MODIFIED FRACTIONAL BROWNIAN MOTION AND OPTION PRICING
MODIFIED FRACTIONAL BROWNIAN MOTION AND OPTION PRICING

By

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Abstract

The Black-Scholes model introduced by Black and Scholes (1973) and Merton (1973) has become synonymous with modern finance theory. It assumes that the dynamics of stock prices is well described by exponential Brownian motion, which is not consistent with empirical stock price returns, and then the dependence structure of stock price returns has been at the center of intense scrutiny for the last 30 or more years. This project studies modified fractional Brownian motions and shows that two different classes of modified fractional Brownian motions are equivalent to Brownian motion. We discuss option pricing under the hypothesis that the underlying asset price process satisfies a stochastic differential equation driven by a modified fractional Brownian motion. Parameter estimation and simulation methods are given. In particular, we investigate the ability of the self-similarity parameter $H$ to explain the discrepancy between the Black-Scholes model and the reality of the market. The proposed method is applied to a real data set. The empirical results indicate that the model is better than the Black-Scholes model.
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Chapter 1

Introduction

1.1 Black-Scholes Model

We consider a market consisting of one bank account and one stock that pays no dividends. Borrowing and short-selling are allowed, the borrowing rate is equal to the lending rate, and it is possible to buy and sell any fraction of stock shares. Moreover, there exist no transaction costs and stock shares can be bought and sold at the same price. The bank account is a riskless security, where one always gets back the investment, plus interest which can be fixed, or which can vary with time. Shares of stock can be bought or sold in the market. Their price is subject to a large number of factors and it can go up or go down. The stock is considered a risky asset since we cannot be sure if the price will go up or down. The price of the stock at time $t$, $S_t$, is modeled as a random variable.

We are going to consider options on the stock. An option is security which gives its holder the right to sell or to buy the underlying asset (for example shares, currency)
within a stated period (American option), or at the end of the stated period (European option) at a predetermined price. The options providing the right to buy are referred to as call options. If a contract gives the right to sell, it is called as put option.

We consider a standard European call option which gives its holder the right to buy at a given time $T$ (expiry date) fixed in the future, a stock for price $K$ (strike) pointed out in the contract. If the random price of the stock is described by the process $(S_t)$, then the gain of the holder at time $T$ equals $\max(S_T - K, 0)$. (If $S_T > K$, the holder buys the stock for the price $K$ and sells it for the spot price $S_T$. In this case, he gains $S_T - K$. If $S_T < K$, it makes no sense to use the right given to him by the option of the contract.)

Let us start with the basic results of Black and Scholes (1973). Assume that the Bank account $S^0_t$ and the stock price $S_t$ satisfy the following stochastic differential equation

$$dS^0_t = rS^0_t dt, \quad S^0_0 = 1,$$  \hspace{1cm} (1.1)  

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$  \hspace{1cm} (1.2)

where $r$ denotes the riskless interest rate which is assumed to be constant, $\mu$ denotes the average rate of stock return, $\sigma$ denotes the volatility coefficient which is assumed to be a positive constant, and $B$ is the standard Brownian motion.

The solution to the above equation is given by

$$S^0_t = e^{rt},$$  \hspace{1cm} (1.3)  

$$S_t = S_0 \exp \left\{ \mu t + \sigma B_t - \frac{\sigma^2}{2} t \right\}.$$  \hspace{1cm} (1.4)
and thus the discounted stock price
\[
\tilde{S}_t = e^{-rt}S_t
\]
\[
= S_0 \exp \left\{ (\mu - r - \frac{\sigma^2}{2})t + \sigma B_t \right\}.
\]

Let
\[
R_T = \exp \left\{ aB_t - \frac{1}{2}a^2T \right\}, \quad \dot{B}_t = B_t - at
\]
where \( a = \frac{r - \mu}{\sigma} \). Define \( Q \) by \( dQ = R_T dP \). Then by Girsanov's theorem, \( Q \) is a probability measure equivalent to \( P \). We then have
\[
\tilde{S}_t = S_0 e^{\dot{B}_t - \frac{1}{2} \sigma^2 t}, \quad d\tilde{S}_t = \sigma \tilde{S}_t d\dot{B}_t,
\]
and \( \dot{B}_t \) is a Brownian motion under \( Q \). Then, it can be shown that the market consisting of the geometric Brownian motion and a bank account is complete.

Consider a European call option in this market with striking price \( K \) and terminal time \( T \). Then completeness of the market tells us (see Kallianpur and Karandikar, 1999, Theorem 9.2) that the price \( C_t \) of the call option at time \( t \) is given by the following expression:
\[
C_t = E^*_t \left[ e^{-r(T-t)} \max(S_T - K, 0) \right], \quad (1.5)
\]
where \( E^*_t \) is the conditional expectation under the equivalent martingale measure to \( P \) given all information available at time \( t \). In such a model, one may obtain an explicit option price \( C_t \) at time \( t \) as the following
\[
C_t = S_t \phi(h(S_t, t, \sigma)) - Ke^{-r(T-t)} \phi \left( h(S_t, t, \sigma) - \sigma \sqrt{T-t} \right), \quad (1.6)
\]
where
\[
h(S_t, t, \sigma) = \frac{\log S_t/K + (r + \frac{\sigma^2}{2}) (T-t)}{\sigma \sqrt{T-t}},
\]
and $\phi$ is the standard normal distribution function:

$$
\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.
$$

1.2 Dependence Structure of Stock Returns

The Black-Scholes model introduced by Black and Scholes (1973) and Merton (1973) has become synonymous with modern finance theory. It assumes that the dynamics of stock prices is well described by exponential Brownian motion; that is, stock returns behave like a sequence of independently identical distributed (i.i.d.) Gaussian random variables. However, that is not consistent with empirical stock returns, and so the dependence structure of stock returns has been at the center of intense scrutiny for the last 30 or more years. Early investigations into the dependence structure of asset returns (e.g., by Fama (1965, 1970)) concluded that successive returns could be assumed to be independent. Later on, Lo and Mackinlay (1988) revisited this random walk hypothesis; after a careful analysis of market from a 25-year period (1962-1987), they found substantial short-range dependence in the data and strongly rejected the hypothesis that asset returns are i.i.d. Mandelbrot (1967) and Greene and Fielitz (1977) presented the empirical findings of long-term memory in common stock returns. Huang and Yang (1995) discussed the fractal structure in multinational stock returns. Recently, using the CRSP (Center for Research in Security Prices) daily stock return data, Willinger, Taqqa and Teverovsky (1999) revisited the question of whether or not actual stock market prices exhibit long-range dependence. They found empirical evidence of long-range dependence in stock returns.

Hurst parameter $H$ is commonly used to measure long-range dependence. The frac-
fractional Brownian motion is a typical long-range dependent process with Hurst parameter $H$. It is a natural generalization of well-known Brownian motion. Correspondingly, fractional Gaussian noise generated by fractional Brownian motion is a generalization of white Gaussian noise (i.e., a sequence of independently identical distributed Gaussian random variables).

Since the stock return processes are long-term dependent, it is not reliable for us to assume that the underlying asset prices follow geometric Brownian motions for option pricing. Based on this fact, we assume that asset price process follows a log Gaussian process with long-range dependence. That is, we replace the sequence of i.i.d. with a dependent sequence for underlying asset price return model. A natural choice of such a dependent process is a fractional Brownian motion. However, the fractional Brownian motion admits arbitrage. Therefore, we consider a modification of the fractional Brownian motion.

### 1.3 Fractional Brownian Motion

We let $\{B_t, t \in \mathbb{R}\}$ be a standard Brownian motion, then the fractional Brownian motion $\{B_t^H, t \in \mathbb{R}\}$ with self-similarity parameter $H \in (0, 1)$ is defined by

$$B_t^H = C_H \left[ \int_{-\infty}^{t} \varphi_H(t-s)dB_s - \int_{-\infty}^{0} \varphi_H(-s)dB_s \right],$$

where

$$\varphi_H(t) = t^{H-\frac{1}{2}}1_{\{t>0\}}, \quad t \in \mathbb{R}$$

and

$$C_H = \left\{ \frac{1}{2H} + \int_0^\infty (\varphi_H(1+s) - \varphi_H(s))^2ds \right\}^{-\frac{1}{2}}.$$
The process $B^H$ is clearly a zero-mean Gaussian process, and the constant $C_H$ chosen to normalize the covariance structure neatly:

$$E \left[ B_t^H - B_s^H \right]^2 = |t - s|^{2H}. \quad (1.10)$$

The correlation of successive increments is given as

$$\frac{E[-B_s^H B_t^H]}{E[(B_s^H)^2]} = 2^{2H-1} - 1. \quad (1.11)$$

When $H = \frac{1}{2}$, the correlation is zero, as expected for the independent increments of classical Brownian motion. When $H < \frac{1}{2}$, the correlation is negative. When $H > \frac{1}{2}$, the correlation is positive.

Obviously, the fractional Brownian motion can be defined by a zero-mean Gaussian process such that (1.10) holds. Also the fractional Brownian motion path is a fractal curve with dimension $(2 - H)$.

It is inconvenient that the fractional Brownian motion does not have a derivative. The derivative of smoothed Brownian motion is just the sequence of uncorrelated Gaussian random variables referred to as white Gaussian noise. Similarly, the derivative of smoothed fractional Brownian motion will be referred to as fractional Gaussian noise.

Let $B^H$ be a fractional Brownian motion and let $\delta > 0$. Define

$$B_t^H(\delta) = \delta^{-1} \int_t^{t+\delta} B_u^H \, du \quad (1.12)$$

and

$$(B_t^H)'(\delta) = \delta^{-1} [B_{t+\delta}^H - B_t^H]. \quad (1.13)$$

Then we call $\{(B_t^H)'(\delta), t \in \mathbb{R}\}$ a fractional Gaussian noise. It is a stationary process with the autocovariance below

$$Cov \left( (B_{t+h}^H)'(\delta), (B_t^H)'(\delta) \right) = \frac{1}{2} \delta^{2H-2} \left\{ \left( \frac{h}{\delta} + 1 \right)^{2H} + \left| \frac{h}{\delta} - 1 \right|^{2H} - 2 \left( \frac{h}{\delta} \right)^{2H} \right\}. \quad (1.14)$$
Other properties of fractional Brownian motion can be seen in Mandelbrot and Van Ness (1968).

### 1.4 Arbitrage in Fractional Brownian Motion Models

In Lispster and Shiryaev (1989), Lin (1995), Rogers (1997) and Cheridito (2003) it is shown for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, fractional Brownian motion is not a semimartingale. Moreover Rogers (1997), Shiryaev (1998), Salopek (1998), Kallianpur and Karandikar (1999), Cheridito (2001) presented arbitrage strategies for fractional Brownian motion models.

Rogers (1997) constructed arbitrage for the following model:

$$ S_t^0 = 1, S_t = S_0 + \sigma B_t^H. \quad (1.15) $$

His strategy consists of a combination of buy and hold strategies and works for all parameters $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. The drawbacks of this model are that $S_t$ can become negative and the returns are lower for higher stock prices.

In Shiryaev (1998) only the case $H \in (\frac{1}{2}, 1)$ was treated. For the model:

$$ S_t^0 = 1, S_t = S_0 + \mu t + \sigma B_t^H, \quad (1.16) $$

one can choose a $c > 0$ and set

$$ \pi^0(t) = -c \left( \mu t + \sigma B_t^H \right)^2 - 2cS_0 \left( \mu t + \sigma B_t^H \right), \quad \pi(t) = 2c \left( \mu t + \sigma B_t^H \right), \quad (1.17) $$

to obtain a self-financing arbitragy.

7
For the model:

\[ S_t^0 = e^{rt}, \quad S_t = S_0 e^{\mu t + \sigma B_t^H}, \]  

(1.18)

one can set for all \( c > 0 \),

\[ \pi^0(t) = c S_0 \left( 1 - \exp \left\{ 2(\mu - r)t + 2\sigma B_t^H \right\} \right) \quad \pi(t) = 2c \left( \exp \left\{ (\mu - r)t + \sigma B_t^H \right\} - 1 \right) \]  

(1.19)

to obtain a self-financing arbitrage.

In Kallianpur and Karandikar (1999), for \( H \in (\frac{1}{2}, 1) \), the integral with respect to fractional Brownian motion is defined in terms of \( L^1 \) convergence of Riemann sums. Assume that the bank account \( S_t^0 \) and the stock price \( S_t \) satisfy the following stochastic differential equation

\[ dS_t^0 = r S_t^0 dt, \]  

(1.20)

\[ dS_t = \mu S_t dt + \sigma S_t dB_t^H. \]  

(1.21)

The solution to the above equation is given by

\[ S_t^0 = e^{rt}, \quad S_t = S_0 e^{\mu t + \sigma B_t^H}. \]  

(1.22)

(1.23)

Set

\[ \pi(t) = 2e^{\mu t + \sigma B_t^H} \left( e^{\sigma B_t^H} - 1 \right). \]  

(1.24)

Then \( \pi \) is a self-financing arbitrage strategy.

Cheridito (2003) constructed strong arbitrage strategies for the following models:

\[ \frac{S_t}{S_t^0} = \nu(t) + \sigma B_t^H, \quad H \in \left( 0, \frac{1}{2} \right) \cup \left( \frac{1}{2}, 1 \right) \]  

(1.25)
and
\[ \frac{S_t}{S_0} = e^{\nu(t) + \sigma B_H^t}, \quad H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right). \] (1.26)

1.5 Fractional Black-Scholes Option Pricing Model

In Black-Scholes model, if the ordinary Brownian motion is replaced by fractional Brownian motion, then the model is called fractional Black-Scholes model. That is, the stock price satisfies the following equation

\[ dS_t = \mu S_t dt + \sigma S_t dB^H_t. \]

Since $B_H^t$ is not a semimartingale, the usual Itô integral cannot be used. As shown in Section 1.4, the models based on fractional Brownian motion and related pathwise integral will admit arbitrage. However, some authors such as Hu and Øksendal (2003) and Elliot and van der Hoek (2003) insist on using fractional Black-Scholes model. In order to overcome arbitrage, Hu and Øksendal (2003) and Elliot and van der Hoek (2003) defined the stochastic integral based on the Wick product. In Hu and Øksendal (2003), only the case $H \in \left(\frac{1}{2}, 1\right)$ was considered. Elliot and van der Hoek (2003) extend that of Hu and Øksendal (2003) to the case $H \in (0, 1)$. The framework of their theory is to take the underlying probability space to be $(\mathcal{S}'(\mathbb{R}), \mathcal{F})$ the space of tempered distributions, with $\mathcal{F}$ the Borel field. A probability measure $P$ is given on $(\mathcal{S}'(\mathbb{R}), \mathcal{F})$ by the Bochner-Minlos theorem. Fractional Brownian motion $B_H^t$ is defined for $H \in \left(\frac{1}{2}, 1\right)$ by the Bochner-Minlos theorem in Hu and Øksendal (2003), and for $H \in (0, 1)$ by fractional transforms in Elliot and van der Hoek (2003). The fractional Black-Scholes market is shown to be complete in Hu and Øksendal (2003) and Elliot and van der Hoek (2003). Therefore, a fractional Black-Scholes formula for a European
call with strike price \( K \) is given by

\[
C_t = E_t^* \left[ e^{-r(T-t)} \max(S_T - K, 0) \right] = S_t \phi(h_H(S_t, t, \sigma)) - Ke^{-r(T-t)} \phi \left( h_H(S_t, t, \sigma) - \sigma \sqrt{T^{2H} - t^{2H}} \right) \tag{1.27}
\]

where \( E_t^* \) is the conditional expectation under the equivalent martingale measure to \( P \) given all information available at time \( t \),

\[
h_H(S_t, t, \sigma) = \frac{\log \frac{S_t}{K} + r(T - t) + \frac{\sigma^2}{2} (T^{2H} - t^{2H})}{\sigma \sqrt{T^{2H} - t^{2H}}},
\]

and \( \phi \) is the standard normal distribution function. Similar results have also been obtained by Benth (2003), Biagini and Øksendal (2003) and Biagini et al. (2002).
Chapter 2

Modified Fractional Brownian Motion and Option Pricing

2.1 Introduction

Why do we consider the modified fractional Brownian motion? According to earlier literature cited in Section 1.4, arbitrage opportunities exist with fractional Brownian motion, and then fractional Brownian motion is an absurd candidate for stock price process. On the other hand, as cited in Section 1.5, fractional Black-Scholes market has no arbitrage, and therefore a fractional Black-Scholes formula is obtained as the Black-Scholes formula. Clearly, the latter is not compatible with the former. More recently, Björk and Hult (2005) resolved this contradiction and concluded that the fractional Black-Scholes theory are not economically meaningful by pointing out that the definition of the self-financing trading strategies and/or the definition of the value of a portfolio used by the authors does not have a reasonable economic interpretation.
We need to modify fractional Brownian motion.

## 2.2 Modified Fractional Brownian Motion

To avoid arbitrage, we can change the convolution kernel in the Mandelbrot-van Ness representation of fractional Brownian motion such that the process is a semimartingale with a distribution similar to the one of fractional Brownian motion. Rogers (1997) constructed such a Gaussian process with the same long-range dependence as fractional Brownian motion. We define

\[ M_t^\varphi = C_\varphi \left\{ \int_{-\infty}^{t} \varphi(t-s)dB_s - \int_{-\infty}^{0} \varphi(-s)dB_s \right\} \quad (2.1) \]

where \( \varphi(0) \neq 0 \) and \( \varphi(t) = 0 \) for \( t < 0 \) such that \( \frac{M_t^\varphi}{C_\varphi \varphi(0)} \) is equivalent to Brownian motion, and

\[ C_\varphi = \left\{ \int_{0}^{1} \varphi^2(1-s)ds + \int_{-\infty}^{0} (\varphi(1-s) - \varphi(-s))^2ds \right\}^{-\frac{1}{2}}, \quad (2.2) \]

then \( M_t^\varphi \) is called a modified fractional Brownian motion associated with the kernel \( \varphi \). Clearly, \( M_t^\varphi \) is a centered Gaussian process. The increment \( M_{t+\delta}^\varphi - M_t^\varphi \) is a stationary Gaussian process with the variance and the autocovariance below, respectively

\[ E|M_{t+\delta}^\varphi - M_t^\varphi|^2 = V_\varphi(\delta) \quad (2.3) \]

where

\[ V_\varphi(\delta) = C_\varphi^2 \left( \int_{-\infty}^{0} (\varphi(\delta-s) - \varphi(-s))^2ds + \int_{0}^{\delta} \varphi^2(\delta-s)ds \right), \quad (2.4) \]

and

\[ E[(M_{t+\delta}^\varphi - M_t^\varphi)(M_{s+\delta}^\varphi - M_s^\varphi)] = \gamma_\varphi(t-s, \delta) \quad (2.5) \]

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where
\[
\gamma_\varphi(t-s, \delta) = \frac{1}{2}(V_\varphi(|t-s-\delta|) + V_\varphi(|t-s+\delta|) - 2V_\varphi(|t-s|)).
\]  
(2.6)

If \( \varphi \in C^2(\mathbb{R}_+) \), \( \varphi(0) \neq 0 \), \( \varphi'(0) = 0 \) and \( \lim_{t \to -\infty} \varphi''(t)t^{\frac{3}{2}-H} < \infty \), then \( M^{\varphi} \) is a Gaussian process with the same long-range dependence as the fractional Brownian motion and becomes a semimartingale because in this case, \( M^{\varphi} \) can be expressed as
\[
\frac{M_t^{\varphi}}{C^{\varphi}} = \varphi(0)B_t + \int_0^t \left( \int_{-\infty}^s \varphi''(s-v)B_v dv \right) ds.
\]  
(2.7)

So, we have the following theorem.

**Theorem 2.2.1.** Assume that \( \varphi \in C^2(\mathbb{R}_+) \), \( \varphi(0) \neq 0 \), \( \varphi'(0) = 0 \) and \( \lim_{t \to -\infty} \varphi''(t)t^{\frac{3}{2}-H} < \infty \). Set
\[
h(s) = \frac{1}{\varphi(0)} \int_{-\infty}^s \varphi''(s-v)B_v dv.
\]

Let the probability measure \( Q \) satisfy
\[
\frac{dQ}{dP} = \exp \left\{ \int_0^T h(s)dB_s - \frac{1}{2} \int_0^T h^2(s)ds \right\}.
\]

Then \( \frac{M_t^{\varphi}}{C^{\varphi}(0)} \) is a standard Brownian motion under \( Q \) for \( 0 \leq t \leq T \).

**Proof.** Note that
\[
\frac{M_t^{\varphi}}{C^{\varphi}(0)} = \int_0^t h(s)ds + B_t
\]  
(2.8)

and \( h(s) \) has a normal distribution with mean 0 and variance \( \sigma_s^2 \) where for \( s \geq 0 \),
\[
\sigma_s^2 \leq (\varphi(0))^{-2} \int_{-\infty}^s |\varphi''(s-v)|^2|v|dv
= (\varphi(0))^{-2} \int_0^\infty |\varphi''(u)|^2|s-u|du
\leq (\varphi(0))^{-2} \left( s \int_0^\infty |\varphi''(u)|^2du + \int_0^\infty |\varphi''(u)|^2udu \right)
\]

From \( \lim_{t \to -\infty} \varphi''(t)t^{\frac{3}{2}-H} < \infty \), there exists a positive number \( N \) such that for \( t \geq N \)
\[
|\varphi''(t)| \leq c_1t^{-\frac{5}{2}}
\]
for some constant $c_1$, and then we have that

$$
\int_0^\infty [\varphi''(u)]^2 du \leq \int_0^N [\varphi''(u)]^2 du + c_1 \int_N^\infty u^{2H-5} du
= \int_0^N [\varphi''(u)]^2 du + c_1(4 - 2H)N^{2H-4} < \infty
$$

and

$$
\int_0^\infty [\varphi''(u)]^2 u du \leq \int_0^N [\varphi''(u)]^2 u du + c_1 \int_N^\infty u^{2H-4} du
= \int_0^N [\varphi''(u)]^2 u du + c_1(3 - 2H)N^{2H-3} < \infty.
$$

Hence,

$$
\sigma_s^2 \leq c_2, \quad 0 \leq s \leq T
$$

for some constant $c_2$. Since for $\delta < \frac{1}{2c_2}$ and $0 \leq s \leq T$, 

$$
E\left[e^{\delta h^2(s)}\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_s^2}} e^{-\frac{x^2}{2\sigma_s^2}} dx
= \frac{1}{\sqrt{1 - 2\delta\sigma_s^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\leq \frac{1}{\sqrt{1 - 2\delta c_2}},
$$

then there exists $\delta > 0$ and a positive constant $c_3$ such that

$$
E\left[e^{\delta h^2(s)}\right] \leq c_3
$$

for each $s \in [0, T]$.

Note that

$$
E\left[\int_0^T h^2(s) ds\right] \leq (\varphi(0))^{-2} \int_0^T \int_{-\infty}^{\varphi''(s-u)} [\varphi''(s-u)]^2 v du ds < \infty.
$$
From Corollary 7.2.2. and Theorem 7.1.3 (Girsanov) in Kallianpur (1980), we complete the proof of this theorem.

A natural choice of $\varphi$ is as follows:

$$\varphi_{H}^{\varepsilon}(t) = (\varepsilon + t^2)^{\frac{2H-1}{4}}1_{\{t \geq 0\}}.$$  

In the special case of $\varepsilon = 0$ and $H = \frac{1}{2}$, we recover the well-known Black and Scholes model.

We have shown that this class of processes is equivalent to Brownian motion. However, Cheridito (2001) pointed out that such modification of fractional Brownian motion is not enough to ensure that the model is arbitrage-free. By regularising fractional Brownian motion, Cheridito gave a class of processes which have a unique equivalent martingale measure. If the kernel function $\varphi$ satisfies the following condition:

$$\varphi(t) = \varphi(0) + \int_0^t \psi(s)ds, t \geq 0, \text{ for some } \varphi \in L^2(\mathbb{R}_+),$$  

then the Gaussian process generated by the kernel $\varphi$ is equivalent to Brownian motion. Set for $H \in (0, 1)$, $a \in \mathbb{R}$ and $b > 0$,

$$\varphi_{H}^{a,b}(t) = \begin{cases} 
  a + \frac{\varphi(b)-a}{b}t, & t \in [0, b] \\
  \varphi_H(t), & t \in (-\infty, 0) \cup (b, \infty).
\end{cases}$$

The Gaussian process generated by the kernel $\varphi_{H}^{a,b}$ is called as regularized fractional Brownian motion. Cheridito (2001) figured out that the functions $\varphi_{H}^{a,b}(t)$ satisfy (2.9), and concluded that $M_{\varphi_{H}^{a,b}}$ is equivalent to Brownian motion for the case $a \neq 0$. Let’s give another proof for this case.

**Theorem 2.2.2.** Assume that $a \neq 0$. Then

$$\frac{M_{\varphi_{H}^{a,b}}}{C_{\varphi_{H}^{a,b}}(0)} = B_t + \int_0^t h(s)ds$$  

(2.10)
where
\[
\begin{align*}
\varphi(s) &= \frac{1}{\varphi_{H}^{a,b}(0)} \left\{ \int_{-\infty}^{s-b} (H - \frac{1}{2})(s - v)^{H-3/2} dB_v + \frac{\varphi_H(b) - a}{b} (B_s - B_{s-b}) \right\}. 
\end{align*}
\]  

(2.11)

**Proof.** Note that the derivative of \( \varphi_{H}^{a,b}(t) \) is given by
\[
(\varphi_{H}^{a,b})'(t) = \begin{cases} 
\frac{\varphi_H(b) - a}{b}, & t \in (0, b) \\
\varphi_H(t), & t \in (-\infty, 0) \cup (b, \infty).
\end{cases}
\]

For convenience of composition, we write \( \varphi_{H}^{a,b} \) as \( \varphi \) without confusing.

For \( 0 \leq t \leq b \),
\[
\begin{align*}
\frac{1}{C_p} M_{t}^{p} &= \int_{-\infty}^{0} (\varphi(t - s) - \varphi(-s)) dB_s + \int_{0}^{t} \varphi(t - s) dB_s \\
&= \int_{-\infty}^{t} (\varphi(t - s) - \varphi(-s)) dB_s + \int_{0}^{t} \varphi(t - s) dB_s \\
&\quad + \int_{t}^{t-b} \varphi(t - s) dB_s + \int_{0}^{t} \varphi(t - s) dB_s \\
&= \int_{-\infty}^{t} \varphi'(v - s) dB_s + \int_{0}^{t} \varphi'(v - s) dB_s + \int_{0}^{v} \varphi'(v - s) dB_s + \varphi(0)B_t \\
&\quad + \int_{0}^{t} \varphi'(v - s) dB_s + \varphi(0)B_t \\
&= \int_{0}^{t} \varphi'(v - s) dB_s + \varphi(0)B_t \\
&= \int_{0}^{t} \varphi'(v - s) dB_s + \varphi(0)B_t \\
&= \varphi(0) \left\{ B_t + \int_{0}^{t} h(s) ds \right\}
\end{align*}
\]

since
\[
\int_{v-b}^{v} \varphi'(v - s) dB_s = \frac{\varphi_H(b) - a}{b} (B_s - B_{s-b}),
\]
where $h$ is defined by (2.11).

For $t > b$,

\[
\frac{1}{C_{\varphi}} M_t^c = \int_{-\infty}^{0} (\varphi(t-s) - \varphi(-s)) dB_s + \int_{0}^{t} \varphi(t-s) dB_s \\
= \int_{-\infty}^{-b} (\varphi(t-s) - \varphi(-s)) dB_s + \int_{-b}^{0} (\varphi(t-s) - \varphi(-s)) dB_s \\
+ \int_{0}^{t-b} (\varphi(t-s) - \varphi(-s)) dB_s + \int_{t-b}^{t} \varphi(t-s) dB_s \\
= \int_{-\infty}^{-b} \left( \int_{0}^{t} \varphi(v-s) dv \right) dB_s \\
+ \int_{-b}^{0} \left( \int_{0}^{t} \varphi(v-s) dv + \int_{0}^{b+s} \varphi'(v-s) dv \right) dB_s \\
+ \int_{0}^{t-b} \left( \int_{0}^{t} \varphi(v-s) dv + \varphi(b) \right) dB_s + \int_{t-b}^{t} \left( \int_{0}^{t} \varphi(v-s) dv + \varphi(0) \right) dB_s \\
= \int_{0}^{t} \left( \int_{-\infty}^{-b} \varphi'(v-s) dB_s \right) dv + \int_{0}^{b} \left( \int_{-\infty}^{v-b} \varphi'(v-s) dB_s \right) dv \\
+ \int_{0}^{t} \left( \int_{-b}^{0} \varphi'(v-s) dB_s \right) dv + \int_{0}^{b} \left( \int_{-b}^{v-b} \varphi'(v-s) dB_s \right) dv \\
+ \int_{0}^{b} \left( \int_{v-b}^{0} \varphi'(v-s) dB_s \right) dv + \varphi(b) B_{t-b} \\
+ \int_{t-b}^{t} \left( \int_{t-b}^{v} \varphi'(v-s) dB_s \right) dv + \varphi(0)(B_t - B_{t-b}) \\
= \int_{b}^{t} \left( \int_{-\infty}^{-b} \varphi'(v-s) dB_s \right) dv + \int_{0}^{b} \left( \int_{-\infty}^{-b} \varphi'(v-s) dB_s \right) dv \\
+ \int_{b}^{t} \left( \int_{-b}^{0} \varphi'(v-s) dB_s \right) dv + \int_{0}^{b} \left( \int_{-b}^{v-b} \varphi'(v-s) dB_s \right) dv \\
+ \int_{0}^{b} \left( \int_{v-b}^{0} \varphi'(v-s) dB_s \right) dv + \varphi(b) B_{t-b} \\
+ \int_{t-b}^{t} \left( \int_{t-b}^{v} \varphi'(v-s) dB_s \right) dv + \varphi(0)(B_t - B_{t-b}) \\
= \int_{0}^{t} \left( \int_{v-b}^{v} \varphi'(v-s) dB_s \right) dv \\
+ \int_{0}^{b} \frac{\varphi_{H}(b)}{b} (-B_{v-b}) dv + \varphi(b) B_{t-b} \\
+ \int_{t-b}^{t} \frac{\varphi_{H}(b)}{b} (B_v - B_{t-b}) dv + a(B_t - B_{t-b})
\]
\[
= \int_0^t \left( \int_{-\infty}^{v-b} \varphi'(v-s)dB_s \right) dv \\
- \int_0^b \frac{\varphi_H(b) - a}{b} B_{v-b}dv + \frac{\varphi_H(b) - a}{b} \int_{t-b}^t B_vdv \\
+ \varphi_H(b)B_{t-b} - (\varphi_H(b) - a)B_{t-b} + a(B_t - B_{t-b}) \\
= \int_0^t \left( \int_{-\infty}^{v-b} \varphi'(v-s)dB_s \right) dv \\
+ \frac{\varphi_H(b) - a}{b} \left( \int_{t-b}^t B_vdv - \int_0^b B_{v-b}dv \right) + aB_t \\
= aB_t + \int_0^t \left( \int_{-\infty}^{v-b} \varphi'(v-s)dB_s \right) dv \\
+ \frac{\varphi_H(b) - a}{b} \left( \int_0^t (B_v - B_{v-b})dv - \int_0^b B_vdv + \int_b^t B_{v-b}dv \right) \\
= aB_t + \int_0^t \left( \int_{-\infty}^{v-b} \varphi'(v-s)dB_s + \frac{\varphi_H(b) - a}{b} (B_v - B_{v-b}) \right) dv \\
= \varphi(0) \left\{ B_t + \int_0^t h(s)ds \right\}.
\]

**Theorem 2.2.3.** Let \( \varphi_H^{a,b}(0) \neq 0 \). Define \( Q^{\varphi_H^{a,b},b} \) by

\[
\frac{dQ^{\varphi_H^{a,b},b}}{dP} = \exp \left\{ \int_0^T h(s)dB_s - \frac{1}{2} \int_0^T h^2(s)ds \right\}.
\]  
(2.12)

Then \( \frac{M^{\varphi_H^{a,b},b}}{C^{\varphi_H^{a,b},b}(0)} \) is a standard Brownian motion under \( Q^{\varphi_H^{a,b},b} \) for \( 0 \leq t \leq T \).

**Proof.** By Theorem 2.2.2,

\[
\frac{M^{\varphi_H^{a,b},b}}{C^{\varphi_H^{a,b},b}(0)} = B_t + \int_0^t h(s)ds.
\]

Note that \( h(s) \) has a normal distribution with mean 0 and variance \( \sigma_s^2 \) where for \( s \geq 0 \),

\[
\sigma_s^2 \leq 2(\varphi(0))^{-2} \left\{ \int_{-\infty}^{s-b} [\varphi'(s-v)]^2 dv + \left( \frac{\varphi_H(b) - a}{b} \right)^2 b \right\} \\
= 2(\varphi(0))^{-2} \left\{ \int_{-\infty}^{s-b} \left( H - \frac{1}{2} \right)^2 (s-v)^{2(H-\frac{1}{2})} dv + \left( \frac{\varphi_H(b) - a}{b} \right)^2 b \right\}
\]
\[= 2(\varphi(0))^{-2} \left\{ \int_{b}^{\infty} \left( H - \frac{1}{2} \right)^2 u^{2H-3} dv + \left( \frac{\varphi_H(b) - a}{b} \right)^2 \right\} \]

\[= 2(\varphi(0))^{-2} \left\{ (H - \frac{1}{2})^2 \frac{b^{2H-2}}{2 - 2H} + \left( \frac{\varphi_H(b) - a}{b} \right)^2 \right\} \]

\[= c_4 \]

Since for \( \delta < \frac{1}{2c_4} \) and \( s \geq 0 \),

\[E\left[e^{\delta h^2(s)}\right] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_s} e^{\delta x^2} \cdot e^{-\frac{x^2}{2\sigma_s^2}} dx \]

\[= \frac{1}{\sqrt{1 - 2\delta\sigma^2_s}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \]

\[\leq \frac{1}{\sqrt{1 - 2\delta c_4}}, \]

then there exists \( \delta > 0 \) and a positive constant \( c_5 \) such that

\[E\left[e^{\delta h^2(s)}\right] \leq c_5 \]

for each \( s \in [0, T] \).

Note that for \( T < \infty \),

\[E\left[\int_{0}^{T} h^2(s) ds\right] = \int_{0}^{T} E[h^2(s)] ds < \infty. \]

From Corollary 7.2.2. and Theorem 7.1.3 (Girsanov) in Kallianpur (1980), we complete the proof of this theorem.
2.3 Option Pricing with Modified Fractional Brownian Motion

Let the bank account $S^0_t$ and the stock price $S_t$ satisfy the following stochastic differential equation:

\begin{align}
  dS^0_t &= rS^0_t dt, \\
  dS_t &= \mu S_t dt + \sigma S_t dM^\varphi_t,
\end{align}

(2.13, 2.14)

where $r$ denotes the interest rate (for the sake of simplicity and to be able to compare this model with the Black-Scholes model, $r$ is assumed to be constant); $\mu$ denotes the average rate of stock return; $\sigma$ denotes the volatility coefficient of stock return (assumed to be positive and constant) and $S_0$ is a given constant.

By Theorem 2.2.1 and 2.2.3, for both $\varphi = \varphi^H$ and $\varphi = \varphi^{a,b}_H, \frac{M^\varphi}{C_t^\varphi(0)}$ are equivalent to Brownian motion. Therefore, for each case, there exists a unique probability measure $P^*$ which is equivalent to $P$ such that the discount stock price process $\{\hat{S}_t, 0 \leq t \leq T\}$ is a martingale under $P^*$.

We consider now a standard European call option with the payment function $f_T = \max(S_T - K, 0)$, where $T$ is the expiration time, and $K$ is the striking price. Then the price $C_t$ of the call option at time $t$ is given by

\begin{align}
  C_t &= E_t^* \left[ e^{-r(T-t)} \max(S_T - K, 0) \right] \\
  &= S_t \phi(h(S_t, t, \sigma)) - K e^{-r(T-t)} \phi \left( h(S_t, t, \sigma) - \sigma \sqrt{T-t} \right)
\end{align}

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where $E_t^*$ denotes the conditioned expectation under the risk neutral probability $P^*$ given all information available at time $t$, $\sigma_\varphi = \sigma C_\varphi(0)$, and $h$ is defined as in Section 1.1.
Chapter 3

Parameter Estimation and Simulation Methods

3.1 Parameter Estimation

Assume that the underlying asset price process $S_t$ follows the model:

$$dS_t = \mu S_t dt + \sigma S_t dM_{t}^{\varepsilon,H}$$

(3.1)

where $M_{t}^{\varepsilon,H}$ is a modified fractional Brownian motion generated by the kernel $\varphi_H$. We focus on estimating parameters $\mu, \sigma, H$. A discrete approximation of (3.1) is given by

$$\Delta \log(S_t) = \mu \Delta t + \sigma \Delta M_{t}^{\varepsilon,H}.$$ 

Choose $\varepsilon$ to be a small positive number such as 0.1, 0.01, 0.001 and so on. We need to estimate $\mu, \sigma$ and $H$ based on the data set $\{S_{t_i}, i = 1, \cdots, n\}$. 
Let \( n \) be number of annual trading days and \( N \) be days in year. Let

\[
    r_i = \Delta \log(S_{t,i})
\]

and

\[
    \bar{r}_k = \frac{1}{k} \sum_{i=1}^{k} r_i.
\]

Then

\[
    r_i \sim N(\mu/N, \sigma^2 V_\delta^2(\delta)) \tag{3.2}
\]

where \( \delta = 1/N. \)

Method I: R/S Analysis and Moment Estimation.

\( H \) can be estimated by the R/S analysis. Let

\[
    R_k = \max_{1 \leq i \leq k} \sum_{j=1}^{i} (r_j - \bar{r}_k) - \min_{1 \leq i \leq k} \sum_{j=1}^{i} (r_j - \bar{r}_k) \tag{3.3}
\]

and

\[
    S_k = \sqrt{\frac{1}{k} \sum_{i=1}^{k} (r_i - \bar{r}_k)^2}. \tag{3.4}
\]

Then

\[
    Q_k = \frac{R_k}{S_k} = O(k^H). \tag{3.5}
\]

We have an estimator of \( H \):

\[
    \hat{H} = \frac{\sum_{k=m+1}^{n} (\log k - \frac{1}{n-m} \sum_{i=m+1}^{n} \log i) (\log(Q_k) - \frac{1}{n-m} \sum_{i=m+1}^{n} \log(Q_k))}{\sum_{k=m+1}^{n} (\log k - \frac{1}{n-m} \sum_{i=m+1}^{n} \log i)^2}. \tag{3.6}
\]

Using method of moments, we have that

\[
    \hat{\mu} = N \bar{r}_n \tag{3.7}
\]

and

\[
    \hat{\sigma} = \left( V_\hat{H}^2 \left( \frac{1}{N} \right) \right)^{-\frac{1}{2}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (r_i - \bar{r}_n)^2}. \tag{3.8}
\]
Method II: Maximum Likelihood Estimation.

Note that the joint distributed density of $r_1, r_2, \cdots, r_n$ is

$$f(r_1, r_2, \cdots, r_n|\mu, \sigma, H) = (2\pi \sigma^2)^{-\frac{n}{2}} |\Sigma_H|^{-\frac{1}{2}} \exp \left\{ - \frac{1}{2\sigma^2} \left( r - \frac{\mu}{N} 1 \right)^T \Sigma^{-1}_H \left( r - \frac{\mu}{N} 1 \right) \right\},$$

(3.9)

where

$$r = (r_1, \cdots, r_n)^T, \quad 1 = (1, \cdots, 1)^T$$

$$\Sigma_H = (\sigma_{ij}), \quad \sigma_{ij} = \gamma_H^a \left( \frac{i-j}{N}, \frac{1}{N} \right) = \sigma_{ji}.$$  

The log likelihood function is

$$l_n(\mu, \sigma, H) = - \frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2) - \frac{1}{2} \log(|\Sigma_H|) - \frac{1}{2\sigma^2} \left( r - \frac{\mu}{N} 1 \right)^T \Sigma^{-1}_H \left( r - \frac{\mu}{N} 1 \right).$$

(3.10)

Solving the likelihood equations

$$\frac{\partial l_n}{\partial \mu} = - \frac{1}{2\sigma^2} \left\{ -2 \frac{1}{N} r^T \Sigma^{-1}_H 1 + \frac{2\mu}{N^2} 1^T \Sigma^{-1}_H 1 \right\} = 0$$

and

$$\frac{\partial l_n}{\partial \sigma^2} = - \frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \left( r - \frac{\mu}{N} 1 \right)^T \Sigma^{-1}_H \left( r - \frac{\mu}{N} 1 \right) = 0$$

gives

$$\mu^* = \frac{N r^T \Sigma^{-1}_H 1}{1^T \Sigma^{-1}_H 1}$$

and

$$\sigma^* = \left\{ \left( r - \frac{\mu^*}{N} 1 \right)^T \Sigma^{-1}_H \left( r - \frac{\mu^*}{N} 1 \right) \right\}^{\frac{1}{2}}.$$

Substituting $\mu^*$ and $\sigma^*$ into the expression of $l_n$, we get

$$l_n(H) = - \frac{n}{2} \left[ \log(2\pi) + 1 - \log n \right] - \frac{n}{2} \log \left\{ \left( r - \frac{\mu^*}{N} 1 \right)^T \Sigma^{-1}_H \left( r - \frac{\mu^*}{N} 1 \right) \right\} - \frac{1}{2} \log |\Sigma_H|. $$

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Hence, the likelihood estimators of $\mu, \sigma^2, H$ are given by

$$\hat{\mu} = \frac{N(r^T \Sigma_{H}^{-1} 1)}{1^T \Sigma_{H}^{-1} 1},$$

$$\hat{\sigma}^2 = \frac{(r - \bar{X})^T \Sigma_{H}^{-1} (r - \bar{X})}{n},$$

and

$$\hat{H} = \arg\min_{0 < H < 1} \{-l_n(H)\}.$$ 

**Remark:** If we replace $\mathcal{M}_{H}^\varepsilon$ with $\mathcal{M}_{H}^{\varepsilon_a, b}$, then for fixed $a$ and $b$ we can derive maximum likelihood estimators for parameters $\mu, \sigma, H$. For example, take $a = \varphi_H(b)$ and $b = \delta$.

### 3.2 Simulation Methods

**Step 1:** Generate the long-range dependent path by simulating the multivariate normal distribution with the specified covariance matrix.

Let $t_0 = 0, t_1 = \Delta t, \cdots, t_i = i\Delta t, \cdots, t_n = T, n = T/\Delta t$, and $\delta = \Delta t$. Let $\varepsilon_i = M_t^\varepsilon - M_{t_{i-1}}^\varepsilon, i = 1, \cdots, n$. Then vector $(\varepsilon_1, \cdots, \varepsilon_n)^T$ follows n-dimensional normal distribution $N(0, \Sigma)$ where $\Sigma = (\sigma_{ij})$ with $\sigma_{ij} = \gamma_{H}(i - j|\delta, \delta)$. One can use the Cholesky decomposition for $\Sigma$ to simulate $(\varepsilon_1, \cdots, \varepsilon_n)^T$. Note that the Cholesky decomposition for $\Sigma$ produces a matrix $L$ which is lower triangular satisfying $LL^T = \Sigma$. Let $Z$ follow $N(0, I)$. Then $LZ$ follows $N(0, \Sigma)$.

**Step 2:** Generate underlying asset price path.

Direct discretization gives

$$S_{t_i} = S_{t_{i-1}} \exp(\mu\delta + \sigma \varepsilon_i).$$  \hspace{1cm} (3.11)
Step 3: Simulate option price (in term of the risk-neutral measure) by the standard Monte Carlo method.

For example, to estimate a European call option $C_0$, we simulate $S_T$ repeatedly $N$ times, then

$$\hat{C}_0 = \frac{1}{N} \sum_{k=1}^{N} e^{-rT} \max(S_T^k - K, 0).$$

To evaluate our numerical method, we compare the difference between the price obtained by Black-Scholes formula and that obtained by the proposed method.

Price a European call. Let expiration time $T = 1$, riskless interest rate $r = 0.1$, drift $\mu = r$, volatility $\sigma = 0.2$. Consider different $S_0$ and $K$: $S_0 = 50, 60, 70, 80, 90, 100$, $K = 55, 68, 75, 85, 100, 112$. The results are shown in Table 3.1 based on 50 replications.

Table 3.1: Comparison between Black-Scholes Formula and Simulated Option Pricing

| $S_0$ | $K$ | B-S    | $H = 0.5$    | $|e|$        |
|------|-----|--------|--------------|-------------|
| 50   | 55  | 4.091526 | 4.088164     | 0.003361736 |
| 60   | 68  | 4.113892 | 4.061760     | 0.052132364 |
| 70   | 75  | 6.625005 | 6.547680     | 0.077324623 |
| 80   | 85  | 7.914147 | 7.907729     | 0.006418274 |
| 90   | 100 | 6.948979 | 6.928210     | 0.020769289 |
| 100  | 112 | 7.365854 | 7.266830     | 0.099024434 |

In all cases, $|e| < 0.1$, so we conclude that our numerical method works well.
Chapter 4

Simulation Studies

4.1 Option Prices and Parameter $H$

We investigate the relationship between option price and parameter $H$. Consider a European call. Let expiration time $T = 1$, strike price $K = 55$, $S_0 = 50$, riskless interest rate $r = 0.1$, volatility $\sigma = 0.2$. Assume that the underlying asset price process follows as

$$dS_t = \mu S_t dt + \sigma S_t dM_t^\varepsilon, H.$$

Consider different $H$ for the model. Applying the modified fractional Black-Scholes option pricing formula to these cases, we obtain results shown in Table 4.1 and Figure 4.1. One can see that the option price is decreasing on $H$. Therefore, the Black-Scholes option pricing overestimates option in comparison with the modified fractional Black-Scholes option pricing with $H > 1/2$. 
Figure 4.1: Option Price against Parameter $H$
Table 4.1: Option Prices and Parameter $H$ ($S_0 = 50, K = 55$)

<table>
<thead>
<tr>
<th>$H$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_0$</td>
<td>6.02</td>
<td>5.60</td>
<td>5.15</td>
<td>4.64</td>
<td>4.09</td>
<td>3.49</td>
<td>2.84</td>
<td>2.15</td>
<td>1.41</td>
</tr>
</tbody>
</table>

4.2 Implied Volatility

A key implication of the Black-Scholes formula is that all standard European options on the same underlying asset with the same time-to-expiration should have the same implied volatility. Indeed, this idea is so ingrained in practice that options are commonly sold not by quoting price but by quoting implied volatility. And it becomes a way of testing the validity of the Black-Scholes formula to see if implied volatilities are the same independent of strike price. The relation between implied volatility and strike price is termed the implied volatility smile or skew. In practice, uncertainty about future volatility and potential jump movements in the underlying asset price are probably the most important reasons why option prices have volatility smiles which are inconsistent with the Black-Scholes formula.

Now we consider a European call option pricing where the underlying price process satisfies a stochastic differential equation driven by a modified fractional Brownian motion. Next we calculate European call option prices based on this model and use it to explain the skew and smile effect of the implied volatilities.

As Belkacem (1997) mentioned, the implied volatility of a given European call option price is the volatility that equals the European call option price given by the Black-Scholes model to that price. It is a volatility implicitly reflected in the option
price. In the Black and Scholes model, if we know the option price, the stock price, the expiration time and the interest rate, we can extract from formula (1.6) the volatility of the stock return implied to the call option. Unfortunately, the Black-Scholes formula cannot be inverted analytically to solve for implied volatility. Nonetheless, the formula can be quickly solved with numerical techniques to obtain a good approximation. In particular, we can implement a Newton-Raphson search, which typically converges in about three guesses to a close approximation of the true volatility.

We consider European calls. Let $S_0 = 100$, expiration time $T = 1$, riskless interest rate $r = 0.1$, volatility $\sigma = 0.2$. Assume that the underlying asset price process follows as

$$dS_t = \mu S_t dt + \sigma S_t dM_t^{e,H}.$$  

Calculate the option pricing for different $H \in (0,1)$ with different striking prices, and then estimate implied volatilities. The results are shown in Tables 4.2-9 and Figures 4.2-9. One can see that the implied volatilities display skew and smile which frequently occur in the market.
Table 4.2: Option Prices and Implied Volatilities for $H = 0.1$

<table>
<thead>
<tr>
<th>$K$</th>
<th>Option Price</th>
<th>Implied Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>37.300775</td>
<td>0.2979360</td>
</tr>
<tr>
<td>75</td>
<td>33.243189</td>
<td>0.2979360</td>
</tr>
<tr>
<td>80</td>
<td>29.393087</td>
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Table 4.3: Option Prices and Implied Volatilities for $H = 0.2$

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Table 4.4: Option Prices and Implied Volatilities for $H = 0.3$

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Table 4.5: Option Prices and Implied Volatilities for $H = 0.4$

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Table 4.6: Option Prices and Implied Volatilities for $H = 0.6$

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Table 4.7: Option Prices and Implied Volatilities for $H = 0.7$

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Table 4.8: Option Prices and Implied Volatilities for $H = 0.8$

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Table 4.9: Option Prices and Implied Volatilities for $H = 0.9$

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Figure 4.2: Implied Volatilities when $H = 0.1$
Figure 4.3: Implied Volatilities when $H = 0.2$
Figure 4.4: Implied Volatilities when $H = 0.3$
Figure 4.5: Implied Volatilities when $H = 0.4$
Figure 4.6: Implied Volatilities when $H = 0.6$
Figure 4.7: Implied Volatilities when $H = 0.7$
Figure 4.8: Implied Volatilities when $H = 0.8$
Figure 4.9: Implied Volatilities when $H = 0.9$
We consider the following four call options studied by Belkacem (1997):

(1) Some stock option, $H = 0.8$, $r = 8\%$, $\sigma = 15\%$, $S_0 = 70$, $T = 1$, $K = 60, 65, 68, 70, 73, 75, 77, 80, 84, 88$:

(2) NYSE index option, $H = 0.75$, $r = 7\%$, $\sigma = 46\%$, $S_0 = 100$, $T = 0.5$, $K = 80, 85, 90, 95, 100, 105, 110, 115, 118, 120$:

(3) CAC40 index option, $H = 0.85$, $r = 8\%$, $\sigma = 52\%$, $S_0 = 100$, $T = 0.25$, $K = 85, 90, 95, 100, 105, 108, 110, 113, 115, 118$:

(4) CAC40 index option, $H = 0.85$, $r = 8\%$, $\sigma = 52\%$, $S_0 = 100$, $T = 0.5$, $K = 85, 90, 95, 100, 105, 108, 110, 113, 115, 118$.

The results are shown in Tables 4.10-13 and Figures 10-13.
Table 4.10: Option Prices and Implied Volatilities for Some Stock

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Table 4.11: Option Prices and Implied Volatilities for NYSE Index

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Table 4.12: Option Prices and Implied Volatilities for CAC40 Index (T=0.25)

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Table 4.13: Option Prices and Implied Volatilities for CAC40 Index (T=0.5)

<table>
<thead>
<tr>
<th>$K$</th>
<th>Option Price</th>
<th>Implied Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>85</td>
<td>18.954260</td>
<td>0.34978663</td>
</tr>
<tr>
<td>90</td>
<td>14.866740</td>
<td>0.27173000</td>
</tr>
<tr>
<td>95</td>
<td>11.250754</td>
<td>0.16944628</td>
</tr>
<tr>
<td>100</td>
<td>8.201666</td>
<td>0.13893899</td>
</tr>
<tr>
<td>105</td>
<td>5.757023</td>
<td>0.22565800</td>
</tr>
<tr>
<td>108</td>
<td>4.572831</td>
<td>0.25476486</td>
</tr>
<tr>
<td>110</td>
<td>3.893410</td>
<td>0.26969079</td>
</tr>
<tr>
<td>113</td>
<td>3.026122</td>
<td>0.28659331</td>
</tr>
<tr>
<td>115</td>
<td>2.540330</td>
<td>0.03036273</td>
</tr>
<tr>
<td>118</td>
<td>1.934125</td>
<td>0.04095779</td>
</tr>
</tbody>
</table>
Figure 4.10: Implied Volatilities for Some Stock Option
Figure 4.11: Implied Volatilities for NYSE Index Option
Figure 4.12: Implied Volatilities for CAC40 Index Option (T=0.25)
Figure 4.13: Implied Volatilities for CAC40 Index Option (T=0.5)
Chapter 5

Application to Stock Study

To assess the performance of the modified fractional Black-Scholes formula, we compare the option pricing for the modified fractional model with the Black-Scholes formula using stock data.

5.1 Data

The stock and option data are from http://quote.yahoo.com. The data consist of historical prices of the AMZN stock from Feb. 8, 2005 to Feb. 9, 2006, shown in Figure 5.1, and the call options on the stock on Feb. 9, 2006, shown in Table 5.1.
Figure 5.1: Time Series Plot of AMZN Stock Prices, from Feb. 08, 2005 to Feb. 9, 2006
Table 5.1: AMZN Call Options on Feb. 9, 2006

<table>
<thead>
<tr>
<th>Strike</th>
<th>30</th>
<th>32.5</th>
<th>35</th>
<th>37.5</th>
<th>40</th>
<th>42.5</th>
<th>45</th>
<th>47.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matured on Feb. 17, 2006</td>
<td>-</td>
<td>-</td>
<td>3.1</td>
<td>1</td>
<td>0.15</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>Matured on Mar. 17, 2006</td>
<td>8.9</td>
<td>-</td>
<td>3.8</td>
<td>1.9</td>
<td>0.75</td>
<td>0.2</td>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>Matured on Apr. 21, 2006</td>
<td>9.3</td>
<td>5.6</td>
<td>4.2</td>
<td>2.5</td>
<td>1.34</td>
<td>0.8</td>
<td>0.3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

5.2 Examination of Black-Scholes Assumption

Let $r_1, r_2, \cdots, r_n$ be defined as in Section 3.1. Then, under Black-Scholes assumption, $r_1, r_2, \cdots, r_n$ are i.i.d. In this section, our goal is to test $H_0$: $r_1, r_2, \cdots, r_n$ are i.i.d.

The time series plot of AMZN stock return and its ACF are shown in Figures 5.2-3, respectively. For the AMZN stock return sequence, using Box-Pierce test, we get that $X$-squared = 7.69 and p-value = 0.005553. From the figure and testing results, one can see that we reject the hypothesis $H_0$. We conclude that the Black-Scholes assumption doesn’t hold for the AMZN stock price process.

5.3 Comparison with Black-Scholes Formula

To assess the proposed models and methods, we do comparison the modified fractional Black-Scholes formula with Black-Scholes formula. To estimate options, we first need to estimate parameters. Using the method presented in Section 3, we get

$$\hat{H} = 0.67, \hat{\sigma} = 0.58$$
Figure 5.2: Time Series Plot of AMZN Stock Return, from Feb. 08, 2005 to Feb. 09, 2006
Figure 5.3: ACF of AMZN Stock Return, from Feb. 08, 2005 to Feb. 09, 2006
based on the stock data. Then, using the methods presented in Section 2.3, we consider pricing AMZN stock call options on Feb. 9, 2006. The option prices computed by Black-Scholes formula and the modified fractional Black-Scholes formula are presented in Tables 5.2-4, respectively. Figures 5.4-6 show actual prices, Black-Scholes’ pricing and modified fractional Black-Scholes pricing for AMZN stock call options. From the figures, one can see that the modified fractional option pricing model is better than the Black-Scholes model based on our studies.

Table 5.2: AMZN Stock Call Options on Feb. 9, 2006 (Matured on Feb. 17, 2006)

<table>
<thead>
<tr>
<th>Strike</th>
<th>30</th>
<th>32.5</th>
<th>35</th>
<th>37.5</th>
<th>40</th>
<th>42.5</th>
<th>45</th>
<th>47.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>-</td>
<td>-</td>
<td>3.1</td>
<td>1</td>
<td>0.15</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>B-S</td>
<td>-</td>
<td>-</td>
<td>3.14</td>
<td>1.20</td>
<td>0.25</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>H=0.67</td>
<td>-</td>
<td>-</td>
<td>3.08</td>
<td>0.97</td>
<td>0.10</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 5.3: AMZN Stock Call Options on Feb. 9, 2006 (Matured on Mar. 17, 2006)

<table>
<thead>
<tr>
<th>Strike</th>
<th>30</th>
<th>32.5</th>
<th>35</th>
<th>37.5</th>
<th>40</th>
<th>42.5</th>
<th>45</th>
<th>47.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>8.9</td>
<td>-</td>
<td>3.8</td>
<td>1.9</td>
<td>0.75</td>
<td>0.2</td>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>B-S</td>
<td>8.32</td>
<td>-</td>
<td>3.93</td>
<td>2.32</td>
<td>1.22</td>
<td>0.57</td>
<td>0.24</td>
<td>0.09</td>
</tr>
<tr>
<td>H=0.67</td>
<td>8.29</td>
<td>-</td>
<td>3.60</td>
<td>1.84</td>
<td>0.75</td>
<td>0.24</td>
<td>0.06</td>
<td>0.01</td>
</tr>
</tbody>
</table>
Table 5.4: AMZN Stock Call Options on Feb. 9, 2006 (Matured on Apr. 21, 2006)

<table>
<thead>
<tr>
<th>Strike</th>
<th>30</th>
<th>32.5</th>
<th>35</th>
<th>37.5</th>
<th>40</th>
<th>42.5</th>
<th>45</th>
<th>47.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>9.3</td>
<td>5.6</td>
<td>4.2</td>
<td>2.5</td>
<td>1.34</td>
<td>0.8</td>
<td>0.3</td>
<td>0.1</td>
</tr>
<tr>
<td>B-S</td>
<td>8.75</td>
<td>6.61</td>
<td>4.76</td>
<td>3.25</td>
<td>2.11</td>
<td>1.30</td>
<td>0.77</td>
<td>0.44</td>
</tr>
<tr>
<td>H=0.67</td>
<td>8.60</td>
<td>6.29</td>
<td>4.23</td>
<td>2.57</td>
<td>1.41</td>
<td>0.70</td>
<td>0.31</td>
<td>0.13</td>
</tr>
</tbody>
</table>
Figure 5.4: AMZN Stock Call Option Pricing on Feb. 9, 2006 (Matured on 17/02/06)
Figure 5.5: AMZN Stock Call Option Pricing on Feb. 9, 2006 (Matured on 17/03/06)
Figure 5.6: AMZN Stock Call Option Pricing on Feb. 9, 2006 (Matured on 21/04/06)
Chapter 6

Conclusion

This study provides an extension of the classical Black and Scholes model for option pricing. We have studied modifications of fractional Brownian motion and option pricing under the hypothesis that the underlying asset price satisfies a stochastic differential equation driven by a modified fractional Brownian motion with long-term dependence. We have obtained modified pricing formula for European call option and an estimation of the implied volatility due to a misspecification of the underlying stock's return that fits well empirical results obtained on the market. We have investigated in particular the ability of the self-similarity parameter $H$ to explain the discrepancy between the Black-Scholes model and the reality of the market. We conclude that the modified fractional model is better than the Black-Scholes model for option pricing based on our study.
Bibliography


