Statistical Inference for r-out-of-n F-system Based on Birnbaum-Saunders Distribution

STATISTICAL INFERENCE FOR r-OUT-OF-n F-SYSTEM BASED ON BIRNBAUM-SAUNDERS DISTRIBUTION

BY

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To my grandparents

Abstract

The r-out-of-n F-system and load-sharing system are very common in industrial engineering. Statistical inference has been developed here for an equal-load sharing r-out-of-n F-system on Birnbaum-Sauders (BS) lifetime distribution. A simulation study is carried out with different parameter values and different censoring rates in order to examine the performance of the proposed estimation method. Moreover, to find maximum likelihood estimates numerically, three methods of finding initial values for the parameters - pseudo complete sample method, Type-II modified moment estimators of BS distribution method and stochastic approximation method - are developed. These three methods are then compared based on the number of iterations and simulation time. Two real data sets and one simulated data set are used for illustrative purposes. Finally, some concluding comments are made including possible future directions for investigation.

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Chapter 1

Introduction

1.1 Background

A system is called an r-out-of-n F-system if the system fails when at least r components have failed. An r-out-of-n F-system was first applied in military and industrial engineering. Since then, it has found other applications, such as the multiengine system in a car, the multicylinder system in a power system, and the multipump system in a hydraulic control system. Now, it is widely used in software engineering and electrical engineering as well. The simplest cases are the one-out-of-n F-system and n-out-of-n F-system. A one-out-of-n F-system is equivalent to a series system, which fails if any one of the components fails. On the other hand, an n-out-of-n F-system is equivalent to a parallel system with n components, which functions as long as any one of the components function.

Of particular interest is a system wherein the reliability structure changes as components fail. For instance, in an equal load-sharing system, if one component fails, the load is equally redistributed to all surviving components. The load sharing system is commonly encountered in applications. For example, current load sharing in DC-DC converters in power engineering and wood load sharing in civil engineering are both common examples. Many inferential methods have been developed for load sharing systems based on different distributions. Kim and Kvam (2004) considered load-sharing systems in which the failure rate of a given component depends on the set of working components at any given time. They applied such systems in biostatistics. Also, they assumed the load share rule to be unknown and derived methods for statistical inference on load-sharing parameters using maximum likelihood method. The failure rate of components was observed in two cases, namely, the equal load sharing system and the system in which the load for each working component increases by an unknown rule when other components fail. They also discussed hypothesis tests for these special load-sharing models. A general likelihood structure for reliability based on load sharing has been developed by Park (2010), who derived the closed-form Maximum Likelihood Estimator (MLE) and Best Unbiased Estimator (BUE) when the lifetime distribution is exponential. The author also discussed estimation of model parameters when the lifetime distribution is Weibull. Park (2013) subsequently used the Expectation-Maximization (EM) algorithm to obtain the MLEs for load sharing systems when the lifetime distribution of components is lognormal or Normal. Numerical examples illustrating the EM algorithm have also been presented.

For an equal load-sharing model in r-out-of-n systems, many inferential methods have been discussed in the literature. Based on the definition of such a system, we can only observe the first r failures, which means that the sample generated is a Type-II censored data and thus could be modelled using sequential order statistics

(SOS). Cramer and Kamps (1996) introduced SOS as a flexible model to describe 'sequential r-out-of-n systems' in which the failure of any component possibly influences other components such that their corresponding failure rate is adjusted with respect to the number of preceding failures. They then discussed some properties of the MLEs of the model parameters and several tests to verify the appropriateness of this model. Moreover, they discussed the MLEs of model parameters when the lifetime distributions are exponential and Weibull. Various estimators, including the MLE, Minimum-Variance Unbiased Estimator (UMVUE) and Best Linear Unbiased Estimator (BLUE) of the location and scale parameters, based on exponential distribution have also been presented by Cramer and Kamps (2001). Furthermore, BLUEs and best linear estimators based on generalized order statistics from Generalized Pareto distribution have been derived by Burkschat (2010). Kvam and Pena (2011) discussed a semiparametric estimator for the load-share parameters in an equal load-share model based on r-out-of-n F-systems. The asymptotic limit process of the estimator was shown to be a Gaussian process. These results are applicable in materials testing, software reliability, and power plant safety assessment. Balakrishnan *et al.* (2011) derived the likelihood function based on SOS, and modelled the parameters of an equal load-sharing system by using different link functions in SOS models. They further discussed the MLEs of these parameters based on a simple proportional link function and a linear link function, and established some properties of the estimators through analytical methods as well as Monte Carlo simulations. Balakrishnan *et al.* (2015) discussed statistical inference for composite dynamic systems based on a Burr Type-XII distribution. They assumed the lifetimes of components of a r-out-of-n F-system to have a Burr Type-XII distribution with a new hazard rate model called power trend conditionally proportional hazard model (PTCPHM). Point estimates and interval estimates of parameters based on Burr Type-XII distribution have then been developed and the MLEs of the three parameters in PTCPHM have been derived. They also carried out a hypothesis test to test if the baseline hazard rate changes upon each component failure.

In this thesis, we consider a SOS model with a Birnbaum-Saunders distribution as baseline for a r-out-of-n F-system. Using a link function to capture the increase in stress on surviving components due to the load caused by failures, the maximum likelihood estimation of model parameters is discussed here. Three different methods are proposed to provide initial values for the parameters required for the Newton-Raphson iterative process for determining the MLEs. An extensive Monte Carlo simulation study proposed estimation method as well as to compare the three methods for providing initial values. Finally, some examples are presented to illustrate the usefulness of the proposed model as well as the inferential results developed here.

Chapter 2

The Model

2.1 Birnbaum-Saunders Distribution

The Birnbaum-Saunders(BS) distribution was first derived by Birnbaum and Saunders (1969a) as a model for fatigue studies. They proposed this lifetime distribution with two parameters and then discussed some properties of this distribution. Subsequently, Birnbaum and Saunders (1969b) derived the MLEs of the parameters of this distribution. The BS distribution is a positively skewed distribution and is useful for analyzing lifetime data, and has been widely used in reliability analysis. Let us denote the Birnbaum-Saunders distribution by BS (α, β), where α is the shape parameter and β is the scale parameter. The probability density function (PDF), cumulative density function (CDF) and hazard rate function are, respectively, as follows:

$$f(x;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{x}\right)^{\frac{1}{2}} + \left(\frac{\beta}{x}\right)^{\frac{3}{2}}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right], \tag{2.1}$$

$$F(x;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{x}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{x}\right)^{\frac{1}{2}}\right\}\right], \tag{2.2}$$

$$\text{and } h(x;\alpha,\beta) = \frac{\frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{x}\right)^{\frac{1}{2}} + \left(\frac{\beta}{x}\right)^{\frac{3}{2}}\right] exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right]}{1-\Phi\left[\frac{1}{\alpha}\left\{\left(\frac{x}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{x}\right)^{\frac{1}{2}}\right\}\right]}, \tag{2.3}$$

for $0 < x < \infty, \alpha, \beta > 0$, and where $\Phi()$ denotes the standard normal CDF. A plot of these three is presented in Figure 2.1 when $\alpha = 0.5$ and $\beta = 1$.

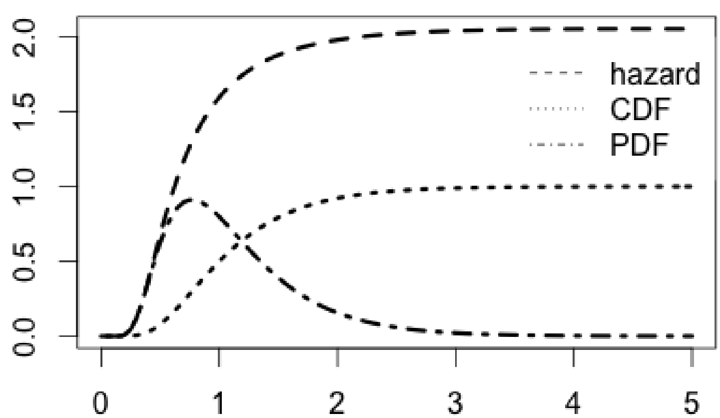

Figure 2.1: Graphs of the PDF, CDF and hazard function of the BS distribution.

The BS distribution has been shown to have an increasing and decreasing hazard function by Kundu *et al.* (2008).

The hazard functions, plotted in Figure 2.2 for different values of the shape parameter α support this characteristic.

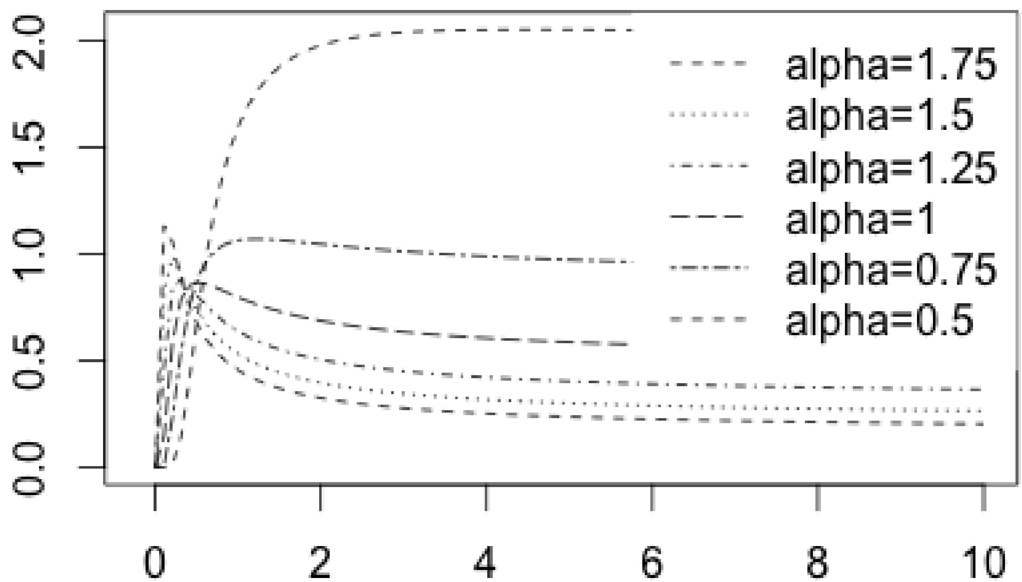

Figure 2.2: Hazard functions of the BS distribution for different values of α, when $\beta = 1$.

The BS distribution is closely related to the normal distribution. Specifically, when $X \sim \text{BS}(\alpha, \beta)$, consider the monotone transformation

$$X = \frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{1/2} + \left(\frac{T}{\beta}\right)^{-1/2}\right], \tag{2.4}$$

or

$$T = \beta \left(1 + 2X^2 + 2X(1 + X^2)^{1/2}\right). \tag{2.5}$$

Then, T is normally distributed with mean zero and variance $\frac{1}{4}\alpha^2$.

2.2 Model Assumptions

In this thesis, we assume the lifetimes of components in a r-out-of-n F-system are identical and have the same distribution function. Let $X_1, X_2, ..., X_n$ denote the component lifetimes; then, the lifetime of a r-out-of-n F-system corresponds to the r^{th} order statistic, denoted by $X_{r:n}$. Since the system fails when the r^{th} component fails, we can only observe the first r failure times. If the i^{th} failure is observed at time x, the remaining components are assumed to have the same lifetime distribution. The 1^{st} failure is denoted by $X_{1:n}^{(0)}$, 2^{nd} failure is denoted by $X_{1:n-1}^{(1)}$..., and so on. Then, $X_{1:n}^{(0)}$,$X_{1:n-1}^{(1)}$,$X_{1:n-2}^{(2)}$... are the so-called SOS of component lifetimes from a r-out-of-n F-system.

As previously mentioned, in an equal load sharing model, when each component fails, the resulting load is equally distributed to the surviving components in the system. Thus, as more components fail, the remaining components will face more stress, meaning that there is a higher likelihood of failure for surviving units to fail after each failure. For this reason, if we let h_j denote the hazard rate function of components at the j^{th} failure, it will be natural to assume that

$$h_1(x) < h_2(x) < ... < h_{n-1}(x) < h_n(x). \tag{2.6}$$

So, the lifetime distribution of the surviving components is modified due to the increasing hazard form in (2.6). Barlow *et al.* (1963) related the properties of the distribution function to properties of the corresponding hazard rate function. The hazard rate could be either non-increasing or non-decreasing. Hollander and Pena (1995) explored a stochastic approach to model complex systems with conditional proportional hazards. Deshpande *et al.* (2010) developed a general semiparametric multivariate family of distributions based on load sharing model through proportional conditional hazards. Here, we assume that the survival function of each component changes upon each failure and that the survival function of all surviving components is assumed to be the same until the next failure due to the equal load-share rule.

We introduce a baseline hazard function, $h_0(x)$, which is the hazard function of each component before any failure has occurred. Furthermore, we make the assumption

$$h_j(x) = \eta_j h_0(x),\ j = 1, 2, .., n, \tag{2.7}$$

where the η_js are positive constants. Eq. (2.7) can also be stated equivalently as

$$S_j(t) = [S_0(t)]^{\eta_j}. \tag{2.8}$$

Then, according to (2.6), we must have the condition $\eta_1 < \eta_2 < ....$ So, it is natural to consider

$$\eta_j = e^{j\eta}. \tag{2.9}$$

With this choice of link function, η_j is an increasing function of j, and since $h_j(x)$ is an increasing function of η_j, this readily implies that the hazard rate increases upon each component failure, which is consistent with the definition of a load-sharing system.

Chapter 3

Likelihood Inference

3.1 Notation

In this section, we derive the likelihood function based on an equal load sharing system and under the assumptions provided in Section 2.2. For our purpose, we let F_0, S_0, f_0 denote the baseline distribution, baseline survival and baseline density functions, respectively, of each component when no failures have occurred. After the the i^{th} failure occurs, each surviving component has the same distribution function, survival and density functions, denoted by F_i, S_i, f_i.

Based on the assumptions made, we observe the following:

$$\Pr(X_{1:n}^{(0)} \leq x_1) = 1 - (1 - F_0(x_1))^n$$
$$\Pr(X_{1:n-1}^{(1)} \leq x_2 | X_{1:n}^{(0)} = x_1) = 1 - \left(\tfrac{1-F_1(x_2)}{1-F_1(x_1)}\right)^{n-1}, \; x_2 > x_1,$$
$$\Pr(X_{1:n-2}^{(2)} \leq x_3 | X_{1:n-1}^{(1)} = x_2) = 1 - \left(\tfrac{1-F_2(x_3)}{1-F_2(x_2)}\right)^{n-2}, \; x_3 > x_2,$$

......

......

3.2 Joint PDF

We can readily express the joint PDF of the r component lifetimes as

$$\begin{aligned}
&f(x_1, x_2, ..., x_r)\\
=&f(x_r \mid x_1, .., x_{r-1})f(x_1, x_2, ..., x_{r-1})\\
=&f_r(x_r \mid x_{r-1})f(x_1, x_2, ..., x_{r-1})\\
=&f_r(x_r \mid x_{r-1})f(x_{r-1} \mid x_1, x_2, ..., x_{r-2})f(x_1, x_2, ..., x_{r-2})\\
&...\\
&...\\
&...\\
=&f_r(x_r \mid x_{r-1})f_{r-1}(x_{r-1} \mid x_{r-2})...f_2(x_2 \mid x_1)f_1(x_1) \qquad (3.1)
\end{aligned}$$

To obtain an expression from (3.1), let us look at each term individually.

First, from the conditional specification given above, we have

$$f(x_r \mid x_{r-1}) = (n-r+1)f_r(x_r)\frac{[1-F_r(x_r)]^{n-r}}{[1-F_r(x_{r-1})]^{n-r+1}}, x_r > x_{r-1}. \qquad (3.2)$$

Similarly,

$$f(x_{r-1} \mid x_{r-2}) = (n-r+2)f_{r-1}(x_{r-1})\frac{[1-F_{r-1}(x_{r-1})]^{n-r+1}}{[1-F_{r-1}(x_{r-2})]^{n-r+2}}, x_{r-1} > x_{r-2}, \qquad (3.3)$$

$$...$$

$$f(x_2 \mid x_1) = (n-1)f_2(x_2)\frac{[1-F_2(x_2)]^{n-2}}{[1-F_2(x_1)]^{n-1}}, x_2 > x_1,$$

and finally,

$$f(x_1) = nf_1(x_1)[1 - F_1(x_1)]^{n-1}. \tag{3.4}$$

Clearly then the joint PDF is the product of Eqns.(3.2)-(3.4). Now, to make the derivation neat and clean, we look at similar terms.
First, write that the product of constants is given by

$$(n - r + 1)(n - r + 2)...(n - 1)n = \frac{n!}{(n - r)!}. \tag{3.5}$$

Then, the product of $f_i(x_i)$'s is

$$\begin{aligned}
& f_r(x_r)f_{r-1}(x_{r-1})...f_2(x_2)f_1(x_1) \\
&= \eta_r f_0(x_r)[1 - F_0(x_r)]^{\eta_r - 1}\eta_{r-1}f_0(x_{r-1})[1 - F_0(x_{r-1})]^{\eta_{r-1}-1}... \\
&\quad \eta_2 f_0(x_2)[1 - F_0(x_2)]^{\eta_2 - 1}\eta_1 f_0(x_1)[1 - F_0(x_1)]^{\eta_1 - 1} \\
&= \eta_r\eta_{r-1}...\eta_2\eta_1 f_0(x_r)f_0(x_{r-1})...f_0(x_2)f_0(x_1)[1 - F_0(x_r)]^{\eta_r - 1})[1 - F_0(x_{r-1})]^{\eta_{r-1}-1}... \\
&\quad [1 - F_0(x_2)]^{\eta_2 - 1}[1 - F_0(x_1)]^{\eta_1 - 1} \\
&= \prod_{j=1}^{r}\eta_j \prod_{j=1}^{r} f_0(x_j) \prod_{j=1}^{r}[1 - F_0(x_j)]^{\eta_j - 1}.
\end{aligned} \tag{3.6}$$

The terms of the form $[1 - F_i(x_i)]^{n-i}$ can be modified by the assumption in Eq.(2.8), giving

$$\begin{aligned}
&[1 - F_r(x_r)]^{n-r} = [1 - (1 - [1 - F_0(x_r)]^{\eta_r})]^{n-r} = [1 - F_0(x_r)]^{\eta_r(n-r)}\\
&[1 - F_{r-1}(x_{r-1})]^{n-r+1} = [1 - F_0(x_r)]^{\eta_{r-1}(n-r+1)}\\
&...\\
&[1 - F_2(x_2)]^{n-2} = [1 - F_0(x_2)]^{\eta_2(n-2)}\\
&[1 - F_1(x_1)]^{n-1} = [1 - F_0(x_1)]^{\eta_1(n-1)}. \qquad (3.7)
\end{aligned}$$

The product of these terms can be written more compactly as

$$\begin{aligned}
&[[1 - F_0(x_r)]^{\eta_r(n-r)}[1 - F_0(x_r)]^{\eta_{r-1}(n-r+1)}...[1 - F_0(x_2)]^{\eta_2(n-2)}[1 - F_0(x_1)]^{\eta_1(n-1)}\\
=&\prod_{j=1}^{r}[1 - F_0(x_j)]^{\eta_j(n-j)}. \qquad (3.8)
\end{aligned}$$

The terms $[1 - F_i(x_{i-1})]^{n-i+1}$ can also be modified by the assumption in Eq.(2.8), giving

$$\begin{aligned}
&[1 - F_r(x_{r-1})]^{n-r+1} = [1 - F_0(x_{r-1})]^{\eta_r(n-r+1)}\\
&[1 - F_{r-1}(x_{r-2})]^{n-r+2} = [1 - F_0(x_{r-2})]^{\eta_{r-1}(n-r+2)}\\
&...\\
&[1 - F_2(x_1)]^{n-1} = [1 - F_0(x_1)]^{\eta_2(n-1)}. \qquad (3.9)
\end{aligned}$$

The product of these terms $[1 - F_i(x_{i-1})]^{n-i+1}$ can be written as

$$\begin{aligned}&[1 - F_0(x_{r-1})]^{\eta_r(n-r+1)}[1 - F_0(x_{r-2})]^{\eta_{r-1}(n-r+2)}...[1 - F_0(x_1)]^{\eta_2(n-1)}\\ =&\prod_{j=1}^{r-1}[1 - F_0(x_j)]^{\eta_{j+1}(n-j)}.\end{aligned} \tag{3.10}$$

By combining Eqns.(3.5),(3.6),(3.8) and (3.10), the joint PDF is then

$$\frac{n!}{(n-r)!}\frac{(\prod_{j=1}^{r}\eta_j)\prod_{j=1}^{r}f_0(x_j)\prod_{j=1}^{r}[1 - F_0(x_j)]^{\eta_j - 1}\prod_{j=1}^{r}[1 - F_0(x_j)]^{\eta_j(n-j)}}{\prod_{j=1}^{r-1}[1 - F_0(x_j)]^{\eta_{j+1}(n-j)}} \tag{3.11}$$

Under the model assumptions in Section 2.2, namely,

$$\eta_j = e^{j\eta}, \tag{3.12}$$

our likelihood function becomes

$$
\begin{aligned}
L(x;\alpha,\beta,\eta) =& \frac{n!}{(n-r)!}(e^{\eta\frac{r(r+1)}{2}})f_0(x_r)[1-F_0(x_r)]^{e^{r\eta}-1}\prod_{j=1}^{r-1}f_0(x_j)\prod_{j=1}^{r-1}[1-F_0(x_j)]^{e^{j\eta}-1} \\
& [1-F_0(x_r)]^{e^{r\eta}(n-r)}\prod_{j=1}^{r-1}[1-F_0(x_j)]^{e^{j\eta}(n-j)}\left(\prod_{j=1}^{r-1}[1-F_0(x_j)]^{e^{(j+1)\eta}(n-j)}\right)^{-1} \\
=& \frac{n!}{(n-r)!}(e^{\eta\frac{r(r+1)}{2}})f_0(x_r)[1-F_0(x_r)]^{e^{r\eta}-1}[1-F_0(x_r)]^{e^{r\eta}(n-r)} \\
& \left(\prod_{j=1}^{r-1}f_0(x_j)[1-F_0(x_j)]^{e^{j\eta}-1}[1-F_0(x_j)]^{e^{j\eta}(n-j)}[1-F_0(x_j)]^{-e^{(j+1)\eta}(n-j)}\right) \\
=& \frac{n!}{(n-r)!}(e^{\eta\frac{r(r+1)}{2}})f_0(x_r)[1-F_0(x_r)]^{e^{r\eta}-1}[1-F_0(x_r)]^{e^{r\eta}(n-r)} \\
& \left(\prod_{j=1}^{r-1}f_0(x_j)[1-F_0(x_j)]^{e^{j\eta}-1+e^{j\eta}(n-j)-e^{(j+1)\eta}(n-j)}\right) \\
=& \frac{n!}{(n-r)!}(e^{\eta\frac{r(r+1)}{2}})f_0(x_r)[1-F_0(x_r)]^{e^{r\eta}(n-r+1)-1} \\
& \left(\prod_{j=1}^{r-1}f_0(x_j)[1-F_0(x_j)]^{e^{j\eta}(n-j+1)-e^{(j+1)\eta}(n-j)-1}\right) \\
=& \frac{n!}{(n-r)!}(e^{\eta\frac{r(r+1)}{2}})\left(\prod_{j=1}^{r-1}f_0(x_j)[1-F_0(x_j)]^{e^{j\eta}(n-j+1)-e^{(j+1)\eta}(n-j)-1}\right) \\
& f_0(x_r)[1-F_0(x_r)]^{e^{r\eta}(n-r+1)-1} \\
=& \frac{n!}{(n-r)!}e^{\frac{r(1+r)}{2}\eta}\left\{\prod_{j=1}^{r-1}[1-F_0(x_j)]^{m_j-m_{j+1}-1}f_0(x_j)\right\}[1-F_0(x_r)]^{m_r-1}f_0(x_r),
\end{aligned}
\tag{3.13}
$$

where $m_j = (n-j+1)e^{j\eta}$.

3.3 Log-likelihood Function

Based on the likelihood derived in Section 3.2, we obtain the log-likelihood as follows:

$$
\begin{aligned}
&\ln L(x;\alpha,\beta,\eta)\\
=&\ln(n!)-\ln((n-r)!)+\frac{r(1+r)}{2}\eta+\sum_{j=1}^{r-1}\{(m_j-m_{j+1}-1)\ln(1-F_0(x_j))+\ln f_0(x_j)\}\\
&+(m_r-1)\ln(1-F_0(x_r))+\ln f_0(x_r)\\
=&\ln(n!)-\ln((n-r)!)+\frac{r(1+r)}{2}\eta\\
&+\sum_{j=1}^{r-1}\left\{[(n-j+1)e^{j\eta}-(n-j)\beta_0 e^{(j+1)\eta}-1]\ln(1-F_0(x_j))+\ln f_0(x_j)\right\}\\
&+[(n-r+1)e^{r\eta}-1]\ln(1-F_0(x_r))+\ln f_0(x_r)\\
=&\ln(n!)-\ln((n-r)!)+\frac{r(1+r)}{2}\eta\\
&+\sum_{j=1}^{r-1}\left\{(n-j+1)e^{j\eta}\ln\frac{1-F_0(x_j)}{1-F_0(x_{j-1})}\right\}-\sum_{j=1}^{r-1}\ln(1-F_0(x_j))+\sum_{j=1}^{r-1}\ln f_0(x_j)\\
&+[(n-r+1)e^{r\eta}-1]\ln(1-F_0(x_r))+\ln f_0(x_r)\\
=&\ln(n!)-\ln((n-r)!)+\frac{r(1+r)}{2}\eta\\
&+\sum_{j=1}^{r}\left\{(n-j+1)e^{j\eta}\ln\frac{1-F_0(x_j)}{1-F_0(x_{j-1})}\right\}-\sum_{j=1}^{r}\ln(1-F_0(x_j))+\sum_{j=1}^{r}\ln f_0(x_j)\\
=&\ln(n!)-\ln((n-r)!)+\frac{r(1+r)}{2}\eta\\
&+\sum_{j=1}^{r}\left\{(n-j+1)e^{j\eta}\ln\frac{1-F_0(x_j)}{1-F_0(x_{j-1})}\right\}-\sum_{j=1}^{r}\ln(1-F_0(x_j))+\sum_{j=1}^{r}\ln f_0(x_j),
\end{aligned}
\tag{3.14}
$$

where we assume $1 - F_0(x_0) = 1$.

3.4 First Derivatives

We derive here the first derivatives of the log-likelihood function with respect to our three model parameters.

First, we have

$$\begin{aligned}
\frac{\partial \ln L}{\partial \eta} =& \frac{r(1+r)}{2} + \sum_{j=1}^{r} \left\{ (n-j+1) j e^{j\eta} \ln \frac{1 - F_0(x_j)}{1 - F_0(x_{j-1})} \right\} \\
=& \frac{r(1+r)}{2} + \sum_{j=1}^{r} \left\{ (n-j+1) j e^{j\eta} \ln \frac{S_0(x_j)}{S_0(x_{j-1})} \right\}.
\end{aligned} \tag{3.15}$$

Next, for the parameter β, we have

$$\begin{aligned}
\frac{\partial \ln L}{\partial \beta} =& \sum_{j=1}^{r} \left\{ (n-j+1) e^{j\eta} \frac{\partial}{\partial \beta} \ln(1 - F_0(x_j)) - \frac{\partial}{\partial \beta} \ln(1 - F_0(x_{j-1})) \right\} \\
& - \sum_{j=1}^{r} \frac{\partial}{\partial \beta} \ln(1 - F_0(x_j)) + \sum_{j=1}^{r} \frac{\partial}{\partial \beta} \ln f_0(x_j) \\
=& \sum_{j=1}^{r} (n-j+1) e^{j\eta} \left\{ \frac{\partial}{\partial \beta} \ln S_0(x_j) - \frac{\partial}{\partial \beta} \ln S_0(x_{j-1}) \right\} \\
& - \sum_{j=1}^{r} \frac{\partial}{\partial \beta} \ln S_0(x_j) + \sum_{j=1}^{r} \frac{\partial}{\partial \beta} \ln f_0(x_j),
\end{aligned} \tag{3.16}$$

where

$$\frac{\partial}{\partial \beta} \ln S_0(x_j) = \frac{\frac{\partial}{\partial \beta} S_0(x_j)}{S_0(x_j)}, \tag{3.17}$$

$$\begin{aligned}
\frac{\partial}{\partial \beta} S_0(x_j) =& -\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j}+\frac{x_j}{\beta}-2\right)\right\} \frac{1}{2\alpha}\left(-\beta^{-\frac{3}{2}} x_j^{\frac{1}{2}} - \beta^{-\frac{1}{2}} x_j^{-\frac{1}{2}}\right) \\
=& -\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j}+\frac{x_j}{\beta}-2\right)\right\} \frac{1}{2\alpha x_j}\left(-\beta^{-\frac{3}{2}} x_j^{\frac{3}{2}} - \beta^{-\frac{1}{2}} x_j^{\frac{1}{2}}\right) \\
=& \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j}+\frac{x_j}{\beta}-2\right)\right\} \frac{1}{2\alpha x_j}\left[\left(\frac{x_j}{\beta}\right)^{\frac{3}{2}} + \left(\frac{x_j}{\beta}\right)^{\frac{1}{2}}\right], \quad (3.18)
\end{aligned}$$

and

$$\frac{\partial}{\partial \beta} \ln f_0(x_j) = -\frac{1}{\beta} + \frac{x_j + 3\beta}{2x_j\beta + 2\beta^2} - \frac{1}{2\alpha^2}\left(\frac{1}{x_j} - \frac{x_j}{\beta^2}\right). \quad (3.19)$$

Finally, for α, we have

$$\begin{aligned}
\frac{\partial \ln L}{\partial \alpha} =& \sum_{j=1}^{r}(n-j+1)e^{j\eta}\left\{\frac{\partial}{\partial \alpha}\ln(1-F_0(x_j)) - \frac{\partial}{\partial \alpha}\ln(1-F_0(x_{j-1}))\right\} \\
& - \sum_{j=1}^{r}\frac{\partial}{\partial \alpha}\ln(1-F_0(x_j)) + \sum_{j=1}^{r}\frac{\partial}{\partial \alpha}\ln f_0(x_j) \\
=& \sum_{j=1}^{r}(n-j+1)e^{j\eta}\left\{\frac{\partial}{\partial \alpha}\ln S_0(x_j) - \frac{\partial}{\partial \alpha}\ln S_0(x_{j-1})\right\} \\
& - \sum_{j=1}^{r}\frac{\partial}{\partial \alpha}\ln S_0(x_j) + \sum_{j=1}^{r}\frac{\partial}{\partial \alpha}\ln f_0(x_j), \quad (3.20)
\end{aligned}$$

where

$$\frac{\partial \ln S_0(x_j)}{\partial \alpha} = \frac{\frac{\partial}{\partial \alpha} S_0(x_j)}{S_0(x_j)}, \quad (3.21)$$

$$\frac{\partial S_0(x_j)}{\partial \alpha} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j}+\frac{x_j}{\beta}-2\right)\right\} \frac{1}{\alpha^2}\left\{\left(\frac{x_j}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{x_j}\right)^{\frac{1}{2}}\right\}, \quad (3.22)$$

and

$$\frac{\partial \ln f_0(x_j)}{\partial \alpha} = -\frac{1}{\alpha} + \frac{1}{\alpha^3}\left(\frac{x_j}{\beta} + \frac{\beta}{x_j} - 2\right). \tag{3.23}$$

3.5 Information Matrix

We obtain the information matrix through the second derivatives.

First, we have

$$\frac{\partial^2 \ln L}{\partial \eta^2} = \sum_{j=1}^{r}\left\{(n-j+1)j^2 e^{j\eta} \ln \frac{S_0(x_j)}{S_0(x_{j-1})}\right\}. \tag{3.24}$$

Next,

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \beta^2} =& \sum_{j=1}^{r}(n-j+1)e^{j\eta}\left\{\frac{\partial^2}{\partial \beta^2}\ln S_0(x_j) - \frac{\partial^2}{\partial \beta^2}\ln S_0(x_{j-1})\right\} \\ &- \sum_{j=1}^{r}\frac{\partial^2}{\partial \beta^2}\ln S_0(x_j) + \sum_{j=1}^{r}\frac{\partial^2}{\partial \beta^2}\ln f_0(x_j),\end{aligned} \tag{3.25}$$

where

$$\frac{\partial^2}{\partial \beta^2}\ln S_0(x_{j-1}) = \frac{\partial}{\partial \beta}\frac{\frac{\partial}{\partial \beta}S_0(x_j)}{S_0(x_j)} = \frac{[\frac{\partial^2}{\partial \beta^2}S_0(x_j)]S_0(x_j) - [\frac{\partial S_0(x_j)}{\partial \beta}]^2}{S_0(x_j)^2}. \tag{3.26}$$

Here,

$$\begin{aligned}\frac{\partial^2 S_0(x_j)}{\partial \beta^2} =&\frac{1}{2\sqrt{2\pi}\alpha x_j}\exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j}+\frac{x_j}{\beta}-2\right)\right\}\\&\left\{[-\frac{3}{2}x_j^{\frac{3}{2}}\beta^{-\frac{5}{2}}-\frac{1}{2}x_j^{\frac{1}{2}}\beta^{-\frac{3}{2}}]-\frac{1}{2\alpha}\left(\frac{1}{x_j}-\frac{x_j}{\beta^2}\right)\left[\left(\frac{x_j}{\beta}\right)^{\frac{3}{2}}+\left(\frac{x_j}{\beta}\right)^{\frac{1}{2}}\right]\right\},\end{aligned} \tag{3.27}$$

where

$$\frac{\partial \ln f_0(x_j)}{\partial \beta} = -\frac{1}{\beta}+\frac{x_j+3\beta}{2x_j\beta+2\beta^2}-\frac{1}{2\alpha^2}\left(\frac{1}{x_j}-\frac{x_j}{\beta^2}\right) \tag{3.28}$$

and

$$\begin{aligned}\frac{\partial^2 \ln f_0(x_j)}{\partial \beta^2} =&\frac{1}{\beta^2}+\frac{(x_j+3\beta)'(2x_j\beta+2\beta^2)-(2x_j\beta+2\beta^2)'(x_j+3\beta)}{(2x_j\beta+2\beta^2)^2}-\frac{1}{2\alpha^2}\left(\frac{2x_j}{\beta^3}\right)\\=&\frac{1}{\beta^2}+\frac{(2x_j+4\beta)-(x_j+3\beta)(2x_j+4\beta)}{(2x_j\beta+2\beta^2)^2}-\frac{x_j}{\alpha^2\beta^3}\\=&\frac{1}{\beta^2}+\frac{-4x_j\beta-6\beta^2-2x_j^2}{4(x_j\beta+\beta^2)^2}-\frac{x_j}{\alpha^2\beta^3}\\=&\frac{1}{\beta^2}-\frac{2x_j\beta+3\beta^2+x_j^2}{2(x_j\beta+\beta^2)^2}-\frac{x_j}{\alpha^2\beta^3}.\end{aligned} \tag{3.29}$$

We also have

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \alpha^2} =& \sum_{j=1}^{r}(n-j+1)e^{j\eta}\left\{\frac{\partial^2}{\partial \alpha^2}\ln S_0(x_j)-\frac{\partial^2}{\partial \alpha^2}\ln S_0(x_{j-1})\right\}\\&-\sum_{j=1}^{r}\frac{\partial^2}{\partial \alpha^2}\ln S_0(x_j)+\sum_{j=1}^{r}\frac{\partial^2}{\partial \alpha^2}\ln f_0(x_j),\end{aligned} \tag{3.30}$$

where

$$\frac{\partial^2}{\partial\alpha^2}\ln S_0(x_{j-1}) = \frac{\partial}{\partial\alpha}\frac{\frac{\partial}{\partial\alpha}S_0(x_j)}{S_0(x_j)} = \frac{[\frac{\partial^2}{\partial\alpha^2}S_0(x_j)]S_0(x_j) - [\frac{\partial S_0(x_j)}{\partial\alpha}]^2}{S_0(x_j)^2}, \tag{3.31}$$

$$\begin{aligned}\frac{\partial^2 S_0(x_j)}{\partial\alpha^2} =& \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j}+\frac{x_j}{\beta}-2\right)\right\}\frac{1}{\alpha^3}\left(\frac{\beta}{x_j}+\frac{x_j}{\beta}-2\right)\frac{1}{\alpha^2}\left\{\left(\frac{x_j}{\beta}\right)^{\frac{1}{2}}-\left(\frac{\beta}{x_j}\right)^{\frac{1}{2}}\right\}\\ &-\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j}-\frac{x_j}{\beta}-2\right)\right\}\frac{2}{\alpha^3}\left\{\left(\frac{x_j}{\beta}\right)^{\frac{1}{2}}-\left(\frac{\beta}{x_j}\right)^{\frac{1}{2}}\right\}\\ =&\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j}+\frac{x_j}{\beta}-2\right)\right\}\left\{\left(\frac{x_j}{\beta}\right)^{\frac{1}{2}}-\left(\frac{\beta}{x_j}\right)^{\frac{1}{2}}\right\}\\ &\left\{\frac{1}{\alpha^5}\left(\frac{\beta}{x_j}+\frac{x_j}{\beta}-2\right)-\frac{2}{\alpha^3}\right\}.\end{aligned} \tag{3.32}$$

Furthermore, we have

$$\frac{\partial \ln L}{\partial\beta\partial\eta} = \sum_{j=1}^{r}(n-j+1)je^{j\eta}\left\{\frac{\partial}{\partial\beta}\ln S_0(x_j) - \frac{\partial}{\partial\beta}\ln S_0(x_{j-1})\right\}, \tag{3.33}$$

where

$$\frac{\partial}{\partial\beta}\ln S_0(x_j) = \frac{\frac{\partial}{\partial\beta}S_0(x_j)}{S_0(x_j)}. \tag{3.34}$$

We furthermore have

$$\frac{\partial^2 \ln L}{\partial\eta\partial\alpha} = \sum_{j=1}^{r}(n-j+1)je^{j\eta}\{\frac{\partial}{\partial\alpha}\ln S_0(x_j) - \frac{\partial}{\partial\alpha}\ln S_0(x_{j-1})\},$$

where

$$\frac{\partial \ln S_0(x_j)}{\partial \alpha} = \frac{\frac{\partial}{\partial \alpha} S_0(x_j)}{S_0(x_j)} \tag{3.35}$$

and

$$\frac{\partial S_0(x_j)}{\partial \alpha} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j} + \frac{x_j}{\beta} - 2\right)\right\} \frac{1}{\alpha^2} \left\{\left(\frac{x_j}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{x_j}\right)^{\frac{1}{2}}\right\}. \tag{3.36}$$

Finally, we have

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} =& \sum_{j=1}^{r} (n-j+1) e^{j\eta} \left\{\frac{\partial^2 \ln S_0(x_j)}{\partial \alpha \partial \beta} - \frac{\partial^2 \ln S_0(x_{j-1})}{\partial \alpha \partial \beta}\right\} \\ &- \sum_{j=1}^{r} \frac{\partial^2 \ln S_0(x_j)}{\partial \alpha \partial \beta} + \sum_{j=1}^{r} \frac{\partial^2 \ln S_0(x_j)}{\partial \alpha \partial \beta},\end{aligned}$$

where

$$\frac{\partial^2 \ln S_0(x_j)}{\partial \alpha \partial \beta} = \frac{\frac{\partial^2 S_0(x_j)}{\partial \alpha \partial \beta} S_0(x_j) - \frac{\partial \ln S_0(x_j)}{\partial \beta} \frac{\partial \ln S_0(x_j)}{\partial \alpha}}{[S_0(x_j)]^2} \tag{3.37}$$

and

$$\begin{aligned}\frac{\partial^2 S_0(x_j)}{\partial \alpha \partial \beta} =& \frac{1}{\alpha^2 \sqrt{2\pi}} \exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j} + \frac{x_j}{\beta} - 2\right)\right\} \\ &- \frac{1}{2\sqrt{2\pi}\alpha^4}\left(\frac{\beta}{x_j} - \frac{x_j}{\beta^2}\right)\left\{\left(\frac{x_j}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{x_j}\right)^{\frac{1}{2}}\right\} \\ &- \frac{1}{2\sqrt{2\pi}\alpha^2} \exp\left\{-\frac{1}{2\alpha^2}\left(\frac{\beta}{x_j} + \frac{x_j}{\beta} - 2\right)\right\} (x_j^{\frac{1}{2}} \beta^{-\frac{3}{2}} + x_j^{-\frac{1}{2}} \beta^{-\frac{3}{2}}).\end{aligned} \tag{3.38}$$

By using the above second derivatives, we obtain the information matrix as

$$I_{3\times3}\text{= -}\begin{bmatrix} \frac{\partial^2 \ln L}{\partial\alpha^2} & \frac{\partial^2 \ln L}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln L}{\partial\alpha\partial\eta} \\ \frac{\partial^2 \ln L}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln L}{\partial\beta^2} & \frac{\partial^2 \ln L}{\partial\beta\partial\eta} \\ \frac{\partial^2 \ln L}{\partial\alpha\partial\eta} & \frac{\partial^2 \ln L}{\partial\beta\partial\eta} & \frac{\partial^2 \ln L}{\partial\eta^2} \end{bmatrix}.$$

With the use of the expression of the information matrix above, we can determine the estimated variance-covariance matrix of the MLEs $(\hat{\alpha}, \hat{\beta}, \hat{\eta})$ as $I|_{(\hat{\alpha}=\alpha,\hat{\beta}=\beta,\hat{\eta}=\eta)}$.

This can be used, along with the asymptotic normality of the MLEs, to construct confidence intervals for the model parameters or to carry out hypotheses tests.

3.6 Inference for multiple r-out-of-n system

The inference of reliability of the singular r-out-of-n system can be extended to the multiple r-out-of-n systems. Suppose there are m r-out-of-n systems work independently, let each r-out-of-n system denote as r_j-out-of-n_j system, and the corresponding likelihood function denote as $L_{r_j-out-of-n_j}$ where $j = 1, 2...m$. Let the likelihood function of such a multiple r-out-of-n systems denote as $L_{multiple}$, and it can be written as:

$$L_{multiple} = \prod_{j=1}^{m} L_{r_j-out-of-n_j}. \tag{3.39}$$

Specifically, if m r-out-of-n systems are identical, let the likelihood function of these singular identical r_j-out-of-$n{-}j$ system denote as $L_{r-out-of-n}$. Then, let the likelihood function of the m multiple identical r-out-of-n systems denote as $L_{identical}$ can be simplified as

$$L_{identical} = \prod_{j=1}^{m} L_{r-out-of-n}. \tag{3.40}$$

Moreover, the multiple r-out-of-n system has more accurate estimation than the singular r-out-of-n system. Let the log-likelihood function, Fisher Information matrix, Variance-Corvariance and standard error for such a multiple system denote as $lnL_{identical}$, $I_{identical}$, $\Sigma_{identical}$ and $SE_{identical}$, respectively. For a singular r-out-of-n system, let the log-likelihood function, Fisher Information matrix, Variance-Corvariance and standard error denote as lnL, I, Σ and SE, respectively.

Then, we have

$$lnL = \sum_{j=1}^{m} lnL_{identical} = m \times lnL_{identical}, \tag{3.41}$$

$$I = m \times I_{identical}, \tag{3.42}$$

$$\Sigma = \frac{1}{m}\Sigma_{identical}, \tag{3.43}$$

$$SE = \frac{1}{\sqrt{m}} \times SE_{identical}. \tag{3.44}$$

Chapter 4

Numerical Computations

4.1 Data Generation Algorithm

To generate SOS data from the proposed r-out-of-n F-system with the baseline BS distribution under our model, the following algorithm can be used:

Step 1: Generate $X_{(1)}$.

For this, we note

$$Pr(X_{1:n}^{(0)} \leq x_1) = 1 - (1 - F_0(x_1))^n. \tag{4.1}$$

Generate u as a random variate from the uniform distribution on (0,1). Then, x_1 is obtained by solving for x in $u = 1 - (1 - F_0(x))^n$.

Step 2: Generate $X_{(j)}$, conditional on $X_{(j-1)} = x_{j-1}$.

To do this, we make use of the left-truncated distribution, since x_j is always greater than the left truncation point $x_{(j-1)}$. Then, we have

$$\begin{aligned}
&Pr(X_{1:n}^{(j-1)} \leq x \mid X_{1:n}^{(j-1)} \geq x_{j-1}) \\
=&1 - \frac{[S_0(x)]^{\eta_j(n-j+1)}}{[S_0(x_{j-1})]^{\eta_j(n-j+1)}} \\
=&1 - \left\{ \frac{[S_0(x)]}{[S_0(x_{j-1})]} \right\}^{\eta_j(n-j+1)}, x > x_{j-1}.
\end{aligned}$$

Then, generate a random uniform variate u, with which x_j is obtained by solving for x in the equation $u = 1 - \left\{ \frac{[S_0(x)]}{[S_0(x_{j-1})]} \right\}^{\eta_j(n-j+1)}$.

4.2 Methods for Initial Values

In order to use the Newton-Raphson method for finding the MLEs of the model parameters, the initial values need to be chosen carefully to start the numerical iterative process. The choice of initial values is critical to saving computational time as well as to ensure convergence of the algorithm. Here, three different ways of providing initial values for the parameters α and β are discussed. For all three methods, we chose the initial value of η to be zero. The performance of the three methods is examined in this chapter through Monte Carlo simulations.

4.2.1 Pseudo Complete Sample Method

In this method, we "complete" our Type-II censored sample. To complete the censored data, we first replace the last n-r missing observations with the r^{th} ordered failure time. Then, based on this complete sample, we find the Modified Moment Estimators (MMEs) which were first derived by Ng *et al.* (2003). To be specific, let

$\{x_1, x_2, ..., x_n\}$ be a random sample of size n from the Birnbaum-Saunders distribution BS(α, β). Define

$$s = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad r = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{-1}\right]^{-1}. \tag{4.2}$$

Then, the MMEs, $\tilde{\alpha}$ and $\tilde{\beta}$, of Ng *et al.* (2003) are given by

$$\tilde{\alpha} = \left\{2\left[\left(\frac{s}{r}\right)^{1/2} - 1\right]\right\}^{1/2} \quad \text{and} \quad \tilde{\beta} = (sr)^{1/2}. \tag{4.3}$$

These are provided as initial values in the Newton-Raphson method for determining the MLEs. We refer to this as Method 1.

4.2.2 Type-II Right Censored Method

The estimation of parameters of BS distribution based on Type-II censored samples was first discussed by Ng *et al.* (2006). Upon equating the derivative of log-likelihood with respect to β to 0, the estimate of β can be found numerically. Then, since the MLE of α is a pure function of the MLE of β, the estimate of α can be obtained directly from this relationship between the MLEs of these two parameters. The derivative of log-likelihood with respect to β and the relationship between the MLEs of the parameters α and β has been simplified by Balakrishnan and Zhu (2014). Let us denote a Type-II right censored data of size k from the BS distribution by $(x_{1:n}, ..., x_{k:n})$. Then, Balakrishnan and Zhu (2014) have shown that the derivative of log-likelihood with respect to β is given by

$$\frac{\partial lnL}{\partial\beta} = -\frac{k}{\beta} + \sum_{i=1}^{k}\frac{x_{i:n}+3\beta}{2\beta(x_{i:n}+\beta)} - \frac{1}{2\alpha^2}\sum_{i=1}^{k}\left(\frac{1}{x_{i:n}} - \frac{x_{i:n}}{\beta^2}\right) + \frac{(n-k)A}{2\alpha x_{k:n}}\left[\left(\frac{x_{k:n}}{\beta}\right)^{1/2} + \left(\frac{x_{k:n}}{\beta}\right)^{3/2}\right], \tag{4.4}$$

where $A = \phi(\delta)/(1-\Phi(\delta))$, $\phi(\cdot)$ is the PDF of the standard normal distribution and $\delta = (1/\alpha)(\sqrt{x_{k:n}/\beta} - \sqrt{\beta/x_{k:n}})$, and the relationship between the MLEs of the parameters α and β is given by

$$\tilde{\alpha} = \left[\frac{\sum_{i=1}^{k}(x_{k:n-x_{i:n}})((1/x_{i:n}) - (1/\tilde{\beta}))}{\sum_{i=1}^{k}(x_{k:n}+x_{i:n})/(x_{i:n}+\tilde{\beta})}\right]^{1/2}. \tag{4.5}$$

Upon equating (4.4) to zero and then solving numerically by using Newton-Raphson method, the estimate of β can be obtained. Then, $\tilde{\alpha}$ can be obtained from (4.6). These can be provided as initial values in the Newton-Raphson method for determining the MLEs. We refer to this as Method 2.

4.2.3 Stochastic Approximation Method

The third method obtains initial values through a stochastic approximation. The first step involves using the pseudo complete sample method described in Section 4.2 to obtain $\tilde{\alpha}_0$ and $\tilde{\beta}_0$. From there, $\tilde{\alpha}_0$ and $\tilde{\beta}_0$ are used to generate the remaining $n-r$ observations from the left-truncated BS distribution. Based on this complete data, the estimates of α and β, denoted by $\tilde{\alpha}_1$ and $\tilde{\beta}_1$, are obtained. With these new estimates, the new missing data and the new estimates are once again obtained. This process is repeated, say, k times. In the end, we find the mean of the generated MLEs, which are then used as initial values in the Newton-Raphson method for determining the MLEs. We refer to this as Method 3.

Chapter 5

Simulation Study

In this chapter, we present the simulation results on the numerical maximum likelihood estimates of the model parameters of r-out-of-n F-system based on BS distribution. Tables 5.1-5.27 provide estimates of Bias, MSE, and coverage probabilities of 95% confidence interval and 90% confidence interval for all three parameters, and average number of iterations and average computational time, based on 10000 simulations, for r/n 100%, 80% and 60%. Here, we chose α as 0.25, 0.5 and 0.75; η as 0.005, 0.01 and 0.015; sample size as 60, 100 and 150. Also, β, as a scale parameter, was chosen to be 1 without loss of generality.

Since the two parameters of the BS distribution have to be positive, to keep the MLEs positive in the Newton-Raphson method, a logarithm transformation was implemented in the simulation study. For this, we first exponentiated the parameters in the log-likelihood, then, used logarithm of initial values in the Newton-Raphson method, and finally, the estimates were exponentiated to get the MLEs. Further, for Method 3, we used k=300.

For simplification and notation, we have used some shortened notation in the tables, which are as follows:

Bias $=10^3\times$Bias, MSE $= 10^3\times$MSE, Time $= 10^{-4}\times$Time(sec),

95% = coverage probability of 95% confidence interval,

90% = coverage probability of 90% confidence interval,

Iter MLE = the average number of iterations for the convergence to MLEs,

Iter Initial = the average number of iterations for the convergence to initial values,

r/n = the number of observations/ system size.

Generally, in these tables, negative bias is observed in the estimates of all three parameters. With a larger sample size, smaller MSE and smaller bias is observed and the coverage probability of 95% confidence interval and 90% confidence interval get closer to the nominal levels. With a greater r, smaller MSE and smaller bias is also observed and here again the coverage probabilities of the confidence intervals get closer to the nominal levels.

Upon comparing the three methods for initial values, we observe that Bias, MSE and coverage probabilities are all almost the same. Also, the estimates are also seen to be quite close to the parameter setting. For most of the cases considered, the pseudo complete sample method is seen to take the least time to get the MLEs. There are two exceptions, when α is 0.75, and when sample sizes are 100 and 150, in which case Type-II right censored method takes the least time to get MLEs. This suggests that Type-II right censored method could be a good choice for initial values for large α. Stochastic approximation method takes a long time to get initial values, and so the number of iterations to approach MLEs is not small, and for this reason this method is not recommended for use.

Table 5.1: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.25, \beta = 1, n = 60$ AND DIFFERENT r BY METHOD 1

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$					
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iterations	Time
r/n=100%														
0.005	-12.84	2.93	87.70	83.10	-7.27	4.42	89.49	85.22	-3.05	0.13	92.94	87.56	20.60	371
0.01	-13.32	3.00	87.44	82.83	-7.77	4.53	89.26	85.04	-3.12	0.13	92.86	87.50	20.54	373
0.015	-13.79	3.07	87.30	82.66	-8.29	4.64	89.06	84.88	-3.19	0.14	92.83	87.36	20.77	381
r/n=80%														
0.005	-16.43	3.82	85.57	81.09	-11.55	6.60	86.81	82.55	-4.95	0.28	92.06	86.58	23.54	334
0.01	-16.82	3.89	85.46	80.98	-12.03	6.73	86.54	82.30	-5.01	0.28	92.06	86.59	23.55	326
0.015	-17.24	3.94	85.28	80.86	-12.61	6.82	86.27	82.14	-5.07	0.28	92.03	86.67	23.54	324
r/n=60%														
0.005	-21.95	5.30	82.64	78.01	-18.33	10.91	83.32	79.06	-8.98	0.72	91.49	85.41	23.61	250
0.01	-22.31	5.37	82.47	77.77	-18.82	11.10	83.14	78.86	-9.05	0.72	91.50	85.42	23.75	250
0.015	-22.65	5.44	82.35	77.63	-19.29	11.29	83.00	78.68	-9.10	0.72	91.55	85.40	23.88	249

Table 5.2: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.25, \beta = 1, n = 60$ AND DIFFERENT r BY METHOD 2

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-12.84	2.93	87.71	83.10	-7.27	4.42	89.49	85.22	-3.04	0.13	92.94	87.56	6.14	20.55	409
0.01	-13.32	3.00	87.44	82.84	-7.77	4.53	89.26	85.03	-3.12	0.13	92.86	87.50	5.90	20.51	430
0.015	-13.79	3.07	87.30	82.67	-8.29	4.64	89.06	84.88	-3.19	0.14	92.83	87.36	5.79	20.75	426
r/n=80%															
0.005	-16.42	3.82	85.58	81.09	-11.54	6.60	86.81	82.55	-4.95	0.28	92.07	86.58	5.05	29.47	400
0.01	-16.82	3.88	85.46	80.99	-12.03	6.73	86.54	82.30	-5.01	0.28	92.06	86.61	4.72	28.56	393
0.015	-17.19	3.95	85.28	80.87	-12.50	6.87	86.28	82.13	-5.06	0.28	92.00	86.64	4.51	27.65	385
r/n=60%															
0.005	-21.93	5.29	82.66	78.02	-18.31	10.91	83.33	79.08	-8.97	0.72	91.50	85.42	6.04	26.77	295
0.01	-22.31	5.37	82.47	77.77	-18.82	11.10	83.14	78.86	-9.05	0.72	91.50	85.42	6.06	26.79	296
0.015	-22.65	5.44	82.35	77.63	-19.29	11.29	83.00	78.68	-9.10	0.72	91.55	85.40	6.10	26.76	295

Table 5.3: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.25, \beta = 1, n = 60$ AND DIFFERENT CR BY METHOD 3

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-12.82	2.93	87.71	83.11	-7.24	4.42	89.51	85.24	-3.04	0.13	92.96	87.58	0	20.60	422
0.01	-13.33	3.00	87.43	82.82	-7.79	4.53	89.25	85.03	-3.12	0.14	92.85	87.49	0	20.54	420
0.015	-13.79	3.07	87.30	82.66	-8.29	4.64	89.06	84.88	-3.19	0.14	92.83	87.36	0	20.77	418
r/n=80%															
0.005	-16.43	3.82	85.58	81.09	-11.54	6.60	86.81	82.55	-4.95	0.28	92.07	86.59	300	24.37	1573
0.01	-16.82	3.89	85.46	80.98	-12.03	6.73	86.54	82.29	-5.01	0.28	92.06	86.62	300	24.36	1588
0.015	-17.19	3.95	85.28	80.86	-12.50	6.87	86.26	82.12	-5.06	0.28	92.00	86.64	300	24.37	1609
r/n=60%															
0.005	-21.95	5.30	82.65	78.01	-18.33	10.91	83.32	79.06	-8.98	0.72	91.49	85.41	300	24.51	1497
0.01	-22.31	5.37	82.47	77.77	-18.82	11.10	83.14	78.86	-9.05	0.72	91.49	85.42	300	24.51	1496
0.015	-22.65	5.44	82.35	77.64	-19.29	11.29	83.00	78.68	-9.10	0.72	91.55	85.40	300	24.51	1533

Table 5.4: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.5, \beta = 1, n = 60$ AND DIFFERENT r BY METHOD 1

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$					
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iterations	Time
r/n=100%														
0.005	-26.76	13.52	86.88	82.56	-11.06	20.97	87.55	83.99	-3.42	0.15	92.51	87.01	22.87	390
0.01	-27.67	13.89	86.68	82.27	-11.82	21.49	87.34	83.75	-3.49	0.16	92.39	86.90	22.91	391
0.015	-28.39	14.21	86.45	81.94	-12.42	22.00	87.09	83.43	-3.53	0.15	92.31	86.83	23.29	394
r/n=80%														
0.005	-33.23	18.12	84.69	80.20	-15.85	32.50	84.68	80.63	-5.43	0.32	91.69	86.21	25.61	329
0.01	-33.82	18.46	84.51	80.03	-16.36	33.23	84.54	80.45	-5.46	0.32	91.70	86.19	25.56	350
0.015	-34.37	18.81	84.40	79.95	-16.83	33.98	84.41	80.30	-5.47	0.32	91.70	86.20	25.53	334
r/n=60%														
0.005	-43.33	26.58	81.20	76.69	-22.04	65.06	80.72	76.73	-9.75	0.80	91.15	84.97	25.10	254
0.01	-43.80	27.08	81.10	76.50	-22.29	68.18	80.52	76.46	-9.78	0.80	91.14	85.00	25.19	257
0.015	-44.47	26.98	80.98	76.39	-23.45	62.01	80.39	76.31	-9.80	0.80	91.17	85.10	25.31	256

Table 5.5: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.5, \beta = 1, n = 60$ AND DIFFERENT r BY METHOD 2

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-26.79	13.54	86.86	82.54	-11.08	20.98	87.52	83.98	-3.43	0.16	92.49	86.99	7.21	22.88	481
0.01	-27.66	13.89	86.67	82.26	-11.81	21.49	87.32	83.74	-3.49	0.16	92.38	86.89	6.98	22.90	482
0.015	-28.39	14.21	86.44	81.94	-12.42	22.00	87.09	83.43	-3.53	0.15	92.32	86.83	6.75	23.28	487
r/n=80%															
0.005	-33.22	18.12	84.69	80.20	-15.84	32.50	84.69	80.63	-5.43	0.32	91.69	86.21	6.55	21.58	343
0.01	-33.82	18.46	84.52	80.04	-16.36	33.23	84.54	80.45	-5.46	0.32	91.70	86.19	6.16	21.57	382
0.015	-34.37	18.81	84.40	79.95	-16.83	33.98	84.41	80.30	-5.47	0.32	91.70	86.21	5.69	21.80	381
r/n=60%															
0.005	-43.34	26.59	81.20	76.69	-22.04	65.06	80.72	76.73	-9.76	0.80	91.15	84.97	6.15	22.17	296
0.01	-43.81	27.08	81.10	76.51	-22.29	68.18	80.53	76.46	-9.78	0.80	91.14	85.00	6.08	21.96	299
0.015	-44.47	26.98	80.98	76.39	-23.46	62.01	80.40	76.31	-9.80	0.80	91.16	85.10	6.03	21.83	294

Table 5.6: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.5, \beta = 1, n = 60$ AND DIFFERENT r BY METHOD 3

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-26.72	13.52	86.89	82.57	-11.02	20.97	87.56	84.00	-3.42	0.15	92.52	87.02	0	22.87	436
0.01	-27.67	13.89	86.68	82.27	-11.82	21.49	87.34	83.75	-3.49	0.16	92.39	86.90	0	22.91	441
0.015	-28.39	14.21	86.45	81.94	-12.42	22.00	87.09	83.43	-3.53	0.15	92.31	86.83	0	23.29	414
r/n=80%															
0.005	-33.22	18.12	84.69	80.18	-15.84	32.50	84.69	80.63	-5.43	0.32	91.69	86.19	300	26.18	1455
0.01	-33.82	18.46	84.51	80.04	-16.35	33.23	84.54	80.44	-5.46	0.32	91.70	86.19	300	26.07	1447
0.015	-34.37	18.81	84.40	79.95	-16.82	33.98	84.41	80.30	-5.47	0.32	91.69	86.21	300	26.08	1459
r/n=60%															
0.005	-43.33	26.58	81.20	76.69	-22.04	65.08	80.72	76.73	-9.75	0.80	91.15	84.97	300	26.48	1429
0.01	-43.80	27.08	81.10	76.50	-22.29	68.20	80.53	76.45	-9.78	0.80	91.14	85.00	300	26.39	1434
0.015	-44.47	26.98	80.98	76.39	-23.46	62.01	80.39	76.31	-9.80	0.80	91.16	85.10	300	26.31	1447

Table 5.7: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.75, \beta = 1, n = 60$ AND DIFFERENT r BY METHOD 1

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$					
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iterations	Time
r/n=100%														
0.005	-43.75	38.33	85.20	80.75	-11.26	63.24	84.87	81.09	-4.18	0.19	91.93	86.31	23.96	399
0.01	-44.66	39.58	85.07	80.51	-11.42	66.12	84.70	80.92	-4.21	0.19	91.83	86.21	23.78	397
0.015	-45.20	40.77	84.80	80.35	-11.24	69.42	84.63	80.76	-4.20	0.19	91.82	86.22	24.11	399
r/n=80%														
0.005	-50.41	56.63	82.76	78.22	-6.37	133.50	81.76	77.75	-6.39	0.39	91.13	85.36	26.96	334
0.01	-50.31	58.77	82.65	78.02	-4.77	146.81	81.66	77.57	-6.35	0.38	91.18	85.40	26.98	333
0.015	-50.40	60.34	82.53	77.90	-3.89	155.14	81.50	77.55	-6.30	0.37	91.24	85.50	27.03	334
r/n=60%														
0.005	-65.23	83.56	79.10	74.60	-4.31	259.62	77.43	73.44	-11.42	0.93	90.46	84.57	26.16	255
0.01	-65.51	84.76	78.99	74.45	-4.12	260.49	77.31	73.37	-11.36	0.92	90.55	84.66	26.28	259
0.015	-66.15	85.25	78.85	74.33	-4.93	253.94	77.19	73.27	-11.32	0.91	90.62	84.83	26.42	259

Table 5.8: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.75, \beta = 1, n = 60$ AND DIFFERENT r BY METHOD 2

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-43.76	38.35	85.20	80.75	-11.27	63.25	84.87	81.09	-4.19	0.19	91.92	86.31	8.00	23.69	443
0.01	-44.63	39.56	85.07	80.53	-11.42	66.11	84.73	80.92	-4.21	0.19	91.84	86.22	7.68	23.58	414
0.015	-45.20	40.77	84.81	80.35	-11.24	69.42	84.64	80.76	-4.20	0.19	91.82	86.22	7.40	24.00	412
r/n=80%															
0.005	-50.40	56.62	82.76	78.21	-6.37	133.45	81.77	77.75	-6.39	0.39	91.13	85.37	7.12	22.65	395
0.01	-50.31	58.77	82.64	78.02	-4.77	146.81	81.67	77.58	-6.35	0.38	91.18	85.39	7.00	22.87	390
0.015	-50.41	60.34	82.55	77.91	-3.89	155.10	81.52	77.55	-6.30	0.37	91.23	85.51	6.79	23.32	390
r/n=60%															
0.005	-65.23	83.56	79.11	74.60	-4.30	259.62	77.43	73.44	-11.42	0.93	90.46	84.57	6.51	23.31	306
0.01	-65.50	84.75	78.99	74.46	-4.12	260.43	77.31	73.37	-11.36	0.92	90.55	84.65	6.45	23.18	300
0.015	-66.14	85.25	78.85	74.33	-4.93	253.91	77.19	73.27	-11.32	0.91	90.61	84.83	6.36	23.08	304

Table 5.9: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.75, \beta = 1, n = 60$ AND DIFFERENT r BY METHOD 3

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-43.72	38.32	85.22	80.77	-11.24	63.23	84.89	81.11	-4.18	0.19	91.95	86.33	0	23.95	435
0.01	-44.69	39.58	85.07	80.51	-11.46	66.13	84.71	80.91	-4.22	0.19	91.83	86.21	0	23.78	443
0.015	-45.20	40.77	84.80	80.35	-11.24	69.42	84.63	80.76	-4.20	0.19	91.82	86.22	0	24.11	437
r/n=80%															
0.005	-50.40	56.63	82.76	78.20	-6.36	133.46	81.76	77.75	-6.39	0.39	91.13	85.36	300	27.62	1290
0.01	-50.30	58.77	82.65	78.02	-4.76	146.79	81.67	77.57	-6.35	0.38	91.18	85.38	300	27.53	1362
0.015	-50.40	60.34	82.53	77.90	-3.88	155.14	81.52	77.55	-6.29	0.37	91.23	85.51	300	27.52	1405
r/n=60%															
0.005	-65.22	83.56	79.10	74.60	-4.30	259.58	77.43	73.44	-11.41	0.93	90.46	84.57	300	27.97	1353
0.01	-65.50	84.76	79.00	74.48	-4.11	260.54	77.31	73.37	-11.36	0.92	90.56	84.65	300	27.95	1385
0.015	-66.14	85.24	78.85	74.33	-4.93	253.90	77.19	73.27	-11.32	0.91	90.61	84.83	300	27.93	1384

Table 5.10: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.25, \beta = 1, n = 100$ AND DIFFERENT r BY METHOD 1

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$					
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iterations	Time
r/n=100%														
0.005	-7.55	1.78	90.58	85.85	-3.97	2.66	91.89	87.19	-1.04	0.03	93.96	88.56	23.23	655
0.01	-8.01	1.85	90.39	85.60	-4.46	2.76	91.55	86.89	-1.08	0.03	93.81	88.54	24.56	655
0.015	-8.42	1.91	90.16	85.49	-4.92	2.86	91.29	86.70	-1.10	0.03	93.87	88.61	25.67	663
r/n=80%														
0.005	-9.78	2.27	89.42	84.15	-6.74	3.86	90.18	85.82	-1.71	0.06	93.50	87.96	25.99	558
0.01	-10.16	2.34	89.32	83.92	-7.21	3.98	90.06	85.53	-1.74	0.06	93.54	87.95	26.02	561
0.015	-10.59	2.39	89.10	83.68	-7.77	4.08	89.90	85.28	-1.76	0.05	93.53	87.92	26.15	537
r/n=60%														
0.005	-13.52	3.12	87.37	82.84	-11.59	6.20	88.03	83.71	-3.21	0.14	92.91	87.55	26.00	413
0.01	-13.87	3.18	87.24	82.69	-12.06	6.35	87.90	83.47	-3.23	0.14	92.95	87.56	26.34	416
0.015	-14.18	3.24	87.10	82.58	-12.49	6.51	87.71	83.27	-3.24	0.14	92.98	87.62	26.80	427

Table 5.11: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.25, \beta = 1, n = 100$ AND DIFFERENT r BY METHOD 2

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-7.54	1.78	90.58	85.86	-3.97	2.66	91.90	87.18	-1.04	0.03	93.96	88.55	6.02	23.24	740
0.01	-8.01	1.85	90.40	85.61	-4.46	2.76	91.55	86.90	-1.08	0.03	93.83	88.54	5.73	24.58	770
0.015	-8.42	1.91	90.14	85.49	-4.92	2.86	91.29	86.70	-1.10	0.03	93.87	88.61	5.70	25.68	786
r/n=80%															
0.005	-9.78	2.27	89.43	84.15	-6.74	3.86	90.18	85.83	-1.71	0.06	93.51	87.96	4.71	23.13	608
0.01	-10.16	2.34	89.34	83.91	-7.20	3.98	90.07	85.52	-1.74	0.06	93.54	87.96	4.28	23.74	626
0.015	-10.49	2.40	89.11	83.68	-7.62	4.10	89.91	85.28	-1.75	0.05	93.47	87.84	4.31	24.46	637
r/n=60%															
0.005	-13.52	3.12	87.39	82.84	-11.58	6.20	88.03	83.71	-3.21	0.14	92.92	87.56	6.01	20.12	444
0.01	-13.87	3.18	87.24	82.69	-12.06	6.35	87.89	83.47	-3.23	0.14	92.94	87.57	6.05	19.97	438
0.015	-14.18	3.24	87.11	82.58	-12.49	6.51	87.71	83.25	-3.24	0.14	92.98	87.62	6.10	20.16	445

Table 5.12: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.25, \beta = 1, n = 100$ AND DIFFERENT r BY METHOD 3

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-7.55	1.78	90.58	85.85	-3.97	2.66	91.89	87.19	-1.04	0.03	93.96	88.56	0	23.22	691
0.01	-8.01	1.85	90.39	85.60	-4.46	2.76	91.55	86.89	-1.08	0.03	93.81	88.54	0	24.56	712
0.015	-8.42	1.91	90.16	85.49	-4.92	2.86	91.29	86.70	-1.10	0.03	93.87	88.61	0	25.67	560
r/n=80%															
0.005	-9.78	2.27	89.42	84.15	-6.74	3.86	90.17	85.82	-1.71	0.06	93.51	87.96	300	27.26	2186
0.01	-10.16	2.34	89.33	83.92	-7.20	3.98	90.06	85.54	-1.74	0.06	93.54	87.96	300	27.12	2200
0.015	-10.49	2.40	89.11	83.68	-7.62	4.10	89.91	85.28	-1.75	0.05	93.48	87.84	300	26.94	2144
r/n=60%															
0.005	-13.52	3.12	87.40	82.84	-11.58	6.20	88.03	83.70	-3.21	0.14	92.92	87.56	300	27.40	2012
0.01	-13.87	3.18	87.24	82.69	-12.05	6.35	87.90	83.47	-3.23	0.14	92.94	87.57	300	27.22	2038
0.015	-14.18	3.24	87.11	82.58	-12.49	6.51	87.71	83.27	-3.24	0.14	92.98	87.62	300	26.77	2036

Table 5.13: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.5, \beta = 1, n = 100$ AND DIFFERENT r BY METHOD 1

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$					
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iterations	Time
r/n=100%														
0.005	-15.70	8.26	90.00	85.29	-5.80	12.34	90.47	86.40	-1.18	0.03	93.75	88.41	25.20	665
0.01	-16.45	8.55	89.79	85.12	-6.41	12.74	90.19	86.13	-1.20	0.03	93.78	88.44	27.24	686
0.015	-17.07	8.83	89.66	85.04	-6.95	13.14	90.04	85.94	-1.20	0.03	93.68	88.39	28.34	700
r/n=80%														
0.005	-19.78	10.73	88.69	83.65	-9.33	18.13	88.81	84.29	-1.88	0.06	93.37	87.77	27.93	543
0.01	-20.35	11.02	88.49	83.50	-9.85	18.65	88.55	84.09	-1.88	0.06	93.33	87.79	28.08	550
0.015	-20.90	11.28	88.34	83.37	-10.37	19.14	88.40	83.82	-1.88	0.06	93.39	87.85	28.45	554
r/n=60%														
0.005	-26.82	15.35	86.54	82.06	-15.19	31.83	85.82	81.67	-3.48	0.16	92.58	87.26	27.96	426
0.01	-27.31	15.70	86.44	81.99	-15.60	32.84	85.76	81.53	-3.48	0.15	92.64	87.29	28.06	432
0.015	-27.73	16.04	86.31	81.84	-15.94	33.91	85.67	81.36	-3.46	0.15	92.71	87.27	28.16	434

Table 5.14: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.5, \beta = 1, n = 100$ AND DIFFERENT r BY METHOD 2

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-15.70	8.26	90.00	85.29	-5.79	12.34	90.46	86.40	-1.18	0.03	93.75	88.41	7.24	25.21	728
0.01	-16.45	8.55	89.79	85.12	-6.42	12.74	90.19	86.13	-1.20	0.03	93.75	88.43	6.88	27.22	709
0.015	-17.07	8.83	89.66	85.04	-6.95	13.14	90.04	85.94	-1.20	0.03	93.68	88.40	6.44	28.33	711
r/n=80%															
0.005	-19.76	10.72	88.68	83.64	-9.30	18.13	88.81	84.29	-1.88	0.06	93.37	87.77	6.53	25.02	637
0.01	-20.34	11.02	88.47	83.52	-9.83	18.65	88.55	84.09	-1.88	0.06	93.34	87.81	5.57	25.66	650
0.015	-20.82	11.30	88.35	83.36	-10.26	19.19	88.42	83.83	-1.87	0.06	93.38	87.82	4.69	26.43	658
r/n=60%															
0.005	-26.83	15.35	86.54	82.07	-15.19	31.83	85.84	81.68	-3.48	0.16	92.58	87.27	6.12	22.66	465
0.01	-27.31	15.70	86.44	81.99	-15.61	32.84	85.75	81.54	-3.48	0.15	92.65	87.29	6.02	22.62	470
0.015	-27.73	16.04	86.31	81.84	-15.94	33.91	85.69	81.38	-3.46	0.15	92.71	87.27	6.00	22.89	467

Table 5.15: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.5, \beta = 1, n = 100$ AND DIFFERENT r BY METHOD 3

True η	α				β				η						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-15.70	8.26	90.00	85.29	-5.80	12.34	90.47	86.40	-1.18	0.03	93.75	88.41	0	25.20	725
0.01	-16.45	8.55	89.79	85.12	-6.41	12.74	90.19	86.13	-1.20	0.03	93.78	88.44	0	27.24	754
0.015	-17.07	8.83	89.66	85.04	-6.95	13.14	90.04	85.94	-1.20	0.03	93.68	88.39	0	28.34	763
r/n=80%															
0.005	-19.77	10.73	88.68	83.64	-9.31	18.13	88.81	84.30	-1.88	0.06	93.37	87.77	300	28.90	2038
0.01	-20.34	11.02	88.49	83.50	-9.83	18.65	88.54	84.08	-1.88	0.06	93.34	87.79	300	28.87	2093
0.015	-20.83	11.30	88.35	83.36	-10.27	19.19	88.39	83.83	-1.87	0.06	93.38	87.84	300	28.81	2060
r/n=60%															
0.005	-26.82	15.35	86.54	82.06	-15.18	31.83	85.84	81.69	-3.48	0.16	92.58	87.28	300	29.28	1819
0.01	-27.31	15.69	86.44	81.99	-15.61	32.83	85.76	81.53	-3.48	0.15	92.64	87.29	300	29.28	1950
0.015	-27.73	16.04	86.31	81.84	-15.95	33.90	85.67	81.36	-3.46	0.15	92.71	87.28	300	29.25	1964

Table 5.16: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.75, \beta = 1, n = 100$ AND DIFFERENT r BY METHOD 1

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$					
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iterations	Time
r/n=100%														
0.005	-25.26	23.98	88.86	84.20	-4.65	37.23	88.27	84.43	-1.44	0.04	93.29	88.04	25.91	662
0.01	-25.84	24.91	88.63	84.09	-4.68	39.18	88.14	84.24	-1.43	0.04	93.37	87.99	28.00	684
0.015	-26.14	25.78	88.64	84.05	-4.45	41.42	88.08	84.11	-1.40	0.04	93.48	88.10	30.13	714
r/n=80%														
0.005	-29.94	32.29	87.01	82.45	-4.83	57.65	86.31	81.78	-2.21	0.08	92.85	87.48	28.98	560
0.01	-30.08	33.27	86.96	82.28	-4.37	59.80	86.20	81.70	-2.17	0.07	92.95	87.55	29.22	560
0.015	-30.07	34.24	86.87	82.29	-3.77	62.11	86.13	81.58	-2.12	0.07	93.08	87.64	29.21	560
r/n=60%														
0.005	-38.51	49.89	84.74	80.17	-3.05	119.94	83.05	79.06	-4.03	0.19	92.25	86.78	29.31	432
0.01	-38.29	51.53	84.74	80.15	-1.68	127.71	83.01	78.99	-3.96	0.18	92.31	86.91	29.36	434
0.015	-37.99	53.23	84.69	80.12	-0.16	136.71	82.98	78.87	-3.88	0.18	92.31	86.96	29.48	431

Table 5.17: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.75, \beta = 1, n = 100$ AND DIFFERENT r BY METHOD 2

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-25.27	23.98	88.86	84.22	-4.67	37.23	88.27	84.44	-1.45	0.04	93.29	88.04	7.98	25.75	665
0.01	-25.84	24.90	88.64	84.09	-4.68	39.18	88.14	84.23	-1.43	0.04	93.37	88.03	7.45	28.00	691
0.015	-26.09	25.80	88.64	84.05	-4.39	41.46	88.08	84.11	-1.40	0.04	93.48	88.09	7.01	30.05	721
r/n=80%															
0.005	-29.91	32.29	87.01	82.43	-4.79	57.64	86.27	81.77	-2.21	0.08	92.85	87.48	7.15	25.55	542
0.01	-30.07	33.27	86.95	82.32	-4.36	59.80	86.23	81.73	-2.17	0.07	92.93	87.55	6.87	26.57	556
0.015	-30.06	34.24	86.87	82.32	-3.76	62.10	86.14	81.59	-2.12	0.07	93.12	87.64	6.12	27.64	568
r/n=60%															
0.005	-38.50	49.88	84.75	80.19	-3.04	119.90	83.06	79.06	-4.02	0.19	92.26	86.77	6.60	24.12	406
0.01	-38.29	51.52	84.74	80.18	-1.67	127.69	83.01	79.01	-3.96	0.18	92.31	86.91	6.42	24.16	407
0.015	-37.99	53.23	84.68	80.12	-0.15	136.73	82.97	78.86	-3.88	0.18	92.31	86.96	6.18	24.54	414

Table 5.18: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.75, \beta = 1, n = 100$ AND DIFFERENT r BY METHOD 3

True η	α				β				η						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-25.26	23.98	88.86	84.20	-4.65	37.23	88.27	84.43	-1.44	0.04	93.29	88.04	0	25.90	643
0.01	-25.84	24.91	88.63	84.09	-4.68	39.18	88.14	84.24	-1.43	0.04	93.37	87.99	0	28.00	667
0.015	-26.14	25.78	88.64	84.05	-4.45	41.42	88.08	84.11	-1.40	0.04	93.48	88.10	0	30.13	694
r/n=80%															
0.005	-29.91	32.29	87.01	82.44	-4.78	57.64	86.32	81.78	-2.21	0.08	92.85	87.48	300	30.42	1859
0.01	-30.05	33.26	86.97	82.30	-4.33	59.80	86.21	81.71	-2.17	0.07	92.94	87.55	300	30.27	1777
0.015	-30.06	34.24	86.87	82.31	-3.76	62.10	86.13	81.58	-2.12	0.07	93.08	87.66	300	30.18	1825
r/n=60%															
0.005	-38.50	49.88	84.75	80.20	-3.05	119.92	83.06	79.07	-4.02	0.19	92.26	86.79	300	30.89	1698
0.01	-38.29	51.52	84.74	80.16	-1.68	127.71	83.01	79.01	-3.96	0.18	92.31	86.92	300	30.86	1702
0.015	-37.99	53.23	84.69	80.12	-0.17	136.68	82.98	78.88	-3.88	0.18	92.31	86.96	300	30.67	1714

Table 5.19: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.25, \beta = 1, n = 150$ AND DIFFERENT r BY METHOD 1

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$					
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iterations	Time
r/n=100%														
0.005	-4.77	1.24	92.07	87.13	-2.08	1.82	92.70	87.89	-0.41	0.01	94.10	88.79	26.33	1007
0.01	-5.19	1.30	91.89	86.88	-2.53	1.92	92.50	87.62	-0.43	0.01	94.02	88.73	26.92	1007
0.015	-5.50	1.35	91.70	86.78	-2.90	2.01	92.32	87.42	-0.43	0.01	94.04	88.94	26.50	1003
r/n=80%														
0.005	-6.09	1.54	91.01	86.48	-3.71	2.59	92.08	87.29	-0.67	0.02	94.23	88.60	27.80	856
0.01	-6.43	1.60	90.90	86.36	-4.11	2.70	91.82	87.31	-0.68	0.02	94.27	88.58	28.05	824
0.015	-6.74	1.65	90.80	86.19	-4.53	2.80	91.62	87.01	-0.68	0.01	94.33	88.63	26.43	808
r/n=60%														
0.005	-8.88	2.09	89.87	84.87	-7.41	4.11	90.15	85.77	-1.37	0.04	93.84	88.47	28.04	643
0.01	-9.19	2.14	89.74	84.71	-7.83	4.25	90.01	85.50	-1.37	0.04	93.92	88.52	28.38	675
0.015	-9.51	2.20	89.66	84.58	-8.28	4.39	89.83	85.36	-1.37	0.04	93.99	88.56	28.42	646

Table 5.20: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.25, \beta = 1, n = 150$ AND DIFFERENT r BY METHOD 2

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-4.78	1.24	92.06	87.11	-2.09	1.82	92.73	87.89	-0.41	0.01	94.09	88.75	5.83	26.31	1158
0.01	-5.19	1.30	91.89	86.88	-2.53	1.92	92.50	87.62	-0.43	0.01	94.02	88.73	5.69	26.90	1165
0.015	-5.50	1.35	91.70	86.78	-2.90	2.01	92.32	87.42	-0.43	0.01	94.04	88.94	5.31	26.51	1151
r/n=80%															
0.005	-6.08	1.54	91.01	86.54	-3.70	2.59	92.08	87.31	-0.67	0.02	94.23	88.63	4.31	23.72	896
0.01	-6.42	1.60	90.90	86.36	-4.11	2.70	91.82	87.30	-0.68	0.02	94.27	88.58	4.23	24.77	908
0.015	-6.68	1.65	90.80	86.19	-4.43	2.81	91.63	87.01	-0.67	0.01	94.31	88.59	4.82	25.21	919
r/n=60%															
0.005	-8.87	2.09	89.87	84.86	-7.41	4.11	90.15	85.77	-1.37	0.04	93.84	88.47	6.01	22.73	665
0.01	-9.19	2.14	89.74	84.71	-7.83	4.25	90.01	85.50	-1.37	0.04	93.92	88.53	6.05	24.21	691
0.015	-9.45	2.20	89.66	84.60	-8.20	4.39	89.83	85.37	-1.36	0.04	93.99	88.54	6.13	24.84	698

Table 5.21: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.25, \beta = 1, n = 150$ AND DIFFERENT r BY METHOD 3

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-4.77	1.24	92.07	87.13	-2.08	1.82	92.70	87.89	-0.41	0.01	94.10	88.79	0	26.33	1043
0.01	-5.19	1.30	91.89	86.88	-2.53	1.92	92.50	87.62	-0.43	0.01	94.02	88.73	0	26.92	1055
0.015	-5.50	1.35	91.70	86.78	-2.90	2.01	92.32	87.42	-0.43	0.01	94.04	88.94	0	26.50	1047
r/n=80%															
0.005	-6.08	1.54	91.03	86.48	-3.70	2.59	92.06	87.30	-0.67	0.02	94.23	88.61	300	28.92	2628
0.01	-6.42	1.60	90.90	86.36	-4.10	2.70	91.83	87.31	-0.68	0.02	94.27	88.58	300	28.74	2619
0.015	-6.71	1.65	90.81	86.21	-4.47	2.80	91.64	87.03	-0.68	0.01	94.33	88.61	300	28.59	2645
r/n=60%															
0.005	-8.87	2.09	89.87	84.87	-7.41	4.11	90.15	85.78	-1.37	0.04	93.84	88.47	300	28.49	2450
0.01	-9.19	2.14	89.74	84.71	-7.84	4.25	90.01	85.50	-1.37	0.04	93.92	88.51	300	28.34	2462
0.015	-9.45	2.20	89.66	84.60	-8.20	4.39	89.83	85.37	-1.36	0.04	93.99	88.54	300	28.44	2614

Table 5.22: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.5, \beta = 1, n = 150$ AND DIFFERENT r BY METHOD 1

True η	$\hat{alpha}$				$\hat{\beta}$				$\hat{\eta}$					
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iterations	Time
r/n=100%														
0.005	-9.81	5.76	91.49	86.68	-2.54	8.37	91.77	87.23	-0.47	0.01	94.05	87.84	28.00	1032
0.01	-10.41	6.02	91.51	86.76	-3.02	8.73	91.73	87.06	-0.47	0.01	93.92	88.70	30.05	1065
0.015	-10.83	6.25	91.38	86.74	-3.38	9.09	91.64	86.97	-0.46	0.01	94.01	88.80	30.32	1070
r/n=80%														
0.005	-12.13	7.27	90.69	86.43	-4.45	11.97	90.87	86.65	-0.74	0.02	94.04	88.52	29.90	831
0.01	-12.63	7.53	90.57	86.37	-4.90	12.41	90.72	86.49	-0.73	0.02	94.15	88.40	30.75	848
0.015	-13.10	7.76	90.42	86.27	-5.36	12.86	90.60	86.38	-0.72	0.02	94.24	88.56	31.49	874
r/n=60%														
0.005	-17.55	10.17	89.15	84.60	-9.59	19.68	88.55	84.50	-1.48	0.04	93.71	88.27	29.62	645
0.01	-17.98	10.46	89.06	84.48	-9.98	20.35	88.35	84.41	-1.47	0.04	93.81	88.29	29.94	645
0.015	-18.30	10.74	88.91	84.41	-10.24	21.03	88.24	84.24	-1.44	0.04	93.91	88.35	29.97	645

Table 5.23: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.5, \beta = 1, n = 150$ AND DIFFERENT r BY METHOD 2

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-9.82	5.77	91.48	86.66	-2.54	8.37	91.79	87.18	-0.47	0.01	94.03	87.82	7.18	28.03	1054
0.01	-10.40	6.02	91.51	86.76	-3.02	8.73	91.73	87.06	-0.47	0.01	93.92	88.70	6.59	30.05	1087
0.015	-10.83	6.25	91.38	86.74	-3.38	9.09	91.64	86.97	-0.46	0.01	94.01	88.80	6.01	30.32	1091
r/n=80%															
0.005	-12.11	7.27	90.70	86.42	-4.43	11.97	90.88	86.65	-0.74	0.02	94.02	88.53	6.22	26.02	910
0.01	-12.60	7.53	90.56	86.39	-4.84	12.43	90.72	86.48	-0.73	0.02	94.14	88.38	4.54	27.59	856
0.015	-12.94	7.78	90.43	86.28	-5.12	12.90	90.60	86.38	-0.71	0.02	94.22	88.53	4.67	28.31	863
r/n=60%															
0.005	-17.55	10.17	89.18	84.62	-9.59	19.69	88.56	84.51	-1.48	0.04	93.71	88.26	6.05	23.78	593
0.01	-17.97	10.46	89.03	84.49	-9.96	20.35	88.35	84.41	-1.47	0.04	93.79	88.31	6.00	25.72	520
0.015	-18.29	10.74	88.93	84.41	-10.23	21.03	88.23	84.25	-1.44	0.04	93.91	88.37	6.00	27.00	635

Table 5.24: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.5, \beta = 1, n = 150$ AND DIFFERENT r BY METHOD 3

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-9.81	5.76	91.49	86.68	-2.54	8.37	91.77	87.23	-0.47	0.01	94.05	87.84	0	28.00	1174
0.01	-10.41	6.02	91.51	86.76	-3.02	8.73	91.73	87.06	-0.47	0.01	93.92	88.70	0	30.05	1212
0.015	-10.83	6.25	91.38	86.74	-3.38	9.09	91.64	86.97	-0.46	0.01	94.01	88.80	0	30.32	1201
r/n=80%															
0.005	-12.10	7.27	90.69	86.44	-4.41	11.96	90.87	86.65	-0.74	0.02	94.04	88.53	300	30.73	2915
0.01	-12.59	7.53	90.57	86.39	-4.84	12.43	90.72	86.48	-0.73	0.02	94.14	88.38	300	30.56	2922
0.015	-12.94	7.78	90.43	86.28	-5.13	12.90	90.60	86.38	-0.71	0.02	94.22	88.53	300	30.64	2868
r/n=60%															
0.005	-17.53	10.16	89.16	84.61	-9.57	19.68	88.55	84.50	-1.48	0.04	93.71	88.27	300	31.53	2700
0.01	-17.97	10.45	89.06	84.48	-9.97	20.35	88.36	84.41	-1.47	0.04	93.81	88.29	300	31.54	2669
0.015	-18.30	10.74	88.93	84.41	-10.24	21.03	88.24	84.25	-1.44	0.04	93.91	88.37	300	31.52	2678

Table 5.25: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.75, \beta = 1, n = 150$ AND DIFFERENT r BY METHOD 1

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$					
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iterations	Time
r/n=100%														
0.005	-15.56	16.73	90.44	86.03	-0.75	24.34	89.74	85.69	-0.58	0.01	93.76	89.40	27.42	1010
0.01	-15.61	17.31	90.62	86.51	-0.39	25.09	90.12	86.21	-0.55	0.01	93.79	88.62	30.81	1082
0.015	-15.58	17.91	90.63	86.53	0.00	25.98	90.13	86.25	-0.52	0.01	93.90	88.72	32.16	1192
r/n=80%														
0.005	-17.83	21.60	89.64	85.93	-0.25	35.39	88.94	85.00	-0.87	0.02	93.83	88.23	31.03	874
0.01	-17.87	22.34	89.62	85.88	0.17	36.64	88.89	84.94	-0.83	0.02	93.92	88.35	31.69	872
0.015	-17.74	23.09	89.64	85.92	0.76	38.07	88.89	84.90	-0.79	0.02	94.01	88.53	31.19	866
r/n=60%														
0.005	-24.91	32.20	87.58	83.27	-1.66	65.88	86.22	82.21	-1.71	0.05	93.48	87.89	31.02	653
0.01	-24.75	33.28	87.59	83.21	-0.77	69.32	86.20	82.22	-1.65	0.05	93.58	88.03	31.25	661
0.015	-24.46	34.42	87.59	83.20	0.30	73.41	86.22	82.14	-1.59	0.05	93.64	88.12	31.35	666

Table 5.26: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.75, \beta = 1, n = 150$ AND DIFFERENT r BY METHOD 2

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-15.54	16.72	90.48	86.07	-0.74	24.34	89.78	85.71	-0.58	0.01	93.76	89.38	7.84	27.41	1200
0.01	-15.61	17.31	90.62	86.51	-0.39	25.09	90.12	86.21	-0.55	0.01	93.79	88.62	7.13	30.81	1282
0.015	-15.61	17.90	90.63	86.53	-0.02	25.97	90.13	86.25	-0.52	0.01	93.90	88.72	6.44	32.12	1307
r/n=80%															
0.005	-17.92	21.58	89.59	85.92	-0.35	35.36	88.97	85.10	-0.87	0.02	93.84	88.25	7.06	26.40	965
0.01	-17.94	22.30	89.62	85.90	0.08	36.58	88.89	84.95	-0.83	0.02	93.94	88.41	6.24	29.10	1016
0.015	-17.72	23.09	89.64	85.92	0.78	38.07	88.89	84.90	-0.79	0.02	94.01	88.53	4.49	30.02	1028
r/n=60%															
0.005	-24.88	32.19	87.59	83.26	-1.62	65.87	86.24	82.20	-1.70	0.05	93.47	87.91	6.62	23.98	695
0.01	-24.74	33.28	87.58	83.21	-0.76	69.32	86.22	82.24	-1.65	0.05	93.58	88.09	6.20	26.09	727
0.015	-24.45	34.41	87.62	83.20	0.31	73.40	86.23	82.14	-1.59	0.05	93.65	88.12	6.01	28.21	757

Table 5.27: BIAS AND MSE OF MLES $\hat{\alpha}$, $\hat{\beta}$ AND $\hat{\eta}$ for $\alpha = 0.75, \beta = 1, n = 150$ AND DIFFERENT r BY METHOD 3

True η	$\hat{\alpha}$				$\hat{\beta}$				$\hat{\eta}$						
	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Bias	MSE	95%	90%	Iter Initial	Iter MLE	Time
r/n=100%															
0.005	-15.56	16.73	90.44	86.03	-0.75	24.34	89.74	85.69	-0.58	0.01	93.76	89.40	0	27.42	1136
0.01	-15.61	17.31	90.62	86.51	-0.39	25.09	90.12	86.21	-0.55	0.01	93.79	88.62	0	30.81	1212
0.015	-15.58	17.91	90.63	86.53	0.00	25.98	90.13	86.25	-0.52	0.01	93.90	88.72	0	32.16	1232
r/n=80%															
0.005	-17.78	21.59	89.64	85.93	-0.19	35.39	88.94	85.01	-0.86	0.02	93.83	88.23	300	32.36	2771
0.01	-17.85	22.33	89.63	85.91	0.19	36.64	88.89	84.95	-0.83	0.02	93.92	88.39	300	31.97	2880
0.015	-17.72	23.09	89.64	85.91	0.78	38.07	88.89	84.90	-0.79	0.02	94.01	88.53	300	32.00	2840
r/n=60%															
0.005	-24.89	32.20	87.59	83.28	-1.63	65.87	86.23	82.21	-1.70	0.05	93.49	87.92	300	32.88	2659
0.01	-24.74	33.27	87.59	83.21	-0.77	69.30	86.21	82.22	-1.65	0.05	93.58	88.04	300	32.82	2633
0.015	-24.47	34.41	87.61	83.20	0.28	73.40	86.23	82.14	-1.59	0.05	93.65	88.12	300	32.34	2625

Chapter 6

Illustrative Examples

In this Chapter, we provide three examples to illustrate the model and the inferential methods developed in the preceding chapters.

6.1 Lifetimes of small electric carts

Data on times (in months) to first failure of 20 small electric carts have been presented by Zimmer *et al.* (1998), and these are presented in Table 6.1. These small electric carts were used for internal transportation and delivery in a large manufacturing facility. This manufacturing company has 20 small electric carts for full service and we assume that theses electric carts form a load sharing system, meaning that, when an electric cart fails, the remaining carts have an increased shared load. Moreover, the small electric carts system shuts down if 10 carts fail. So, this can be considered as a 10-out-of-20 system.

Table 6.1: Times to first failure of electric carts

0.9	1.5	2.3	3.2	3.9	5.0	6.2	7.5	8.3	10.4
11.1	12.6	15.0	16.3	19.3	22.6	24.8	31.5	38.1	53

Under this setup, based on the data in Table 6.1, we determined the MLEs, standard errors and confidence intervals for α, β and η in Table 6.2. Here, the estimate of η is negative, which means that there is no stress added when failures occur in this data. Though this is not an example of a r-out-of-n system, we use it here as an illustration.

Table 6.2: MLEs and standard error of the parameters and corresponding 95% and 90% CIs in electric carts example

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\eta}$
MLE	0.815	4.481	-0.167
Standard Error	0.168	0.022	0.006
95% CI	(0.486,1.144)	(4.437,4.525)	(-0.179,-0.154)
90% CI	(0.539,1.091)	(4.444,4.518)	(-0.177,-0.156)

We then employed parametric bootstrap procedure using these MLEs' (based on 10000 bootstrap runs) to construct a quantile-quantile plot (QQ plot), which is a plot of the sample values versus the average values determined from the bootstrap runs. This QQ plot is presented in Figure 6.1:

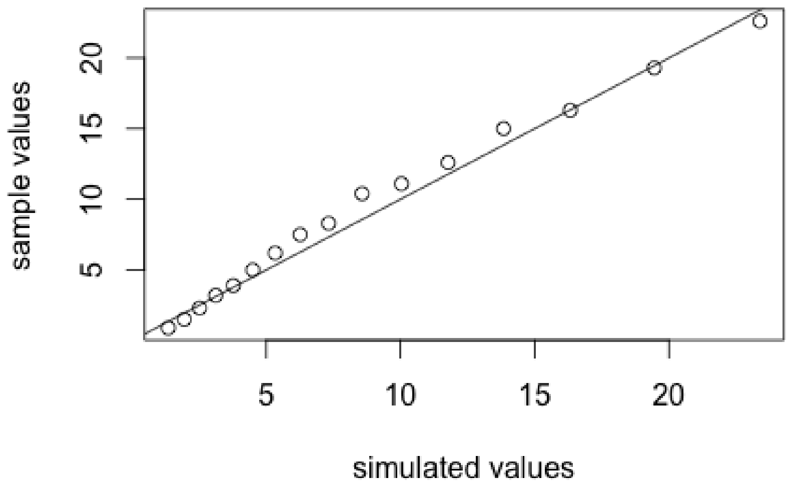

Figure 6.1: QQ plot of first failures of electric carts data

We observe from Figure 6.1 that all of the points lie quite close to a straight line to reveal that the BS r-out-of-n model is quite reasonable for these data. In fact, the correlation coefficient between the simulated values and the sample values is 0.994. The P-value, (values of correlation falling determined from 10000 simulations in [0,0.994]) turns out to be 0.983. This provides a strong evidence for the model proposed in Section 2.2 for these data.

6.2 Lifetimes of 101 Strips of Aluminum Coupon

Birnbaum and Saunders (1958) reported the lifetime of aluminum strips, and these are given in Table 6.3. In their study, periodic loading was applied to the strips with a frequency of 18 hertz (cycles per second). Also, a stress of 21,000 psi (pounds per square inch) was continuously applied to the strips. Suppose we assume all these strips to collectively form a system, and that it would function until 80 of the strips fail.

Table 6.3: Lifetimes of 101 Strips of Aluminum Coupon

370	706	716	746	785	797	844	855	858	886
930	960	988	990	1000	1010	1010	1016	1018	1020
1055	1085	1102	1102	1108	1115	1120	1134	1140	1199
1200	1200	1203	1222	1235	1238	1252	1258	1262	1269
1270	1290	1293	1200	1310	1313	1318	1330	1355	1390
1416	1419	1420	1420	1450	1452	1457	1458	1481	1485
1502	1505	1513	1522	1522	1530	1540	1560	1567	1578
1594	1602	1604	1608	1630	1642	1647	1730	1750	1750
1763	1768	1781	1782	1792	1820	1868	1881	1890	1893
1895	1910	1923	1940	1945	2023	2100	2130	2215	2268
2240									

We fitted the r-out-of-n system model based on the BS distribution for the MLEs, standard error and confidence interval in Table 6.4:

Table 6.4: MLEs and standard error of the parameters and corresponding 95% and 90% CIs in aluminum coupon example

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\eta}$
MLE	0.4285	1534.991	0.0100851
Standard Error	0.0377	0.0008	4.906×10^{-7}
95% CI	(0.3546,0.5024)	(1534.935,1535.047)	(0.0100841,0.0100861)
90% CI	(0.3664,0.4905)	(1534.990,1534.992)	(0.0100843,0.0100859)

Here again, we used these estimates to carry out a parametric bootstrap method (based on 10000 bootstrap runs) to construct a QQ plot, which is presented in Figure 6.2 below:

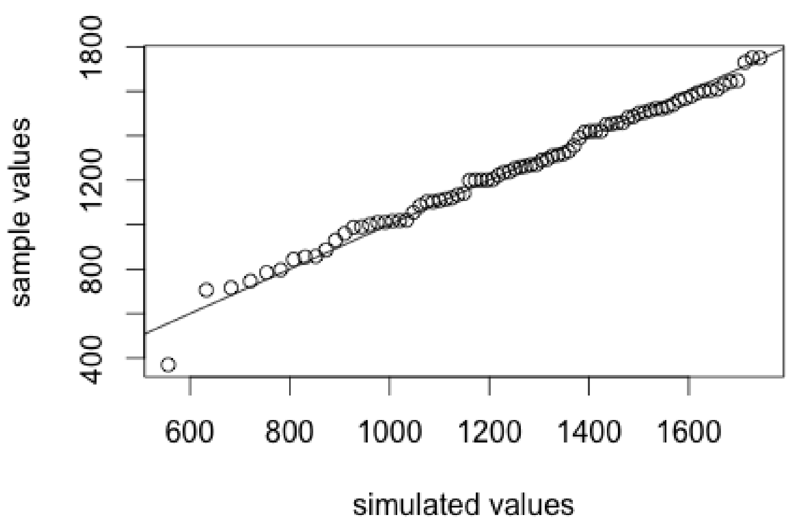

Figure 6.2: QQ plot of lifetimes of aluminum strips data

All the points lie quite close to a straight line, except for one point at the left end of the plot. The correlation coefficient is 0.994, and the corresponding P value determined from 10000 simulations is 0.957. This provides a strong evidence for the suitability of the model in Section 2.2 for these data.

6.3 Type-II SOS data

Type-II SOSs of size 80 were generated from r-out-of-n F-system having a baseline distribution function as BS(0.4,1) with η=0.03. For convenience, these data are presented in Table 6.5.

Table 6.5: TYPE-II SOS data of n=100, r=80, generated from BS(0.4,1) at η=0.03

0.3114176	0.3577712	0.3684494	0.3961078	0.4346105	0.4379220	0.4835562	0.5199460
0.5219281	0.5258875	0.5760557	0.5812905	0.5848958	0.5944515	0.5962252	0.6114193
0.6201812	0.6236542	0.6384021	0.6415310	0.6488707	0.6550551	0.6767554	0.6990025
0.7040638	0.7062574	0.7095626	0.7205109	0.7257680	0.7265536	0.7285592	0.7291695
0.7294989	0.7348153	0.7375594	0.7379760	0.7478291	0.7492575	0.7507252	0.7509777
0.7543550	0.7543629	0.7689378	0.7727302	0.7729789	0.7794213	0.7893797	0.7932303
0.7959455	0.7990022	0.8007871	0.8043308	0.8049593	0.8076435	0.8095390	0.8113607
0.8132980	0.8178583	0.8199980	0.8204087	0.8217829	0.8222121	0.8242837	0.8253971
0.8272018	0.8307985	0.8325746	0.8351007	0.8364394	0.8372908	0.8375984	0.8430772
0.8437732	0.8439731	0.8440389	0.8448930	0.8506242	0.8507807	0.8508000	0.8514420

Here we use the proposed method to perform simulations and determine the MLEs, standard error and confidence interval in Table 6.6.

Table 6.6: MLEs and standard error of the parameters and corresponding 95% and 90% CIs in TYPE-II SOS data

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\eta}$
MLE	0.669	1.389	0.048
Standard Error	0.068	0.004	0.0004
95% CI	(0.537,0.802)	(1.382,1.396)	(0.0467,0.0483)
90% CI	(0.558,0.780)	(1.383,1.395)	(0.0468,0.0482)

We carried out a parametric bootstrap method (based on 10000 bootstrap runs) to construct the QQ plot:

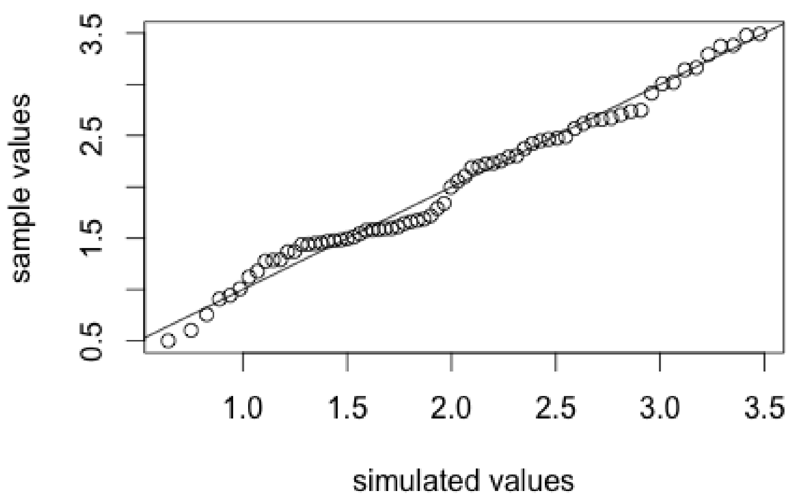

Figure 6.3: QQ plot of TYPE-II SOS data from Balakrishnan *et al.* (2015)

We observe a reasonable fit for this data. The correlation coefficient is 0.990 and the corresponding P value is 0.3882. This suggests that the model in Section 2.2 is very good to fit the data.

Chapter 7

Conclusions and Remarks

In this thesis, inference has been developed for r-out-of-n F-system with equal load-sharing based on BS lifetime distribution. The components of such a system fail sequentially, and the system failure occurs when the rth component fails. Failure of a component often induces a higher load on surviving components, and increases the hazard rate. In developing a model, we have assumed the load to be equally redistributed to all the surviving components, which is referred to as an equal load sharing system.

With the increasing load due to each failure, the surviving components are more likely to fail. By assuming the relationship between baseline hazard function and the hazard function upon each failure as $h_j(x) = \eta_j h_0(x)$ with positive η_j, we assumed a link function of the form $\eta_j = e^{j\eta}$. Thus, we get a model consisting of three parameters, with parameters α and β coming from the BS distribution and one additional parameter η coming from the link function.

To determine the MLEs of the model parameters using the Newton-Raphson method, we have discussed three different methods for providing initial values for

α, β and η. The pseudo complete sample method replaces the censored data with the r^{th} value, then finds the MMEs to provide them based on the complete sample as initial values. The Type-II right censored estimator is numerically obtained by equating the first derivative of the loglikelihood with respect to β, and then calculates the initial value of α as a pure function of $\tilde{\beta}$. Stochastic approximation method, based on the censored sample, uses pseudo complete sample method to obtain the first estimates, and then generates the remaining $n-r$ observations from the left-truncated BS distribution and finds new MLEs. This process is repeated many times, and then the mean of the generated MLEs are given as the initial values.

Based on the extensive Monte Carlo simulation study carried out, the pseudo complete sample method takes least time and least number of iterations to converge to MLEs. However, Type-II right censored method is better in a few cases for large values of α and large sample sizes. Thus, overall, we would recommend the use of the pseudo complete sample method for providing the initial values for the numerical iterative procedure to determine the MLEs.

Finally, though the BS distribution is a flexible lifetime distribution, there are some generalizations of it that can be used in this context. For example, it would be of great interest to extend the proposed method to the case of Generalized Birnbaum-Sauders distributions discussed by Sanhueza *et al.* (2008). This is a possible future research problem that one may consider.

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