MALLIAVIN CALCULUS AND ITS APPLICATION IN FINANCE

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Abstract

In recent years, some efficient methods have been developed for calculating derivative price sensitivities, or the Greeks, using Monte Carlo simulation. However, the slow convergence, especially for discontinuous payoff functions, is well known for Monte Carlo simulation. In this project, we investigate the Malliavin calculus and its application in computation of the Greeks. Malliavin calculus and Wiener Chaos theory are introduced. The theoretical framework of the Malliavin weighted scheme of computation of the Greeks is explored in details, and the numerical implementation of the one-dimensional case and an example of the two-dimensional case are presented. Finally, the results are compared with those of finite difference scheme.

Chapter 1

Introduction

The growth of derivatives markets is one of the exciting developments over the last 25 years. Derivatives are used for various purposes such as hedging and insurance. As such, the process of their pricing becomes essential.

A derivative or contingent claim is a financial instrument whose value depends on the value of other, more basic underlying variables. They are mainly used for speculation and hedging purposes. There are all kinds of derivatives, such as options, futures, forward contracts, swaps and other derivatives. Financial derivatives are also applied in various industries. For example, to meet various needs such as hedging and speculation, interest rate derivatives, foreign exchange derivatives, credit derivatives, electricity derivatives, weather derivatives, insurance derivatives and energy derivatives have been created, and are actively traded.

The history of modern pricing techniques for derivatives dates back to 1877, when Charles Castelli wrote a book entitled The Theory of Options in Stocks and Shares [13]. Castelli's book introduced the hedging and speculation aspects of options to the public, but lacked any solid theoretical base. Twenty three years later, Louis Bachelier offered the earliest known analytical valuation for options in his mathematics dissertation at the Sorbonne. He was on the right track, but he used a process to generate share price that allowed both negative security prices and option prices that exceeded the price of the underlying asset. In 1955, a professor at MIT named Paul Samuelson wrote an unpublished paper entitled "Brownian Motion in the Stock Market". During that same year, Richard Kruizenga, one of Samuelson's students, cited Bachelier's work in his dissertation entitled "Put and Call Options: A Theoretical and Market Analysis". In 1962, another dissertation written by A. James Boness, focused on options. In his work, entitled "A Theory and Measurement of Stock Option Value", Boness developed a pricing model that made a significant theoretical jump from that of his predecessors. More significantly, his work served as a precursor to that of Fischer Black and Myron Scholes, who in 1973 introduced their landmark option pricing model. The work of Fischer Black, Myron Scholes, and Robert Merton made a major breakthrough in the stock option pricing. They developed the well-known Black-Scholes model, which has a huge influence on the way that traders price and hedge options. It has also made a big contribution to the growth and success of financial engineering.

There have been different fields in option pricing literature. Option pricing theory can be developed from a probabilistic method referred to as the martingale point of view [7], a partial differential equation method, and a lattice-based method. All these techniques are closely tied to the Feynman Kac formula. For example, the Black Scholes option pricing formula can be derived by various approaches including the martingale representation and probabilistic approach with the help of the Feynman Kac formula. On the other hand, the Black Scholes equation can be derived by solving the heat equation.

In this project, we review some new developments of the probabilistic theory, namely Malliavin Calculus and Wiener chaos. They provide new tools to address the sensitivity of option pricing problems. The Malliavin calculus, named after Paul Malliavin, is an infinite-dimensional differential calculus on the Wiener space. It is called stochastic calculus of variations, in other words it provides the mechanics to compute derivatives of random variables. It was initially tailored to investigate regularity properties of the law of Wiener functionals such as solutions of stochastic differential equations [11], with further contributions by Stroock, Bismut, Watanabe and others.

Later, more applications have emerged. One of the important applications is the existence of the adjoint operator of the Malliavin derivative called the "Skorohod integral". The Skorohod integral is an extension of the Ito integral for nonadapted processes, and is a starting point in developing the stochastic calculus for nonadapted processes. Hence this calculus has allowed mathematicians to explore the stochastic differential equations where the solution is not adapted to the Brownian filtration.

In recent years, Malliavin calculus has been applied in many works in the area of finance and economics. Serrat [14] has used the Malliavin derivatives in his model of dynamic equilibrium for two-country exchange economy with non-traded goods and complete financial markets. Øksendal [12] suggested the use of Malliavin calculus in economics and the option pricing literature. Fournié et al. [6] suggested the use of Malliavin calculus for the faster computation of the Greeks.

The Greeks, or sensitivities, are important measure of market risk to analyze the impact of a misspecification of some stochastic model on the expected payoff function. The traditional method to compute the Greeks is the finite difference method. However, its convergence is very slow, especially for discontinuous payoff functions. To overcome this inefficiency, Broadie and Glasserman [4] introduced the likelihood ratio method. The biggest drawback of this method is that it only works when the density function of the underlying variable is explicitly known. To solve this problem, the integration by parts technique in Malliavin calculus allows the computation of derivatives sensitivities while avoiding a direct differentiation of the complex payoff functions. Ben-hamou [1][2][3] conducted research on the fast computation of the Greeks by means of Malliavin calculus and generalized the approach. He also explicitly computed the sensitivities for continuous time Asian options using the Malliavin weighted scheme. In 2005, Davis and Johansson [5] extended the application of Malliavin calculus for Levy processes. Their work allows for derivation of similar stochastic weights as in the continuous case for a certain class of jump-diffusion processes.

The aim of this project is to study Malliavin calculus and its application in computation can be used to compute the sensitivities of financial derivatives. In Chapter 2, we will introduce the basics of Malliavin calculus, including the Malliavin derivative, Skorohod integral, Wiener chaos, integration by parts formula, and the Malliavin derivative of a diffusion. Then the application of Malliavin calculus in computing the Greeks for the one-dimensional cases will be given in chapter 3. The delta, vega and gamma will be estimated using the Malliavin weighted scheme for different types of options. The results will be compared with those of the finite difference method. Having priced the sensitivities for the one dimensional cases, we proceed to Chapter 4 to explore more application of Malliavin weighted scheme for two-dimensional cases. At last, we will summarize the advantages of Malliavin calculus in computation of the Greeks, as well as some of its limitations.

Chapter 2

Malliavin Calculus

The Malliavin Calculus is an infinite-dimensional differential calculus on the Wiener Space. It was initially developed by Malliavin to study the smoothness of the densities of the solutions of stochastic differential equations. For many years, Malliavin Calculus has been considered highly theoretical and technical from the mathematical point of view. In 1991, Karatzas and Ocone showed how the representation theorem that Ocone had formulated some years earlier in terms of the Malliavin derivative could be used in finance. This theorem is now known as the Clark-Ocone formula. It provides the methodology of fiding replicating portfolios in complete markets driven by Brownian motion. This application has led to increasing interests in Malliavin calculus both from mathematicians and practitioners. Since then more research in this area has been conducted, among which is the important application of estimating the Greeks in finance using Malliavin calculus suggested by Fournié et al [6]. The key idea is to apply the integration by parts formula. In the following part of this chapter, we will study the Malliavin derivative operator, Skorohod integral, and Wiener chaos theory. Then we will derive a generalized duality formula and introduce the Malliavin derivative of a diffusion.

2.1 The Derivative Operator

Let W_t be a 1-dimensional Wiener process (Brownian motion) on the probability space $(\Omega, \mathfrak{F}, P)$ and let $(\mathfrak{F}_t)$ be the filtration generated by $\{W_s, 0 \leq s \leq t\}$. In what follows, we need to distinguish between three different Hilbert spaces associated with this probabilistic setup. First let $L^2[0,T]$ denote the Hilbert space of functions on the interval $[0,T]$ satisfying

$$||h||_{L^2[0,T]} := \int_0^T |h(t)|^2 dt < \infty.$$

Secondly, let $L^2(\Omega)$ denote the set of random variables on Ω satisfying

$$||F||_{L^2(\Omega)} := E[|F|^2] < \infty.$$

Finally, we denote by $L^2(\Omega \times [0,T])$ the set of processes defined on $\Omega \times [0,T]$ satisfying

$$||u_t||_{L^2(\Omega \times T)} := E\left[\int_0^T |u(t,w)|^2 dt\right] < \infty.$$

Further, let us denote $W(h) := \int_0^T h(t) dW_t$, $h \in L^2[0,T]$.

For the definition of the Malliavin derivative, we are going to use as *test functions* the class $C_p^\infty(\mathbb{R}^n)$ of infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ with partial derivatives of polynomial growth.

We denote by $\mathcal{S}$ the class of smooth random variables of the form

$$F = f(W(h_1), ..., W(h_n))$$

where $f \in C_p^\infty(\mathbb{R}^n)$ and $h_1, ..., h_n \in L^2([0,T])$, for arbitrary n. Note that $\mathcal{S}$ is then a dense subspace of $L^2(\Omega)$.

Definition 2.1 The Malliavin derivative of a smooth random variable $F \in \mathcal{S}$ is given by the stochastic process

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))h_i(t). \tag{2.1.1}$$

So far we have obtained a linear operator that takes random variables into processes, that is, $D : \mathcal{S} \subset L^2(\Omega) \to L^2(\Omega \times T)$. We can now extend the domain of D by considering the norm

$$||F||_{1,2} = ||F||_{L^2(\Omega)} + ||D_t F||_{L^2(\Omega \times T)}$$

We then define $\mathbb{D}^{1,2}$ as the closure of $\mathcal{S}$ in the norm $|| \cdot ||_{1,2}$. Then

$$D : \mathbb{D}^{1,2} \subset L^2(\Omega) \to L^2(\Omega \times T)$$

is a closed unbounded operator with a dense domain $\mathbb{D}^{1,2}$.

It follows easily from definition (2.1.1) that this operator satisfies all the properties of a derivative:

1. Linearity: $D_t(aF + G) = aD_t F + D_t G, \quad \forall F, G \in \mathbb{D}^{1,2}$
2. Chain Rule: $D_t(f(F)) = \sum \frac{\partial f}{\partial x_i}(F) D_t F_i, \quad \forall f \in C_p^\infty(\mathbb{R}^n)$,
   $F = (F_1, \ldots, F_n), \quad F_i \in \mathbb{D}^{1,2}$
3. Product Rule: $D_t(FG) = F(D_t G) + G(D_t F), \quad \forall F, G \in \mathbb{D}^{1,2}$ s.t. $FG \in \mathbb{D}^{1,2}$.

To illustrate the intuition behind the derivative operator, we give the following examples [12]:

1. $D_t(\int_0^T h(t)dW_t) = h(t)$

2. $D_t W_s = \mathbf{1}_{\{t \leq s\}}$

3. $D_t f(W_s) = f'(W_s)\mathbf{1}_{\{t \leq s\}}$

4. $D_t(exp(W(t_0))) = exp(W(t_0)) \cdot \mathbf{1}_{[0,t_0]}(t)$

where $\mathbf{1}_{(\cdot)}$ is an indicator function. Example 1 follows directly from equation (2.1.1) by setting $f(x) = x$. Example 2 is a special case of example 1 since W_s can be written as $\int_0^s \mathbf{1}_{\{t \leq s\}} dW_t$. Example 3 can be proved by the chain rule and the result of example 2. Example 4 can be treated as a special case of example 3 by setting the function f to be the exponential function.

2.2 The Skorohod Integral

We now define the Skorohod integral of a process $u \in L^2[T \times \Omega]$ as the adjoint to the Malliavin derivative operator.

As with any operator on Hilbert spaces, we define the domain of the adjoint of $D : L^2(\Omega) \longrightarrow L^2(\Omega \times T)$ as the set

$$Dom(\delta) = \left\{ u \in L^2[\Omega \times T] : \left| E\left(\int_0^T D_t F u_t dt \right) \right| \leq c(u) ||F||_{L^2(\Omega)}, \forall F \in D^{1,2} \right\}.$$

On this set, we can then define the operator $\delta : Dom(\delta) \subset L^2[\Omega \times T] \to L^2(\Omega)$ by the adjoint property

$$< F, \delta(u) >_{L^2} = < D_t F, u >_{L^2[\Omega \times T]},$$

that is,

$$E[F\delta(u)] = E\left[\int_0^T D_t F u_t dt \right]. \quad (2.2.2)$$

We then see that δ is a closed, unbounded operator taking square integrable processes to square integrable random variables. This formula is also known as the duality formula. Note that $E[\delta(u)] = 0$ for all $u \in Dom(\delta)$, since the Malliavin derivative of a constant is zero. A crucial result is the following integration by parts formula:

Proposition 2.1: Let $u \in Dom(\delta), F \in D^{1,2}$ such that $Fu \in L^2[\Omega \times T]$. Then

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F u_t dt, \tag{2.2.3}$$

which means, in particular, that (Fu) is Skorohod integrable if and only if the right hand side belongs to $L^2(\Omega)$.

Proof: Let $G = g(W(G_1), ..., W(G_n))$ be smooth with g of compact support. Then from the definition of the Skorohod integral, followed by the product rule we get

$$\begin{aligned} E[GF\delta(u)] &= E\left[\int_0^T D_t(GF)u_t dt\right] \\ &= E\left[G\int_0^T D_t F u_t dt\right] + E\left[\int_0^T (D_t G) F u_t dt\right] \\ &= E\left[G\int_0^T D_t F u_t dt\right] + E\left[G\delta(Fu)\right] \end{aligned}$$

which completes the proof, since G is arbitrary.

This formula is very useful for calculating the Skorohod integral for processes given by a single random variable, as shown in the next two examples:

For $t_0 \in [0,T]$,

$$\begin{aligned}\delta(W(t_0)) &= W(t_0)\delta(1) - \int_0^T D_t(W(t_0))dt \\ &= W(t_0)W(T) - \int_0^T \mathbf{1}_{\{t\le t_0\}}dt \\ &= W(t_0)W(T) - t_0\end{aligned}$$

$$\begin{aligned}\delta(W^2(t_0)) &= W(t_0)\delta(W(t_0)) - \int_0^T D_t(W(t_0))W(t_0)dt \\ &= W(t_0)[W(t_0)W(T) - t_0] - \int_0^T \mathbf{1}_{\{t\le t_0\}}W(t_0)dt \\ &= W^2(t_0)W(T) - 2t_0W(t_0)\end{aligned}$$

As the previous examples show, the Skorohod integral can be calculated for processes which are not adapted to the filtration $\mathfrak{F}_t$, such as the process W_{t_0} for a fixed $t_0 \in [0,T]$. For adapted processes, however, we have the following important property of the Skorohod integral:

Theorem 2.1 Let $u(t,\omega)$ be a stochastic process such that

$$E[\int_0^T u^2(t,\omega)dt] < \infty$$

and suppose that $u(t,\omega)$ be $\mathfrak{F}_t$-measurable for all $t \in [0,T]$.

Then $u \in Dom(\delta)$ and

$$\delta(u) = \int_0^T u(t,\omega)dW(t).$$

Proof: See [12].

Let us now compute the Malliavin derivative of a given Skorohod integral:

$$\begin{aligned}D_t(\delta(W(t_0))) &= D_t(W(t_0)W(T) - t_0) \\ &= W(t_0)\mathbf{1}_{[0,T]}(t) + W(T)\mathbf{1}_{[0,t_0]}(t) \\ &= W(t_0) + W(T)\mathbf{1}_{[0,t_0]}(t)\end{aligned}$$

where $\mathbf{1}_{(\cdot)}$ is an indicator function.

As we know from elementary calculus, the derivative of an integral of a function is the function itself. However, from the above example, this is not the case for Malliavin calculus. The Malliavin derivative of a Skorohod integral is given by the general result of Theorem 2.5. In order to establish this result, we need to introduce elements of Wiener Chaos theory first.

2.3 Wiener Chaos

Denote by $H_n(x)$ the nth Hermite polynomial, which is defined by

$$H_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2}; \qquad n = 0, 1, 2, \ldots$$

and $H_0(x) = 1$. The polynomials are the coefficients of the expansion in powers of t of the function $F(x,t) = e^{xt-\frac{t^2}{2}}$:

$$\begin{aligned} F(x,t) &= e^{xt-\frac{t^2}{2}} \\ &= e^{\frac{x^2}{2}} e^{-\frac{1}{2}(x-t)^2} \\ &= e^{\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\frac{d^n}{dt^n} e^{-\frac{1}{2}(x-t)^2})|_{t=0} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \end{aligned}$$

It then follows that we have the recurrence relation

$$(x - \frac{d}{dx}) H_n(x) = H_{n+1}(x).$$

$$
\begin{cases}
H_0(x) = 1 \\
H_1(x) = (x) \\
H_2(x) = x^2 - 1 \\
H_3(x) = x^3 - 3x \\
H_4(x) = x^4 - 6x^2 + 3 \\
\vdots
\end{cases}
$$

Using Hermite polynomials we can construct families of random variables with special orthogonality properties, as given by the following lemma.

Lemma 2.1 Let X, Y be two random variables with joint Gaussian distribution such that $E(X) = E(Y) = 0$ and $E(X^2) = E(Y^2) = 1$. Then for all $n, m \geq 0$, we have

$$
E[H_n(X)H_m(Y)] = \begin{cases} 0, & \text{if } n \neq m \\ n!(E(XY))^n & \text{if } n = m \end{cases}
$$

Proof: From the characteristic function of a joint Gaussian random variable we obtain

$$
E[e^{sX+vY}] = e^{\frac{s^2}{2}+svE[XY]+\frac{v^2}{2}}.
$$

Hence,

$$
E[e^{sX-\frac{s^2}{2}}e^{vY-\frac{v^2}{2}}] = e^{svE[XY]}.
$$

Taking the partial derivative $\frac{\partial^{n+m}}{\partial s^n \partial v^m}$ on both sides, we obtain

$$
E[H_n(X)H_m(Y)] = \begin{cases} 0, & \text{if } n \neq m \\ n!(E(XY))^n & \text{if } n = m. \end{cases}
$$

We will denote by $\mathcal{H}_n$ the closed linear subspace of $L^2(\Omega)$ generated by the random variables $H_n(W(h)), h \in L^2[0,T]$. By Lemma 2.1 we deduce that $\mathcal{H}_n$ and $\mathcal{H}_m$ are orthogonal subspaces when $n \neq m$. The space $\mathcal{H}_n$ is called the Wiener chaos of order n. We then have the following theorem.

Theorem 2.2 The space $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{H}_n$:

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

Proof [11]: Let $X \in L^2(\Omega)$ be such that X is orthogonal to $\mathcal{H}_n$ for all $n \geq 0$, i.e. $E[XH_n(W(h))] = 0, \forall n, \forall h \in L^2[0,T]$. Then

$$E[XW(h)^n] = 0, \quad \forall n, \forall h,$$

and therefore

$$E[Xe^{W(h)}] = 0, \quad \forall h.$$

Finally we have

$$E[Xe^{\sum_{i=1}^{m} t_i W(h_i)}] = 0, \quad \forall t_1, \ldots, t_m \in \mathbb{R} \quad \forall h_1, \ldots, h_m \in L^2[0,T].$$

We then can deduce that $X = 0$, which completes the proof.

From the previous theorem, we conclude that any square integrable random variable can be decomposed as a convergent sum of components belonging to each of the orthogonal subspaces $\mathcal{H}_n$. Such decomposition is called the Wiener chaos expansion of a square integrable random variable.

The Wiener chaos expansion is intimately related to Ito integrals and can be done in two alternative ways: a time-ordered expansion and a symmetric expansion.

For the first type of expansion, let

$$\mathcal{S}_n = \{(t_1, \ldots, t_n) \in [0,T]^n; 0 \le t_1 \le t_2 \le \ldots \le t_n \le T\}.$$

We then define the (n-fold) iterated Ito integral for a deterministic function $f \in L^2[\mathcal{S}_n]$ as

$$J_n(f) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \ldots, t_n) dW_{t_1} \cdots dW_{t_{n-1}} dW_{t_n}.$$

Theorem 2.3 (Time-ordered Wiener chaos): Let $F \in L^2(\Omega)$. Then

$$F = \sum_{m=0}^{\infty} J_m(f_m)$$

for (unique) deterministic function $f_m \in L^2[\mathcal{S}_m]$. Moreover,

$$||F||_{L^2[\Omega]} = \sum_{m=0}^{\infty} ||f_m||_{L^2[\mathcal{S}_m]}.$$

Proof: Observe that, due to Ito isometry,

$$||J_n(f)||_{L^2(\Omega)} = ||f||_{L^2[\mathcal{S}_n]},$$

so the image of $L^2[\mathcal{S}_n]$ under J_n is closed in $L^2(\Omega)$. Moreover, in the special case where $f(t_1, \ldots, t_n) = h(t_1) \ldots h(t_n)$, $h \in L^2[0,T]$, we have

$$n! J_n(h(t_1)h(t_2) \ldots h(t_n)) = ||h||^n H_n(\frac{W(h)}{||h||}),$$

as can be seen by induction. Therefore, $\mathcal{H}_n \subset J_n(L^2[\mathcal{S}_n])$. Finally, a further application of Ito isometry shows that

$$E[J_m(g)J_n(h)] = \begin{cases} 0, & \text{if } n \neq m \\ < g, h >_{L^2[\mathcal{S}_n]} & \text{if } n = m \end{cases}$$

Therefore $J_m(L^2[\mathcal{S}_n])$ is orthogonal to $\mathcal{H}_n$ for all $n \neq m$. But this implies that $L^2(\Omega) = \bigoplus_{n=0}^{\infty} J_n(L^2[\mathcal{S}_n])$, which completes the proof.

We now give a example of the time-ordered Wiener chaos expansion. Let $F = W_T^2$. We use the fact that $H_2(x) = x^2 - 1$. By writing $W_T = \int_0^T 1_{\{t \le T\}} dW_t = \int_0^T h(t) dW_t$, we obtain

$$||h|| = \left(\int_0^T 1^2_{\{t \le T\}} dt \right)^{1/2} = T^{1/2},$$

so

$$H_2\left(\frac{W_T}{||h||} \right) = \frac{W_T^2}{T} - 1.$$

From

$$2 \int_0^T \int_0^{t_2} 1_{\{t \le T\}} 1_{\{t \le T\}} dW_{t_1} dW_{t_2} = T \left(\frac{W_T^2}{T} - 1 \right),$$

we find

$$W_T^2 = T + 2J_2(1).$$

For the second type of Wiener chaos expansion, let us say that a real function $g : [0,T]^n \longrightarrow \mathbb{R}$ is symmetric if

$$g(X_{\sigma_1}, \ldots, X_{\sigma_n}) = g(X_1, \ldots, X_n)$$

for all permutations σ of the set $\{1, \ldots, n\}$. Moreover, if

$$||g||^2_{L^2[0,T]^n} = \int_0^T \int_0^T \cdots \int_0^T g^2(t_1, \ldots, t_n) dt_1 dt_n < \infty$$

we say that $g \in L_s^2([0,T]^n)$, the space of symmetric square integrable functions on $[0,T]^n$. Since $\mathcal{S}_n$ only occupies the fraction $\frac{1}{n!}$ of the n-dimensional box $[0,T]^n$, for a symmetric function we have

$$\begin{aligned}
||g||^2_{L^2[0,T]^n} &= \int_0^T \int_0^T \cdots \int_0^T g^2(t_1,\ldots,t_n)dt_1\cdots dt_n \\
&= n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g^2(t_1,\ldots,t_n)dt_1\cdots dt_n \\
&= n!||g||^2_{L^2[S_n]}
\end{aligned}$$

Now we can extend the definition of the iterated Ito integral to symmetric functions as follows

$$\begin{aligned}
I_n(g) &\equiv \int_0^T \int_0^T \cdots \int_0^T g(t_1,\ldots,t_n)dW_{t_1}\cdots dW_{t_n} \\
&:= n!J_n(g) \\
&= n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1,\ldots,t_n)dW_{t_1}\cdots dW_{t_n},
\end{aligned}$$

for all $g \in L^2_s[0,T]^n$. It then follows that

$$E[I_n^2(g)] = E[(n!)^2 J_n^2(g)] = (n!)^2||g||_{L^2[S_n]} = n!||g||_{L^2[0,T]^n}.$$

For a special function of the form

$$g(t_1,t_2,\ldots,t_n) = h(t_1)h(t_2)\cdots h(t_n), \quad for \quad h \in L^2[0,T]$$

we can use the Hermite polynomial we introduced above, and get the useful forumla

$$\begin{aligned}
I_n(h(t_1)\ldots h(t_n)) &= n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} h(t_1)h(t_2)\cdots h(t_n)dW_{t_1}\cdots dW_{t_{n-1}}dW_{t_n} \\
&= ||h||^n H_n\left(\frac{W(h)}{||h||}\right).
\end{aligned}$$

Following exactly the same steps used in the proof of Theorem 2.3, we can show that:

Theorem 2.4 (Symmetric Wiener chaos): Let $F \in L^2(\Omega)$. Then

$$F = \sum_{m=0}^{\infty} I_m(g_m)$$

for (unique) deterministic functions $g_m \in L_s^2[0,T]^n$. Moreover

$$||F||_{L^2(\Omega)} = \sum_{m=0}^{\infty} m!||g_m||_{L^2[0,T]^n}.$$

We now introduce two propositions relating Wiener chaos with the Malliavin derivative and the Skorohod integral. Suppose that $F \in L^2(\Omega)$ with expansion

$$F = \sum_{m=0}^{\infty} I_m(g_m), \quad g_m \in L_s^2[0,T]^m. \tag{2.3.4}$$

Proposition 2.2 $F \in \mathbb{D}^{1,2}$ if and only if

$$\sum_{m=1}^{\infty} mm!||g_m||^2_{L^2(T^m)} < \infty \tag{2.3.5}$$

and in this case

$$D_t F = \sum_{m=1}^{\infty} m I_{m-1}(g_m(\cdot, t)).$$

Moreover, $||D_t F||^2_{L^2(\Omega\times T)} = \sum_{m=1}^{\infty} mm!||g_m||^2_{L^2}$.

This proposition shows that when condition (2.3.5) is satisfied, then if we take the Malliavin derivative of F, the order of each integral of its Wiener chaos expansion (2.3.4) is decreased by one.

Before we proceed to Proposition 2.3, we define the symmetrization of the function $f_n(t_1, \ldots, t_n, t)$ to be

$$\begin{aligned}\widetilde{f}_n(t_1, \ldots, t_n, t) &= \frac{1}{n+1}[f_n(t_1, \ldots, t_n, t) \\ &= +\sum_{i=1}^{n} f_n(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n, t_i)].\end{aligned}$$

Now let $u \in L^2(\Omega \times T)$. Then it follows from the Wiener-Ito expansion that

$$u_t = \sum_{m=0}^{\infty} I_m(g_m(\cdot, t)), \quad g_m(\cdot, t) \in L_s^2[0, T]^m \tag{2.3.6}$$

Moreover

$$||u_t||_{L^2(\Omega \times T)}^2 = \sum_{m=0}^{\infty} m! ||g_m||_{L^2[0,T]^{m+1}}$$

Proposition 2.3 $u \in Dom(\delta)$ if and only if

$$\sum_{m=0}^{\infty} (m+1)! ||\tilde{g}_m||_{L^2(T^{m+1})}^2 < \infty \tag{2.3.7}$$

in which case

$$\delta(u) = \sum_{m=0}^{\infty} I_{m+1}(\tilde{g}_m).$$

Proposition 2.3 implies that when condition (2.3.7) is satisfied, then when we take the Skorohod integral of u, the order of each integral of its Wiener chaos expanion (2.3.6) is increased by one.

Making use of Wiener chaos expansion, we can now prove the following general result concerning the Malliavin derivative of a Skorohod integral.

Theorem 2.5 Let $u \in L^2(\Omega \times T)$ be a process such that $u_s \in D^{1,2}$ for each $s \in [0, T]$. Assume further that, for each fixed t, the process $D_t u_s$ is Skorohod Integrable i.e.($D_t u_s \in Dom(\delta)$). Furthermore, suppose that $\delta(D_t u) \in L^2(\Omega \times T)$. Then $\delta(u) \in D^{1,2}$ and we have

$$D_t(\delta(u)) = u_t + \delta(D_t u).$$

Proof:

Let $u_s = \sum_{m=0}^{\infty} I_m(f_m(\cdot, s))$.

Then

$$\delta(u) = \sum_{m=0}^{\infty} I_{m+1}(\tilde{f}_m)$$

where $\tilde{f}_m$ is the symmetrization of $f_m(\cdot, s)$. Then

$$D_t(\delta(u)) = \sum_{m=0}^{\infty}(m+1)I_m(\tilde{f}_m(\cdot, t))$$

Now note that

$$\begin{aligned}\tilde{f}_m(t_1, \dots, t_m, t) &= \frac{1}{m+1}[f_m(t_1, \dots, t_m, t) \\ &= +\sum_{i=1}^{m} f_m(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_m, t_i)].\end{aligned}$$

Hence

$$D_t(\delta(u)) = \sum_{m=0}^{\infty} I_m(f_m(\cdot, t)) + \sum_{m=0}^{\infty} mI_m(\tilde{f}_m(\cdot, t, \cdot)).$$

On the other hand

$$\begin{aligned}\delta(D_t u) &= \delta\left[D_t\left(\sum_{m=0}^{\infty} I_m(f_m(\cdot, s))\right)\right] \\ &= \delta\left[\sum_{m=0}^{\infty} mI_{m-1}(f_m(\cdot, t, s))\right] \\ &= \sum_{m=0}^{\infty} mI_m(\tilde{f}_m(\cdot, t, \cdot))\end{aligned}$$

Comparing the two expressions gives the result.

2.4 Malliavin Derivative of a Diffusion

Now suppose that

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.4.8}$$

with the initial condition $X_0 = x$. The drift function $b : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ and the volatility structure $\sigma : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ of the process $(X_t)_{t\in[0,T]}$ are assumed to be continuously differentiable with bounded derivatives and Lipschitz conditions, i.e. there exists a constant $K < +\infty$ such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K|x-y|$$

$$|b(t,x)| + |\sigma(t,y)| \le K(1+|x|).$$

These two conditions are the usual conditions to ensure existence and uniqueness of a continuous and strong solution of the SDE (2.4.8) with its initial condition.

Having the above conditions satisfied, we can write the solution of SDE (2.4.8) as

$$X_t = X_0 + \int_0^t b(u,X_u)du + \int_0^t \sigma(u,X_u)dW_u$$

Then the Malliavin derivative of the process $X(t)_{t\in[0,T]}$ satisfies the following linear equation

$$\begin{aligned}
D_s(X_t) &= D_s\left(\int_0^t b(u,X_u)du\right) + D_s\left(\int_0^t \sigma(u,X_u)dW_u\right) \\
&= \int_0^t b^{'}(u,X_u)D_sX_u du + \int_0^t \sigma^{'}(u,X_u)D_sX_u dW_u + \int_0^t \sigma(u,X_u)D_s(dW_u) \\
&= \int_0^t b^{'}(u,X_u)D_sX_u 1_{\{s\le u\}}du + \int_0^t \sigma^{'}(u,X_u)D_sX_u 1_{\{s\le u\}}dW_u + \sigma(s,X_s) \\
&= \int_s^t b^{'}(u,X_u)D_sX_u du + \int_s^t \sigma^{'}(u,X_u)D_sX_u dW_u + \sigma(s,X_s)
\end{aligned}$$

We can solve this equation and obtain the solution:

$$D_sX_t = \sigma(s,X_s)exp\left[\int_s^t (b^{'} - \frac{1}{2}(\sigma^{'})^2)du + \int_s^t \sigma^{'}dW_u\right].$$

Now we define the first variation process $(Y_t)_{t\in[0,T]}$ as the derivative of X_t with respect to the initial value $Y_t = \frac{\partial X_t}{\partial x}$. The process $(Y_t)_{t\in[0,T]}$ satisfies the SDE

$$dY_t = b^{'}(t,X_t)Y_t dt + \sigma^{'}(t,X_t)Y_t dW_t, \quad Y_0 = 1.$$

Solving this SDE we obtain

$$Y_t = exp\left[\int_0^t (b' - \frac{1}{2}(\sigma')^2)du + \int_0^t \sigma' dW_u\right].$$

Therefore we can express the $D_s X_t$ in terms of the first variation process Y_t:

$$D_s X_t = \frac{Y_t}{Y_s}\sigma(s, X_s)1_{\{s \le t\}}. \tag{2.4.9}$$

We now present two examples to study the application of (2.4.9).

For a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x,$$

we have that the first variation process satisfies

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t, \quad Y_0 = 1.$$

Therefore,

$$Y_t = \frac{X_t}{x} \tag{2.4.10}$$

and

$$D_s X_t = \frac{X_t}{X_s}\sigma X_s = \sigma X_t.$$

For an Ornstein-Uhlenbeck process

$$dX_t = -kX(t)dt + \sigma dW_t, \quad X_0 = x,$$

we have that the first variation process satisfies

$$dY_t = -kY_t dt, \quad Y_0 = 1.$$

Therefore,

$$Y_t = e^{-kt}$$

and

$$D_s X_t = e^{-k(t-s)}\sigma.$$

2.5 Generalized Duality Formula

We now consider differentiation under the expectation operator and establish a generalization of the duality formula (2.2.2). Suppose $F \in \mathbb{D}^{1,2}$ and denote by $\mathcal{W}$ the set of random variables π such that

$$E[\phi'(F)G] = E[\phi(F)\pi], \quad \forall \quad \phi \in C_p^{\infty}(\mathbb{R}). \tag{2.5.11}$$

Theorem 2.6 A necessary and sufficient condition for a weight to be of the form $\pi = \delta(u)$ where $u \in Dom(\delta)$, is that

$$E\left[\int_0^T D_t F u_t dt | \mathfrak{F}(F)\right] = E[G|\mathfrak{F}(F)]. \tag{2.5.12}$$

Moreover, $\pi_0 = E[\pi|\mathfrak{F}(F)]$ is the minimum over all weights in $\mathcal{W}$ of the functional

$$Var^{\pi} = E[(\phi(F)\pi - E[\phi'(F)G])^2]$$

Proof: Suppose that $u \in Dom(\delta)$ satisfies

$$E\left[\int_0^T D_t F u_t dt | \mathfrak{F}\right] = E[G|\mathfrak{F}],$$

then

$$\begin{aligned} E[\phi'(F)G] &= E[E[\phi'(F)G|\mathfrak{F}]] \\ &= E[\phi'(F)E[G|\mathfrak{F}]] \\ &= E\left[\phi'(F)E\left[\int_0^T D_t F u_t dt | \mathfrak{F}\right]\right] \\ &= E\left[\int_0^T D_t \phi(F) u_t dt\right] \\ &= E[\phi(F)\delta(u)] \end{aligned}$$

so $\pi = \delta(u)$ is a weight.

Conversely, if $\pi = \delta(u)$ for some $u \in Dom(\delta)$ is a weight, then

$$\begin{aligned} E[\phi'(F)G] &= E[\phi(F)\delta(u)] \\ &= E\left[\int_0^T D_t\phi(F)u_t dt\right] \\ &= E\left[\phi'(F)\int_0^T D_t F u_t dt\right] \end{aligned}$$

Therefore,

$$E\left[\int_0^T D_t F u_t dt|\sigma(F)\right] = E[G|\sigma(F)].$$

To prove the minimal variance claim, observe first that for any two weights π_1, π_2 we must have $E[\pi_1|\sigma(F)] = E[\pi_2|\sigma(F)]$. Therefore, setting $\pi_0 = E[\pi|\sigma(F)]$ for a generic weight π we obtain

$$\begin{aligned} Var^{\pi} &= E[(\phi(F)\pi - E[\phi'(F)G])^2] \\ &= E[(\phi(F)(\pi - \pi_0) + \phi(F)\pi_0 - E[\phi'(F)G])^2] \\ &= E[(\phi(F)(\pi - \pi_0))^2] + E[(\phi(F)\pi_0 - E[\phi'(F)G])^2] \\ &+ 2E[\phi(F)(\pi - \pi_0)(\phi(F)\pi_0 - E[\phi'(F)G])] \end{aligned}$$

But

$$\begin{aligned} & E[\phi(F)(\pi - \pi_0)(\phi(F)\pi_0 - E[\phi'(F)G])] \\ = \; & E[E[\phi(F)(\pi - \pi_0)(\phi(F)\pi_0 - E[\phi'(F)G])]|\sigma(F)] = 0 \end{aligned}$$

Therefore the minimum must be achieved for $\pi = \pi_0$.

Chapter 3

Malliavin Weighted Scheme for Computation of the Greeks

The growing emphasis on risk management issues as well as the development of more and more complicated financial derivatives have urged practitioners to come up with efficient ways to compute the prices sensitivities, or "the Greeks", with respect to model parameters. In this chapter we will present a methodology of computing the Greeks based on Malliavin calculus. We will show how to apply the generalized duality formula in the Malliavin weighted scheme and present explicit examples of computation of the Greeks for European options, binary options and Asian options. We will also investigate how this new method differs from the finite difference method in calculation of the sensitivities.

3.1 Calculating the Greeks

As important measures of risk, the price sensitivities are mathematically defined as the derivatives of a price with respect to various model parameters, and are generally known as Greeks, such as the delta, vega, and gamma.

The delta of a financial derivative is defined as the rate of change of the derivative price with respect to the price of the underlying asset. We denote by $P(x)$ the price of an option with the initial value of the underlying equal to x, then the delta is given by

$$\triangle = \frac{\partial P}{\partial x}$$

The value of delta measures how sensitive the derivative price is to the underlying security. It is a crucial parameter for portfolio hedging purpose.

The vega of a financial derivative is the rate of change of the derivative price with respect to the volatility of the underlying security:

$$\nu = \frac{\partial P}{\partial \sigma}.$$

If the absolute value of vega is high, then the derivative price is very sensitive to small changes in volatility. If the value is low, then volatility changes have relatively little impact on the value of the derivative.

The gamma of a financial derivative is the second partial derivative of the derivative price with respect to the underlying. It measures the rate of change of the derivative's delta with respect to the price of the underlying asset:

$$\gamma = \frac{\partial^2 P}{\partial x^2}.$$

The value of gamma measures how fast the delta changes when the price of underlying asset changes.

The delta and the vega can be approximated by three different schemes of the finite difference approximation [15], while the gamma is usually approximated by the central difference scheme. That is, using a forward difference scheme, we have

$$\triangle = \frac{P(x+\epsilon) - P(x)}{\epsilon}, \qquad \nu = \frac{P(\sigma+\epsilon) - P(\sigma)}{\epsilon}, \qquad \epsilon > 0,$$

while for the backward difference scheme we obtain

$$\triangle = \frac{P(x) - P(x-\epsilon)}{\epsilon}, \qquad \nu = \frac{P(\sigma) - P(\sigma-\epsilon)}{\epsilon}, \qquad \epsilon > 0.$$

Finally, with the central difference scheme, we have

$$\triangle = \frac{P(x+\epsilon) - P(x-\epsilon)}{2\epsilon}, \qquad \nu = \frac{P(\sigma+\epsilon) - P(\sigma-\epsilon)}{2\epsilon},$$

$$\gamma = \frac{P(x+\epsilon) + P(x-\epsilon) - 2P(x)}{(\epsilon)^2}, \qquad \epsilon > 0.$$

In all cases, each term in the differences can be calculated by a Monte Carlo simulation. Taking the delta as an example, in the case of forward and backward difference schemes, if the Monte Carlo simulations of the two estimators of $P(x+\epsilon)$ and $P(x)$ or $P(x)$ and $P(x-\epsilon)$ are drawn independently, then it is proved in Glynn (1989) that the best possible convergence rate is typically $n^{-\frac{1}{4}}$. Replacing the forward and backward difference schemes by the central difference scheme improves the optimal convergence rate to $n^{-\frac{1}{3}}$. By taking common random numbers, one can recover the best convergence rate that can be expected from Monte Carlo methods, that is $n^{-\frac{1}{2}}$. This result was described by Glasserman and Yao (1992), Glynn (1989) and L'Ecuyer and Perron (1994).

In practice, due to the discontinuous payoff function, the convergence rate is much lower than those theoretical bounds. In 1996, Broadie and Glasserman suggested the likelihood ratio method to overcome the inefficiency of finite difference method. The main purpose is to come up with an efficient way of avoiding taking derivative of the payoff function, by expressing the Greeks as the expectation of the payoff function times a weight. Let $\Phi(X_T)$ be the payoff function of an option, and suppose we want to compute the Greek with respect to the parameter θ. If the density function of the underlying variable is explicitly known as $p(x, \theta)$ [3], then we can compute the Greek by

$$
\begin{aligned}
\text{Greek} &= \frac{\partial}{\partial\theta} E[\Phi(X_T)] \\
&= \frac{\partial}{\partial\theta} \int \Phi(x) p(x, \theta) dx \\
&= \int \Phi(x) \frac{\partial}{\partial\theta} \log p(x, \theta) p(x) dx \\
&= E[\Phi(X_T) \frac{\partial}{\partial\theta} \log p(X_T, \theta)] \\
&= E[\Phi(X_T) \text{weight}]
\end{aligned}
$$

The key restriction of this method is that it requires the explicit form of the density function of the underlying variable. In order to overcome this restriction, Fournié *et al* (1999) showed how any Greek can be expressed as the expectation of the payoff function times a weight that could be expressed in terms of a Malliavin derivative without knowing the density function explicitly. Benhamou (2000) then proved that any weight could be expressed as the Skorohod integral of a generating process satisfying some necessary and sufficient conditions.

In this chapter, we review these results in the context of European and Asian options making use of simplified and straightward proofs.

3.2 European Options

We consider a continuous time model under the risk neutral measure Q where underlying asset price is assumed to follow a geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_{t_0} = S_0,$$

where r is the constant risk-free interest rate, σ is the constant volatility. We also assume that the payoff function of a path-independent option on the underlying S_t is $\Phi(S_T)$. Since the asset price S_t is given by

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t},$$

the delta of the option is

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial S_0} E[e^{-rT}\phi(S_T)] \\
&= E[e^{-rT}\phi'(S_T)\frac{\partial S_T}{\partial S_0}] \\
&= \frac{e^{-rT}}{S_0} E[\phi'(S_T)S_T].
\end{aligned} \tag{3.2.1}$$

The expectations in equation (3.2.1) and in the following context are under the risk neutral measure Q. Now by comparing equation (3.2.1) with equation (2.5.11), we see that $G = S_T$ and that

$$u = \frac{S_T}{\int_0^T D_s S_T ds}$$

satisfies the condition (2.5.12). It then follows from (2.5.11) that

$$\begin{aligned} E\left[\phi'(S_T)\, S_T\right] &= E\left[\phi(S_T)\,\delta\left(\frac{S_T}{\int_0^T D_s S_T ds}\right)\right] \\ &= E\left[\phi(S_T)\,\delta\left(\frac{S_T}{\int_0^T \sigma S_T 1_{\{s\le T\}} ds}\right)\right] \\ &= E\left[\phi(S_T)\,\delta\left(\frac{1}{\sigma T}\right)\right] \\ &= E\left[\phi(S_T)\,\frac{W_T}{\sigma T}\right] \end{aligned}$$

where the second equality follows from equation (2.4.9). Therefore the final expression of delta is given by

$$\Delta = \frac{e^{-rT}}{S_0} E\left[\phi(S_T)\,\frac{W_T}{\sigma T}\right] \tag{3.2.2}$$

For the vega, we have

$$\begin{aligned} \nu &= \frac{\partial}{\partial \sigma} E\left[e^{-rT}\phi(S_T)\right] \\ &= E\left[e^{-rT}\phi'(S_T)\,\frac{\partial S_T}{\partial \sigma}\right] \\ &= e^{-rT} E\left[\phi'(S_T)\,(W_T - \sigma T)\, S_T\right] \end{aligned}$$

By setting $G = (W_T - \sigma T)S_T$ in equation (2.5.11), we have that

$$u = \frac{(W_T - \sigma T)S_T}{\int_0^T D_s S_T ds}$$

satisfies equation (2.5.12). Therefore,

$$
\begin{aligned}
E\left[\phi'(S_T)(W_T-\sigma T)S_T\right] &= E\left[\phi(S_T)\,\delta\left(\frac{(W_T-\sigma T)S_T}{\int_0^T D_s S_T ds}\right)\right] \\
&= E\left[\phi(S_T)\,\delta\left(\frac{(W_T-\sigma T)S_T}{\int_0^T \sigma S_T 1_{\{s\le T\}} ds}\right)\right] \\
&= E\left[\phi(S_T)\,\delta\left(\frac{W_T-\sigma T}{\sigma T}\right)\right] \\
&= E\left[\phi(S_T)\left(\frac{1}{\sigma T}\delta(W_T)-W_T\right)\right] \\
&= E\left[\phi(S_T)\left(\frac{W_T^2-T}{\sigma T}-W_T\right)\right] \\
&= E\left[\phi(S_T)\left(\frac{W_T^2}{\sigma T}-W_T-\frac{1}{\sigma}\right)\right]
\end{aligned}
$$

So

$$
\nu = e^{-rT}E\left[\Phi(S_T)\left(\frac{W_T^2}{\sigma T}-W_T-\frac{1}{\sigma}\right)\right] \qquad (3.2.3)
$$

Finally we will give the expression for gamma:

$$
\begin{aligned}
\gamma &= \frac{\partial^2}{\partial S_0^2}E\left[e^{-rT}\phi(S_T)\right] \\
&= \frac{e^{-rT}}{S_0^2}E\left[\phi''(S_T)S_T^2\right]
\end{aligned}
$$

Since it involves the second derivative, we need to apply the duality formula twice. First,

$$
\begin{aligned}
E\left[\phi''(S_T)S_T^2\right] &= E\left[\phi'(S_T)\delta\left(\frac{S_T^2}{\int_0^T \sigma S_T 1_{\{s\le T\}} ds}\right)\right] \\
&= E\left[\phi'(S_T)\delta\left(\frac{S_T^2}{\sigma T S_T}\right)\right] \\
&= E\left[\phi'(S_T)\delta\left(\frac{S_T}{\sigma T}\right)\right] \\
&= E\left[\phi'(S_T)\left(\frac{S_T W_T}{\sigma T}-S_T\right)\right]
\end{aligned}
$$

Applying the duality formula again and we have

$$
\begin{aligned}
E\left[\phi'(S_T)\left(\frac{S_T W_T}{\sigma T} - S_T\right)\right] &= E\left[\phi(S_T)\delta\left(\frac{\frac{W_T}{\sigma T} - 1}{\sigma T}\right)\right] \\
&= E\left[\Phi(S_T)\left(\delta\left(\frac{W_T}{\sigma^2 T^2}\right) - \delta\left(\frac{1}{\sigma T}\right)\right)\right] \\
&= E\left[\Phi(S_T)\left(\frac{W_T^2 - T}{\sigma^2 T^2} - \frac{W_T}{\sigma T}\right)\right]
\end{aligned}
$$

Hence

$$
\gamma = \frac{e^{-rT}}{S_0^2 \sigma T} E\left[\phi(S_T)\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)\right]. \tag{3.2.4}
$$

We observe that the relationship between ν and γ can be summarized as

$$
\nu = S_0^2 \sigma T \gamma \tag{3.2.5}
$$

As expected, we see from equation (3.2.2), (3.2.3), and (3.2.4) that these Greeks can be computed, without differentiating the payoff function, through expected values involving weights which are independent of Φ. In the next subsections we apply these formulas to different choices of payoff functions.

3.2.1 European Call Options

We consider the case of a European call option on a non-dividend paying stock. Its payoff function is $\Phi(S_T) = [S_T - K]^+$. We compute the Greeks delta, gamma and vega with the Antithetic Variate Monte Carlo simulation method. We will also display the computation results by using the finite difference method and the exact values of the Greeks. Finally we will compare the three sets of results. More precisely, we choose the parameters values $S_0 = 100$, $K = 75$, $r = 5\%$, $\sigma = 20\%$, $T = 1$ year.

The exact values of the Greeks [9] for the European call option are given by

$$\Delta = N(d_1) \tag{3.2.6}$$

$$\nu = S_0\sqrt{T}N'(d_1) \tag{3.2.7}$$

$$\gamma = \frac{N'(d_1)}{S_0\sigma\sqrt{T}} \tag{3.2.8}$$

where $d_1 = \frac{ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$, $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$ and $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

Figure 3.1, 3.2 and 3.3 show that in the case of European options, Malliavin simulation underperforms compared to the finite difference method.

Denote by $e(n)$ the error in the calculation of the Greeks by either the Malliavin or the finite difference method with n Monte Carlo simulations, we are interested in their convergence rates in the form $e(n) = kn^{-b}$. From a linear regression for the errors in figures 3.1, 3.2, and 3.3 we obtain the results summarized in Table 3.1.

	Malliavin scheme	Finite difference scheme
Delta	$b = 0.5000, k = 0.6739$	$b = 0.5201, k = 0.0522$
Gamma	$b = 0.4333, k = 0.0393$	$b = 0.4974, k = 0.0260$
Vega	$b = 0.4861, k = 115.4226$	$b = 0.5630, k = 40.0288$

Table 3.1: Linear Regression for the Errors of Convergence of the Greeks

It is slightly lower than that of the finite difference scheme. This example shows that for derivatives with smooth payoff functions, Malliavin weighted scheme is not an efficient method to compute the sensitivities. This is because the finite difference method includes an antithetic variate variance reduction method, while the Malliavin scheme uses random variables with large variances, such as the product $[S_T - K]^+ W_T$ for the calculation of the delta.

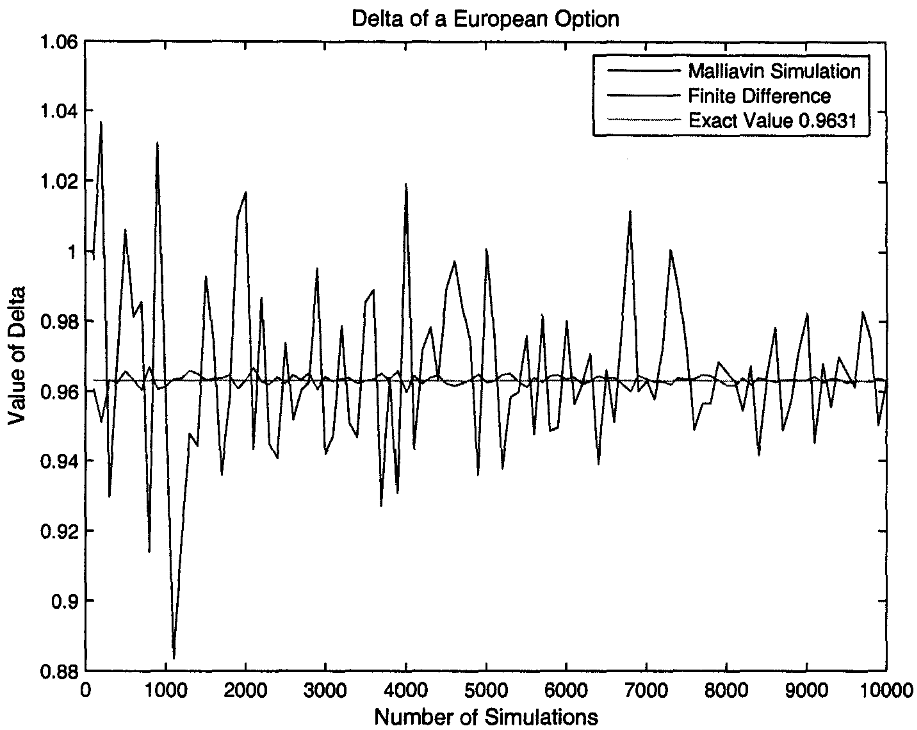

Figure 3.1: Comparison of the computation of the delta of a European call option by finite difference method and by Malliavin weighted scheme.

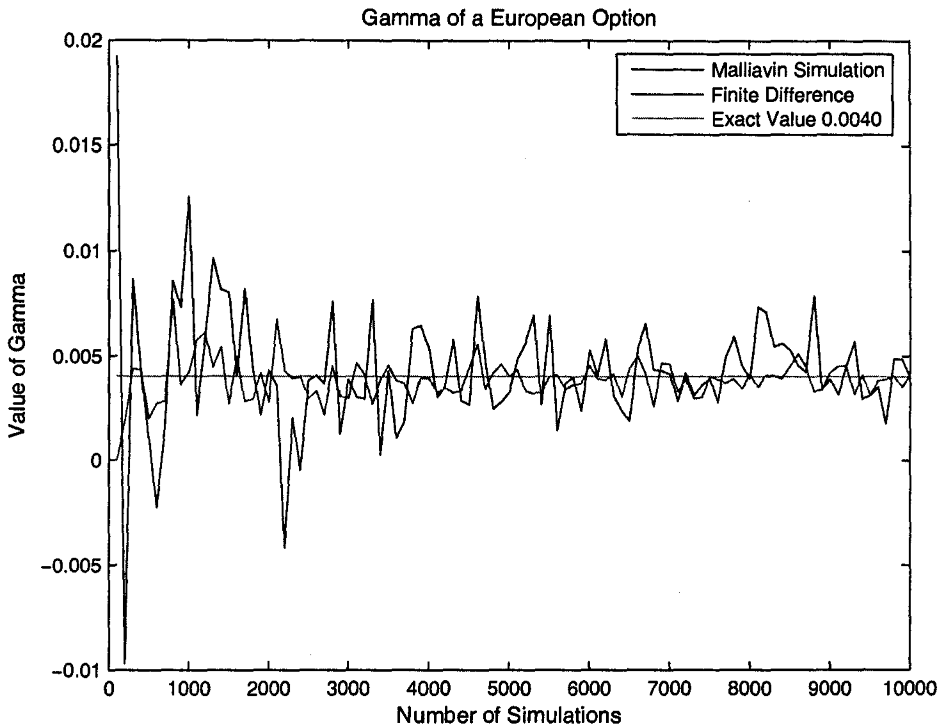

Figure 3.2: Comparison of the computation of the gamma of a European call option by finite difference method and by Malliavin weighted scheme.

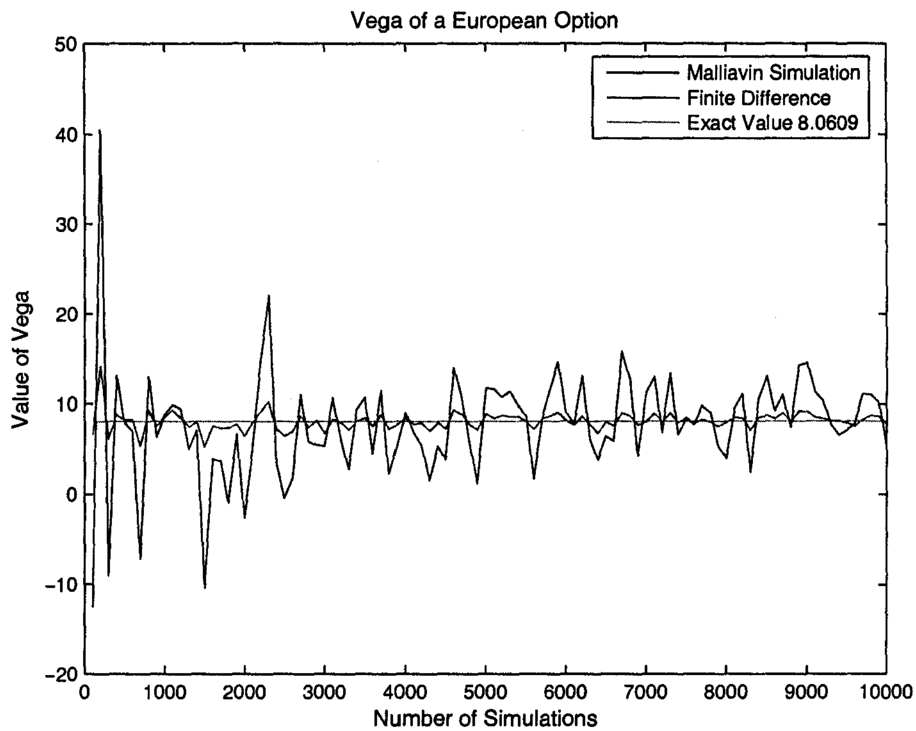

Figure 3.3: Comparison of the computation of the vega of a European call option by finite difference method and by Malliavin weighted scheme.

3.2.2 European Binary Options

A binary option (or digital option) is an option that has a discontinuous payoff. There are many forms of binary options, among which the cash-or-nothing and asset-or-nothing are two basic forms. Here we consider a simple example of a binary call option whose payoff $\Phi(S_T)$ is of the form $1_{\{S_T \geq K\}}$. The option pays off nothing if the asset price ends up below the strike price K at maturity T and pays 1, if it ends up above the strike price. We again compute delta ,gamma and vega by both the Antithetic Variate Monte Carlo simulation method and finite difference method and compare the results. We use the same set of parameters: $S_0 = 100$, $K = 75$, $r = 5\%$, $\sigma = 20\%$, $T = 1$ year.

In Figure 3.4, 3.5 and 3.6, we observe that the Malliavin simulations perform much better than finite difference method. This is because by avoid differentiating as a result of integration by parts technique, the Malliavin method smoothens the payoff of the binary option at the discontinuity point.

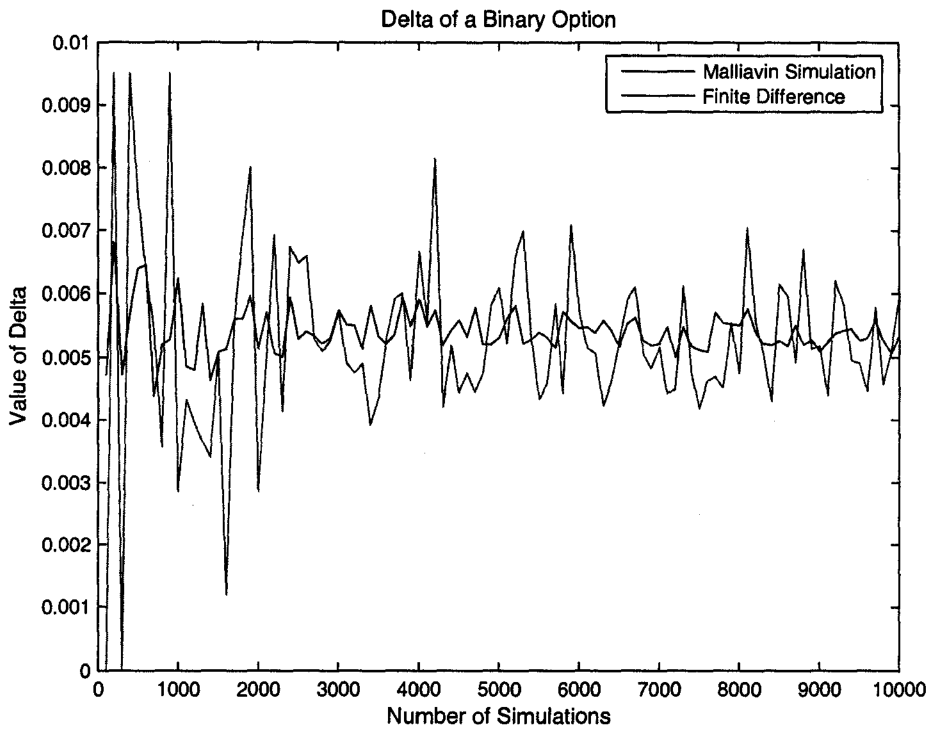

Figure 3.4: Comparison of the computation of the delta of a binary option by finite difference method and by Malliavin weighted scheme.

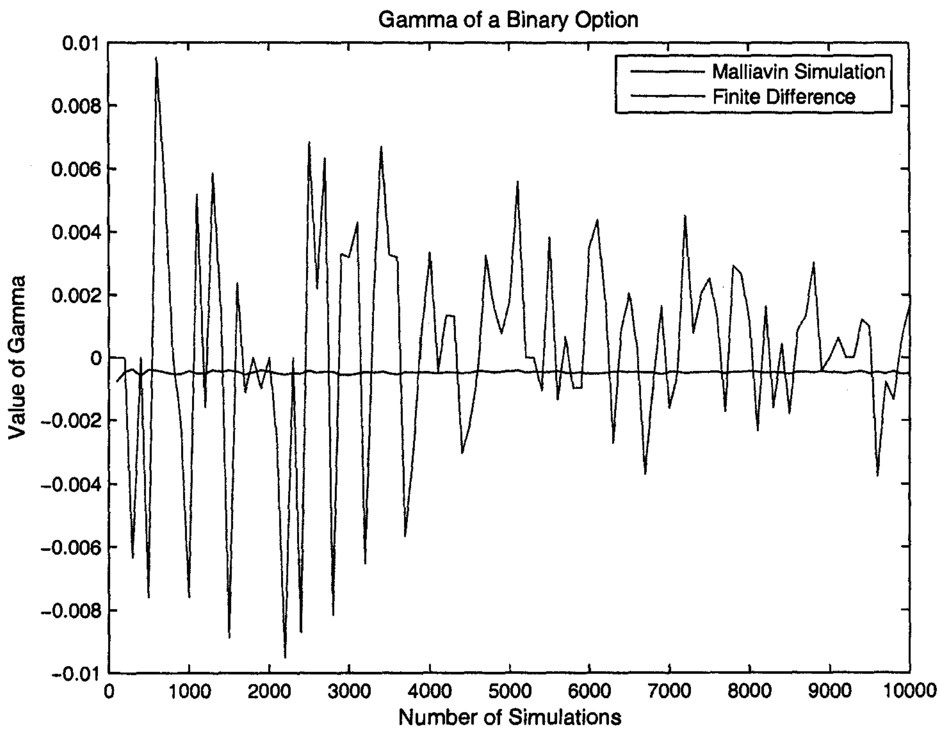

Figure 3.5: Comparison of the computation of the gamma of a binary option by finite difference method and by Malliavin weighted scheme.

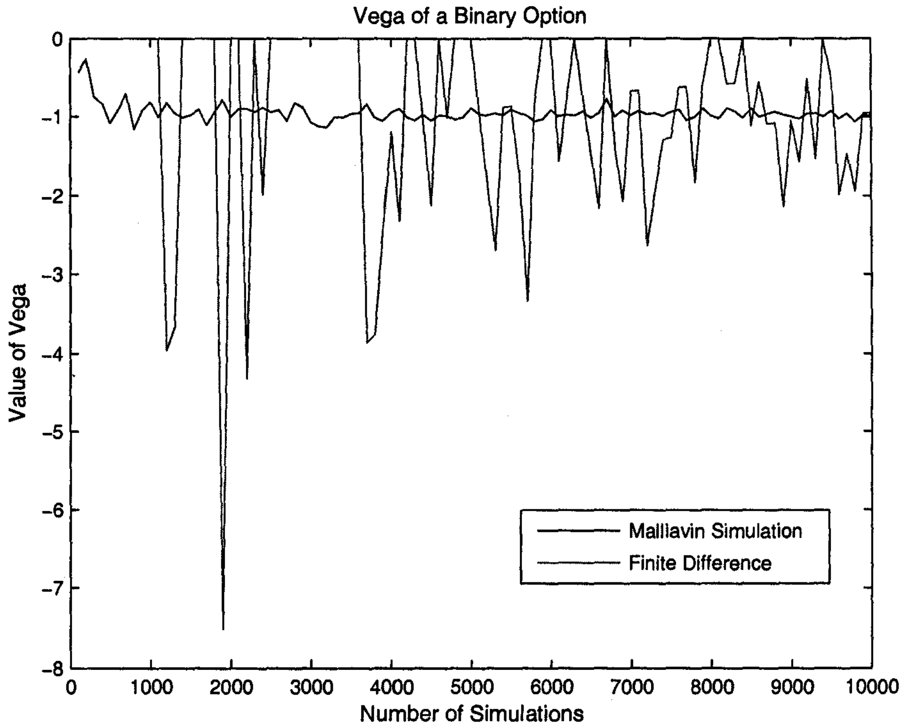

Figure 3.6: Comparison of the computation of the vega of a binary option by finite difference method and by Malliavin weighted scheme.

3.3 Asian Options

An important type of exotic options are Asian options. Asian options are options where the payoff depends on some form of averaging of the underlying asset prices over a part or the whole of the life of the option [15]. To study the performance of Malliavin scheme for path-dependent options, we will give an example of a corridor Asian option whose payoff $\Phi(\int_0^T S_t dt)$ is $1_{\{K_1 < \int_0^T S_t dt < K_2\}}$. We again assume that S follows lognormal process under the Black-Scholes framework. Then the corridor Asian option price is given by

$$V_0 = e^{-rT} E^Q \left[\Phi \left(\int_0^T S_t dt \right) \right] \tag{3.3.9}$$

where the expectation is under the risk neutral measure Q.

For the delta of such option, we need to compute

$$\frac{\partial}{\partial S_0} E \left[\Phi \left(\int_0^T S_t dt \right) \right] = E \left[\Phi' \left(\int_0^T S_t dt \right) \int_0^T Y_t dt \right],$$

where we used the definition of the first variation process Y_t.

Setting $F = \int_0^T S_t dt$ and $G = \int_0^T Y_t dt$, we see that a sufficient condition for equation (2.5.12) is that

$$\int_0^T Y_t \left(\int_0^t \frac{\sigma(s, S_s)}{Y_s} u_s ds \right) dt = \int_0^T Y_t dt. \tag{3.3.10}$$

We can then verify that the process

$$u_s = \frac{2Y_s^2}{\sigma(s, S_s) \int_0^T Y_t dt} = \frac{2S_s}{\sigma S_0 \int_0^T S_t dt}$$

satisfies the condition (3.3.10). Moreover, we can perform an explicit Skorohod integration and obtain

$$\delta(u) = \frac{2\int_0^T S_u dW_u}{\sigma S_0 \int_0^T S_u du} + \frac{1}{S_0}. \tag{3.3.11}$$

Therefore, the delta of the corridor Asian option is

$$\Delta = e^{-rT} E^Q \left[1_{\{K_1 < \int_0^T S_t dt < K_2\}} \times \left(\frac{2\int_0^T S_u dW_u}{\sigma S_0 \int_0^T S_u du} + \frac{1}{S_0} \right) \right] \tag{3.3.12}$$

In Figure 3.7, we notice that the Malliavin simulation slightly outperforms finite difference method, and the Malliavin deltas are smaller than the finite difference deltas.

For the vega of the Asian option (3.3.9), we have that

$$\begin{aligned} \frac{\partial}{\partial \sigma} E\left[\Phi\left(\int_0^T S_t dt\right)\right] &= E\left[\Phi'\left(\int_0^T S_t dt\right)\int_0^T \frac{\partial S_t}{\partial \sigma} dt\right] \\ &= E\left[\Phi'\left(\int_0^T S_t dt\right)\int_0^T \left(W_t - \sigma t\right) S_t dt\right]. \end{aligned}$$

Setting $F = \int_0^T S_t dt$ and $G = \int_0^T (W_t - \sigma t) S_t dt$, we see that a sufficient condition for equation (2.5.12) is that

$$\int_0^T Y_t \left(\int_0^t \frac{\sigma(s, S_s)}{Y_s} u_s ds \right) dt = \int_0^T (W_t - \sigma t) S_t dt.$$

We can then verify that

$$u = \frac{\int_0^T S_t W_t dt}{\sigma \int_0^T t S_t dt} - 1$$

satisfies this condition. Moreover, another explicit Skorohod integration leads to

$$\delta(u) = \frac{\int_0^T (\int_0^T S_t W_t dt) dW_s}{\sigma \int_0^T t S_t dt} + \frac{1}{S_0} \int_0^T S_t W_t dt \frac{\int_0^T t^2 S_t dt}{(\int_0^T t S_t dt)^2} - W_T \tag{3.3.13}$$

Hence the vega of the corridor Asian option is given by

$$\begin{aligned}\nu &= e^{-rT}E^Q\left[1_{\{K_1<\int_0^T S_tdt<K_2\}} * \left(\frac{\int_0^T(\int_0^T S_tW_tdt)dW_s}{\sigma\int_0^T tS_tdt}\right.\right.\\ &+ \left.\left.\frac{1}{S_0}\int_0^T S_tW_tdt\frac{\int_0^T t^2S_tdt}{(\int_0^T tS_tdt)^2} - W_T\right)\right].\end{aligned}$$

Figure 3.9 shows that in the case of vega, the Malliavin simulation performs slightly better than finite difference, and the Malliavin vegas are slightly higher than finite difference vegas.

Since S_t and Y_t are proportional, it follows from [1] that the relationship between the gamma and the vega is the same as equation (3.2.5), that is

$$\gamma = \frac{\nu}{S_0^2\sigma T}. \tag{3.3.14}$$

We can see clearly in Figure 3.8 that for the case of gamma, the Malliavin weighted scheme dramatically outperform finite difference. The Malliavin weighted scheme converges very fast with almost no oscillations, whereas the finite difference estimator fluctates drastically.

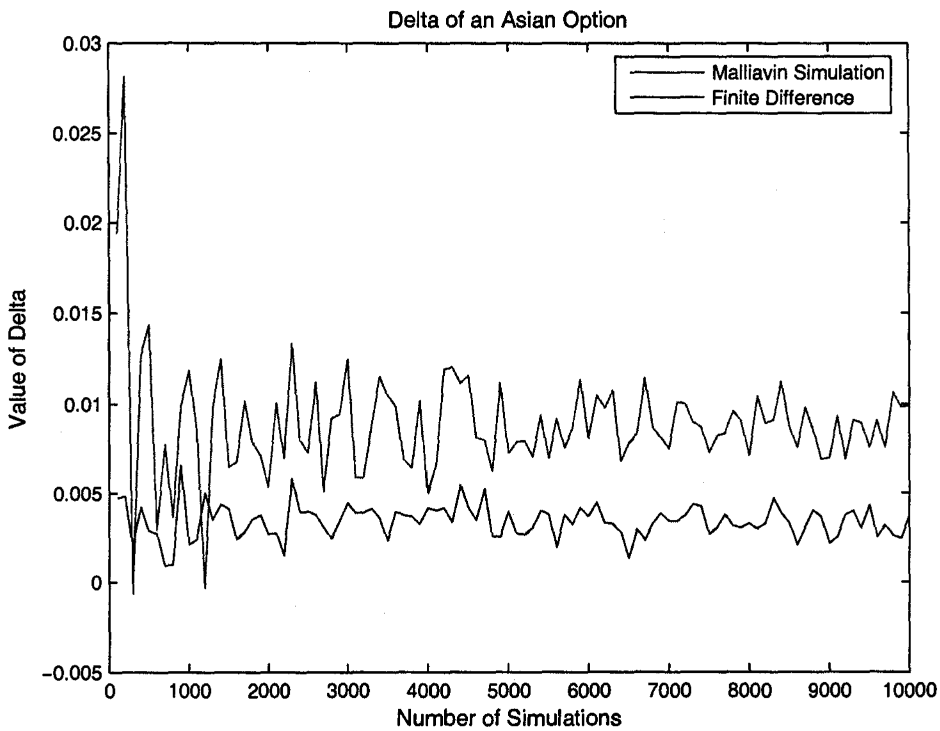

Figure 3.7: Comparison of the computation of the delta of a Corridor Asian option by finite difference method and by Malliavin weighted scheme.

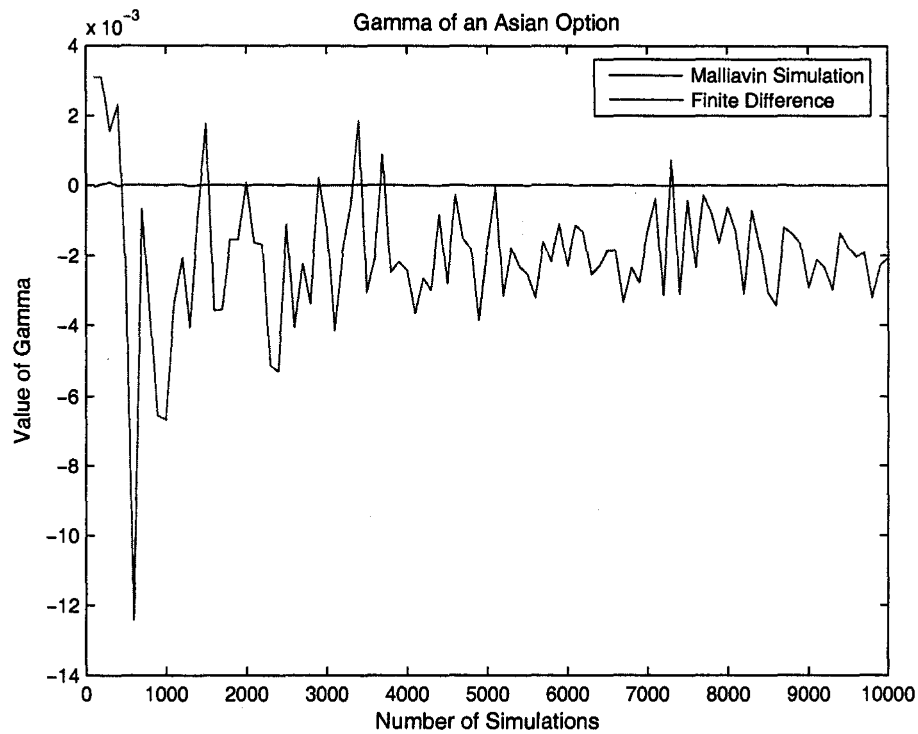

Figure 3.8: Comparison of the computation of the gamma of a Corridor Asian option by finite difference method and by Malliavin weighted scheme.

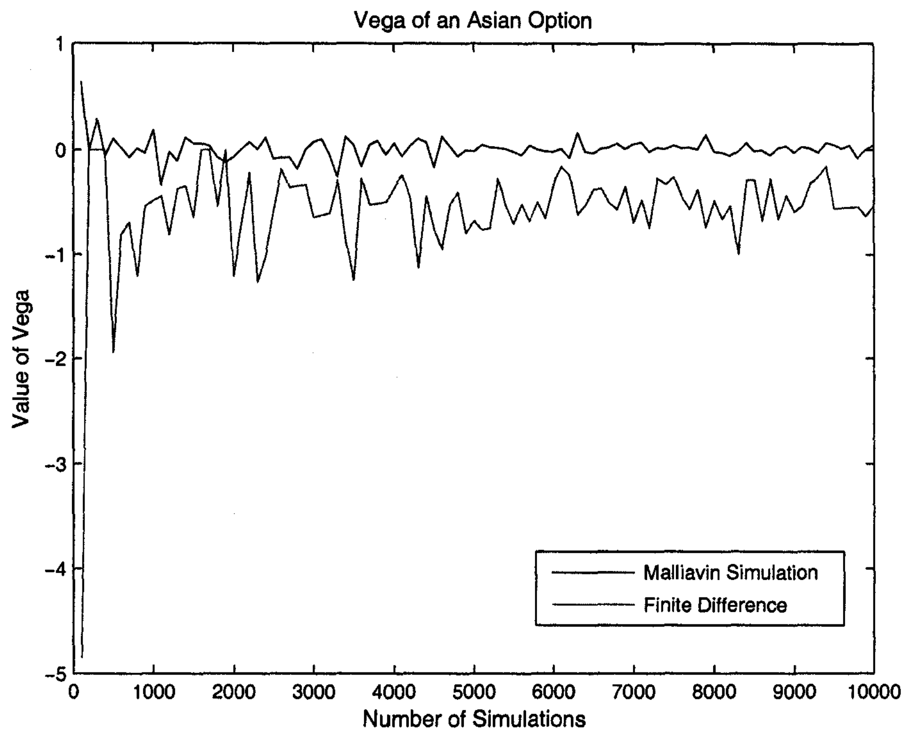

Figure 3.9: Comparison of the computation of the vega of a Corridor Asian option by finite difference method and by Malliavin weighted scheme.

Chapter 4

Malliavin Weighted Scheme in Two Dimensions

In Chapter 3 we presented the one-dimensional Malliavin weighted scheme for the computation of the Greeks. We also presented the numerical results of European options, binary options and Asian options. In this chapter, we will investigate the two-dimensional Malliavin weighted scheme and then display numerical results for the exchange options. We will first extend the setting for Malliavin calculus to the multidimensional case.

4.1 Multidimensional Malliavin Calculus

Let $W_t = (W_t^1, \cdots, W_t^d), 0 \leq t \leq T$ be a d-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathfrak{F}, P)$. Denote by $(\mathfrak{F}_t)$ the filtration generated by

W_t. Then the Malliavin derivative operative is similarly defined as in Section 2.1:

$$D_t^j F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n)) h_i^j(t), \qquad j = 1, \dots, d$$

where $h_i : [0,T] \longrightarrow \mathbb{R}^d$, $h_i = (h_i^1(t), \dots, h_i^d(t))$, $W(h) = \int_0^T h(t)dW_t$, $F = f(W(h_1), ..., W(h_n))$, and $f \in C_p^\infty(\mathbb{R}^n)$. That is, the Malliavin derivative

$$D : \mathcal{S} \subset L^2(\Omega) \longrightarrow L^2(\Omega \times T)^d$$

maps a scalar random variable F to the vector processes $(D_t^1 F, \dots, D_t^d F)$. In addition, the norm is again defined as

$$||F||_{1,2} = ||F||_{L^2(\Omega)} + ||D_t F||_{L^2(\Omega \times T)^d}.$$

Proposition 4.1 Let $\{X_t, t \geq 0\}$ be an $\mathbb{R}^d$ valued Ito process whose dynamics are driven by the stochastic differential equation[6]:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{4.1.1}$$

where $b : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ are supposed to be continuously differentiable functions with bounded derivatives. Let $Y(t), t \geq 0$ be the associated first variation process defined by the stochastic differential equation:

$$dY_t = b'(t, X_t)Y_t dt + \sum_{i=1}^{d} \sigma_i'(t, X_t) Y_t dW_t^i, \quad Y_0 = I_d \tag{4.1.2}$$

where I_d is the identity matrix in $\mathbb{R}^{d\times d}$, primes denote derivatives and σ_i is the i-th column vector of σ. Then each component of the process $\{X_t, t \geq 0\}$ belongs to $\mathbb{D}^{1,2}$, and its Malliavin derivative is given by:

$$D_s X_t = Y_t Y_s^{-1} \sigma(X_s) 1_{\{s \leq t\}}, \quad s \geq 0 \quad a.s. \tag{4.1.3}$$

The Skorohod integral is similarly defined as in the one-dimensional case, that is, we define the operator $\delta : Dom(\delta) \subset L^2[\Omega \times T]^d \to L^2(\Omega)$ with domain

$$Dom(\delta) = \left\{ u \in L^2[\Omega \times T]^d : \left| E\left(\int_0^T D_t F u_t dt \right) \right| \leq c(u) ||F||_{L^2(\Omega)}, \forall F \in \mathbb{D}^{1,2} \right\}$$

characterized by

$$E[F\delta(u)] = E\left[\int_0^T D_t F u_t dt \right], \tag{4.1.4}$$

where in the right hand side we implicitly perform a scalar product between the vector processes $(D_t^1 F, \ldots, D_t^d F)$ and $(u_t^1, \ldots, u_t^d)$. That is, the Skorohod integral maps vector processes in $L^2[\Omega \times T]^d$ back to scalar random variables. Moreover, both Theorem 2.1 and Proposition 2.1 generalize easily to the multidimensional context.

4.2 Multidimensional Malliavin Weighted Scheme

In this section, we will consider the variations in the initial condition and the variations in the diffusion coefficient for multidimensional diffusion processes and show how they can be related to Malliavin weights.

4.2.1 Variation in the Initial Condition

We provide the calculation of the derivatives of the expectation with respect to the initial condition $x \in \mathbb{R}^d$ in the form of a weighted expectation. Let the payoff function be a map $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ with

$$E[\Phi(X_t)^2] < \infty$$

for $0 < t \leq T$. We denote by $u(x)$ the expectation

$$u(x) = E[\Phi(X_T)], \quad \text{for} \quad X_0 = x.$$

We also make the assumption that the diffusion matrix σ satisfies the uniform ellipticity condition:

$$\exists\epsilon > 0, \quad \xi^*\sigma^*(t,x)\sigma(t,x)\xi \geq \epsilon|\xi|^2 \quad \text{for} \quad \text{any} \quad \xi, x \in \mathbb{R}^d. \tag{4.2.5}$$

where $*$ denotes transposition. Since b' and σ' are ssumed to be Lipschitz and bounded, the first variation process lies in $L^2(\Omega \times T)^d$[10], and therefore the above condition insures that the process $\{\sigma^{-1}(X_t)Y_t, 0 \leq t \leq T\}$ belongs to $L^2(\Omega \times T)^d$.

Proposition 4.2 Under the assumption (4.2.5), for any $x \in \mathbb{R}^d$ we have:

$$\nabla u(x) = E\left[\Phi(X_T)\int_0^T \frac{1}{T}[\sigma^{-1}(t,X_t)Y_t]^* dW_t\right].$$

Proof: Similar to the one-dimensional case, we have

$$\nabla u(x) = E[\nabla\Phi(X_T)Y_T] \tag{4.2.6}$$

We can then observe that, according to equation (4.1.3),

$$Y_T = \frac{1}{T}\int_0^T D_t X_T \sigma^{-1}(t,X_t)Y_t dt.$$

Substituting this into equation (4.2.6) gives

$$\begin{aligned}
\nabla u(x) &= E\left[\nabla\Phi(X_T)\frac{1}{T}\int_0^T D_t X_T \sigma^{-1}(t,X_t)Y_t dt\right] \\
&= E\left[\int_0^T \nabla\Phi(X_T)D_t X_T \frac{\sigma^{-1}(t,X_t)}{T} Y_t dt\right] \\
&= E\left[\int_0^T D_t\Phi(X_T)\frac{\sigma^{-1}(t,X_t)}{T} Y_t dt\right] \\
&= E\left[\Phi(X_T)\delta\left(\frac{\sigma^{-1}(t,X_t)}{T}Y_t\right)\right] \\
&= E\left[\Phi(X_T)\int_0^T \frac{\sigma^{-1}(t,X_t)}{T} Y_t dW_t\right]
\end{aligned}$$

where the third equality holds by the chain rule, the fourth equality holds by equation (4.1.4), and the last equality holds since the Skorohod integral of an adapted process is just the Ito integral of this process.

4.2.2 Variation in the Diffusion Coefficient

Let $\widetilde{\sigma} : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ be continuously differentiable function with bounded derivatives and assume that the diffusion matrix $\sigma + \epsilon\widetilde{\sigma}$ satisfies the uniform ellipticity condition for any ϵ:

$$\exists\eta > 0, \quad \xi^*(\sigma + \epsilon\widetilde{\sigma})^*(t,x)(\sigma + \epsilon\widetilde{\sigma})(t,x)\xi \geq \eta|\xi|^2 \quad \text{for} \quad \text{any} \quad \xi, x \in \mathbb{R}^d. \qquad (4.2.7)$$

We consider the perturbed process $\{X_t^\epsilon, 0 \leq t \leq T\}$ defined by:

$$X_0^\epsilon = x$$

$$dX_t^\epsilon \;=\; b(t, X_t^\epsilon)dt + [\sigma(t, X_t^\epsilon) + \epsilon\widetilde{\sigma}(t, X_t^\epsilon)]dW_t. \qquad (4.2.8)$$

Now we introduce the $\mathbb{R}^d$ valued variation process of the process with respect to ϵ:

$$Z^\epsilon(0) = 0_d$$

$$\begin{aligned} dZ_t^\epsilon \;&=\; b'(t, X_t^\epsilon)Z_t^\epsilon dt + \widetilde{\sigma}(t, X_t^\epsilon)dW_t \\ &+ \sum_{i=1}^{n}[\sigma_i' + \epsilon\widetilde{\sigma}_i'](t, X_t^\epsilon)Z_t^\epsilon dW_t^i \end{aligned} \qquad (4.2.9)$$

where 0_d is the zero column vector of $\mathbb{R}^d$. We use the notation X_t, Y_t and Z_t for the case when $\epsilon = 0$. Then we define the process $\beta_t, 0 \leq t \leq T$:

$$\beta_t = Y_t^{-1}Z_t, \qquad 0 \leq t \leq T \quad a.s. \qquad (4.2.10)$$

where the first variation process Y is defined as before. Define $u(\epsilon) = E\left[\Phi(X_T^\epsilon)\right]$.

Proposition 4.3 Under assumption (4.2.7), we have:

$$\frac{\partial}{\partial\epsilon}u(\epsilon)|_{\epsilon=0} = E\left[\Phi(X_T)\delta\left(\frac{\sigma^{-1}(t,X_t)Y_t}{T}\beta_T\right)\right] \tag{4.2.11}$$

Proof:

$$\frac{\partial}{\partial\epsilon}u(\epsilon)|_{\epsilon=0} = E\left[\Phi'(X_T)Z_T\right] \tag{4.2.12}$$

But

$$Z_T = \frac{1}{T}\int_0^T D_t X_T \sigma^{-1}(t,X_t)Y_t\beta_T dt$$

so that, substituting this into equation (4.2.12) gives

$$\begin{aligned}\frac{\partial}{\partial\epsilon}u(\epsilon)|_{\epsilon=0} &= E\left[\int_0^T \Phi'(X_T)D_tX_T\frac{\sigma^{-1}(t,X_t)Y_t}{T}\beta_T dt\right] \\ &= E\left[\int_0^T D_t\Phi(X_T)\frac{\sigma^{-1}(t,X_t)Y_t}{T}\beta_T dt\right] \\ &= E\left[\Phi(X_T)\delta\left(\frac{\sigma^{-1}(t,X_t)Y}{T}\beta_T\right)\right]\end{aligned}$$

where the second equality holds by the chain rule, and the last equality holds by equation (4.1.4).

4.3 Two-dimensional Examples

In this section we will display an example of calculation of delta and vega for the two-dimensional case. We will show explicitly how they can be computed for an exchange options under the Malliavin weighted scheme.

An exchange option is an option to exchange one asset for another. The first to price these options was Margrabe (1978) and the derivations of the formulas were extensions of the Black-Scholes work. We consider an exchange option to exchange asset 2 for asset 1 at time T, so the payoff function is $\Phi(S_T) = [S_T^1 - S_T^2]^+$. It is assumed that under the risk-neutral measure Q, the price processes of asset 1 and asset 2 satisfy

$$dS_t^1 = rS_t^1 dt + \sigma_1 S_t^1 dW_t^1, \quad S_0^1 = s_1 \tag{4.3.13}$$

$$dS_t^2 = rS_t^2 dt + \sigma_2 S_t^2 d\widetilde{W}_t, \quad S_0^2 = s_2 \tag{4.3.14}$$

where W_t^1, $\widetilde{W}_t$ are Brownian motions with $E[dW_t^1 d\widetilde{W}_t] = \rho dt$. The risk-free rate r is assumed to be constant. σ_1 and σ_2 are the volatilities of asset 1 and asset 2, respectively. The value of the exchange option is then given by

$$C = E[e^{-rT}[S_T^1 - S_T^2]^+].$$

By the Margrabe formula, we get

$$C(s_1, s_2) = s_1 N(d_1) - s_2 N(d_2)$$

where $N(\cdot)$ is the normal distribution function, and

$$d_1 = \frac{ln(s_1/s_2) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

In this two-dimensional case, the delta is a two-dimensional vector. The exact solutions of the delta vector and the vega are given by

$$\Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial C}{\partial s_1} \\ \frac{\partial C}{\partial s_2} \end{pmatrix} = \begin{pmatrix} N(d_1) \\ -N(d_2) \end{pmatrix}, \tag{4.3.15}$$

$$\nu = \frac{\partial C}{\partial \epsilon}|_{\epsilon=0} = \frac{\partial C}{\partial \sigma}\frac{\partial}{\partial \epsilon}\sigma(\epsilon)|_{\epsilon=0} = s_1\sqrt{T}N'(d_1)\frac{\sigma_1+\sigma_2-\rho(\sigma_1+\sigma_2)}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}, \tag{4.3.16}$$

where

$$\sigma(\epsilon) = \sqrt{(\sigma_1+\epsilon)^2+(\sigma_2+\epsilon)^2-2\rho(\sigma_1+\epsilon)(\sigma_2+\epsilon)}.$$

We can rewrite equation (4.3.14) as

$$dS_t^2 = rS_t^2dt + \sigma_2 S_t^2(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) \tag{4.3.17}$$

where W_t^1 and W_t^2 are two independent Brownian motions and $d\widetilde{W}_t = \rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2$, therefore the condition $E[dW_t^1 d\widetilde{W}_t] = \rho dt$ is satisfied.

We can rewrite the SDEs in the matrix form:

$$d\begin{pmatrix} S_t^1 \\ S_t^2 \end{pmatrix} = \begin{pmatrix} rS_t^1 \\ rS_t^2 \end{pmatrix} dt + \begin{pmatrix} \sigma_1 S_t^1 & 0 \\ \sigma_2\rho S_t^2 & \sigma_2\sqrt{1-\rho^2}S_t^2 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} \tag{4.3.18}$$

Therefore in equation (4.1.1),

$$b(t, S_t) = \begin{pmatrix} rS_t^1 \\ rS_t^2 \end{pmatrix} \tag{4.3.19}$$

$$\sigma(t, S_t) = \begin{pmatrix} \sigma_1 S_t^1 & 0 \\ \sigma_2\rho S_t^2 & \sigma_2\sqrt{1-\rho^2}S_t^2 \end{pmatrix} \tag{4.3.20}$$

$$dW_t = \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}. \tag{4.3.21}$$

Then in equation (4.1.2),

$$b'(t, S_t) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \tag{4.3.22}$$

$$\sigma_1'(t,S_t) = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2\rho \end{pmatrix}, \quad \sigma_2'(t,S_t) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} \tag{4.3.23}$$

Substituting the above equations into (4.1.2), we obtain

$$dY_t = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} Y_t dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2\rho \end{pmatrix} Y_t dW_t^1 + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} Y_t dW_t^2 \tag{4.3.24}$$

where

$$Y_t = \begin{pmatrix} Y_t^{11} & Y_t^{12} \\ Y_t^{21} & Y_t^{22} \end{pmatrix}, \quad \text{and} \quad Y_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.3.25}$$

After rearranging, we derive

$$\begin{cases} dY_t^{11} = rY_t^{11}dt + \sigma_1 Y_t^{11} dW_t^1 + \sigma_{12} Y_t^{11} dW_t^2, & Y_0^{11} = 1 \\ dY_t^{12} = rY_t^{12}dt + \sigma_1 Y_t^{12} dW_t^1 + \sigma_{12} Y_t^{12} dW_t^2, & Y_0^{12} = 0 \\ dY_t^{21} = rY_t^{21}dt + \sigma_2\rho Y_t^{21} dW_t^1 + \sigma_2\sqrt{1-\rho^2} Y_t^{21} dW_t^2, & Y_0^{21} = 0 \\ dY_t^{22} = rY_t^{22}dt + \sigma_2\rho Y_t^{22} dW_t^1 + \sigma_2\sqrt{1-\rho^2} Y_t^{22} dW_t^2, & Y_0^{22} = 1 \end{cases}$$

Solving the set of SDEs, we obtain the following solutions

$$\begin{cases} Y_t^{11} = \exp[(r - \frac{1}{2}\sigma_1^2)t + \sigma_1 W_t^1] \\ Y_t^{12} = 0 \\ Y_t^{21} = 0 \\ Y_t^{22} = \exp[(r - \frac{1}{2}\sigma_2^2)t + \sigma_2\rho W_t^1 + \sigma_2\sqrt{1-\rho^2} W_t^2] \end{cases} \tag{4.3.26}$$

4.3.1 Delta of Exchange Options

By Proposition 4.2, we can obtain the delta vector by computing

$$\Delta = E[e^{-rT}\Phi(S_T) \int_0^T \frac{1}{T}[\sigma^{-1}(t,S_t)Y_t]^* dW_t] \tag{4.3.27}$$

where $\Phi(S_T) = [S_T^1 - S_T^2]^+$.

Now we compute the term $\sigma^{-1}(t, S_t)Y_t$:

$$\begin{aligned}
\sigma^{-1}(t,S_t)Y_t &= \begin{pmatrix} \sigma_1 S_t^1 & 0 \\ \sigma_2\rho S_t^2 & \sigma_2\sqrt{1-\rho^2}S_t^2 \end{pmatrix}^{-1} \begin{pmatrix} Y_t^{11} & 0 \\ 0 & Y_t^{22} \end{pmatrix} \\
&= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}S_t^1S_t^2} \begin{pmatrix} \sigma_2\sqrt{1-\rho^2}S_t^2 & 0 \\ -\sigma_2\rho S_t^2 & \sigma_1 S_t^1 \end{pmatrix} \begin{pmatrix} Y_t^{11} & 0 \\ 0 & Y_t^{22} \end{pmatrix} \\
&= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}S_t^1S_t^2} \begin{pmatrix} \sigma_2\sqrt{1-\rho^2}S_t^2Y_t^{11} & 0 \\ -\sigma_2\rho S_t^2Y_t^{11} & \sigma_1 S_t^1Y_t^{11} \end{pmatrix} \qquad (4.3.28)
\end{aligned}$$

Hence by equation (4.3.27), the delta vector becomes

$$\begin{aligned}
\Delta &= \frac{e^{-rT}}{T\sigma_1\sigma_2\sqrt{1-\rho^2}} E\left[\Phi(S_T)\int_0^T \frac{1}{S_t^1S_t^2}\begin{pmatrix} \sigma_2\sqrt{1-\rho^2}S_t^2Y_t^{11} & -\sigma_2\rho S_t^2Y_t^{11} \\ 0 & \sigma_1 S_t^1Y_t^{11} \end{pmatrix}\begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}\right] \\
&= \frac{e^{-rT}}{T\sigma_1\sigma_2\sqrt{1-\rho^2}} E\left[\Phi(S_T)\begin{pmatrix} \int_0^T \frac{\sigma_2\sqrt{1-\rho^2}Y_t^{11}}{S_t^1}dW_t^1 - \int_0^T \frac{\sigma_2\rho Y_t^{11}}{S_t^1}dW_t^2 \\ \int_0^T \frac{\sigma_1 Y_t^{22}}{S_t^2}dW_t^2 \end{pmatrix}\right] \\
&= \frac{e^{-rT}}{T\sigma_1\sigma_2\sqrt{1-\rho^2}} E\left[\Phi(S_T)\begin{pmatrix} \sigma_2\sqrt{1-\rho^2}\int_0^T \frac{Y_t^{11}}{S_t^1}dW_t^1 - \sigma_2\rho\int_0^T \frac{Y_t^{11}}{S_t^1}dW_t^2 \\ \sigma_1\int_0^T \frac{Y_t^{22}}{S_t^2}dW_t^2 \end{pmatrix}\right]
\end{aligned} \qquad (4.3.29)$$

where

$$\begin{cases} Y_t^{11} = exp[(r - \frac{1}{2}\sigma_1^2)t + \sigma_1 W_t^1] \\ Y_t^{22} = exp[(r - \frac{1}{2}\sigma_2^2)t + \sigma_2\rho W_t^1 + \sigma_2\sqrt{1-\rho^2}W_t^2] \end{cases} \qquad (4.3.30)$$

From the SDE (4.3.18), we can solve for S_t^1 and S_t^2:

$$S_t^1 = s_1 e^{(r-\frac{1}{2}\sigma_1^2)t+\sigma_1 W_t^1} \qquad (4.3.31)$$

$$S_t^2 = s_2 e^{(r-\frac{1}{2}\sigma_2^2)t+\sigma_2\rho W_t^1+\sigma_2\sqrt{1-\rho^2}W_t^2} \tag{4.3.32}$$

Now we substitute equation (4.3.31) and equation (4.3.32) into equation (4.3.29) we finally obtain the expression of delta

$$\Delta = \frac{e^{-rT}}{T\sigma_1\sigma_2\sqrt{1-\rho^2}} E\left[\Phi(S_T)\begin{pmatrix}\sigma_2\sqrt{1-\rho^2}\frac{1}{s_1}W_T^1 - \sigma_2\rho\frac{1}{s_1}W_T^2 \\ \sigma_1\frac{1}{s_2}W_T^2\end{pmatrix}\right] \tag{4.3.33}$$

Having obtained the expressions of the delta vector, we can conduct the Monte Carlo simulations. The parameters we choose are:

- Starting values of the stock prices: $s_1 = 100$, $s_2 = 70$;
- Time to maturity: $T = 1$ year;
- Volatilities of underlying assets: $\sigma_1 = 20\%$ and $\sigma_2 = 15\%$;
- Correlation coefficient between the assets: $\rho = 0.5$;
- Risk-free rate: $r = 5\%$.

Figure 4.1 and 4.2 illustrate the deltas of the exchange option for the Malliavin weighted scheme and finite difference method. The delta profile present a big difference between the two methods, as can be seen in the two graphs. Similar to the delta of a European option we presented in Figure 3.1, the finite difference method outperforms the Malliavin simulation for both components of the delta vector of the exchange option. This result is further illustrated by a linear regression for the errors in Figure 4.1 and 4.2. We use the same notations for the parameters as in Chapter 3 and summarize the results in Table 4.1.

	Malliavin scheme	Finite difference scheme
Delta 1	$b = 0.4452, k = 0.7594$	$b = 0.4775, k = 0.1086$
Delta 2	$b = 0.4186, k = 1.1128$	$b = 0.4829, k = 0.1178$

Table 4.1: Linear Regression for the Errors of Convergence of the Delta Vector

The Malliavin convergence is again lower than that of finite difference scheme. This is again due to the fact that the payoff function of the exchange option is continuous. The weighting functions presented in equation (4.3.33) imply that the Brownian motions W_T^1 and W_T^2 add a large variance and therefore the Malliavin simulations converge mucher slower than the finite difference method.

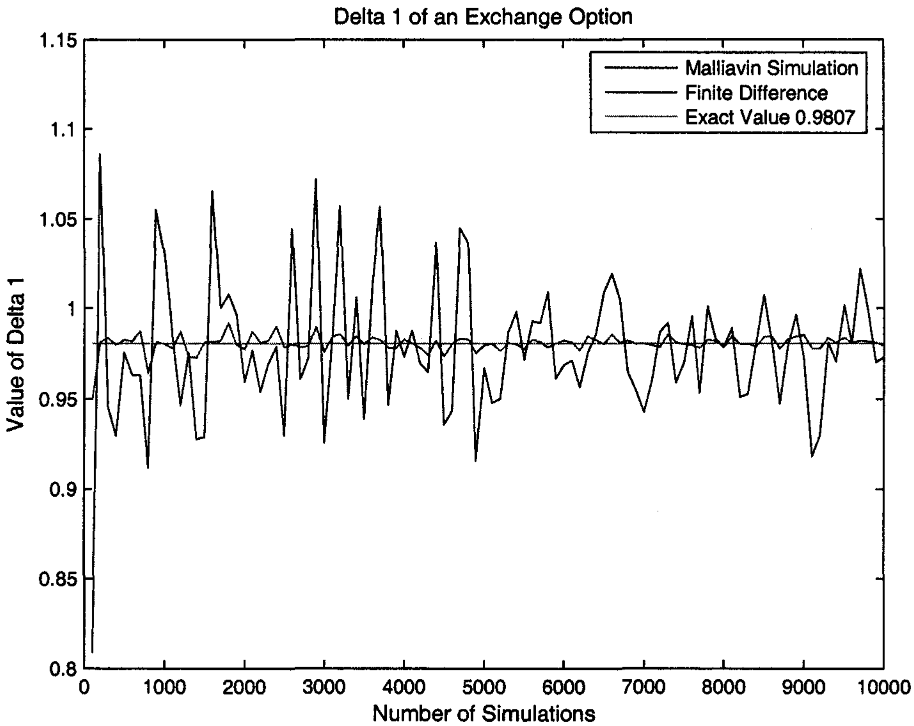

Figure 4.1: Comparison of the computation of delta 1 of an exchange option by finite difference method and by Malliavin weighted scheme.

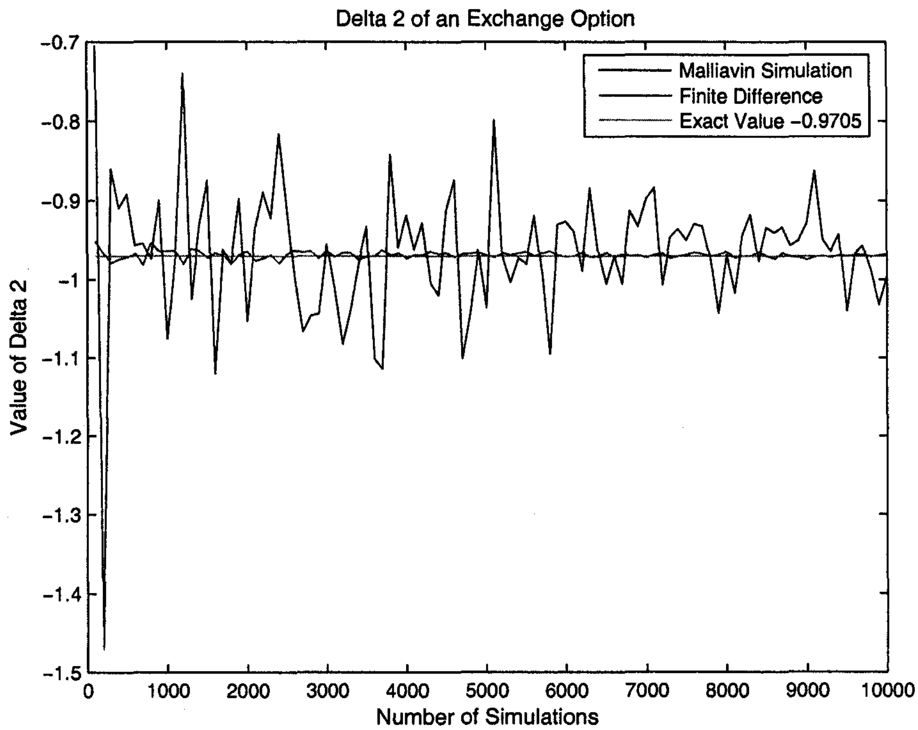

Figure 4.2: Comparison of the computation of delta 2 of an exchange option by finite difference method and by Malliavin weighted scheme.

4.3.2 Vega of Exchange Options

By setting $\epsilon = 0$ in equation (4.2.9), we will have the following Z process for the exchange option:

$$\begin{aligned} dZ_t &= b'(t,S_t)Z_t dt + \widetilde{\sigma}(t,S_t)dW_t \\ &+ \sum_{i=1}^{n} \sigma_i'(t,S_t)Z_t dW_t^i \end{aligned} \tag{4.3.34}$$

where dW_t is defined as in the previous section, and

$$b'(t,S_t) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \quad \widetilde{\sigma}(t,S_t) = \begin{pmatrix} S_t^1 & 0 \\ S_t^2\rho & S_t^2\sqrt{1-\rho^2} \end{pmatrix} \tag{4.3.35}$$

$$\sigma_1'(t,S_t) = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2\rho \end{pmatrix}, \quad \sigma_2'(t,S_t) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} \tag{4.3.36}$$

Hence equation (4.3.34) becomes

$$\begin{aligned} d\begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix} &= \begin{pmatrix} rZ_t^1 \\ rZ_t^2 \end{pmatrix} dt + \begin{pmatrix} S_t^1 dW_t^1 \\ S_t^2\rho dW_t^1 + S_t^2\sqrt{1-\rho^2}dW_t^2 \end{pmatrix} \\ &+ \begin{pmatrix} \sigma_1 Z_t^1 \\ \sigma_2\rho Z_t^2 \end{pmatrix} dW_t^1 + \begin{pmatrix} 0 \\ \sigma_2\sqrt{1-\rho^2}Z_t^2 \end{pmatrix} dW_t^2 \end{aligned} \tag{4.3.37}$$

By equation (4.3.26),

$$Y_t^{-1} = \begin{pmatrix} \frac{1}{Y_t^{11}} & 0 \\ 0 & \frac{1}{Y_t^{22}} \end{pmatrix} \tag{4.3.38}$$

where Y_t^{11} and Y_t^{22} are defined as before. Therefore by equation (4.2.10) we obtain

$$\beta_t = \begin{pmatrix} \beta_t^1 \\ \beta_t^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{Y_t^{11}} & 0 \\ 0 & \frac{1}{Y_t^{22}} \end{pmatrix} \begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix} = \begin{pmatrix} \frac{Z_t^1}{Y_t^{11}} \\ \frac{Z_t^2}{Y_t^{22}} \end{pmatrix} \tag{4.3.39}$$

By the product rule, we finally derive (Appendix A.1)

$$\begin{aligned} d\beta_t^1 &= -\frac{S_t^1}{Y_t^{11}}dt + \frac{S_t^1\sigma_1}{Y_t^{11}}dW_t^1 \\ &= -\sigma_1 S_0^1 dt + S_0^1 dW_t^1 \end{aligned} \tag{4.3.40}$$

Hence

$$\beta_T^1 = -\sigma_1 S_0^1 T + S_0^1 W_T^1 \tag{4.3.41}$$

Similarly, we obtain (Appendix A.2)

$$\begin{aligned} d\beta_t^2 &= -\sigma_2\frac{S_t^1}{Y_t^{22}}dt + \rho\frac{S_t^2}{Y_t^{22}}dW_t^1 + \sqrt{1-\rho^2}\frac{S_t^2}{Y_t^{22}}dW_t^2 \\ &= -\sigma_2 S_0^2 dt + \rho S_0^2 dW_t^1 + \sqrt{1-\rho^2}S_0^2 dW_t^2 \end{aligned} \tag{4.3.42}$$

So

$$\beta_T^2 = -\sigma_2 S_0^2 T + \rho S_0^2 W_T^1 + \sqrt{1-\rho^2}S_0^2 W_T^2. \tag{4.3.43}$$

By equation (4.3.28), we know that

$$\begin{aligned} \sigma^{-1}(t,S_t)Y_t\frac{\beta_T}{T} &= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}S_t^1S_t^2}\begin{pmatrix} \sigma_2\sqrt{1-\rho^2}S_t^2Y_t^{11} & 0 \\ -\sigma_2\rho S_t^2 Y_t^{11} & \sigma_1 S_t^1 Y_t^{11} \end{pmatrix}\frac{\beta_T}{T} \\ &= \begin{pmatrix} \frac{1}{\sigma_1 S_0^1} & 0 \\ -\frac{\rho}{\sqrt{1-\rho^2}\sigma_1 S_0^1} & \frac{1}{\sqrt{1-\rho^2}\sigma_2 S_0^2} \end{pmatrix}\frac{\beta_T}{T} \\ &= \begin{pmatrix} \frac{W_T^1}{\sigma_1 T} - 1 \\ \frac{\rho-1}{\sqrt{1-\rho^2}} + \frac{\sigma_1\rho-\sigma_2\rho}{\sigma_1\sigma_2 T\sqrt{1-\rho^2}}W_T^1 + \frac{1}{\sigma_2 T}W_T^2 \end{pmatrix} \end{aligned} \tag{4.3.44}$$

Finally by Proposition 4.3, we obtain the expression of vega:

$$\begin{aligned}
\nu &= e^{-rT}E\left[\Phi(S_T)\delta\left(\left(\frac{\frac{W_T^1}{\sigma_1 T}-1}{\frac{\rho-1}{\sqrt{1-\rho^2}}+\frac{\sigma_1\rho-\sigma_2\rho}{\sigma_1\sigma_2 T\sqrt{1-\rho^2}}W_T^1+\frac{1}{\sigma_2 T}W_T^2}\right)\right)\right] \\
&= e^{-rT}E\left[\Phi(S_T)\left(-W_T^1+\frac{(W_T^1)^2-T}{\sigma_1 T}+\frac{\rho-1}{\sqrt{1-\rho^2}}W_T^2\right.\right. \\
&+ \left.\left.\frac{(\sigma_1-\sigma_2)\rho}{\sigma_1\sigma_2 T\sqrt{1-\rho^2}}W_T^1 W_T^2+\frac{(W_T^2)^2-T}{\sigma_2 T}\right)\right] \qquad (4.3.45)
\end{aligned}$$

Having obtained the expression of vega, we then conduct the Malliavin simulations for exchange options. We choose the same set of parameters as for computing the delta.

Figure 4.3 compares the vega of the exchange option for the Malliavin weighted scheme and finite difference method. Similar to the case of the delta, the finite difference method again outperforms the Malliavin simulation for the vega of the exchange option. This is also due to the fact that the payoff function of the exchange option is continuous. Again as implied by equation (4.3.45), the two Brownian motions W_T^1 and W_T^2 add a large variance to the vega. By performing a linear regression, we obtain the parameters $b = 0.4365$, $k = 135.2602$ for the Malliavin convergence and $b = 0.5214$, $k = 58.2300$ for the finite difference convergence. Therefore, for both delta and vega of the exchange option, Malliavin simulations converge mucher slower than the finite difference method.

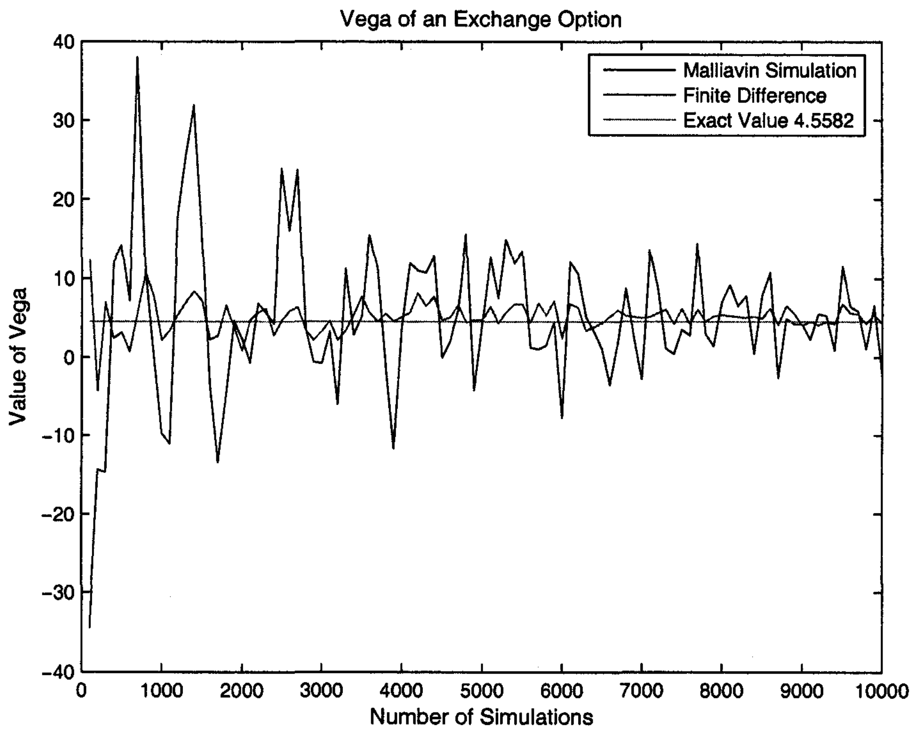

Figure 4.3: Comparison of the computation of vega of an exchange option by finite difference method and by Malliavin weighted scheme.

Chapter 5

Conclusion

In this project, we have introduced the Malliavin calculus and Wiener chaos theory. We first studied the derivative operator. Then we extended the Ito integral to the Skorohod integral, the integral of processes that are not necessarily adapted to the Brownian filtration. The Wiener chaos expansion was proved. Then having explored the essential integration by parts formula in Malliavin calculus, we showed how it can help us to express the derivatives price sensitivities as the expectation of the payoff function times a weighting function which is independent of the payoff function. This is the so-called Malliavin weighted scheme which provides an alternative method to compute the Greeks. It is especially useful when the density function of the underlying diffusion is not explicitly known as is the case of Asian options.

Having derived the Malliavin weighted scheme by the integration by parts formula, we then applied this scheme to price the delta, vega and gamma. It turns out that for the computation of the Greeks for European options, Malliavin weighted scheme underperforms the finite difference method. As the number of Monte Carlo simulations increase, the Greeks under the finite difference scheme converge much faster than under the Malliavin weighted scheme. We observe the same results for the delta vector of exchange options. In general, for smooth payoff functions, finite difference method can handle the computation of the Greeks more efficiently than Malliavin weighted scheme. This is due to the fact that the finite difference method includes an antithetic variate variance reduction method, while the Malliavin scheme involves random variables with large variances.

However, as shown in the calculation of the Greeks for binary options and Asian option, the Malliavin simulations can converge much faster than the finite difference scheme. This is due to the fact that the Malliavin weighted scheme avoids differentiating and smoothens considerably the payoff of the option to simulate. Therefore, we suggest a cautious use of the Malliavin weighted scheme. In future work, we can apply a mixed strategy. For example, we can use a local version of the Malliavin weighted scheme to smoothen the payoff at the discontinuity kink, and use the traditional finite difference scheme elsewhere.

There are many extensions to the Malliavin method presented in this project, such as a systematic study of the local Malliavin method mentioned above. The two-dimensional examples we presented can also be generalized to multidimensional derivatives, such as basket options. Finally, extensions of the Malliavin calculus to Levy processes will allow the development of a weighted scheme for derivatives whose underlying processes exhibit jumps. In conclusion, the Malliavin method is a powerful tool to compute the Greeks and it is going to have an increasing influence over the next years.

Appendix A

Proof of Equations

A.1 Proof of Equation (4.3.40)

By equation (4.3.30), we know that

$$dY_t^{11} = rY_t^{11}dt + \sigma_1 Y_t^{11}dW_t^1$$

and therefore,

$$\begin{aligned}
d\frac{1}{Y_t^{11}} &= -\frac{1}{(Y_t^{11})^2}dY_t^{11} + \frac{1}{(Y_t^{11})^3}(dY_t^{11})^2 \\
&= -\frac{1}{(Y_t^{11})^2}(rY_t^{11}dt + \sigma_1 Y_t^{11}dW_t^1) + \frac{1}{(Y_t^{11})^3}\sigma_1^2(Y_t^{11})^2dt \\
&= -\frac{r-\sigma_1^2}{Y_t^{11}}dt - \frac{\sigma_1}{Y_t^{11}}dW_t^1
\end{aligned}$$

By equation (4.3.37) we have

$$dZ_t^1 = rZ_t^1dt + \sigma_1 Z_t^1 dW_t^1 + S_t^1 dW_t^1.$$

Now by the product rule, we obtain

$$\begin{aligned} d\beta_t^1 &= d\frac{Z_t^1}{Y_t^{11}} \\ &= \frac{1}{Y_t^{11}}dZ_t^1 + Z_t^1 d(\frac{1}{Y_t^{11}}) + (\sigma_1 Z_t^1 + S_t^1)(-\frac{\sigma_1}{Y_t^{11}})dt \\ &= -\frac{S_t^1}{Y_t^{11}}dt + \frac{S_t^1\sigma_1}{Y_t^{11}}dW_t^1 \\ &= -\sigma_1 S_0^1 dt + S_0^1 dW_t^1 \end{aligned}$$

which completes the proof.

A.2 Proof of Equation (4.3.42)

By equation (4.3.30), we have

$$dY_t^{22} = rY_t^{22}dt + \sigma_2\rho Y_t^{22}dW_t^1 + \sigma_2\sqrt{1-\rho^2}Y_t^{22}dW_t^2$$

Then

$$d(\frac{1}{Y_t^{22}}) = -\frac{r-\sigma_2^2}{Y_t^{22}}dt - \frac{\sigma_2\rho}{Y_t^{22}}dW_t^1 - \frac{\sigma_2\sqrt{1-\rho^2}}{Y_t^{22}}dW_t^2$$

From equation (4.3.37) we know

$$dZ_t^2 = rZ_t^2dt + S_t^2\rho dW_t^1 + \sigma_2\rho Z_t^2 dW_t^1 + S_t^2\sqrt{1-\rho^2}dW_t^2 + \sigma_2\sqrt{1-\rho^2}Z_t^2 dW_t^2.$$

Hence,

$$\begin{aligned} d\beta_t^2 &= d\frac{Z_t^2}{Y_t^{22}} \\ &= \frac{1}{Y_t^{22}}dZ_t^2 + Z_t^2 d(\frac{1}{Y_t^{22}}) - (\frac{\sigma_2 S_t^2\rho^2}{Y_t^{22}} + \frac{\sigma_2^2\rho^2 Z_t^2}{Y_t^{22}})dt \\ &- (\frac{\sigma_2 S_t^2(1-\rho^2)}{Y_t^{22}} + \frac{\sigma_2^2(1-\rho^2)Z_t^2}{Y_t^{22}})dt \\ &= \rho\frac{S_t^2}{Y_t^{22}}dW_t^1 + \sqrt{1-\rho^2}\frac{S_t^2}{Y_t^{22}}dW_t^2 - \sigma_2\frac{S_t^2}{Y_t^{22}}dt \\ &= -\sigma_2 S_0^2 dt + \rho S_0^2 dW_t^1 + \sqrt{1-\rho^2}S_0^2 dW_t^2. \end{aligned}$$

Appendix B

Notation Index

Symbol	Description
$\mathbb{R}$	Set of real numbers
$L^2([0,T])$	Hilbert space of deterministic square integrable functions, defined on $[0,T]$
$L^2(\Omega)$	Set of square integrable random variables that belong to Ω
$L^2(\Omega \times [0,T])$	Set of square integrable processes defined on $\Omega \times [0,T]$
$< \cdot, \cdot >$	Canonical scalar product of $L^2([0,T])$
$\|\| \cdot \|\|$	Canonical norm of $L^2([0,T])$
$C_p^\infty(\cdot)$	Set of infinitely differentiable functions with all their partial derivatives with polynomial growth
C_b^1	Set of differentiable functions with their partial derivative bounded
$\mathbf{1}_A$	$\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise

$W(t)$	Brownian motion
$(\Omega, \mathfrak{F}, P)$	Complete probability space
$\mathfrak{F}_t$	Augmented filtration generated by the Brownian motion $W(t)$
$H_n(\cdot)$	n-th Hermite polynomial
S_n	The set defined by $\mathcal{S}_n = \{(t_1, \ldots, t_n) \in [0,T]^n; 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq T\}$
$\mathcal{H}_n$	Space of of Weiner chaos of ordern n
$\mathbb{D}^{1,2}$	Banach space, completion of S with respect to the norm $\|\|F\|\|_{1,2} = \|\|F\|\|_{L^2(\Omega)} + \|\|D_tF\|\|_{L^2(\Omega\times T)}$
S	The class of smooth random variables of the form $F = f(W(h_1), ..., W(h_n))$
D_tF	Malliavin derivative
δ	Skorohod integral
$Dom(\delta)$	The set defined by $Dom(\delta) = \{u \in L^2[T \times \Omega] : \|E(\int_0^T D_tFu_tdt)\| \leq c(u)\|\|F\|\|_{L^2(\Omega)}, \forall F \in D^{1,2}\}$
Δ	Derivative of the option price with respect to the price of the underlying asset
ν	Derivative of the option price with respect to the volatility of the underlying asset
Γ	The second order derivative of the option price with respect to the price of the underlying asset
$N(\cdot)$	Cumulative distribution function of normal distribution

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