MODIFIED FRACTIONAL BROWNIAN MOTION AND OPTION PRICING

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Abstract

The Black-Scholes model introduced by Black and Scholes (1973) and Merton (1973) has become synonymous with modern finance theory. It assumes that the dynamics of stock prices is well described by exponential Brownian motion, which is not consistent with empirical stock price returns, and then the dependence structure of stock price returns has been at the center of intense scrutiny for the last 30 or more years. This project studies modified fractional Brownian motions and shows that two different classes of modified fractional Brownian motions are equivalent to Brownian motion. We discuss option pricing under the hypothesis that the underlying asset price process satisfies a stochastic differential equation driven by a modified fractional Brownian motion. Parameter estimation and simulation methods are given. In particular, we investigate the ability of the self-similarity parameter H to explain the discrepancy between the Black-Scholes model and the reality of the market. The proposed method is applied to a real data set. The empirical results indicate that the model is better than the Black-Scholes model.

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Chapter 1

Introduction

1.1 Black-Scholes Model

We consider a market consisting of one bank account and one stock that pays no dividends. Borrowing and short-selling are allowed, the borrowing rate is equal to the lending rate, and it is possible to buy and sell any fraction of stock shares. Moreover, there exist no transaction costs and stock shares can be bought and sold at the same price. The bank account is a riskless security, where one always gets back the investment, plus interest which can be fixed, or which can vary with time. Shares of stock can be bought or sold in the market. Their price is subject to a large number of factors and it can go up or go down. The stock is considered a risky asset since we cannot be sure if the price will go up or down. The price of the stock at time t, S_t, is modeled as a random variable.

We are going to consider options on the stock. An option is security which gives its holder the right to sell or to buy the underlying asset (for example shares, currency) within a stated period (American option), or at the end of the stated period (European option) at a predetermined price. The options providing the right to buy are referred to as call options. If a contract gives the right to sell, it is called as put option.

We consider a standard European call option which gives its holder the right to buy at a given time T (expiry date) fixed in the future, a stock for price K (strike) pointed out in the contract. If the random price of the stock is described by the process (S_t), then the gain of the holder at time T equals $\max(S_T - K, 0)$. (If $S_T > K$, the holder buys the stock for the price K and sells it for the spot price S_T. In this case, he gains $S_T - K$. If $S_T < K$, it makes no sense to use the right given to him by the option of the contract.)

Let us start with the basic results of Black and Scholes (1973). Assume that the Bank account S_t^0 and the stock price S_t satisfy the following stochastic differential equation

$$dS_t^0 = rS_t^0 dt, \quad S_0^0 = 1, \tag{1.1}$$

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1.2}$$

where r denotes the riskless interest rate which is assumed to be constant, μ denotes the average rate of stock return, σ denotes the volatility coefficient which is assumed to be a positive constant, and B is the standard Brownian motion.

The solution to the above equation is given by

$$S_t^0 = e^{rt}, \tag{1.3}$$

$$S_t = S_0 \exp\left\{\mu t + \sigma B_t - \frac{\sigma^2}{2} t\right\}. \tag{1.4}$$

and thus the discounted stock price

$$\begin{aligned}\tilde{S}_t &= e^{-rt}S_t \\ &= S_0 \exp\left\{(\mu - r - \frac{\sigma^2}{2})t + \sigma B_t\right\}.\end{aligned}$$

Let

$$R_T = \exp\left\{aB_t - \frac{1}{2}a^2T\right\}, \quad \hat{B}_t = B_t - at$$

where $a = \frac{r-\mu}{\sigma}$. Define Q by $dQ = R_T dP$. Then by Girsanov's theorem, Q is a probability measure equivalent to P. We then have

$$\tilde{S}_t = S_0 e^{\hat{B}_t - \frac{1}{2}\sigma^2 t}, \quad d\tilde{S}_t = \sigma \tilde{S}_t d\hat{B}_t,$$

and $\hat{B}_t$ is a Brownian motion under Q. Then, it can be shown that the market consisting of the geometric Brownian motion and a bank account is complete.

Consider a European call option in this market with striking price K and terminal time T. Then completeness of the market tells us (see Kallianpur and Karandikar, 1999, Theorem 9.2) that the price C_t of the call option at time t is given by the following expression:

$$C_t = E_t^*\left[e^{-r(T-t)}\max(S_T - K, 0)\right], \tag{1.5}$$

where E_t^* is the conditional expectation under the equivalent martingale measure to P given all information available at time t. In such a model, one may obtain an explicit option price C_t at time t as the following

$$C_t = S_t\phi(h(S_t, t, \sigma)) - Ke^{-r(T-t)}\phi\left(h(S_t, t, \sigma) - \sigma\sqrt{T-t}\right), \tag{1.6}$$

where

$$h(S_t, t, \sigma) = \frac{log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

and ϕ is the standard normal distribution function:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

1.2 Dependence Structure of Stock Returns

The Black-Scholes model introduced by Black and Scholes (1973) and Merton (1973) has become synonymous with modern finance theory. It assumes that the dynamics of stock prices is well described by exponential Brownian motion; that is, stock returns behave like a sequence of independently identical distributed (i.i.d.) Gaussian random variables. However, that is not consistent with empirical stock returns, and so the dependence structure of stock returns has been at the center of intense scrutiny for the last 30 or more years. Early investigations into the dependence structure of asset returns (e.g., by Fama (1965, 1970)) concluded that successive returns could be assumed to be independent. Later on, Lo and Mackinlay (1988) revisited this random walk hypothesis; after a careful analysis of market from a 25-year period (1962-1987), they found substantial short-range dependence in the data and strongly rejected the hypothesis that asset returns are i.i.d. Mandelbrot (1967) and Greene and Fielitz (1977) presented the empirical findings of long-term memory in common stock returns. Huang and Yang (1995) discussed the fractal structure in multinational stock returns. Recently, using the CRSP (Center for Research in Security Prices) daily stock return data, Willinger, Taqqe and Teverovsky (1999) revisited the question of whether or not actual stock market prices exhibit long-range dependence. They found empirical evidence of long-range dependence in stock returns.

Hurst parameter H is commonly used to measure long-range dependence. The fractional Brownian motion is a typical long-range dependent process with Hurst parameter H. It is a natural generalization of well-known Brownian motion. Correspondingly, fractional Gaussian noise generated by fractional Brownian motion is a generalization of white Gaussian noise (i.e., a sequence of independently identical distributed Gaussian random variables).

Since the stock return processes are long-term dependent, it is not reliable for us to assume that the underlying asset prices follow geometric Brownian motions for option pricing. Based on this fact, we assume that asset price process follows a log Gaussian process with long-range dependence. That is, we replace the sequence of i.i.d. with a dependent sequence for underlying asset price return model. A natural choice of such a dependent process is a fractional Brownian motion. However, the fractional Brownian motion admits arbitrage. Therefore, we consider a modification of the fractional Brownian motion.

1.3 Fractional Brownian Motion

We let $\{B_t, t \in \Re\}$ be a standard Brownian motion, then the fractional Brownian motion $\{B_t^H, t \in \Re\}$ with self-similarity parameter $H \in (0,1)$ is defined by

$$B_t^H = C_H \left[\int_{-\infty}^{t} \varphi_H(t-s)dB_s - \int_{-\infty}^{0} \varphi_H(-s)dB_s \right], \tag{1.7}$$

where

$$\varphi_H(t) = t^{H-\frac{1}{2}} 1_{\{t>0\}}, \quad t \in \Re \tag{1.8}$$

and

$$C_H = \left\{ \frac{1}{2H} + \int_0^\infty (\varphi_H(1+s) - \varphi_H(s))^2 ds \right\}^{-\frac{1}{2}}. \tag{1.9}$$

The process B^H is clearly a zero-mean Gaussian process, and the constant C_H chosen to normalize the covariance structure neatly:

$$E\left|B_t^H - B_s^H\right|^2 = |t-s|^{2H}. \tag{1.10}$$

The correlation of successive increments is given as

$$\frac{E[-B_{-s}^H B_s^H]}{E[(B_s^H)^2]} = 2^{2H-1} - 1. \tag{1.11}$$

When $H = \frac{1}{2}$, the correlation is zero, as expected for the independent increments of classical Brownian motion. When $H < \frac{1}{2}$, the correlation is negative. When $H > \frac{1}{2}$, the correlation is positive.

Obviously, the fractional Brownian motion can be defined by a zero-mean Gaussian process such that (1.10) holds. Also the fractional Brownian motion path is a fractal curve with dimension $(2-H)$.

It is inconvenient that the fractional Brownian motion does not have a derivative. The derivative of smoothed Brownian motion is just the sequence of uncorrelated Gaussian random variables referred to as white Gaussian noise. Similarly, the derivative of smoothed fractional Brownian motion will be referred to as fractional Gaussian noise.

Let B^H be a fractional Brownian motion and let $\delta > 0$. Define

$$B_t^H(\delta) = \delta^{-1} \int_t^{t+\delta} B_u^H du \tag{1.12}$$

and

$$\left(B_t^H\right)'(\delta) = \delta^{-1}\left[B_{t+\delta}^H - B_t^H\right]. \tag{1.13}$$

Then we call $\left\{\left(B_t^H\right)'(\delta), t \in \Re\right\}$ a fractional Gaussian noise. It is a stationary process with the autocovariance below

$$Cov\left((B_{t+h}^H)'(\delta), (B_t^H)'(\delta)\right) = \frac{1}{2}\delta^{2H-2}\left\{\left(\frac{h}{\delta}+1\right)^{2H} + \left|\frac{h}{\delta}-1\right|^{2H} - 2\left(\frac{h}{\delta}\right)^{2H}\right\}. \tag{1.14}$$

Other properties of fractional Brownian motion can be seen in Mandelbrot and Van Ness (1968).

1.4 Arbitrage in Fractional Brownian Motion Models

In Lispster and Shiryaev (1989), Lin (1995), Rogers (1997) and Cheridito (2003) it is shown for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, fractional Brownian motion is not a semimartingale. Moreover Rogers (1997), Shiryaev (1998), Salopek (1998), Kallianpur and Karandikar (1999), Cheridito (2001) presented arbitrage strategies for fractional Brownian motion models.

Rogers (1997) constructed arbitrage for the following model:

$$S_t^0 = 1, S_t = S_0 + \sigma B_t^H. \tag{1.15}$$

His strategy consists of a combination of buy and hold strategies and works for all parameters $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. The drawbacks of this model are that S_t can become negative and the returns are lower for higher stock prices.

In Shiryaev (1998) only the case $H \in (\frac{1}{2}, 1)$ was treated. For the model:

$$S_t^0 = 1, S_t = S_0 + \mu t + \sigma B_t^H, \tag{1.16}$$

one can choose a $c > 0$ and set

$$\pi^0(t) = -c\left(\mu t + \sigma B_t^H\right)^2 - 2cS_0\left(\mu t + \sigma B_t^H\right), \quad \pi(t) = 2c\left(\mu t + \sigma B_t^H\right), \tag{1.17}$$

to obtain a self-financing arbitragy.

For the model:

$$S_t^0 = e^{rt}, S_t = S_0 e^{\mu t + \sigma B_t^H}, \tag{1.18}$$

one can set for all $c > 0$,

$$\pi^0(t) = cS_0 \left(1 - \exp\left\{2(\mu - r)t + 2\sigma B_t^H\right\}\right) \quad \pi(t) = 2c\left(\exp\left\{(\mu - r)t + \sigma B_t^H\right\} - 1\right) \tag{1.19}$$

to obtain a self-financing arbitragy.

In Kallianpur and Karandikar (1999), for $H \in (\frac{1}{2}, 1)$, the integral with respect to fractional Brownian motion is defined in terms of L^1 convergence of Riemann sums. Assume that the bank account S_t^0 and the stock price S_t satisfy the following stochastic differential equation

$$dS_t^0 = rS_t^0 dt, \tag{1.20}$$

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H. \tag{1.21}$$

The solution to the above equation is given by

$$S_t^0 = e^{rt}, \tag{1.22}$$

$$S_t = S_0 e^{\mu t + \sigma B_t^H}. \tag{1.23}$$

Set

$$\pi(t) = 2e^{\mu t + \sigma B_t^H}\left(e^{\sigma B_t^H} - 1\right). \tag{1.24}$$

Then π is a self-financing arbitrage strategy.

Cheridito (2003) constructed strong arbitrage strategies for the following models:

$$\frac{S_t}{S_t^0} = \nu(t) + \sigma B_t^H, \quad H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \tag{1.25}$$

and

$$\frac{S_t}{S_t^0} = e^{\nu(t)+\sigma B_t^H}, \quad H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right). \tag{1.26}$$

1.5 Fractional Black-Scholes Option Pricing Model

In Black-Scholes model, if the ordinary Brownian motion is replaced by fractional Brownian motion, then the model is called fractional Black-Scholes model. That is, the stock price satisfies the following equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H.$$

Since B_H is not a semimaringale, the usual Itô integral cannot be used. As shown in Section 1.4, the models based on fractional Brownian motion and related pathwise integral will admit arbitrage. However, some authors such as Hu and Øksendal (2003) and Elliot and van der Hoek (2003) insist on using fractional Black-Scholes model. In order to overcome arbitrage, Hu and Øksendal (2003) and Elliot and van der Hoek (2003) defined the stochastic integral based on the Wick product. In Hu and Øksendal (2003), only the case $H \in (\frac{1}{2}, 1)$ was considered. Elliot and van der Hoek (2003) extend that of Hu and Øksendal (2003) to the case $H \in (0, 1)$. The framework of their theory is to take the underlying probability space to be $(\mathcal{S}'(\Re), \mathcal{F})$ the space of tempered distributions, with $\mathcal{F}$ the Borel field. A probability measure P is given on $(\mathcal{S}'(\Re), \mathcal{F})$ by the Bochner-Minlos theorem. Fractional Brownian motion B_H is defined for $H \in (\frac{1}{2}, 1)$ by the Bochner-Minlos theorem in Hu and Øksendal (2003), and for $H \in (0, 1)$ by fractional transforms in Elliot and van der Hoek (2003). The fractional Black-Scholes market is shown to be complete in Hu and Øksendal (2003) and Elliot and van der Hoek (2003). Therefore, a fractional Black-Scholes formula for a European call with strike price K is given by

$$\begin{aligned} C_t &= E_t^* \left[e^{-r(T-t)} \max(S_T - K, 0)\right] \\ &= S_t \phi(h_H(S_t, t, \sigma)) - K e^{-r(T-t)} \phi\left(h_H(S_t, t, \sigma) - \sigma\sqrt{T^{2H} - t^{2H}}\right) \end{aligned} \tag{1.27}$$

where E_t^* is the conditional expectation under the equivalent martingale measure to P given all information available at time t,

$$h_H(S_t, t, \sigma) = \frac{log\frac{S_t}{K} + r(T-t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}},$$

and ϕ is the standard normal distribution function. Similar results have also been obtained by Benth (2003), Biagini and Øksendal (2003) and Biagini et al. (2002).

Chapter 2

Modified Fractional Brownian Motion and Option Pricing

2.1 Introduction

Why do we consider the modified fractional Brownian motion? According to earlier literature cited in Section 1.4, arbitrage opportunities exist with fractional Brownian motion, and then fractional Brownian motion is an absurd candidate for stock price process. On the other hand, as cited in Section 1.5, fractional Black-Scholes market has no arbitrage, and therefore a fractional Black-Scholes formula is obtained as the Black-Scholes formula. Clearly, the latter is not compatible with the former. More recently, Björk and Hult (2005) resolved this contradiction and concluded that the fractional Black-Scholes theory are not economically meaningful by pointing out that the definition of the self-financing trading strategies and/or the definition of the value of a portfolio used by the authors does not have a reasonable economic interpretation.

We need to modify fractional Brownian motion.

2.2 Modified Fractional Brownian Motion

To avoid arbitrage, we can change the convolution kernel in the Mandelbrot-van Ness representation of fractional Brownian motion such that the process is a semimartingale with a distribution similar to the one of fractional Brownian motion. Rogers (1997) constructed such a Gaussian process with the same long-range dependence as fractional Brownian motion. We define

$$M_t^{\varphi} = C_{\varphi}\left\{\int_{-\infty}^{t} \varphi(t-s)dB_s - \int_{-\infty}^{0} \varphi(-s)dB_s\right\} \tag{2.1}$$

where $\varphi(0) \neq 0$ and $\varphi(t) = 0$ for $t < 0$ such that $\frac{M^{\varphi}}{C_{\varphi}\varphi(0)}$ is equivalent to Brownian motion, and

$$C_{\varphi} = \left\{\int_0^1 \varphi^2(1-s)ds + \int_{-\infty}^{0} (\varphi(1-s) - \varphi(-s))^2 ds\right\}^{-\frac{1}{2}}, \tag{2.2}$$

then M^{φ} is called a modified fractional Brownian motion associated with the kernel φ. Clearly, M^{φ} is a centered Gaussian process. The increment $M_{t+\delta}^{\varphi} - M_t^{\varphi}$ is a stationary Gaussian process with the variance and the autocovariance below, respectively

$$E|M_{t+\delta}^{\varphi} - M_t^{\varphi}|^2 = V_{\varphi}(\delta) \tag{2.3}$$

where

$$V_{\varphi}(\delta) = C_{\varphi}^2\left(\int_{-\infty}^{0} (\varphi(\delta-s) - \varphi(-s))^2 ds + \int_0^{\delta} \varphi^2(\delta-s)ds\right), \tag{2.4}$$

and

$$E[(M_{t+\delta}^{\varphi} - M_t^{\varphi})(M_{s+\delta}^{\varphi} - M_s^{\varphi})] = \gamma_{\varphi}(t-s, \delta) \tag{2.5}$$

where

$$\gamma_\varphi(t-s,\delta) = \frac{1}{2}(V_\varphi(|t-s-\delta|) + V_\varphi(|t-s+\delta|) - 2V_\varphi(|t-s|)). \tag{2.6}$$

If $\varphi \in C^2(\Re_+)$, $\varphi(0) \neq 0$, $\varphi'(0) = 0$ and $\lim_{t\to\infty} \varphi''(t)t^{\frac{5}{2}-H} < \infty$, then M^φ is a Gaussian process with the same long-range dependence as the fractional Brownian motion and becomes a semimartingale because in this case, M^φ can be expressed as

$$\frac{M_t^\varphi}{C_\varphi} = \varphi(0)B_t + \int_0^t \left(\int_{-\infty}^s \varphi''(s-v)B_v dv\right) ds. \tag{2.7}$$

So, we have the following theorem.

Theorem 2.2.1. *Assume that* $\varphi \in C^2(\Re_+)$, $\varphi(0) \neq 0$, $\varphi'(0) = 0$ *and* $\lim_{t\to\infty} \varphi''(t)t^{\frac{5}{2}-H} < \infty$. *Set*

$$h(s) = \frac{1}{\varphi(0)} \int_{-\infty}^s \varphi''(s-v)B_v dv.$$

Let the probability measure Q *satisfy*

$$\frac{dQ}{dP} = \exp\left\{\int_0^T h(s)dB_s - \frac{1}{2}\int_0^T h^2(s)ds\right\}.$$

Then $\frac{M^\varphi}{C_\varphi\varphi(0)}$ *is a standard Brownian motion under* Q *for* $0 \leq t \leq T$.

Proof. Note that

$$\frac{M_t^\varphi}{C_\varphi\varphi(0)} = \int_0^t h(s)ds + B_t \tag{2.8}$$

and $h(s)$ has a normal distribution with mean 0 and variance σ_s^2 where for $s \geq 0$,

$$\begin{aligned}
\sigma_s^2 &\leq (\varphi(0))^{-2} \int_{-\infty}^s [\varphi''(s-v)]^2 |v| dv \\
&= (\varphi(0))^{-2} \int_0^\infty [\varphi''(u)]^2 |s-u| du \\
&\leq (\varphi(0))^{-2} \left(s\int_0^\infty [\varphi''(u)]^2 du + \int_0^\infty [\varphi''(u)]^2 u du\right)
\end{aligned}$$

From $\lim_{t\to\infty} \varphi''(t)t^{\frac{5}{2}-H} < \infty$, there exists a positive number N such that for $t \geq N$

$$|\varphi''(t)| \leq c_1 t^{H-\frac{5}{2}}$$

for some constant c_1, and then we have that

$$\begin{aligned}\int_0^\infty [\varphi''(u)]^2 du &\le \int_0^N [\varphi''(u)]^2 du + c_1 \int_N^\infty u^{2H-5} du \\ &= \int_0^N [\varphi''(u)]^2 du + c_1(4-2H)N^{2H-4} < \infty\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty [\varphi''(u)]^2 u du &\le \int_0^N [\varphi''(u)]^2 u du + c_1 \int_N^\infty u^{2H-4} du \\ &= \int_0^N [\varphi''(u)]^2 u du + c_1(3-2H)N^{2H-3} < \infty.\end{aligned}$$

Hence,

$$\sigma_s^2 \le c_2, \quad 0 \le s \le T$$

for some constant c_2. Since for $\delta < \frac{1}{2c_2}$ and $0 \le s \le T$,

$$\begin{aligned}& E\left[e^{\delta h^2(s)}\right] \\ =& \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma_s} e^{\delta x^2} \cdot e^{-\frac{x^2}{2\sigma_s^2}} dx \\ =& \frac{1}{\sqrt{1-2\delta\sigma_s^2}} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ \le& \frac{1}{\sqrt{1-2\delta c_2}},\end{aligned}$$

then there exists $\delta > 0$ and a positive constant c_3 such that

$$E\left[e^{\delta h^2(s)}\right] \le c_3$$

for each $s \in [0, T]$.

Note that

$$E\left[\int_0^T h^2(s) ds\right] \le (\varphi(0))^{-2} \int_0^T \int_{-\infty}^s [\varphi''(s-v)]^2 v dv ds < \infty.$$

From Corollary 7.2.2. and Theorem 7.1.3 (Girsanov) in Kallianpur (1980), we complete the proof of this theorem.

A natural choice of φ is as follows:

$$\varphi_H^{\varepsilon}(t) = (\varepsilon + t^2)^{\frac{2H-1}{4}} 1_{\{t \geq 0\}}.$$

In the special case of $\varepsilon = 0$ and $H = \frac{1}{2}$, we recover the well-known Black and Scholes model.

We have shown that this class of processes is equivalent to Brownian motion. However, Cheridito (2001) pointed out that such modification of fractional Brownian motion is not enough to ensure that the model is arbitrage-free. By regularising fractional Brownian motion, Cheridito gave a class of processes which have a unique equivalent martingale measure. If the kernel function φ satisfies the following condition:

$$\varphi(t) = \varphi(0) + \int_0^t \psi(s)ds, t \geq 0, \text{for some } \varphi \in L^2(\Re_+), \tag{2.9}$$

then the Gaussian process generated by the kernel φ is equivalent to Brownian motion. Set for $H \in (0,1)$, $a \in \Re$ and $b > 0$,

$$\varphi_H^{a,b}(t) = \begin{cases} a + \frac{\varphi_H(b)-a}{b}t, & t \in [0,b] \\ \varphi_H(t), & t \in (-\infty, 0) \cup (b, \infty). \end{cases}$$

The Gaussian process generated by the kernel $\varphi_H^{a,b}$ is called as regularized fractional Brownian motion. Cheridito (2001) figured out that the functions $\varphi_H^{a,b}(t)$ satisfy (2.9), and concluded that $M^{\varphi_H^{a,b}}$ is equivalent to Brownian motion for the case $a \neq 0$. Let's give another proof for this case.

Theorem 2.2.2. *Assume that $a \neq 0$. Then*

$$\frac{M^{\varphi_H^{a,b}}}{C_{\varphi_H^{a,b}} \varphi_H^{a,b}(0)} = B_t + \int_0^t h(s)ds \tag{2.10}$$

where

$$h(s) = \frac{1}{\varphi_H^{a,b}(0)} \left\{ \int_{-\infty}^{s-b} (H - \frac{1}{2})(s-v)^{H-3/2} dB_v + \frac{\varphi_H(b) - a}{b}(B_s - B_{s-b}) \right\}. \qquad (2.11)$$

Proof. Note that the derivative of $\varphi_H^{a,b}(t)$ is given by

$$\left(\varphi_H^{a,b}\right)'(t) = \begin{cases} \frac{\varphi_H(b)-a}{b}, & t \in (0,b) \\ \varphi_H'(t), & t \in (-\infty, 0) \cup (b, \infty). \end{cases}$$

For convenience of composition, we write $\varphi_H^{a,b}$ as φ without confusing.

For $0 \le t \le b$,

$$\begin{aligned}
\frac{1}{C_\varphi} M_t^\varphi &= \int_{-\infty}^{0} (\varphi(t-s) - \varphi(-s))\, dB_s + \int_0^t \varphi(t-s) dB_s \\
&= \int_{-\infty}^{-b} (\varphi(t-s) - \varphi(-s))\, dB_s + \int_{-b}^{t-b} (\varphi(t-s) - \varphi(-s))\, dB_s \\
&\quad + \int_{t-b}^{0} (\varphi(t-s) - \varphi(-s))\, dB_s + \int_0^t \varphi(t-s) dB_s \\
&= \int_{-\infty}^{-b} \left(\int_0^t \varphi'(v-s) dv \right) dB_s + \int_{-b}^{t-b} \left(\int_{s+b}^{t} \varphi'(v-s) dv + \int_0^{s+b} \varphi'(v-s) dv \right) dB_s \\
&\quad + \int_{t-b}^{0} \left(\int_0^t \varphi'(v-s) dv \right) dB_s + \int_0^t \left(\int_s^t \varphi'(v-s) dv \right) dB_s + \varphi(0) B_t \\
&= \int_0^t \left(\int_{-\infty}^{-b} \varphi'(v-s) dB_s \right) dv + \int_0^t \left(\int_{-b}^{v-b} \varphi'(v-s) dB_s \right) dv \\
&\quad + \int_0^t \left(\int_{v-b}^{t-b} \varphi'(v-s) dB_s \right) dv + \int_0^t \left(\int_{t-b}^{0} \varphi'(v-s) dB_s \right) dv \\
&\quad + \int_0^t \left(\int_0^v \varphi'(v-s) dB_s \right) dv + \varphi(0) B_t \\
&= \int_0^t \left(\int_{-\infty}^{v-b} \varphi'(v-s) dB_s \right) dv + \int_0^t \left(\int_{v-b}^{v} \varphi'(v-s) dB_s \right) dv + \varphi(0) B_t \\
&= \int_0^t \left(\int_{-\infty}^{v-b} \varphi'(v-s) dB_s + \int_{v-b}^{v} \varphi'(v-s) dB_s \right) dv + \varphi(0) B_t \\
&= \varphi(0) \left\{ B_t + \int_0^t h(s) ds \right\}
\end{aligned}$$

since

$$\int_{v-b}^{v} \varphi'(v-s) dB_s = \frac{\varphi_H(b) - a}{b}(B_s - B_{s-b}),$$

where h is defined by (2.11).

For $t > b$,

$$\begin{aligned}
\frac{1}{C_\varphi} M_t^\varphi &= \int_{-\infty}^{0} \left(\varphi(t-s) - \varphi(-s)\right) dB_s + \int_0^t \varphi(t-s) dB_s \\
&= \int_{-\infty}^{-b} \left(\varphi(t-s) - \varphi(-s)\right) dB_s + \int_{-b}^{0} \left(\varphi(t-s) - \varphi(-s)\right) dB_s \\
&\quad + \int_0^{t-b} \left(\varphi(t-s) - \varphi(-s)\right) dB_s + \int_{t-b}^{t} \varphi(t-s) dB_s \\
&= \int_{-\infty}^{-b} \left(\int_0^t \varphi'(v-s) dv \right) dB_s \\
&\quad + \int_{-b}^{0} \left(\int_{b+s}^{t} \varphi'(v-s) dv + \int_0^{b+s} \varphi'(v-s) dv \right) dB_s \\
&\quad + \int_0^{t-b} \left(\int_{b+s}^{t} \varphi'(v-s) dv + \varphi(b) \right) dB_s + \int_{t-b}^{t} \left(\int_s^t \varphi'(v-s) dv + \varphi(0) \right) dB_s \\
&= \int_0^t \left(\int_{-\infty}^{-b} \varphi'(v-s) dB_s \right) dv + \int_0^b \left(\int_{-b}^{v-b} \varphi'(v-s) dB_s \right) dv \\
&\quad + \int_b^t \left(\int_{-b}^{0} \varphi'(v-s) dB_s \right) dv + \int_0^b \left(\int_{v-b}^{0} \varphi'(v-s) dB_s \right) dv \\
&\quad + \int_b^t \left(\int_0^{v-b} \varphi'(v-s) dB_s \right) dv + \varphi(b) B_{t-b} \\
&\quad + \int_{t-b}^{t} \left(\int_{t-b}^{v} \varphi'(v-s) dB_s \right) dv + \varphi(0)(B_t - B_{t-b}) \\
&= \int_b^t \left(\int_{-\infty}^{-b} \varphi'(v-s) dB_s \right) dv + \int_0^b \left(\int_{-\infty}^{-b} \varphi'(v-s) dB_s \right) dv \\
&\quad + \int_b^t \left(\int_{-b}^{v-b} \varphi'(v-s) dB_s \right) dv + \int_0^b \left(\int_{-b}^{v-b} \varphi'(v-s) dB_s \right) dv \\
&\quad + \int_0^b \left(\int_{v-b}^{0} \varphi'(v-s) dB_s \right) dv + \varphi(b) B_{t-b} \\
&\quad + \int_{t-b}^{t} \left(\int_{t-b}^{v} \varphi'(v-s) dB_s \right) + \varphi(0)(B_t - B_{t-b}) \\
&= \int_0^t \left(\int_{-\infty}^{v-b} \varphi'(v-s) dB_s \right) dv \\
&\quad + \int_0^b \frac{\varphi_H(b) - a}{b} (-B_{v-b}) dv + \varphi_H(b) B_{t-b} \\
&\quad + \int_{t-b}^{t} \frac{\varphi_H(b) - a}{b} (B_v - B_{t-b}) dv + a(B_t - B_{t-b})
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \left(\int_{-\infty}^{v-b} \varphi'(v-s)dB_s \right) dv \\
&\quad - \int_0^b \frac{\varphi_H(b)-a}{b} B_{v-b}dv + \frac{\varphi_H(b)-a}{b} \int_{t-b}^t B_v dv \\
&\quad + \varphi_H(b)B_{t-b} - (\varphi_H(b)-a)B_{t-b} + a(B_t - B_{t-b}) \\
&= \int_0^t \left(\int_{-\infty}^{v-b} \varphi'(v-s)dB_s \right) dv \\
&\quad + \frac{\varphi_H(b)-a}{b} \left(\int_{t-b}^t B_v dv - \int_0^b B_{v-b}dv \right) + aB_t \\
&= aB_t + \int_0^t \left(\int_{-\infty}^{v-b} \varphi'(v-s)dB_s \right) dv \\
&\quad + \frac{\varphi_H(b)-a}{b} \left(\int_0^t (B_v - B_{v-b})dv - \int_0^{t-b} B_v dv + \int_b^t B_{v-b}dv \right) \\
&= aB_t + \int_0^t \left(\int_{-\infty}^{v-b} \varphi'(v-s)dB_s + \frac{\varphi_H(b)-a}{b}(B_v - B_{v-b}) \right) dv \\
&= \varphi(0) \left\{ B_t + \int_0^t h(s)ds \right\}.
\end{aligned}
$$

Theorem 2.2.3. *Let $\varphi_H^{a,b}(0) \neq 0$. Define $Q^{\varphi_H^{a,b}}$ by*

$$
\frac{dQ^{\varphi_H^{a,b}}}{dP} = \exp\left\{ \int_0^T h(s)dB_s - \frac{1}{2}\int_0^T h^2(s)ds \right\}. \tag{2.12}
$$

Then $\frac{M^{\varphi_H^{a,b}}}{C_{\varphi_H^{a,b}}\varphi_H^{a,b}(0)}$ is a standard Brownian motion under $Q^{\varphi_H^{a,b}}$ for $0 \le t \le T$.

Proof. By Theorem 2.2.2,

$$
\frac{M^{\varphi_H^{a,b}}}{C_{\varphi_H^{a,b}}\varphi_H^{a,b}(0)} = B_t + \int_0^t h(s)ds.
$$

Note that $h(s)$ has a normal distribution with mean 0 and variance σ_s^2 where for $s \ge 0$,

$$
\begin{aligned}
\sigma_s^2 &\le 2(\varphi(0))^{-2} \left\{ \int_{-\infty}^{s-b} [\varphi'(s-v)]^2 dv + \left(\frac{\varphi_H(b)-a}{b} \right)^2 b \right\} \\
&= 2(\varphi(0))^{-2} \left\{ \int_{-\infty}^{s-b} \left(H - \frac{1}{2} \right)^2 (s-v)^{2(H-\frac{3}{2})} dv + \left(\frac{\varphi_H(b)-a}{b} \right)^2 b \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= 2(\varphi(0))^{-2}\left\{\int_b^\infty \left(H-\frac{1}{2}\right)^2 u^{2H-3}dv + \left(\frac{\varphi_H(b)-a}{b}\right)^2 b\right\} \\
&= 2(\varphi(0))^{-2}\left\{(H-\frac{1}{2})^2\frac{b^{2H-2}}{2-2H} + \left(\frac{\varphi_H(b)-a}{b}\right)^2 b\right\} \\
&= c_4
\end{aligned}
$$

Since for $\delta < \frac{1}{2c_4}$ and $s \geq 0$,

$$
\begin{aligned}
& E\left[e^{\delta h^2(s)}\right] \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_s} e^{\delta x^2} \cdot e^{-\frac{x^2}{2\sigma_s^2}} dx \\
&= \frac{1}{\sqrt{1-2\delta\sigma_s^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&\leq \frac{1}{\sqrt{1-2\delta c_4}},
\end{aligned}
$$

then there exists $\delta > 0$ and a positive constant c_5 such that

$$
E\left[e^{\delta h^2(s)}\right] \leq c_5
$$

for each $s \in [0, T]$.

Note that for $T < \infty$,

$$
E\left[\int_0^T h^2(s)ds\right] = \int_0^T E[h^2(s)]ds < \infty.
$$

From Corollary 7.2.2. and Theorem 7.1.3 (Girsanov) in Kallianpur (1980), we complete the proof of this theorem.

2.3 Option Pricing with Modified Fractional Brownian Motion

Let the bank account S_t^0 and the stock price S_t satisfy the following stochastic differential equation:

$$dS_t^0 = rS_t^0 dt, \tag{2.13}$$

$$dS_t = \mu S_t dt + \sigma S_t dM_t^\varphi, \tag{2.14}$$

where r denotes the interest rate (for the sake of simplicity and to be able to compare this model with the Black-Scholes model, r is assumed to be constant); μ denotes the average rate of stock return; σ denotes the volatility coefficient of stock return (assumed to be positive and constant) and S_0 is a given constant.

By Theorem 2.2.1 and 2.2.3, for both $\varphi = \varphi_H^\epsilon$ and $\varphi = \varphi_H^{a,b}$, $\frac{M^\varphi}{C_\varphi \varphi(0)}$ are equivalent to Brownian motion. Therefore, for each case, there exists a unique probability measure P^* which is equivalent to P such that the discount stock price process $\{\tilde{S}_t, 0 \leq t \leq T\}$ is a martingale under P^*.

We consider now a standard European call option with the payment function $f_T = max(S_T - K, 0)$, where T is the expiration time, and K is the striking price. Then the price C_t of the call option at time t is given by

$$\begin{aligned} C_t &= E_t^* \left[e^{-r(T-t)} max(S_T - K, 0) \right] \\ &= S_t \phi(h(S_t, t, \sigma_\varphi)) - K e^{-r(T-t)} \phi\left(h(S_t, t, \sigma_\varphi) - \sigma_\varphi \sqrt{T-t} \right) \end{aligned}$$

where E_t^* denotes the conditioned expectation under the risk neutral probability P^* given all information available at time t, $\sigma_\varphi = \sigma C_\varphi \varphi(0)$, and h is defined as in Section 1.1.

Chapter 3

Parameter Estimation and Simulation Methods

3.1 Parameter Estimation

Assume that the underlying asset price process S_t follows the model:

$$dS_t = \mu S_t dt + \sigma S_t dM_t^{\varepsilon,H} \tag{3.1}$$

where $M_t^{\varepsilon,H}$ is a modified fractional Brownian motion generated by the kernel φ_H^{ε}. We focus on estimating parameters μ, σ, H. A discrete approximation of (3.1) is given by

$$\Delta \log(S_t) = \mu \Delta t + \sigma \Delta M_t^{\varepsilon,H}.$$

Choose ε to be a small positive number such as $0.1, 0.01, 0.001$ and so on. We need to estimate μ, σ and H based on the data set $\{S_{t_i}, i = 1, \cdots, n\}$.

Let n be number of annual trading days and N be days in year. Let

$$r_i = \Delta \log(S_{t_i})$$

and

$$\bar{r}_k = \frac{1}{k}\sum_{i=1}^{k} r_i.$$

Then

$$r_i \sim N(\mu/N, \sigma^2 V_H^{\varepsilon}(\delta)) \tag{3.2}$$

where $\delta = 1/N$.

Method I: R/S Analysis and Moment Estimation.

H can be estimated by the R/S analysis. Let

$$R_k = \max_{1\le i\le k}\sum_{j=1}^{i}(r_j - \bar{r}_k) - \min_{1\le i\le k}\sum_{j=1}^{i}(r_j - \bar{r}_k) \tag{3.3}$$

and

$$S_k = \sqrt{\frac{1}{k}\sum_{i=1}^{k}(r_i - \bar{r}_k)^2}. \tag{3.4}$$

Then

$$Q_k = \frac{R_k}{S_k} = O(k^H). \tag{3.5}$$

We have an estimator of H:

$$\tilde{H} = \frac{\sum_{k=m+1}^{n}(\log k - \frac{1}{n-m}\sum_{i=m+1}^{n} log_i)(\log(Q_k) - \frac{1}{n-m}\sum_{i=m+1}^{n} log(Q_k))}{\sum_{k=m+1}^{n}(\log k - \frac{1}{n-m}\sum_{i=m+1}^{n} log_i)^2}. \tag{3.6}$$

Using method of moments, we have that

$$\tilde{\mu} = N\bar{r}_n \tag{3.7}$$

and

$$\tilde{\sigma} = \left(V_{\tilde{H}}^{\varepsilon}\left(\frac{1}{N}\right)\right)^{-\frac{1}{2}}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r}_n)^2}. \tag{3.8}$$

Method II: Maximum Likelihood Estimation.

Note that the joint distributed density of $r_1, r_2, \cdots, r_n$ is

$$f(r_1, r_2, \cdots, r_n|\mu, \sigma, H) = (2\pi\sigma^2)^{-\frac{n}{2}}|\Sigma_H|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left(\mathbf{r}-\frac{\mu}{N}\mathbf{1}\right)^T \Sigma_H^{-1}\left(\mathbf{r}-\frac{\mu}{N}\mathbf{1}\right)\right\}, \tag{3.9}$$

where

$$\mathbf{r} = (r_1, \cdots, r_n)^T, \quad \mathbf{1} = (1, \cdots, 1)^T$$

$$\Sigma_H = (\sigma_{ij}), \quad \sigma_{ij} = \gamma_H^\varepsilon\left(\frac{i-j}{N}, \frac{1}{N}\right) = \sigma_{ji}.$$

The log likelihood function is

$$l_n(\mu, \sigma, H) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2}\log(|\Sigma_H|) - \frac{1}{2\sigma^2}\left(\mathbf{r}-\frac{\mu}{N}\mathbf{1}\right)^T \Sigma_H^{-1}\left(\mathbf{r}-\frac{\mu}{N}\mathbf{1}\right). \tag{3.10}$$

Solving the likelihood equations

$$\frac{\partial l_n}{\partial \mu} = -\frac{1}{2\sigma^2}\left\{-2\frac{1}{N}\mathbf{r}^T\Sigma_H^{-1}\mathbf{1} + \frac{2\mu}{N^2}\mathbf{1}^T\Sigma_H^{-1}\mathbf{1}\right\} = 0$$

and

$$\frac{\partial l_n}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\left(\mathbf{r}-\frac{\mu}{N}\mathbf{1}\right)^T \Sigma_H^{-1}\left(\mathbf{r}-\frac{\mu}{N}\mathbf{1}\right) = 0$$

gives

$$\mu^* = \frac{N\mathbf{r}^T\Sigma_H^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma_H^{-1}\mathbf{1}}$$

and

$$\sigma^* = \left\{\frac{\left(\mathbf{r}-\frac{\mu^*}{N}\mathbf{1}\right)^T \Sigma_H^{-1}\left(\mathbf{r}-\frac{\mu^*}{N}\mathbf{1}\right)}{n}\right\}^{\frac{1}{2}}.$$

Substituting μ^* and σ^* into the expression of l_n, we get

$$l_n(H) = -\frac{n}{2}[(\log(2\pi) + 1 - \log n] - \frac{n}{2}\log\left\{\left(\mathbf{r}-\frac{\mu^*}{N}\mathbf{1}\right)^T \Sigma_H^{-1}\left(\mathbf{r}-\frac{\mu^*}{N}\mathbf{1}\right)\right\} - \frac{1}{2}\log|\Sigma_H|.$$

Hence, the likelihood estimators of μ, σ^2, H are given by

$$\hat{\mu} = \frac{N\mathbf{r}^T\Sigma_{\hat{H}}^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma_{\hat{H}}^{-1}\mathbf{1}},$$

$$\hat{\sigma}^2 = \frac{\left(\mathbf{r} - \frac{\hat{\mu}}{N}\mathbf{1}\right)^T \Sigma_{\hat{H}}^{-1} \left(\mathbf{r} - \frac{\hat{\mu}}{N}\mathbf{1}\right)}{n},$$

and

$$\hat{H} = \text{argmin}_{0<H<1}\{-l_n(H)\}.$$

Remark: If we replace $M^{\varepsilon,H}$ with $M^{\varphi_H^{a,b}}$, then for fixed a and b we can derive maximum likelihood estimators for parameters μ, σ, H. For example, take $a = \varphi_H(b)$ and $b = \delta$.

3.2 Simulation Methods

Step 1: Generate the long-range dependent path by simulating the multivariate normal distribution with the specified covariance matrix.

Let $t_0 = 0, t_1 = \Delta t, \cdots, t_i = i\Delta t, \cdots, t_n = T, n = T/\Delta t$, and $\delta = \Delta t$. Let $\xi_i = M_{t_i}^{\varepsilon,H} - M_{t_{i-1}}^{\varepsilon,H}$, $i = 1, \cdots n$. Then vector $(\xi_1, \cdots, \xi_n)^T$ follows n-dimensional normal distribution $\mathbf{N}(\mathbf{0}, \mathbf{\Sigma})$ where $\mathbf{\Sigma} = (\sigma_{ij})$ with $\sigma_{ij} = \gamma_H^{\varepsilon}(|i-j|\delta, \delta)$. One can use the Cholesky decomposition for $\mathbf{\Sigma}$ to simulate $(\xi_1, \cdots, \xi_n)^T$. Note that the Cholesky decomposition for $\mathbf{\Sigma}$ produces a matrix L which is lower triangular satisfying $\mathbf{L}\mathbf{L}^\mathbf{T} = \mathbf{\Sigma}$. Let $\mathbf{Z}$ follow $\mathbf{N}(\mathbf{0}, \mathbf{I})$. Then $\mathbf{LZ}$ follows $\mathbf{N}(\mathbf{0}, \mathbf{\Sigma})$.

Step 2: Generate underlying asset price path.

Direct discretization gives

$$S_{t_i} = S_{t_{i-1}} \exp(\mu\delta + \sigma\xi_i). \tag{3.11}$$

Step 3: Simulate option price (in term of the risk-neutral measure) by the standard Monte Carlo method.

For example, to estimate a European call option C_0, we simulate S_T repeatedly N times, then

$$\hat{C}_0 = \frac{1}{N} \sum_{k=1}^{N} e^{-rT} \max(S_T^k - K, 0).$$

To evaluate our numerical method, we compare the difference between the price obtained by Black-Sholes formula and that obtained by the proposed method.

Price a European call. Let expiration time $T = 1$, riskless interest rate $r = 0.1$, drift $\mu = r$, volatility $\sigma = 0.2$. Consider different S_0 and K: $S_0 = 50, 60, 70, 80, 90, 100$, $K = 55, 68, 75, 85, 100, 112$. The results are shown in Table 3.1 based on 50 replications.

Table 3.1: Comparison between Black-Scholes Formula and Simulated Option Pricing

S_0	K	B-S	$H = 0.5$	$\lvert e \rvert$
50	55	4.091526	4.088164	0.003361736
60	68	4.113892	4.061760	0.052132364
70	75	6.625005	6.547680	0.077324623
80	85	7.914147	7.907729	0.006418274
90	100	6.948979	6.928210	0.020769289
100	112	7.365854	7.266830	0.099024434

In all cases, $|e| < 0.1$, so we conclude that our numerical method works well.

Chapter 4

Simulation Studies

4.1 Option Prices and Parameter H

We investigate the relationship between option price and parameter H. Consider a European call. Let expiration time $T = 1$, strike price $K = 55$, $S_0 = 50$, riskless interest rate $r = 0.1$, volatility $\sigma = 0.2$. Assume that the underlying asset price process follows as

$$dS_t = \mu S_t dt + \sigma S_t dM_t^{\varepsilon,H}.$$

Consider different H for the model. Applying the modified fractional Black-Scholes option pricing formula to these cases, we obtain results shown in Table 4.1 and Figure 4.1. One can see that the option price is decreasing on H. Therefore, the Black-Scholes option pricing overestimates option in comparison with the modified fractional Black-Scholes option pricing with $H > 1/2$.

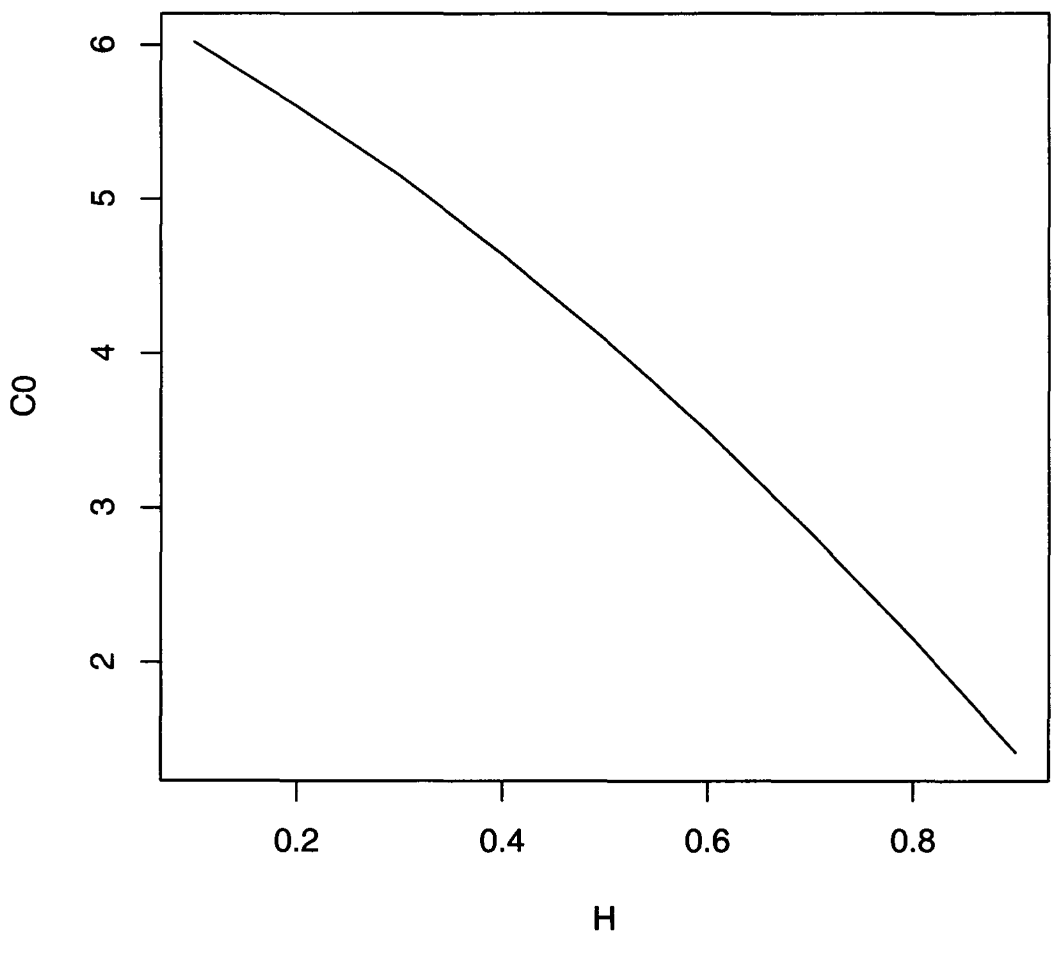

Figure 4.1: Option Price against Parameter H

Table 4.1: Option Prices and Parameter H ($S_0 = 50, K = 55$)

H	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
C_0	6.02	5.60	5.15	4.64	4.09	3.49	2.84	2.15	1.41

4.2 Implied Volatility

A key implication of the Black-Scholes formula is that all standard European options on the same underlying asset with the same time-to-expiration should have the same implied volatility. Indeed, this idea is so ingrained in practice that options are commonly sold not by quoting price but by quoting implied volatility. And it becomes a way of testing the validity of the Black-Scholes formula to see if implied volatilities are the same independent of strike price. The relation between implied volatility and strike price is termed the implied volatility smile or skew. In practice, uncertainty about future volatility and potential jump movements in the underlying asset price are probably the most important reasons why option prices have volatility smiles which are inconsistent with the Black-Scholes formula.

Now we consider a European call option pricing where the underlying price process satisfies a stochastic differential equation driven by a modified fractional Brownian motion. Next we calculate European call option prices based on this model and use it to explain the skew and smile effect of the implied volatilities.

As Belkacem (1997) mentioned, the implied volatility of a given European call option price is the volatility that equals the European call option price given by the Black-Scholes model to that price. It is a volatility implicitly reflected in the option price. In the Black and Scholes model, if we know the option price, the stock price, the expiration time and the interest rate, we can extract from formula (1.6) the volatility of the stock return implied to the call option. Unfortunately, the Black-Scholes formula cannot be inverted analytically to solve for implied volatility. Nonetheless, the formula can be quickly solved with numerical techniques to obtain a good approximation. In particular, we can implement a Newton-Raphson search, which typically converges in about three guesses to a close approximation of the true volatility.

We consider European calls. Let $S_0 = 100$, expiration time $T = 1$, riskless interest rate $r = 0.1$, volatility $\sigma = 0.2$. Assume that the underlying asset price process follows as

$$dS_t = \mu S_t dt + \sigma S_t dM_t^{\varepsilon,H}.$$

Calculate the option pricing for different $H \in (0, 1)$ with different striking prices, and then estimate implied volatilities. The results are shown in Tables 4.2-9 and Figures 4.2-9. One can see that the implied volatilities display skew and smile which frequently occur in the market.

Table 4.2: Option Prices and Implied Volatilities for $H = 0.1$

K	Option Price	Implied Volatility
70	37.300775	0.2979360
75	33.243189	0.2979360
80	29.393087	0.2979356
85	25.786693	0.2979358
90	22.452038	0.2979357
95	19.407378	0.2979359
100	16.660891	0.2979359
105	14.211405	0.2979361
110	12.049802	0.2979360
115	10.160765	0.2979361
120	8.524599	0.2979359
125	7.118926	0.2979359
130	5.920138	0.2979359
135	4.904562	0.2979360
140	4.049334	0.2979360

Table 4.3: Option Prices and Implied Volatilities for $H = 0.2$

K	Option Price	Implied Volatility
70	37.111441	0.2766129
75	32.964320	0.2766128
80	29.012791	0.2766126
85	25.300566	0.2766131
90	21.863759	0.2766127
95	18.727993	0.2766127
100	15.907170	0.2766127
105	13.403718	0.2766130
110	11.209961	0.2766131
115	9.310133	0.2766128
120	7.682635	0.2766130
125	6.302231	0.2766133
130	5.141961	0.2766132
135	4.174698	0.2766132
140	3.374299	0.2766131

Table 4.4: Option Prices and Implied Volatilities for $H = 0.3$

K	Option Price	Implied Volatility
70	36.945586	0.2533419
75	32.704523	0.2533422
80	28.640852	0.2533425
85	24.806842	0.2533419
90	21.249033	0.2533421
95	18.003324	0.2533423
100	15.092076	0.2533421
105	12.523357	0.2533424
110	10.292009	0.2533422
115	8.381932	0.2533423
120	6.768976	0.2533419
125	5.423889	0.2533423
130	4.314982	0.2533422
135	3.410289	0.2533420
140	2.679168	0.2533424

Table 4.5: Option Prices and Implied Volatilities for $H = 0.4$

K	Option Price	Implied Volatility
70	36.812889	0.2278669
75	32.477690	0.2278673
80	28.292852	0.2278672
85	24.319395	0.2278669
90	20.616975	0.2278674
95	17.236037	0.2278672
100	14.211927	0.2278670
105	11.561973	0.2278673
110	9.285587	0.2278672
115	7.366693	0.2278671
120	5.777533	0.2278668
125	4.482882	0.2278673
130	3.443987	0.2278672
135	2.621777	0.2278668
140	1.979207	0.2278670

Table 4.6: Option Prices and Implied Volatilities for $H = 0.6$

K	Option Price	Implied Volatility
70	36.6760740	0.1696962
75	32.1905858	0.1696965
80	27.7704161	0.1696963
85	23.4792997	0.1696961
90	19.4039654	0.1696961
95	15.6416437	0.1696966
100	12.2813752	0.1696961
105	9.3862020	0.1696965
110	6.9824860	0.1696964
115	5.0588194	0.1696961
120	3.5729944	0.1696963
125	2.4632686	0.1696963
130	1.6600401	0.1696964
135	1.0952789	0.1696962
140	0.7086218	0.1696964

Table 4.7: Option Prices and Implied Volatilities for $H = 0.7$

K	Option Price	Implied Volatility
70	36.6626118	0.13709611
75	32.1450046	0.13709601
80	27.6490242	0.13709577
85	23.2164529	0.13709569
90	18.9283047	0.13709603
95	14.9052048	0.13709603
100	11.2860953	0.13709607
105	8.1914953	0.13709579
110	5.6894055	0.13709608
115	3.7802338	0.13709640
120	2.4047357	0.13709585
125	1.4669616	0.13709582
130	0.8600349	0.03079045
135	0.4857826	0.13709618
140	0.2650602	0.13709606

Table 4.8: Option Prices and Implied Volatilities for $H = 0.8$

K	Option Price	Implied Volatility
70	36.66138757	0.10237136
75	32.13734517	0.10237120
80	27.61489821	0.10237156
85	23.10351633	0.10237092
90	18.64173193	0.05621663
95	14.33265455	0.10237135
100	10.36329874	0.10237095
105	6.96370865	0.10237131
110	4.31090471	0.10237150
115	2.44684514	0.10237158
120	1.27195712	0.10237129
125	0.60662912	0.10237121
130	0.26641528	0.02594019
135	0.10826449	0.10237125
140	0.04093211	0.10237120

Table 4.9: Option Prices and Implied Volatilities for $H = 0.9$

K	Option Price	Implied Volatility
70	3.666138e+01	0.07627822
75	3.213719e+01	0.06495548
80	2.761301e+01	0.06474575
85	2.308885e+01	0.06474574
90	1.856583e+01	0.06205170
95	1.406016e+01	0.06474568
100	9.679152e+00	0.06474574
105	5.762641e+00	0.06474596
110	2.817184e+00	0.06474565
115	1.088338e+00	0.06474552
120	3.261592e-01	0.06474572
125	7.570083e-02	0.02438365
130	1.373425e-02	0.02383378
135	1.977474e-03	0.06474565
140	2.299734e-04	0.06474566

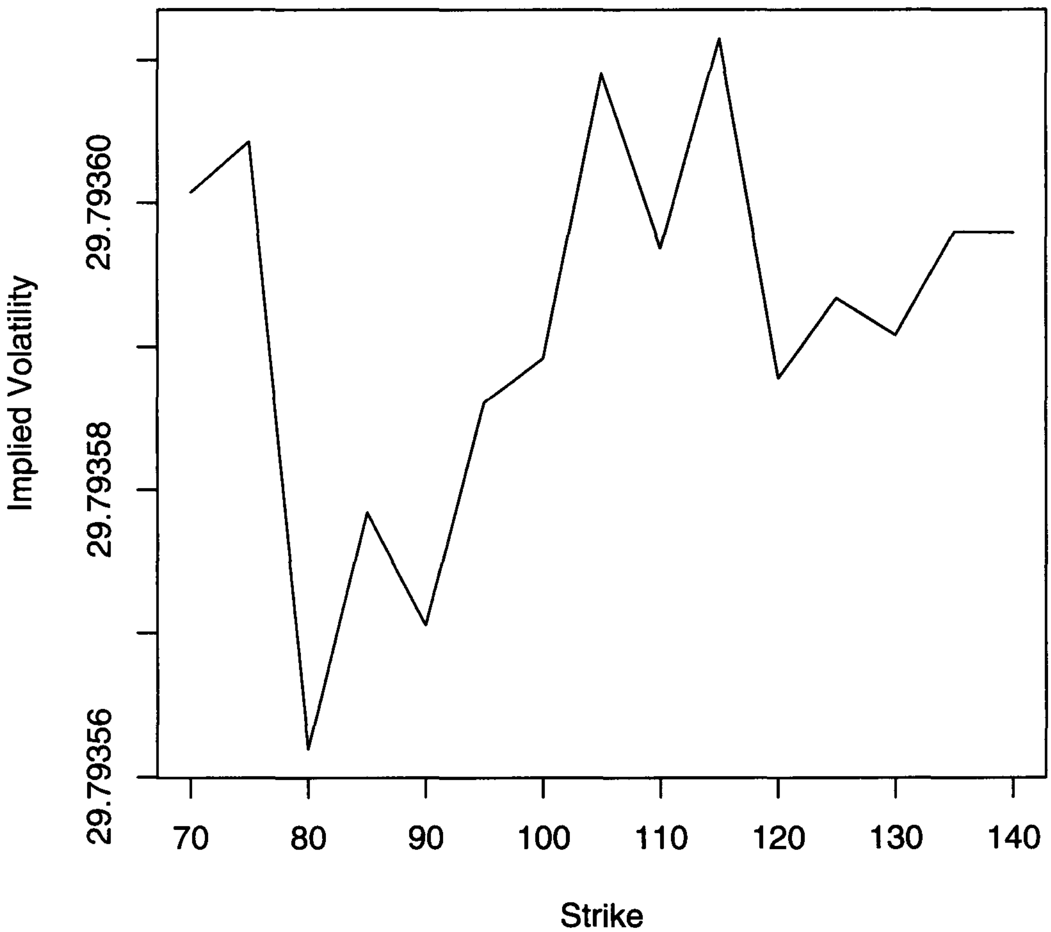

Figure 4.2: Implied Volatilities when $H = 0.1$

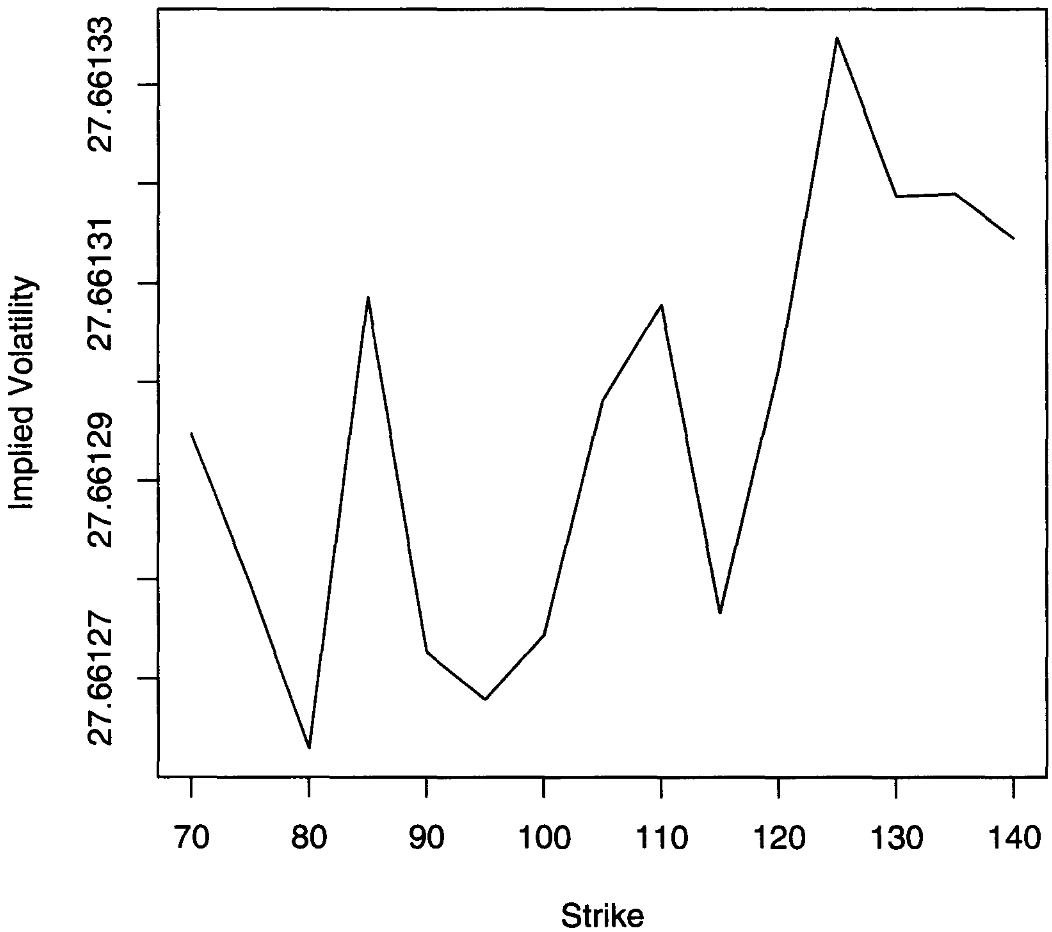

Figure 4.3: Implied Volatilities when $H = 0.2$

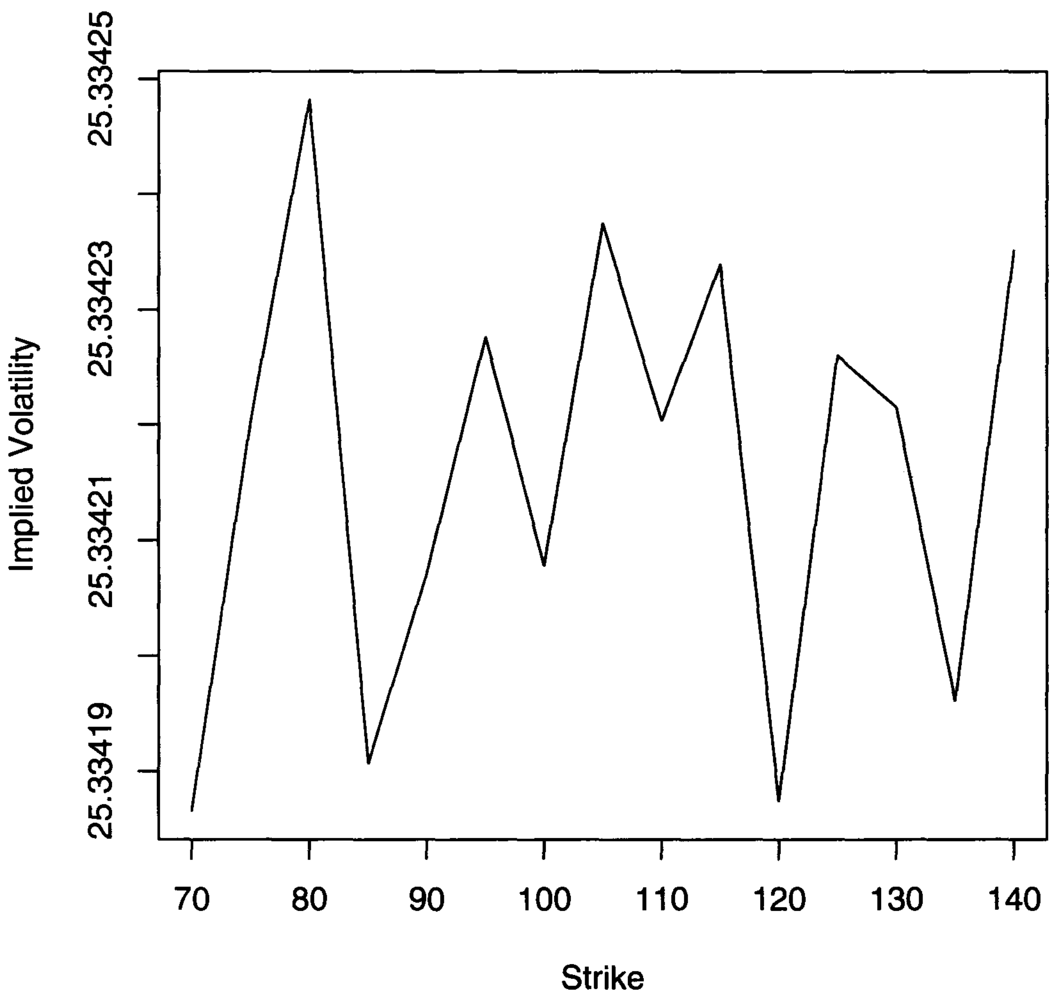

Figure 4.4: Implied Volatilities when $H = 0.3$

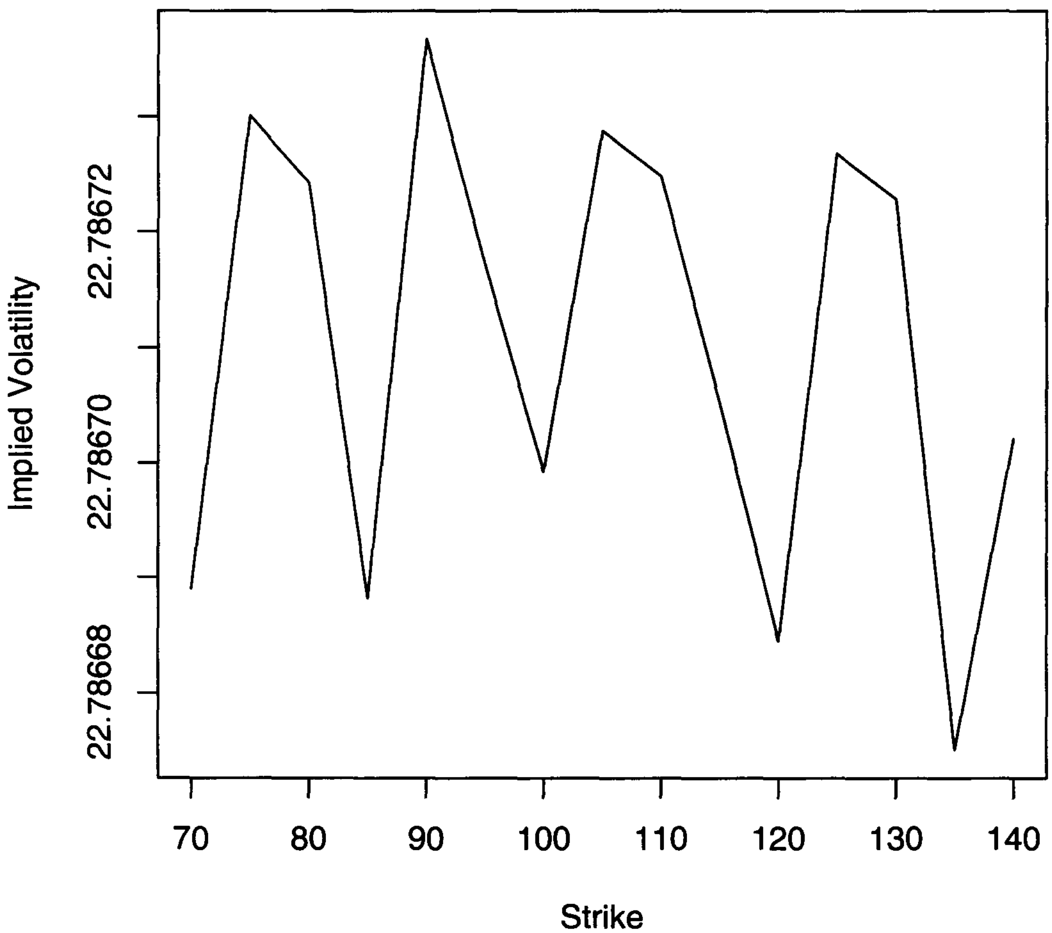

Figure 4.5: Implied Volatilities when $H = 0.4$

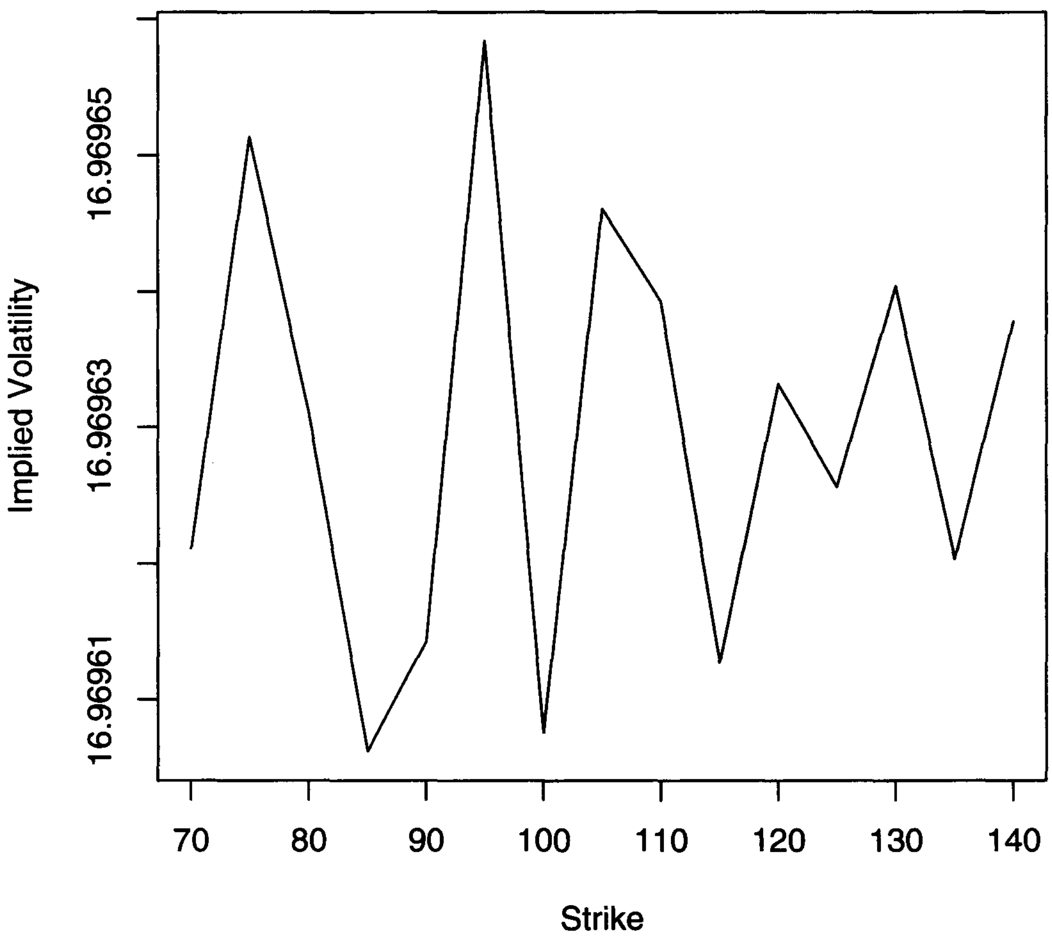

Figure 4.6: Implied Volatilities when $H = 0.6$

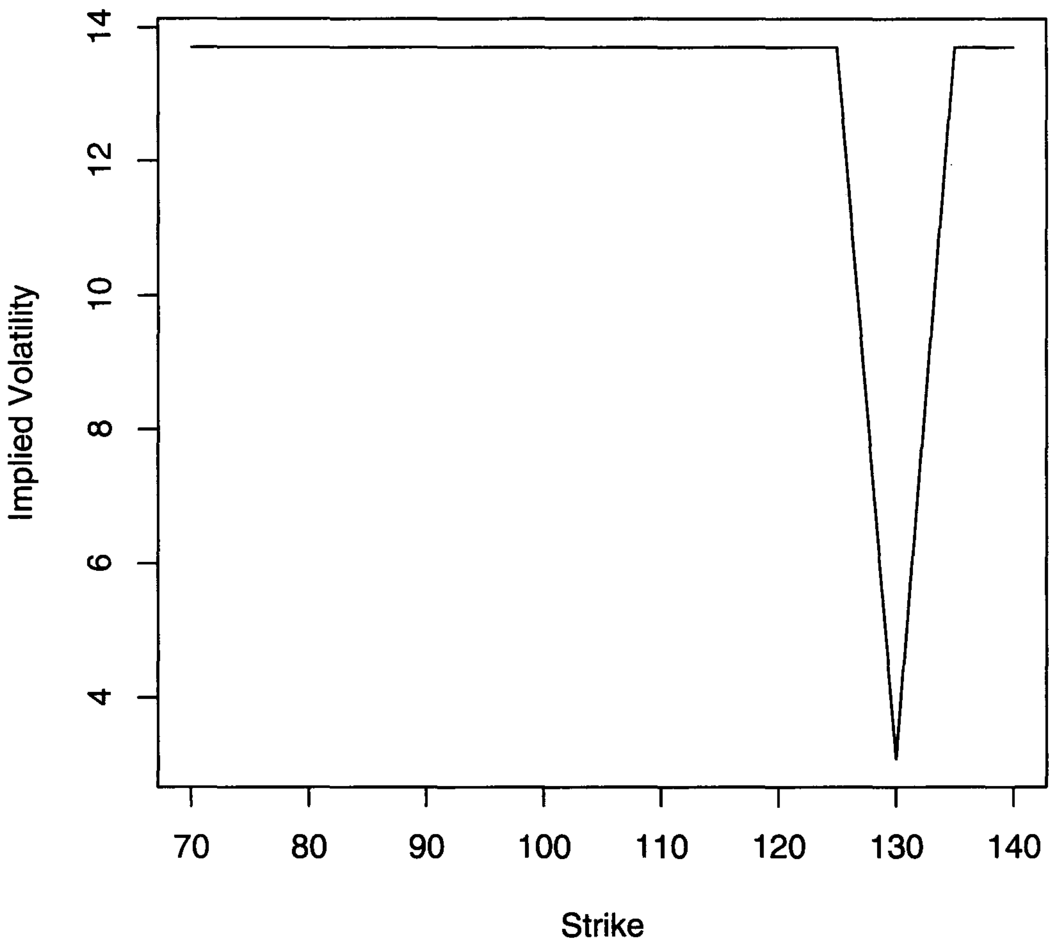

Figure 4.7: Implied Volatilities when $H = 0.7$

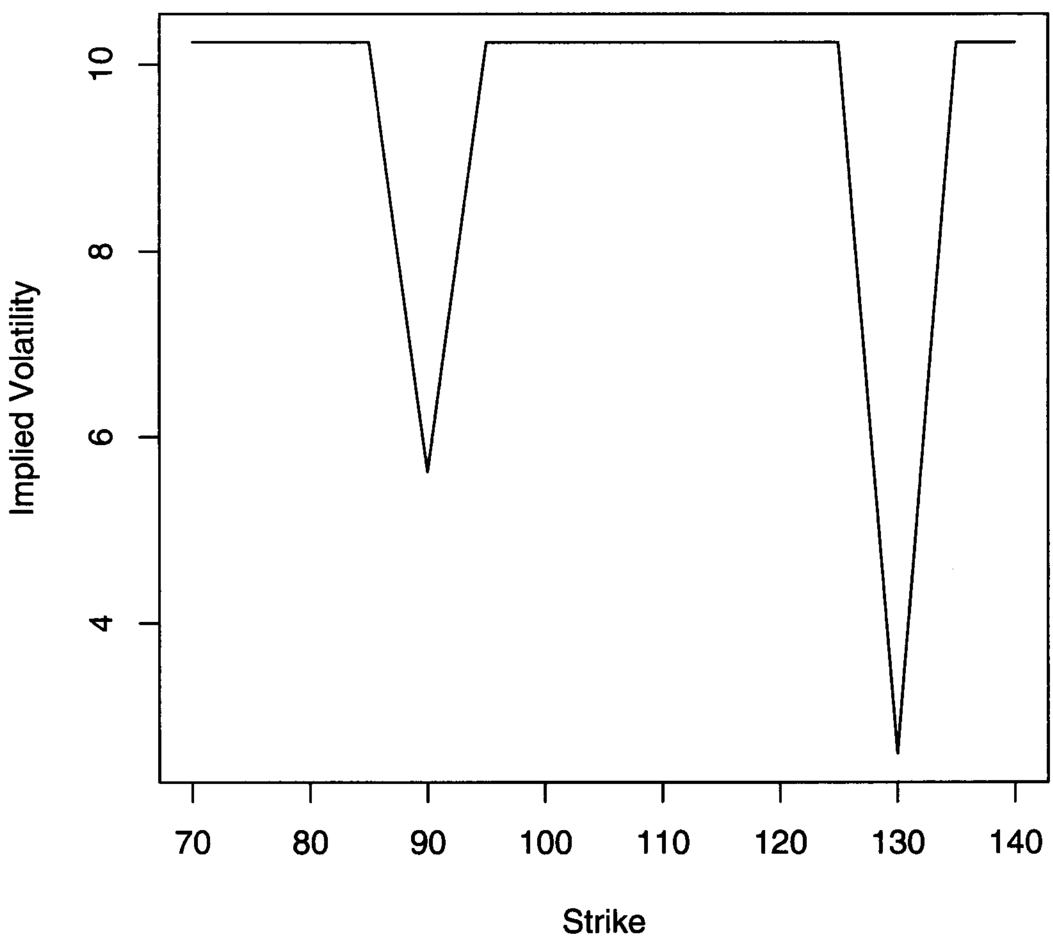

Figure 4.8: Implied Volatilities when $H = 0.8$

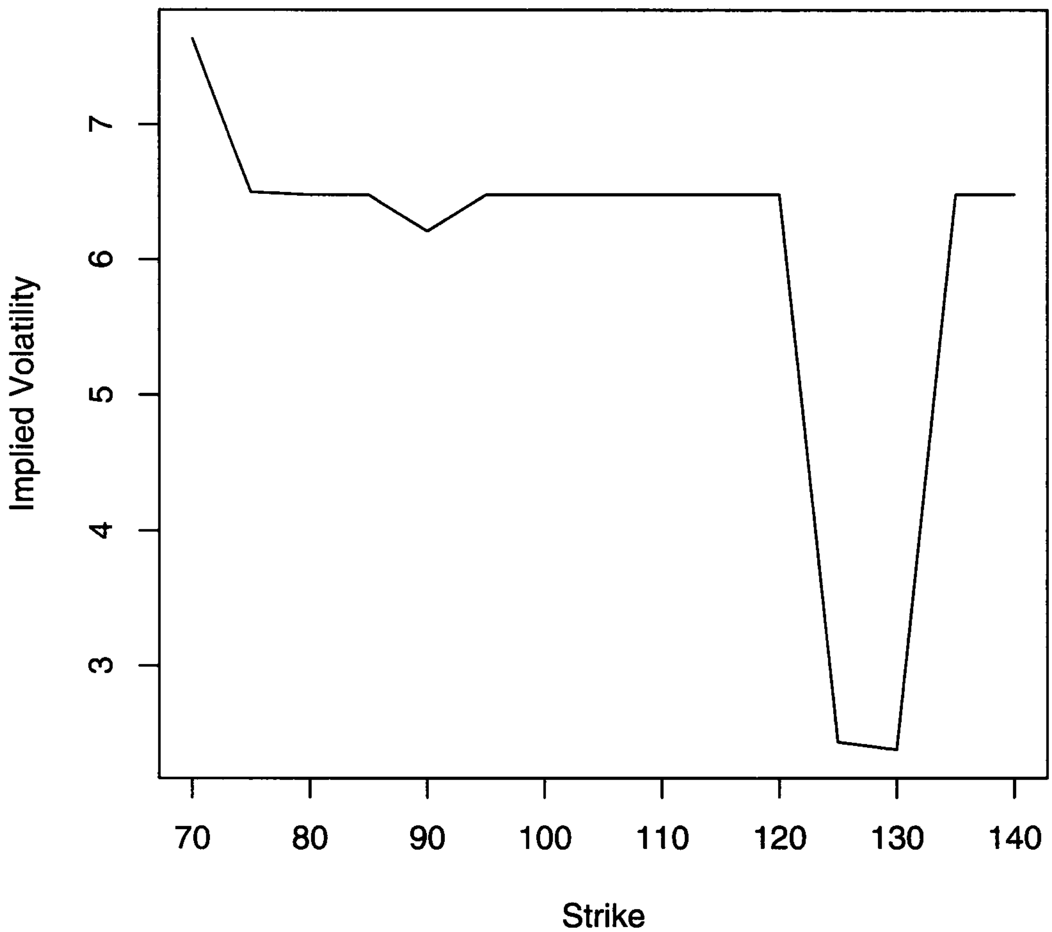

Figure 4.9: Implied Volatilities when $H = 0.9$

We consider the following four call options studied by Belkacem (1997):

(1) Some stock option, $H = 0.8$, $r = 8\%$, $\sigma = 15\%$, $S_0 = 70$, $T = 1$, $K = 60, 65, 68, 70, 73, 75, 77, 80, 84, 88$;

(2) NYSE index option, $H = 0.75$, $r = 7\%$, $\sigma = 46\%$, $S_0 = 100$, $T = 0.5$, $K = 80, 85, 90, 95, 100, 105, 110, 115, 118, 120$;

(3) CAC40 index option, $H = 0.85$, $r = 8\%$, $\sigma = 52\%$, $S_0 = 100$, $T = 0.25$, $K = 85, 90, 95, 100, 105, 108, 110, 113, 115, 118$;

(4) CAC40 index option, $H = 0.85$, $r = 8\%$, $\sigma = 52\%$, $S_0 = 100$, $T = 0.5$, $K = 85, 90, 95, 100, 105, 108, 110, 113, 115, 118$.

The results are shown in Tables 4.10-13 and Figures 10-13.

Table 4.10: Option Prices and Implied Volatilities for Some Stock

K	Option Price	Implied Volatility
60	14.61454943	0.07677805
65	10.03885711	0.02633231
68	7.40674245	0.07677850
70	5.77859859	0.07677857
73	3.66265356	0.07677847
75	2.53685076	0.07677874
77	1.66292636	0.07677833
80	0.79212830	0.07677841
84	0.24002439	0.02173405
88	0.05782948	0.07677834

Table 4.11: Option Prices and Implied Volatilities for NYSE Index

K	Option Price	Implied Volatility
80	23.493111	0.4470354
85	19.358743	0.3825094
90	15.604741	0.3157222
95	12.299976	0.2334549
100	9.481735	0.1346165
105	7.152723	0.2185467
110	5.285239	0.2615025
115	3.829786	0.2839489
118	3.128973	0.2869092
120	2.724994	0.2811354

Table 4.12: Option Prices and Implied Volatilities for CAC40 Index (T=0.25)

K	Option Price	Implied Volatility
85	16.8792990	0.518257930
90	12.4232598	0.388540804
95	8.5128191	0.215600834
100	5.3777236	0.006350021
105	3.1144201	0.345782029
108	2.1507616	0.395300951
110	1.6510519	0.422045823
113	1.0820129	0.454842889
115	0.8026180	0.024694220
118	0.5002408	0.492970153

Table 4.13: Option Prices and Implied Volatilities for CAC40 Index(T=0.5)

K	Option Price	Implied Volatility
85	18.954260	0.34978663
90	14.866740	0.27173000
95	11.250754	0.16944628
100	8.201666	0.13893899
105	5.757023	0.22565800
108	4.572831	0.25476486
110	3.893410	0.26969079
113	3.026122	0.28659331
115	2.540330	0.03036273
118	1.934125	0.04095779

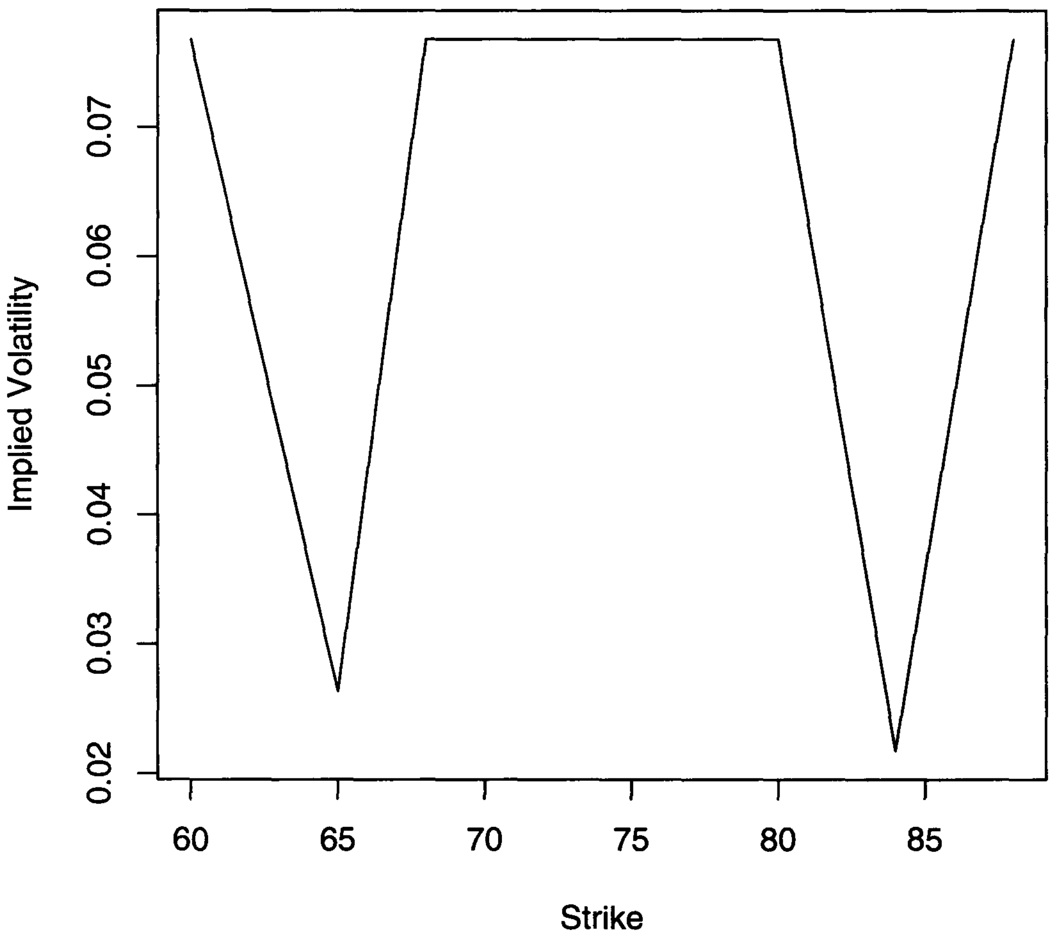

Figure 4.10: Implied Volatilities for Some Stock Option

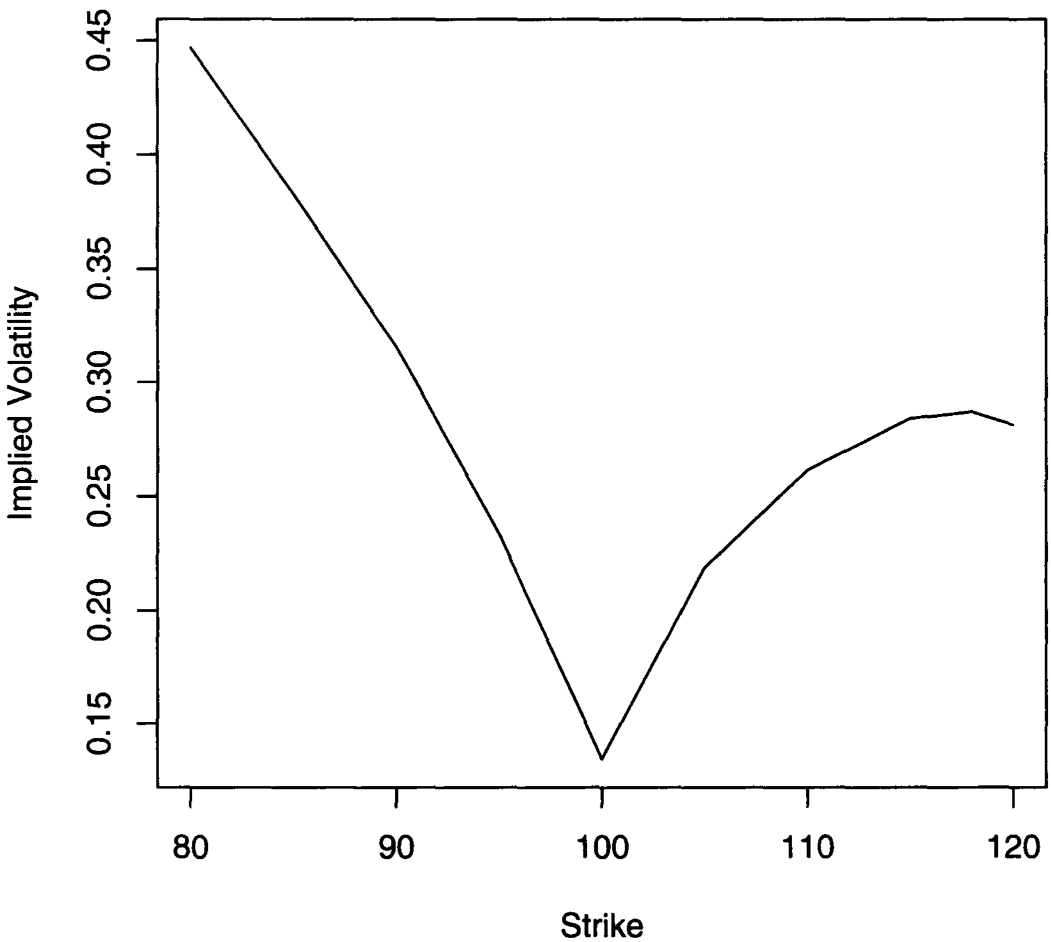

Figure 4.11: Implied Volatilities for NYSE Index Option

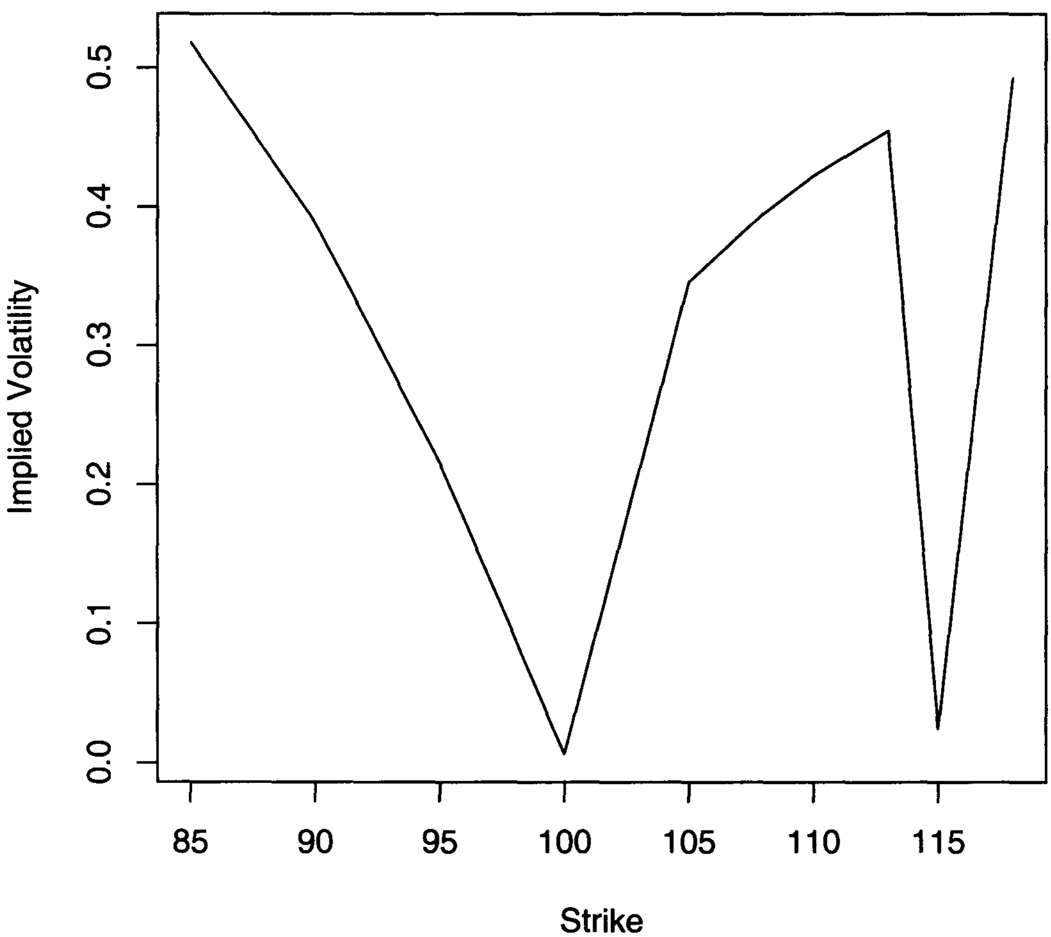

Figure 4.12: Implied Volatilities for CAC40 Index Option (T=0.25)

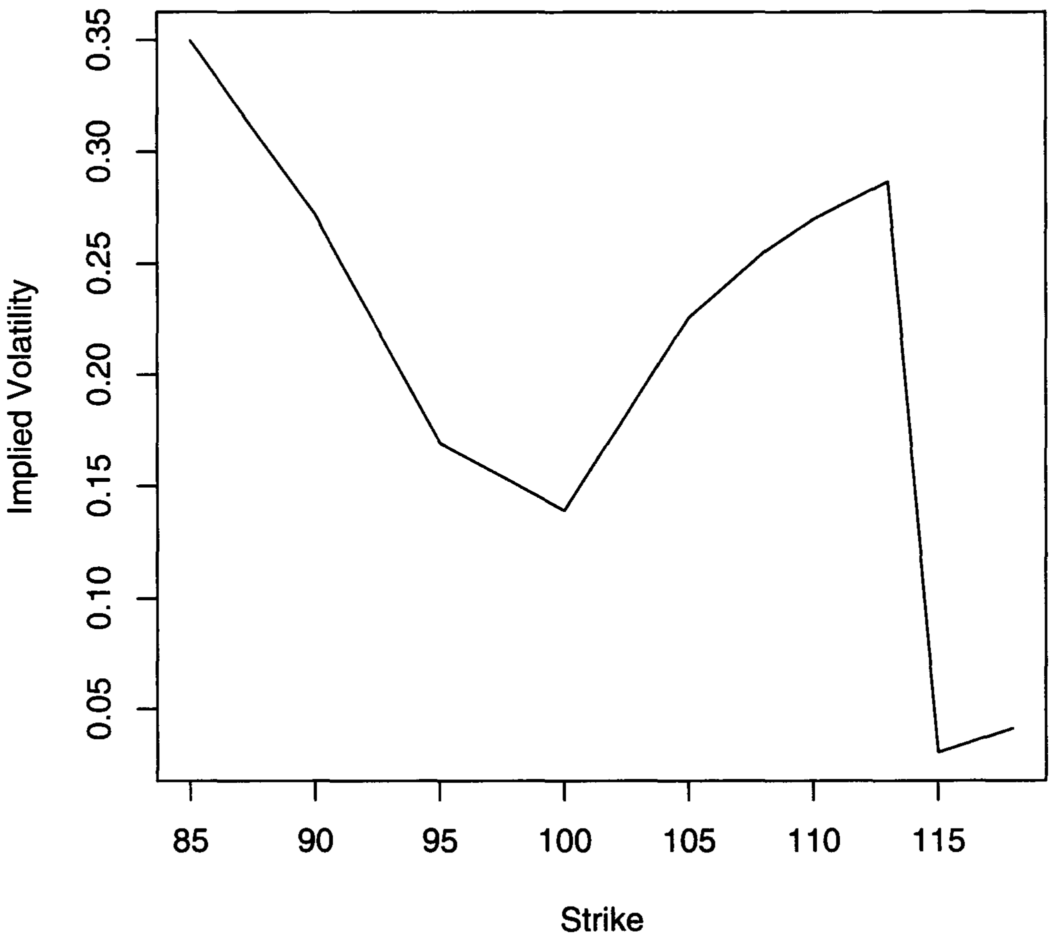

Figure 4.13: Implied Volatilities for CAC40 Index Option (T=0.5)

Chapter 5

Application to Stock Study

To assess the performance of the modified fractional Black-Scholes formula, we compare the option pricing for the modified fractional model with the Black-Scholes formula using stock data.

5.1 Data

The stock and option data are from http://quote.yahoo.com. The data consist of historical prices of the AMZN stock from Feb. 08, 2005 to Feb. 9, 2006, shown in Figure 5.1, and the call options on the stock on Feb. 9, 2006, shown in Table 5.1.

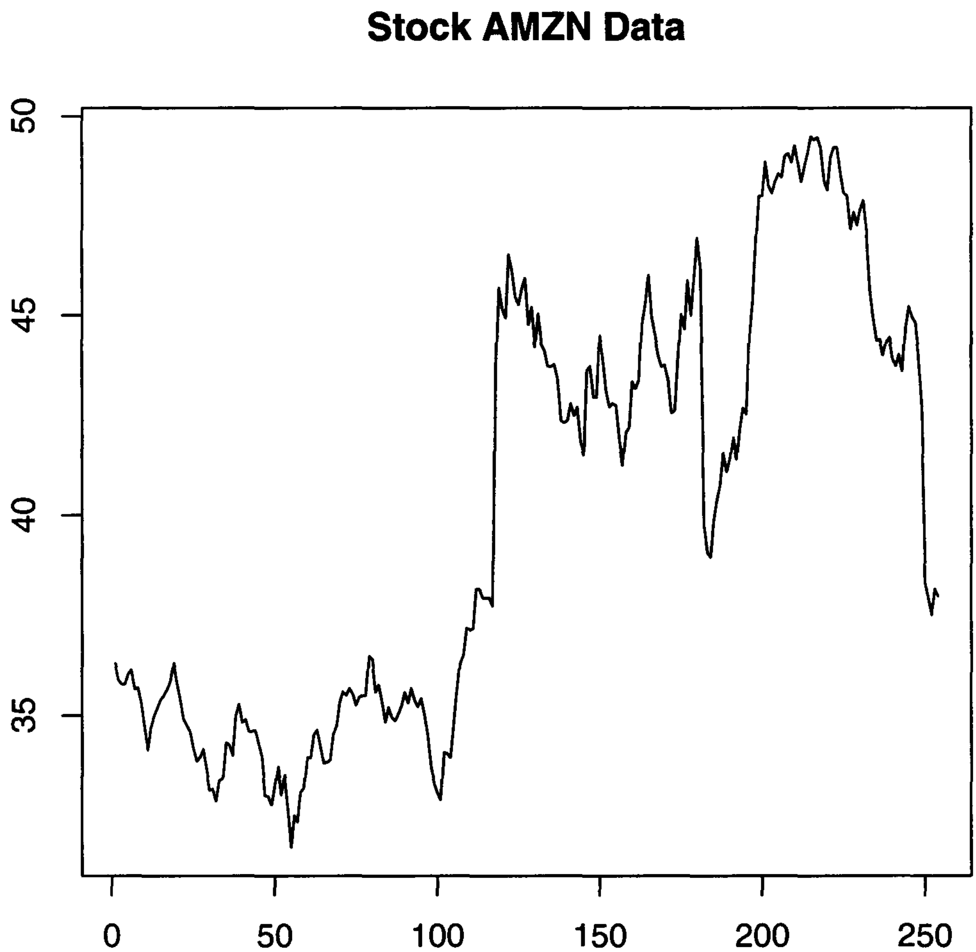

Figure 5.1: Time Series Plot of AMZN Stock Prices, from Feb. 08, 2005 to Feb. 9, 2006

Table 5.1: AMZN Call Options on Feb. 9, 2006

Strike	30	32.5	35	37.5	40	42.5	45	47.5
Matured on Feb. 17, 2006	-	-	3.1	1	0.15	0.05	0.05	0.05
Matured on Mar. 17, 2006	8.9	-	3.8	1.9	0.75	0.2	0.1	0.05
Matured on Apr. 21, 2006	9.3	5.6	4.2	2.5	1.34	0.8	0.3	0.1

5.2 Examination of Black-Scholes Assumption

Let $r_1, r_2, \cdots, r_n$ be defined as in Section 3.1. Then, under Black-Scholes assumption, $r_1, r_2, \cdots, r_n$ are i.i.d. In this section, our goal is to test H_0: $r_1, r_2, \cdots, r_n$ are i.i.d. The time series plot of AMZN stock return and its ACF are shown in Figures 5.2-3, respectively. For the AMZN stock return sequence, using Box-Pierce test, we get that X-squared = 7.69 and p-value = 0.005553. From the figure and testing results, one can see that we reject the hypothesis H_0. We conclude that the Black-Scholes assumption doesn't hold for the AMZN stock price process.

5.3 Comparison with Black-Scholes Formula

To assess the proposed models and methods, we do comparison the modified fractional Black-Scholes formula with Black-Scholes formula. To estimate options, we first need to estimate parameters. Using the method presented in Section 3, we get

$$\hat{H} = 0.67, \ \hat{\sigma} = 0.58$$

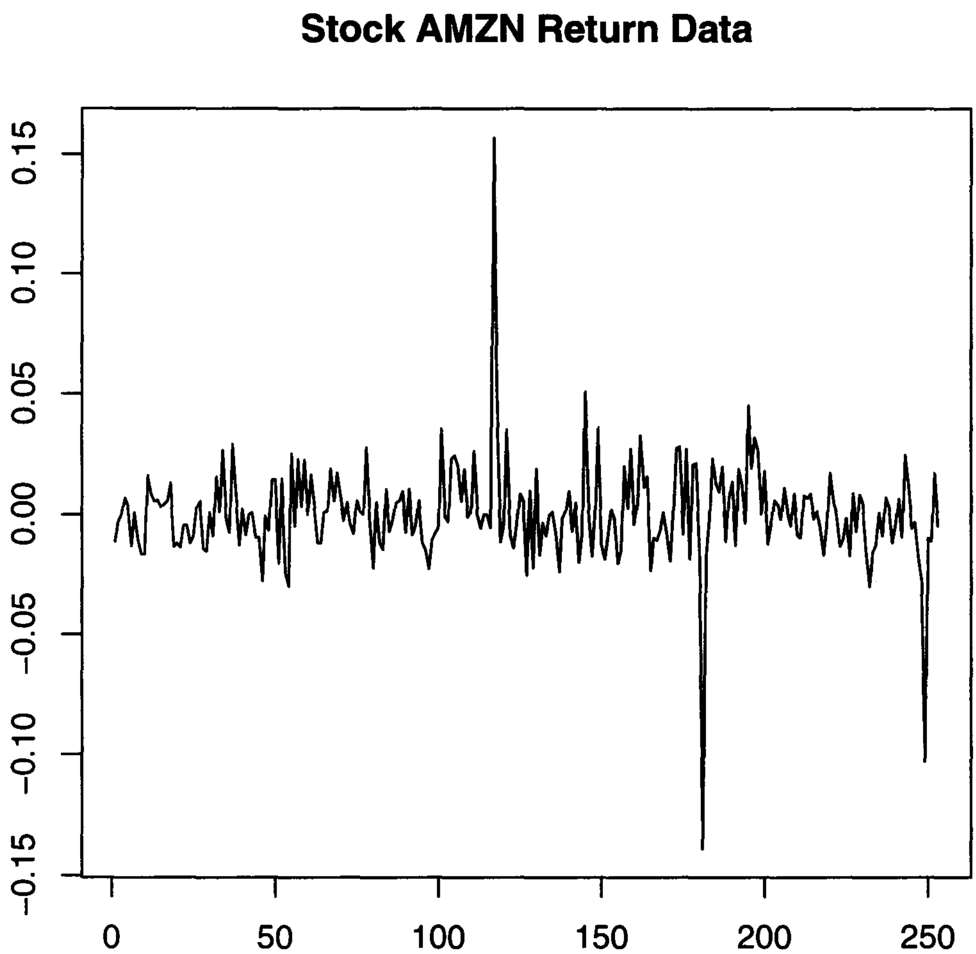

Figure 5.2: Time Series Plot of AMZN Stock Return, from Feb. 08, 2005 to Feb. 09, 2006

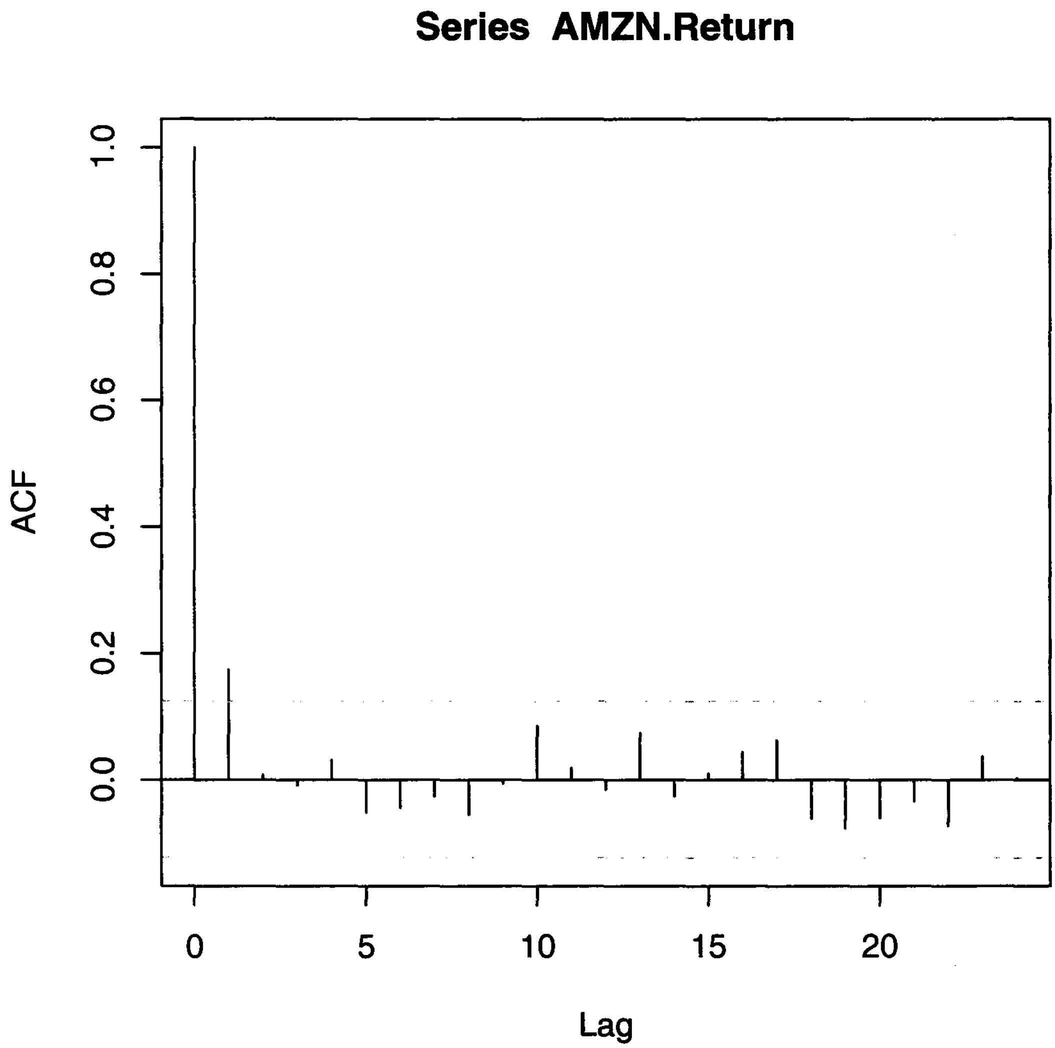

Figure 5.3: ACF of AMZN Stock Return, from Feb. 08, 2005 to Feb. 09, 2006

based on the stock data. Then, using the methods presented in Section 2.3, we consider pricing AMZN stock call options on Feb. 9, 2006. The option prices computed by Black-Scholes formula and the modified fractional Black-Scholes formula are presented in Tables 5.2-4, respectively. Figures 5.4-6 show actual prices, Black-Scholes' pricing and modified fractional Black-Sholes pricing for AMZN stock call options. From the figures, one can see that the modified fractional option pricing model is better than the Black-Scholes model based on our studies.

Table 5.2: AMZN Stock Call Options on Feb. 9, 2006 (Matured on Feb. 17, 2006)

Strike	30	32.5	35	37.5	40	42.5	45	47.5
Price	-	-	3.1	1	0.15	0.05	0.05	0.05
B-S	-	-	3.14	1.20	0.25	0.03	0.00	0.00
H=0.67	-	-	3.08	0.97	0.10	0.03	0.00	0.00

Table 5.3: AMZN Stock Call Options on Feb. 9, 2006 (Matured on Mar. 17, 2006)

Strike	30	32.5	35	37.5	40	42.5	45	47.5
Price	8.9	-	3.8	1.9	0.75	0.2	0.1	0.05
B-S	8.32	-	3.93	2.32	1.22	0.57	0.24	0.09
H=0.67	8.29	-	3.60	1.84	0.75	0.24	0.06	0.01

Table 5.4: AMZN Stock Call Options on Feb. 9, 2006 (Matured on Apr. 21, 2006)

Strike	30	32.5	35	37.5	40	42.5	45	47.5
Price	9.3	5.6	4.2	2.5	1.34	0.8	0.3	0.1
B-S	8.75	6.61	4.76	3.25	2.11	1.30	0.77	0.44
H=0.67	8.60	6.29	4.23	2.57	1.41	0.70	0.31	0.13

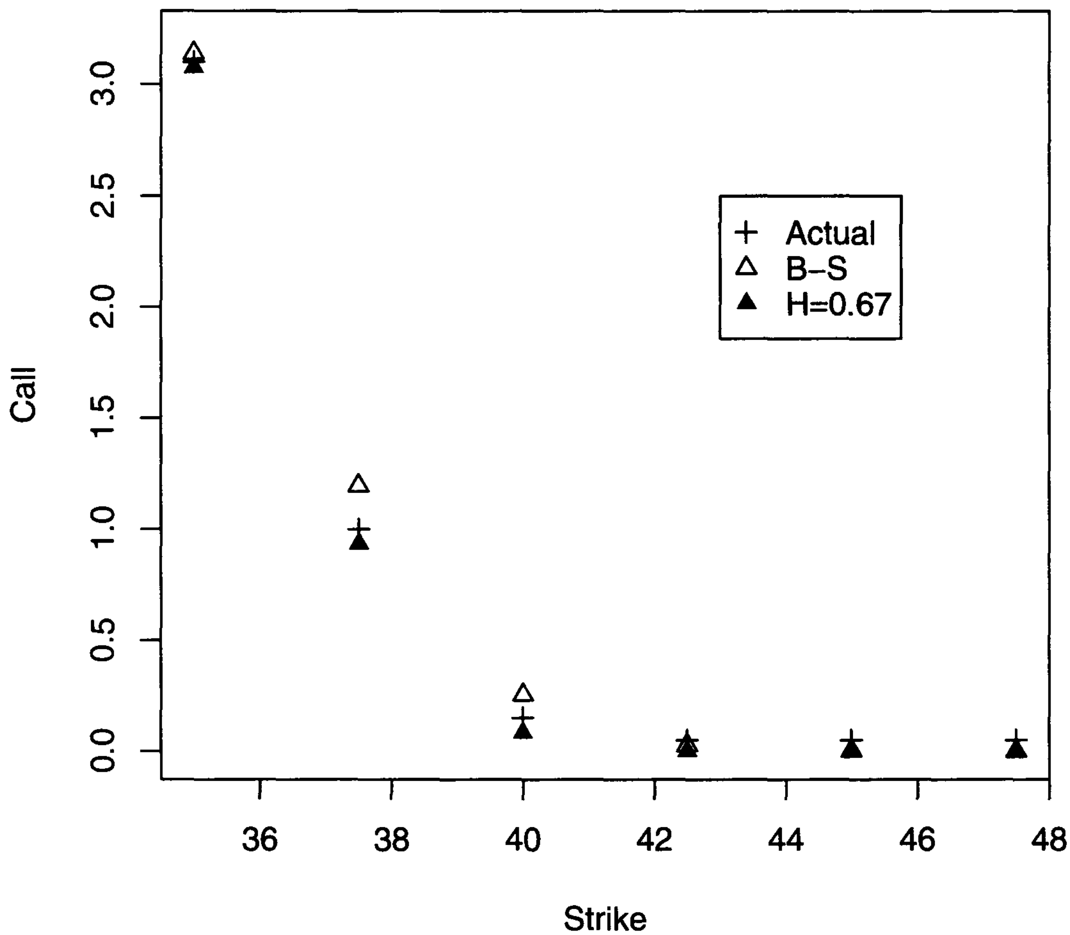

Figure 5.4: AMZN Stock Call Option Pricing on Feb. 9, 2006 (Matured on 17/02/06)

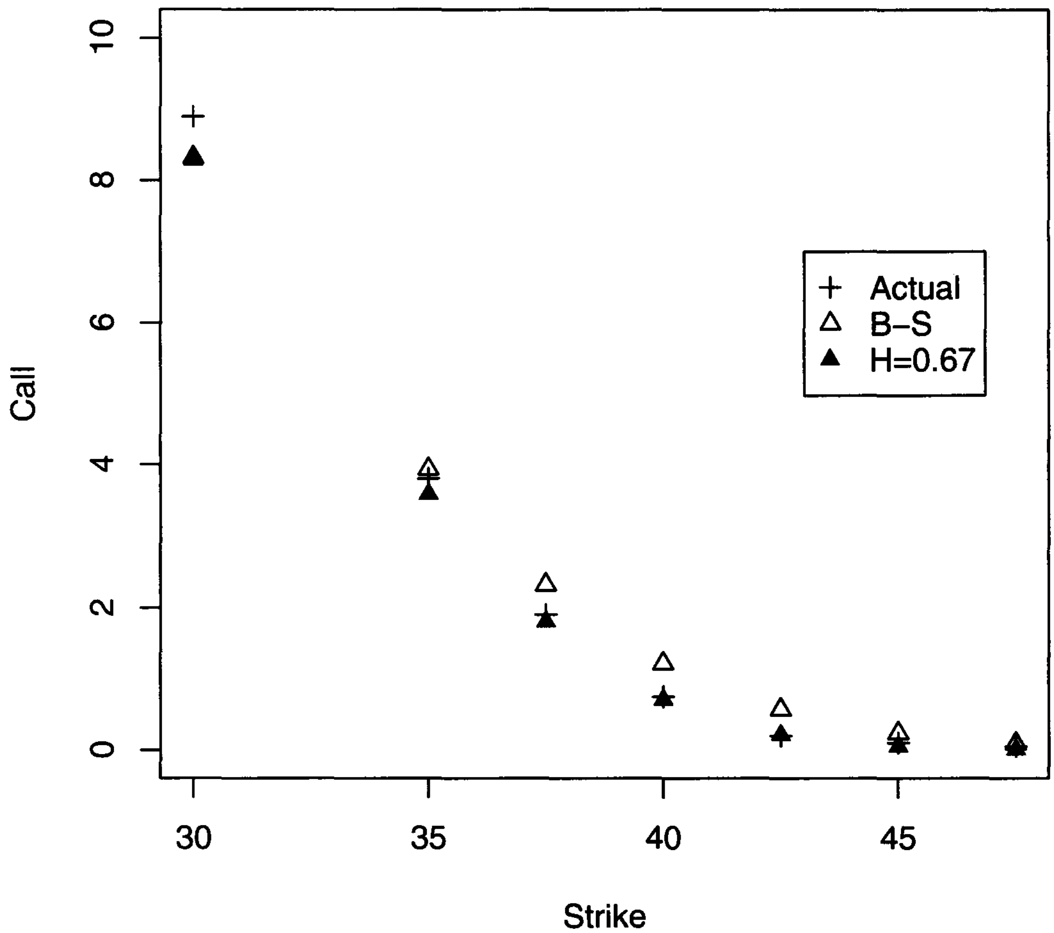

Figure 5.5: AMZN Stock Call Option Pricing on Feb. 9, 2006 (Matured on 17/03/06)

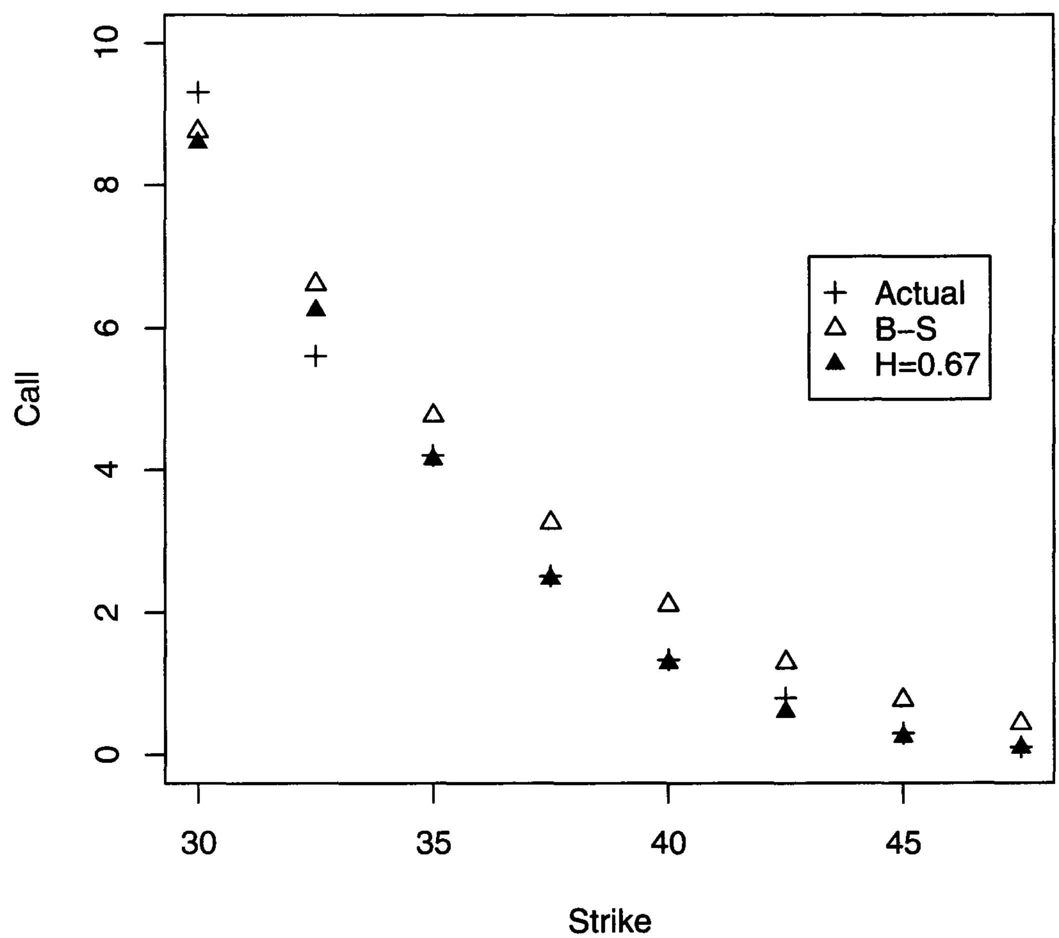

Figure 5.6: AMZN Stock Call Option Pricing on Feb. 9, 2006 (Matured on 21/04/06)

Chapter 6

Conclusion

This study provides an extension of the classical Black and Scholes model for option pricing. We have studied modifications of fractional Brownian motion and option pricing under the hypothesis that the underlying asset price satisfies a stochastic differential equation driven by a modified fractional Brownian motion with long-term dependence. We have obtained modified pricing formula for European call option and an estimation of the implied volatility due to a misspecification of the underlying stock's return that fits well empirical results obtained on the market. We have investigated in particular the ability of the self-similarity parameter H to explain the discrepancy between the Black-Scholes model and the reality of the market. We conclude that the modified fractional model is better than the Black-Scholes model for option pricing based on our study.

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