things using Equation (5):

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}(\frac{1}{v}\frac{\partial}{\partial u}).$$

By the chain rule, this is

$$(\frac{\partial}{\partial t}\frac{1}{v})\frac{\partial}{\partial u} + \frac{1}{v}(\frac{\partial}{\partial t}\frac{\partial}{\partial u}).$$

Now u is time independent, so

$$\frac{\partial}{\partial t}\frac{\partial}{\partial u} = \frac{\partial}{\partial u}\frac{\partial}{\partial t}$$

and from the previous lemma

$$\frac{\partial}{\partial t}\frac{1}{v} = -\frac{1}{v^2}\frac{\partial v}{\partial t} = \frac{v\gamma\kappa^2}{v^2} = \gamma\kappa^2\frac{1}{v}.$$

Thus

$$(\frac{\partial}{\partial t}\frac{1}{v})\frac{\partial}{\partial u} + \frac{1}{v}(\frac{\partial}{\partial t}\frac{\partial}{\partial u}) = \gamma\kappa^2(\frac{1}{v}\frac{\partial}{\partial u}) + (\frac{1}{v}\frac{\partial}{\partial u})\frac{\partial}{\partial t} = \gamma\kappa^2\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}$$

which proves the lemma.

Lemma 3.5. *The time derivatives of the unit tangent and inward pointing unit normal vectors are*

$$\frac{\partial \boldsymbol{T}}{\partial t} = (\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s})\boldsymbol{N} \tag{25}$$
$$\frac{\partial \boldsymbol{N}}{\partial t} = -(\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s})\boldsymbol{T}. \tag{26}$$

Proof. The Frenet Equation

$$\mathbf{T} = \frac{\partial \mathbf{F}}{\partial s},$$

means that

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \mathbf{F}}{\partial s} = (\frac{\partial}{\partial t}\frac{\partial}{\partial s})\mathbf{F}.$$

Using Lemma 3.4 this becomes

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial}{\partial s}\frac{\partial \mathbf{F}}{\partial t} + \gamma\kappa^2\frac{\partial \mathbf{F}}{\partial s}.$$

By our evolution rule (21)

$$\frac{\partial \mathbf{F}}{\partial t} = \gamma\kappa\mathbf{N},$$

and the Frenet Equation

$$\mathbf{T} = \frac{\partial \mathbf{F}}{\partial s},$$

we get

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \mathbf{F}}{\partial t} + \gamma\kappa^2\frac{\partial \mathbf{F}}{\partial s} &= \frac{\partial}{\partial s}(\gamma\kappa\mathbf{N}) + \gamma\kappa^2\mathbf{T} \\ &= (\frac{\partial}{\partial s}(\gamma\kappa))\mathbf{N} + \gamma\kappa\frac{\partial \mathbf{N}}{\partial s} + \gamma\kappa^2\mathbf{T}\end{aligned}$$

where the last line comes from applying the chain rule. By the Frenet Equation (12), the last two terms will cancel, and we are left with

$$(\frac{\partial}{\partial s}(\gamma\kappa))\mathbf{N} = (\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s})\mathbf{N}$$

which proves the first equality.

The second equation follows from

$$0 = \frac{\partial}{\partial t}\langle \mathbf{T}, \mathbf{N}\rangle = \langle \frac{\partial \mathbf{T}}{\partial t}, \mathbf{N}\rangle + \langle \mathbf{T}, \frac{\partial \mathbf{N}}{\partial t}\rangle$$
$$= (\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s}) + \langle \mathbf{T}, \frac{\partial \mathbf{N}}{\partial t}\rangle.$$

Since $\mathbf{N}$ has constant length 1, it must be that

$$\frac{\partial \mathbf{N}}{\partial t} \perp \mathbf{N}$$

or

$$\frac{\partial \mathbf{N}}{\partial t} = x\mathbf{T}$$

for some function x. Then

$$(\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s}) + \langle \mathbf{T}, \frac{\partial \mathbf{N}}{\partial t}\rangle$$
$$= (\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s}) + \langle \mathbf{T}, x\mathbf{T}\rangle$$
$$= (\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s}) + x = 0$$

which does imply Equation (26) as desired.

Let α denote the angle between the tangent vector to the curve C and the horizontal axis, taken in the positive sense. Then

$$\mathbf{T}(\alpha) = (\cos\alpha, \sin\alpha)$$

and

$$\mathbf{N}(\alpha) = (-\sin\alpha, \cos\alpha).$$

The next lemma concerns α as a function of the variables s and t.

Lemma 3.6.

$$\begin{aligned} \frac{\partial \alpha}{\partial s} &= \kappa & (27)\\ \frac{\partial \alpha}{\partial t} &= \frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s}. & (28)\end{aligned}$$

Proof. Well,

$$\frac{\partial \mathbf{T}}{\partial u} = \frac{\partial}{\partial u}(\cos\alpha, \sin\alpha) = (-\sin\alpha\frac{\partial \alpha}{\partial u}, \cos\alpha\frac{\partial \alpha}{\partial u}) = \frac{\partial \alpha}{\partial u}\mathbf{N}.$$

On the other hand from the Frenet Equations,

$$\frac{\partial \mathbf{T}}{\partial u} = v\frac{\partial \mathbf{T}}{\partial s} = v\kappa\mathbf{N}.$$

Then

$$\begin{aligned} \frac{\partial \alpha}{\partial u}\mathbf{N} &= v\kappa\mathbf{N}\\ \frac{\partial \alpha}{\partial s}\frac{\partial s}{\partial u}\mathbf{N} &= v\kappa\mathbf{N}\\ \frac{\partial \alpha}{\partial s}v\mathbf{N} &= v\kappa\mathbf{N}\\ \frac{\partial \alpha}{\partial s} &= \kappa.\end{aligned}$$

To prove the second equation, recall from the previous lemma that,

$$\frac{\partial \mathbf{T}}{\partial t} = (\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s})\mathbf{N}.$$

Also,

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial}{\partial t}(\cos\alpha, \sin\alpha) = (-\sin\alpha\frac{\partial \alpha}{\partial t}, \cos\alpha\frac{\partial \alpha}{\partial t}) = \frac{\partial \alpha}{\partial t}\mathbf{N}.$$

Comparing these calculations gives the desired equality.

The next and last technical lemma hardly deserves that title-the generalization to anisotropy loses the simplicity of the isotropic case. However in the spirit of tradition, since we are following Gage and Hamilton's strategy we will proceed.

Lemma 3.7.

$$\frac{\partial \kappa}{\partial t} = \frac{\partial^2 \gamma}{\partial s^2}\kappa + 2\frac{\partial \gamma}{\partial s}\frac{\partial \kappa}{\partial s} + \gamma\frac{\partial^2 \kappa}{\partial s^2} + \gamma\kappa^3$$

Proof. Using the previous result and Lemma 3.4,

$$\frac{\partial \kappa}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \alpha}{\partial s} = \frac{\partial}{\partial s}\frac{\partial \alpha}{\partial t} + \gamma\kappa^2\frac{\partial \alpha}{\partial s} =$$

$$\frac{\partial}{\partial s}\left(\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s}\right) + \gamma\kappa^3 =$$

$$\frac{\partial^2 \gamma}{\partial s^2}\kappa + 2\frac{\partial \gamma}{\partial s}\frac{\partial \kappa}{\partial s} + \gamma\frac{\partial^2 \kappa}{\partial s^2} + \gamma\kappa^3.$$

Now we are ready to prove the following generalization of Theorem 2.2:

Theorem 3.8. *The time derivative of the area enclosed by a curve C which is evolving by anisotropic mean curvature is*

$$\frac{\partial A}{\partial t} = -\int_C \gamma(\alpha)d\alpha. \tag{29}$$

Proof. Equation (19) for the area enclosed by a plane curve gives,

$$\frac{\partial A}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 \langle \mathbf{F}, v\mathbf{N}\rangle du =$$

$$-\frac{1}{2}\int_0^1 \left(\langle \frac{\partial \mathbf{F}}{\partial t}, v\mathbf{N}\rangle + \langle \mathbf{F}, \frac{\partial v}{\partial t}\mathbf{N}\rangle + \langle \mathbf{F}, v\frac{\partial \mathbf{N}}{\partial t}\rangle\right)du.$$

From our anisotropic evolution law, the first term is

$$-\frac{1}{2}\int_0^1 \langle \frac{\partial \mathbf{F}}{\partial t}, v\mathbf{N}\rangle = -\frac{1}{2}\int_0^1 \langle \gamma\kappa\mathbf{N}, v\mathbf{N}\rangle du. \tag{30}$$

From Lemma 1.3 the second term is

$$-\frac{1}{2}\int_0^1 \langle \mathbf{F}, \frac{\partial v}{\partial t}\mathbf{N}\rangle = -\frac{1}{2}\int_0^1 \langle \mathbf{F}, -\gamma\kappa^2 v\mathbf{N}\rangle du. \tag{31}$$

Using Lemma 3.5, the third term becomes

$$-\frac{1}{2}\int_0^1 \langle \mathbf{F}, v\frac{\partial \mathbf{N}}{\partial t}\rangle du \tag{32}$$

$$= -\frac{1}{2}\int_0^1 \langle \mathbf{F}, -v(\frac{\partial \gamma}{\partial s}\kappa + \gamma\frac{\partial \kappa}{\partial s})\mathbf{T}\rangle du \tag{33}$$

$$= \frac{1}{2}\int_0^1 (\frac{\partial}{\partial u}(\gamma\kappa))\langle \mathbf{F}, \mathbf{T}\rangle du. \tag{34}$$

This last expression we can integrate by parts to get

$$\frac{1}{2}[\gamma\kappa\langle \mathbf{F}, \mathbf{T}\rangle]_0^1 - \frac{1}{2}\int_0^1 (\gamma\kappa\langle \frac{\partial \mathbf{F}}{\partial u}, \mathbf{T}\rangle + \gamma\kappa\langle \mathbf{F}, \frac{\partial \mathbf{T}}{\partial u}\rangle)du. \tag{35}$$

But C is closed so the boundary term is zero and we are left with

$$-\frac{1}{2}\int_0^1 (\gamma\kappa\langle \frac{\partial \mathbf{F}}{\partial u}, \mathbf{T}\rangle + \gamma\kappa\langle \mathbf{F}, \frac{\partial \mathbf{T}}{\partial u}\rangle)du \tag{36}$$

If we simplify

$$\frac{\partial \mathbf{F}}{\partial u} \text{ and } \frac{\partial \mathbf{T}}{\partial u}$$

using The Frenet Equations, and collect the terms from (30), (31) and (36) then

$$\frac{\partial A}{\partial t} = -\frac{1}{2}\int_0^1 (\gamma\kappa v - \gamma\kappa^2 v\langle \mathbf{F}, \mathbf{N}\rangle + \gamma\kappa v\langle \mathbf{T}, \mathbf{T}\rangle + \gamma\kappa^2 v\langle \mathbf{F}, \mathbf{N}\rangle)du.$$

Two of the terms cancel, and $\langle \mathbf{T}, \mathbf{T}\rangle = 1$ so

$$\frac{\partial A}{\partial t} = -\frac{1}{2}\int_0^1 2\gamma\kappa v du$$

$$= -\int_0^L \gamma\kappa ds.$$

By Lemma 3.6

$$\kappa = \frac{\partial \alpha}{\partial s}$$

whence

$$\begin{aligned}\frac{\partial A}{\partial t} &= -\int_0^L \gamma \kappa ds \\ &= -\int_0^L \gamma \frac{\partial \alpha}{\partial s} ds \\ &= -\int_C \gamma(\alpha) d\alpha\end{aligned}$$

which proves the theorem.

If we let $\gamma \equiv 1$ then Theorem 2.2 follows as a corollary since the net rotation of the tangent vector around the curve is 2π.

4 Mullins' Law part 1

In 1956, W. Mullins, then a research engineer in Pittsburgh, published a paper titled *Two-Dimensional Motion of Idealized Grain Boundaries* [15] in which he analyzes the physical migration of grain boundaries in recrystallized metals. Mullins' set up is as follows:

> consider a network of arbitrary curves dividing the plane into polygon-like cells. Let each vertex be the terminus of three such curves meeting at equal angles of $\frac{2\pi}{3}$ radians. If these curves move according to (mean curvature)...the rate of area loss or gain of a given cell is determined solely by the number of its sides.

Our restatement of Mullins' finding is the following theorem.

Theorem 4.1 (Mullins' Law). *Let $\boldsymbol{F}(\cdot, t)$ be a piecewise smooth simple closed curve with n sides such that the interior angle at each vertex is $\frac{2\pi}{3}$ while each side is evolving according to mean curvature then the change in the area enclosed by the curve is given by*

$$\frac{\partial A}{\partial t} = \frac{\pi}{3}(n-6). \tag{37}$$

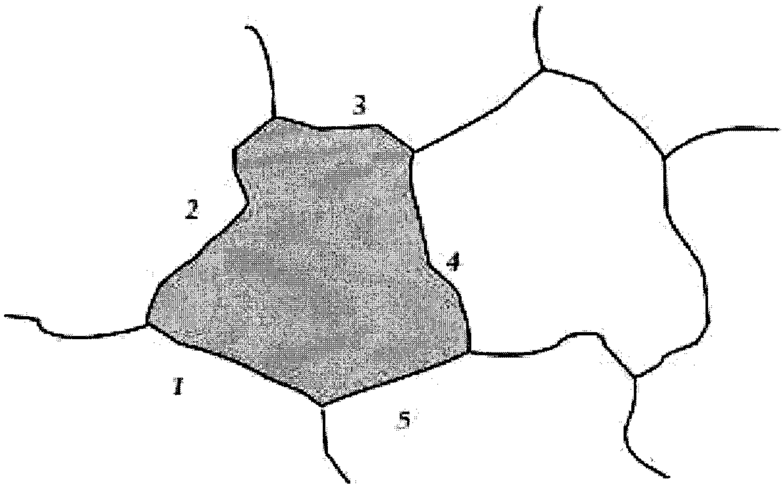


Figure 5: Mullins' original setup.

This is an elegant and suprising result. The only factor which determines the change in the area enclosed by a curve is the number of its sides. Specifically any curve with more than six sides will increase in area while a curve with six sides will evolve while maintaining a fixed area. This section will lay the groundwork for a more general theorem (Theorem 4.4) which we will analyze to determine what assumptions must be made in order to arrive at Mullins' specific formula. To start with, some new notation needs to be introduced.

Consider a closed, piecewise smooth, embedded plane curve given by the vector function $\mathbf{F}(s,t)$, and again take the standard orientation. We will only consider one curve, or grain, instead of a network of them as Mullins [15] and Bronsard et al. [5] did because, as we will show, the short term changes in length and area do not depend on neighboring cells. We agreed to consider

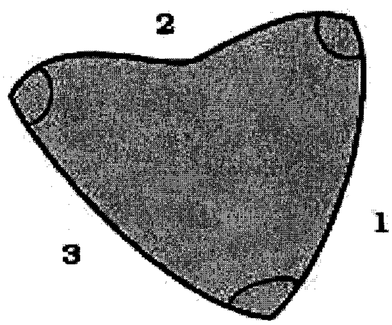


Figure 6: This curve has three sides so Mullins' Law dictates that it's area will decrease initially.

curves with a finite number of sides, and now we will also assume that each curve has at least two sides. Label each side $1, \ldots, n$ in a counterclockwise manner, and label the vertices by their arclength. That is, pick one vertex as a starting point and label it $s_0 = 0$, then label the vertices $s_0, s_1, \ldots, s_n$ in a counterclockwise manner so that $s_n = L$ the total length of the curve, and $\mathbf{F}(s_0, t) = \mathbf{F}(s_n, t)$. Now the ith side should begin at s_{i-1} and end at s_i.

We again need to overcome the dependence of the arclength parameter s on time, and again we will introduce a time independent parameter $u \epsilon [0, 1]$. Consider the vector functions $\mathbf{F}_i(u, t)$ for $i = 1, \ldots, n$ which parameterize each side of our curve independently. Now the image of $\mathbf{F}_i$ is the ith side of

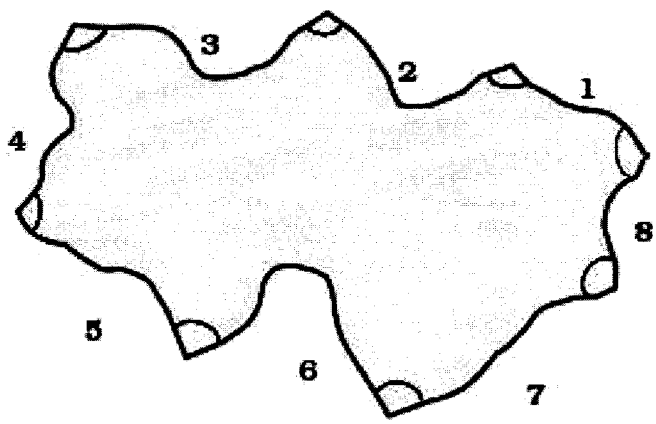


Figure 7: This curve has more than six sides so it's area will increase initially.

the curve with

$$\mathbf{F}_i(0,t) = \mathbf{F}(s_{i-1},t)$$

and

$$\mathbf{F}_i(1,t) = \mathbf{F}(s_i,t).$$

In general a subscript will indicate this situation. For example, $\mathbf{N}_i(u,t)$ is the inward pointing unit normal along the ith side. Be aware that in many cases, as with the unit tangent vectors $\mathbf{T}_i(u,t)$, the ith and (i+1)th functions will *not* agree at a vertex (see Figure 9), and some functions, for example the curvature functions $\kappa_i(u,t)$, are not even defined at vertices.

Now we are ready to define the evolution rules for piecewise smooth curves with specific angle conditions.

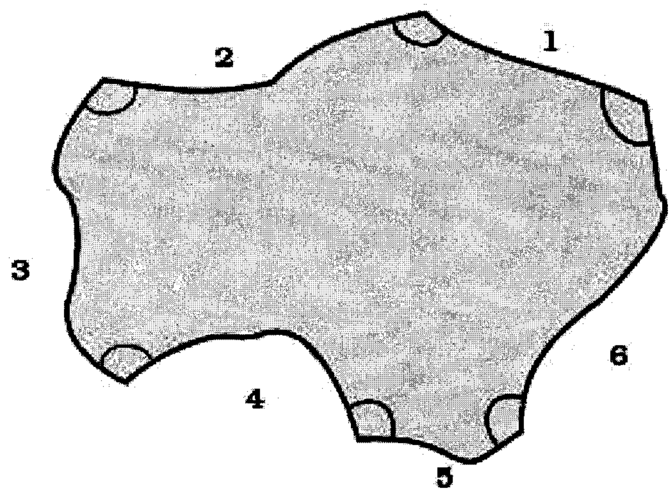


Figure 8: This curve has exactly six sides so Mullins' Law states that it's area will remain unchanged during it's inital evolution.

Definition 4.2. *The sides evolve by anisotropic mean curvature: for* $i = 1, \ldots, n$ *and all* $u \in (0,1)$,

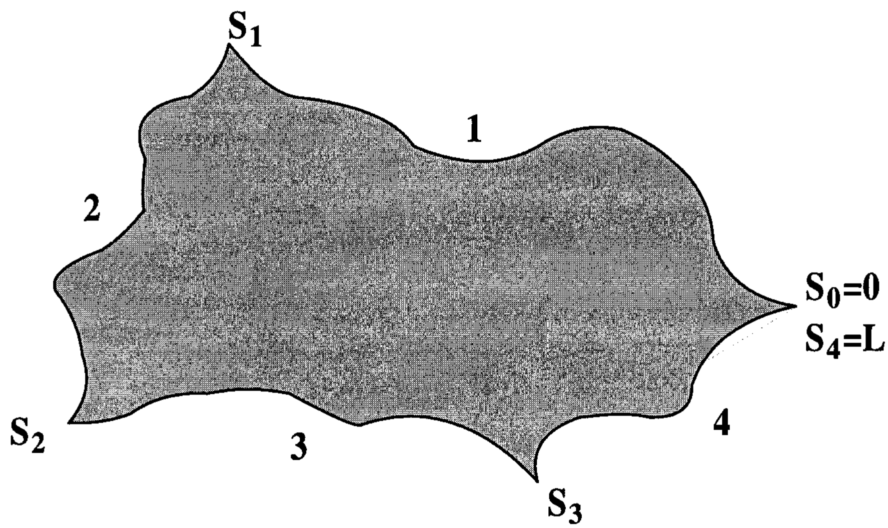
$$\langle \frac{\partial \boldsymbol{F}_i}{\partial t}, \boldsymbol{N}_i \rangle = \gamma \kappa_i \tag{38}$$

where again $\gamma(\alpha(u))$ *is a positive function, as in Definition 3.1, of the angle between the tangent vector and the horizontal axis.*
The sides do not separate:
for $i = 1, \ldots, n-1$,

$$\boldsymbol{F}_i(1,t) = \boldsymbol{F}_{i+1}(0,t) \tag{39}$$

and

$$\boldsymbol{F}_n(1,t) = \boldsymbol{F}_1(0,t). \tag{40}$$



Finally, the angle at each vertex is specified:
for $i = 1, \ldots, n-1$,

$$\langle \boldsymbol{T}_i(1,t), \boldsymbol{T}_{i+1}(0,t) \rangle = \cos(\pi - \beta_i) \tag{41}$$

and

$$\langle \boldsymbol{T}_n(1,t), \boldsymbol{T}_1(0,t) \rangle = \cos(\pi - \beta_n) \tag{42}$$

where

$$0 < \beta_i < \pi \textit{ for } i = 1, \ldots, n. \tag{43}$$

The reader may be worried when they recall Theorem 2.7 which says that a piecewise smooth initial curve will smooth out immediately, in which case it should obey the formula in Theorem 2.2 and shrink irrespective of it's original number of sides. This is not a trivial concern and it brings us

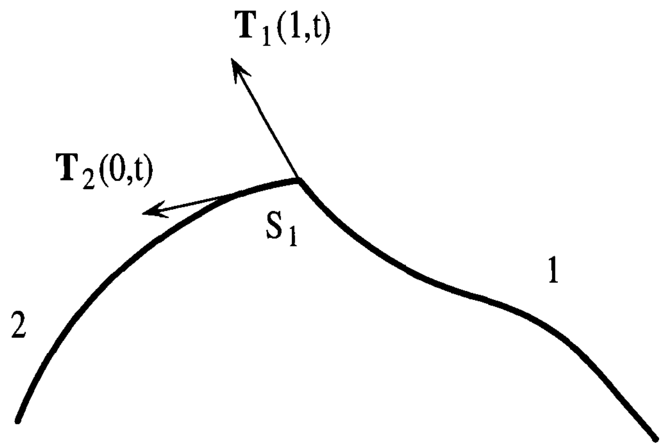
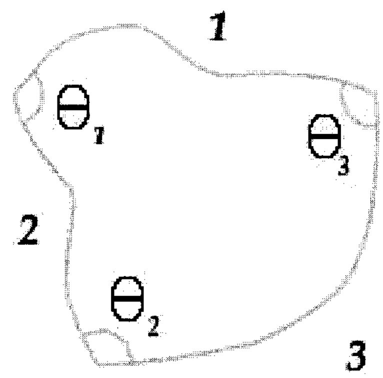


Figure 9: The two unit tangent vectors are *not* equal.

to a very important point. The kinks in Mullins' grains are not the same as DeTurck's imperfections because Mullins is *implicitly* requiring that his vertices *do not* smooth out but retain their original angles, and moreover he is requiring that the sides remain stuck together and do not separate at those points. If we were to write down the system of PDE's corresponding to this situation, it would be different from the systems analyzed by Gage, Hamilton and DeTurck, and consequently we need some new existence type results to carry on with a meaningful analysis. Mullins' recognized this situation and addressed it briefly in his paper:

> There is a possible question...of the consistency of the curvature rule of motion with the rule of equal intersection angles at the vertices. Since, however, the tendency of the curves to shorten themselves underlies both rules, their mutual consistency seems plausible.



Luckily for everyone, Bronsard and Reitich [5] proved the short-time existence of solutions to the system of PDE's defined by Definition 4.2. A word of caution is required here: the results in Section 2 are "long-time" solutions which describe the behavior of curves until they dissappear. However, Mullins' Law is a "short-time result" and any conclusions drawn from it should be viewed in that context.

Now we have some notation to work with, and confidence that the curves we are interested in do in fact exist.

Remember in the case of smooth curves, the time derivative of both the length and area could be calculated explicitly, and both quantities were negative. In the case of piecewise smooth curves with angle conditions the situation is more complicated - first we shall find that the time derivative of the length is no longer obviously negative.

The length of a piecewise smooth curve is

$$L = \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} ds = \sum_{i=1}^{n} \int_0^1 v_i du, \tag{44}$$

so

$$\frac{\partial L}{\partial t} = \sum_{i=1}^{n} \int_0^1 \frac{\partial v_i}{\partial t} du.$$

Lemma 1.3 applies to each v_i:

$$\frac{\partial v_i}{\partial t} = v_i \langle \mathbf{T}_i, \frac{\partial}{\partial u}(\frac{\partial \mathbf{F}_i}{\partial t}) \rangle$$

and so

$$\frac{\partial v_i}{\partial t} = v_i \langle \mathbf{T}_i, \frac{\partial}{\partial u}(a_i \mathbf{T}_i + \gamma \kappa_i \mathbf{N}_i) \rangle$$

for some C^1 real valued functions a_i. Now

$$\begin{aligned} \frac{\partial L}{\partial t} &= \sum_{i=1}^{n} \int_0^1 v_i \langle \mathbf{T}_i, \frac{\partial}{\partial u}(a_i \mathbf{T}_i + \gamma \kappa_i \mathbf{N}_i) \rangle du \\ &= \sum_{i=1}^{n} \int_0^1 \langle \mathbf{T}_i, \frac{\partial a_i}{\partial u} \mathbf{T}_i \rangle du \\ &+ \sum_{i=1}^{n} \int_0^1 \langle \mathbf{T}_i, a_i \frac{\partial \mathbf{T}_i}{\partial u} \rangle du \\ &+ \sum_{i=1}^{n} \int_0^1 \langle \mathbf{T}_i, \frac{\partial \gamma \kappa_i}{\partial u} \mathbf{N}_i \rangle du \\ &+ \sum_{i=1}^{n} \int_0^1 \langle \mathbf{T}_i, \gamma \kappa_i \frac{\partial \mathbf{N}_i}{\partial u} \rangle du. \end{aligned}$$

Since

$$\frac{\partial \mathbf{T}_i}{\partial u} = v_i \kappa_i \mathbf{N}_i$$

and $\mathbf{T}_i \perp \mathbf{N}_i$, the second and third terms are zero and we are left with

$$\frac{\partial L}{\partial t} = \sum_{i=1}^{n} \int_0^1 \langle \mathbf{T}_i, \frac{\partial a_i}{\partial u} \mathbf{T}_i \rangle du$$

$$+\sum_{i=1}^{n}\int_0^1 \langle \mathbf{T}_i, \gamma\kappa_i \frac{\partial \mathbf{N}_i}{\partial u}\rangle du.$$

Well,

$$\frac{\partial \mathbf{N}_i}{\partial u} = -v_i\kappa_i\mathbf{T}_i$$

whence

$$\frac{\partial L}{\partial t} = \sum_{i=1}^{n}\int_0^1 \frac{\partial a_i}{\partial u}du$$

$$+\sum_{i=1}^{n}\int_0^1 \langle \mathbf{T}_i, \gamma\kappa_i(-v_i\kappa_i\mathbf{T}_i)\rangle du$$

$$= \sum_{i=1}^{n}\int_0^1 da_i - \sum_{i=1}^{n}\int_0^1 \gamma\kappa_i^2 v_i du.$$

By simplifying we arrive at the following formula:

$$\frac{\partial L}{\partial t} = \sum_{i=1}^{n}[a_i(1) - a_i(0)] - \sum_{i=1}^{n}\int_{s_{i-1}}^{s_i} \gamma\kappa^2 ds. \tag{45}$$

From this formula we see that in the piecewise smooth case with $\gamma = 1$, the integral of the curvature squared is still present, but there is a new term which does not have an obvious sign.

Example 4.3. *Consider the symmetric football shape in Figure (10). If the functions a_i are π-periodic in the sense that*

$$a_1(0) = a_2(0) \textit{ and } a_1(1) = a_2(1),$$

then because of the symmetry of the shape the first term in Equation (45) equals zero. In fact we don't need the entire football to be symmetric, but only some small neighborhoods of the two vertices. However, without at least some symmetry or some explicit knowledge about the functions a_i the sign of $\frac{\partial L}{\partial t}$ is difficult to determine.

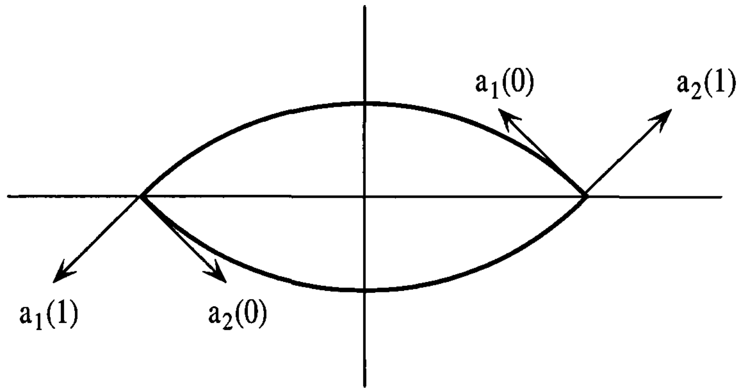


Figure 10: A symmetric football shape

The time derivative of the area is also less straightforward than in the smooth case, but inspired by Mullins work there is some interesting analysis to be done.

Theorem 4.4. *Recall that $\alpha_i(u,t)$ is the function whose output is the angle between the tangent vector $\boldsymbol{T}_i(u,t)$ and the horizontal axis along the ith side. We'll write $\alpha_i(0)$ and $\alpha_i(1)$ for this angle at the initial and final vertices of the ith side respectively. Then*

$$\frac{\partial A}{\partial t} = -\sum_{i=1}^{n} \int_{\alpha_i(0)}^{\alpha_i(1)} \gamma(\alpha) d\alpha. \tag{46}$$

We will include two proofs of this theorem. The first proof is reminiscent of Gage and Hamilton's approach to smooth curves, it develops Equation (46) as an intrinsic property of a curve.

Proof 1. From Equation (19) the time derivative of the area A enclosed by the piecewise smooth curve $\mathbf{F}(s,t)$ is

$$\frac{\partial A}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial t} \frac{1}{2} \int_{s_{i-1}}^{s_i} \langle \mathbf{F}, \mathbf{N} \rangle ds.$$

Rewriting everything in terms of the time independent parameter u allows the time derivative to come inside the integral,

$$\frac{\partial A}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial t} \frac{1}{2} \int_0^1 \langle \mathbf{F}_i, v_i \mathbf{N}_i \rangle du$$

$$= -\sum_{i=1}^{n} \frac{1}{2} \int_0^1 \frac{\partial}{\partial t} \langle \mathbf{F}_i, v_i \mathbf{N}_i \rangle du$$

$$= -\sum_{i=1}^{n} \frac{1}{2} \int_0^1 \langle \frac{\partial \mathbf{F}_i}{\partial t}, v_i \mathbf{N}_i \rangle du \tag{47}$$

$$-\frac{1}{2} \sum_{i=1}^{n} \int_0^1 \langle \mathbf{F}_i, \frac{\partial}{\partial t}(v_i \mathbf{N}_i) \rangle du. \tag{48}$$

From Equation (38), the first term becomes

$$-\sum_{i=1}^{n} \frac{1}{2} \int_0^1 \langle \frac{\partial \mathbf{F}_i}{\partial t}, v_i \mathbf{N}_i \rangle du = -\sum_{i=1}^{n} \frac{1}{2} \int_0^1 \langle \gamma \kappa_i \mathbf{N}_i + a_i \mathbf{T}_i, v_i \mathbf{N}_i \rangle du$$

but $\mathbf{T}_i \perp \mathbf{N}_i$ so we have

$$-\sum_{i=1}^{n} \frac{1}{2} \int_0^1 \gamma \kappa_i v_i du = -\sum_{i=1}^{n} \frac{1}{2} \int_{s_{i-1}}^{s_i} \gamma \kappa ds. \tag{49}$$

We could proceed here as in the proof of Theorem 3.8, but looking back at line (34) of that proof we see that we cannot integrate by parts in the same place because the functions κ_i are *not* continuous on the closed interval $[0, 1]$. Our approach will be to instead write things in terms of component functions. Let

$$\mathbf{F}_i(u, t) = (x_i(u, t), y_i(u, t)),$$

then

$$\mathbf{T}_i = \frac{1}{v_i} \frac{\partial \mathbf{F}_i}{\partial u} = \frac{1}{v_i} (\frac{\partial x_i}{\partial u}, \frac{\partial y_i}{\partial u})$$

and, for planar curves,

$$\mathbf{N}_i = \frac{1}{v_i}(-\frac{\partial y_i}{\partial u}, \frac{\partial x_i}{\partial u}).$$

So

$$\frac{\partial}{\partial t}(v_i \mathbf{N}_i) = \frac{\partial}{\partial t}(-\frac{\partial y_i}{\partial u}, \frac{\partial x_i}{\partial u}),$$

but we can interchange differentiation by t and u,

$$\frac{\partial}{\partial t}(v_i \mathbf{N}_i) = \frac{\partial}{\partial u}(-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}).$$

Now the second term (48) becomes

$$-\sum_{i=1}^{n} \frac{1}{2} \int_0^1 \langle (x_i, y_i), \frac{\partial}{\partial u}(-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}) \rangle du.$$

This *is* a continuous function on $[0, 1]$ because by Bronsard and Reitich's results [5] $\mathbf{F}_i$ is C^1 in t, and we have assumed that it is C^∞ in u. So integration by parts gives

$$-\sum_{i=1}^{n} \frac{1}{2} [\langle (x_i, y_i), (-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}) \rangle]_0^1$$

$$+\sum_{i=1}^{n} \frac{1}{2} \int_0^1 \langle (\frac{\partial x_i}{\partial u}, \frac{\partial y_i}{\partial u}), (-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}) \rangle du.$$

Notice that the integrand of the second term can be rearranged,

$$\langle (\frac{\partial x_i}{\partial u}, \frac{\partial y_i}{\partial u}), (-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}) \rangle = -\frac{\partial x_i}{\partial u}\frac{\partial y_i}{\partial t} + \frac{\partial y_i}{\partial u}\frac{\partial x_i}{\partial t}$$

$$= -(\frac{\partial x_i}{\partial t}(-\frac{\partial y_i}{\partial u}) + \frac{\partial y_i}{\partial t}\frac{\partial x_i}{\partial u}) = -\langle \frac{\partial \mathbf{F}_i}{\partial t}, v_i \mathbf{N}_i \rangle$$

and from the evolution equation (38) this is simply

$$-\langle \gamma \kappa_i \mathbf{N}_i, v_i \mathbf{N}_i \rangle = -\gamma \kappa_i v_i.$$

Let us regroup here-using this last calculation together with equation (49) we have

$$\frac{\partial A}{\partial t} = -\sum_{i=1}^{n} \frac{1}{2} \int_{s_{i-1}}^{s_i} \gamma\kappa ds$$

$$-\sum_{i=1}^{n} \frac{1}{2} [\langle (x_i, y_i), (-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}) \rangle du]_0^1$$

$$-\sum_{i=1}^{n} \frac{1}{2} \int_{s_{i-1}}^{s_i} \gamma\kappa_i v_i du$$

$$= -\sum_{i=1}^{n} \frac{1}{2} [\langle (x_i, y_i), (-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}) \rangle]_0^1$$

$$-\sum_{i=1}^{n} \frac{1}{2} \int_{s_{i-1}}^{s_i} \gamma\kappa ds - \sum_{i=1}^{n} \frac{1}{2} \int_{s_{i-1}}^{s_i} \gamma\kappa ds$$

$$= -\sum_{i=1}^{n} \frac{1}{2} [\langle (x_i, y_i), (-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}) \rangle]_0^1 - \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} \gamma\kappa ds. \tag{50}$$

The first term, the boundary term, we can also rearrange,

$$-\textstyle\sum_{i=1}^{n} \frac{1}{2} [\langle (x_i, y_i), (-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}) \rangle]_0^1$$

$$= \tfrac{1}{2} \textstyle\sum_{i=1}^{n} (\langle (x_i, y_i), (-\frac{\partial y_i}{\partial t}, \frac{\partial x_i}{\partial t}) \rangle \mid_{u=0}$$

$$-\langle (x_{i-1}, y_{i-1}), (-\tfrac{\partial y_{i-1}}{\partial t}, \tfrac{\partial x_{i-1}}{\partial t}) \rangle \mid_{u=1})$$

$$+\tfrac{1}{2} (\langle (x_n, y_n), (-\tfrac{\partial y_n}{\partial t}, \tfrac{\partial x_n}{\partial t}) \rangle \mid_{u=1}$$

$$-\langle (x_1, y_1), (-\tfrac{\partial y_1}{\partial t}, \tfrac{\partial x_1}{\partial t}) \rangle \mid_{u=0}).$$

By definition, for $i = 1, \ldots, n$,

$$\begin{aligned} (x_i, y_i)(0, t) &= \mathbf{F}_i(0, t) \\ = \mathbf{F}_{i-1}(1, t) &= (x_{i-1}, y_{i-1})(1, t) \end{aligned}$$

and

$$(x_n, y_n)(1, t) = (x_1, y_1)(0, t).$$

Similarly, from Equations (39) and (40), for $i = 1, \ldots, n$,

$$(-\tfrac{\partial y_i}{\partial t}, \tfrac{\partial x_i}{\partial t})(0,t) = (-\tfrac{\partial y_{i-1}}{\partial t}, \tfrac{\partial x_{i-1}}{\partial t})(1,t)$$

and

$$(-\tfrac{\partial y_n}{\partial t}, \tfrac{\partial x_n}{\partial t})(1,t) = (-\tfrac{\partial y_1}{\partial t}, \tfrac{\partial x_1}{\partial t})(0,t).$$

Since all these expressions are equal, the boundary term from Equation (50) is in fact zero, hence

$$\frac{\partial A}{\partial t} = -\sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} \gamma \kappa ds.$$

To finish things off, recall that one of the results proved in Lemma 3.6 was

$$\kappa = \frac{\partial \alpha}{\partial s}$$

whence

$$\begin{aligned} \tfrac{\partial A}{\partial t} &= -\textstyle\sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} \gamma(\alpha) \tfrac{\partial \alpha}{\partial s} ds \\ &= -\textstyle\sum_{i=1}^{n} \int_{\alpha_i(0)}^{\alpha_i(1)} \gamma(\alpha) d\alpha \end{aligned}$$

which was the desired result.

The second proof is Mullins' original approach using polar coordinates. However, we need that the parameter θ be time independent for this proof to work, and one consequence of this is that it imposes a particular tangential component to the flow of the curve and there is a question of consistency with any tangential component imposed by the angle conditions.

Proof 2. Pick a point p outside the curve as the origin. Starting at the point s_o which corresponds to some angle θ_0, traverse the curve in the counter-clockwise direction and split each curve $\mathbf{F}_i$ into subsections at the "inflection points" of θ, that is, the points where the angle θ changes between increasing and decreasing as the curve is traversed. Label each point where the curve is divided, including the vertices, by it's angle with respect to the origin p:

$$\theta_0, \ldots, \theta_m$$

where $n \leq m$, and some of the angles may be the same. Why should there be only finitely many such points? By supposition there are only finitely many vertices and because the curvature along each side is bounded the number of inflection points must be finite. Moreover, Grayson [12] proved that additional inflection points are not created during a curves evolution, which guarantees that $m < \infty$. Recall Equation (20), the area formula in polar coordinates. From that formula, the area inside the piecewise smooth curve is given by

$$A = \sum_{j=1}^{m} \frac{1}{2} \int_{\theta_{j-1}}^{\theta_j} r^2 d\theta.$$

We take θ to be time independent so that the differential operator can be taken inside the integral. When we differentiate, we will also have boundary terms,but they will cancel out thanks to the short time existence of the flow. We now have

$$\begin{aligned} \frac{\partial A}{\partial t} &= \sum_{j=1}^{m} \frac{1}{2} \int_{\theta_{j-1}}^{\theta_j} \frac{\partial}{\partial t}(r^2) d\theta \\ &= \sum_{j=1}^{m} \frac{1}{2} \int_{\theta_{j-1}}^{\theta_j} 2r \frac{\partial r}{\partial t} d\theta. \end{aligned} \tag{51}$$

Next we can compute $\frac{\partial r}{\partial t}$. In polar coordinates

$$\frac{\partial \mathbf{F}}{\partial t} = (\frac{\partial r}{\partial t} \cos\theta, \frac{\partial r}{\partial t} \sin\theta)$$

and

$$\mathbf{T} = \frac{\partial \mathbf{F}}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$= \frac{\partial \theta}{\partial s}(\frac{\partial r}{\partial \theta}\cos\theta - r\sin\theta, \frac{\partial r}{\partial \theta}\sin\theta + r\cos\theta)$$

so

$$\mathbf{N} = \frac{\partial \theta}{\partial s}(-\frac{\partial r}{\partial \theta}\sin\theta - r\cos\theta, \frac{\partial r}{\partial \theta}\cos\theta - r\sin\theta).$$

From these calculations,

$$\langle\frac{\partial \mathbf{F}}{\partial t}, \mathbf{N}\rangle = -r\frac{\partial r}{\partial t}\frac{\partial \theta}{\partial s}$$

but on the other hand, from our evolution equation (38),

$$\langle\frac{\partial \mathbf{F}}{\partial t}, \mathbf{N}\rangle = \gamma\kappa = \gamma\frac{\partial \alpha}{\partial s}.$$

Equating the right hand sides of these equations, and solving for $\frac{\partial r}{\partial t}$ gives

$$\frac{\partial r}{\partial t} = -\frac{1}{r}\gamma\frac{\partial \alpha}{\partial \theta}$$

whence Equation (51) becomes

$$\frac{\partial A}{\partial t} = -\sum_{j=1}^{m}\int_{\theta_{j-1}}^{\theta_j}\gamma\frac{\partial \alpha}{\partial \theta}d\theta.$$

We would like to cancel $d\theta$ and leave everything in terms of α, but what happens to the limits of integration? If θ_j is a smooth point then it has a well defined tangent vector and one specific value of α associated with it. The only points we will be left with are the original n vertices:

$$\frac{\partial A}{\partial t} = -\sum_{i=1}^{n}\int_{\alpha_i(0)}^{\alpha_i(1)}\gamma(\alpha)d\alpha. \tag{52}$$

This completes the second proof.

This formula is a nice generalization of Theorem 2.2 to the anisotropic evolution of piecewise smooth curves, but it is not particularly enlightening. The analysis we would like to carry out involves understanding why a piecewise smooth curve could potentially evolve so differently from a smooth curve, and more specifically, what assumptions did Mullins make to arrive at his law.

5 Mullins' Law part 2

First let us introduce some notation that will make our job easier. Define

$$\Gamma(x) := \int_0^x \gamma(\alpha)d\alpha. \tag{53}$$

Corollary 5.1.

$$\frac{\partial A}{\partial t} = -\Gamma(2\pi) + \sum_{i=1}^{n} \int_{\alpha_i(1)}^{\alpha_i(1)+\beta_i} \gamma(\alpha)d\alpha. \tag{54}$$

Proof. First, from Theorem 4.4 we have

$$\frac{\partial A}{\partial t} = -\sum_{i=1}^{n}(\Gamma(\alpha_i(1)) - \Gamma(\alpha_i(0)))$$

$$= -(\Gamma(\alpha_n(1)) - \Gamma(\alpha_1(0))) - \sum_{i=1}^{n-1}(\Gamma(\alpha_{i+1}(0)) - \Gamma(\alpha_i(1)))$$

$$= -\int_{\alpha_1(0)}^{\alpha_n(1)} \gamma(\alpha)d\alpha + \sum_{i=1}^{n-1} \int_{\alpha_i(1)}^{\alpha_{i+1}(0)} \gamma(\alpha)d\alpha.$$

Well,

$$-\int_{\alpha_1(0)}^{\alpha_n(1)} \gamma(\alpha)d\alpha$$

$$= -(\int_{\alpha_1(0)}^{0} \gamma(\alpha)d\alpha + \int_0^{2\pi} \gamma(\alpha)d\alpha + \int_{2\pi}^{\alpha_n(1)} \gamma(\alpha)d\alpha)$$

$$= -\Gamma(2\pi) + \int_0^{\alpha_1(0)} \gamma(\alpha)d\alpha + \int_{\alpha_n(1)}^{2\pi} \gamma(\alpha)d\alpha,$$

but γ is 2π-periodic so

$$- \int_{\alpha_1(0)}^{\alpha_n(1)} \gamma(\alpha)d\alpha$$

$$= -\Gamma(2\pi) + \int_0^{\alpha_1(0)} \gamma(\alpha)d\alpha + \int_{\alpha_n(1)-2\pi}^{2\pi-2\pi} \gamma(\alpha)d\alpha$$

$$= -\Gamma(2\pi) + \int_{\alpha_n(1)-2\pi}^{\alpha_1(0)} \gamma(\alpha)d\alpha$$

$$= -\Gamma(2\pi) + \int_{\alpha_n(1)-2\pi+2\pi}^{\alpha_1(0)+2\pi} \gamma(\alpha)d\alpha.$$

Now we have

$$\frac{\partial A}{\partial t} = -\Gamma(2\pi) + \sum_{i=1}^{n-1} \int_{\alpha_i(1)}^{\alpha_{i+1}(0)} \gamma(\alpha)d\alpha + \int_{\alpha_n(1)}^{\alpha_1(0)+2\pi} \gamma(\alpha)d\alpha. \tag{55}$$

The angle conditions (41) and (42) in Definition 4.2 imply that for $i = 1, \ldots, n-1$

$$\alpha_{i+1}(0) = \alpha_i(1) + \beta_i,$$

and

$$\alpha_n(1) = \alpha_1(0) + 2\pi - \beta_n.$$

These two identities allow us to rewrite (55),

$$\frac{\partial A}{\partial t} = -\Gamma(2\pi) + \sum_{i=1}^{n-1} \int_{\alpha_i(1)}^{\alpha_i(1)+\beta_i} \gamma(\alpha)d\alpha + \int_{\alpha_n(1)}^{\alpha_n(1)+\beta_n} \gamma(\alpha)$$

$$= -\Gamma(2\pi) + \sum_{i=1}^{n} \int_{\alpha_i(1)}^{\alpha_i(1)+\beta_i} \gamma(\alpha)d\alpha$$

which proves the claim.

This formula is very nice for two reasons. Fist, it demonstrates independence of the choice of the starting vertex, and second it shows that the formula has two competing terms. The negative term is $-\Gamma(2\pi)$ and the positive term is the sum of integrals

$$\int_{\alpha_i(1)}^{\alpha_i(1)+\beta_i} \gamma(\alpha)d\alpha.$$

This seems to imply that the vertices, and specifically the jump that the tangent vector encounters at each vertex, play the most important role in determining how the area will change.

The natural periodicity of γ allows us to reduce the required integration to the interval $[0, 3\pi]$.

Lemma 5.2.

$$\frac{\partial A}{\partial t} = -\Gamma(2\pi) + \sum_{i=1}^{n} \int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i} \gamma(\alpha)d\alpha, \tag{56}$$

where $\bar{\alpha}_i(1) = \alpha_i(1)$ *mod* $2\pi\boldsymbol{Z}$*.*

Proof. It suffices to show that for $i = 1, \ldots, n$,

$$\int_{\alpha_i(1)}^{\alpha_i(1)+\beta_i} \gamma(\alpha)d\alpha = \int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i} \gamma(\alpha)d\alpha.$$

Well, $\alpha_i(1) \in [2n\pi, 2(n+1)\pi)$ for some $n \in \mathbf{Z}$, and either

$$\text{Case 1: } (\alpha_i(1) + \beta_i) \in [2n\pi, 2(n+1)\pi). \tag{57}$$

or

$$\text{Case 2: } (\alpha_i(1) + \beta_i) \in [2(n+1)\pi, 2(n+2)\pi). \tag{58}$$

In the first case

$$\begin{aligned} \alpha_i(1) &= 2n\pi + \bar{\alpha}_i(1) \\ &\text{and} \\ \alpha_i(1) + \beta_i &= 2n\pi + \overline{\alpha_i(1) + \beta_i}, \end{aligned}$$

so that

$$\overline{\alpha_i(1)+\beta_i} = \bar{\alpha}_i(1) + \beta_i. \tag{59}$$

In Case 2 then again

$$\alpha_i(1) \;=\; 2n\pi + \bar{\alpha}_i(1)$$

but this time

$$\alpha_i(1) + \beta_i \;=\; 2n\pi + 2\pi + \overline{\alpha_i(1)+\beta_i},$$

so that

$$\overline{\alpha_i(1)+\beta_i} = \bar{\alpha}_i(1) + \beta_i - 2\pi. \tag{60}$$

Now, γ is a 2π-periodic function so we can freely translate integrals of γ over intervals of 2π. Hence, in Case 1,

$$\begin{aligned}\int_{\alpha_i(1)}^{\alpha_i(1)+\beta_i} \gamma(\alpha)d\alpha &= \int_{\alpha_i(1)-2n\pi}^{\alpha_i(1)+\beta_i-2n\pi} \gamma(\alpha)d\alpha \\ = \int_{\bar{\alpha}_i(1)}^{\overline{\alpha_i(1)+\beta_i}} \gamma(\alpha)d\alpha &= \int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i} \gamma(\alpha)d\alpha.\end{aligned}$$

In Case 2 we have,

$$\begin{aligned}\int_{\alpha_i(1)}^{\alpha_i(1)+\beta_i} \gamma(\alpha)d\alpha &= \int_{\alpha_i(1)-2n\pi}^{\alpha_i(1)+\beta_i-2n\pi} \gamma(\alpha)d\alpha \\ = \int_{\bar{\alpha}_i(1)}^{\overline{\alpha_i+\beta_i}+2\pi} \gamma(\alpha)d\alpha &= \int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i-2\pi+2\pi} \gamma(\alpha)d\alpha.\end{aligned}$$

So in both cases

$$\int_{\alpha_i(1)}^{\alpha_i(1)+\beta_i} \gamma(\alpha)d\alpha = \int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i} \gamma(\alpha)d\alpha \tag{61}$$

which is exactly what needed to be shown.

This leads us to a first result about the change in area.

Proposition 5.3. *If*

$$\frac{\Gamma(2\pi)}{\min_{i=1,\dots,n}\{\Gamma(\bar{\alpha}_i(1)+\beta_i)-\Gamma(\bar{\alpha}_i(1))\}} \le n \tag{62}$$

$$\textit{then } \frac{\partial A}{\partial t} \ge 0, \ \textit{and if}$$

$$\frac{\Gamma(2\pi)}{\max_{i=1,\dots,n}\{\Gamma(\bar{\alpha}_i(1)+\beta_i)-\Gamma(\bar{\alpha}_i(1))\}} > n \tag{63}$$

$$\textit{then } \frac{\partial A}{\partial t} < 0.$$

Proof. Well certainly

$$n(\min_{i=1,\dots,n}\{\Gamma(\bar{\alpha}_i(1)+\beta_i)-\Gamma(\bar{\alpha}_i(1))\}) \le \sum_{i=1}^{n}\int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i}\gamma(\alpha)d\alpha$$

so by the hypothesis (62),

$$\Gamma(2\pi) \le n(\min_{i=1,\dots,n}\{\Gamma(\bar{\alpha}_i(1)+\beta_i)-\Gamma(\bar{\alpha}_i(1))\}) \le \sum_{i=1}^{n}\int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i}\gamma(\alpha)d\alpha.$$

Then

$$0 \le \sum_{i=1}^{n}\int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i}\gamma(\alpha)d\alpha - \Gamma(2\pi)$$

but by the previous result, Lemma 5.2, the right hand side of the inequality is exactly $\frac{\partial A}{\partial t}$.

Similarly

$$\sum_{i=1}^{n}\int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i}\gamma(\alpha)d\alpha \le n(\max_{i=1,\dots,n}\{\Gamma(\bar{\alpha}_i(1)+\beta_i)-\Gamma(\bar{\alpha}_i(1))\})$$

together with (63) implies that

$$\frac{\partial A}{\partial t} = -\Gamma(2\pi) + \sum_{i=1}^{n}\int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i}\gamma(\alpha)d\alpha < 0$$

and this proves the proposition.

Remark *Note that division by* $\Gamma(\bar{\alpha}_i(1)+\beta_i)-\Gamma(\bar{\alpha}_i(1)$, *for any* $i = 1, \ldots, n$ *is allowed because it was specified that* $\beta_i > 0$, *which implies that each integral is strictly greater than zero.*

This is a first step, but to get an idea about the change in area enclosed by a curve we need to calculate n integrals on the interval $[0, 3\pi]$ which is considerably more work than simply counting to n. So how did Mullins arrive at a formula which depended only on the number of sides? By making two important assumptions we will finally obtain a result which is strikingly similar to Mullins' Law. We can count the number of sides of our curve and know whether the area enclosed by C will grow, shrink, or remain constant, without computing the actual value of $\Gamma(\beta)$ (we know that that it is positive) provided that all of the interior angles at the vertices are equal and γ is periodic with respect to that angle.

Theorem 5.4. *Suppose that each interior angle is the same, that is*

$$\beta_i = \beta \textit{ for } i = 1, \ldots, n$$

for some fixed $0 < \beta < \pi$. *If* γ *is* β*-periodic then*

$$\frac{\partial A}{\partial t} = (n-k)\Gamma(\beta) \tag{64}$$

where $k\beta = 2\pi$.

Proof. First, γ is already 2π-periodic, so if it is also β-periodic then there exists some integer k (positive since $\beta < \pi < 2\pi$) such that $k\beta = 2\pi$. Then

$$-\Gamma(2\pi) = -\Gamma(k\beta) = -k\Gamma(\beta)$$

and

$$\sum_{i=1}^{n} \int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta_i} \gamma(\alpha) d\alpha = \sum_{i=1}^{n} \int_{\bar{\alpha}_i(1)}^{\bar{\alpha}_i(1)+\beta} \gamma(\alpha) d\alpha$$

because all the angles are equal. Because of the periodicity of γ this becomes

$$\begin{aligned} \sum_{i=1}^{n} \int_0^{\beta} \gamma(\alpha) d\alpha &= n \int_0^{\beta} \gamma(\alpha) d\alpha \\ &= n\Gamma(\beta). \end{aligned}$$

Adding the two pieces back together gives

$$\frac{\partial A}{\partial t} = (n-k)\Gamma(\beta)$$

which proves the theorem.

Mullins' specific result follows as a corollary to this theorem.

Corollary 5.5. *Let $\boldsymbol{F}(\cdot, t)$ describe a piecewise smooth curve with n sides whose interior angle at each vertex is $\frac{2\pi}{3}$ and which is evolving according to Definition 4.2 with $\gamma \equiv 1$ then*

$$\frac{\partial A}{\partial t} = \frac{\pi}{3}(n-6).$$

Proof. In this case, the angle condition of $\frac{2\pi}{3} = \pi - \beta$ corresponds to $\beta = \frac{\pi}{3}$. So $6\frac{\pi}{3} = 2\pi$, or $k = 6$. If $\gamma \equiv 1$ then

$$\Gamma(\beta) = \int_0^{\beta} \gamma(\alpha) d\alpha = \int_0^{\frac{\pi}{3}} d\alpha = \frac{\pi}{3}.$$

So by Theorem 5.4

$$\frac{\partial A}{\partial t} = (n-k)\Gamma(\beta) = (n-6)\frac{\pi}{3},$$

which is Mullins' Law.

Mullins specific angle condition is very reasonable given the geometric situation he is considering. In a system of simultaneously evolving grains, where each vertex is the terminus of three curves it makes sense to require that all three interior angles are equal. The three angles of course must add up to 2π hence the angle condition of $\frac{2\pi}{3}$ and the corresponding value of $\beta = \frac{\pi}{3}$. Also, Mullins chose the simplest $\frac{\pi}{3}$-periodic function possible, the constant function $\gamma \equiv 1$. So in fact, although Mullins' Law extends very nicely to the anisotropic case via Equation (64), Mullins considered only the isotropic flow in his original work.

6 Related questions

Thus far, this paper has concerned itself with presenting results about the evolution of smooth plane curves by mean curvature flow, and then exploring and generalizing some results to piecewise smooth plane curves, but there are many other generalizations, and related topics of interest. Much of the inspiration for studying mean curvature flow came from physical phenomena, such as Mullins' original interest in the migration of "grain boundaries of a recrystallized metal." Another closely related physical problem is the study of crystal formation and evolution. However, the classical mean curvature flow is somewhat unsatisfying in this context because most crystals are polygons which we have noted do not change under Equation (1). In response to this, J. Taylor proposed in [17] a "theory for moving polygonal curves...by their crystalline curvature," which is "analogous to ordinary curvature." Basically, the side of a polygon evolving by crystalline curvature will move in the direction of it's inward pointing normal vector, at a speed inversely proportional to it's length. A convex polygon will shrink, and as it shrinks, the speed of it's collapse will increase. Further work on this topic can be found in [16], [17], [18]. Another natural generalization (mathematically speaking) of smooth curves evolving by mean curvature is the evolution of smooth, closed surfaces, or hypersurfaces. In fact the word "mean" in mean curvature flow takes on geometric meaning only in dimensions higher than 1. A surface is evolving by mean curvature flow if each point moves in the direction of the inward pointing surface normal with speed proportional to the *mean* curvature at that point, the average curvature at that point of all the curves on the surface passing through it. This particular generalization seems to be the most popular because it is related to R. Hamilton's promising strategy for tackling the Poincare Conjecture. A good reference is G. Huisken's paper [13] in which he proves that convex surfaces evolve by mean curvature into spheres, which then shrink to round points.

The third related question we will spend a bit more time on, it is the isotropic evolution of curves on surfaces. In this context the flow of a smooth curve on a surface that we will consider is given by

$$\langle \frac{\partial \mathbf{F}}{\partial t}, \mathbf{N} \rangle = \kappa_g \tag{65}$$

where, as before, $\mathbf{F}(\cdot, t)$ is the vector function describing the curve, $\mathbf{N}$ is the normal to the curve (*not* the surface normal), and κ_g is the geodesic curvature. M. Grayson includes a epilogue in his paper [12] titled *Curves on surfaces* from which we will include a part:

> The fact that embedded curves in the plane evolve nicely is a strong argument for the niceness of curves evolving on a Riemannian surface. [A]bresch and Langer [1] show(ed) that curvature bounded for all time implies convergence to a closed geodesic. If our techniques were generalized slightly, they should be able to show that either the curve would become convex and shrink to a point, or its curvature would remain bounded for all time.

Our principal reference for this subject is M. Gage's paper [8] (for those interested, Gage examines the same subject with a slightly different flow in a second paper [9]) The major results of this paper are in the direction of Grayson's conjecture. First, Gage proves an existence theorem: if the surface has bounded Gauss curvature, then a simple closed curve evolves nicely so long as it's curvature remains bounded. The other major result is that if the curvature remains bounded forever, so that the curve can evolve indefinitely, then the geodesic curvature decreases uniformly to zero. That means that, under these certain conditions, curves on surfaces evolve either to geodesics, or to points, just as Grayson predicted. Gage's last section is an application to the unit sphere. He shows that closed curves which divide the unit sphere into two regions of equal area, and whose total space curvature is less than 3π, evolve into a great circle. He notes that the result is "significant because *a priori* one must allow the possibility that the curve converges to a slowly rotating great circle." He proves this by showing that "the entire one parameter family (of curves), not just a subsequence, converges to a single geodesic." Gage's more general results, outlined above, further imply that a curve confined inside one hemisphere of the unit circle will collapse to a point. This means that both the length of, and area inside such a curve will decrease. As we have concerned ourselves with area thus far, we will present here some analysis of precisely how the area in such a situation will decrease. Consider a smooth closed curve $\mathbf{F}(\cdot, t)$ on the unit sphere which can be parameterized in the spherical coordinates $(1, \theta, \phi)$. Then we can express the curve as the graph of a function $\phi(\theta, t)$ so that the area enclosed is given by

$$A = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\phi(\theta,t)} \sin \phi d\phi d\theta$$

which can be integrated so that,

$$A = \int_{\theta=0}^{2\pi} (1 - \cos\phi(\theta, t))d\theta.$$

If θ is time independent then

$$\begin{aligned}\frac{\partial A}{\partial t} &= \frac{\partial}{\partial t}\int_{\theta=0}^{2\pi} (1 - \cos\phi(\theta, t))d\theta \\ &= \int_{\theta=0}^{2\pi} \frac{\partial \phi}{\partial t}\sin\phi d\theta. \end{aligned} \tag{66}$$

We can calculate $\frac{\partial \phi}{\partial t}$ more explicitly from the equation

$$\langle \frac{\partial \mathbf{F}}{\partial t}, \mathbf{N} \rangle = \kappa_g \tag{67}$$

again, where $\mathbf{N}$ is the "inward pointing" unit normal to the curve, that is, the projection of the vector onto the xy-plane is inward pointing with respect to the projection of the curve. Any point (x, y, z) on the curve can be expressed as

$$(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

so that

$$\frac{\partial \mathbf{F}}{\partial t} = (\frac{\partial \phi}{\partial t}\cos\phi\cos\theta, \frac{\partial \phi}{\partial t}\cos\phi\sin\theta, -\frac{\partial \phi}{\partial t}\sin\phi).$$

On the other hand

$$\mathbf{T} = \frac{\partial \mathbf{F}}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$\begin{aligned} &= \tfrac{\partial \theta}{\partial s}(\tfrac{\partial \phi}{\partial \theta}\cos\phi\cos\theta - \sin\phi\sin\theta, \\ &\quad \tfrac{\partial \phi}{\partial \theta}\cos\phi\sin\theta + \sin\phi\cos\theta, \\ &\quad -\tfrac{\partial \phi}{\partial \theta}\sin\phi), \end{aligned}$$

and the normal to the curve is

$$\mathbf{N} = \mathbf{n} \times \mathbf{T}$$

where $\mathbf{n}$ is the unit *surface* normal. For the unit sphere, the unit surface normal is the position vector $\mathbf{F}$, and we have just calculated the unit tangent vector $\mathbf{T}$, so

$$\begin{aligned}\mathbf{N} = \frac{\partial \theta}{\partial s}(&-\frac{\partial \phi}{\partial \theta}\sin\theta - \sin\phi\cos\phi\cos\theta,\\ &\frac{\partial \phi}{\partial \theta}\cos\theta - \sin\phi\cos\phi\sin\theta,\\ &\frac{\partial \phi}{\partial \theta}\sin^2\phi).\end{aligned}$$

Now

$$\begin{aligned}&\langle\frac{\partial \mathbf{F}}{\partial t}, \mathbf{N}\rangle\\ &= \frac{\partial \theta}{\partial s}(-\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial \theta}\sin\theta\cos\theta\cos\phi - \frac{\partial \phi}{\partial t}\cos^2\phi\cos^2\theta\sin\phi\\ &\quad + -\frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial \theta}\cos\theta\cos\phi\sin\theta - \frac{\partial \phi}{\partial t}\cos^2\phi\sin^2\theta\sin\phi\\ &\quad -\frac{\partial \phi}{\partial t}\sin^3\phi)\\ &= \frac{\partial \theta}{\partial s}(-\frac{\partial \phi}{\partial t}(\cos^2\phi\sin\phi + \sin^3\phi))\\ &= -\frac{\partial \theta}{\partial s}\frac{\partial \phi}{\partial t}\sin\phi.\end{aligned}$$

From Equation (67) we now have

$$-\frac{\partial \theta}{\partial s}\frac{\partial \phi}{\partial t}\sin\phi = \kappa_g,$$

or

$$\frac{\partial \phi}{\partial t}\sin\phi = -\kappa_g\frac{\partial s}{\partial \theta}.$$

So (66) becomes

$$\frac{\partial A}{\partial t} = \int_{\theta=0}^{2\pi} \frac{\partial \phi}{\partial t} \sin \phi d\theta$$

$$= -\int_{\theta=0}^{2\pi} \kappa_g \frac{\partial s}{\partial \theta} d\theta$$

whence

$$\frac{\partial A}{\partial t} = -\int_0^L \kappa_g ds. \tag{68}$$

By the Gauss-Bonnett Formula

$$\frac{\partial A}{\partial t} = -\int_0^L \kappa_g ds$$

$$= \int\int_{A(t)} K dA - 2\pi.$$

Since $K \equiv 1$ on the unit sphere

$$\int\int_{A(t)} K dA = A(t)$$

and the result is the ODE

$$\frac{\partial A}{\partial t} = A(t) - 2\pi \tag{69}$$

whose solutions are

$$A(t) = 2\pi + A_0 e^t. \tag{70}$$

Notice that the initial area inside the curve is thus given by $A(0) = 2\pi - A_0$ and that

$$\frac{\partial A}{\partial t} = A_0 e^t.$$

This complements Gage's results nicely. The "interior" of a closed curve on the unit sphere is somewhat ambiguous, nevertheless there are only two cases to consider: either the curve divides the sphere into two portions of equal area, or unequal areas. In the latter case, the smaller portion will have an area less than 2π which means it will shrink in correspondence with Gage's proof that the curve will shrink to a round point. In the case that the initial curve separates the unit sphere into two regions, each of area 2π Gage's theorem tells us that the curve will evolve to a great circle. Formula (70) adds to this result. Notice that if the initial area inside the curve is 2π, then since

$$A(0) = 2\pi - A_0$$

we know that $A_0 = 0$. Then $\frac{\partial A}{\partial t} = 0$, which proves that during the evolution of the initial curve into a great circle the two divided portions of the sphere have constant area.

Though the literature does not seem to contain any existence theorems for piecewise smooth curves on the unit sphere we can, under the assumption that short time solutions do exist, undertake a brief investigation of the behavior of such curves under the flow given by Equation (65). Equation (66) holds for piecewise smooth curves with a finite number of vertices since the integrand would exist almost everywhere along the curve. In addition, the calculations leading to Equation (68) hold along each smooth section of curve, and so we arrive at

$$\frac{\partial A}{\partial t} = -\int_C \kappa_g ds. \tag{71}$$

Suppose the curve C has n vertices. We choose one portion of the sphere enclosed by C to be the "interior," then the interior angles are given by β_i $i = 1, \ldots, n$. The Gauss-Bonnet formula again leads us to an ODE:

$$\frac{\partial A}{\partial t} = A(t) - (2\pi - \sum_{i=1}^{n} \beta_i) \tag{72}$$

which has solutions

$$A(t) = 2\pi - \sum_{i=1}^{n} \beta_i + A_0 e^t. \tag{73}$$

Example 6.1. *Suppose that C had an initial interior area $A(0) = \pi(2-\sqrt{2})$, and four vertices, each interior angle measuring $\frac{2\pi}{3}$. Then*

$$\sum_{i=1}^{n} \beta_i = \frac{8\pi}{3}$$

and we can solve for A_0:

$$A(0) = \pi(2-\sqrt{2}) = 2\pi - \frac{8\pi}{3} + A_0$$

so $A_0 = \pi(\frac{8-3\sqrt{2}}{3})$. This means that

$$\frac{\partial A}{\partial t} = \pi(\frac{8-3\sqrt{2}}{3})e^t$$

which is greater than zero, despite the fact that the initial interior area was less than 2π.

In the smooth case, the time derivative of the area depended only on the initial interior area, but as the above example illustrates, in the piecewise smooth case, the number of vertices, and the interior angles play an important role. The presence of enough vertices will cause even a very short curve to behave in a manner opposite what we would expect from the results about smooth curves.

Proposition 6.2. *Suppose C is a piecewise smooth curve on the unit sphere with n vertices. Choose one portion of the sphere to be the interior of C so that the interior angles measure β_i for each $i = 1, \ldots, n$. Also suppose that C evolves according to Equation (65) in such a way that the interior angles at* *each vertex do not change. Then the interior area of C will increase provided that the initial area plus the sum of the interior angles is greater than 2π. That is, if*

$$A(0) + \sum_{i=1}^{n} \beta_i \quad > \quad 2\pi$$

then

$$\frac{\partial A}{\partial t} \quad > \quad 0.$$

Proof. First note that, exactly like the smooth case,

$$\frac{\partial A}{\partial t} = A_0 e^t$$

so the sign of $\frac{\partial A}{\partial t}$ depends only on A_0. Then the result follows from Equation (73) because

$$A(0) = 2\pi - \sum_{i=1}^{n} \beta_i + A_0$$

means that

$$A_0 = (A(0) + \sum_{i=1}^{n} \beta_i) - 2\pi.$$

So the inequality

$$A(0) + \sum_{i=1}^{n} \beta_i > 2\pi$$

implies that the sign of A_0 and thus the sign of $\frac{\partial A}{\partial t}$ is positive.

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