BERGE METRICS
FOR BINARY STRINGS

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By

SANDRA GREGOV, B.A.

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AUTHOR: Sandra Gregov
B.A. with Honours (York University)

SUPERVISOR: Dr. Antoine Deza

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Abstract

We consider a number of generalizations of the following question originally posed by Claude Berge in 1966. Let $\mathbf{S_n}$ denote the set of all strings made of $\lceil \frac{n}{2} \rceil$ white coins and $\lfloor \frac{n}{2} \rfloor$ black coins. Berge asked what is the minimum number of moves required to sort an alternating string of $\mathbf{S_n}$ by taking 2 adjacent coins to 2 adjacent vacant positions on a one-dimensional board of infinite length such that the sorted string has all white coins immediately followed by all black coins (or visa versa).

We survey and present results dealing with the first generalization of Berge sorting which allows Berge *k-moves*, i.e., taking k adjacent coins to k adjacent vacant positions. We then explore a further generalization which asks for any pair of strings in $\mathbf{S_n}$ what is the minimum number of Berge *k-moves* needed to transform one string into the other. This induces a natural metric on the set $\mathbf{S_n}$ called the Berge *k-metric*. We examine bounds for the diameter of $\mathbf{S_n}$ allowing Berge *k-moves*. In particular, we present lower and upper bounds for Berge *1-metric* and explore some aspects of Berge *2-metric* along with computational results.

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Notations

$\mathbf{S_n}$: the set of all strings made of $\lceil \frac{n}{2} \rceil$ white coins and $\lfloor \frac{n}{2} \rfloor$ black coins.

S_n: a generic string in $\mathbf{S_n}$.

$T_{n,\circ}$: the string made of $\lceil \frac{n}{2} \rceil$ white coins followed by $\lfloor \frac{n}{2} \rfloor$ black coins.

$T_{n,\bullet}$: the sorted string made of $\lfloor \frac{n}{2} \rfloor$ black coins followed by $\lceil \frac{n}{2} \rceil$ white coins.

T_n: either $T_{n,\circ}$ or $T_{n,\bullet}$.

$A_{n,\circ}$: the string made of alternating $\lceil \frac{n}{2} \rceil$ white coins and $\lfloor \frac{n}{2} \rfloor$ black coins beginning with a white coin.

$A_{n,\bullet}$: the string made of alternating $\lfloor \frac{n}{2} \rfloor$ black coins followed by $\lceil \frac{n}{2} \rceil$ white coins beginning with a black coin.

A_n: either $A_{n,\circ}$ or $A_{n,\bullet}$.

$S^1 \rightarrow^k S^2$: transform S^1 into S^2 using *Berge k-moves*.

$h_{n,k}(S^1, S^2)$: minimum number of Berge *k-moves* needed to transform S^1 into S^2.

A solution in $h_{n,k}(S^1, S^2)$ move(s) is called *optimal.*

$B_{n,k}(S^1, S^2)$: a solution, i.e., an ordered set of *Berge k-moves* to transform S^1 into S^2.

$H_{n,k}(\mathbf{S_n}) = \max_{(S^1,S^2)\in\mathbf{S_n}} h_{n,k}(S^1, S^2)$: *diameter* of the set $\mathbf{S_n}$.

D_{S_n}: disorder of S_n, i.e. the number of coins with a right neighbour of a different colour or empty.

$D^i_{S_n}$: disorder of S_n after i Berge *k-moves* are performed.

$m_i(S^1, S^2)$: the number of matching coins between strings $S^1, S^2 \in \mathbf{S_n}$ for some shift i ($i = -n, \ldots, n$) of S^1.

Chapter 1

Introduction

1.1 Original Berge Problem

Claude Berge (1926 - 2002) was a well recognized French mathematician who is considered to be one of the modern founders of combinatorics and graph theory. Among his many contributions, Claude Berge edited a series of open problems which appeared in the Revue Française de Recherche Opérationelle (French Journal of Operations Research), under the title *Problèmes plaisans et délectables* (Pleasant and delectable problems) between 1962 and 1966. This series was in tribute to the 17th century work of Bachet [3].

In the last issue of the series, Berge [4] stated problem 41, the problem of interest in this thesis, which is defined as follows. Given a string of alternating $\lceil \frac{n}{2} \rceil$ white coins and $\lfloor \frac{n}{2} \rfloor$ black coins on a one-dimensional board of infinite length, we are required to sort the coins into a string consisting of $\lceil \frac{n}{2} \rceil$ white coins followed immediately by $\lfloor \frac{n}{2} \rfloor$ black coins (or visa versa) by taking 2 adjacent coins to 2 adjacent vacant positions on the board.

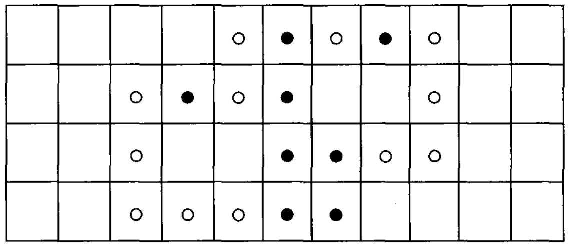

Figure 1.1: A solution for sorting A_5 using Berge *2-moves.*

Let A_n be the string made of alternating $\lceil \frac{n}{2} \rceil$ white and $\lfloor \frac{n}{2} \rfloor$ black coins, and let T_n be the string made of $\lceil \frac{n}{2} \rceil$ white coins followed by $\lfloor \frac{n}{2} \rfloor$ black coins or visa versa. Let $h_{n,2}(A_n, T_n)$ denote the minimum number of moves taking 2 adjacent coins to 2 adjacent vacant positions, called Berge *2-moves*, needed to sort A_n (into T_n) up to translation.

Berge [4] noted that $h_{5,2}(A_5, T_5) = h_{6,2}(A_6, T_6) = 3$ and $h_{7,2}(A_7, T_7) = 4$. Figure 1.1 demonstrates a solution for sorting A_5 in 3 Berge *2-moves.* Moreover, Berge asked if $h_{n,2}(A_n, T_n)$ is a increasing function.

This was the last article of the series with no solution published by the journal. Given the popularity of Berge and his problems, problem 41 was most likely examined and solved within the last forty years. However, the first published solution to Berge's original question might have been given by Avis and Deza [2] in 2006 proving that for $k = 2$ and $n \geq 5$, $h_{n,2}(A_n, T_n) = \lceil \frac{n}{2} \rceil$.

This problem appeared in the *12th Prolog Programming Contest* [14] held in Seattle in 2006. In the statement of the problem, it is noted that this result is surprising given that half of the white coins and half of the black coins are incorrectly positioned. Indeed, this is an unexpected result. Moreover, as we will see in the next section, Deza and Hua [6] conjectured that the minimum number of moves needed to sort A_n is independent of the number of adjacent coins being moved. This leads us to the first generalization of Berge sorting.

1.2 Generalized Berge *k-moves*

We begin with a one-dimensional board of infinite length where the string initially lies in positions 1 through n. A Berge *k-move* takes k adjacent coins to k adjacent vacant positions on the board. More specifically, a single Berge *k-move* will be denoted as $\{\ j\ \ i\ \}$ which indicates that adjacent coins in positions $i, i+1, \ldots, i+k-1$ are moved to vacant positions $j, j+1, \ldots, j+k-1$. In Figure 1.2, the first Berge *2-move* used in sorting A_n takes the pair of coins beginning in position 3 and moves the coins to the beginning of position -1 which is denoted as $\{\ -1\ \ 3\ \}$.

...	-2	-1	0	1	2	3	4	5	6	7	...
				○	●	○	●	○			
		○	●	○	●			○			

Figure 1.2: The first Berge *2-move* in this solution is expressed as $\{\ -1\ \ 3\ \}$.

A sequence of successive moves are joined via union symbol $\{\ j\ \ i\ \} \cup \{\ l\ \ k\ \}$ indicating that $\{\ j\ \ i\ \}$ is the first move and $\{\ l\ \ k\ \}$ is the next move. If a move fills vacant positions which were created by the previous move, that is, $\{\ j\ \ i\ \} \cup \{\ i\ \ k\ \}$, then we can simplify the notation as $\{\ j\ \ i\ \ k\ \}$. For example, the solution for sorting A_5 using Berge *2-moves* in Figure 1.3 is expressed as $\{\ -1\ \ 3\ \} \cup \{\ 3\ \ 0\ \} \cup \{\ 0\ \ 4\ \} = \{\ -1\ \ 3\ \ 0\ \ 4\ \}$.

...	-2	-1	0	1	2	3	4	5	6	7	...
				○	●	○	●	○			
		○	●	○	●			○			
		○			●	●	○	○			
		○	○	○	●	●					

Figure 1.3: A solution for sorting A_5 using Berge *2-moves* is $\{\ -1\ 3\ 0\ 4\ \}$.

The generalized Berge problem asks what is $h_{n,k}(A_n, T_n)$, that is, what is the minimum number of Berge *k-moves* needed to sort A_n. A solution in $h_{n,k}(A_n, T_n)$ move(s) is called *optimal.* Here, we briefly outline some results for $h_{n,k}(A_n, T_n)$ which are explored in greater depth in Chapter 2.

Deza and Hua [6] showed for $k \geq 1$ and $n \geq 3$, $h_{n,k}(A_n, T_n) \geq \lfloor \frac{n}{2} \rfloor$. The authors showed that for $k = 1$ and $n \geq 3$,

$$h_{n,1}(A_n, T_n) = \left\lceil \frac{n}{2} \right\rceil \text{ for } n \not\equiv 3 \pmod 4$$

and

$$h_{n,1}(A_n, T_n) = \left\lfloor \frac{n}{2} \right\rfloor \text{ for } n \equiv 3 \pmod 4$$

where the latter result implies that lower bound previously stated is tight. They improved the lower bound to the following:

$$\text{for } k \geq 2 \text{ and } n \geq 5,\ h_{n,k}(A_n, T_n) \geq \left\lceil \frac{n}{2} \right\rceil.$$

As already noted, for $k = 2$ and $n \geq 5$, Avis and Deza [2] showed

$$h_{n,2}(A_n, T_n) = \left\lceil \frac{n}{2} \right\rceil.$$

For $k = 3$ and $n \geq 5$, Deza and Hua [6] proved that

$$h_{n,3}(A_n, T_n) = \left\lceil \frac{n}{2} \right\rceil \text{ for } n \not\equiv 0 \pmod 4$$

and Deza and Xie [7] showed

$$h_{n,3}(A_n, T_n) = \left\lceil \frac{n}{2} \right\rceil \text{ for } n \equiv 0 \pmod 4$$

except for $h_{12,3}(A_{12}, T_{12}) = 7$ and $h_{16,3}(A_{16}, T_{16}) = 9$.

Deza and Hua [6] conjectured that for $k \geq 2$ and $n \geq 2k + 11$,

$$h_{n,k}(A_n, T_n) = \left\lceil \frac{n}{2} \right\rceil.$$

This is substantiated in [6] where the authors computed values of $h_{n,k}(A_n, T_n)$ for $k \leq 14$ and $k + 2 \leq n \leq 50$. The computed values taken from [8] are shown in Table 1.1. As mentioned earlier, this conjecture, substantiated by significantly large values of n and k, which states that sorting A_n using Berge *k-moves* is independent of k is quite surprising. A natural extension of the generalized Berge sorting is to examine the behaviour of transforming generic strings into other generic strings using Berge *k-moves*. This question was originally raised in [6] and brings us to the primary focus of this thesis.

1.3 Berge *k-metric*

Let $\mathbf{S_n}$ be the set of all strings made of $\lceil \frac{n}{2} \rceil$ white coins and $\lfloor \frac{n}{2} \rfloor$ black coins. A further generalization of the Berge sorting problem asks for any pair of strings $S^1, S^2 \in \mathbf{S_n}$ what is $h_{n,k}(S^1, S^2)$, that is, what is the minimum number of Berge *k-moves* needed to transform S^1 into S^2 (up to translation).

		n																																											
		5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	3	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	4		3	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5			2	2	1	1	1	1	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	6				3	2	1	1	1	1	2	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
k	7					3	3	2	2	1	2	1	0	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	8						6	4	3	2	2	1	2	1	1	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	9							6	5	3	2	2	2	1	1	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	10								10	4	4	2	3	2	2	1	2	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	11									11	6	4	3	2	3	1	2	1	2	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	12										15	7	5	3	3	2	3	2	1	1	1	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	13											14	9	5	4	3	3	2	2	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	14												21	9	7	4	4	3	3	2	2	2	2	1	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 1.1: Table represents $h_{n,k}(A_n, T_n) - \lceil \frac{n}{2} \rceil$ for $k \leq 14$ and $k + 2 \leq n \leq 50$. Grey highlighted entries substantiate the conjecture that $h_{n,k}(A_n, T_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 2k + 11$.

For example, if $S^1 = \{\bullet \bullet \circ \circ \circ\bullet\}$ and $S^2 = \{\circ \bullet \circ \circ \bullet\bullet\}$, then what is $h_{n,2}(S^1, S^2)$? For illustrative purposes, we place the string we wish to achieve, S^2, directly above the string we wish to transform, S^1, with both strings initially lying in positions 1 through 6. For general n, both strings initially lying in positions 1 through n. A possible solution is illustrated in Figure 1.4 where $h_{n,2}(S^1, S^2) = 2$ and this is the minimum as clearly one move is not enough.

	...	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	...
S^2					○	●	○	○	●	●						
S^1					●	●	○	○	○	●						
							○	○	○	●			●	●		
									○	●	○	○	●	●		

Figure 1.4: Solution for transforming S^1 into S^2 using Berge *2-moves*.

A natural setting is to view $h_{n,k}(S^1, S^2)$ as the shortest path between any pair of strings of $\mathbf{S_n}$. Let $S^1, S^2, S^3 \in \mathbf{S_n}$, and $h_{n,k}(S^1, S^2) = \min\{S^1 \rightarrow^k S^2\}$ be the minimum number of Berge *k-moves* required to transform S^1 into S^2. We have

$$\begin{aligned} h_{n,k}(S^1, S^2) &= 0 \Longleftrightarrow S^1 = S^2 && (0)\\ h_{n,k}(S^1, S^2) &\geq 0 && (1)\\ h_{n,k}(S^1, S^2) &= h_{n,k}(S^2, S^1) && (2)\\ h_{n,k}(S^1, S^2) &\leq h_{n,k}(S^1, S^3) + h_{n,k}(S^3, S^2) && (3). \end{aligned}$$

The first two properties are trivial. If we transform S^1 to S^2, then we can always transform S^2 back to S^1. This gives us property (2). The final property is the triangle inequality.

We first verify that we can go from one string to another in a finite number of moves, i.e, $h_{n,k}(S^1, S^2) < \infty$, for $k = 1$ and $k = 2$. This can be verified by showing that any string of $\mathbf{S_n}$ can be sorted (into T_n) in a finite number of moves. Then by symmetry of $h_{n,k}(S^1, S^2)$, we have that T_n can be transformed into any string of $\mathbf{S_n}$. Thus, using the triangle inequality, we can transform any pair of strings of $\mathbf{S_n}$ into one another by going through T_n in a finite number of moves. Thus, we consider the associated Berge *k-metric* between a pair of strings of $\mathbf{S_n}$ which is the minimum number of Berge *k-moves* required to transform one string into other. In particular, we show that for strings $S^1, S^2 \in \mathbf{S_n}$, $h_{n,1}(S^1, S^2) \leq \frac{\sqrt{2}}{2}n$ and $h_{n,2}(S^1, S^2) \leq 2n$ in Chapter 3 and Chapter 4, respectively.

In thesis, we examine the *diameter* of $\mathbf{S_n}$ allowing Berge *k-moves*, denoted $H_{n,k}(\mathbf{S_n})$, which is defined as

$$H_{n,k}(\mathbf{S_n}) = \max_{(S^1,S^2)\in\mathbf{S_n}} h_{n,k}(S^1, S^2).$$

1.4 Intuition on Bounds for $H_{n,k}(\mathbf{S_n})$

Deza and Hua [6] showed that the lower bound for sorting A_n into T_n using Berge *k-moves* is $\lceil \frac{n}{2} \rceil$, i.e., $h_{n,k}(A_n, T_n) \geq \lceil \frac{n}{2} \rceil$. By the lower bound of $h_{n,k}(A_n, T_n)$ and the definition of $H_{n,k}(\mathbf{S_n})$, we have that

$$H_{n,k}(\mathbf{S_n}) \geq \left\lceil \frac{n}{2} \right\rceil.$$

In terms of an upper bound for $H_{n,k}(\mathbf{S_n})$, we are interested in exploring the question if $H_{n,k}(\mathbf{S_n}) = \lceil \frac{n}{2} \rceil$, that is, are A_n and T_n antipodal? In order to address this question, we focus on $h_{n,1}(S^1, S^2)$ for $S^1, S^2 \in \mathbf{S_n}$ in Chapter 3 and explore some aspects of $h_{n,2}(S^1, S^2)$ for $S^1, S^2 \in \mathbf{S_n}$ in Chapter 4.

With this thesis, we hope to encourage further study of the Berge metrics from both theoretical and computational approaches.

Chapter 2

Berge Moves

2.1 Lower Bound for Sorting A_n

In order to establish a lower bound on sorting A_n, we need the following definition. Define the **disorder** of a string, $S_n \in \mathbf{S_n}$, after the $i-th$ move, denote $D^i_{S_n}$, as the number of coins whose right neighbour is not a coin of the same colour or empty. For example, $D^0_{A_n} = n$ and $D^0_{T_n} = 2$.

The disorder of a string between successive moves can either increase by at most 2 or decrease by at most 2. In other words, $\left|D^i_{S_n} - D^{i+1}_{S_n}\right| \leq 2$. This property can be verified by examining all possible moves. In general, for Berge *k-moves*, the disorder is only affected at the ends of the k adjacent coins. For instance, if we are moving coins in positions $i, i+1, \ldots, i+k-1$, then the coin in position i may change the disorder depending on the coin in position $i-1$ and the coin in position $i+k-1$ may change the disorder depending on the coin in position $i+k$. Next we examine the change in disorder for moving coins in positions $i, i+1, \ldots, i+k-1$.

1. Assume the $(i-1)-th$ and $i-th$ coins are different and the $(i+k-1)-th$ and $(i+k)-th$ coins are different.

   a. Move coins in positions $i, i+1, \ldots, i+k-1$ to $n+1, \ldots, n+k$: if the $n-th$ coin and the $i-th$ coin are same, then the disorder decreases by one; otherwise the disorder stays the same.

   b. Move coins in positions $i, i+1, \ldots, i+k-1$ to left end of the string, i.e., to positions $1-k, \ldots, 0$: similar analysis as in 1a.

   c. Move coins in positions $i, i+1, \ldots, i+k-1$ to the right or left of the string such that there are vacant positions between the moved coins and the original string: the disorder does not change.

   d. Move coins in positions $i, i+1, \ldots, i+k-1$ to fill vacant positions $j, j+1, \ldots, j+k-1$:

      * if the $(j-1)-th$ coin and the $i-th$ coin are same and the $(j+k)-th$ coin and if the $(i+k-1)-th$ coin are the same, then the disorder decreases by two.

      * if either one of them is different, then the disorder decreases by one.

      * if they are both different, then the disorder stays the same.

2. Assume the $(i-1)-th$ and $i-th$ coins are different and the $(i+k-1)-th$ and $(i+k)-th$ coins are of the same colour.

   a. Move coins in positions $i, i+1, \ldots, i+k-1$ to right end of the string, i.e., in positions $n+1, \ldots, n+k$: if the $n-th$ coin and the $i-th$ coin are same, then the disorder doesn't change and if they are different, then the disorder increases by one.

b. Move coins in positions $i, i+1, \dots, i+k-1$ to left end of the string, i.e., to positions $1-k, \dots, 0$: similar analysis as in 2a.

c. Move coins in positions $i, i+1, \dots, i+k-1$ to the right or left of the string such that there are vacant positions between the moved coins and the original string: the disorder does not change.

d. Move coins in positions $i, i+1, \dots, i+k-1$ to fill vacant positions $j, j+1, \dots, j+k-1$:

   * if the $(j-1)-th$ coin and the $i-th$ coin are same and the $(i+k-1)-th$ coin and if the $(j+k)-th$ coin are the same, then the disorder decreases by one.
   * if one of them is different, then the disorder stays the same.
   * if they are both different, then the disorder increases by one.

3. Assume the $(i-1)-th$ and $i-th$ coins are the same colour and the $(i+k-1)-th$ and $(i+k)-th$ coins are different. (Similar to 2.)

4. Assume the $(i-1)-th$ and $i-th$ coins are the same colour and the $(i+k-1)-th$ and $(i+k)-th$ coins are the same colour.

   a. Move coins in positions $i, i+1, \dots, i+k-1$ to right end of the string, i.e., in positions $n+1, \dots, n+k$: if the $n-th$ coin and the $i-th$ coin are same, then the disorder increases by one and if they are different, then the disorder increases by two.

   b. Move coins in positions $i, i+1, \dots, i+k-1$ to left end of the string, i.e., to positions $1-k, \dots, 0$: similar analysis as in 4a.

   c. Move coins in $i, i+1, \dots, i+k-1$ to the right or left of the string such that there are vacant positions between the moved coins and the original string: the disorder increases by two.

d. Move coins in positions $i, i+1, \ldots, i+k-1$ to fill vacant positions $j, j+1, \ldots, j+k-1$:

  * if the $(j-1)-th$ coin and the $i-th$ coin are same and the $(i+k-1)-th$ coin and if the $(j+k)-th$ coin are the same, then the disorder does not change.
  * if one of them is different, then the disorder increases by one.
  * if they are both different, then the disorder increases by two.

In sorting A_n, we are required to decrease the disorder from n to 2 in the least number of moves. Adopting common terminology from [2] and [6], we say that $D^i_{A_n} - D^{i+1}_{A_n} = 2$ is an optimal move, $D^i_{A_n} - D^{i+1}_{A_n} = 1$ is a suboptimal move, and $D^i_{A_n} - D^{i+1}_{A_n} = 0$ is a neutral move.

Lemma 2.1. *[6] For $k \geq 1$ and $n \geq 3$, $h_{n,k}(A_n, T_n) \geq \lfloor \frac{n}{2} \rfloor$.*

We recall the proof for Lemma 2.1. First note that the initial disorder is n, i.e., $D^0_{T_n} = n$, and the final disorder when i equals the minimum number of moves is 2, i.e., $D^{h_{n,k}(A_n,T_n)}_{A_n} = 2$. The first move can at most decrease the disorder by one because the only reasonable option is to take coins from the interior of the string and adjoin them to one of the ends. Since we have that $|D^i_{A_n} - D^{i+1}_{A_n}| \leq 2$, each of the other moves satisfy $D^i_{A_n} - D^{i+1}_{A_n} \leq 2$. This implies that $h_{n,k}(A_n, T_n) \geq \lfloor \frac{n}{2} \rfloor$.

Later in this chapter, we will see that $h_{n,1}(A_n, T_n) = \lfloor \frac{n}{2} \rfloor$ for $n \equiv 3 \pmod 4$, implying Lemma 2.1 is tight. Moreover, the lower bound can be tightened to the following lemma.

Lemma 2.2. *[6] For $k \geq 2$ and $n \geq 5$, $h_{n,k}(A_n, T_n) \geq \lceil \frac{n}{2} \rceil$.*

We recall the proof of Lemma 2.2 for $k = 2$ [2] for completeness. If n is even, then $\lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. Assume n is odd and $h_{n,2}(A_n, T_n) = \lfloor \frac{n}{2} \rfloor$. This implies that moves $i = 2$ to $i = h_{n,2}(A_n, T_n)$ are all optimal, and thus, decrease the disorder by 2. Without loss of generality, assume the first move is to the right. After the first move which is suboptimal, we have the board illustrated in Figure 2.1.

	...	1	2	3	...	$i-1$	i	$i+1$	$i+2$	...	$n-1$	n	$n+1$	$n+2$	...
$D^0_{A_n} = n$		○	●	○	...	●	○	●	○	...	●	○			
$D^1_{A_n} = n-1$		○	●	○	...	●			○	...	●	○	○	●	

Figure 2.1: Board after the first suboptimal move for Berge *2-moves* on A_n for n odd.

Since the subsequent moves are optimal, we need fill the vacant positions created by the previous move. The vacant positions will alternate between ● _ _ ○ and ○ _ _ ●. The coins in the last 3 positions after the first move, ○ ○ ●, cannot fill ● _ _ ○ with an optimal move. Thus, another suboptimal move is necessary to sort the last 2 coins. This implies that we need $\lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$ Berge 2-moves.

2.2 Optimal Solutions for Sorting A_n

In this section, we exhibit optimal solutions for Berge *1-moves*, *2-moves*, and *3-moves*. These optimal solutions for sorting A_n are constructed inductively, and thus, are depend on the colour of the first coin.

Let $A_{n,\circ} \in \mathbf{S_n}$ be an alternating string whose first coin is white and let $A_{n,\bullet} \in \mathbf{S_n}$ be an alternating string beginning with a black coin. Similarly, let $T_{n,\circ} \in \mathbf{S_n}$ to be the string with $\lceil \frac{n}{2} \rceil$ white coins followed by $\lfloor \frac{n}{2} \rfloor$ black coins and $T_{n,\bullet} \in \mathbf{S_n}$ to be the string with $\lfloor \frac{n}{2} \rfloor$ black coins followed by $\lceil \frac{n}{2} \rceil$ white coins. Let $B_{n,k}(A_n, T_n)$ denote a solution, that is, an ordered set of Berge *k-moves* needed to sort A_n (into T_n).

2.2.1 Optimal Solutions for Sorting A_n by Berge *1-moves*

The optimal solutions presented in this section can be found in [6].

Case 1: For $n \equiv 3 \pmod 4$, the following is a solution in $\lfloor \frac{n}{2} \rfloor$ Berge *1-moves.*

The base case is $B_{3,1}(A_{3,\circ}, T_3) = \{\ 4\ \ 1\ \}$. Let $n = 4i + 3$ for $i \geq 1$. Assume we have a solution $B_{4i-1,1}(A_{4i-1,\circ}, T_{4i-1})$ taking $\lfloor \frac{4i-1}{2} \rfloor$ moves. We ignore coins in positions 1, 2, $4i + 2$, and $4i + 3$, and sort the remaining $4i - 1$ coins using the solution of $B_{4i-1,1}(A_{4i-1,\circ}, T_{4i-1})$. We complete the solution by the following 2 moves: $\{\ 3\ \ 4i+2\ \ 1\ \}$. Thus, the total number of moves needed is $\lfloor \frac{4i-1}{2} \rfloor + 2 = \lfloor \frac{4i+3}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. Figure 2.2 illustrates this constructive induction. We can apply $B_{4i-1,1}(A_{4i-1,\circ}, T_{4i-1})$ while ignoring the coins in positions 1, 2, $4i + 2$, and $4i + 3$ because by induction these coins are not among the entries of $B_{4i-1,1}(A_{4i-1,\circ}, T_{4i-1})$ in the first $2i - 1$ moves for $i \geq 1$.

Case 2: For $n \equiv 1 \pmod 4$, the following is a solution in $\lceil \frac{n}{2} \rceil$ Berge *1-moves.*

...	0	1	2	3	4	5	6	7	8	...
		○	●	○	●	○	●	○		

⇓

Ignore coins in 1, 2, 6 and 7, and re-number the positions to use the solution of $B_{3,1}(A_{3,\circ}, T_3)$.

...	0	~~1~~	~~2~~	~~3~~ 1	~~4~~ 2	~~5~~ 3	~~6~~	~~7~~	~~8~~ 4	...
		~~○~~	~~●~~	○	●	○	~~●~~	~~○~~		
		~~○~~	~~●~~		●	○	~~●~~	~~○~~	○	

⇓

Complete the solution with { 3 6 1 }.

...	0	1	2	3	4	5	6	7	8	...
		○	●	○	●	○	●	○		
		○	●		●	○	●	○	○	
		○	●	●	●	○		○	○	
			●	●	●	○	○	○	○	

Figure 2.2: Finding $B_{7,1}(A_{7,\circ}, T_7)$ using $B_{3,1}(A_{3,\circ}, T_3)$.

The base case is $B_{5,1}(A_{5,\circ}, T_5) = \{\ 6\ 3\ 4\ 1\ \}$. Let $n = 4i+1$ for $i > 1$. Assume we have a solution $B_{4i-3,1}(A_{4i-3,\circ}, T_{4i-3})$ taking $\lceil \frac{4i-3}{2} \rceil$ moves. Ignore coins in positions 1, 2, $4i$, and $4i+1$. Sort the remaining $4i-3$ coins using the solution of $B_{4i-3,1}(A_{4i-3,\circ}, T_{4i-3})$. We complete the solution by the following 2 moves: $\{\ 3\ 4i\ 1\ \}$. Thus, the total number of Berge *1-moves* required is $\lceil \frac{4i-3}{2} \rceil + 2 = \lceil \frac{4i+1}{2} \rceil = \lceil \frac{n}{2} \rceil$.

Case 3: For $n \equiv 2 \pmod 4$, the following is a solution in $\lceil \frac{n}{2} \rceil$ Berge *1-moves*.

The base case is $B_{6,1}(A_{6,\circ}, T_6) = \{\ 7\ 3\ 6\ 1\ \}$. Let $n = 4i+2$ for $i > 1$. Assume we have a solution $B_{4i-2,1}(A_{4i-2,\circ}, T_{4i-2})$ taking $\lceil \frac{4i-2}{2} \rceil$ moves. Ignore coins in positions 1, 2, $4i+1$, and $4i+2$, and sort the remaining $4i-2$ coins using the solution of $B_{4i-2,1}(A_{4i-2,\circ}, T_{4i-2})$. Finally, we complete the solution by the following 2 moves: $\{\ 3\ 4i+2\ 1\ \}$. Thus, the total number of moves needed is $\lceil \frac{4i-2}{2} \rceil + 2 = \lceil \frac{4i+2}{2} \rceil = \lceil \frac{n}{2} \rceil$.

Case 4: For $n \equiv 0 \pmod 4$, the following is a solution in $\lceil \frac{n}{2} \rceil$ Berge *1-moves*.

The base case is $B_{4,1}(A_{4,\circ}, T_4) = \{\ 5\ 2\ 1\ \}$. Let $n = 4i+4$ for $i \geq 1$. Assume we have a solution $B_{4i,1}(A_{4i,\circ}, T_{4i})$ which takes $\lceil \frac{4i}{2} \rceil$ moves. We ignore coins in positions 1, 2, $4i+3$, and $4i+4$, sort the remaining $4i$ coins using the first $\frac{4i}{2} - 1 = 2i - 1$ moves of $B_{4i,1}(A_{4i,\circ}, T_{4i})$, and then complete the solution by the following 3 moves: $\{\ 4\ 4i+3\ 2\ 1\ \}$. Thus, the total number of moves needed is $\lceil \frac{4i-2}{2} \rceil + 3 = \lceil \frac{4i-2+6}{2} \rceil = \lceil \frac{n}{2} \rceil$.

In summary, for $k = 1$ and $n \geq 3$,

$$h_{n,1}(A_n, T_n) = \left\lceil \frac{n}{2} \right\rceil \text{ for } n \not\equiv 3 \pmod 4$$

and

$$h_{n,1}(A_n, T_n) = \left\lfloor \frac{n}{2} \right\rfloor \text{ for } n \equiv 3 \pmod 4.$$

2.2.2 Optimal Solutions for Sorting A_n by Berge *2-moves*

In this section, we develop recursive expressions from [2] for optimal solutions of sorting A_n using Berge *2-moves*. For simplicity, let $B_{n,k}(A_{n,\circ}, T_n)$ be denote by $B_{n,k}$. Define $B^i_{n,k}$ as the $i-th$ entry in the solution for sorting $A_{n,\circ}$.

Case 1: For $n = 4j, j > 2$, the following is a solution in $\frac{n}{2}$ Berge *2-moves.*

The base case is $B_{8,2} = \{\ 9\ 2\ 5\ 8\ 1\ \}$. Define $B_{4j,2}$ as follows:

- $B^1_{4j,2} = 4j + 1$ and $B^2_{4j,2} = 2$
- For $2 < i < \frac{n}{2}$,
    - if $i < j + 1$ and
        - i is even, then $B^i_{4j,2} = B^i_{4j-4,2} + 2$
        - i is odd, then $B^i_{4j,2} = B^i_{4j-4,2}$
    - if $i = j + 1$ and
        - j is even, then $B^i_{4j,2} = 2j + 1$
        - j is odd, then $B^i_{4j,2} = 4j - 1$
    - if $i = j + 2$ and

...	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
		○	●	○	●	○	●	○	●	○	●	○	●					
		○			●	○	●	○	●	○	●	○	●	●	○			
		○	○	●	●			○	●	○	●	○	●	●	○			
		○	○	●	●	●	○	○	●	○			●	●	○			
		○	○	●	●	●			●	○	○	○	●	●	○			
		○	○	●	●	●	●	●	●	○	○	○			○			
				●	●	●	●	●	●	○	○	○	○	○	○			

Figure 2.3: An optimal solution for sorting $A_{12,\circ}$ using Berge *2-moves.*

    - $*$ j is even, then $B^i_{4j,2} = 4j - 3$
    - $*$ j is odd, then $B^i_{4j,2} = 2j$

  - if $i > j + 2$ and

    - $*$ i is even, then $B^i_{4j,2} = B^{(i-2)}_{4j-4,2} + 2$
    - $*$ i is odd, then $B^i_{4j,2} = B^{(i-2)}_{4j-4,2}$

- $B^{(\frac{n}{2})}_{4j,2} = 4j$ and $B^{(\frac{n}{2})+1}_{4j,2} = 1$

For example, if we want to find $B_{12,2}$, then we have $B^1_{12,2} = (4)(3) + 1 = 13$ and $B^2_{12,2} = 2$. Next for $2 < i < 6$, we have $B^3_{12,2} = B^3_{12-4,2} = B^3_{8,2} = 5$, $B^4_{12,2} = 4(3) - 2$ and $B^5_{12,2} = 2(3)$. Finally, we can find $B^6_{12,2} = 4(3) = 12$ and $B^7_{12,2} = 1$. This gives $B_{12,2} = \{\ 13\ 2\ 5\ 10\ 6\ 12\ 1\ \}$ whose solution is illustrated in Figure 2.3.

$B_{8,2}$	=	{ 9 2 5 8 1 }
$B_{12,2}$	=	{ 13 2 5 10 6 12 1 }
$B_{16,2}$	=	{ 17 2 5 12 9 13 6 16 1 }
$B_{20,2}$	=	{ 21 2 5 14 9 18 10 15 6 20 1 }
$B_{24,2}$	=	{ 25 2 5 16 9 20 13 21 10 17 6 24 1 }
$B_{28,2}$	=	{ 29 2 5 18 9 22 13 26 14 23 10 19 6 28 1 }

Table 2.1: Optimal solutions for sorting $A_{n,\circ}$ in $\frac{n}{2}$ Berge *2-moves* for $n = 4j$, $2 \leq j \leq 7$.

This recursive expression can be used to find solutions for the first 6 values of j given in Table 2.1. Reading one column at a time in Table 2.1 perhaps gives a better sense of how to recursively construct the next solution from the previous solution. The middle entries which form the *V-shape* within the solution can also be found by adding 4 to the diagonal entry of $j-2$ if j is even and by adding 8 to the diagonal entry of $j-2$ if j is odd for entries along the left diagonal (entries highlighted in light grey in Table 2.1). To complete the right diagonal (entries highlighted in medium grey in Table 2.1), we add 5 to the diagonal of entry $j-2$ for even j and add 5 to the diagonal of entry $j-2$ for odd values of j.

Case 2: For $n = 4j+2, j > 2$, the following is a solution in $\frac{n}{2}$ Berge *2-moves*.

The base case is $B_{10,2} = \{\ 11\ 2\ 7\ 4\ 10\ 1\ \}$. Define $B_{4j+2,2}$ as follows:

- $B^1_{4j+2,2} = 4j+3$, $B^2_{4j+2,2} = 2$ and $B^3_{4j+2,2} = B^3_{4j-2,2} + 2$

- For $3 < i \leq \frac{n}{2}$,
  - if $i < j+2$ and
    - i is even and $B^i_{4j-2,2} = 2j-2$, then $B^i_{4j+2,2} = B^i_{4j-2,2} + 2$; otherwise $B^i_{4j+2,2} = B^i_{4j-2,2}$,
    - i is odd, then $B^i_{4j+2,2} = B^i_{4j-2,2} + 4$
  - if $i = j+2$ and
    - j is even, then $B^i_{4j+2,2} = 2j$
    - j is odd, then $B^i_{4j+2,2} = 2j+5$
  - if $i = j+3$ and
    - j is even, then $B^i_{4j+2,2} = 2j+6$
    - j is odd, then $B^i_{4j,2} = 2j-1$
  - if $i > j+3$ and
    - i is even, then $B^i_{4j+2,2} = B^{(i-2)}_{4j-2,2}$
    - i is odd, then $B^i_{4j+2,2} = B^{(i-2)}_{4j-2,2} + 4$
- $B^{(\frac{n}{2})+1}_{4j,2} = 1$

Table 2.2 has the first few solutions of A_n for $n = 4j+2$ where $2 \leq j \leq 7$. The diagonal entries (highlighted in light and medium grey) for $n = 4j+2$ can be constructed by adding 4 to the $j-2$ diagonal entry of the corresponding diagonal.

The following properties were observed by Avis and Deza [2]. The first property for n even is that the solutions given in Tables 2.1 and 2.2 are always

...	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	...
		○	●	○	●	○	●	○	●	○	●	○	●	○	●					
		○			●	○	●	○	●	○	●	○	●	○	●	●	○			
		○	○	●	●	○	●	○	●			○	●	○	●	●	○			
		○	○	●	●	○			●	●	○	○	●	○	●	●	○			
		○	○	●	●	○	○	●	●	●	○			○	●	●	○			
		○	○	●	●			●	●	●	○	○	○	○	●	●	○			
		○	○	●	●	●	●	●	●	●	○	○	○	○			○			
				●	●	●	●	●	●	●	○	○	○	○	○	○	○			

Figure 2.4: An optimal solution for sorting $A_{14,\circ}$ using Berge *2-moves*.

$B_{10,2}$	=	{	11	2	7	4										10	1	}
$B_{14,2}$	=	{	15	2	9	6	11								5	14	1	}
$B_{18,2}$	=	{	19	2	11	6	15	8						14	5	18	1	}
$B_{22,2}$	=	{	23	2	13	6	19	10	15				9	18	5	22	1	}
$B_{26,2}$	=	{	27	2	15	6	23	10	19	12		18	9	22	5	26	1	}
$B_{30,2}$	=	{	31	2	17	6	27	10	23	14	19	13	22	9	26	5	30	1	}

Table 2.2: Optimal solutions for sorting $A_{n,\circ}$ in $\frac{n}{2}$ Berge *2-moves* for $n = 4j+2$, $2 \leq j \leq 7$.

shifted to the right by two positions. A second property is that all of the black coins are positioned to the immediate left of all the white coins. A third property is that first $\lceil \frac{n}{4} \rceil$ moves create pairs of black coins and pairs of white coins. In Tables 2.1 and 2.2, these moves are the ones to the left of and including the light grey diagonal. The remaining $\lfloor \frac{n}{4} \rfloor$ moves position the black pairs of coins among the black sequence of coins positioned in 3 to $\frac{n+4}{2}$ and the white pairs among positions $\frac{n+6}{2}$ through $n+2$. These moves correspond to the entries that are to the right of and including the medium grey diagonal presented in Tables 2.1 and 2.2. Another important property of the recursion for n even is that the coins in positions $n-1$ and n are never moved which ensures that the following recursion for n odd holds true.

Case 3: For $n \geq 9$ odd, the following is a solution in $\lceil \frac{n}{2} \rceil$ Berge *2-moves.*

The following definition is taken from [2]. Define $B_{n,2} = \{\ n+1\quad 1\ \} \cup \Sigma_{n-1}$ where

$$\Sigma^i_{n-1} = B^i_{n-1,2} + 2 \text{ for } B^i_{n-1,2} \leq n-3$$

and

$$\Sigma^i_{n-1} = B^i_{n-1,2} + 3 \text{ for } B^i_{n-1_2} \geq n-1.$$

For example, $B_{11,2} = \{\ 12\ \ 1\ \} \cup \Sigma_{10}$. If $B^i_{10,2} \leq 8$, then $\Sigma^i_{10} = B^i_{10,2} + 2$. As well, if $B^i_{10,2} \geq 10$, then $\Sigma^i_{10} = B^i_{10,2} + 3$. Using these facts and the solution of $B_{10,2}$ given in Table 2.2, we have $B_{11,2} = \{\ 12\ \ 1\ \} \cup \{\ 14\ \ 4\ \ 9\ \ 6\ \ 13\ \ 3\ \}$. The solution of $B_{11,2}$ is illustrated in Figure 2.5. Since this recursion uses the solution for $n-1$ which is even, the coins in positions $n-2$ and $n-1$ are not

...	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
				○	●	○	●	○	●	○	●	○	○	●				
		○	●	○	●	○	●	○	●	○	●	○						
				○			●	○	●	○	●	○	○	●	●	○		
				○	○	●	●	○	●			○	○	●	●	○		
				○	○	●			●	●	○	○	○	●	●	○		
				○	○	●	●	●	●	●	○	○	○			○		
						●	●	●	●	●	○	○	○	○	○	○		

Figure 2.5: An optimal solution for sorting $A_{11,\circ}$ using Berge *2-moves*.

used in the solution of $B_{n-1,2}$ according to the property mentioned above for n even, and therefore, makes the solution $B_{n-1,2}$ possible.

2.2.3 Optimal Solutions for Sorting A_n by Berge *3-moves*

Cases 1 to 3 are optimal solutions given by Deza and Hua [6] for sorting A_n where $n \not\equiv 0 \pmod 4$ using Berge *3-moves*. Case 4 is an optimal solution given by Deza and Xie [7] for $n \equiv 0 \pmod 4$.

Case 1: For $n \equiv 1 \pmod 4$, the following is a solution in $\lceil \frac{n}{2} \rceil$ Berge *3-moves*.

The base case is $B_{5,3}(A_{5,\circ}, T_5) = \{\ 6\ 2\ 5\ 1\ \}$. Let $n = 4i + 1$ for $i \geq 2$, and assume $B_{4i-3,3}(A_{4i-3,\circ}, T_{4i-3})$ has a solution in $\lceil \frac{4i-3}{2} \rceil$ moves. Ignore coins in positions 1, 2, $2i + 3$ and $2i + 4$. Sort the remaining coins using $B_{4i-3,3}(A_{4i-2,\circ}, T_{4i-2})$ and complete the solution with the following 2 moves:

$\{\ 3\ 2i+4\ 1\ \}$. Therefore, the total number of Berge *3-moves* needed is $\lceil \frac{4i-3}{2} \rceil + 2 = \lceil \frac{4i-3+4}{2} \rceil = \lceil \frac{4i+1}{2} \rceil$ as desired. Moreover, we can use the solution $B_{4i-3,3}(A_{4i-3,\circ}, T_{4i-3})$ while ignoring coins in positions 1, 2, $2i+3$ and $2i+4$ because $B_{4i+1,3}(A_{4i+1,\circ}, T_{4i+1})$ does not have any of -1, 0, $2j+1$ or $2i+2$ among its entries in the first $2j-1$ moves for $i > 0$.

Case 2: For $n \equiv 2 \pmod 4$, the following is a solution in $\lceil \frac{n}{2} \rceil$ Berge ***3-moves.***

The base case is $B_{6,3}(A_{6,\circ}, T_6) = \{7\ 2\ 6\ 1\ \}$. Let $n = 4i+2$ for $i \geq 2$, and assume $B_{4i-2,3}(A_{4i-2,\circ}, T_{4i-2})$ has a solution in $\lceil \frac{4i-2}{2} \rceil$ moves. We ignore coins in positions 1 ,2, $2i+3$ and $2i+4$ and sorting the remaining coins using the solution of $B_{4i-2,3}(A_{4i-2,\circ}, T_{4i-2})$ together with the following three moves $\{\ 3\ 2i+4\ 1\ \}$ to complete the solution. In total, we have $\lceil \frac{4i-2}{2} \rceil + 2 = \lceil \frac{4i+2}{2} \rceil$ moves are needed. The steps in this case are demonstrated in Figure 2.6. As in the previous case, we can use the solution $B_{4i-2,3}(A_{4i-2,\circ}, T_{4i-2})$ while ignoring coins in positions $1, 2, 2i+3$ and $2i+4$ because $B_{4i+2,3}(A_{4i+2,\circ}, T_{4i+2})$ does not have any of $-1, 0, 2j+1$ or $2i+2$ among its entries in the first $2j-1$ moves for $i > 0$.

The following lemma is needed in finding an optimal solution for the case $n \equiv 3 \pmod 4$ because it uses $B_{n,3}(A_{n,\circ}, T_n)$ for $n \equiv 3 \pmod 4$. Lemma 2.3 can be checked by induction.

Lemma 2.3. *[6] (i) For $n \equiv 2 \pmod 4$, the solutions $B_{n,3}(A_n, T_n)$ shift the final string three positions to the right.*
(ii) For $n \equiv 2 \pmod 4$, the solutions $B_{n,3}(A_n, T_n)$ sort $A_{n,\bullet}$ into $T_{n,\bullet}$ and $A_{n,\circ}$ into $T_{n,\circ}$.

...	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
		○	●	○	●	○	●	○	●	○	●				

⇓

Ignore coins in 1, 2, 7 and 8, and re-number the positions to use the solution of $B_{6,1}(A_{6,\circ}, T_6)$.

...	0	~~1~~	~~2~~	~~3~~ 1	~~4~~ 2	~~5~~ 3	~~6~~ 4	~~7~~	~~8~~	~~9~~ 5	~~10~~ 6	~~11~~ 7	~~12~~ 8	~~13~~ 9	...
		~~○~~	~~●~~	○	●	○	●	~~○~~	~~●~~	○	●				
		~~○~~	~~●~~	○				~~○~~	~~●~~	○	●	●	○	●	
		~~○~~	~~●~~	○	●	●	○	~~○~~	~~●~~	○				●	
		~~○~~	~~●~~				○	~~○~~	~~●~~	○	○	●	●	●	

⇓

Complete the solution with { 3 8 1 }.

...	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
		○	●	○	●	○	●	○	●	○	●				
		○	●	○				○	●	○	●	●	○	●	
		○	●	○	●	●	○	○	●	○				●	
		○	●				○	○	●	○	○	●	●	●	
		○	●	●	○	○	○	○				●	●	●	
					○	○	○	○	○	●	●	●	●	●	

Use original positions for the solution:

$B_{10,3}(A_{10,\circ}, T_{10}) = \{\ 11\ 4\ 10\ 3\ 8\ 1\ \}$.

Figure 2.6: Example of sorting $A_{10,\circ}$ using Berge *3-moves*.

Case 3: For $n \equiv 3 \pmod 4$, the following is a solution in $\lceil \frac{n}{2} \rceil$ Berge *3-moves.*

The base case is $B_{7,3}(A_{7,\circ}, T_7) = \{\ -2\ 4\ \ -1\ 3\ \ -2\ \}$. Let $n = 4i+3$ for $i \geq 2$. The first move is $\{\ -2\ 4i\ \}$. Next we ignore the coin in position $4i+3$ and use the solution of $B_{4i+3,3}(A_{4i+3,\bullet}, T_{4i+3,\bullet})$ to complete the solution. Lemma 2.3 ensures the validity of $B_{4i+2,3}(A_{4i+2,\circ}, T_{4i+2})$. The number of moves need is $\lceil \frac{4i+2}{2} \rceil + 1 = \lceil \frac{n-1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil = \lceil \frac{n}{2} \rceil$.

Case 4: For $n \equiv 0 \pmod 4$, the following is a solution in $\lfloor \frac{n}{2} \rfloor$ Berge *3-moves.*

A solution for $n \equiv 0 \pmod 4$ taking $\lceil \frac{n}{2} \rceil$ moves was established by Deza and Xie [7]. The induction technique used in this case differs from the previous cases by incorporating extra moves at various stages. The first stage moves coins of alternating colours, i.e., $\circ \bullet \circ$, $\bullet \circ \bullet$. The second stage moves coins of mixed colours such as $\circ \circ \bullet$, $\circ \bullet \bullet$, $\bullet \bullet \circ$, and $\bullet \circ \circ$. The final stage moves coins of the same colour, $\circ \circ \circ$ and $\bullet \bullet \bullet$. There are two key coin positions which are important in the sorting process. One of them are called *pivots* which are the coins that remain fixed (never moved in the optimal solution). The other is the *anchor point* which refers to k consecutive coins ($k = 3$ in this case) which are surrounded by pivots such that they are vacant at the end of the first stage and filled at the beginning of stage 3. For example, Figure 2.7 shows the pivots of $A_{20,\circ}$ which are in positions 5, 11, 16, and 20 and the anchor point which includes positions 17, 18, and 19.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	Stages
○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●							
○				○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	●	○	●				1
○	○	●	○	○	●				●	○	●	○	●	○	●	○	●	○	●	●	○	●				1
○	○	●	○	○	●	●	○	●	●	○				○	●	○	●	○	●	●	○	●				1
○	○	●	○	○	●	●	○	●	●	○	○	●	○	○	●				●	●	○	●				1
○	○	●	○	○	●	●	○	●	●	○	○				●				●	●	○	●	●	○	○	2
○	○	●	○	○	●	●	○	●	●	○	○	○	●	●	●				●	●				○	○	2
○	○	●	○	○				●	●	○	○	○	●	●	●				●	●	●	●	○	○	○	2
			○	○	○	○	●	●	●	○	○	○	●	●	●				●	●	●	●	○	○	○	2
			○	○	○	○				○	○	○	●	●	●	●	●	●	●	●	●	●	○	○	○	3
			○	○	○	○	○	○	○	○	○	○	●	●	●	●	●	●	●	●	●	●				3

Figure 2.7: The solution with the different stages for $A_{20,\circ}$. The pivots are highlighted in medium grey and the anchor point is highlighted in light grey.

Given the solution of $A_{20,\circ}$, we can use it to find the solution of $A_{28,\circ}$. We use stage 1 of $A_{20,\circ}$ plus one extra move to vacate the anchor point. Next we use the first 2 moves of stage 2 for $A_{20,\circ}$ with two additional moves. Then we finish stage 2 with the last two moves of $A_{20,\circ}$. The first move of stage 3 vacates the anchor point and the next move refills it. This adds an additional two moves. Finally, we finish the solution with the last moves of $A_{20,\circ}$ in stage 3. The solution of $A_{28,\circ}$ is illustrated in Figure 2.8 and the extra moves needed at each stage are indicated by an asterisk in the stage column.

In general,

$$h_{20+8t,3}(A_{20+8t,\circ}, T_n) = 10 + 4t$$

and

$$h_{32+8t,3}(A_{32+8t,\circ}, T_n) = 16 + 4t$$

for $t \geq 0$. We have omitted the proofs for these cases because they are similar to previous proofs. The basis of this technique is to first find a base case with the proper anchor point and pivots. For larger n, we ignore proper coins and use the solution from previous case. The extra moves are needed to fill the anchor point at the end of stage 1, for the extra moves needed to obtain the correct anchor coin in stage 2, and to vacate the anchor point at the beginning of stage 3. How to choose the *proper* coins are explained further in [7], but they essential depend on the value of t.

An advantage of this technique is that it may be extended to larger values of k since many solutions for sorting A_n exhibit this recurrence with pivots, anchor points and stages. However, a further refinement of this technique is still needed in order to have a general solution for larger k.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	Stage
○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●							
○				○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	●	○	●				1
○	○	●	○	○	●				●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	○	●	●	○	●				1
○	○	●	○	○	●	●	○	●	●	○	●	○	●	○				○	●	○	●	○	●	○	●	○	●	●	○	●				1
○	○	●	○	○	●	●	○	●	●	○	●	○	●	○	○	●	○	○	●	○	●				●	○	●	●	○	●				1
○	○	●	○	○	●	●	○	●	●	○				○	○	●	○	○	●	○	●	●	○	●	●	○	●	●	○	●				1*
○	○	●	○	○	●	●	○	●	●	○				○	○				●	○	●	●	○	●	●	○	●	●	○	●	●	○	○	2
○	○	●	○	○	●	●	○	●	●	○				○	○	○	●	●	●	○	●	●	○	●	●	○	●	●				○	○	2
○	○	●	○	○	●	●	○	●	●	○				○	○	○	●	●	●	○	●	●	○				●	●	●	●	○	○	○	2*
○	○	●	○	○	●	●	○	●	●	○				○	○	○	●	●	●				○	○	●	●	●	●	●	●	○	○	○	2*
○	○	●	○	○				●	●	○				○	○	○	●	●	●	●	●	○	○	○	●	●	●	●	●	●	○	○	○	2
			○	○	○	○	●	●	●	○				○	○	○	●	●	●	●	●	○	○	○	●	●	●	●	●	●	○	○	○	2
			○	○	○	○	●	●	●	○	○	○	○	○	○	○	●	●	●	●	●				●	●	●	●	●	●	○	○	○	3*
			○	○	○	○				○	○	○	○	○	○	○	●	●	●	●	●	●	●	●	●	●	●	●	●	●	○	○	○	3
			○	○	○	○	○	○	○	○	○	○	○	○	○	○	●	●	●	●	●	●	●	●	●	●	●	●	●	●				3

Figure 2.8: The solution with the different stages for $A_{28,\circ}$. The ignored coins are highlighted in very light grey, the pivots are highlighted in medium grey and the anchor point is highlighted in dark grey.

Chapter 3

Berge *1-metric*

3.1 Matching up to Shifting

Transforming a string by moving one coin at a time is equivalent to moving the misplaced coins. Let $m(S^1, S^2) = \max_i m_i(S^1, S^2)$ where $m_i(S^1, S^2)$ is the number of matching coins between a pair of strings $S^1, S^2 \in \mathbf{S_n}$ for some shift i, $(i = -n, \ldots, n)$ of S^1. If $m(S^1, S^2) = p$, then $h_{n,1}(S^1, S^2) = n - p$. See Figure 3.1. If $m(S^1, S^2) = m_0(S^1, S^2) = p$, then $h_{n,1}(S^1, S^2) = (n - p) + 1$ (an extra move is required). See Figure 3.2. In general, finding $m(S^1, S^2)$ is equivalent to finding the minimum number of moves needed to transform a generic string of $\mathbf{S_n}$ into another.

3.2 Sorting a Generic String

The generalized Berge sorting deals with sorting A_n using Berge *k-moves*. Recall that $h_{n,1}(A_n, T_n) = \lceil \frac{n}{2} \rceil$ for $n \not\equiv 3 \pmod 4$ and $h_{n,1}(A_n, T_n) = \lfloor \frac{n}{2} \rfloor$ for $n \equiv 3 \pmod 4$. In this section, we evaluate the upper bound on $h_{n,1}(S_n, T_n)$

	...	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
S^2				○	○	●	○	●	○	●	●	●	○				
S^1					○	●	○	●	●	○	●	○	○	●			
				○	○	●	○	●	●		●	○	○	●			
				○	○	●	○	●		●	●	○	○	●			
				○	○	●	○	●	○	●	●		○	●			
				○	○	●	○	●	○	●	●	●	○				

Figure 3.1: A solution in 4 moves for $m(S^1, S^2) = 6$.

	...	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
S^2				○	○	●	●	○	●	○	○	●	●				
S^1				○	○	●	●	○	●	○	●	○	●				
				○	○	●	●	○	●	○	●		●	○			
				○	○	●	●	○	●	○		●	●	○			
				○	○	●	●	○	●	○	○	●	●				

Figure 3.2: A solution in 3 moves for $m(S^1, S^2) = m_0(S^1, S^2) = 8$.

for any string $S_n \in \mathbf{S_n}$.

Lemma 3.1. *Let $S_n \in \mathbf{S_n}$. If n is even, then $h_{n,1}(S_n, T_n) \leq \frac{n}{2}$.*

Proof. *Assume n is even. This implies that the number of matching coins between any pair of strings is always even for shift $i = 0$. If $\frac{n}{2}$ is odd, then there is no initial matching of $\frac{n}{2}$ coins. Assume $m_0(S_n, T_{n,\circ}) \geq \frac{n}{2} + 1$. Then $h_{n,1}(S_n, T_{n,\circ}) \leq n - \left(\frac{n}{2} + 1\right) + 1 = \frac{n}{2}$. Next, assume $m_0(S_n, T_{n,\circ}) \leq \frac{n}{2} - 1$. Then $m_0(S_n, T_{n,\bullet}) > n - \left(\frac{n}{2} - 1\right) = \frac{n}{2} + 1$. Thus,*

$$h_{n,1}(S_n, T_{n,\bullet}) \leq n - \left(\frac{n}{2} + 1\right) + 1 = \frac{n}{2}.$$

Next assume $\frac{n}{2}$ is even. If $m_0(S_n, T_{n,\circ}) > \frac{n}{2}$, then this implies that $m_0(S_n, T_{n,\circ}) \geq \frac{n}{2} + 2$ since there is always an even number of matching coins for $i = 0$. Moreover, we have $h_{n,1}(S_n, T_{n,\circ}) \leq n - \left(\frac{n}{2} + 2\right) + 1 = \frac{n}{2} - 1$. If $m_0(S_n, T_{n,\circ}) < \frac{n}{2} \leq \frac{n}{2} - 2$, then $m_0(S_n, T_{n,\bullet}) > n - \left(\frac{n}{2} - 2\right) > \frac{n}{2} + 2$ implying

$$h_{n,1}(S_n, T_{n,\bullet}) \leq n - \left(\frac{n}{2} - 2\right) + 1 = \frac{n}{2} - 1.$$

If $m_0(S_n, T_{n,\circ}) = \frac{n}{2}$, then we also have that $m_0(S_n, T_{n,\bullet}) = \frac{n}{2}$. So, shift S_n to the right one position. Now there are $n - 1$ coins to compare. If $m_1(S_n, T_{n,\circ}) = p$, then $m_1(S_n, T_{n,\bullet}) = (n-1) - p$. Thus, for $p < \left\lfloor \frac{n-1}{2} \right\rfloor$,

$$m_1(S_n, T_{n,\bullet}) = (n-1) - p \geq (n-1) - \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{n-1}{2} \right\rceil = \frac{n}{2}$$

since n is even implying $h_{n,1}(S_n, T_{n,\bullet}) \leq \frac{n}{2}$. **Q.E.D.**

Lemma 3.2. *Let $S_n \in \mathbf{S_n}$. If n is odd, then $h_{n,1}(S_n, T_n) \leq \left\lceil \frac{n}{2} \right\rceil$.*

Proof. *Assume n is odd. Note for n odd, the initial matching will always be odd. If $m_0(S_n, T_{n,\circ}) \geq \left\lceil \frac{n}{2} \right\rceil$, then*

$$h_{n,1}(S_n, T_{n,\circ}) \leq n - \left\lceil \frac{n}{2} \right\rceil + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n}{2} \right\rceil.$$

Next assume $m_0(S_n, T_{n,\circ}) < \lceil \frac{n}{2} \rceil \leq \lfloor \frac{n}{2} \rfloor$. *If* $n = 4i + 5$, *then* $\lfloor \frac{n}{2} \rfloor$ *is even implying* $m_0(S_n, T_{n,\circ}) \leq \lfloor \frac{n}{2} \rfloor - 1$. *Then*

$$m_0(S_n, T_{n,\bullet}) \geq n - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) = \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil + 1$$

which implies that

$$h_{n,1}(S_n, T_{n,\bullet}) \leq n - \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) + 1 = \left\lfloor \frac{n}{2} \right\rfloor .$$

If $n = 4i+3$, *then* $\lfloor \frac{n}{2} \rfloor$ *is odd. We separate* $m_0(S_n, T_{n,\circ}) \leq \lfloor \frac{n}{2} \rfloor$ *into two cases. If* $m_0(S_n, T_{n,\circ}) < \lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor - 2$ *(since the initial matching is always odd), then*

$$m_0(S_n, T_{n,\bullet}) > n - \left(\left\lfloor \frac{n}{2} \right\rfloor - 2\right) = \left\lceil \frac{n}{2} \right\rceil + 2.$$

This implies $h_{n,1}(S_n, T_{n,\bullet}) \leq \lceil \frac{n}{2} \rceil$ *moves. If* $m_0(S_n, T_{n,\circ}) = \lfloor \frac{n}{2} \rfloor$, *then*

$$m_0(S_n, T_{n,\bullet}) = n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$$

However, this contradicts the fact the initial matching is odd because $\lceil \frac{n}{2} \rceil$ *is even. Thus we need to split up the analysis as follows. If* $m_0(S_n, T_{n,\circ}) = \lfloor \frac{n}{2} \rfloor$ *and the* $\lceil \frac{n}{2} \rceil - th$ *coin of* S_n *is white (implying it is a matching in both* $T_{n,\circ}$ *and* $T_{n,\bullet}$*), then*

$$m_0(S_n, T_{n,\bullet}) = (n-1) - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + 1 = \left\lceil \frac{n}{2} \right\rceil + 1.$$

If $m_0(S_n, T_{n,\circ}) = \lfloor \frac{n}{2} \rfloor$ *and the* $\lceil \frac{n}{2} \rceil - th$ *coin of* S_n *is black, then*

$$m_0(S_n, T_{n,\bullet}) = (n-1) - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor .$$

Thus we need to shift S_n *to the right one position.*

If $m_1(S_n, T_{n,\circ}) \geq \lceil \frac{n}{2} \rceil$, *then we have a solution is at most* $\lceil \frac{n}{2} \rceil$ *moves. If* $m_1(S_n, T_{n,\circ}) < \lceil \frac{n}{2} \rceil \leq \lfloor \frac{n}{2} \rfloor$, *then*

$$m_0(S_n, T_{n,\bullet}) > n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$T_{15,\circ}$	∘	∘	∘	∘	∘	∘	∘	∘	•	•	•	•	•	•	•
S^1	•	•	∘	∘	∘	∘	•	•	•	∘	∘	∘	∘	•	•
$T_{14,\bullet}$	•	•	•	•	•	•	•	∘	∘	∘	∘	∘	∘	∘	∘

Figure 3.3: An example of a string in $\mathbf{S_{15}}$ requiring $\lceil \frac{15}{2} \rceil = 8$ Berge *1-moves.*

giving a desired matching that ensures a solution in at most $\lceil \frac{n}{2} \rceil$ *Berge 1-moves.*
Q.E.D

Recall that $h_{4i+3,1}(A_n, T_n) = \lfloor \frac{n}{2} \rfloor$. However, for $n = 15, 19$, and 23, computational results shows there exists strings in $\mathbf{S_n}$ which require $\lceil \frac{n}{2} \rceil$ Berge *1-moves.* For example, take the string S^1 shown in Figure 3.3 which has $m(S^1, T_{n,\circ}) = m(S^1, T_{n,\bullet}) = 7$, thus requires at least 8 moves.

Corollary 3.1. *For any pair of strings* $S^1, S^2 \in \mathbf{S_n}$, $h_{n,1}(S^1, S^2) \leq n + 1$.

Proof. Let $S^1, S^2 \in \mathbf{S_n}$. *Using triangle inequality of* $h_{n,1}(S^1, S^2)$, *we have*

$$h_{n,1}(S^1, S^2) \leq h_{n,1}(S^1, T_n) + h_{n,1}(T_n, S^2).$$

By symmetry, we obtain

$$h_{n,1}(S^1, S^2) \leq h_{n,1}(S^1, T_n) + h_{n,1}(S^2, T_n).$$

Lemma 3.1 and 3.2 imply

$$h_{n,1}(S^1, S^2) \leq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil = n + 1.$$

Therefore, for any pair of strings in $\mathbf{S_n}$, *we can transform one string into another in a finite number of Berge 1-moves.* **Q.E.D.**

3.3 Upper Bound for $H_{n,1}(\mathbf{S_n})$

The following proof was given by Péter Sziklai [13].

Lemma 3.3. $H_{n,1}(\mathbf{S_n}) \leq \lfloor \frac{3n}{4} \rfloor$.

Proof. Let $S^1, S^2 \in \mathbf{S_n}$. *Since each white coin is matched* $\lceil \frac{n}{2} \rceil$ *times and each black coin is matched* $\lfloor \frac{n}{2} \rfloor$ *times, we have*

$$\sum_{i=-n+1}^{n-1} m_i(S^1, S^2) = \left(\left\lceil \frac{n}{2} \right\rceil \times \left\lceil \frac{n}{2} \right\rceil\right) + \left(\left\lfloor \frac{n}{2} \right\rfloor \times \left\lfloor \frac{n}{2} \right\rfloor\right) \geq \frac{n^2}{2}.$$

For at least one i_0, $m_{i_0}(S^1, S^2) > \frac{n^2}{2} \times \frac{1}{2n} = \frac{n}{4}$. *Since* $m_{i_0}(S^1, S^2)$ *is an integer, we have* $m_{i_0}(S^1, S^2) \geq \lceil \frac{n}{4} \rceil$. *Thus,* $h_{n,1}(S^1, S^2) \leq n - \lceil \frac{n}{4} \rceil = \lfloor \frac{3n}{4} \rfloor$ *Therefore,* $H_{n,1}(\mathbf{S_n}) \leq \lfloor \frac{3n}{4} \rfloor$. **Q.E.D.**

Noticing that for $|i| \geq \frac{n}{2}$, the value of $m_i(S^1, S^2)$ is at most $(n - |i|)$, we consider $k \leq \frac{n}{2}$ and write

$$\frac{n^2}{2} = \sum_{i=-n}^{-(n-k)} m_i(S^1, S^2) + \sum_{i=-(n-k)}^{(n-k)} m_i(S^1, S^2) + \sum_{i=(n-k)}^{n} m_i(S^1, S^2).$$

In the first and third term, $m_i(S^1, S^2)$ is less than $(n - |i|)$. Thus we have

$$\frac{n^2}{2} \leq \sum_{i=-(n-k)}^{(n-k)} m_i(S^1, S^2) + 2 \sum_{i=(n-k)}^{n} i.$$

This simplifies to

$$\frac{n^2}{2} - k\,(k+1) \leq \sum_{i=-(n-k)}^{(n-k)} m_i(S^1, S^2).$$

The right hand side of the inequality has $2\,(n-k)+1$ terms. It yields that its smallest term is at least

$$\frac{\frac{n^2}{2} - k\,(k+1)}{2\,(n-k)}.$$

If we choose $k = \frac{n}{2}$, we get that the smallest term is at least $\frac{n}{4}$. Choosing $k = \frac{n}{4}$ gives

$$\frac{\frac{n^2}{2} - \frac{n^2}{16}}{2\left(n - \frac{n}{2}\right)} = \frac{\frac{7n^2}{16}}{\frac{3n}{2}} = \frac{7n}{24}.$$

The following was suggested by Pólik [11] to find the best fraction (ignoring that n is an integer). We begin by letting $k = f(n)$. We wish to show that

$$\frac{n^2 - 2k^2}{4(n-k)} = \frac{n^2 - 2f^2(n)}{4(n - f(n))} \sim cn.$$

Dividing by n^2,

$$\frac{n^2 - 2f^2(n)}{4n(n - f(n))} = \frac{1 - 2\left(\frac{f(n)}{n}\right)^2}{4\left(1 - \left(\frac{f(n)}{n}\right)\right)}.$$

Note $f(n) < n$ and $0 < \frac{f(n)}{n} < 1$. Assume $\lim_{n\to\infty} \frac{f(n)}{n} = \alpha$. Then

$$\frac{1 - 2\left(\frac{f(n)}{n}\right)^2}{4\left(1 - \left(\frac{f(n)}{n}\right)\right)} \to \frac{1 - 2\alpha^2}{4(1-\alpha)} \text{ as } n \to \infty.$$

To maximize, we take its derivative in α and solve when equal to zero. After simplification, we have

$$\frac{4(1 - 4\alpha + 2\alpha^2)}{16(1 - 2\alpha - \alpha^2)} = 0$$

giving

$$1 - 4\alpha + 2\alpha^2 = 0$$

with a maximum at $\alpha = 1 - \frac{\sqrt{2}}{2}$. If $f(n) = log(n)$, then $\lim_{n\to\infty} \frac{f(n)}{n} = 0$ implies

$$\frac{1 - 2\left(\frac{f(n)}{n}\right)^2}{4\left(1 - \left(\frac{f(n)}{n}\right)\right)} = \frac{1}{4}.$$

If $f(n) = an^b$ with $b < 1$, then we also get $\frac{1}{4}$. For $b = 1$, we get $a = 1 - \frac{\sqrt{2}}{2}$ and plugging this in gives the ratio

$$\frac{n^2 - 2\left(1 - \frac{\sqrt{2}}{2}\right)^2 n^2}{4\left(n - \left(1 - \frac{\sqrt{2}}{2}\right)n\right)} = \frac{n^2 - 2n^2 + 2\sqrt{2}n^2 - n^2}{2\sqrt{2}n} = \left(1 - \frac{\sqrt{2}}{2}\right)n.$$

By choosing $k = \left(1 - \frac{\sqrt{2}}{2}\right)n$, we get that there exists a shift i such that $m_i(S^1, S^2) > \left(1 - \frac{\sqrt{2}}{2}\right)n$ implying $H_{n,1}(\mathbf{S_n}) \leq n - \left(1 - \frac{\sqrt{2}}{2}\right)n = \frac{\sqrt{2}}{2}n$.

This approach proposed by Péter Sziklai could easily be extended to a set of strings made of xn white coins and $(1-x)\,n$ black coins where $0 \leq x \leq 1$. Then for any pair of such strings P^1, P^2,

$$\sum_{i=-n+1}^{n-1} m_i(P^1, P^2) = (xn)^2 + ((1-x)\,n)^2.$$

3.4 Lower Bound for $H_{n,1}(\mathbf{S_n})$

In this section, we are trying to identify a class of strings requiring the most number of moves. In doing so, we construct strings which have the least number of matching coins between each other for any shift i. We first note that transforming $T_{n,\circ}$ into $T_{n,\bullet}$, we need exactly $\lceil \frac{n}{2} \rceil$ moves because $m(T_{n,\circ}, T_{n,\bullet}) = \lceil \frac{n}{2} \rceil$ for shift $i = \lfloor \frac{n}{2} \rfloor$.

Now imagine we slice $T_{n,\circ}$ and $T_{n,\bullet}$ in thirds such that the outer left and right slices have $\lfloor \frac{n}{3} \rfloor$ coins respectively and the middle slice has $\lceil \frac{n}{3} \rceil$ coins. Now we swap the position of the black and white coins in the middle slice of $T_{n,\circ}$ as shown in Figure 3.4.

no. of coins	$\lfloor \frac{n}{3} \rfloor$	$\lceil \frac{n}{3} \rceil$	$\lfloor \frac{n}{3} \rfloor$
$T_{n,\bullet}$	$\bullet \ \dots \ \bullet$	$\bullet \ \dots \ \bullet \ \bullet \ \circ \ \circ \ \dots \ \circ$	$\circ \ \dots \ \circ$
$T_{n,\circ}$	$\circ \ \dots \ \circ$	$\bullet \ \dots \ \bullet \ \bullet \ \circ \ \circ \ \dots \ \circ$	$\bullet \ \dots \ \bullet$

Figure 3.4: Slicing the board into thirds

In the middle slice, we already have $\lceil \frac{n}{3} \rceil$ matching coins for shift $i = 0$. The matching in the middle slice is the best matching of the three slice. As well, if we shift $T_{n,\circ}$, then the best matching is still $\lceil \frac{n}{3} \rceil$. Using this example, our goal will be to minimize the number of matching coins by minimizing the number of coins in the outer slices and minimizing the number of matching coins in the middle slice.

3.4.1 Lower Bound for $H_{n,1}(\mathbf{S_n})$ for n even

We divide up the values of n into 3 groups: $n = 6 + 6i$, $n = 8 + 6i$ and $n = 16 + 6i$ for $i \geq 0$.

For $n = 6 + 6i$, $i \geq 0$, we will show that there exist strings $S_*^1, S_*^2 \in \mathbf{S_n}$ such that $m(S_*^2, S_*^1) = \lfloor \frac{n}{3} \rfloor - \lfloor \frac{i}{5} \rfloor$. We begin by letting string $S_*^1, S_*^2 \in \mathbf{S_n}$ as shown in Figure 3.5(A). If $\lfloor \frac{i}{5} \rfloor = 0$, then we need to make one swap of the white and black coin in positions $\frac{n}{2}$ and $\frac{n}{2} + 1$ of string S_*^1 as shown in Figure 3.5(B). This will ensure that no matter how we shift S_*^2 we can only get a $\frac{n}{3}$ matching.

In general for $n = 6 + 6i$, $i \geq 0$, we need to make $3 \times \lfloor \frac{i}{5} \rfloor$ pairs of alternating white and black coins in S_*^1 about the center of the string giving an alternating sequence of white and black coins of length $2 \times 3 \times \lfloor \frac{i}{5} \rfloor$ (Figure 3.5(C)). This reduces the number of initial matchings in the center slice to be less than or equal to the number of coins in the outer slices of S_*^2. As a consequence, this also restricts the number of matching coins we can obtain for any shift i of S_*^2. For instance, the maximum number of matching coins among the center slice is achieved by shifting S_*^2 to the left or right by $3 \times \lfloor \frac{i}{5} \rfloor$ positions in which case $m(S_*^2, S_*^1) = \frac{n}{3} - \lfloor \frac{i}{5} \rfloor$ matchings.

In a similar manner, we can construct strings $S_*^1, S_*^2 \in \mathbf{S_n}$ for $n = 16 + 6i$, $i \geq 0$, such that $m(S_*^2, S_*^1) = \lfloor \frac{n}{3} \rfloor - \lfloor \frac{i}{5} \rfloor$. The only difference is that we need to have $1 + \left(3 \times \lfloor \frac{i}{5} \rfloor\right)$ pairs of alternating white and black coins in the center slice about the $\frac{n}{2} - th$ position of S_*^1 as illustrated in Figure 3.7(A).

To find the value of a in Figure 3.7(A), we note that the center slice has a length of

$$2a + 2\left(1 + \left(3 \times \left\lfloor \frac{i}{5} \right\rfloor\right)\right) = \left\lceil \frac{n}{3} \right\rceil + 2 \times \left\lfloor \frac{i}{5} \right\rfloor .$$

Solving for a, we get

$$a = \left\lfloor \frac{n}{6} \right\rfloor - \left(2 \times \left\lfloor \frac{i}{5} \right\rfloor\right) .$$

Note that for $n = 10$, the least number of matching coins is $\lfloor \frac{10}{3} \rfloor + 1 = 4$ for strings S_*^1 and S_*^2 given in Figure 3.6. For this reason, we begin this case at $n = 16$.

Next we examine $n = 8 + 6i$, $i \geq 0$. The string S_*^2 has $\lfloor \frac{n}{3} \rfloor - \lfloor \frac{i-3}{5} \rfloor$ white coins, followed by $\lceil \frac{n}{6} \rceil + \lfloor \frac{i-3}{5} \rfloor$ black coins, followed by $\lceil \frac{n}{6} \rceil + \lfloor \frac{i-3}{5} \rfloor$ white coins, and finally, followed by the remaining $\lfloor \frac{n}{3} \rfloor - \lfloor \frac{i-3}{5} \rfloor$ black coins. Note the for $i = 0, 1, 2$, $\lfloor \frac{i-3}{5} \rfloor$ is negative. The reason for the $i - 3$ is that the first three values of n need to have $\lfloor \frac{n}{3} \rfloor + 1$ coins in the outer slices to minimize the number of matchings. For $i = 0, 1, 2$, $S_*^1 = T_{n,\bullet}$. If we start at values of n which satisfy

positions	$1 \ \dots \ \frac{n}{3} - \lfloor \frac{i}{5} \rfloor$	$\dots \ \frac{n}{2}-1 \ \ \frac{n}{2} \ \ \frac{n}{2}+1 \ \ \frac{n}{2}+2 \ \dots$	$n - \frac{n}{3} - \lfloor \frac{i}{5} \rfloor \ \dots \ n$
S_*^1	● ... ●	● ... ● ● ○ ○ ... ○	○ ... ○
S_*^2	○ ... ○	● ... ● ● ○ ○ ... ○	● ... ●

(A)

positions	$1 \ \dots \ \frac{n}{3} - \lfloor \frac{i}{5} \rfloor$	$\dots \ \frac{n}{2}-1 \ \ \frac{n}{2} \ \ \frac{n}{2}+1 \ \ \frac{n}{2}+2 \ \dots$	$n - \frac{n}{3} - \lfloor \frac{i}{5} \rfloor \ \dots \ n$
S_*^1	● ... ●	● ... ● ○ ● ○ ... ○	○ ... ○
S_*^2	○ ... ○	● ... ● ● ○ ○ ... ○	● ... ●

(B)

no. of coins	$\lfloor \frac{n}{3} \rfloor - \lfloor \frac{i}{5} \rfloor$	$\frac{n}{6} - (3 \times \lfloor \frac{i}{5} \rfloor)$	$2 \times 3 \times \lfloor \frac{i}{5} \rfloor$	$\frac{n}{6} - (3 \times \lfloor \frac{i}{5} \rfloor)$	$\lfloor \frac{n}{3} \rfloor - \lfloor \frac{i}{5} \rfloor$
S_*^1	● ... ●	● ... ●	○ ● ... ○ ● ... ○ ●	○ ... ○	○ ... ○
S_*^2	○ ... ○	● ... ●	● ● ... ● ○ ... ○ ○	○ ... ○	● ... ●

(C)

Figure 3.5: (A) shows the initial layout of the two strings, (B) shows the swap of the two middle coins in S_*^1, and (C) gives the general setting for larger n where $n = 6 + 6i, i \geq 0$.

	1	2	3	4	5	6	7	8	9	10
S_*^1	●	●	●	●	●	○	○	○	○	○
S_*^2	○	○	○	○	●	○	●	●	●	●

Figure 3.6: Two strings, $S_*^1, S_*^2 \in \mathbf{S_{10}}$ such that $m(S_*^2, S_*^1) = 4$.

$\lfloor \frac{i-3}{5} \rfloor = 0$, then we need have two pairs of alternating white and black coins about the center position of S_*^1. For values of i that satisfy $\lfloor \frac{i-3}{5} \rfloor = 1$, we need to have 4 pairs of alternating white and black coins about the $\frac{n}{2} - th$ position in S_*^1. Continuing in this manner, we need have have $\left(3 \times \lfloor \frac{i-3}{5} \rfloor\right) + 2$ pairs of alternating white and black coins in the center slice. Thus, we can construct the following strings S_*^1 and S_*^2 as illustrated in Figure 3.7(B).

The center slice has $\lceil \frac{n}{3} \rceil + 2 \times \lfloor \frac{i-3}{5} \rfloor$ white and black coins. To calculate the value of b, we set-up the following equation and solve for b:

$$2b + 2\left(\left(3 \times \left\lfloor \frac{i-3}{5} \right\rfloor\right) + 2\right) = \left\lceil \frac{n}{3} \right\rceil + 2 \times \left\lfloor \frac{i-3}{5} \right\rfloor.$$

After simplification of the above equation, we get that

$$b = \left\lceil \frac{n}{6} \right\rceil - 2 \times \left\lfloor \frac{i-3}{5} \right\rfloor + 2.$$

3.4.2 Lower Bound for $H_{n,1}(\mathbf{S_n})$ for n odd

To construct strings S_*^1 and S_*^2 for $n = 5 + 6i, i \geq 0$, we use the strings from $n = 6 + 6i$ and delete the first black coin in S_*^1 and the last black coin in S_*^2. This gives us the correct number of black and white coins in both strings. As well, since the first one third slice of $n = 5 + 6i$ has the same number of white coins as $n = 6 + 6i$, we have that for $n = 5 + 6i$, $m(S_*^2, S_*^1) = \lfloor \frac{n}{3} \rfloor - \lfloor \frac{i}{5} \rfloor$.

no. of coins	$\lfloor \frac{n}{3} \rfloor - \lfloor \frac{i}{5} \rfloor$	a	$2 \times \left(1 + 3 \times \lfloor \frac{i}{5} \rfloor\right)$	a	$\lfloor \frac{n}{3} \rfloor - \lfloor \frac{i}{5} \rfloor$
S_*^1	● … ●	● … ●	○ ● … ○ ● … ○ ●	○ … ○	○ … ○
S_*^2	○ … ○	● … ●	● ● … ● ○ … ○ ○	○ … ○	● … ●

(A)

no. of coins	$\lfloor \frac{n}{3} \rfloor - \lfloor \frac{i-3}{5} \rfloor$	b	$2 \times \left(\left(3 \times \lfloor \frac{i-3}{5} \rfloor\right) + 2\right)$	b	$\lfloor \frac{n}{3} \rfloor - \lfloor \frac{i-3}{5} \rfloor$
S_*^1	● … ●	● … ●	○ ● … ○ ● … ○ ●	○ … ○	○ … ○
S_*^2	○ … ○	● … ●	● ● … ● ○ … ○ ○	○ … ○	● … ●

(B)

Figure 3.7: (A) has the worst case for $n = 16 + 6i$ where $i = 0, 1, 2, \ldots$, and (B) shows the worst case for $n = 8 + 6i$ for $i = 0, 1, 2, \ldots$ for Berge *1-moves*.

The same idea also holds for $n = 7 + 6i, i \geq 0$. Deleting the first coin and last coin of S^1_* and S^2_* constructed for $n = 8 + 6i$, respectively, gives $m(S^1_*, S^2_*) = \lfloor \frac{n}{3} \rfloor - \lfloor \frac{i-3}{5} \rfloor$ for $n = 7 + 6i, i \geq 0$.

As well, for $n = 15 + 6i$, we begin by using strings S^1_* and S^2_* constructed for $n = 16 + 6i$ and delete the first coin of S^1 and last coin of S^2. Then $m(S^2_*, S^1_*) = \lfloor \frac{n}{3} \rfloor - \lfloor \frac{i}{5} \rfloor$ for $15 + 6i, i \geq 0$.

If we express i in terms of n for each $n = 5 + 6i$, $6 + 6i$, $7 + 6i$, $8 + 6i$, $15 + 6i$, and $16 + 6i$, $i \geq 0$ and take the minimum over all values of i, we have the following lemma.

Lemma 3.4. *There exist strings $S^1_*, S^2_* \in \mathbf{S_n}$ such that $m(S^1_*, S^2_*) = \lfloor \frac{n}{3} \rfloor - \lfloor \frac{n}{30} \rfloor$ which implies $H_{n,1}(\mathbf{S_n}) \geq \lceil \frac{2n}{3} \rceil + \lfloor \frac{n}{30} \rfloor$.*

Combining the upper and lower bounds for $H_{n,1}(\mathbf{S_n})$, we have

$$0.700n \leq H_{n,1}(\mathbf{S_n}) \leq 0.707n.$$

Chapter 4

Berge *2-metric*

4.1 Upper Bound for $H_{n,2}(\mathbf{S_n})$

We begin by considering the associated Berge *2-metric* of $\mathbf{S_n}$. Assume that we can transform any string $S_n \in \mathbf{S_n}$ into $T_{n,\circ}$ in a finite number of moves, i.e., $h_{n,2}(S_n, T_{n,\circ}) \leq an$ for some a. Fix b coins in the first b positions, and transform the remaining $n - b$ coins. Then

$$h_{n,2}(S_n, T_{n,\circ}) \leq a\,(n-b) + c \leq a$$

where c is the number of moves need to merge the fixed b coins into the sorted string on $n - b$ coins. Taking $a = 1, b = 4$ and $c = 4$ gives us the following lemma.

Lemma 4.1. *Let $S_n \in \mathbf{S_n}$. Then $h_{n,2}(S_n, T_{n,\circ}) \leq n$.*

Proof. Inductive hypothesis: It takes less than n to sort any string $S_n \in \mathbf{S_n}$ into $T_{n,\circ}$ for $n \leq 4m$ and the result is shift by two positions to the right. Base Case: For $m = 2$, any string $S_n \in \mathbf{S_n}$ for $n = 5, 6, 7, 8$ can be

1	2	3	4	5	6	7	8	...	$\lceil\frac{n-4}{2}\rceil - 2$	$\lceil\frac{n-4}{2}\rceil - 1$	$\lceil\frac{n-4}{2}\rceil$	$\lfloor\frac{n-4}{2}\rfloor$	...	$n+1$	$n+2$
$\circ$	$\circ$	$\bullet$	$\bullet$			$\circ$	$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
$\circ$	$\circ$	$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$			$\bullet$	...	$\bullet$	$\bullet$
$\circ$	$\circ$			$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$
		$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$

1	2	3	4	5	6	7	8	...	$\lceil\frac{n-4}{2}\rceil - 2$	$\lceil\frac{n-4}{2}\rceil - 1$	$\lceil\frac{n-4}{2}\rceil$	$\lfloor\frac{n-4}{2}\rfloor$	...	$n+1$	$n+2$
$\bullet$	$\bullet$	$\circ$	$\circ$			$\circ$	$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$			$\bullet$	...	$\bullet$	$\bullet$
		$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$

Figure 4.1: Solutions when the first 4 coins are $\circ\circ\bullet\bullet$ or $\bullet\bullet\circ\circ$.

sorted into $T_{n,\circ}$ in at most n moves shifted two positions to the right. The solutions of these strings are given in Appendix A. Induction: $m \to m+1$. Ignore the first 4 coins of S_n and sort S_{n-4} into $T_{n-4,\circ}$ using the induction (i.e. in $n-4$ moves and it is shifted by 2).

Next we merge the first 4 untouched coins into the $T_{n-4,\circ}$. For this, we have six cases to consider. The first 4 coins can be positioned as follows: $\bullet\bullet\circ\circ$, $\circ\circ\bullet\bullet$ $\circ\bullet\bullet\circ$, $\bullet\circ\circ\bullet$, $\bullet\circ\bullet\circ$ or $\circ\bullet\circ\bullet$. The first two cases are shown in Figure 4.1 with solutions taking less than 4 moves and shifted to 2 positions to the right.

The solutions with $\circ\bullet\bullet\circ$ and $\bullet\circ\circ\bullet$ as the first four coins are given in Figure 4.2 and need at most 4 moves with the final string shifted to the right by two positions.

The last two cases need 4 moves such that the final strings are shifted to the right by two positions. Their solutions are given in Figure 4.3.

1	2	3	4	5	6	7	8	...	$\lceil\frac{n-4}{2}\rceil - 2$	$\lceil\frac{n-4}{2}\rceil - 1$	$\lceil\frac{n-4}{2}\rceil$	$\lfloor\frac{n-4}{2}\rfloor$	...	$n+1$	$n+2$
$\circ$	$\bullet$	$\bullet$	$\circ$			$\circ$	$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
$\circ$	$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$			$\bullet$	...	$\bullet$	$\bullet$
$\circ$			$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$
$\circ$	$\circ$	$\circ$	$\circ$			$\circ$	$\circ$	...	$\circ$	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$
		$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$

1	2	3	4	5	6	7	8	...	$\lceil\frac{n-4}{2}\rceil - 2$	$\lceil\frac{n-4}{2}\rceil - 1$	$\lceil\frac{n-4}{2}\rceil$	$\lfloor\frac{n-4}{2}\rfloor$	...	$n+1$	$n+2$
$\bullet$	$\circ$	$\circ$	$\bullet$			$\circ$	$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
		$\circ$	$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
$\circ$	$\circ$	$\circ$	$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	...	$\circ$			$\bullet$	...	$\bullet$	$\bullet$
$\circ$	$\circ$	$\circ$			$\circ$	$\circ$	$\circ$	...	$\circ$	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$
		$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$

Figure 4.2: Solutions when the first 4 coins are $\circ\bullet\bullet\circ$ and $\bullet\circ\circ\bullet$.

1	2	3	4	5	6	7	8	...	$\lceil\frac{n-4}{2}\rceil - 2$	$\lceil\frac{n-4}{2}\rceil - 1$	$\lceil\frac{n-4}{2}\rceil$	$\lfloor\frac{n-4}{2}\rfloor$	...	$n+1$	$n+2$
$\bullet$	$\circ$	$\bullet$	$\circ$			$\circ$	$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
$\bullet$			$\circ$	$\circ$	$\bullet$	$\circ$	$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$			$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$			$\bullet$	...	$\bullet$	$\bullet$
		$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\circ$	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$

1	2	3	4	5	6	7	8	...	$\lceil\frac{n-4}{2}\rceil - 2$	$\lceil\frac{n-4}{2}\rceil - 1$	$\lceil\frac{n-4}{2}\rceil$	$\lfloor\frac{n-4}{2}\rfloor$	...	$n+1$	$n+2$
$\circ$	$\bullet$	$\circ$	$\bullet$			$\circ$	$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
$\circ$			$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	...	$\circ$	$\circ$	$\circ$	$\bullet$	...	$\bullet$	$\bullet$
$\circ$	$\circ$	$\circ$	$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	...	$\circ$			$\bullet$	...	$\bullet$	$\bullet$
$\circ$	$\circ$	$\circ$			$\circ$	$\circ$	$\circ$	$\circ$	...	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$
		$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	...	$\bullet$	$\bullet$	$\bullet$	...	$\bullet$	$\bullet$

Figure 4.3: Solutions when the first 4 coins are $\bullet\circ\bullet\circ$ and $\circ\bullet\circ\bullet$.

Therefore, at most 4 moves are needed to merge the first four coins giving a total of $(n-4)+4=n$ *Berge 2-moves.* **Q.E.D.**

By Lemma 4.1, we have the following upper bound on $H_{n,2}(\mathbf{S_n})$.

Lemma 4.2. $H_{n,2}(\mathbf{S_n}) \leq 2n$.

Proof. Let $S^1, S^2 \in \mathbf{S_n}$. *By definition, we have*

$$H_{n,2}(\mathbf{S_n}) = \max_{(S^1,S^2 \in \mathbf{S_n})} h_{n,2}(S^1, S^2).$$

Using the triangle inequality of $h_{n,2}(S^1, S^2)$, *we obtain*

$$H_{n,2}(\mathbf{S_n}) \leq \max_{(S^1,S^2 \in \mathbf{S_n})} [h_{n,2}(S^1, T_{n,\circ}) + h_{n,2}(T_{n,\circ}, S^2)].$$

By distribution and using symmetry of $h_{n,2}(S^1, S^2)$, *we get*

$$\begin{aligned} H_{n,2}(\mathbf{S_n}) &\leq \max_{(S^1 \in \mathbf{S_n})} h_{n,2}(S^1, T_{n,\circ}) + \max_{(S^2 \in \mathbf{S_n})} h_{n,2}(T_{n,\circ}, S^2) \\ &= \max_{(S^1 \in \mathbf{S_n})} h_{n,2}(S^1, T_{n,\circ}) + \max_{(S^2 \in \mathbf{S_n})} h_{n,2}(S^2, T_{n,\circ}) \end{aligned}$$

By Lemma 4.1, $\max_{(S^1 \in \mathbf{S_n})} h_{n,2}(S^1, T_{n,\circ}) \leq n$ *and* $\max_{(S^2 \in \mathbf{S_n})} h_{n,2}(S^2, T_{n,\circ}) \leq n$ *implying*

$$\begin{aligned} H_{n,2}(\mathbf{S_n}) &\leq n + n \\ &= 2n. \end{aligned}$$

Q.E.D.

4.2 Exploiting Symmetries of $\mathbf{S_n}$

Unlike Berge *1-metric*, it is much more difficult to compute $H_{n,k}(\mathbf{S_n})$ for $k > 1$. The size of $\mathbf{S_n}$ is

$$\binom{n}{\lceil \frac{n}{2} \rceil}.$$

This implies that for $H_{n,k}(\mathbf{S_n})$ we need to make

$$\begin{pmatrix} \begin{pmatrix} n \\ \lceil \frac{n}{2} \rceil \end{pmatrix} \\ 2 \end{pmatrix}$$

computations. Thus, we have a combinatorial explosion. As a result, $H_{n,k}(\mathbf{S_n})$ is only computable for small values of n.

For n even, we can compute $H_{n,k}(\mathbf{S_n})$ for strings beginning with a white coin. Then by interchanging the white and black coins, we will get a solution for transforming strings beginning with a black coin. For example, if we can transform ○●○○●● into ○○●○●●, then we can transform ●○●●○○ into ●●○●○○. However, exploiting the symmetry of $\mathbf{S_n}$ is not enough to significantly reduce the computation time.

4.3 Preliminary Computations

4.3.1 Computing $H_{6,2}(\mathbf{S_6})$, $H_{8,2}(\mathbf{S_8})$ and $H_{10,2}(\mathbf{S_{10}})$

The computational results for the first few even values of n give interesting insight into Berge *2-metric* (Table 4.1). In particular, the extreme case for $n = 6$ is transforming $A_{6,\bullet}$ into $A_{6,\circ}$ which requires a minimum of 6 Berge *2-moves*. For example, one solution is

$$B_{6,2}(A_{6,\bullet}, A_{6,\circ}) = \{-1 \ \ 3 \ \ 0\} \cup \{7 \ \ 4\} \cup \{0 \ \ 6\} \cup \{4 \ \ -1\} \cup \{6 \ \ 1\}.$$

This may be due to the fact that there is small set of possible moves required to correct 6 misplaced coins. In contrast to $k = 1$, $h_{n,1}(A_{n,\bullet}, A_{n,\circ}) = 1$ for all n.

n	$H_{n,2}(\mathbf{S_n})$
6	6
8	6
10	6

Table 4.1: Computed $H_{n,2}(\mathbf{S_n})$ for even values of n, $6 \leq n \leq 10$.

As n increases, transforming $A_{n,\circ}$ into $A_{n,\bullet}$ has a nice inductive solution requiring $\frac{n}{2}+1$ Berge *2-moves* which is given in section 4.3.2. Given that $H_{10,2}(\mathbf{S_{10}}) = 6 = \frac{10}{2}+1$, $H_{n,k}(\mathbf{S_n})$ does not tend to fluctuate up and down, and $H_{n,2}(\mathbf{S_n})$ should be close for both odd and even values of n, it is probable that $H_{n,2}(\mathbf{S_n}) = \frac{n}{2}+1$. However, this is merely speculation and further computation is necessary.

4.3.2 Transforming $A_{n,\bullet}$ into $A_{n,\circ}$

Case 1: For $n = 8+4i, i \geq 0$, the following is a solution in $\frac{n}{2}+1$ Berge *2-moves*.

The base case is $B_{8,2}(A_{8,\bullet}, A_{8,\circ}) = \{\ -1\ 2\ 5\ 1\ 4\ 7\ \}$. Let $n = 8+4i, i > 0$ and assume $B_{4+4i,2}(A_{4+4i,\bullet}, A_{4+4i,\circ})$ has a solution in $\frac{4i+4}{2}+1 = 2i+3$ moves. Ignore coins in positions $2i+3$, $2i+4$, $4i+7$, and $4i+8$ of $A_{n,\bullet}$. Next, transform the remaining coins using the first $2i+1$ moves of $B_{4+4i,2}(A_{4+4i,\bullet}, A_{4+4i,\circ})$. We complete the solution with the following 4 moves:

1. For $i=1$, we complete the solution with $\{\ 1\ \ 2i{+}2\ \ 4i{+}5\ \ 2i{+}4\ \ 4i{+}7\ \}$.

2. For $i > 1$, we use $\{\ 4i+3\ \ 2i+2\ \ 4i+5\ \ 2i+4\ \ 4i+7\ \}$.

$B_{8,2}(A_{8,\bullet}, A_{8,\circ})$	=	{	−1	2	5	1	4	7 }						
$B_{12,2}(A_{12,\bullet}, A_{12,\circ})$	=	{	−1	2	7	1	4	9	6	11 }				
$B_{16,2}(A_{16,\bullet}, A_{16,\circ})$	=	{	−1	2	9	1	4	11	6	13	8	15 }		
$B_{20,2}(A_{20,\bullet}, A_{20,\circ})$	=	{	−1	2	11	1	4	13	6	15	8	17	10	19 }

Table 4.2: Solutions for transforming $A_{n,\bullet}$ into $A_{n,\circ}$ using Berge *2-moves* for $n = 8 + 4i$, $0 \leq i \leq 3$.

The solution of $A_{8+4i,\bullet}$ takes $(2i + 1) + 4$ moves which equals $\frac{n}{2} + 1$ (take $i = \frac{n-8}{4}$ and plug into $(2i + 1) + 4$). The first few values of $n = 8 + 4i, i \geq 0$, are given in Table 4.2. We also demonstrate this induction for transforming $A_{12,\bullet}$ into $A_{12,\circ}$ using the solution of $A_{8,\bullet}$ in Figure 4.4.

Case 2: For $n = 10 + 4i, i \geq 0$, the following is a solution in $\frac{n}{2} + 1$ Berge *2-moves*.

The base case is $B_{10,2}(A_{10,\bullet}, A_{10,\circ}) = \{ -1\ 2\ 5\ 0\ 7\ 4\ 9 \}$. For $i > 0$, we let $n = 10 + 4i$ and assume $B_{n-4,2}(A_{n-4,\bullet}, A_{n-4,\circ})$ has a solution in

$$\frac{n-4}{2} + 1 = 4 + 2i$$

moves. Ignore coins in positions $4i+1$, $4i+2$, $4i+9$, and $4i+10$. Then we transform the remaining coins using the first $2i+2$ moves of $B_{n-4,2}(A_{n-4,\bullet}, A_{n-4,\circ})$. Then we complete the solution with the following 4 moves:

$$\{\ 4i+5\ \ 2i+2\ \ 4i+7\ \ 2i+4\ \ 4i+9\ \}.$$

Thus, to transform $A_{10+4i,\bullet}$ into $A_{10+4i,\circ}$ we need $(2i+2)+4$ moves, or equivalently, $\frac{n}{2} + 1$ moves. Table 4.3 has the first few solutions for $n = 10 + 4i$ where $i \geq 0$.

...	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	...
			●	○	●	○	●	○	●	○	●	○	●	○	

⇓

Ignore coins in 5, 6,11 and 12, and re-number the positions to use the first three moves of $B_{8,2}(A_{8,\bullet}, A_{8,\circ})$.

...	−1	0	1	2	3	4	~~5~~	~~6~~	~~7~~ → 5	~~8~~ → 6	~~9~~ → 7	~~10~~ → 8	~~11~~	~~12~~	...
			●	○	●	○	~~●~~	~~○~~	●	○	●	○	~~●~~	~~○~~	
	○	●	●			○	~~●~~	~~○~~	●	○	●	○	~~●~~	~~○~~	
	○	●	●	●	○	○	~~●~~	~~○~~			●	○	~~●~~	~~○~~	
	○	●			○	○	~~●~~	~~○~~	●	●	●	○	~~●~~	~~○~~	

⇓

Complete the solution with { 1 4 9 6 11 }.

...	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	...
			●	○	●	○	●	○	●	○	●	○	●	○	
	○	●	●			○	●	○	●	○	●	○	●	○	
	○	●	●	●	○	○	●	○			●	○	●	○	
	○	●			○	○	●	○	●	●	●	○	●	○	
	○	●	○	●	○			○	●	●	●	○	●	○	
	○	●	○	●	○	●	○	○	●	●			●	○	
	○	●	○	●	○	●	○			●	○	●	●	○	
	○	●	○	●	○	●	○	●	○	●	○	●			

Use original positions for the solution:

$B_{12,2}(A_{12,\bullet}, A_{12,\circ}) = \{\ -1\ 2\ 7\ 1\ 4\ 9\ 6\ 11\ \}$.

Figure 4.4: A solution transforming $A_{12,\bullet}$ into $A_{12,\circ}$ using Berge *2-moves*.

$$
\begin{aligned}
B_{10,2}(A_{10,\bullet}, A_{10,\circ}) &= \{\ -1\ \ 2\ \ 5\ \ 0\ \ 7\ \ 4\ \ 9\ \} \\
B_{14,2}(A_{14,\bullet}, A_{14,\circ}) &= \{\ -1\ \ 2\ \ 7\ \ 0\ \ 9\ \ 4\ \ 11\ \ 6\ \ 13\ \} \\
B_{18,2}(A_{18,\bullet}, A_{18,\circ}) &= \{\ -1\ \ 2\ \ 9\ \ 0\ \ 11\ \ 4\ \ 13\ \ 6\ \ 15\ \ 8\ \ 17\ \} \\
B_{22,2}(A_{22,\bullet}, A_{22,\circ}) &= \{\ -1\ \ 2\ \ 11\ \ 0\ \ 13\ \ 4\ \ 15\ \ 6\ \ 17\ \ 8\ \ 19\ \ 10\ \ 21\ \}
\end{aligned}
$$

Table 4.3: Solutions for transforming $A_{n,\bullet}$ into $A_{n,\circ}$ using Berge *2-moves* for $n = 10 + 4i$, $0 \leq i \leq 3$.

Next, we examine the lower bound for transforming $A_{n,\bullet}$ into $A_{n,\circ}$ using Berge *2-moves*. First observe that initially all black coins of $A_{n,\bullet}$ lie in an odd position and all white coins of $A_{n,\bullet}$ lie in an even position. However, for $A_{n,\circ}$, it is the opposite. Initially, all white coins of $A_{n,\circ}$ lie in an odd position and all black coins of $A_{n,\circ}$ lie in an even position. In order to transform $A_{n,\bullet}$ into $A_{n,\circ}$, we need to move all of black coins of $A_{n,\bullet}$ to an even position and all white coins of $A_{n,\bullet}$ to an odd position. Since $k = 2$, we can correct the position of at most two coins of $A_{n,\bullet}$ with each Berge *2-move*. Given that we need to correct the position of n coins of $A_{n,\bullet}$, this indicates that $h_{n,2}(A_{n,\bullet}, A_{n,\circ}) \geq \frac{n}{2}$. Combining this observation with the solution of transforming $A_{n,\bullet}$ into $A_{n,\circ}$, it seems to indicate that for even values of $n \geq 8$

$$\frac{n}{2} \leq h(A_{n,\bullet}, A_{n,\circ}) \leq \frac{n}{2} + 1.$$

4.3.3 Computing $H_{5,2}(\mathbf{S_5})$, $H_{7,2}(\mathbf{S_7})$ and $H_{9,2}(\mathbf{S_9})$

For odd values of $n \leq 9$, $H_{n,2}(\mathbf{S_n}) = \lceil \frac{n}{2} \rceil$ (Table 4.4). These values of n are relatively small and leave open the possibility of $H_{n,2}(\mathbf{S_n})$ increasing with larger n. However, this seems unlikely since there is a greater degree of freedom in choosing which coins to move for larger n which aids in reducing

n	$H_{n,2}(\mathbf{S_n})$
5	3
7	4
9	5

Table 4.4: Computed $H_{n,2}(\mathbf{S_n})$ for odd values of n, $5 \leq n \leq 9$.

the number of moves required. It seems probable that $H_{n,2}(\mathbf{S_n}) = \lceil \frac{n}{2} \rceil$ for $n > 9$, as extreme cases tend to appear for small n.

Chapter 5

Conclusion

In this thesis, we present a framework for Berge metrics for binary strings. In particular, we consider the associated Berge *k-metric*, $h_{n,k}(S^1, S^2)$, between a pair of strings $S^1, S^2 \in \mathbf{S_n}$ which is the minimum number of Berge *k-moves* needed to transform one string into another.

We begin by surveying the first generalization of the original Berge problem which allows Berge *k-moves* for sorting the alternating string A_n; that is, for transforming A_n into the sorted string T_n. Since the conjecture stating that

$$h_{n,k}(A_n, T_n) = \left\lceil \frac{n}{2} \right\rceil \text{ for } k \geq 2 \text{ and } n \geq 2k + 11$$

holds for $k = 1, 2, 3$ and for $k \leq 14$ and $k + 2 \leq n \leq 50$, we believe that $h_{n,k}(A_n, T_n)$ is independent of k. We address the following additional question: are A_n and T_n antipodal, i.e., the furtherest away pair in $\mathbf{S_n}$?

For Berge *1-metric*, A_n and T_n are not antipodal. In particular, we exhibit strings $S^1_*, S^2_* \in \mathbf{S_n}$ such that $h_{n,1}(S^1_*, S^2_*) \geq 0.700n$. Observing that transforming a pair of strings in $\mathbf{S_n}$ using Berge *1-moves* amounts to finding the maximum number of matching coins between two strings up to shifting, we have $H_{n,1}(\mathbf{S_n}) \leq 0.707n$.

For Berge *2-metric*, we have $\left\lceil \frac{n}{2} \right\rceil \leq H_{n,2}(\mathbf{S_n}) \leq 2n$. From preliminary computations, we conjecture that $H_{n,2}(\mathbf{S_n}) \leq \left\lceil \frac{n}{2} \right\rceil + 1$ for $n \geq 10$ since the degree of freedom in choosing moves increases as n increases. If this conjecture holds, then A_n and T_n are close to being antipodal for $H_{n,2}(\mathbf{S_n})$.

As observed in this thesis, $H_{n,1}(\mathbf{S_n})$ and $H_{n,2}(\mathbf{S_n})$ behaviors are quite different. This leads us to believe that $H_{n,k}(\mathbf{S_n})$ may be k dependent. More specifically, we believe that $H_{n,k}(\mathbf{S_n}) \leq \left\lceil \frac{n}{2} \right\rceil + c_k$ for some c_k dependent only on k.

Appendix A

$B_{n,2}(S_n, T_{n,\circ}) \leq n$, $S_n \in \mathbf{S_n}$ for $n = 5, 6, 7, 8$

Solutions for transforming general strings of $\mathbf{S_n}$ into $T_{n,\circ}$ using Berge *2-moves* in at most n moves and shifted to the right by 2 positions. A white coin is represented by 1 and a black coin is represented by 0.

$S_5 \in \mathbf{S_5}$	no. of moves	$B_{5,2}(S_5, T_{5,\circ})$
00111	1	{6 1}
01011	5	{ 6 4 1 } ∪ { 8 3 6 8 }
01101	4	{6 2 4 6 1}
01110	5	{6 4 1 3 5 1}
10011	3	{6 2 4 1}
10101	3	{6 3 5 1}
10110	5	{6 4 2 5 3 1}
11001	2	{6 3 1}
11010	3	{6 2 5 1}
11100	2	{6 4 1}

$S_6 \in \mathbf{S_6}$	no. of moves	$B_{6,2}(S_6, T_{6,\circ})$
000111	5	{7 1 4 2 5 1}
001011	3	{7 3 6 1}
001101	3	{7 4 6 1}
001110	1	{7 1}
010011	4	{7 2 4 6 1}
010101	5	{7 2 5 3 6 1}
010110	3	{7 2 6 1}
011001	5	{7 3 5 2 6 1}
011010	5	{7 5 1 4 6 1}
011100	4	{7 4 2 6 1}
100011	3	{7 3 5 1}
100101	3	{7 2 5 1}
100110	3	{7 2 4 1}
101001	4	{7 1 4 6 1}
101010	5	{7 1 4 2 6 1}
101100	2	{7 5 1}
110001	4	{7 2 6 4 1}
110010	2	{7 3 1}
110100	4	{7 4 6 3 1}
111000	2	{7 4 1}

$S_7 \in \mathbf{S_7}$	no. of moves	$B_{7,2}(S_7, T_{7,\circ})$
0001111	5	{8 1 5 2 6 1}
0010111	3	{8 3 7 1}
0011011	3	{8 4 7 1}
0011101	3	{8 5 7 1}
0011110	1	{8 1}
0100111	4	{8 2 4 7 1}
0101011	4	{8 2 5 7 1}
0101101	5	{8 2 6 3 7 1}
0101110	3	{8 2 7 1}
0110011	5	{8 4 1 3 6 1}
0110101	5	{8 3 6 2 7 1}
0110110	5	{8 3 5 2 7 1}
0111001	5	{8 5 1 4 6 1}
0111010	4	{8 4 2 7 1}
0111100	4	{8 5 2 7 1}
1000111	3	{8 3 6 1}
1001011	4	{8 2 6 4 1}
1001101	3	{8 2 6 1}
1001110	3	{8 2 4 1}
1010011	3	{8 4 6 1}
1010101	5	{8 5 2 7 4 1}
1010110	5	{8 1 4 2 7 1}
1011001	4	{8 1 5 7 1}
1011010	5	{8 1 5 2 7 1}
1011100	2	{8 6 1}
1100011	4	{8 2 7 4 1}
1100101	4	{8 5 7 3 1}
1100110	2	{8 3 1}
1101001	4	{8 2 7 5 1}
1101010	4	{8 4 7 3 1}
1101100	4	{8 6 2 4 1}
1110001	4	{8 3 7 5 1}
1110010	2	{8 4 1}
1110100	3	{8 6 3 1}
1111000	2	{8 5 1}

$S_8 \in \mathbf{S_8}$	no. of moves	$B_{8,2}(S_8, T_{8,o})$
00001111	3	{9 3 7 1}
00010111	4	{9 2 5 7 1}
00011011	4	{9 2 6 8 1}
00011101	3	{9 2 7 1}
00011110	4	{9 7 2 8 1}
00100111	3	{9 4 7 1}
00101011	5	{9 3 8 5 7 1}
00101101	3	{9 3 8 1}
00101110	4	{9 1 3 6 1}
00110011	3	{9 5 7 1}
00110101	3	{9 4 8 1}
00110110	4	{9 1 4 6 1}
00111001	3	{9 5 8 1}
00111010	5	{9 1 4 2 6 1}
00111100	1	{9 1}
01000111	5	{9 4 1 3 7 1}
01001011	5	{9 2 6 3 8 1}
01001101	4	{9 2 4 8 1}
01001110	5	{9 2 8 3 7 1}
01010011	5	{9 2 6 4 8 1}
01010101	4	{9 2 5 8 1}
01010110	5	{9 2 8 4 7 1}
01011001	4	{9 6 2 7 1}
01011010	5	{9 2 8 5 7 1}
01011100	3	{9 2 8 1}

$S_8 \in \mathbf{S_8}$	no. of moves	$B_{8,2}(S_8, T_{8,o})$
01100011	5	{9 4 2 6 8 1}
01100101	4	{9 4 2 7 1}
01100110	5	{9 4 1 3 6 1}
01101001	5	{9 5 1 4 8 1}
01101010	5	{9 3 6 2 8 1}
01101100	5	{9 3 5 2 8 1}
01110001	4	{9 5 2 7 1}
01110010	5	{9 5 1 4 6 1}
01110100	4	{9 4 2 8 1}
01111000	4	{9 5 2 8 1}
10000111	4	{9 2 7 4 1}
10001011	5	{9 2 5 7 3 1}
10001101	4	{9 1 3 8 1}
10001110	3	{9 3 6 1}
10010011	4	{9 2 7 5 1}
10010101	5	{9 2 4 7 3 1}
10010110	4	{9 2 6 4 1}
10011001	5	{9 2 4 8 5 1}
10011010	3	{9 2 6 1}
10011100	3	{9 2 4 1}
10100011	5	{9 4 1 5 7 1}
10100101	4	{9 1 4 8 1}

$S_8 \in \mathbf{S_8}$	no. of moves	$B_{8,2}(S_8, T_{8,\circ})$
10100110	3	{9 4 6 1}
10101001	4	{9 3 8 6 1}
10101010	5	{9 3 8 2 6 1}
10101100	4	{9 7 3 6 1}
10110001	4	{9 1 5 8 1}
10110010	4	{9 5 3 6 1}
10110100	4	{9 7 4 6 1}
10111000	2	{9 6 1}
11000011	4	{9 3 7 5 1}
11000101	4	{9 2 8 4 1}
11000110	4	{9 3 6 4 1}
11001001	4	{9 5 8 3 1}
11001010	5	{9 3 6 2 4 1}
11001100	2	{9 3 1}
11010001	4	{9 2 8 5 1}
11010010	4	{9 5 2 6 1}
11010100	4	{9 4 8 3 1}
11011000	4	{9 2 8 6 1}
11100001	4	{9 3 8 5 1}
11100010	4	{9 4 2 6 1}
11100100	2	{9 4 1}
11101000	3	{9 6 3 1}
11110000	2	{9 5 1}

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