PRICING STRATEGY WITH REFERENCE PRICES

PRICING STRATEGY WITH REFERENCE PRICES

By

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To my wife, Sheryl, who believed I could and helped me believe it too.

To my sons, Marcus and Zachary, who inspire me every day

Abstract

Price and inventory decisions are key levers of profit for firms. A manager needs to understand the impacts of pricing, ordering and stocking decisions not only on today's operations but also on future demand. In this dissertation we investigate these intertwining decisions by incorporating inter-temporal effects of pricing decisions through reference prices. We introduce three significant extensions to reference price models to provide more meaningful insight into pricing, inventory and ordering decisions.

We first present a threshold reference model. The threshold model incorporates zones of insensitivity around expected price that moderate the reference impacts on demand. This provides a rigourous model that is flexible enough to handle different pricing strategies such as single everyday low pricing (EDLP), high-low pricing (HiLo) and other general price cycles. We develop two solution approaches and provide computational results.

We next introduce a reference model with stochastic demand. There is considerable previous research supporting the consideration of variability in pricing and inventory decisions and this is especially true in the context of inter-temporal demand interactions based on pricing decisions. We find that the introduction of stochastic elements can actually increase or decrease the length of the price cycle for some consumers in a reference model depending on the parameters of the model. This extends the stochastic demand model and bridges to reference models for improved managerial insight.

The final model presented is the dynamic lot sizing model. When prices and production decisions or order quantities are determined simultaneously the interactions need to be considered to optimize profits. The reference model incorporates the inter-temporal price effects to provide a clearer picture of the optimal decision. The inclusion of reference effects does change the optimal decision.

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Chapter 1

Introduction

1.1 Motivation

The coordination of marketing and operations activities can be a significant challenge for firms. The choice of price, order size and inventory level are three key levers for profitability. Operations research has contributed significant insight for managers for making decisions on all three of these factors for firms. This includes planning over time, including stochastic demand and the dynamic lot sizing problem. A lack of consideration of costs or impacts of operations decisions in marketing or marketing decisions in operations can lead to decisions that may be optimal in one element of the supply chain but is less than optimal for the whole firm. This is more complicated when there are inter-temporal effects of decisions such as future demand impacts due to current pricing decision as well as inventory carrying costs or shortage costs, if inventories are insufficient. There is significant empirical evidence in the marketing literature that prices in the current period can affect demand in future periods. The consideration of inter-temporal demand effects of price within the context of pricing and inventory/ordering decisions has not received any significant attention in the literature.

Researchers have explicitly recognized the increasing complexity of managerial decisions in the supply chain (Melnyk et al. [2009], Melnyk et al. [2010]). Supply chain researchers have begun attempting to incorporate marketing concepts into operations models to allow for a more holistic modeling approach with richer insights on supply chain performance (Malhotra and Sharma [2002], Tang [2010]). It is, however, worth noting that the bulk of the pricing research has been largely focused on single period pricing decisions and that any multiple period models have not considered inter-temporal demand effects. Understanding the impact of inter-temporal demand impacts of pricing decisions and the interaction with inventory and ordering decisions could provide significant insight to managers.

Yano and Gilbert [2005] provide a recent review of joint pricing and inventory decisions. The research on multi-period problems has, for the most part, been focused on multi-period inventory effects given random demand rather than the inter-temporal impacts of pricing decisions. This gap in the research provides an opportunity to provide additional insight into this complex managerial problem.

One of the best approaches to incorporating inter-temporal effects of pricing decisions on demand is to incorporate reference pricing. Reference price is a concept that represents the price a consumer (or any purchaser) uses to compare observed prices in the market. The reference price is used to determine whether the observed price is a "good deal" or not. Reference price models have been widely studied in the marketing literature. However, it is only very recently that operations researchers have begun to incorporate reference price approaches into their models. The reference approach offers the potential for rich managerial insight. It is also worth noting that although it has not happened to date, Gimpl-Heersink et al. [2008] suggest that the incorporation of reference prices into simultaneously solved pricing and inventory models will yield valuable insight.

It is reasonable that a pricing action in this period would be expected to affect demand in this period and in the subsequent periods and reference price models use the mechanism of an expected price to incorporate that. Forecasting and inventory planning in a supply chain require a good understanding of current and future demand and the impacts of marketing and other activities on that demand. Reference price models can provide that inter-temporal linkage to improve inventory and marketing/pricing planning. Effective analysis of the linkages between marketing activities, including pricing actions, inventory and other supply chain factors can lead to a more refined and accurate insight into decision making. The more completely the linkages between marketing, demand and the other components of the supply chain are incorporated into analytical models, the richer the managerial insight will become.

While we frequently see price cycles in real retail settings, the results of the application of reference price models in operations contexts have not been flexible enough to evaluate if and when price cycles might be optimal. Previous work has evaluated the impact of price promotion (in this context referring to a weekly special or sale) without considering whether: a) cycles are optimal and b) which prices were optimal (Fader and Lodish [1990], Oliveira-Castro et al. [2005], Bridges et al. [2006], Srinivasan et al. [2008]). The work of Kopalle et al. [1996] found that for loss seeking consumers (those for whom response to lower than expected price is greater than for a higher than expected price) a price cycle is always optimal. They also show that for loss averse consumers (those for whom the response to a higher than expected price is greater than the response to a lower than expected price) a two or three period cycle is never better than a one period cycle. We show later that a one period cycle is always optimal for loss averse consumers. We then introduce a threshold model, which is based on empirical work done in marketing and provides a theoretically sound framework within which price cycles can exist and be modeled. It is also a more robust model as it does not require cyclical pricing but will allow it under specific conditions. It allows for price to cycle for the empirically validated loss-averse consumer as well.

The reference threshold models are evaluated in a monopolistic setting. It is worth noting that reference price models are most relevant for frequently purchased items. Greenleaf [1995] used peanut butter for his empirical analysis and the determination of optimal prices. There is research that suggests for frequently purchased items such as groceries, the first decision a consumer makes is the store and then will make within store decisions on specific items or brands (Rajiv et al. [2002], Gijsbrechts et al. [2008], Sands et al. [2009]). It is, therefore, reasonable to model the demand process in a monopolistic context. These can be extended into a oligopolistic context but these initial robust threshold models provide interesting and relevant insight into price paths for loss averse consumers. The threshold model has some appeal as it is sufficiently robust to offer either cyclical or every day low pricing (EDLP) strategies depending on the parameters of the demand function. Previously, reference price models (and most other models of demand) have imposed either cyclical or EDLP strategies depending on loss aversion or loss seeking behaviour on the part of consumers.

We then evaluate the impact of reference prices (for both loss averse and loss seeking customers) on optimal pricing decisions with stochastic demand. Previous work with reference prices has only considered deterministic demand models. There is a rich body of literature which considers inventory decisions when demand is random (e.g. see Petruzzi and Dada [1999] and Porteous [2002]). This includes analysis which considers price as exogenous and when price is a decision variable. It is reasonable to consider the impact of stochastic demand on pricing and inventory decisions in a reference price context. We highlight that while Gimpl-Heersink et al. [2008] suggest that simultaneous models of price and inventory decisions do not generally provide significant incremental benefits as compared to sequential models, the incorporation of reference prices would still require simultaneous solutions to determine accurate results. We develop both loss averse and loss seeking reference price models and explore the results analytically and computationally to determine the interaction between stochastic demand and inter-temporal reference effects.

The application of reference models has, to date, been limited to determining optimal price paths and the associated individual period demands. We also propose evaluating approaches to analyzing reference price models and the associated optimal pricing paths in the context of setup or ordering costs and inter-temporal holding costs for inventory. When prices and production or ordering quantities are determined simultaneously, the interactions need to be considered to optimize profits. Early work (Wagner and Whitin [1958]) provided dynamic lots sizing approaches for exogenous prices. Subsequently Thomas [1970] developed the problem with price as a decision variable. The ordering patterns and pricing decisions that come out of optimal solutions will clearly be impacted if there are inter-temporal effects of pricing decisions. It is worth exploring what the impact will be on optimal decisions given reference prices. This will provide valuable insight for managers. This goes well beyond any of the applications in the operations literature to date.

1.2 Contribution

The work in this dissertation represents a significant contribution in the area of pricing strategies as it is the first work, to our knowledge, to comprehensively build multi-period models with inter-temporal reference effects. We show that the consideration of inter-temporal effects is important as the relationship between past prices and current demand does affect the optimal decisions.

Chapter 3 provides the first comprehensive analysis of the reference price model. It builds on the work of Kopalle et al. [1996] and presents a proof that the optimal price strategy for the loss averse model is a single price. This is also the first model to explicitly incorporate non-negativity conditions within the demand model. This chapter also has an analytical solution for the two period loss seeking reference model. We know that a multi-period cycle will always be better than a single price strategy. While we would still have to solve the model using the dynamic programming recursion, this two period analytical result does provide some insight into the model and serves as the foundation for a later result in the model with stochastic demand.

The development of the threshold reference model in Chapter 4 is the first time that the threshold model has been brought into an optimization context to determine pricing strategy for frequently purchased items. It builds on the empirical foundation of the marketing literature and bridges to the pricing analysis from operations. This is the first multi-period model optimization model that has the flexibility to have optimal strategies that include both single price strategies and price cycles. It is theoretically sound and we provide computational results that include both single and multi-price strategies depending on the specific parameters of the model. We also present a number of propositions that reduce the search space for the dynamic programming model to increase the efficiency with which it can be solved. Given the advances in nonlinear integer programming approaches, we show that this model can be formulated and effectively solved as a mixed integer programming model. This approach has never been used before in the reference pricing literature, either with our without thresholds.

The work in Chapter 5 is the first to incorporate stochastic demand into reference price models. We develop two models, the first with loss averse consumers and the second with loss seeking consumers. We introduce a recursion to maximize expected profit in order to determine the optimal long term pricing strategy. We show that for a loss averse customer, incorporating reference pricing into a stochastic demand context yields the same optimal pricing decision as without reference pricing. In the loss seeking case, the incorporation of stochastic demand is likely to change the optimal pricing decision but the specific nature of the change on the optimal pricing decision depends on the specific parameters of demand and the distribution of the stochastic element of demand. We present computational results for both loss averse and loss seeking consumers.

Petruzzi and Dada [1999] highlight the problem of inventory in excess of the optimal stocking quantity. They highlight that previous research has used the assumption of costly disposal to simplify the pricing choice to the price which yields a stocking factor equal to starting inventory within the expected profit function. This clearly has implications in a reference pricing context. We introduce an approach to determining the optimal price in a period where starting inventory is greater than the optimal stocking factor. We first introduce a result to determine the optimal price without reference prices that does not require the assumption of costly disposal and then extend that result to the reference context.

We investigate the dynamic lot sizing problem in a reference pricing model. This has not been done previously. We provide a general solution approach and some basic results that narrow the search space for optimal solutions. We show that the risk averse model without setup and holding costs provides an upper bound on cycle length and develop lower bounds for price in each pricing period. We then provide an algorithm for solving the dynamic lot sizing problem and provide some computational results. The managerial insight for loss averse consumers is that optimal price cycles will likely be shorter and price increases smaller than had been previously thought without reference prices.

1.3 Overview

This thesis is broken into seven chapters. This first chapter provides an introduction and overview of the problems and the research conducted. We set the broad context for the research and the motivation and contribution of the work. Chapter 2 reviews the relevant literature. The first sections provide an overview of the literature on reference prices generally. The early research on reference prices focused primarily on individual consumers and is published largely in the marketing literature. We then discuss the emerging published research in the area of firm level models. This translates the existence of reference prices into models that allow for the development of strategic optimal pricing decisions to optimize profits given different types of consumers. The final portion of the reference price section introduces the literature relative to the concept of a threshold on reference prices. The threshold has been found in marketing literature but never, to our knowledge, been incorporated into firm level pricing models to provide insight into the optimal price path for firms facing this type of demand.

We also provide an overview of the literature relative to :

1. the incorporation of stochastic demand into price determination models, and

2. ordering decision with setup/order and holding costs.

A stochastic element for demand changes the problem to maximizing expected demand based on expected shortage and overage costs. It has been shown (see for example Petruzzi and Dada [1999]) that the optimal price with uncertain demand is lower than in the certain demand case. We outline this literature to set the foundation for the incorporation of uncertainty into reference price models.

The joint inventory and pricing decision given setup/order costs and holding costs is a difficult one that has received attention in the literature. The inter-temporal effects of reference prices add a new element to these problems. Once again, we set the foundation for this work in the final section of the literature review.

Chapter 3 introduces the basic reference model that will serve as the foundation for all of the subsequent work. The model presented is similar to that in some of the seminal work in this area (Greenleaf [1995], Kopalle et al. [1996]). We do present the model in more detail than previously published and explicitly acknowledge elements such as non-negativity constraints. We offer a proof of the optimality of a single price strategy (EDLP) in the loss averse reference model. We also present some analytical results for the price in a two period cycle for the loss seeking case.

The threshold model is introduced first in Chapter 4. A threshold implies a zone above or below the expected (reference) price within which there is no reference effect. The specific size of the threshold as well as the nature of demand response outside of that threshold, allow the model the flexibility to have optimal price paths that either cycle or have a single price. The model is built on a solid empirical foundation from a rich marketing literature (see for example Kalwani and Yim [1992], Moon and Voss [2009] and Krider and Han [2004]). We offer some basic results attributable to threshold models. We also offer a number of propositions that reduce the search space for optimal price paths in the dynamic programming formulation and some computational results based on that formulation. We then offer an approximation of reference price formation that allows us to formulate a mixed integer nonlinear programming model to determine optimal price paths. This performs very well providing accurate results efficiently.

The stochastic demand model follows the threshold model. In Chapter 5, we outline the basic stochastic demand model in the absence of reference prices and then extend it to both the loss averse and loss seeking reference demand cases. We present some computational results to demonstrate the impact of different demand parameters and distributions for the stochastic portion of demand. A loss averse additive reference demand model behaves the same as the regular demand does with the incorporation of stochasticity. The loss seeking case, however, will either have larger or smaller price spreads given stochasticity. The specific response depends on the parameters and the distribution of the stochastic element of demand.

The final contribution is the analysis of the reference model with setup/ordering costs and holding costs. This combines the optimal pricing decision with inventory decisions over time. We show how the problem can be solved in the context of stable demand and generally and present computational results in Chapter 6.

The thesis concludes with a summary and overview of the results as well as a discussion of potential extensions to the research.

Chapter 2

Literature review

2.1 Introduction

We provide a review of the relevant literature with respect to the specific areas of interest in reference price models as well as the foundation for the application of these models to different problems. The first section of this review provides an overview of the evolution of modelling in the marketing/operations interface to set the broad context within which we conduct our analysis. We then provide a discussion of previous reference price research, with a focus on price determination models. We then present a discussion on the research into thresholds on reference prices. This is largely grounded in the marketing literature and serves as the empirical foundation for the robust threshold reference pricing model that represents the first contribution of this research - a reference pricing model with the flexibility to either cycle or be constant depending on the parameters of the model. We then present a review of relevant literature in the area of joint pricing and inventory with stochastic demand and pricing and production models with setup and holding costs. Models in these areas will set the foundation for our contributions in Chapters 5 and 6.

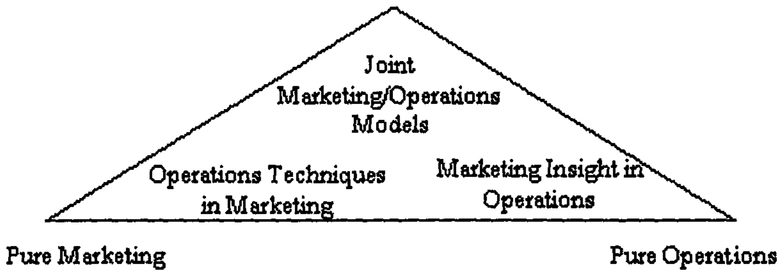
2.2 The Marketing-Operations Interface

There has been an evolution in the study of business problems. As early as 1977, authors such as Shapiro [1977] highlighted the often conflicting objectives between the marketing and operations functions within a firm. He suggests greater integration in decision making in order to leverage the relative strengths of the different components of the firm. This strategic impetus could as easily have been given to some researchers to bridge the gap and provide managers with analytical insight for the firm that includes elements of both operations and marketing strategy. Subsequent research (Pauwels [2004]) has also found that effectiveness in one area (in this case marketing) depends not only on marketing related factors but also on broader company characteristics. There is, therefore, an ongoing need for research that brings together elements of marketing and operations strategy and tactics.

Tang [2010] provides a review of marketing operations models and their evolution. He highlights that the integration between marketing and operations has moved from coexistence to increased collaboration and coordination. This is largely based on an increased understanding of the value proposition for consumers. The supply chain has become an integral part of the customer value proposition (Lilien et al. [2010], and Melnyk et al. [2009]). The work of academic researchers is paralleling the evolution of supply chain managers in practice, who seek to maximize profit across the whole enterprise rather than in individual departments and account for the fact that the activities of disparate departments can affect customer satisfaction and profit. Walters and Rainbird [2004] argue that supply chain models need to incorporate demand factors to effectively provide operations insight. They acknowledge that efficiency and cost reduction can be important but cannot be considered in the absence of an understanding of the consumer impacts.

Tang [2010] proposes a framework within which to consider marketing-operations interface research and provides a comprehensive review in several areas. Fundamentally the marketing function involves creating value for and extracting value from customers. This considers the customer wants and needs and frames an offer at a

Figure 2.1: Marketing/Operations Interface



specific price that they believe will best meet the needs of the customer. Operations is responsible for getting the product or service to the customer. He provides an overview of the different types of research that have been undertaken that link those marketing and operations functions explicitly.

In the context of the pricing research in this dissertation, it is worth a discussion of the integration of pricing and operations specifically. We would suggest there are three ways that marketing and operations research interact. Our proposed framework is presented in Figure 2.1. The framework builds from completely separate research streams of pure marketing and pure operations. There are joint marketing and operations models like the ones presented later in this dissertation (represented as the tip of the triangle in Figure 2.1). There are also studies that do not completely integrate marketing and operations into the models but rather either apply operations techniques to marketing problems or apply marketing insight into operations models (represented as the extremes of the triangle base in Figure 2.1).

Thomas [1970] extended the operations research approach of Wagner and Whitin [1958] which considered production decisions with price as exogenous, to include price as a decision variable. This work was published in 1970 and preceded much of the more recent discussion of marketing-operations interface. This is clearly a transition from purely marketing work to joint models to purely operations research and is well covered in previous reviews such as Tang [2010].

There is also a body of research that has applied optimization and other operations techniques to marketing problems. This research takes the behavioural results from marketing to develop decision insight for managers. The marketing results often provides perspective on consumers response when firms take a specific approach and then the operations tools can provide insight into which approach might be best suited to the company.

An example of purely marketing research would be the work of Tellis [1988a]. They look at consumer response in the face of loyalty programs and advertising exposure. It is an important insight but gives no perspective on the level of advertising to choose. They find advertising works more to keep existing customers (prevent switching) than to promote switching from other brands. Similarly, Bridges et al. [2006] looks at previous use and promotion level to evaluate consumer response. Gaur and Fisher [2005] estimate a demand curve based on an experimental exposure to different prices in a store context. There is a rich literature in pure marketing with respect to consumer behaviour or response and the work cited here is merely an introduction for context.

We now consider the next step in the continuum, the case where operations or optimization tools are applied in a marketing context. The work of Prinzie and Van den Poel [2006] applies Markov chain analysis in an effort to better anticipate the next financial service a particular customer is most likely to want. This takes an operations framework and applies it to a customer relationship management problem. Voina [2004] optimizes marketing mix using the theory of controlled stochastic processes and develops a solution algorithm. Similarly, Green and Krieger [1985] optimize the producer and consumer welfare in order to make product line choices. They also propose heuristics for solving the problem. Baltas [2004] exploits stochastic interdependencies to develop a model for multiple brand choice.

Esteban-Bravo et al. [2005] evaluate optimal promotion duration in the magazine industry. They explicitly model expected response based on duration and develop a model to provide insight for publishers to maximize return on promotional investment. Nguyen and Shi [2006] are able to incorporate both market share and market size dynamics into an optimization model for advertising strategy. They develop an empirical demand model to provide the parameters of the model.

In the pricing area, Turui and Dahana [2006] uses a Markov chain Monte Carlo method to evaluate producer price sensitivity and the existence of price thresholds. Greenleaf [1995] and Kopalle et al. [1996] both use optimization tools in reference price models to develop optimal pricing decisions.

We now consider some of the literature that brings marketing and operations considerations in an operations context. While this analysis is often the most complex, it provides more meaningful managerial insight as it helps provide strategic insight into decisions. This work is often built on the foundation of empirical marketing research to provide insight into consumer response. There are several different types of approaches and factors considered. Taaffe et al. [2008] examine a newsvendor problem with consideration of target market and marketing effort. It is a market allocation and marketing effort problem optimizing both ordering (operations) and marketing efforts. Bottani and Montanari [2010] explicitly model the interaction between supply chain parameters such as logistics costs and times and demand using simulation to assess optimal supply chain structure. Aydin and Porteus [2008] evaluate joint pricing and inventory decisions for multiple products with stochastic demand in a newsvendor model. They show that they can find unique optimal solutions despite some issues with the objective function. Petruzzi and Dada [1999] evaluate the joint pricing and stocking decision and summarize the previous work in this area. They consolidate the work relative to additive and multiplicative components of stochastic demand. They do spend some time talking about the multi-period model and that due to its complexity it has received very little attention in the literature.

There is an increasing understanding that building models that reflect both operations and marketing decisions and provide insight into optimal decisions can contribute to the literature and to managers. There is a building body of literature that considers pricing and inventory decisions, with both deterministic and stochastic demand. There has been little research in pricing and inventory in multi-period models and this is a gap we begin to address in Chapters 5 and 6.

2.3 Reference Prices

The concept of reference price has been widely studied in the literature (Winer [1986], Biswas and Sherrell [1993], Biswas et al. [1999], Shirai [2003], Shirai [2004], Alvarez and Casielles [2006], Chandrashekaran and Grewal [2006], Danziger and Segev [2006], Hu [2007], Campo and Yague [2007], Natter et al. [2007], Wolk and Spann [2008]). Mazumdar et al. [2005] provide a recent comprehensive review of reference price research and particularly highlight the value of understanding how prices would cycle in the face of loss-averse customers and recommend that models that consider intertemporal effects of pricing and promotional decisions continue to be developed. Ideally these models will explicitly incorporate the profit maximization problem. They suggest that this is a limitation of the marketing research approaches that have focused on individual consumers rather than at the firm level and highlight this as a significant opportunity for further research.

Kalyanaram and Winer [1995] propose three empirical generalizations from previous reference price research. The first is that reference prices are real. There is clear empirical evidence that consumers have an expected price and that the relationship between expected price and the observed price has an impact on purchase behaviour. The second generalization is that historical price plays a role in establishing the consumers' reference price. This is reflected in the commonly implemented reference formation specifications. Their final generalization is that consumers tend to be loss averse; that they respond more significantly to a higher than expected price than a lower than expected one.

The predominant formulation for reference demand models (Rajendran and Tellis [1994], Mazumdar and Papatla [2000], Erdem et al. [2001], Mazumdar et al.

[2005], Moon et al. [2006]) is the linear formulation. We use this formulation to introduce the basic model Demand is then:

$$D_t = a - bp_t + \beta(r_t - p_t), a, b, \beta > 0, t = 1, 2, ..., T.$$

where D_t is demand in period t;
a is the intercept;
b is a parameter on retail price representing the direct price effect;
p_t is the retail price in period t;
β is a parameter on the difference between observed price and the reference price; and
r_t is the reference price in period t.

Demand in period t, D_t, is a function of price, p_t and customer reference price, r_t. The parameter β models the impact of the transactional utility. That is, it models the impact on demand of observing a higher or lower than expected price at the time of purchase. A constant β means that consumers respond similarly when price is higher or lower than expected. Demand is decreasing in price and increasing in reference price. We assume that the demand parameters (a, b, β) are not time dependent. Reference models apply best to frequently purchased items such as groceries. Pricing periods at grocery stores with printed flyers are usually pricing weekly. It is reasonable to assume that the demand parameters for many grocery items such as milk or peanut butter would be stable from week to week.

We generally expect that the direct price impacts are greater than the reference impacts but this need not be the case. The results summarized in Kalyanaram and Winer [1995] suggest that consumers are loss averse so we need to specify that:

$$\beta = \begin{cases} \beta_G \text{ if } p \leq \text{r} \\ \beta_L \text{ if } p > r \end{cases} \quad \text{with } \beta_G < \beta_L$$

Winer [1986] uses the most simple of historical price relationship for reference price, $r_t = p_{t-1}$. He cites abundant early reference price research which found empirically that a lagged single price was a good indicator for reference price and performed as well as other more sophisticated measures. The bulk of reference price studies have used an adaptive expectations model based on the seminal work in Nerlove [1958]. This uses an exponential smoothing approach to reflect both historic prices and the most recent pricing experience.

$$r_t = \alpha r_{t-1} + (1 - \alpha)p_{t-1} : \ 0 \leq \alpha \leq 1, t = 2, 3, ..., \infty.$$

r_1 is assumed to be known. In practice, an estimate of the starting reference price would be available from a variety of potential sources. Market research may indicate a "willingness to pay" value for new products. For an existing product with a change in demand parameters or cost, consumers would have experience with the historical price. In our models, the value of the starting reference price is not critical as we allow for a period of adjustment before we determine the steady state price.

The parameter, α , reflects the degree to which the previous reference price is "remembered" in forming a new reference price based on the most recent price. Mazumdar et al. [2005] find strong support in the literature for this exponential smoothing specification. They find that the α parameter is generally between 0.2 and 0.35.

2.3.1 Reference Prices in Firm Level Pricing Models

There has been considerably less work with reference price models at the firm level to determine pricing strategy. Popescu and Wu [2007] suggest that "with few exceptions, the dynamic pricing literature is oblivious of such behavioural aspects underlying demand." Most studies have assumed that demand is exogenous and independent of past pricing decisions. It would seem that, despite an obvious increase in the complexity of the models, there is an opportunity for more realistic and robust insight from models that build in this dynamic demand interaction. It is only very recently that researchers have begun to incorporate reference price approaches into pricing models of the supply chain in order to gain insight into optimal pricing strategies (Greenleaf [1995], Kopalle et al. [1996], Miao-Shen and Chuan-Biau [1999], Fibich et al. [2003], Anderson et al. [2005], Fibich et al. [2007], Natter et al. [2007], Popescu and Wu [2007], Urban [2008]).

The initial work by Greenleaf [1995] was an attempt to get firm level pricing insight using a reference price model. He empirically estimates demand for peanut butter and looks recursively at multi-period promotion strategies and finds that it is optimal to have a cyclical approach with a phase during which the retailer promotes and a subsequent phase during which the retailer does not promote. Greenleaf's data suggests that consumers are loss seeking which is at odds with the earlier findings that consumers tend to be loss averse and so respond more to loss than to gain. He suggests this may be due to problems with aggregation and consumer heterogeneity. Finally, Greenleaf also evaluates the potential for response to a distributor deal announced in advance. The impact of a lower cost increases deal frequency during the time that the retailer has lower cost inventory. In the end, different factors probably contribute to the irregularity in retailer promotions.

Kopalle et al. [1996] also attempt to model the impact of promotions on profits. In the case where the impact of gains exceeds that of losses, a cyclical promotional strategy is optimal which is consistent with the results of Greenleaf [1995]. The authors prove and also show numerically that the optimal pricing strategy for loss-seeking consumers (those for whom the impact of a lower than expected price is larger than that of a higher than expected price) cycles. While it is appealing to have found a model that seems to support the HiLo pricing that we see in practice, it is built on the assumption that consumers are loss-seeking which has little empirical support. Regardless, we are left wanting for a model that provides a framework for assessing pricing strategies, both cyclical and monotonic which is consistent with loss-averse behaviour for consumers.

Fibich et al. [2003] use a continuous time model. They argue that specifying a continuous time profit function and solving for optimality conditions provides very elegant explicit solutions that are easier to use than the computationally difficult dynamic approaches that have been used in the past. Once again, however, we are left without an explanation of how and when a cyclical pricing strategy might be optimal.

The work of Natter et al. [2007] is particularly interesting. They apply a pricing model which includes reference prices to a European home renovation/do-it-yourself retailer. They develop pricing strategies for the company using the specified model and implement them with dramatic improvements in profitability. They do not report specific model parameters so it is not clear whether consumers are loss averse or loss seeking. They also do not comment on the optimal price paths - whether there is cyclicality or single prices. The models include cross price effects separately from direct demand effects in an effort to maintain computational tractability. It seems to provide good results if somewhat theoretically suspect. It is also worth noting that the demand function includes both reference impacts and discounts which seems to double count. While they do not explicitly address this, they do empirically estimate the demand function and find a good fit with good profit results. It might be argued that the discounts reflect an external reference price effect while the other reference effects reflect an internal reference price.

It is also worth highlighting the results of Gimpl-Heersink et al. [2008]. They evaluate the value of frameworks in which we make joint inventory and pricing decisions. They find that the complexity of the analysis in a simultaneous decision does not often yield significant incremental profits relative to a sequential decision approach. They note, however, that when we incorporate reference prices the benefits that accrue to simultaneous decision analysis is substantial. The incorporation of reference prices increases the impetus to making these decisions jointly which reinforces the approach taken in this research.

2.3.2 Thresholds on Reference Price

The concept of a threshold on reference impacts has been found in a number of studies. Kalwani and Yim [1992] explored reference price formation and found strong evidence of a band of insensitivity around price expectations as well as strong evidence of loss-averse consumers.

There are a number of other papers that have similar findings (Lichtenstein et al. [1988], Gupta and Cooper [1992], Kalyanaram and Little [1994], Janiszewski and Lichtenstein [1999], Krider and Han [2004], Lewis [2004], Campo and Yague [2007], Fibich et al. [2007], Pauwels et al. [2007], Thomas and Menon [2007], Moon and Voss [2009], Marshall and Na [2000]). All of these studies find a range around the reference price within which there is no reference effect.

Turui and Dahana [2006] suggest that we need to consider thresholds on reference price to accurately reflect the response to higher or lower than expected prices. They empirically estimate models with thresholds and find that the reference effects are more pronounced and loss aversions stronger. There are also studies which suggest that the threshold may differ under different circumstances. Pauwels et al. [2007] find explicit evidence of thresholds and that the size of the thresholds may differ between brands and categories. Notably, they find that thresholds may be asymmetric around reference price and that there may be differences in elasticities for gains and losses (asymmetric gain and loss effects). Janiszewski and Jr. [2004] find empirical evidence of different reference thresholds by product. They show that for complements that might form a bundle, the product with the smaller threshold is better to discount to increase the profitability of the bundle. The work of Moon and Voss [2009] finds empirical support for a price range model in explaining the purchase behaviour of consumers of toilet tissue. They find differences between brands which are reflected in the ranges. The authors also find that there is difference in promotional sensitivity (the size of the threshold) depending on how susceptible consumers are to external reference prices. Janiszewski and Lichtenstein [1999] finds that the perception of price in the context of reference price depends on the end points in the range which may also reflect a range around reference price that is "reasonable."

Research by Berkowitz and Walton [1980] found contextual factors contributing to reference price effects. They evaluated the perception of price based on different parameters. They found that store reputation has an impact on price perception. One interesting finding was that discounts at discount stores were less attractive than others. This may suggest that thresholds are bigger for some stores than for others.

Biswas and Sherrell [1993] found that consumers who considered themselves to be well informed about a product were more confident in their price expectations. They suggest that, depending on the product or brand, there are varying degrees of confidence around an expected price. One might interpret this as suggesting that consumers with a lower knowledge of a product may have an expected price range (or threshold) rather than a specific expected price point. Both Shirai [2004] and Thomas and Menon [2007] found evidence of uncertainty relative to reference prices which could contribute to thresholds around reference price. Thomas and Menon [2007] found that the confidence with respect to reference price increased the internal reference price and increased the sensitivity to price changes.

Reference price formation was evaluated by Niedrich et al. [2001] who found evidence to support within category comparisons of price in the formation of reference price. The work finds that the high and low prices within a category play a role in reference price formation. This might establish a type of range for reference price which may be analogous to a threshold.

Briesch et al. [1997] evaluated a number of different reference price approaches. While this is one of very few empirical studies that have found loss seeking behaviour, there are some interesting results. Their specification included a loyalty parameter based on previous purchases of a brand. They found that previous purchase was a significant factor in determining whether a brand was purchased in a subsequent period. The work of Lewis [2004] also considered loyalty and he found that loyalty programs work, that is they increase the amount of a specific brand that consumers will purchase during a year. This suggests that loyal consumers are less susceptible to price changes within a certain range. Within a reference context, then, this would suggest that brands with higher market share would have a threshold on loss if price is higher than expected. Lattin and Bucklin [1989] also found evidence of a loyalty effect in a reference price model. Tellis [1988a] finds that advertising increases preference for current brands (perhaps increasing a threshold) while price and promotion seems to have more impact on quantity purchased.

While the work of Chen et al. [1998] was not related to reference price, they did find that price promotions needed to be larger for high priced products than for lower priced products. They hypothesize that the investment required for a higher priced product is such that a bigger discount is required to incent someone to buy more or buy at all. This could perhaps be interpreted as higher price products having a higher threshold before which no reference impact will occur. Kim and Kramer [2006] find that there is uncertainty with respect to the value of discounts (i.e. 30% off) and the size of a deal which might also contribute to moderating the effects of smaller discounts.

There appears to be strong empirical support for the concept of a threshold around reference price. We address the optimal pricing decisions based on this empirical foundation in Chapter 4.

2.4 Models with Stochastic Demand and Inventory Considerations

There is a wealth of research which considers decisions facing firms with respect to inventory when they face stochastic demand. Porteous [2002] provides a good overview of the problem in both a static and multi-period case. The introduction of the pricing decision for firms facing uncertainty came from the work of Whitin [1955] and subsequently Mills [1959]. The work of Zabel (Zabel [1970], Zabel [1972]) looks at joint pricing and inventory decisions for firms facing uncertainty. Zabel

[1970] considers the monopolist single period case and follows that with Zabel [1972] and the monopolist multi-period case. He characterizes the optimal solutions in finite horizons. This work laid a broad foundation for a significant body of future work. Thowsen [1975] also considers the multi-period problem and he develops sufficient conditions, relative to the specific distributions of the stochastic component of demand, for optimal solutions to exist.

An integration of much of the previous work is introduced by Petruzzi and Dada [1999]. They consider the single period problem (newsvendor) and integrate both additive and multiplicative demand to generalize results. They discuss the characteristics of the optimal solutions for both additive and multiplicative demand and outline approaches to the determination of both optimal stocking factors and prices. They subsequently provide a framework which incorporates either or both demand specifications. It is of particular interest to this research that the authors discuss the multi-period problem generally and the characteristics of the optimal solution under certain conditions.

The work of Gallego and van Ryzin (Gallego and van Ryzin [1994], Gallego and van Ryzin [1997]) considers the problem of pricing a fixed inventory over a finite horizon when the firm faces stochastic demand. They consider both the case of one product and the one for multiple products. They find asymptotically optimal solutions using characteristics of the deterministic problem. They also consider different characteristics within the problem including such things as time varying demand, holding costs and the possibility to reorder among others. Chung et al. [2009] look at a newsvendor problem and allow for an in-season price change. The opportunity to adjust price decreases the uncertainty and increases expected profit. They also show that the initial inventory level is higher. Aydin and Porteus [2008] also consider multiple products and allow for price based substitution but not stockout based substitution. While they acknowledge that the objective is not joint quasi-concave in price, they find that their model as specified can provide a unique optimal solution based on first order conditions.

Chan et al. [2006] evaluates the joint production and pricing problem for a firm facing stochastic demand and with a discrete portfolio of price choices. They have no backlogging (unmet demand is lost) and assume limited production capacity. They assess different policies (delayed pricing and/or delayed production) and evaluate the better approach. They also consider the special case where a single price is chosen for the entire period and discuss solution approaches.

Federgruen and Heching [1999] uses a Markov process model to determine the optimal pricing decision for an infinite or finite horizon problem. Each single period problem is solved optimally with no connection between periods unless inventory is carried forward. They find that a base stock list price policy is optimal if inventory is below the base stock. If inventory is above the base stock they propose discounting to a level that the expected profit is maximized given the stocking level represented by the inventory. The opportunity to promote is included in the model presented by Zhang et al. [2008]. They consider a single product finite horizon problem. Promotion is used at specific thresholds to increase demand in production periods to maximize profit. They also characterize an optimal pricing policy (with some assumptions on the nature of demand). They include setup and holding costs.

Wilhelm and Xu [2002] evaluate pricing and production but add the opportunity to introduce product upgrades. In order to keep the problem tractable they make a substantial number of simplifying assumptions. They evaluate the decisions in a dynamic programming model to develop the decision support tool as to when to upgrade based on previous uncertain outcomes. They limit the production and price choices to specific discrete levels and probabilistic demand levels. This is similar to the work of Damodaran and Wilhelm [2004] who consider a broader range of options such as product design (similar to the upgrades above), pricing, production quantity and marketing activities. They introduce a branch and price algorithm for solving the problem and find it performs quite well and better than most commercial software.

There has not been much work relative to the pricing and inventory decisions in a competitive context. Pan et al. [2009] considers the pricing decision of a dominant retailer (one with market power to set prices, such as Walmart). They evaluate a two period case in which they assume the product will have a declining price. They argue this is the case for many technology products that have a relatively short lifecycle and are subsequently replaced by new technology. They characterize a unique optimal price.

Urban [2008] does consider reference price effects in a single period newsvendor model. This study is limited in that it does not consider the long term effects of a low price on reference price. It does, however, provide a first evaluation of the stocking requirements for a reference model.

There is clear interest in the literature in exploring the approaches to making joint pricing and inventory decisions for firms facing stochastic demand. There has not been any significant exploration of the problem with respect to reference prices.

2.5 Models with Set-up and Holding Costs

There has, to date, been no research on the dynamic lot sizing problem with reference prices. The basic dynamic lot sizing problem was introduced by Wagner and Whitin [1958]. They introduce the dynamic lot sizing problem with known price and, therefore, demand. The problem generalizes the economic order quantity by allowing the demand for product to vary over time. Setup/ordering costs and holding costs are given and then optimal order quantities over time are determined over time. Wagner and Whitin [1958] present a dynamic programing algorithm for solving the problem. The work of Williams [1975] develops a dominance relation that includes setup cost and reduces the solution space for the problem.

There has been substantial work undertaken to develop heuristics that can solve the problem efficiently. The Silver Meal heuristic (Silver and Meal [1973]) is a well known one which performs reasonably well in many cases but can result in arbitrarily large errors in some cases. Axsater [2006] also outlines a simple heuristic which often performs well (and is easier than the Silver Meal) but can also result in very large errors. If the planning horizon is relatively short, it is easy to get optimal solutions using the Wagner Whithin algorithm and even for longer horizon problems the optimal solution may not be computationally impractical. Often, though, the heuristics provide easily computed and relatively good solutions.

Thomas [1970] builds on the basic results of Wagner and Whitin [1958] to develop a model in which price is also a choice variable. Thomas [1970] presents four key lemmas for the dynamic lot sizing problem with price as an endogenous variable. They are worth noting in this context because at least three of them will continue to hold for the reference pricing problem. The four key results are:

1. An optimal program exists in which each period has either production or a starting inventory equal to zero but not both: $I_t x_t = 0$. Where I_t is the starting inventory in period t and x_t is a binary variable that equals one if there is production in period t and zero otherwise.

2. Analogous to the first Lemma, an optimal program exists in which the starting inventory is either zero (in which case production occurs) or is exactly equal to $\sum_{t=1}^{k} D_t(p_t)$ for some $k, t < k \leq T$. This means that production covers either all or none of a subsequent period's demand.

3. An optimal program exists such that if demand in period t is met by production in period $t^*, t^* < t$, then all demand in the intervening periods is also met by production in period t^*.

4. In the optimal program where for some period $I_t = 0$ for some period t, it is optimal to consider periods 1 through $t-1$ alone.

Thomas [1970] also provides results for lower and upper bounds for the problem to simplify the computations. This work set the foundation for a body of subsequent dynamic lot sizing research. This is not intended to be a comprehensive review of the dynamic lot sizing literature. Rather we present some of the extension of Thomas

[1970] in order to establish the general context of our extension with the inclusion of reference prices.

This continues to be a problem that garners research attention. Geunes et al. [2006] evaluates the joint retail pricing and manufacturing quantity decision in an uncapacitated context and present a polynomial time solution approach. Geunes et al. [2009] builds on this work in a production capacity context and provide another polynomial time solution approach. Capacity constraints, either constant or time varying, are incorporated in Deng and Yano [2006]. It is worth noting that they find that optimal prices may actually increase with capacity. Special cases of the capacitated production problem are also evaluated in Chan et al. [2006]. Their problem evaluates the planning process with stochastic demand and a discrete price vector. Incorporating stochastic demand gives the opportunity to take two approaches. Firms can make the pricing and production decisions at the beginning of the planning horizon or adjust the decisions later in the horizon as past demand is known and inventories may change.

The work of Bhattacharjee and Ramesh [2000] introduces another special case of the dynamic lot sizing problem with price as a choice variable. They consider the case where the product has a fixed life (such as a perishable grocery product). This shortens the feasible time horizon for any production/order cycle. They present efficient search heuristics for solving the problem within reasonable parameters viably.

Zhang et al. [2008] include the opportunity to promote products in production periods. They consider a single product finite horizon problem. Promotion is used at specific thresholds to increase demand in production periods to maximize profit. They also characterize an optimal pricing policy (with some assumptions on the nature of demand). They include setup and holding costs.

Merzifonluoglu and Geunes [2006] consider the case where there is delivery flexibility and no capacity limitation. While price is not directly a decision variable in this case, they have the secondary decision of delivery timing and the associated profitability affects. The problem NP-hard and they evaluate a number of heuristics for solving it. They also present an optimal solution approach given certain specific cost functions.

The problem becomes more complex in a supply chain context as is seen in Zhao and Wang [2002]. In this paper, the authors consider a decentralized two stage supply chain in which both a manufacturer and retailer make optimal pricing and production/ordering decisions. They develop a manufacturer's pricing schedule that induces the retailer to behave in a way consistent with a centralized supply chain which optimizes supply chain returns.

In Martel and Gascon [1998], a model is proposed with price changes and price dependent holding cost. This is essentially closer to Wagner and Whitin [1958] in that price changes are given but there are price changes that can affect the production/ordering decisions. They propose a number of heuristics to solve special cases of the problem such as a single price change in the planning horizon and either a permanent price increase or decrease. They note that intuitive solutions yield near optimal decisions. This includes decisions such as stocking up when price is low or a price increase is imminent or holding off ordering when a price decrease is expected.

Once again, there is clear interest in the literature in expanding the scope and sophistication of problems in the dynamic lot sizing model with endogenous pricing decision. This problem, while broadly studied, has not to date, incorporated reference prices.

Chapter 3

The Basic Reference Model

3.1 Introduction

This chapter lays the foundation for the extensions presented in subsequent chapters. We begin with the specification of a simple loss averse reference demand model. This is similar to that specified in both Greenleaf [1995] and Kopalle et al. [1996]. We generally assume that consumers are loss averse based on the overwhelming empirical evidence in the literature (Kalyanaram and Winer [1995], Mazumdar et al. [2005]). In the interest of completeness we also present results for loss seeking consumers (see for example: Briesch et al. [1997]). The reference price model incorporates consumer expectations of purchase price as well as a direct price effect. This results in inter-temporal price effects because the expected price is a function of past prices. Determinations of optimal price paths, then, must consider the impact of the current period pricing decision on future profits. A myopic single period decision making framework (see for example is only meaningful if the firm is no longer planning to participate in the market for the product in question. Our objective is to characterize optimal pricing decisions for current periods with consideration of the implications of those decisions in the long term. We explore the long term stable cycles in order to understand the optimal decisions for current periods.

This chapter first outlines the structure of the reference model, for both loss averse and loss seeking consumers, and presents some basic results that arise from the model. These results will serve as the benchmark against which the model extensions will be evaluated in subsequent chapters.

3.2 The Reference Model

The basic demand model is:

$$
\begin{aligned}
D_t &= \begin{cases} a - bp_t + \beta_G(r_t - p_t);\ \text{if } r_t > p_t \\ a - bp_t + \beta_L(r_t - p_t);\ \text{if } r_t \le p_t \end{cases} \quad t = 1, 2, \ldots\infty \\
c &\le p_t \le \frac{a + \beta_G r_t}{b + \beta_G};\ \text{ if } r_t \le \frac{a}{b};\ \text{else } c \le p_t \le \frac{a + \beta_L r_t}{b + \beta_L}
\end{aligned}
$$

We reiterate that we use the linear model as it is the predominant one in the literature. The upper bound on p_t simply ensures that demand is not negative. It is worth noting here that prior studies (for example Greenleaf [1995], Kopalle et al. [2009] and Fibich et al. [2007]) did not include such conditions. If these are not handled explicitly in a solution procedure, unrealistic prices which lead to negative demand can result. It is reasonable to assume that consumers have a realistic expectation of what a product will cost. Consumers are exposed to prices for similar products or previous models of the same product. It is, therefore, reasonable to assume that that the initial reference price is bound by $r_1 \le \frac{a}{b}$, where $\frac{a}{b}$ is the price at which the non-reference portion of demand, $a - bp_t$, vanishes. In this case, $r_t = \alpha r_{t-1} + (1 - \alpha)p_{t-1} \le \frac{a}{b}$ since $\left(\frac{a+\beta_L r_t}{b+\beta_L}\right) \to \frac{a}{b}$ as $r_t \to \frac{a}{b}$, so we need only have one constraint on price and that is that $p_t \le \frac{a+\beta_L r_t}{b+\beta_L}$ which simplifies the constraints above. This simplified upper bound narrows the feasible range (and reduces the search space). There is some intuition for this. Since initial reference price, r_1, is below $\frac{a}{b}$, any price above $\frac{a+\beta_L r_t}{b+\beta_L}$ will yield a negative demand due to the reference loss. We note that the demand function is identical whether consumers are loss averse or loss seeking. When consumers are loss averse we use $\beta_G < \beta_L$ and when they are loss seeking we use $\beta_G \geq \beta_L$.

The constraint that price not be less than cost merits a discussion. It is a common constraint in defining the parameters of demand. The proof of that in the single period case is trivial. We must consider, however, the case where prices cycle (as it has been shown in the loss seeking case (Kopalle et al. [1996])). The benefit of this constraint in this context is that it reduces the search space for optimal multi-period problems because we do not need in consider prices lower than cost for any period. We offer a Proposition in section 3.4 which suggests that in a two or a three period cycle a price lower than cost is never optimal and then submit a conjecture that this is true regardless of cycle length. We show in section 3.3 that in the loss averse case a single price is optimal and the $c \leq p_t$ constraint is valid again.

A brief discussion on a demand function aggregated across consumers is also warranted.

Based on this demand function we specify a profit function:

$$\pi_t(r_t, p_t) = \sum_{t=1}^{T}(p_t - c)D_t(r_t, p_t)$$

where

c is a unit cost that is assumed to be constant,

D_t is the demand as specified above,

$c \leq p_t \leq \frac{a+\beta_L r_t}{b+\beta_L}$,

$r_1 \leq \frac{a}{b}$.and

T is the planning horizon.

We are interested in the optimal pricing strategy over time and formulate a forward recursion:

$$f_t(r_t) = \max_{p_t \in J_t}\left[\theta^{t-1}\pi_t(r_t, p_t) + f_{t-1}(r_{t-1})\right] \tag{3.2.1}$$

where

$t = 2, 3, .., T$;

$r_t = \alpha r_{t-1} + (1-\alpha)p_{t-1}, t = 2, \ldots, T$, and r_1 is given;

$f_1 = 0$;

J_t is a vector of discrete prices which goes from c to $\frac{a+\beta_L r_t}{b+\beta_L}$ in increment δ;

p_t is the discrete price in period t from vector J_t;

π_t is the profit function as specified above; and

θ is a discount factor, $0 < \theta \leq 1$.

The length of the horizon, T, can affect the results as it reflects the time in which current pricing decisions can continue to have impact. In order to determine the optimum level of pricing in a steady state computationally, we need to specify a horizon of sufficient length to allow results to stabilize and cycles to repeat. Conceptually, the time horizon can be infinite ($T = \infty$), but from a practical computational perspective in the recursion, the time horizon must be finite. We note that the vector of discrete prices increases in increments δ. One would normally expect that this would be increments of one cent (or the smallest denomination of the relevant currency). There is research to suggest that specific price points are more likely to entice a consumer purchase. Notably, prices ending in nine are often considered to be better choices (Bray and Harris [2006], Jianping and Kanetkar [2006], Baumgartner and Steiner [2007], Nguyen et al. [2007], Schindler [2009]). The vector, J_t, can consider any series of prices that a firm wants to consider and need not increase in specific increments or in the lowest possible increments. The subsequent discussion is valid regardless of the specific structure of the price vector. Our specification allows for any vector.

We use a forward recursion rather than the more commonly used backward recurion. In a backward recursion $f_t(r_t)$ is interpretted as the "cost to go." That is it represents the optimal path from the end point to that point in the recurion. Similarly in a forward recursion we interpret $f_t(r_t)$ as the optimal path from the start point to that point in the recursion. We choose a forward recursion in the interest of computational efficiency. This approach has been used before to improve computational efficiency (see for example Psaraftis [1983]). In our recursion, the reference price is the state variable. Reference price is a function of previous prices charged. By working with a forward recursion, we need only consider the states which result from prices charged since the beginning reference price. Using a backward recursion would significantly increase complexity as we would consider some states that are not practical because they cannot be reached given the start point - the initial reference price.

The discount factor also merits a mention. It is common to discount flows in future periods Our specification allows us to do that. We are most interested in developing pricing strategies that reflect the inter-temporal effects. We are interested in identifying repeating optimal price cycles that allow us to leverage reference effects to optimize average profits over time. In that case, we specify a discount factor of 1 because we are looking at a sequence of equally weighted periods to develop a repeating price strategy that optimizes average profits over time and achieves higher per period average profits then a myopic constant pricing strategy. While many of our Propositions are not restricted to the average profit criterion (i.e. $0 < \theta \leq 1$), we will not use θ in subsequent formulations to avoid confusion and maintain clarity for our specific objective.

We also note here that we are using an aggrregated linear demand curve. The aggregation of the demand of individual consumers is a topic that has received significant discussion relative to the empirical estimation of demand and generalizable theoretical results (see for example Nicholson [1975], Henderson and Quandt [1980] and Philips [1982]). Specifically, Philips [1982] highlights several specific conditions required for theoretically sound aggregation of consumers. The first is theoretical plausibility. We do not intend to revisit the extensive discussion relative to aggregation and theoretical plausibility generally and refer the interested reader to the previously cited microeconomic texts. We do, however, highlight that the reference effects is based on utility theory and is validated empirically. It is, therefore, reasonable to assume that this element of consumer utility is plausible and available for aggregation just as price response as a reflection of utility is. The second issue cited is identification. Identification relates to reflecting what the specific demand curve reflects. This is a common challenge when estimating the demand for infrequently purchased items such as, for example, cars. In a given year it is not reasonable to assume that all consumers will by a new car so aggregation of all consumers into a market demand function is somewhat more tenuous than, for example, the weekly demand for a frequently purchased staple item at a grocery store. As is the case in other applied studies with demand aggregated across multiple consumers we are making some general assumptions but our specification is well within the parameters of previous research. Identification also requires a degree of homogeneity across consumers. This is a somewhat more difficult assumption but again we are well within the context of previous work in this regard. We are considering low cost frequently purchased items such as groceries. The demand for these products is usually limited to the store as consumers make the store choice first but the cost of switching stores is high so demand is formed by the price of the item in the store. There is work (Berkowitz and Walton [1980]) that suggests there are differences between stores in pricing expectations and responses but that there is some homogeneity among consumers at a particular store. This also adds support to our approach. Finally we note that Mazumdar et al. [2005] found a relatively narrow band of memory parameter (α) so again the assumption of homogeity seems reasonable. The third condition identified is *ceterus paribus* - that other factors will not change during the time period under consideration. We evaluate short time periods (such as a week at a grocery store) in which it is reasonable to assume factors such as preferences, incomes and other prices are stable. While there is an ongoing debate about aggregated consumer demand and the required assumptions, we are aware of the concerns and our specification is well within the parameters of accepted specifications and provide the opportunity to gain some insight into firm level pricing strategies.

We use the linear demand specification to explore some results in reference models. Our computations use linear models, as do the individual specifications in which we derived closed form solutions to illustrate behaviour. We note that this is consistent with a broad range of economic and operations analysis generally (for example Petruzzi and Dada [1999]) and for reference models specifically (for example Greenleaf [1995] and Kopalle et al. [2009]). Our work expands the insight gained from reference models and the linear specification is validated in previous literature. We acknowledge that there is value in exploring other downward sloping demand specifications as well as undertaking further empirical work in this area.

This recursion allows for the inter-temporal price effects and models the pricing decision facing a firm. Before we begin outlining some key results, we define a function, $g_t(p_t)$, the non-reference component of the profit function. We use an additive reference effect regardless of the form of the demand function. For example, the function $g_t(p_t)$ for a simple linear specification is:

$$g_t(p_t) = (p_t - c)(a - bp_t) = (a + cb)p_t - bp_t^2 - ca$$

The total profit function would be:

$$\pi_t(r_t, p_t) = g_t(p_t) + (p_t - c) \begin{cases} \beta_L(r_t - p_t), \text{ if } r_t - p_t \leq 0 \\ \beta_G(r_t - p_t), \text{ if } 0 < r_t - p_t \end{cases}$$

In the sequel we will omit the subscript t in g, unless it becomes ambiguous.

3.3 The Loss Averse Case

The preponderance of empirical work in reference prices suggests that consumers are loss averse. That means that they respond more significantly to a higher than expected price than they do to a lower than expected price ($\beta_L > \beta_G$). We now consider the optimal pricing strategy for a firm facing loss averse consumers. Having established the function $g(p)$, we have:

Proposition 3.3.1 *For $0 \leq \alpha \leq 1, \beta_L \geq \beta_G$ with a very long planning horizon $T, T \to \infty$, and if g is concave, continuous and differentiable, then a single price is optimal and the optimal price is p^*, the maximizer of g.*

Proof. We first assume that we have a one period cycle and show by contradiction that only p^* will maximize the profit. Assume that price charged $\bar{p} \neq p^*$. We consider two cases.

Case 1: $\bar{p} < p^*$.

Since price is constant we have $\bar{p} = \bar{r}$. We know by definition that $g(\bar{p}) < g(p^*)$. There is a finite loss in profit due to increasing price from $\bar{p}$ to p^* which is equal to $(p^* - c)\beta_L(\bar{r} - p^*)$. On the other hand, there is a long term loss in profit is $\sum_{t=1}^{T} \{g(p^*) - g_t(\bar{p})\} > 0$ since $p^* = \arg\max_p g(p)$. Clearly, $\sum_{t=1}^{T} g(p^*) - g_t(\bar{p}) \geq |(p^* - c)\beta_L(\bar{r} - p^*)|$, i.e. the short term loss in profit of moving to p^* is less than the long term loss in profit of staying at $\bar{p}$. Thus a single price strategy will never stabilize below p^*.

Case 2: When $\bar{p} > p^*$.

In this case decreasing the price to p^* generates a gain equal to $(p^* - c)\beta_G(\bar{r} - p^*) > 0$ and since $g(p^*) > g(\bar{p})$ we conclude that a single price strategy will never stabilize above p^*.

This establishes the first part of the proof: that the optimal single price strategy is to charge p^*.

We now show that for any M-period cycle, $M > 1$, the profit is not superior to that obtained for a 1-period cycle charging p^*. We do not consider the initial adjustment that might be required to achieve the steady state cycle. We consider the long term steady state pricing policy and the associated cycle of repeating prices. Let the vectors $(p_{c1}, p_{c2}, ..., p_{cM})$ and $(r_{c1}, r_{c2}, ..., r_{cM})$ denote the prices and reference prices in an M-period cycle. Note that p_{cM}, the last price in the cycle, will be followed by p_{c1}, the first price in the next cycle. We also define $x^+ = \max(x, 0)$ and $x^- = \min(x, 0)$. Our goal is to show that $Mg(p^*) \geq \sum_{m=1}^{M} g(p_{cm}) + \sum_{m=1}^{M} (p_m - c)\left(\beta_G(r_{cm} - p_{cm})^+ + \beta_L(r_{cm} - p_{cm})^-\right)$ for $M > 1$.

We first present three results which we will use later in the proof.

Result 1: *For any cycle of length* M, $\sum_{m=1}^{M}(r_{cm}-p_{cm})=0$, *i.e., the sum of all the gains is equal to the sum of all of the losses.*

This can be shown as follows. Now:

$$\sum_{m=1}^{M}(r_{cm}-p_{cm})=\sum_{m=1}^{M}r_{cm}-\sum_{m=1}^{M}p_{cm}$$

We note that

$$\sum_{m=1}^{M}r_{cm} = \begin{pmatrix} (1-\alpha)\left(p_{cM}+\alpha p_{c(M-1)}+\alpha^2 p_{c(M-2)}+\ldots+\alpha^{\infty}p_{c(M-1)}\right)+ \\ (1-\alpha)\left(p_{c1}+\alpha p_{cM}+\alpha^2 p_{c(M-1)}+\ldots+\alpha^{\infty}p_{cM}\right)+ \\ \ldots \\ (1-\alpha)\left(p_{c(M-1)}+\alpha p_{c(M-2)}+\alpha^2 p_{c(M-3)}+\ldots+(1-\alpha)^{\infty}p_{c(M-2)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} (1-\alpha)\left(p_{cM}+\alpha p_{cM}+\alpha^2 p_{cM}+\ldots+\alpha^{\infty}p_{cM}\right)+ \\ (1-\alpha)\left(p_{c1}+\alpha p_{c1}+\alpha^2 p_{c1}+\ldots+\alpha^{\infty}p_{c1}\right)+ \\ \ldots \\ (1-\alpha)\left(p_{c(M-1)}+\alpha p_{c(M-1)}+\alpha^2 p_{c(M-1)}+\ldots+(1-\alpha)^{\infty}p_{c(M-1)}\right) \end{pmatrix}.$$

Therefore,

$$\sum_{m=1}^{M}r_{cm}-\sum_{m=1}^{M}p_{cm}=\sum_{m=1}^{M}\frac{(1-\alpha)p_{cm}}{1-\alpha}-\sum_{m=1}^{M}p_{cm}=0.$$

Result 2: *Assume in an* M*-period cycle there are* K *gains and* J *losses such that* $K+J=M$. *Let* $r_{c1}, r_{c2}, \ldots, r_{cM}$ *be the increasing order for the* M *cyclic reference prices and* $p_{c1}, p_{c2}, \ldots, p_{cm}$, *be their corresponding period prices. We can show that*

1. *there exists at least one price* $p_{cm} < r_{c1}$
2. *there exists at least one price* $p_{cm} > r_{cM}$
3. *for any* m *there exist* $i \neq m$ *and* $j \neq m$ *such that* $p_{ci} \leq r_{cm} < p_{cj}$, *where* $1 \leq i, j, m \leq M$

4. $r_{c1} - p_{c1} \leq 0$, *and* $r_{cM} - p_{cM} \geq 0$.

This result is based on the definition of a reference price. We use $p_{c(i)-1}$ and $r_{c(i)-1}, i = 1, \ldots, M$ to denote the price and reference price in the period that immediately precedes the period where $p_{c(i)}$ and $r_{c(i)}$ occurred, respectively. By definition of the reference price we have $\alpha r_{c(1)-1} = r_{c(1)} - (1-\alpha)p_{c(1)-1}$ and by the ordering we have $\alpha r_{c(1)} < \alpha r_{c(1)-1}$. It follows that $\alpha r_{c(1)} < r_{c(1)} - (1-\alpha)p_{c(1)-1}$, or $p_{c(1)-1} < r_{c(1)}$. Thus, establishing part (1) of Result 2. Similar arguments can be used to show part (2). Part (3) follows immediately from parts (1) and (2). For part (4), $r_{c(1)} \leq p_{c(1)}$ has to hold, for otherwise the resulting reference price in the subsequent period would be smaller that $r_{c(1)}$, contradicting the fact that $r_{c(1)} = \min_{1 \leq m \leq M} r_{cm}$. A similar argument can be used to show that $r_{c(M)} \geq p_{c(M)}$. Thus the period corresponding to $r_{c(1)}$ and $r_{c(M)}$ correspond to a loss and gain, respectively.

Result 3: *Given* k *consecutive gains,* $G_1, G_2, \ldots G_K$ *and* J *consecutive losses,* $L_1, L_2, \ldots L_J$, *we have that* $r_{G_1} \geq r_{G_2} \geq \ldots \geq r_{G_K}$ *and* $r_{L_1} \leq r_{L_2} \leq \ldots \leq r_{L_J}$.

This result follows easily from the fact that the reference price in period t is a convex combination of the reference price and price in period $t-1$ and that $r_t \leq p_t$ and $r_t \geq p_t$ for a loss and gain periods, respectively.

Knowing that $Mg(p^*) \geq \sum_{m=1}^{M} g(p_{cm})$ it suffices to establish that

$$\sum_{k=1}^{K} p_{ck}\beta_G G_k + \sum_{j=1}^{J} p_{cj}\beta_L L_j \leq 0$$

to show $Mg(p^*) \geq \sum_{m=1}^{M} g(p_{cm}) + \sum_{m=1}^{M} (p_m - c)\left(\beta_G(r_{cm} - p_{cm})^+ + \beta_L(r_{cm} - p_{cm})^-\right)$.

The remainder of the proof involves showing that, for any cycle length and configuration, all gains are dominated by all the losses. This means that a single price is optimal. We partition the gains and losses into equal pieces and for each combination of equal pieces, the size of the profit loss exceeds the profit gain.

We now begin by considering an arbitrary cycle of length M, with K consecutive gains followed by J consecutive losses. We note that before we finish we generalize this to any M period cycle so that any cycle is shown to be less profitable than the constant price strategy. We define all gains, $(r_{cm} - p_{cm})^+$, G_k, and losses $(r_{cm} - p_{cm})^-$, L_j.

We know from Result 3 that the losses will occur in a sequence with ascending reference prices such that $r_{L_1} \leq r_{L_2} \leq ... \leq r_{L_J}$, and similarly that the gains will occur in order of descending reference prices and specify them such that $r_{G_1} \geq r_{G_2} \geq ... \geq r_{G_K}$. We know from Result 1 that $\sum_J L_j + \sum_K G_k = 0$.

We begin the process of partitioning by matching gains with losses. We do this by beginning at the loss with the lowest reference price and the gain with the lowest reference point. In this case we have exclusively losses which are sequential and gains which are sequential which means we are starting with the last gain and the first loss (the bottom of the price cycle). We begin with L_1 (the loss with the lowest reference price) and G_K (the gain with the lowest reference price). As L_1 is a loss, we know that $p_{L_1} > r_{L_1}$ and using a similar logic we get $p_{G_K} < r_{G_K}$. We assume, without loss of generality, that c=0. We recall that $\beta_L \geq \beta_G$.

For an $M = 2$ cycle we know that $|L_1| = G_1$ and can easily see that $|p_{L_1}\beta_L L_1| = p_{L_1}\beta_L G_1 > p_{G_1}\beta_G G_1$ since $p_{G_1} < r_{L_1} < p_{L_1}$ and $\beta_L > \beta_G$. There is, therefore, no case where $M = 2$ is optimal. It is easy to apply that same logic to $M > 2$.

When $M > 2$, we have three mutually exclusive and exhaustive possible outcomes when we compare L_1 to G_k. Case 1 is when $|L_1| = G_k$. We further specify Case 2 as $|L_1| > G_k$ and Case 3 as $|L_1| < G_k$. In each case we have a specific approach to partitioning and showing that the gains exceed the losses. We follow through the exercise in each individual case, take the loss or gain we have remaining and evaluate which case we have at that point. After each iteration, we have a loss and a gain which we compare and then proceed with the appropriate case.

Case 1) If $|L_1| = G_K$ we know from above that the lost profit from L_1 exceeds the profit gained from G_K.

As we have dealt with L_1 and G_K we now consider L_2 and G_{K-1}. It is worth noting that since $|L_1| = G_K$,

$$|r_{L_1} - p_{L_1}| = r_{G_K} - p_{G_K}$$

and

$$r_{L_1} = \alpha r_{G_K} + (1-\alpha)p_{G_K}$$

we get

$$p_{L_1} = (1+\alpha)r_{G_K} - \alpha p_{G_K}$$

then

$$\begin{aligned} r_{L_2} &= \alpha(\alpha r_{G_K} + (1-\alpha)p_{G_K}) + (1-\alpha)\left((1+\alpha)r_{G_K} - \alpha p_{G_K}\right) \\ &= r_{G_K} \end{aligned}$$

This means that $p_{G_{K-1}} < r_{L_2} < p_{L_2}$, which is similar to what we had for L_1 and G_K. We now can start this process again by comparing L_2 and G_{K-1} which are the next loss and gain in the sequence. We also see that $r_{L_2} < r_{G_{K-1}}$ so we have the same starting condition as we did where the first loss under consideration has a lower reference price than the first gain under consideration. We evaluate which of the cases we have now with L_2 and G_{K-1}.

Case 2) If $|L_1| > G_K$ we consider S additional gains such that $|L_1| \leq \sum_{i=o}^{S} G_{K-i}$ but $|L_1| > \sum_{i=1}^{S-1} G_{K-i}$. We know that $S < K$ due to Result 1. This process takes the number of gains required to provide an absolute value greater than the first loss. We have $|L_1| < \sum_{i=1}^{S} G_{K-i}$ so we partition the gain G_{K-S} into G'_{K-S} and G''_{K-S} such that $|L_1| = \sum_{i=1}^{S-1} G_{K-S} + G''_{K-S}$. This means we have a series of gains and a partial gain that is exactly equal to the size of the first loss. This will always be possible as we know from Result 1 that the total of all the gains is equal to the total of all the losses.

We know that since they are gains $p_{G_i} < r_{G_i}\ \forall i = (K-S)\dots K$. That is for each gain the price is less than the associated reference price ($p_{G_i} < r_{G_i}$). It follows that $r_{L_1} < r_{G_K} < \dots < r_{G_{K-S}}$. Now we can expand

$$|L_1| = \sum_{i=1}^{S-1} G_{K-S} + G''_{K-S}.$$

to give us

$$p_{L_1} - r_{L_1} = r_{G_K} - p_{G_K} + ... + \frac{G''_{K-S}}{G_{K-S}}(r_{G_{K-s}} - p_{G_K}) \tag{3.3.2}$$

The first terms in the equation above are simply the losses and gains expanded as per their definition. The last term is simply the proportion of the specific gain, G_{K-S}, required to exactly cover the first loss, L_1. The prices and associated reference prices for the partitions of gain G_{K-S}, G'_{K-S} and GH''_{K-S} are the same ($r_{G_{K-s}}$ and p_{G_K}) but we have partitioned the total gain into two elements. Now

$$p_{L_1} + p_{G_K} + p_{G_{K-1}} + ... + \frac{G''_{K-S}}{G_{K-S}} p_{G_{K-s}} = r_{L_1} + r_{G_K} + r_{G_{K-1}} + ... + \frac{G''_{K-S}}{G_{K-S}} r_{G_{K-s}} \tag{3.3.3}$$

which means that $p_{L_1} > p_{G_i}\ \forall i = (K-S)...K$. This gives us $|p_{L_1}\beta_L L_1| > \sum_{i=0}^{S-1} p_{G_{K-i}}\beta_G G_{K-i} + \frac{G''_{K-S}}{G_{K-S}} p_{G_{K-s}}\beta_G G_{K-S}$ and so the profit lost due to the loss is greater than the profit gained due to the matching gains. If we do not partition G_{K-S} (because $|L_1| = \sum_{i=1}^{S-1} G_{K-S}$) the $\frac{G''_{K-S}}{G_{K-S}}$ component disappears but the profit impacts are the same. The next step is to compare L_2 and G'_{K-S} or $G_{K-(s+1)}$, depending on whether partitioning is required or not, and go back to one of the appropriate cases. We note before proceeding to the appropriate case, that

$$p_{L_1} + p_{G_K} + p_{G_{K-1}} + ... + p_{G_{K-S}} \leq r_{L_1} + r_{G_K} + r_{G_{K-1}} + ... + r_{G_{K-S}} \tag{3.3.4}$$

but

$$p_{L_1} + p_{G_K} + p_{G_{K-1}} + ... + p_{G_{K-(S-1)}} > r_{L_1} + r_{G_K} + r_{G_{K-1}} + ... + r_{G_{K-(S-1)}}. \tag{3.3.5}$$

Equations 3.3.4 and 3.3.5 follow directly from the fact that $p_{G_{K-S}} < r_{C_{K-S}}$ since G_{K-S} is a gain so we can easily show that $r_{L_2} \leq r_{G_{K-S}}$ and thus we have the same beginning condition where the first loss under consideration has a lower reference price than the first gain under consideration.

Case 3) If $|L_1| < G_K$ we consider S additional losses such that $G_K \leq \left|\sum_{i=1}^{S} L_i\right|$but $G_K \geq \left|\sum_{i=1}^{S-1} L_i\right|$. If $G_K < \left|\sum_{i=0}^{S} L_i\right|$ we partition L_S into L'_S and L''_S such that $G_K = \sum_{i=1}^{S-1} L_i + L''_S$. We know that since they are losses $p_{L_i} > r_{L_i}\ \forall i = 1 \ldots S$. We also know that $p_{G_K} < r_{L_1}$ because of the definition of reference price. We can take an approach similar to that taken in Case 2 and develop

$$p_{G_K} - r_{G_K} = r_{L_1} - p_{L_1} + ... + \frac{L_S''}{L_S}(r_{L_S} - p_{L_S}). \tag{3.3.6}$$

By the same logic used in Equation 3.3.3 - 3.3.6 we find that $p_{G_K}\beta_G G_K < \sum_{i=1}^{S-1} p_{L_i}\beta_L L_i + \frac{L_S''}{L_S} p_{L_S}\beta_G L_S$. That is, that the profit lost due to the losses is greater than the profit gained due to the gains. If we do not partition L_S, the $\frac{L_S''}{L_S}$ component disappears, as it did above but the profit impacts are the same. Once again, the next step is to compare G_{K-1} and L_S' or L_{s+1}. We can again consider the finishing condition with respect to reference prices:

$$p_{L_1} + p_{L_2} + ... + p_{L_S} + p_{G_K} \leq r_{L_1} + r_{L_2} + ... + r_{L_S} + r_{G_K}$$

but

$$p_{L_1} + p_{L_2} + ... + p_{L_{S-1}} + p_{G_K} > r_{L_1} + r_{L_2} + ... + r_{L_{S-1}} + r_{G_K}.$$

and again it is easily shown that $r_{L_S} < r_{G_{K-1}}$. Thus, we again have the same beginning condition where the first loss under consideration has a lower reference price than the first gain under consideration.

This iterative process where Case 1, 2 or 3 is applied sequentially, as appropriate to the entire M period cycle, results in a definitive proof that there is no cycle $M > 1$ that provides more profit than the single price

In the case where the K gains and/or J losses are not consecutive we approach the problem similarly by considering the sub-cycles of the large price cycle within which one or more gains precede one or more losses. The cases (1 through 3) outlined above are applied to the sub-cycles, beginning with the highest reference price which is formed by a gain but involves a loss. We can show that each sub-cycle, and subsequently the entire cycle, results in a net loss of profit.

There is, therefore, no cycle, $M > 1$, which is superior to $M = 1$, when $\beta_L \geq \beta_G$. ■

It is worth noting from the proof that the increase in profit due to a cyclical price depends on $\beta_L \geq \beta_G$. We show that the sum of the positive reference gaps and the sum of the negative reference gaps are equal. If $\beta_L > \beta_G$ then a single price is optimal. Proposition 3.3.1 extends the result of Kopalle et al. [1996] to include cycles of all periodicity. Kopalle et al. [1996] showed that a single price is better than any cycle of length $M = 2$ or $M = 3$. It is worth reiterating that this result does not depend on a specific cycle length or structure. We make no assumptions about the sequence of price increases or decreases. This proposition has implications for managers. Products with long life-cycles (i.e. the planning horizon is long) with loss averse consumers and no thresholds on reference price, maximize profits by ignoring the reference effect and charging a constant price.

3.4 The Loss Seeking Case

While it is generally thought that consumers are loss averse (Kalyanaram and Little [1994], Mazumdar et al. [2005]), there is some empirical support for loss seeking consumers (see, for example Greenleaf [1995] and Briesch et al. [1997]). In the loss seeking case we have $\beta_L < \beta_G$. Kopalle et al. [1996] offered a proof that for loss seeking consumers the optimal price will cycle. That is, if $\beta_L < \beta_G$ then gains will exceed losses and a cyclical price will result. While the approach for Proposition 3.3.1, is different from that of Kopalle et al. [1996], it follows directly from Proposition 3.3.1 that in the loss seeking case the optimal price will cycle.

Given that a cyclical price is optimal, it is worth illustrating the shape of the demand and profit functions. Recall the deterministic reference demand function with symmetric reference effects:

$$D_t = a - bp_t + \beta(r_t - p_t)$$

If we ignore the inter-temporal effects we would solve the single period problem. While we generally cannot ignore the inter-temporal effects we can consider for illustration the special case of the end of a product's lifecycle or the final period in a seasonal product's selling season. In such a case we have a reference price based on historical price exposure but are not concerned about the impact of the current price on future demand. We could solve the single period problem easily resulting in an optimal price

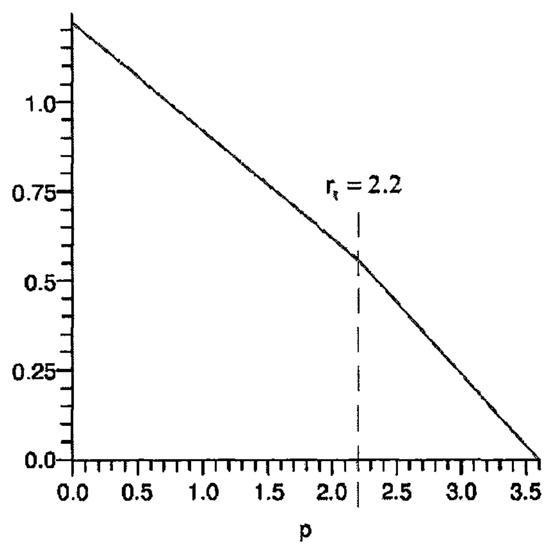
$$p^* = \frac{a + c(b+\beta) + \beta r}{2(b+\beta)} \tag{3.4.7}$$

If we consider the case with asymmetric reference effects we have a demand function which is no longer linear, resulting in a profit function that is no longer as straight forward. The new demand curve is

$$D_t = \begin{cases} a - bp_t + \beta_G(r_t - p_t); \text{ if } r_t > p_t \\ a - bp_t + \beta_L(r_t - p_t); \text{ if } r_t \leq p_t \end{cases}.$$

Figure 3.1 shows what this demand curve looks like in the loss averse case where $r_t = 2.2$ which causes a kink at $r_t = p_t$.

Figure 3.1: Loss Averse Demsand Function



That demand function yields profit function

$$\pi_t = \begin{cases} (p_t - c)(a - bp_t + \beta_G(r_t - p_t)); \text{ if } r_t > p_t \\ (p_t - c)(a - bp_t + \beta_L(r_t - p_t)); \text{ if } r_t \leq p_t \end{cases}$$

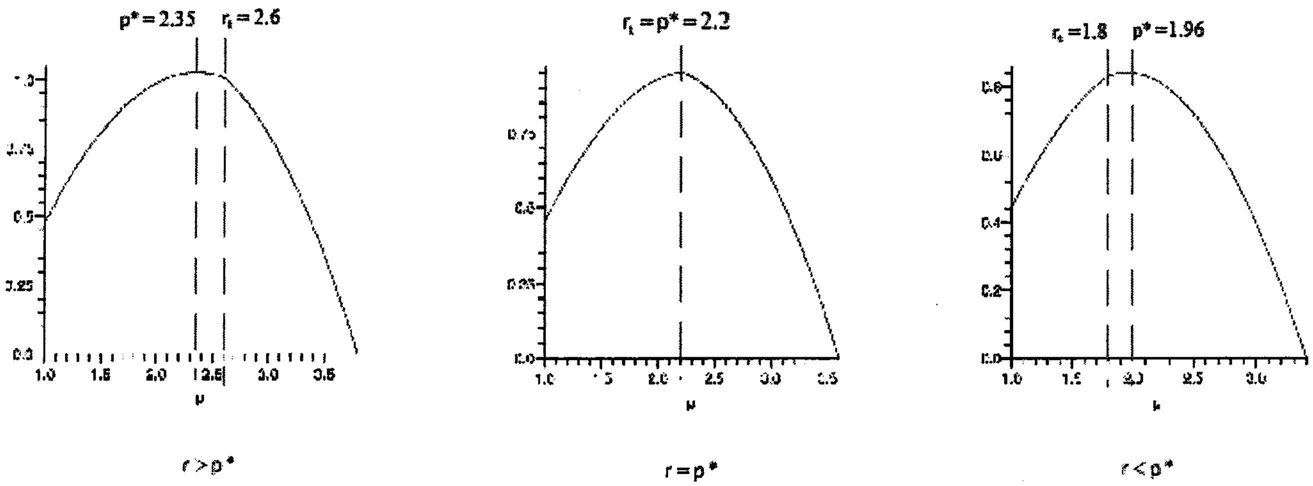
which is continuous and concave but is non-differentiable at $p_t = r_t$. We can illustrate the dynamics of this kinked profit function with a simple example. Consider the demand function

$$D_t = \begin{cases} 1 - 0.2p_t + 0.1(r_t - p_t); \text{ if } r_t > p_t \\ 1 - 0.2p_t + 0.2(r_t - p_t); \text{ if } r_t \leq p_t \end{cases}.$$

The single period maximizer of that function depends on the value of r_t. In the case where $r_t = 2.4$, and $c = 0.5$ the resulting profit function is maximized at $p_t = 2.32$, which is less than $r_t = 2.4$. It is worth noting that this specification results in demand that is less than one unit. This is not a cause for concern as a linear demand curve is completely scalable and this specification is only made for convenience. If we multiply all of the parameters by any common factor we get the same optimal price results, price elasticity results but at higher quantities. Consider equation 3.4.7. If all of the parameters (a, b and β - c is cost and not a parameter) were muliplied by $1,000$ the optimal price would not change. This scalability makes the use of this specification valid and allows for clarity without changing results or insight.

If the reference price is different, however, the resulting profit function is kinked at a different spot (the kink occurs at the reference price). Figure 3.2 shows the profit function when $r_t = 2.6, 2.2$ and 1.8. We note that the other parameters are held constant. When $r_t = 2.2$, the profit function is maximized at $p = r = 2.2$. When $r = 1.8$, the profit function is maximized at $p = 1.95$. In this case, the reference price is so low that a loss is offset by an increase in profit from the non-reference component of the profit function.

Figure 3.2: Loss Averse Reference Profit Functions



The single period maximizer of the reference price problem is then

$$p^* = \begin{cases} \frac{a+c(b+\beta_G)+\beta_G r_t}{2(b+\beta_G)} \text{ if } \frac{a+c(b+\beta_G)+\beta_G r_t}{2(b+\beta_G)} < r_t \\ \frac{a+c(b+\beta_L)+\beta_L r_t}{2(b+\beta_L)} \text{ if } \frac{a+c(b+\beta_L)+\beta_L r_t}{2(b+\beta_L)} > r_t \\ r_t \text{ if } \frac{a+c(b+\beta_L)+\beta_L r_t}{2(b+\beta_L)} < r_t < \frac{a+c(b+\beta_G)+\beta_G r_t}{2(b+\beta_G)} \end{cases} \tag{3.4.8}$$

where the first two elements are the first order conditions for $r_t > p_t$ and $r_t < p_t$, respectively. We note that since $\frac{a+c(b+\beta_L)+\beta_L r_t}{2(b+\beta_L)} < \frac{a+c(b+\beta_G)+\beta_G r_t}{2(b+\beta_G)}$, these three prices are both mutually exclusive and exhaustive.

If the price cycles, as we expect in the loss seeking case, the price in Equation 3.4.8 will not necessarily reflect the optimal price in any of the periods. In this case we need to solve the dynamic programming problem to find the optimal price path. We know that a two period cycle is superior to a single price but we do not know that a two period cycle is optimal. In the work of Kopalle et al. [1996], all of the loss seeking cycles were of periodicty two but we do not know that this will always be the case. We can evaluate a two period cycle analytically.

Proposition 3.4.1 *Given $\beta_G < b$ (which implies also in loss averse case that $\beta_L < b$) and $M = 2$, the profit function is concave in prices and the optimal prices are*

$$p_1^* = \frac{2(b+\beta_G w_1)h_1 + \beta w_1 h_2}{4(b+\beta_L w_1)(b+\beta_G w_1) - \beta^2 w_1^2}$$

and

$$p_2^* = \frac{2(b+\beta_L w_1)h_2 + \beta w_1 h_1}{4(b+\beta_L w_1)(b+\beta_G w_1) - \beta^2 w_1^2}$$

where $\beta = \beta_L + \beta_G$, $h_1 = a + (b + w_1(\beta_L - \beta_G)c$, and $h_2 = a + (b - w_1(\beta_G - \beta_L)c$,$w_1 = (1-\alpha) + (\alpha^2 - \alpha^3) + (\alpha^4 - \alpha^5) + ...$ and $w_2 = (\alpha - \alpha^2) + (\alpha^3 - \alpha^4) + (\alpha^5 - \alpha^6) + ...$.

Proof. Since $M = 2$, we have two optimal prices and two recurring reference prices. We know that:

$$\begin{aligned} r_1 &= \alpha r_2 + (1-\alpha)p_2 \\ &= (1-\alpha)p_2 + (\alpha - \alpha^2)p_1 + (\alpha^2 - \alpha^3)p_2 + ... \\ r_2 &= (1-\alpha)p_1 + (\alpha - \alpha^2)p_2 + (\alpha^2 - \alpha^3)p_1 + ... \end{aligned}$$

We defined

$$w_1 = (1-\alpha) + (\alpha^2 - \alpha^3) + (\alpha^4 - \alpha^5) + ... \tag{3.4.9}$$

and

$$w_2 = (\alpha - \alpha^2) + (\alpha^3 - \alpha^4) + (\alpha^5 - \alpha^6) + ...$$

Note that $w_1 + w_2 = 1$. Thus, we can write

$$r_1 = w_1 p_2 + w_2 p_1$$

and

$$r_2 = w_1 p_1 + w_2 p_2.$$

Note that $r_1 + r_2 = p_1 + p_2$, which means that losses and gains cancel out, as per our earlier result. We assume in the two period cycle that $p_1 > p_2$. We now optimize a repeating two period problem with profit function

$$\pi_{(M=2)} = (p_1 - c)\left[a - bp_1 + \beta_L(w_1 p_2 + w_2 p_1 - p_1)\right] + (p_2 - c)\left[a - bp_2 + \beta_g(w_1 p_1 + w_2 p_2 - p_2)\right] \tag{3.4.10}$$

We can simplify the period 1 reference effect

$$\begin{aligned} w_1 p_2 + w_2 p_1 - p_1 &= w_1 p_2 - (1 - w_2) p_1 \\ &= w_1 (p_2 - p_1). \end{aligned}$$

This gives us a profit for period 1 that is

$$\pi_1 = (p_1 - c)\left[a - bp_1 + \beta_L w_1 (p_2 - p_1)\right].$$

We can do the same for period 2:

$$\pi_2 = (p_2 - c)\left[a - bp_2 + \beta_G w_1 (p_1 - p_2)\right].$$

The two period (repeating) profit function is then

$$\pi_{(M=2)} = (p_1 - c)\left[a - bp_1 + \beta_L w_1 (p_2 - p_1)\right] + (p_2 - c)\left[a - bp_2 + \beta_G w_1 (p_1 - p_2)\right]. \tag{3.4.11}$$

The Hessian of the objective function is

$$\begin{bmatrix} -2\,(b + \beta_L w_1) & (\beta_G + \beta_L) w_1 \\ (\beta_G + \beta_L) w_1 & -2\,(b + \beta_G w_1) \end{bmatrix}$$

which is negative definite when $b > \beta_G$. We can then find optimal prices using first order conditions.

$$p_1 = \frac{a + (b + w_1(\beta_L - \beta_G))c + (\beta_L + \beta_G) w_1 p_2}{2(b + \beta_L w_1)}$$

and

$$p_2 = \frac{a + (b + w_1(\beta_G - \beta_L))c + (\beta_L + \beta_G) w_1 p_1}{2(b + \beta_G w_1)}.$$

Using β, h_1, and h_2 as defined in the proposition, we obtain the desired result. ■

These analytical results for p_1^* and p_2^* will always provide a higher level of profit than will charging p^*, the maximizer of $g(p)$. We reiterate that a two period cycle does not guarantee maximum profit but requires considerably less computation time then solving for the optimal and offers a higher level of profit than does a constant price. Calculating analytical results for $M > 2$, is more complex. For $M = 3$, there are four different options.

1. Two price increases and one price decrease that result in two gains and a loss;
2. Two price increases and one price decrease that result in one gain and two loses;
3. One price increase and two price decreases that result in two gains and a loss; and
4. One price increase and two price decreases that result in one gain and two losses.

We would have to compare the prices from all of the analytical results to the $M = 2$ results to see which is better and then go to $M > 3$ results and continue. In most cases, the dynamic programming approach will be required. The analytical results characterize the prices for a two period cycle.

We now also consider the constraint that price must be greater than or equal to cost.

Proposition 3.4.2 *Given a linear demand function with additive reference component, no two or three period price cycle where $p_t < c$ will ever be optimal.*

Proof. Consider first a two period cycle. When $M = 2$, we have a single reference gain and a single reference loss. We defined the profit function in Equation 3.4.11:

$$\pi_{(M=2)} = (p_1 - c)\left[a - bp_1 + \beta_L w_1(p_2 - p_1)\right] + (p_2 - c)\left[a - bp_2 + \beta_G w_1(p_1 - p_2)\right]$$

We recall that $p_1 > p_2$. The case where both p_1 and p_2 are less than zero is trivial. Now consider the case of an optimal solution in which $p_1 > c$ and $p_2 < c$. We define:

$$\begin{aligned} \pi_1 &= (p_1 - c)\left[a - bp_1 + \beta_L w_1(p_2 - p_1)\right], \text{ and} \\ \pi_2 &= (p_2 - c)\left[a - bp_2 + \beta_G w_1(p_1 - p_2)\right] \end{aligned}$$

If $p_2 < c$ then $\pi_2 < 0$. If we make $p_2 = c$, then $\pi_2 = 0$. We also note that π_1 increases because the reference loss is lower $(w_1(p_2 - p_1))$ as p_2 is larger. There is no case in which $\pi_{(M=2)}$ is larger with $p_2 < c$ with any $p_1 > c$, than it would be with $p_2 = c$ and the same p_1. We could do the analagous $M = 3$ scenarios of which there are the four options outlined above. ∎

Conjecture 3.4.1 *Given a linear demand function with an additive reference component, there will be no cycle length in which any $p_t < c$ will be optimal.*

We note in support of this conjecture that we undertook a number of numerical experiments using the loss averse specification and varying the parameters and had no case in which there was a cycle longer than $M = 2$. We also note that intuitively we would expect that price effects on profitability will be more significant than reference effects. This leads us to this conjecture which suggests it is then reasonable to retain the $p_t \geq c$ constraint to reduce the search space even in the case of loss seeking reference demand. This is an area in which empirical study could add considerable insight and support.

3.5 Summary and Conclusions

In this chapter we introduced the basic reference model. This model serves as the foundation for further analysis relative to the impact of incorporating inter-temporal price effects into decision models. These extensions include the consideration of the effects of stochastic demand on price cycles (in both the loss seeking and loss averse cases), the incorporation of thresholds and consideration of setup/ordering costs and holding costs. Our objective is to understand in more detail the pricing strategies for managers facing demand with reference effects.

We evaluated optimal pricing strategies for the basic model for both loss averse and loss seeking consumers. These will serve as the benchmarks for comparison when we consider other extensions of the basic model. We showed that for loss averse consumers the optimal pricing strategy is a single price that is the same as it would be in the absence of reference effects. We also know that for loss seeking consumers a cyclical price is optimal.

Chapter 4

The Threshold Model

4.1 Introduction

We propose a reference model which incorporates thresholds in order to study the impact of thresholds on a loss averse reference price model. This is consistent with the work of Gupta and Cooper [1992], Kalwani and Yim [1992], Kalyanaram and Winer [1995] and others. We are motivated to examine optimal price cycles in the presence of thresholds in an effort to understand the conditions under which price will cycle and those under which a single price will be optimal. The threshold reference model is more robust in that it allows for both single price and cyclical prices strategies to be optimal. One of the challenges of the loss averse reference model is that, without thresholds, a single price strategy is always optimal. We know that there are retailers who choose to cycle prices and a model which allows for both strategies may provide more practical insights. We can see examples of price cycles regularly in the retail grocery market. We highlight specifically that the cola products (like Coke and Pepsi) are well known to go through a regular cyclical promotion cycle.

4.2 The Threshold Model

We now introduce a model with thresholds within which there is no reference effect. The empirical work in marketing (Kalwani and Yim [1992], Krider and Han [2004], Campo and Yague [2007], Pauwels et al. [2007] and Marshall and Na [2000]) suggests that there is a region of insensitivity around the expected price. We specify a threshold reference model with absolute thresholds as:

$$D_t = \begin{cases} a - bp_t + \beta_G(r_t - \tau - p_t) \ ; \text{if } r_t - \tau - p_t > 0 \\ a - bp_t \ ; \text{if } \rho \leq r_t - p_t \leq \tau \\ a - bp_t + \beta_L(r_t + \rho - p_t) \ ; \text{if } r_t + \rho - p_t < 0 \end{cases}$$

where

$\beta_L \geq \beta_G$,

$\tau \geq 0$ is the gain threshold below the reference price within which there is no gain effect, and

$\rho \geq 0$ is the loss threshold above the reference price within which there is no loss effect.

We specify a similar threshold model with percentage thresholds as:

$$D_t = \begin{cases} a - bp_t + \beta_G((1-\omega)r_t - p_t), \omega r_t < r_t - p_t \\ a - bp_t, -\psi r_t \leq r_t - p_t \leq \omega r_t \\ a - bp_t + \beta_L((1+\psi)r_t - p_t), r_t - p_t < -\psi r_t \end{cases}$$

where

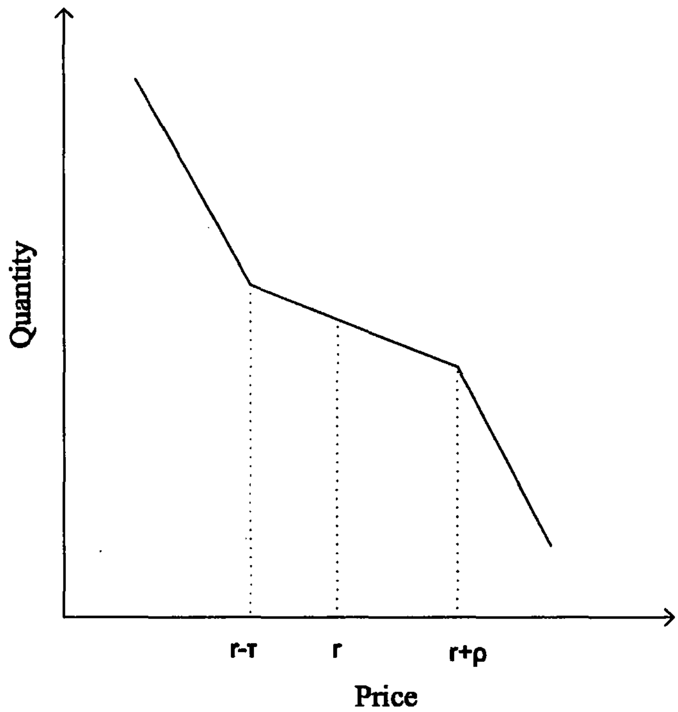
$\beta_L \geq \beta_G$,

ω is the percentage threshold within which there is no gain effect, $0 \leq \omega \leq 1$, and

ψ is the percentage threshold within which there is no loss effect, $0 \leq \psi \leq 1$.

Figure 4.1 shows the shape of the demand curve with reference prices and thresholds. Given two kinks in the curve, the resulting demand curve is neither

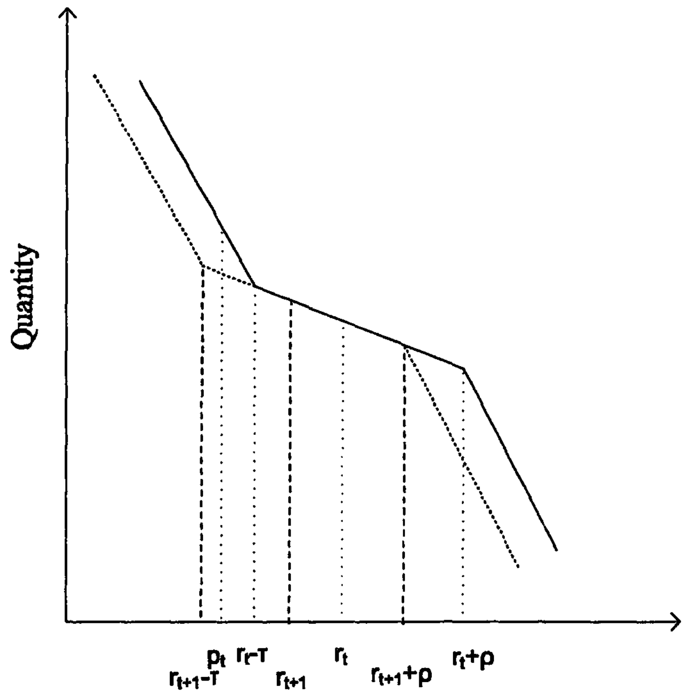
Figure 4.1: Threshold Demand Curve



concave nor convex. It is important to note that the curve shown in Figure 4.1 is only for a given reference price. If a price lower than that reference price is charged in a given period, the resulting reference price for the next period is lower (Figure 4.2) and the demand curve is different. The reverse is true if a price higher than the reference price is charged in the earlier period.

The resulting demand curve is still downward sloping and consistent with economic theory. Tellis [1988b] finds significant evidence of downward sloping demand curves (negative price elasticities) which is consistent with this form. There are examples of "prestige pricing" for some luxury type products or "sweet spot pricing" for selected gift items (Gaur and Fisher [2005], McClure and Kumcu [2008]) and other factors that are counter intuitive when empirically estimated. This specification may

Figure 4.2: Shift in Threshold Demand



provide a context within which some of these anomalies can be explained.

The incorporation of thresholds increases the complexity of the non-negativity conditions. In the linear demand case the non-negativity conditions with fixed thresholds are:

$$1)\text{ If } r_t > \frac{a}{b} + \tau, \text{ then } c \leq p_t \leq \frac{a + \beta_G(r_t - \tau)}{b + \beta_G}, \tag{4.2.1}$$

$$2)\text{ If } \frac{a}{b} - \rho < r_t \leq \frac{a}{b} + \tau, \text{ then } c \leq p_t \leq \frac{a}{b}, \tag{4.2.2}$$

$$3)\text{ If } r_t \leq \frac{a}{b} - \rho, \text{ then } c \leq p_t \leq \frac{a + \beta_L(r_t + \rho)}{b + \beta_L} \tag{4.2.3}$$

We note that in Case 1 in Equation 4.2.1-4.2.3, the reference price is higher than $\frac{a}{b}$ plus the gain threshold and so the upper bound includes a gain threshold and a gain effect. In Case 2 the reference price is within both the thresholds away from $\frac{a}{b}$ and so we use the same bounds that we would in the non-reference model. Finally in Case 3, the reference price is more than the loss threshold away from $\frac{a}{b}$ and so the limit includes a loss threshold and a loss effect.

The first condition represents the third line segment in Figure 4.1. If it intersects zero demand then the gain parameter influences the non-negativity constraints. The point, $a - b(r_t - \tau)$, represents the second kink in the demand curve. If it is below zero, then $a - b(r_t + \rho)$ is also below zero as $a - b(r_t + \rho) < a - b(r_t - \tau)$. The second condition comes into play if the demand curve intersects zero demand. In this case there is no reference impact (as the price is either above or below reference price that causes demand to intersect zero is within the threshold). The final condition is where it is the first segment that intersects with the x-axis and, therefore, we need to be aware of the loss parameter.

For proportional thresholds the non-negativity conditions would be:

$$1)\text{ If } r_r > \frac{a}{b} + \omega, \text{ then } c \leq p_t \leq \frac{a + \beta_G((1 - \omega)r_t)}{b + \beta_G},$$

$$2)\text{ If } \frac{a}{b} - \psi < r_t \leq \frac{a}{b} + \omega, \text{then } c \leq p_t \leq \frac{a}{b}$$

$$3) \text{ If } r_t \leq \frac{a}{b} - \psi, \text{ then } c \leq p_t \leq \frac{a + \beta_L\left((1+\psi)r_t\right)}{b + \beta_L}$$

While the propositions and associated proofs will be consistent with either choice of threshold specification, we choose to focus on the absolute thresholds for the remainder of this discussion. It is easily shown that the results hold regardless of the specific form of the threshold. We choose absolute thresholds as there appears to be some evidence that consumers process price discounts more readily than percentage discounts. The literature is not definitive relative to consumer preference for percentage or absolute price discounts. Hu and Khan [2006] find that consumers prefer absolute price discounts versus percentage discounts for high value services. Their findings suggest that the reverse is true for low end services although the difference is smaller and not consistent for these services. The findings of Chen et al. [1998] were similar. DelVecchio et al. [2007] and Diamond and Campbell [1989] find that percentage discounts do not affect the resultant reference price as much as absolute discounts do. The hypotheses presented suggest that consumers take the time to process percentage discounts for high value items as they require greater expenditure and, therefore, merit greater attention. For lower priced items, a percentage discount may appear bigger and the size of the expenditure does not merit the effort to translate it to an absolute amount. We hypothesize, therefore, that consumers are more likely to have an absolute threshold than a percentage threshold given our application in frequently purchased goods. We reiterate that either approach will work and is consistent with the propositions presented. This may be an avenue for eventual empirical investigation.

The dynamic programming formulation is identical to the one above with the demand function replaced as outlined above. We now evaluate some results under reference pricing with thresholds. These results examine the impact on profitability of gain and loss thresholds individually. We would expect that in most cases both would exist and offset each other but it is worth understanding the specifics of the impacts. First we consider the case where there is no gain threshold ($\tau = 0$) but there is a loss threshold (ρ). In this case there is only a zone of insensitivity to reference price losses.

Proposition 4.2.1 *For* $0 \leq \alpha \leq 1, \beta_L > \beta_G$, *with a very long planning horizon* $T, T \to \infty$,*and a loss threshold only* ($\rho > 0, \tau = 0$ or $\psi > 0, \omega = 0$) *and if* g *is concave, continuous and differentiable, a single price is not optimal.*

Proof. We know from Proposition 3.3.1 that if a single price is optimal, that price is p^* in the absence of thresholds. It is easily shown the same is true with thresholds. Recall that we considered a price policy with a single price, $\bar{p}$, that is not equal to p^*.

Case 1: When $\bar{p} < p^*$, we have $\bar{p} = \bar{r}$, since the price is constant. There is a finite loss in profit due to moving to p^* which is $(p^* - c)\beta_L(\bar{r} + \rho - p^*)^-$. We know $g(\bar{p}) < g(p^*)$. This short term finite loss of profit is less than the long term loss in profit $\sum_{t=1}^{\infty} g(p^*) - g_t(\bar{p})$. Thus a single price strategy will never stabilize below p^* even with the presence of thresholds.

Case 2: When $\bar{p} > p^*$, the proof is straight forward. We once again know that $g(\bar{p}) < g(p^*)$ and $(p^* - c)\beta_G(\bar{r} - \tau - p^*)^+ \geq 0$ so we gain by moving to p^* as we will either have a positive reference gain effect or no reference effect and the price effect is positive.

We have now established that no single price other than p^* will ever be optimal. We now evaluate a cyclical pricing strategy. Consider an $M = 2$ cycle in which we subtract a small increment, δ, from p^* for one price and add the increment to p^* for the second price. We choose a value such that $\delta \leq 0.5\rho$. We use the notation $r_{c,i}$ and r_{ci} ; $i = 1, 2, ..M$ or $p_{c,i}$ and p_{ci} ; $i = 1, 2, ..M$ interchangeably. We specify:

$$p_{c,1} = p^* + \delta$$

and

$$p_{c,2} = p^* - \delta.$$

The reference prices are:

$$r_{c,2} = \alpha r_{c1} + (1-\alpha)p_{c1}$$

and

$$r_{c,1} = \alpha r_{c2} + (1-\alpha)p_{c2}.$$

Solving for r_{c1} and r_{c2} in the above system of equations we find

$$r_{c,1} = p^* - \frac{(1+\alpha^2-2\alpha)\delta}{(1-\alpha^2)}$$

and

$$r_{c,2} = p^* + \frac{(1+\alpha^2-2\alpha)\delta}{(1-\alpha^2)}.$$

We now define Y , the loss effect, such that

$$Y = \begin{cases} r_{c,1} + \rho - (p^* + \delta), \text{ if } r_{c,1} + \rho - (p^* + \delta) < 0 \\ 0 \text{ if } r_{c,1} + \rho - (p^* + \delta) \geq 0 \end{cases}.$$

For $0 \leq \alpha \leq 1$,

$$\frac{(1+\alpha^2-2\alpha)}{(1-\alpha^2)} \leq 1.$$

then

$$\begin{aligned} r_{c,1} + \rho - (p^* + \delta) &\geq p^* - \delta + \rho - p^* - \delta \\ &\geq -2\delta + \rho \\ &\geq 0, \text{ because } \delta \leq 0.5\rho. \end{aligned}$$

It is clear, therefore that $Y = 0$. For the single price strategy to be optimal we need to have

$$2g(p^*) \geq g(p^*+\delta) + g(p^*-\delta) + (p^*+\delta-c)\beta_L Y + (p^*-\delta-c)\beta_G(r_{c,2} - p^* - \delta)$$

but $Y = 0$, so

$$2g(p^*) \geq g(p^*+\delta) + g(p^*-\delta) + (p^*-\delta-c)\beta_G(r_{c,2} - p^* - \delta)$$

$$2g(p^*) - g(p^* + \delta) - g(p^* - \delta) \geq (p^* - \delta - c)\beta_G \left(p^* + \frac{(1 + \alpha^2 - 2\alpha)\delta}{(1 - \alpha^2)} - p^* - \delta\right)$$

$$2g(p^*) - g(p^* + \delta) - g(p^* - \delta) \geq (p^* - \delta - c)\beta_G \delta \left(\frac{(1 + \alpha^2 - 2\alpha)}{(1 - \alpha^2)} + 1\right)$$

$$\frac{g(p^*) - g(p^* + \delta) + g(p^*) - g(p^* - \delta)}{\delta} \geq (p^* - \delta - c)\beta_G \left(\frac{(1 + \alpha^2 - 2\alpha)}{(1 - \alpha^2)} + 1\right)$$

now we take the limit as $\delta \to 0$ and since g is continuous and differentiable

$$\begin{aligned} -g'(p^*) + g'(p^*) &\geq (p^* - c)\beta_G \left(\frac{(1 + \alpha^2 - 2\alpha)}{(1 - \alpha^2)} + 1\right) \\ 0 &\geq (p^* - c)\beta_G \left(\frac{(1 + \alpha^2 - 2\alpha)}{(1 - \alpha^2)} + 1\right) \end{aligned}$$

which is impossible as the right hand side is always positive. Thus with a loss threshold only, the optimal price always cycles. ■

Consider the threshold demand function. We can see that the profit is non-decreasing in ρ. If $r_t - \tau - p_t > 0$, ρ does not affect profit. The parameter ρ comes into play in the next two cases. As ρ increases the portion of the demand curve in which a reference loss occurs is decreased as the second kink in Figure 4.2 shifts to the right. Average profit when continuously charging p^* sets a lower bound for average profit with a loss threshold. If a constant price is charged there are no reference effects.

It is worth noting that the proof of Proposition 4.2.1 shows that there is always a two period cycle that is superior to a constant price. It does not show that a two period cycle will be optimal. This means that a price cycle will increase profit above the lower bound, which is established by the single price policy at p^*. The dynamic program needs to be evaluated to determine the optimal number of periods, M, and the prices to be charged within this price cycle.

We now consider the case where there is only a gain threshold and no loss threshold. Proposition 4.2.2 establishes that if there is a threshold only on the gain, a single price is optimal.

Proposition 4.2.2 *For $0 \leq \alpha \leq 1, \beta_L > \beta_G$ with a very long planning horizon T, $T \to \infty$, and a gain threshold only ($\rho = 0, \tau > 0$ or $\psi = 0, \omega > 0$) and if g is* *concave, continuous and differentiable, a single price policy is optimal for the average per period profit criterion and the single price is p^*, the optimizer of g.*

Proof. We begin with the same arbitrary M period cycle we specified in the proof for Proposition 3.3.1.. The cycle profit is:

$$\pi_M = \sum_{m=1}^{M} g(p_{c,m}) + X$$

where

$$X = \sum_{m=1}^{M} (p_{c,m} - c)(\beta_G(r_{c,m} - \tau - p_{c,m})^+ + \beta_L(r_{c,m} - p_{c,m})^-).$$

we showed in the proof of Proposition 3.3.1 that:

$$\sum_{m=1}^{M} (p_{c,m} - c)(\beta_G(r_{c,m} - p_m)^+ + \beta_L(r_{c,m} - p_{c,m})^-) \leq 0$$

for all cycle lengths. .Knowing that $\tau > 0$ and

$$\begin{aligned}\sum_{m=1}^{M} (p_{c,m} - c)(\beta_G(r_{c,m} - \tau - p_{c,m})^+ + \beta_L(r_{c,m} - p_{c,m})^-) \leq & \sum_{m=1}^{M} (p_{c,m} - c)(\beta_G(r_{c,m} - p_{c,m})^+ \\ & + \beta_L(r_{c,m} - p_{c,m})^-),\end{aligned}$$

we can show using the same logic of the proof of Proposition 3.3.1 that

$$\sum_{m=1}^{M} (p_{c,m} - c)(\beta_G(r_{c,m} - \tau - p_{c,m})^+ + \beta_L(r_{c,m} - p_{c,m})^-) \leq 0$$

and, therefore

$$\sum_{m=1}^{M} g(p_{c,m}) \leq M g(p^*).$$

Thus, in no case where there is a gain threshold only, would a cyclical price be optimal.

Profit is non-increasing in the gain threshold. As the gain threshold increases, profit either stays the same or decreases. If there is only a threshold on gain a single price strategy is optimal. Given Propositions 4.2.1 and 4.2.2 it is clear that loss thresholds and gain thresholds offset each other. The optimal price path then, in the presence of both gain and loss thresholds, will depend on the relative size of the two thresholds, the gain and loss parameters (β_G and β_L). Given that the total profit functions (including reference effects) are neither differentiable nor concave in p, $f_t(r_t)$ is also not concave and cannot provide closed form solutions. ■

For the infinite horizon problem, the lower bound for the average profit is a single price strategy with average profit, $g(p^*)$. Prices will cycle only if it increases average profit (in the undiscounted case) above that lower bound. The key variables that affect profit above this lower bound are the length of the cycle, M, and the price vector $[p_{c1}...p_{cM}]$ where the subscript c before the period denotes that these are sequential prices within a repeating cycle. In this repeating cycle p_{c2} follows p_{c1}. At the end of the M period cycle the cycle repeats and p_{c1} follows p_{cM}. We note also the following key relationships which will impact whether optimal price cycles or is constant.

Key Threshold Results

The profit is non-increasing in gain threshold. As gain threshold grows, the average per period profit will move back to $g(p^*)$.

1. The profit is non-decreasing in loss threshold. As loss threshold grows, average per period profit will increase from $g(p^*)$.

    - The profit can increase with M decreasing.

    - The profit can increase with specific extreme values of p changing.

2. The profit is non-increasing in β_L.

3. The profit is non-decreasing in β_G.

4. The profit is non-increasing in α. If reference price responds more significantly to recent prices, prices are more likely to cycle as larger reference gains can be achieved over smaller price ranges which results in lower losses over the non-reference component of the profit function, $g(p)$.

We use a similar approach to that used in earlier research (Greenleaf [1995], Kopalle et al. [1996]) using discrete time periods and prices and continuous reference prices (rounded to two decimal places) in the dynamic programming framework. While technology has significantly reduced the difficulty in changing prices, there are still many examples in which firms offer a specific price for a given period. Grocery stores with weekly flyers are a good example of this.

4.3 Reducing Computation Time

This dynamic programming approach requires significant computational time to execute to find the optimal price path. We provide the following propositions in order to decrease the computational requirements of the algorithm and improve computational efficiency. Proposition 3.3.1 is a first reduction in the solution space as it eliminates any single price strategy that is not p^*. The Propositions below, more significantly reduce the solution space.

Proposition 4.3.1 *For $0 \leq \alpha \leq 1, \beta_L > \beta_G$ with a very long planning horizon $T, T \rightarrow \infty$, and a gain threshold ($\rho > 0, \tau \geq 0 or \psi > 0, \omega \geq 0$) and if g is concave, continuous and differentiable, the range $[p_{c,\min}, p_{c,\max}]$ (where $p_{c,\min} = \min_{M}, p_{c,m}$ and $p_{c,\max} = \max_{M}, p_{c,m}$) will always include p^*.*

Proof. Consider an optimal solution in which the range $[p_{c,\min}, p_{c,\max}]$ does not include p^*. If $p_{c,\max} < p^*$, each price in the M period price cycle can be increased by $p^* - p_{c,\max}$. In the steady state case, this would increase each of the reference prices in the M period cycle by the same amount. The gaps between price and reference price in each period will be unchanged and, therefore, there would be no change in the reference component of the profit function as the reference gaps for price decreases and price increases would not change. Due to the concavity of $g(p)$ we know that the non-reference component of each period's profit will increase as prices move closer to p^*.

In the case where $p_{c,\min} > p^*$, we similarly subtract $p_{c,\min} - p^*$ from each price in the M period price cycle. In all cases the solution is improved if the range includes p^*. ■

For the subsequent propositions it is convenient to define:

$$A = \sum_{m=1}^{M} (p_{c,m} - c)\beta_G(r_{c,m} - \tau - p_{c,m})^+,$$
$$B = \sum_{m=1}^{M} (g(p^*) - g(p_{c,m})), \text{ and}$$
$$C = \sum_{m=1}^{M} (p_{c,m} - c)\beta_L(r_{c,m} + \rho - p_{c,m})^-.$$

Component A is the total profit gained due to reference gains in an M period cycle. Component B represents the total revenue lost from the non-reference portion of the profit function due to charging a price less than p^*. Component C is the total profit lost due to reference losses during an M period cycle. We note that for $M > 1$:

$$A + B + C > 0 \tag{4.3.4}$$

or price cycling would not be optimal. That is, the profit gains need to exceed the profit losses due to both reference effects and charging a price less than p^*. In the subsequent propositions we will use the absolute threshold example for illustration but the proportional threshold example is analogous.

Proposition 4.3.2 *For $0 \leq \alpha \leq 1, \beta_L > \beta_G$, with loss thresholds ($\rho > 0$, or $\psi > 0$) and concave g and with a very long planning horizon $T, T \to \infty$, for $M > 1$, no price decrease will result in a price greater than $r_{c,m} - \tau$ for absolute thresholds or $r_{c,m} - \omega r_{c,m}$ for proportional thresholds, $m = 1, ..., M$.*

Proof. If Equation 4.3.4 inequality did not hold, then $M = 1$ and a single price strategy would be optimal.

Now consider a portion of an optimal price cycle of length M with sequence, $p_{c,i} > p_{cj} > p_{c,k}$ where $0 < i < j < k \leq M$ and i, j, k are integers. This sequence has two consecutive price decreases and we assume that $r_{c,j} - p_{c,j} < \tau$. The threshold is not exceeded in that interval. If we eliminate $p_{c,j}$ we have a cycle of length $M-1$. Component B will decrease by $g(p^*) - g(p_{c,j}) \geq 0$.

Component C may or may not change. Note the reference price after charging $p_{c,k}$ in the original sequence is $r_{c,k} = \alpha^2 r_{c,j} + (\alpha - \alpha^2)p_{c,j} + (1-\alpha)p_{c,k}$. If we skip the interim price, $p_{c,j}$ the new reference is $\alpha r_{c,j} + (1-\alpha)p_{c,k}$ and

$$\alpha r_{c,j} + (1-\alpha)p_{c,k} > r_{c,k} = \alpha^2 r_{c,j} + (\alpha - \alpha^2)p_{c,j} + (1-\alpha)p_{c,k}. \tag{4.3.5}$$

Given the result in Equation 4.3.5, we know that subsequent reference prices will also be higher than they would be with a cycle including $p_{c,j}$. This means that in all cases the sum of negative reference losses without $p_{c,j}$, $\sum_{m=1}^{M-1}(r_{c,m} - \rho - p_{c,m})^- \leq \sum_{m=1}^{M-1}(r_{c,m} - \rho - p_{c,m})^-$, the sum of the reference losses with $p_{c,j}$. Therefore, in all cases the change in Component C ≥ 0. Given that Component C is, by definition, negative, we know that taking out the interim step leaves profit the same or higher.

Now we consider component A, the one in which the actual pricing took place. The step of eliminating the intermediate price would not change if $r_{c,j} - \tau - p_{c,k} \leq 0$, but would increase if $r_{cj} - \tau - p_{ck} > 0$. Thus, eliminating the interim price in which the reference threshold was not exceeded, we have increased profit. ∎

Proposition 4.3.3 *For $0 \leq \alpha \leq 1, \beta_L > \beta_G$, with loss thresholds and concave g and with a very long planning horizon T, $T \to \infty$, for $M > 1$, if*

$$(p_{c,m} - p_{c,(m+1)})\beta_G(r_{c,m} - \tau - p_{c,(m+1)}) + (p_{c,(m+1)} - c)\beta_G(\alpha(r_{c,m} - p_{c,m}) - \tau) < g(p^*) - g(p_{c,m})$$

for any $p_{c,m} > p_{c,(m+1)}$ there will be no price decrease from $p_{c,m}$ to $p_{c,(m+1)}$, $m - 1, \ldots, M$.

Proof. The proof builds on that for Proposition 4.3.2. We relax the assumption that $r_{c,j} - p_{c,j} < \tau$ so that the threshold may be exceeded in the interim step which is removed. Otherwise our specification of $p_{c,i} > p_{c,j} > p_{c,k}$ remains the same. The impacts on components B and C are identical. We now consider the impact on

Component A. With the interim price the reference gains are $(r_{c,,j} - \tau - p_{c,,j}) + (r_{c,k} - \tau - p_{c,k})$ if $(r_{c,,j} - \tau - p_{c,,j}) > 0$, and $(r_{c,k} - \tau - p_{c,k}) > 0$. Otherwise both are by definition no lower than zero.

Proposition 4.3.2 establishes that we will not consider price paths in which either of the components is less than zero. Without the interim step, the reference gains are $r_{c,,j} - \tau - p_{c,k}$. The difference between the two is the gain effect lost by eliminating the interim step. We find the difference in the reference gaps between the two scenarios (with and without interim step $p_{c,,j}$):

$$(r_{c,,j} - \tau - p_{c,,j}) + (r_{c,k} - \tau - p_{c,k}) - (r_{c,,j} - \tau - p_{c,k})$$

which then can be simplified:

$$\begin{aligned} r_{c,k} - \tau - p_{c,j} &= (\alpha r_{c,,j} + (1-\alpha)p_{c,,j} - \tau - p_{c,,j}) \\ &= \alpha r_{c,,j} - \alpha p_{c,,j} - \tau \\ &= \alpha(r_{c,,j} - p_{c,,j}) - \tau \end{aligned}$$

It is worth noting that $\alpha(r_{c,,j} - p_{c,,j}) - \tau$ may be negative depending on the size of τ. This is because the threshold must be covered twice for two steps and only once when the interim step is removed. Removing the interim step, $p_{c,k}$, reduces component B by $(g(p^*) - g(p_{c,j}))$. The amount lost from component A is then at least $(p_{c,j} - p_{c,k})\beta_G(r_{c,j} - \tau - p_{c,j}) + (p_{c,k} - c)\beta_G(\alpha(r_{c,j} - p_{c,j}) - \tau)$ where the second term may, in fact, be negative. If

$$(p_{c,j} - p_{c,k})\beta_G(r_{c,j} - \tau - p_{c,k}) + (p_{c,k} - c)\beta_G(\alpha(r_{c,j} - p_{c,j}) - \tau) < g(p^*) - g(p_{c,j})$$

for any $p_{c,k} < p_{c,j}$ then there will be no price decrease from $p_{c,j}$ to $p_{c,k}$. ■

Proposition 4.3.4 *For* $0 \leq \alpha \leq 1, \beta_L > \beta_G$, *with loss thresholds* ($\rho > 0$, *or* $\psi > 0$) *and concave* g *and with a very long planning horizon* T, $T \to \infty$, *for* $M > 1$ *and* $r_{c,m'} \leq p^*$, *no price increase will result in a price lower than* $\min(r_{c,m} + \rho, p^*)$, $m = 1, \ldots, M$.

Proof. Consider an optimal price path in which a price increase to price $p_{c,,j}$ falls below the minimum of $r_{c,,j} + \rho$ and p^* in the price path $p_{c,i} < p_{c,j} < p_{c,k}$. Suppose that $p_{c,k}$>min $(r_{c,j} + \rho, p^*)$. Now if we increase $p_{c,,j}$ to the minimum of $r_{c,,j} + \rho$ and p^*, we need to evaluate the impact on each of the components A, B and C. M will not change as we are not inserting or removing a price but rather just increasing $p_{c,,j}$. We have the following cases:

- Component A will increase. A higher reference price, r_{cj}, will result because p_{cj} is higher. Subsequent reference prices will also be higher. We know that $r_{c,t} - \tau - p_{c,t} > 0$ at least once or $M = 1$ and a constant price would be optimal.

- Component B will decrease. An increase in p_{cj} will decrease $g(p^*) - g(p_{c,j})$ due to convexity and the definition of p^*.

- Component C will stay the same or decrease. We remain within the loss threshold for the time period where we charge the higher $p_{c,,j}$. This also results in a higher reference price for the subsequent period. If $r'_{c,k} + \rho - p_{c,k} > 0$ then Component C will decrease. The same is true for subsequent price increases.

In all cases the increase to at least the minimum of $r_{c,j} + \rho$ and p^* increases average profit so the lower price change is not optimal. ∎

These propositions help to reduce the number of price paths that need to be considered in the dynamic programming formulation by defining the limits on potential price decreases with the price paths under consideration.

4.4 Computational Results

We implemented the dynamic program in MatLab. The linear demand function uses parameters $a = 1$, $b = 0.2$. We assume $c = 0.5$. In the absence of reference effects, the optimal price $p^* = \frac{a+bc}{2b} = 2.75$. We use these parameters as they are the ones used by Kopalle et al. [1996] and this allows us to both validate our computations against theirs but also to compare results of the threshold model more explicitly to previous reference price models. We evaluate the impact of changes in gain threshold (τ), loss threshold (ρ), gain and loss reference parameters (β_G, β_L) and memory parameter (α). We evaluate gain and loss thresholds from a low of 0.1 to a high of 0.5. This represents roughly the range of proportional thresholds found in the empirical literature (Gupta and Cooper [1992], Kalwani and Yim [1992]). We evaluate at $\alpha = 0.2$ and 0.35, the range which Mazumdar et al (2000) reported as being found in previous empirical work. Finally we evaluate values of 0.25, 0.2 and 0.1 for β_L and 0.2, 0.1 and 0.05 for β_G.

The dynamic programming formulation is solved with the aid of the propositions which reduce the search space. We are interested in the infinite horizon problem. We use an average profit criterion which means $\theta = 1$ and we treat every period equally and maximize the average profit per period. The pricing decision is made in each period with a view to the impact on future periods so we treat each period equally. We found that 25 periods are sufficient to provide insight into the steady state behaviour of pricing through repeating price cycles. We know that there will be at least one and likely several periods at the end of the horizon that result in price decreases as the future impacts of declining reference prices no longer matter. We also know there will be some adjustment at the beginning of the dynamic program until the price cycles and associated reference prices stabilize. While there is some excellent potential management insight in the price paths at the beginning of the planning horizon (such as the price strategy which optimizes profits in the face of a change in cost or demand) we are most interested in the long term pricing strategy that optimizes prices and, therefore, ignore the first 5 periods during a portion of which this adjustment takes place.

As per Proposition 3.3.1 and the findings of Greenleaf [1995] and Kopalle et al. [1996], we find that in the absence of thresholds:

1. for loss averse consumers a single price strategy is optimal and the single price is equal to p^*.

2. for loss seeking consumers a two period (M=2) cyclical price strategy is optimal.

We now consider the price paths resulting from the implementation of threshold models. Table 4.1 presents the results for one set of parameters. We highlight that the prices are presented in chronological order within an M period cycle. The remaining detailed results from all of the computations are presented in Appendix A.

Response to Loss Threshold

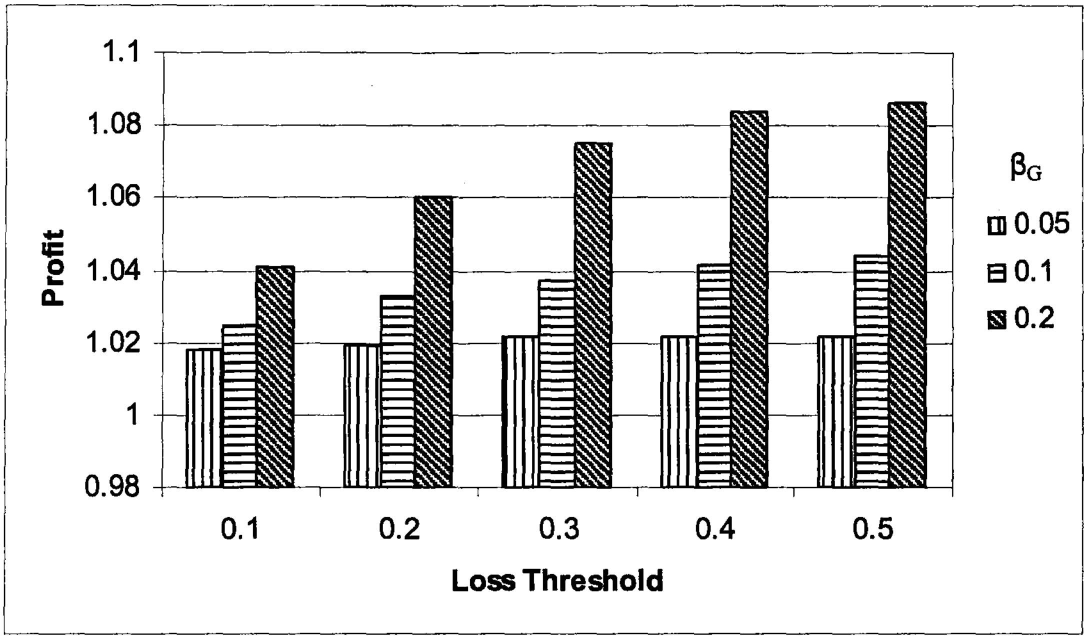
We showed with Proposition 4.2.1 that a loss threshold is required for a price cycle to exist. We also know that profit is non-decreasing in loss threshold. We see in Table 4.1 that profit increases as loss threshold increases. Figure 4.3 shows the results for different levels of gain parameter, β_G. The memory parameter for the profits shown in Figure 4.3 is $\alpha = 0.2$. As expected, as β_G increases the profit increases.

We also note that the increase is not linear. For $\beta_G = 0.05$, the profit gains over the lower bound is minimal and increases only occur until $\rho = 0.3$ at which further profit increases do not occur despite continued increases in threshold. The losses due to moving away from p^* overwhelm the benefit of the larger threshold and the associated gains due to reference effects. This represents meaningful managerial insight. The return to loyalty (in the form of a loss threshold) decreases as loyalty grows. There are diminishing returns to investment in loyalty. The degree to which further increases can be achieved depends on the value of β_L. Increases in profit due to price cycles occur because the gains achieved when the gain threshold is exceeded are greater than the losses due to prices different from p^*. We recall that p^* is the optimizer of $g(p)$ so that individual prices below p^* result in lower profits. A larger loss threshold requires a higher reference price and/or a more significant price decrease to achieve a difference sufficient to have positive reference gains. As a result, M must decrease and or the price variation must be larger to build a higher reference price. By decreasing M, there are fewer steps in the price increases and a higher reference price results. The phenomenon of shorter price cycles and wider price variation is

Table 4.1: Threshold Results

			$\beta_L = 0.25$, $\beta_G = 0.2$, $\alpha = 0.2$		
ρ	τ	M	Prices	Reference Prices	Average Profit
0.1	0	4	2.62, 2.78, 2.86, 2.94	2.92, 2.68, 2.76, 2.84	1.0410
0.2	0	3	2.60, 2.88, 3.04	3.00, 2.68, 2.84	1.0602
0.3	0	3	2.50, 2.92, 3.16	3.10, 2.66, 2.90	1.0752
0.4	0	3	2.44, 3.00, 3.32	3.24, 2.60, 2.92	1.0837
0.5	0	3	2.37, 3.07, 3.47	3.37, 2.57, 2.97	1.0862
0.2	0.1	4	2.46, 2.78, 2.94, 3.10	3.06, 2.58, 2.74, 2.90	1.0493
0.3	0.1	3	2.48, 2.90, 3.14	3.08, 2.60, 2.84	1.0620
0.5	0.1	3	2.35, 3.05, 3.45	3.35, 2.55, 2.95	1.0742
0.2	0.2	5	2.32, 2.68, 2,84, 3.00, 3.16	3.12, 2.48, 2.64, 2.80, 2.96	1.0390
0.3	0.2	5	2.28, 2.76, 2.80, 3.03, 3.28	3.22, 2.47, 2.70, 2.78, 2.98	1.0419
0.5	0.2	3	2.33, 3.03, 3.43	3.33, 2.53, 2.93	1.0623
0.2	0.3	5	2.28, 2.76, 2.80, 3.03, 3.28	3.22, 2.47, 2.70, 2.78, 2.98	1.0227
0.3	0.3	5	2.24, 2.73, 2.81, 3.03, 3.28	3.22, 2.44, 2.67, 2.78, 2.98	1.0349
0.5	0.3	3	2.31, 3.01, 3.41	3.32, 2.51, 2.91	1.0517

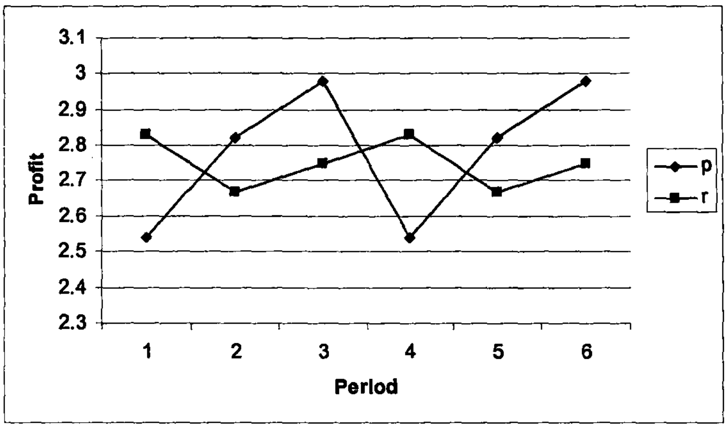
Figure 4.3: Profit as Loss Threshold Increases



demonstrated in Figures 4.4 and 4.5.

In Figure 4.5, there is a two period price cycle. The loss threshold is 0.5 so the high price (3.14) can be charged without incurring a reference loss. In fact, the reference gap in periods 2,4, and 6 is exactly 0.5 so no reference losses occur. For the quantitative model specified and the parameters used, there was never a case in which the loss threshold was exceeded but this need not be the case for all profit functions. The gain effect is then achieved every second period which increases profits relative to the case in Figure 4.4 where the loss threshold is 0.2. In the case where the loss threshold is smaller, two price increases are required to achieve the reference gain that maximizes profit. It is worth noting that for the quantitative model specified, there is always only a single price decrease. Increases in M are always related to the loss threshold and achieving the increase in reference price that optimizes the return

Figure 4.4: Price and Reference Price Cycles ($M = 3$, $\tau = 0.2$)



to a price decrease. This need not be the case for all profit functions.

4.4.1 Response to Gain Threshold

A gain threshold reduces the potential to cycle prices because it removes some of the gain that can result from lowering prices. A portion of the reference gap is lost due to the threshold. In the case where $\beta_G = 0.05$, a gain threshold as low as $\tau = 0.2$ (which represents less than 10% of p^*) precluded profitable price promotion cycles. It is also worth noting that there were no cases within the quantitative model and parameters used, that a value of gain threshold of $\tau = 0.2$ or greater led to a two price cycle. The impact of an increase in gain threshold is not easily predictable and depends on the other parameter values as can be seen in Table 4.2.

Figure 4.5: Price and Reference Price Cycle ($M = 2$, $\tau = 0.5$)

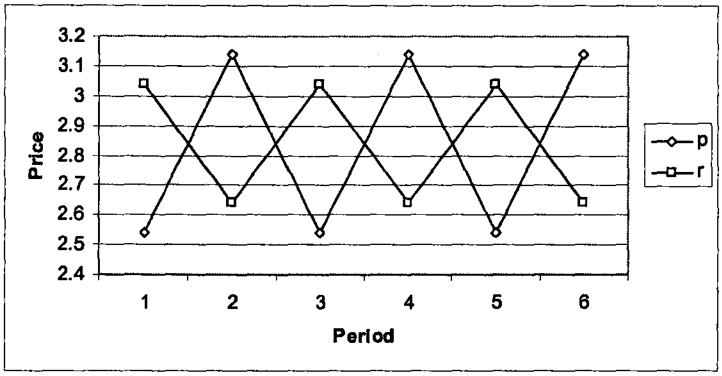


Table 4.2: Impact of Increasing Gain Threshold

$\beta_L = 0.2, \beta_G = 0.1, \alpha = 0.2$					
ρ	τ	M	Prices	Reference Prices	Average Profit
0.3	0	2	2.60, 2.96	2.90, 2.66	1.0373
0.3	0.1	3	2.45, 2.87, 3.11	3.05, 2.57, 2.81	1.0294
0.3	0.2	3	2.45, 2.87, 3.11	3.05, 2.57, 2.81	1.0230
0.3	0.3	3	2.50, 2.83, 3.00	2.96, 2.59, 2.78	1.0147

Given the parameters as specified, in the absence of a gain threshold, a two period cycle is optimal. As the gain threshold, τ, increases profit decreases. In cases with these parameters, M increases to three. For $\tau = 0.1$ the optimal solution involves an $M = 3$ price cycle with a wider spread between the maximum and minimum price and the intermediate price to preclude charging a price that creates a negative reference gap that exceeds the loss threshold. When $\tau = 0.2$, the optimal strategy is identical to that for $\tau = 0.1$. There is no opportunity for further profit increases due to a larger positive reference gap. Profit is lower as a portion of the positive reference gap is lost to the gain threshold. When $\tau = 0.3$, profit decreases more dramatically and the minimum price goes up and maximum price goes down. In this case, the available gain after the gain threshold is not sufficient to offset the profit lost due to lower minimum price and higher maximum price (further away from p^*). It is worth noting that for $\tau = 0.1$ and $\tau = 0.2$ the optimal strategies were identical but profits were lower in the case where gain threshold was bigger. This is because more of the reference gain is lost in the threshold. A wider price dispersion or a larger M would incur greater losses in profits (due to prices further from p^*).

4.4.2 Response to Loss Parameter

We recall that due to loss aversion, $\beta_L \geq \beta_G$, must be true in all cases. For this model and the parameters evaluated, changes in β_L do not affect the outcome because the loss threshold is never exceeded. This can be seen when comparing Table A4 with A6 and A5 with A7. The only difference between the tables is the value of the loss parameter, β_L, and the price path is identical in all cases with similar parameter values. It is worth highlighting that there may be cases in which the loss threshold is exceeded depending on the shape of $g(p)$ and the relative sizes of the gain and loss thresholds ρ and τ. In such a case, an increase in β_L may change the optimal price path by forcing an interim step that does not exceed the threshold.

4.4.3 Response to Gain Parameter

We know that profits are non-decreasing in gain parameter. While loss aversion requires that $\beta_L \geq \beta_G$, the response to changes in the gain parameter can still exist. The gain parameter, along with the thresholds, are the factors which most significantly drive the ability to increase profits by cycling price. A larger gain parameter may allow firms to use a wider price spread (which costs them profits on the $g(p)$ component as the prices move away from p^*) to get larger reference effects. If the price range is not wider, then the profit will be increased because the response to the reference gap is wider. Increased profits do not necessarily mean a reduction in M. The size of the loss threshold, ρ, may require that an interim price increase occurs over the wider price range. This can be seen in Table 4.3. The increases in profits are substantial given the increases in gain parameter used.

Table 4.3: Impact of Change in Gain Parameter

$\beta_L = 0.2, \alpha = 0.2$						
β_G	ρ	τ	M	Prices	Reference Prices	Average Profit
0.1	0.3	0	2	2.60, 2.96	2.90, 2.66	1.0373
0.2	0.3	0	3	2.50, 2.92, 3.16	3.10, 2.66, 2.90	1.0752
0.1	0.5	0.2	3	2.41, 2.83, 3.16	3.03, 2.55, 2.79	1.0233
0.2	0.5	0.2	3	2.33, 3.03, 3.43	3.33, 2.53, 2.93	1.0623

4.4.4 Response to Memory Parameter

We know that profit is non-increasing in α as the reference price responds more slowly to price increases and price decreases. This moderates the size of the reference gap. It appears, however, in this quantitative model, that the changes to the memory parameter have a relatively small impact on profit. We used values of α that were the two extremes of those found by previous empirical work. Table 4.4 highlights some key results. It appears that the impact on profit grows as gain threshold grows. This relates to the moderating impact on reference price of a larger memory parameter. A gain threshold remains constant (absolutely or relatively) and given a smaller reference gap, a more dramatic effect on profits results.

Table 4.4: Impact of Change in Memory Parameter

				$\beta_L = 0.25$, $\beta_G = 0.2$		
α	ρ	τ	M	Prices	Reference Prices	Average Profit
0.20	0.3	0	3	2.50, 2.92, 3.16	3.10, 2.66, 2.90	1.0752
0.35	0.3	0	3	2.44, 2.95, 3.14	3.04, 2.65, 2.85	1.0709
0.20	0.5	0.2	3	2.33, 3.03, 3.43	3.33, 2.53, 2.93	1.0623
0.35	0.5	0.2	3	2.31, 3.06, 3.41	3.23, 2.64, 2.91	1.0510

4.5 A Math Programming Approach

The previous work on determining optimal pricing paths (Greenleaf [1995], Kopalle et al. [1996]), used the dynamic programming formulation to find optimal results. This approach guarantees optimal results but requires significant time to complete the computations. Despite the computational improvements offered by Propositions 4.3.1 through 4.3.4, this has been a barrier to the implementation of this sort of optimization in a managerial decision making threshold pricing context. Advances in nonlinear integer programming approaches, solvers and computational power have made the specification of a nonlinear mixed integer programming threshold pricing model possible. Not only do nonlinear models provide the potential for reduced computation times, but they also provide a model which can incorporate constraints on resources. We have, therefore, explored the potential for developing a nonlinear integer programming optimization reference pricing threshold model.

We review first the basic reference problem without thresholds to provide an approximation of the reference price formation. We extend the formulation later to include thresholds and to solve the threshold problem. We recall that the basic reference problem is :

$$\max \pi = \sum_{t=1}^{T} (p_t - c) \left(a - bp_t + \begin{cases} \beta_G(r_t - p_t); \text{ if } r_t > p_t \\ \beta_L(r_t - p_t); \text{ if } r_t \le p_t \end{cases} \right)$$

subject to

$$\begin{aligned} r_t &= \alpha r_{t-1} + (1-\alpha)\, p_{t-1}, \text{ and} \\ c &\le p_t \le \frac{a + \beta_L r_t}{b + \beta_L}, t = 2, ..., T. \end{aligned}$$

where:

p_t is the price charged in period t (and is the decision variable),

c is a unit cost that is assumed to be constant,

r_1 is given.

We note that we need a large T, to allow for the adjustment period highlighted in Greenleaf [1995] and Kopalle et al. [1996]. For period T, the reference price formation will become:

$$\begin{aligned} r_T &= \alpha r_{T-1} + (1-\alpha) p_{T-1} \\ &= \alpha^{T-1} r_1 + \left(\alpha^{T-2} + \alpha^{T-1}\right) p_1 + ... + \left(\alpha - \alpha^2\right) p_{T-2} + (1-\alpha) p_{T-1} \\ &= \alpha^{T-1} r_1 + \sum_{i=0}^{T-2} (1-\alpha)\alpha^i p_{T-1-i} \text{ where } r_1 \text{ is given} \end{aligned} \tag{4.5.6}$$

We note that as $T \to \infty$, $r_T \to \sum_{i=0}^{\infty}(1-\alpha)e^{-(1-\alpha)i} p_{T-1-i}.$ The above expression for the reference price, together with the piece-wise reference price formulation, makes the objective function complex. Thus, we propose an approximation of the exponential smoothing reference price formation. We know from Mazumdar et al. [2005] that generally $0.2 \le \alpha \le 0.35$. In that case, the impact of past prices on current reference price diminishes very quickly. We propose, therefore, approximating the exponential smoothing reference price formation for $\alpha = 0.2$ using the following formula:

$$r_t = w_1 p_{t-1} + w_2 p_{t-2} + w_3 p_{t-3} \tag{4.5.7}$$

where $w_1 = 0.807$, $w_2 = 0.161$, and $w_3 = 0.032$. The weights are developed by rearranging the terms in Equation 4.5.6 with $\alpha = 0.2$. This yields

$$r_T = 0.8p_{t-1} + 0.16p_{t-2} + 0.032p_{t-3} + .0064p_{t-4} + ...0.2^{T-1}r_1.$$

The impact of the weights beyond the first three are very small. We standardize the first three weights so that they sum to one by dividing by 0.992, the sum of those first three weights, to get the approximate weights. The reference formations (4.5.7) works for all $t = 4, ..., T$. For the second and third periods the reference price formation would be:

$$\begin{aligned} r_2 &= w_1p_1 + (w_2 + w_3)r_1 = w_1p_1 + (1 - w_1)r_1 \\ r_3 &= w_1p_2 + w_2p_3 + w_3r_1 \end{aligned}$$

We can then formulate the threshold reference pricing problem as a non-linear mixed integer programming problem. We have

$$\max \pi = \sum_{t=1}^{T} [(p_t as - c)\,(a - bp_t + (r_t - p_t - \rho Y L_t + \tau Y G_t)(\beta_L Y L_t + \beta_G Y G_t))]$$

subject to:

$$\begin{aligned} r_t &= w_1p_{t-1} + w_2p_{t-2} + w_3p_{t-3} \;\forall t = 4, \ldots, T \\ r_2 &= w_1p_1 + (1 - w_1)r_1 \\ r_3 &= w_1p_2 + w_2p_3 + w_3r_1 \end{aligned} \tag{4.5.8}$$

$$\begin{aligned} (r_1 - p_1 - \tau)YG_1 &\geq 0 \\ ((1 - w_1)r_1 + w_1p_1 - p_2 - \tau)YG_2 &\geq 0 \\ (w_3r_1 + w_2p_1 + w_1p_2 - p_3 - \tau)YG_3 &\geq 0 \\ (w_3p_{t-3} + w_2p_{t-2} + w_1p_{t-1} - p_t - \tau)YG_t &\geq 0 \;\forall t, 4...T \end{aligned} \tag{4.5.9}$$

$$\begin{aligned}
(r_1 - p_1 - \tau)(1 - YG_1) &\leq 0 \\
((1 - w_1)r_1 + w_1 p_1 - p_2 - \tau)(1 - YG_2) &\leq 0 \\
(w_3 r_1 + w_2 p_1 + w_1 p_2 - p_3 - \tau)(1 - YG_3) &\leq 0 \\
(w_3 p_{t-3} + w_2 p_{t-2} + w_1 p_{t-1} - p_t - \tau)(1 - YG_t) &\leq 0 \ \forall t, 4 ... T
\end{aligned} \tag{4.5.10}$$

$$\begin{aligned}
(r_1 - p_1 + \rho)YL_1 &\leq 0 \\
((1 - w_1)r_1 + w_1 p_1 - p_2 + \rho)(1 - YL_2) &\leq 0 \\
(w_3 r_1 + w_2 p_1 + w_1 p_2 - p_3 + \rho)(1 - YL_3) &\leq 0 \\
(w_3 p_{t-3} + w_2 p_{t-2} + w_1 p_{t-1} - p_t + \rho)(1 - YL_t) &\leq 0 \ \forall t, 4 ... T
\end{aligned} \tag{4.5.11}$$

$$\begin{aligned}
(r_1 - p_1 + \rho)(1 - YL_1) &\geq 0 \\
((1 - w_1)r_1 + w_1 p_1 - p_2 + \rho)(1 - YL_2) &\geq 0 \\
(w_3 r_1 + w_2 p_1 + w_1 p_2 - p_3 + \rho)(1 - YL_3) &\geq 0 \\
(w_3 p_{t-3} + w_2 p_{t-2} + w_1 p_{t-1} - p_t + \rho)(1 - YL_t) &\geq 0 \ \forall t, 4 ... T
\end{aligned} \tag{4.5.12}$$

$$YL_t + YG_2 \leq 1 \ \forall t, 1 ... T$$

$$YL_t, YG_2 \in \{0, 1\} \ \forall t, 1 ... T$$

$$c \leq p_t \leq \frac{a + \beta_L r_t}{b + \beta_L}, t = 1, ..., T$$

The majority of the notation has been previously defined. We note that we introduce binary variables, YG_t and YL_t. If there is a gain in period t then YG_t is 1 and otherwise it is 0. Similarly, if there is a loss in period t then YL_t is 1 and otherwise it is 0. It is worth noting that if neither threshold is exceeded, both YG_t and YL_t can be zero. In this case, there is no reference effect as both β_L and β_G are multiplied by zero in the objective function. They cannot both be one.

The first set of constraints (4.5.8) establish the reference price based on the prices. The first three incorporate the original reference price and are different. The last one lags the last three prices weighted to determine the reference price as per the specification. We note that the reference price could easily be built into the objective function but we include it as a constraint for clarity in the objective function presentation.

The next group of contraints (4.5.9 and 4.5.10) establishes whether there is a gain in each individual period. If there is a gain (i.e., the gain threshold is exceeded, YG_t is one and τ and β_G are active in the objective function). We similarly establish the value for YL_t using.the next group of contraints (4.5.11 and 4.5.12).

We require the standard non-negativity constraints as well. This is easily implemented in GAMS. We optimize using the BARON solver, which was found to perform the best for this problem. This needs to run for at least $T = 15$ to provide effective insight into the price cycle because of the adjustments at the beginning and end of the time horizon from the reference effects. The reference price approximation works very well and the results are identical to the full enumeration results from the dynamic program in considerably less time. The time for the MINLP model in BARON was a matter of seconds versus hours or days for the GAMS complete ennumeration. We note that, the total search space for the GAMS model (without consideration for the propositions that reduce the search space) is Y^{15}, where Y is the number of discrete prices between c and the maximimum price constraint.

We evaluated smaller problems ($T = 6, 7, 8$) and BARON quickly found optimal solutions. BARON runs for hours to develop proven optimal solutions for $T = 15$ if time is not constrained. If we constrain the time to 1,000 seconds it finds the optimal solution (as verified by the complete enumeration). In every case in which the GAMS model was evaluated against the dynamic programming model, the optimal solution was found within the 1,000 second limit.

This approach works very well for the linear demand threshold problem that we assessed here and provides a practical approach to finding optimal solutions in a managerial context. The reference price approximation does not compromise the accuracy of the price estimation for the levels of α that are in the empirically validated range.

4.6 Summary and Conclusions

We introduce a reference model with thresholds based on empirical evidence in the marketing literature. The threshold model provides a more robust framework within which an optimal pricing strategy can be studied as it gives the opportunity for both single prices and cyclical strategies for loss averse consumers. This provides a theoretically sound foundation for both price cycles and single price steady state policies.

We develop two solution approaches to the problem and show that it can be effectively implemented to provide optimal price paths for managers. The approach using an approximation of reference price formation performs very well and effectively and efficiently provides optimal solutions to the problem within reasonable computation time. This clears a major hurdle for the effective implementation of this approach in a management context. We summarize the characteristics of the model and extract managerial insight relative to the sensitivity of pricing strategies to the specific parameters of the model.

Chapter 5

The Model with Stochastic Demand and Inventory Considerations

5.1 Introduction

The incorporation of demand uncertainty into optimal pricing strategy has been widely studied. Adding reference prices has the potential to provide additional insight into the stochastic demand pricing and inventory decision. Gimpl-Heersink et al. [2008] evaluate frameworks in which joint inventory and pricing decisions are made. They find that generally the cost in complexity is not worth the small incremental benefit in terms of profit relative to a sequential decision approach. They do highlight, however, that the incorporating reference prices the benefits that accrue to simultaneous decision analysis is substantial. This is what we do here.

Careful consideration must be given to pricing decisions in reference models as the inter-temporal effects of those decisions are important. The reference price (expected price) of a consumer is based on past prices and the optimal pricing and inventory decisions are based in part on reference price. Urban [2008] considers reference price in a stochastic context (and finds an effect relative to the non-reference model) but only considers a single period. This insight has value in the case of a product at the end of a season or at the end of the lifecycle of the product, but will provide suboptimal decisions for the current period as it ignores the impact on future profitability of a lower price (to take advantage of the reference gains) in the current period.

In this chapter we assess the impact of stochastic demand on pricing strategy in the context of a reference price model. To our knowledge there has been no research published in which reference models have included stochastic demand over multiple periods. Firms regularly face uncertain demand. They must choose production or ordering quantities and price which optimizes their expected demand. They face shortage costs if they do not have enough stock to meet demand and holding costs if they order/produce more stock than they can sell during the period. We do not include thresholds in the reference models with stochastic demand. As there has been no work to date on the pricing strategy with multiple periods and reference effects, we focus on highlighting the impacts of incorporating stochastic demand into a reference model. This allows us to specifically investigate the impact of stochastic demand on pricing strategy with reference prices. We acknowledge that it would be interesting in the future to investigate the interaction effects in the presence of reference price thresholds.

The inter-temporal price impacts add complexity to the problem as we do not optimize each period independently. We must determine the stocking factor for each potential price level first and then optimize the prices over the planning horizon. We are focused primarily on finding steady state pricing strategies but also evaluate the pricing strategy in the case where inventory at the beginning of a period exceeds the steady state stocking capacity required for that period. We offer a decision mechanism for determining the price that maximizes the expected profit as the reference price system returns to steady state.

The first section of this chapter outlines the basic stochastic demand model and its associated results. We introduce the mechanism for decision making when price is cycling. We then introduce the reference model with stochastic demand and the characteristics of that model. This is followed by some computational results to demonstrate the impact of different parameters and different distribution functions for the stochastic element of demand. The chapter closes with a summary and discussion of the results.

5.2 The Basic Stochastic Demand Model

We begin with the basic stochastic demand model (using the linear demand function). This is analogous to that presented in Petruzzi and Dada [1999] but using our notation. It serves as a foundation for the further development and introduces our notation explicitly. This model is necessary to lay the foundation for the new propositions we present and the addition of reference models subsequently. We acknowledge that the multiplicative demand case exists and merits consideration. As the reference model has not been evaluated in this context before, we choose to begin with the more tractable additive case to explore the implications of adding reference effects and stochastic demand.

We assume that expected demand in period t is:

$$D_t = a - bp_t + \epsilon$$

where ϵ is a random variable defined on range $[O, H]$ with probability distribution function $\phi(\cdot)$ and cumulative distribution function $\Phi(\cdot)$ and mean μ and variance σ^2. To avoid negative demand we require that $O > -a$. We define a stocking factor, z, a deterministic portion of demand, $y(p_t) = a - bp_t$, and a starting stock, q, such that

$$z_t = q_t - y(p_t).$$

If there is no initial inventory then the starting stock, q, is an order quantity. If there is an initial inventory, x, then the order quantity is $q-x$. In the single period problem we do not require the subscript t. Expected profit for the single period problem is, then:

$$\begin{aligned} E\left[\pi(z,p)\right] &= \int_O^z \left\{p\left(y(p)+u\right)-h(z-u)\right\}\phi(u)du \\ &+\int_z^H \left\{p\left(y(p)+u\right)-s(u-z)\right\}\phi(u)du - c(y(p)+z) \end{aligned}$$

where u is the realized demand, h is a disposal cost for excess units if $z > u$ and s is a shortage cost if $z < u$. We note that h can be negative if it represents a per unit salvage value. In that case, $c+h$ would represent the holding cost for the product for one period. The cost of acquisition is $cq = c(y(p)+z)$. Without loss of generality we will assume that $O = 0$. This maintains the non-negativity condition on demand:

$$p \leq \frac{\alpha+\beta_L r}{b+\beta_L}.$$

If we consider a demand function (D_t) in which $O \neq 0$.. If $O < 0$, we can do a simple transformation such that

$$D_t = a + G - bp_t + \epsilon - O.$$

which yields $a' = a + O$ and a new random variable $\epsilon' = \epsilon - O$ with $\sigma^{2\prime} = \sigma^2$, $\mu' = \mu - O$, $O' = O - O = 0$ and $H' = H - O$. The value of D_t does not change as we have done a simple linear transformation of ϵ with a similar transformation of a. We could do the same if $O > 0$ so lose no generality when we assume that $O = 0$.

We can then restate the expected profit function as:

$$E\left[\pi(z,p)\right] = (p-c)(y(p)+\mu)-(c+h)\int_0^z (z-u)\phi(u)du-(p-c+s)\int_z^H (u-z)\phi(u)du \tag{5.2.1}$$

Taking the first derivatives:

$$\begin{aligned}\frac{\partial E\left[\pi(z,p)\right]}{\partial p} &= a-2bp+bc+\mu-\int_{z}^{H}(u-z)\phi(u)du \\ \frac{\partial E\left[\pi(z,p)\right]}{\partial z} &= -(c+h)+(p+s+h)(1-\Phi(z))\end{aligned}$$

It is easily shown that this profit function is concave in z for a given p. That means for a given level of price we can solve for a unique z. This generates well known critical fractile result (see for example Porteous [2002]) which optimizes the newsvendor problem for a given p.

$$1-\Phi(z^{*}(p))=\frac{c+h}{p+s+h}$$

The critical fractile result is used when p is fixed. In this case, p is a choice variable so we can determine a value of z for each value of p and then search for the optimal price given optimal stocking factor, z. This can be complex depending on the specific characteristics of the expected profit function. In our dynamic programming formulation we search across all feasible prices.

We can also use the fact that the profit function is concave in p for a given z to validate the result of Zabel [1972] which finds the optimal value of p for a given level of z. We derive:

$$\begin{aligned}p^{opt}(z) &= \frac{a+bc+\mu-\int_{z}^{H}(u-z)\phi(u)du}{2b} \\ &= p^{*}+\frac{\mu-\int_{z}^{H}(u-z)\phi(u)du}{2b} \qquad (5.2.2)\end{aligned}$$

where p^* is the optimimizer of $(p-c)y(p)$. Petruzzi and Dada [1999] present a theorem for determining the optimal z for a given level of p. We can then find p^{opt} for each given level of z and optimize the expected profit, $E\left[\pi(z,p(z)\right]$.

We adopt this approach: we determine the optimal value of z, the stocking factor, for each possible price level and then determine the optimal price level over time. The inter-temporal connection between prices in the reference model, and the associated impact on z , and therefore expected profit, preclude the determination of prices and then the optimal stocking factor.

It is, however, clear that, even in the absence of reference effects, we need to consider the implications of the current period's pricing decision in future periods. We must, therefore, consider multi-period models in order to get optimal price paths.

5.3 The Multi-Period Model

The determination of optimal price paths in a dynamic context is complex. We know that, at the start of any period we will have inventory if in the previous period the stocking factor, $z_{t-1} > u_{t-1}$. If $z_{t-1} \leq u_{t-1}$, the ending inventory is zero and we do not need to consider the inventory effects. The general result presented most often is that when disposal is costly, the optimal quantity for a period is (see Petruzzi and Dada [1999]):

$$q_t^* = \begin{cases} q(z_t, p_t); \text{ if } q(z_t, p_t) \geq x_t \\ x_t; \text{ otherwise} \end{cases}$$

where x_t is the starting inventory in any given period. The optimal price then, under this situation, is:

$$p_t^* = \begin{cases} p_t^*; \text{ if } q(z_t, p_t) \geq x_t \\ p_t^{x_t}; \text{ otherwise} \end{cases} \tag{5.3.3}$$

where p_t^* and z_t is price which maximize expected profit (Equation 5.2.1) and $p_t^{x_t}$ is the value of p that optimizes the expected profit when $z_t(x_t, p_t)$ is bound by the constraint $q_t^* = x_t$. The work of Ernst [1970], Thowsen [1975], and Zabel [1972] all explore sufficient conditions for unique solutions to this problem. Ernst was first and the others built on his results. Under select assumptions, they find that if $\Phi(\cdot)$ is from the PF_2 family of distributions (Polya frequency function of order 2) there is a unique solution. These multiple period sufficient conditions become very complex and little recent work has explored this further.

In the case where we have to make pricing decisions over multiple periods we first need to consider the linkages between periods. We take the first step here in the form of inventory. We consider the inter-temporal price effects in the next section when we discuss the reference model with stochastic demand. The multi-period expected profit function would be:

$$\begin{aligned} E\left[\pi_T(z_t, p_t)\right] &= \sum_{t=1}^{T}\{(p_t - c)(y(p_t) + \mu) \\ &\quad -(c+h)\int_0^{z_t}(z_t - u)\phi(u)du - (p_t - c + s)\int_{z_t}^{H}(u - z_t)\phi(u)du\}. \end{aligned} \quad (5.3.4)$$

It is clear that this expected profit function does not include an explicit linkage for carryover of either shortages or excess inventory. We have to account for this. If we have insufficient stock to meet all of the demand we will incur a shortage cost regardless of whether we allow backorders or not. The shortages only have relevance in the subsequent period if backorders are allowed and the demand must be met later. On the other hand, if we have excess demand and can carry the inventory forward (i.e. the product is not perishable) we may have an opportunity to sell the excess inventory in the subsequent period. In any given period the shortages would be $SH_t = (z_{t+1} - \epsilon_t)^-$ and the excess inventory would be $EX_t = (z_{t+1} - \epsilon_{t+1})^+$. We recall that ϵ_t is the random portion of demand. The profit function for any single period, t, with starting inventory x_t, would then be:

$$\begin{aligned} E\left[\pi_t(x_t, z_t, p_t)\right] &= (p_t - c)(y(p_t) + \mu) \\ &\quad -(c+h)\int_0^{z_t}(z - u)\phi(u)du - (p_t - c + s)\int_{z_t}^{H}(u - z_t)\phi(u)du \\ &\quad +\theta\int_0^{z_t} E\left[\pi_{t+1}\left(x_{t+1} = EX_t, z_{t+1}, p_{t+1}\right)\right]\phi(u)du \\ &\quad +\theta\int_{z_t}^{H} E\left[\pi_{t+1}\left(x_{t+1} = SH_t, z_{t+1}, p_{t+1}\right)\right]\phi(u)du \end{aligned} \quad (5.3.5)$$

Equation 5.3.4 includes shortage costs for every period t. If we assume no backorders, then we incur the shortages costs in the current period, t, and simply start the period $t+1$ fresh. For frequently purchased products like groceries, it is reasonable to assume no back orders. Petruzzi and Dada [1999] also note that the case with no backorders is the predominant one in the literature. We then do not need to include any inter-temporal effects of shortages. Therefore, we can eliminate the term in Equation 5.3.5 that relates to shortages $\left[\int_{z_t}^{H} E\left[\pi_{t+1}\left(x_{t+1} = SH_t, z_{t+1}, p_{t+1}\right)\right]\phi(u)du\right]$.

If there are no carryovers (i.e. all ending inventory is lost or salvaged) we can eliminate the term that relates to excess inventory in Equation 5.3.5. It is not as reasonable to assume that all of the excess inventory is lost in the case of frequently purchased products such as groceries. We may have an issue of spoilage (i.e. some of the excess inventory is lost) which adds further complexity. The cost of spoilage can be reflected in the value of h and could also be reflected in a transfer factor between periods. We also note that if $EX_t < z_{t+1}$ in all cases, we can also eliminate the excess inventory term from Equation 5.3.5. This leaves us with the more tractable expected profit function in Equation 5.3.4. The carrying cost $(c + h)$, in the case where h is negative is assigned to period t and the acquisition cost c is transferred to period $t + 1$. We can show that under the relatively mild assumption that the stochastic portion of demand will not be larger than the deterministic one, that $x_t \leq q_t$.

Proposition 5.3.1 *If $H \leq a - bp^{\max}$ (where $p^{\max}$ is the maximum price that occurs in an M period cycle) then $x_t < q_t$ $\forall t$.*

Proof. We know that $q_t = y(p_t) + z_t$ and that $z_t \in [0, H]$ (we assume that $0 = 0$). We also know that $x_t < q_{t-1}$ that is the quantity carried forward in one period cannot be larger than the optimal quantity stocked in previous period. Since $0 \leq z_t \leq y(p^{\max})$ $\forall t$, it follows that $x_t \leq y(p^{\max}) \leq q_t$ $\forall t$. ■

Therefore, in cases where the condition is met that the range of the random variable ϵ is smaller than the smallest deterministic demand $(y(p^{\max}))$, we can always know that the inventory carried forward from the current period will always be smaller than the total inventory required for subsequent period. This makes no assumption at all on what portion of inventory can be carried forward. In this case we can ignore the amount carried over as it will always be smaller than the amount we need, even in the case of a price cycle as long as our original condition is met. We can, therefore, maximize the profit function as represented in Equation 5.3.4.

There is also some question as to whether the assumption of costly disposal is reasonable. This is particularly important in the multi-period case. In many cases, the retailer can carry stock over to subsequent periods. In fact, one might argue that the assumption of costly disposal makes most multi-period analyses moot. It essentially says that inventory can be carried from one period to the next but not to subsequent periods. This may make sense for some products that expire but there are also such inventory policies as first in first out that mean that older stock is used first.

Petruzzi and Dada [1999] provide a general discussion of a multi-period case in which the costly disposal assumption is relaxed. In this case, inventory carried forward need not be disposed of and need not all be discounted. In the stationary demand case, they suggest that $q_t^* \geq x_t$ in all cases. This is true as the order quantity will be the same as long as demand errors are identically and independently distributed and none of the demand parameters change (the stationarity assumption). Even if demand in period $t-1=0$, the order quantity would be zero and there would be no case in which a suboptimal price would have to be charged to get rid of excess inventory (i.e. $.q(z_t, p_t) \geq x_t$ in all cases), In this case, the optimal policy is identical to the single period one and the solution approaches are the same as before. This is true for stationary demand but does not hold if price cycles. If $p_t > p_{t-1}$ in the optimal price path, it is possible that excess inventory will result and that the inventory will exceed the optimal quantity required for this period's price.

Petruzzi and Dada [1999] provide a brief reference to the initial conditions in which $x_t > q_1^*$ and suggest that a temporary sales price could be established to dispose of the excess inventory. They suggest that this would be preferable to using the disposal pricing strategy in 5.3.3. This may or may not be practical depending on the length of the planning horizon and the specific retailer conditions. We offer a more formal pricing policy but first formalize the assumptions and notation. We assume stationary demand and constant h, s, and c for the planning horizon and additive and independent and identically distributed demand between periods. We note that the condition where $x_t > q_1^*$ can only happen at the beginning of the planning horizon but a demand shift or parameter change may lead to this condition as well. It is, therefore, not a trivial problem.

We note that the assumption that any inventory remaining at the end of a planning horizon can be carried forward suggests that h is negative (a salvage value rather than disposal cost) but that $h \geq -c$. The product can not be worth more at the end of the period than it was at the beginning. In this case, $c + h$ would reflect a holding cost for inventory for the period. We also assume that unmet demand is lost (i.e. not deferred). If $s < p_t - c$, there may be some switching to alternatives but we do not have backlogs.

We now present a proposition which provides an optimal price for a period in which starting inventory exceeds the optimal stocking quantity in the case where there is no costly disposal.

Proposition 5.3.2 *For multi-period stationary linear demand with additive and iid error, if $x_t > q_t^*$, the optimal pricing strategy for a period is:*

$$p_t^* = \max\left\{p_t^{x_t}, p^* + \frac{\mu - (c+h) - \int_z^H (u-z)\phi(u)du}{2b}\right\}$$

where $p_t^{x_t}$ is defined as above in Equation 5.3.3.

Proof. We know by definition that $p_t^{x_t}$ is the price that we would charge if we impose the binding constraint $q_t^* = x_t$ on the optimization of expected profit. $\frac{\partial E[\pi(z_t,p_t)]}{\partial p_t}$ is the marginal expected profit with respect to a change in the price. If we choose $p_t^{x_t}$ when $\frac{\partial E[\pi(z_t,p_t)]}{\partial p_t}|p_t = p_t^{x_t} < c + h$ expected profit would be lower than optimal as we could plan to store some of the inventory and do better. Charging $p^* + \frac{\mu-(c+h)-\int_z^H(u-z)\phi(u)du}{2b}$ ensures that the target sales will be those units for which the marginal profit lost is less than the cost of holding them. We recall that p^* is the maximizer of the riskless profit function. In this case, charging $p^* + \frac{\mu-(c+h)-\int_z^H(u-z)\phi(u)du}{2b}$ minimizes the holding cost by only carrying over those units for which the marginal profit would have been less than the cost of carrying them. If $\frac{\partial E[\pi(z_t,p_t)]}{\partial p_t}|p_t = p_t^{x_t} \geq c+h$, then we know that for all units, the cost of storing the additional inventory for one period exceeds the loss in expected profit by charging a price expected to clear that inventory. ■

The incorporation of the potential to choose to charge an intermediate price between $p_t^{x_t}$ and p_t^*, if $q(z_t, p_t) \geq x_t$, adds complexity to the determination of optimal values of p and z. The approach outlined above (from Petruzzi and Dada [1999]) in which an optimal z is determined for each level of price and then an optimal price is found would work well in this context. One would first determine p_t^* from Equation 5.3.3. If $q_t^* = x_t$, (i.e.$p_t^{x_t}$ is chosen) then we would evaluate if $\frac{\partial E[\pi(z_t,p_t)]}{\partial p_t}|p_t = p_t^{x_t} < c+h$. If $\frac{\partial E[\pi(z_t,p_t)]}{\partial p_t}|p_t = p_t^{x_t} < c+h$, then we use $p_t^{x_t}$ and p_t^* as bounds and evaluate prices to determine $p^* + \frac{\mu-(c+h)-\int_z^H(u-z)\phi(u)du}{2b}$.

5.4 The Reference Model with Stochastic Demand

We now consider the reference model with stochastic demand. The demand function is:

$$D_t = \begin{cases} a - bp_t + \beta_G(r_t - p_t) + \epsilon; \text{ if } r_t > p_t \\ a - bp_t + \beta_L(r_t - p_t) + \epsilon; \text{ if } r_t \leq p_t \end{cases} \tag{5.4.6}$$

$$c \leq p_t \leq \frac{a+\beta_G r_t}{b+\beta_G}; \text{ if } a - br_t \leq 0; \text{ else } c \leq p_t \leq \frac{a+\beta_L r_t}{b+\beta_L}$$

We define stochastic demand $y(p_t, r_t) = D_t + \epsilon_t$ where ϵ_t is the random component of demand with mean, μ. The single period expected profit function is the same as Equation 5.2.1 except that the reference effects are included.

$$E[\pi_t(r_t, z_t, p_t)] = (p_t - c)(y(p_t) + \mu) - (c+h)\int_0^z (z-u)\phi(u)du - (p-c+s)\int_z^H (u-z)\phi(u)du \tag{5.4.7}$$

with $y(p_t)$ representing the deterministic part of the demand function in Equation 5.4.6. Taking the first derivative we get:

$$\frac{\partial E\left[\pi(r_t, z_t, p_t)\right]}{\partial p_t} = \begin{cases} a - 2b + \beta_G(r_t - p_t) + \mu - \int_z^B (u - z_t)\phi(u)du; \text{ if } r_t > p_t \\ a - 2b + \beta_L(r_t - p_t) + \mu - \int_z^B (u - z_t)\phi(u)du; \text{ if } r_t \leq p_t \end{cases}$$

$$\frac{\partial E\left[\pi(r_t, z_t, p_t)\right]}{\partial z_t} = -(c+h) + (p_t + s + h)(1 - \Phi(z_t)).$$

It is easily shown that this profit function is concave in z_t for a given p_t and a unique $z_t^*(p_t)$ and maximum profit exists. We also note that the change in derivative of expected profit with respect to z_t is independent of r_t. This is also true if we incorporate multiple periods into the model. In fact the period specific stocking factor, z_t, is independent of reference price and, thus, has no inter-temporal ties. None of the elements of the first derivative of expected profit with respect to z_t, with the exception of price, is period specific. We can solve for the optimal z_t for a given p_t and once again generate the critical fractile result for each level of p_t.

The forward recursion for the multi-period problem is:

$$f_t(r_t) = \max_{p_t \in J_t} \left[\theta^{t-1} E\left[\pi_t(r_t, z_t, p_t)\right] + f_{t-1}(r_{t-1})\right]. \tag{5.4.8}$$

where:

$t = 1, 2, 3, .., T$;

$r_t = \alpha r_{t-1} + (1 - \alpha)p_{t-1}, t = 2, .., T$, and r_1 is given;

$x_t = 0$;

$f_0 = 0$;

J_t is a vector of discrete prices which goes from c to $\frac{a+\beta_L r_t}{b+\beta_L}$ in increment δ;

p_t is the discrete price in period t from vector J_t;

θ is a discount factor and

$E\left[\pi_t(r_t, z_t, p_t)\right]$ is the expected profit function as specified in Equation 5.4.7.

We highlight again that the use of a forward recursion instead of a backward recursion is in the interest of computational tractability. Given that reference price is the state variable and the particular state depends on the previous prices charged, it is more manageable to search the prices forward to optimize the average profit per period. We also reiterate that we use $\theta = 1$ for all of our models to develop an average expected profit per period to optimize steady state pricing strategies.

Each potential value of p_t has a unique value of z_t which satisfies

$$1 - F(z_t) = \frac{c+h}{p_t + s + h}. \tag{5.4.9}$$

As we discussed in the previous section, we assume that there are no backorders and that the inventory carried over is less than or equal to the optimal quantity stocked ($x_t \leq q_t^*$).

We also recall that in the case of excess inventory at the start of a planning horizon or if demand parameters change, we might have the situation in which $x_t > q_t^*$. We can consider the case at the beginning of a planning horizon in which $x_t > q_t^*$ is more complex than in Proposition 5.3.2 because the inter-temporal reference effects will now be a factor and we will have to decide whether to hold some inventory rather than discount to clear the inventory out. If we charge a price $p_t < p^*$ we will have $r_{t+1} = \alpha p^* + (1-\alpha)p_t$. If we charge p^* in period $t+1$, we will incur a reference loss of $(p^* - c)\beta_L(r_{t+1} - p^*)$. The reference losses will, however, not stop there but continue until $r_t = p_t^*$ again. We require an approach to deciding on the price. Before presenting the proposition, in the case where $q_1^* < x_1$ we define q_1^{**} such that $q_1^* < q_1^{**} < x_1$ and p_1^{**} is the price associated with the optimal path that follows given $q_1 = q_1^{**}$. We also define $q_1^{x_1}$ and the associated price, $p_1^{x_1}$, as the quantity and price that arise when the binding constraint $q_1 = x_1$ is imposed. We also reiterate that p_1^* is the price that results if the optimal quantity $q_1^* > x_1$.

Proposition 5.4.1 *For multi-period stationary demand with reference effects and additive iid error, if* $x_1 > q_1^*, \beta_L, \beta_G < b$ *and concave* $g(p)$*, the optimal pricing strategy is chosen such that* $(c+h)(x_1 - q_1^{**}) = f_1(r_1)\mid_{z_t, p_1 = p_t^*} - f_t(r_1)\mid_{p_1 = p_1^{**}}$.

Proof. We can show that for a horizon of any length T, that $f_1(r_t)$ is concave in price using an approach analogous to that in Proposition 3.4.1. Total expected profit at any time given x_1 is $f_t(r_1)\mid_{p_1=p_1} - (c+h)(x_1 - q_1)$. We know when $p_1 = p_1^*$,

$(c+h)(x_1 - q_1^*) \geq 0$ and that $f_t(r_1)\mid_{z_t, p_1 = p_t^*} - f_t(r_1)\mid_{p_1 = p_1^{**}} = 0$. We note that $(c+h)(x_1 - q_1) = 0$ if $(c+h) = 0$ (that is the cost of carrying the inventory forward into the next period is zero) and it is optimal to charge p_1^{xt}. As price decreases, $(c+h)(x_1 - q_1)$ the expected carrying cost, decreases but $f_t(r_1)\mid_{p_1 = p_t^*} - f_t(r_1)\mid_{p_1 = p_1}$, the loss of expected profit increases. When $p_1 = p_1^{xt}$, $(c+h)(x_1 - q_1^{xt}) = 0$, but due to the concavity of $f_t(r_t)$ we have the largest expected profit loss $f_t(r_t)\mid_{p_1 = p_t^*} - f_t(r_1)\mid_{p_1 = p_1^{xt}}$. That is $f_t(r_1)\mid_{p_1 = p_1^{xt}} \leq f_t(r_1)\mid_{p_1 = p_t^{**}}$ for all $p_1^{xt} < p_1^{**} < p_1^*$. It is easy to see, then, that $f_t(r_1)\mid_{p_1 = p_1} - (c+h)(x_1 - q_1)$ is maximized when $(c+h)(x_1 - q_1^{**}) = f_t(r_1)\mid_{p_1 = p_t^*} - f_t(r_1)\mid_{p_1 = p_1^{**}}$. ■

It is easy to see that when carrying cost $(c+h)$ is high we set a price close to p_1^{xt} and when carrying cost is low we will set a price close to p_1^*. The optimal pricing path could be calculated for all potential values of p_1^{**} and q_1^{**}. The level at which the marginal cost of carrying the inventory that reduces the expected profit by exactly the amount that it costs to carry would be the choice of price in this case. It would never be lower than p_1^{xt} or higher than p_1^*. We highlight the fact that we may not actually carry all of the inventory $(x_1 - q_1^{**})$ for the entire period as demand is variable. We simply pick a price and stocking factor that does not force us to plan to sell all of it at a price that would reduce our expected profit given shortage and holding costs.

5.4.1 The Loss Averse Case

We consider first the case where consumers are assumed to be risk averse. We make the same assumptions about demand and the stationarity of the demand parameters as we did in the previous section. We first show that, for expected demand functions for which there is a single optimal value of z_t and p_t, a single price is optimal. Ernst [1970] established that a single z_t and p_t were optimal when the distribution for ϵ was a member of the PF_2 family of distributions. The work of Petruzzi and Dada [1999] established less restrictive conditions under which this would still be the case.

Proposition 5.4.2 *For $0 \leq \alpha \leq 1$, $0 \leq \theta \leq 1$, $\beta_L \geq \beta_G$ with a very long planning horizon T, $T \to \infty$, and unimodal z_t non-reference component of the repeating single period expected profit function, g, then a single price is optimal and the optimal price is p^{opt}, the optimizer of g.*

Proof. The proof follows directly from that of Proposition 3.3.1 due to the unimodality in z of g. ■

We know from Proposition 5.4.2 that for $\beta_L > \beta_G$ (the loss averse case) a single price is optimal. In that case the optimal z and p are:

$$1 - \Phi(z_t^*) = \frac{c+h}{p_t + s + h}$$

$$\begin{aligned} p_t^{opt} &= p(z_t) = \frac{a + bc + \mu - \int_z^B (u-z)\phi(u)du}{2b} \\ &= p^* + \frac{\mu}{2b} - \frac{\int_z^B (u-z)\phi(u)du}{2b} \end{aligned} \tag{5.4.10}$$

The value of p^* is

$$p^* = \frac{a+bc}{2b}$$

which is the maximizer of the non-reference component of the profit function, $g(p_t)$. The values of β_L, β_G and r_t disappear because with a single price there are no reference effects. In this case, then, the long term price and strategy is identical to the case where there are no reference effects. In this case $r_t = p_t^{opt}$ for all t. It is also analogous to the non-reference case in that once the steady state has been attained, there will never be a case where beginning inventory exceeds q_t^*. We also note that at the end of the product lifecycle, when we are no longer concerned about the impact of pricing decisions on future demand, the price decreases and we will also never have a case in which $x_t > q_t^*$. We could calculate z_t for all possible levels of p_t and use the recursion approach to determine the optimal price path.

5.4.2 The Loss Seeking Case

We now consider the loss seeking case. We offer a Proposition which suggests that, under the condition of the unimodality of the expected profit function in z_t, the optimal price cycles in the loss seeking case with stochastic demand.

Proposition 5.4.3 *For $0 \leq \alpha \leq 1$, $0 \leq \theta \leq 1$, $\beta_L \geq \beta_G$ with a very long planning horizon T, $T \longrightarrow \infty$, and unimodal in z_t non-reference component of the repeating single period expected profit function, a single price is not optimal.*

Proof. Given our assumption of unimodality, we have a single optimal z_t and can easily show that the non-reference component of demand is concave in price for the specific z_t. Now consider the optimal price in the absence of reference effects, p^{opt} as in Equation 5.4.10. We follow an approach similar to that for Proposition 4.2.1. If we specify an $M = 2$ period cycle in which we subtract a small increment, δ, from p^{opt} for one price and add the increment to p^{opt} for the second price Once again we use $r_{c,i}$ and r_{ci} interchangeably. We specify:

$$p_{c,1} = p^{opt} + \delta$$

and

$$p_{c,2} = p^{opt} - \delta.$$

The reference prices are:

$$r_{c,2} = \alpha r_{c1} + (1 - \alpha)p_{c1}$$

and

$$r_{c,1} = \alpha r_{c2} + (1 - \alpha)p_{c2}.$$

Solving for r_{c1} and r_{c2} in the above system of equations we find

$$r_{c,1} = p^{opt} - \frac{(1 + \alpha^2 - 2\alpha)\delta}{(1 - \alpha^2)}$$

and

$$r_{c,2} = p^{opt} + \frac{(1 + \alpha^2 - 2\alpha)\delta}{(1 - \alpha^2)}.$$

For $0 \le \alpha \le 1$,

$$\frac{(1+\alpha^2-2\alpha)}{(1-\alpha^2)} \le 1.$$

For the single price strategy to be optimal we need to have

$$2g(p^{opt}) \ge g(p^{opt}+\delta)+g(p^{opt}-\delta)+(p^{opt}+\delta-c)\beta_L(r_{c,1}-p^{opt}-\delta)+(p^{opt}-\delta-c)\beta_G(r_{c,2}-p^{opt}+\delta)$$

$$\begin{aligned}
2g(p^{opt})-g(p^{opt}+\delta)-g(p^{opt}-\delta) &\ge (p^{opt}+\delta-c)\beta_L(p^{opt}-\frac{(1+\alpha^2-2\alpha)\delta}{(1-\alpha^2)}-p^{opt}-\delta) \\
&\quad +(p^{opt}-\delta-c)\beta_G\left(p^{*}+\frac{(1+\alpha^2-2\alpha)\delta}{(1-\alpha^2)}-p^{*}+\delta\right)
\end{aligned}$$

$$\begin{aligned}
2g(p^{*})-g(p^{*}+\delta)-g(p^{*}-\delta) &\ge (p^{opt}+\delta-c)\beta_L\delta\left(\frac{(1+\alpha^2-2\alpha)}{(1-\alpha^2)}-1\right) \\
&\quad +(p^{opt}-\delta-c)\beta_G\delta\left(\frac{(1+\alpha^2-2\alpha)}{(1-\alpha^2)}+1\right)
\end{aligned}$$

$$\begin{aligned}
\frac{g(p^{*})-g(p^{*}+\delta)+g(p^{*})-g(p^{*}-\delta)}{\delta} &\ge (p^{opt}-\delta-c)\beta_G\left(\frac{(1+\alpha^2-2\alpha)}{(1-\alpha^2)}+1\right) \\
&\quad -(p^{opt}+\delta-c)\beta_L\left(\frac{(1+\alpha^2-2\alpha)}{(1-\alpha^2)}+1\right)
\end{aligned}$$

now we take the limit as $\delta \to 0$ and since g is concave

$$\begin{aligned}
0 &\ge (p^{opt}-c)\beta_G\left(\frac{(1+\alpha^2-2\alpha)}{(1-\alpha^2)}+1\right)-(p^{opt}-c)\beta_L\left(\frac{(1+\alpha^2-2\alpha)}{(1-\alpha^2)}+1\right) \\
0 &\ge (\beta_G-\beta_L)(p^{opt}-c)\left(\frac{(1+\alpha^2-2\alpha)}{(1-\alpha^2)}+1\right)
\end{aligned}$$

which is impossible as the right hand side is always positive as $\beta_G > \beta_L$. The gain effect outweighs the loss effect. Thus for loss seeking consumers with stochastic demand, the optimal price always cycles. ■

We highlight again that this does not mean that a two period cycle is optimal but that a single price is not optimal. The general results outlined below do not depend on two phase pricing. The problem can be solved with the dynamic programming recursion outlined in Chapter 3. We evaluate analytically the special case of the loss seeking case in which the optimal approach is a two phase cycle to provide some basic analytical results to compare to the $M = 2$ case without a stochastic element.

We consider the expected profit function with an additive stochastic component, Equation 3.4.11 specified a two stage profit function without stochastic elements. We can add the stochastic components and the resulting expected profit function is:

$$\begin{aligned}
\pi_{M=2} &= (p_1 - c)(a - bp_1 + \mu + \beta_L(w_1p_2 + w_2p_1 - p_1)) \\
&\quad -(c+h)\int_0^{z_1}(z_1 - u)\phi(u)du - (p_1 - c + s)\int_{z_1}^{H}(u - z_1)\phi(u)du \\
&\quad +(p_2 - c)(a - bp_2 + \mu + \beta_L(w_1p_1 + w_2p_2 - p_2) \\
&\quad -(c+h)\int_0^{z_2}(z_2 - u)\phi(u)du - (p_2 - c + s)\int_{z_2}^{H}(u - z_2)\phi(u)du
\end{aligned}$$

The first order conditions are:

$$\begin{aligned}
\frac{\partial \pi}{\partial p_1} &= a + \mu - 2(b + \beta_L w_1)p_1 + (b + \beta_L w_1 - \beta_G w_1)c + \\
&\quad (\beta_L + \beta_G)w_1p_2 - \int_{z_1}^{B}(u - z_1)\phi(u)du \\
\frac{\partial \pi}{\partial p_2} &= a + \mu - 2(b + \beta_G w_1)p_2 + (b + \beta_G w_1 - \beta_L w_1)c + \\
&\quad (\beta_L + \beta_G)w_1p_1 - \int_{z_2}^{B}(u - z_2)\phi(u)du
\end{aligned}$$

We now get

$$\begin{aligned}
p_1 &= \frac{a + \mu + (b + w_1(\beta_L - \beta_G))c + (\beta_L + \beta_G)w_1p_2 - \int_{z_1}^{B}(u - z_1)\phi(u)du}{2(b + \beta_L w_1)} \\
p_2 &= \frac{a + \mu + (b + w_1(\beta_G - \beta_L))c + (\beta_L + \beta_G)w_1p_1 - \int_{z_2}^{B}(u - z_2)\phi(u)du}{2(b + \beta_G w_1)}
\end{aligned}$$

if we define

$$\lambda_1 = \int_{z_1}^{B} (u - z_1)\phi(u)du$$
$$\lambda_2 = \int_{z_2}^{B} (u - z_2)\phi(u)du$$

then

$$\begin{aligned} p_1^{opt} &= \frac{h_1 + \mu + \beta w_1 p_2 - \lambda_1}{2(b + \beta_L w_1)} \\ &= \frac{2(b + \beta_G w_1)h_1 + 2(b + \beta_G w_1)\mu + \beta w_1 h_2 - 2(b + \beta_G w_1)\lambda_1 - 2\beta w_1 \lambda_2}{4(b + \beta_L w_1)(b + \beta_G w_1) - \beta^2 w_1^2} \\ &= p_1^* + \frac{2(b + \beta_G w_1)\mu}{4(b + \beta_L w_1)(b + \beta_G w_1) - \beta^2 w_1^2} - \left(\frac{2(b + \beta_G w_1)\lambda_1 + 2\beta w_1 \lambda_2}{4(b + \beta_L w_1)(b + \beta_G w_1) - \beta^2 w_1^2}\right) \end{aligned}$$

and similarly

$$p_2^{opt} = p_2^* + \frac{2(b + \beta_L w_1)\mu}{4(b + \beta_L w_1)(b + \beta_G w_1) - \beta^2 w_1^2} - \left(\frac{2(b + \beta_L w_1)\lambda_2 + 2\beta w_1 \lambda_1}{4(b + \beta_L w_1)(b + \beta_G w_1) - \beta^2 w_1^2}\right)$$

where $\beta = \beta_L + \beta_G$, $h_1 = a + (b + w_1(\beta_L - \beta_G)c$, and $h_2 = a + (b - w_1(\beta_G - \beta_L)c$.

We know by definition that $p_1^* > p_2^*$ which means by Equation 5.4.9 that $z_1 > z_2$. It is easily shown that $\lambda_1 < \lambda_2$. This leads us directly to the result.

Lemma 5.4.1 *For M=2, with stationary linear demand with an additive stochastic component, the price spread, relative to the case without stochastic demand, between p_1 and p_2 will be:*

a) wider if

$$\frac{(b - \beta_L w_1)}{(b - \beta_G w_1)} < \frac{\lambda_2}{\lambda_1}$$

b) narrower if

$$\frac{(b - \beta_L w_1)}{(b - \beta_G w_1)} > \frac{\lambda_2}{\lambda_1}$$

c) the same if

$$\frac{(b - \beta_L w_1)}{(b - \beta_G w_1)} = \frac{\lambda_2}{\lambda_1}$$

Proof. Consider the component contributed by the stochastic element (we note that the shift caused by the mean, μ, is simply a shift of the linear intercept, a). We want to compare the stochastic components as they will affect whether the price gap widens or narrows. The stochastic component of p_1 is $\frac{2(b+\beta_G w_1)\lambda_1+2\beta w_1\lambda_2}{4(b+\beta_L w_1)(b+\beta_G w_1)-\beta^2 w_1^2}$ and of p_2 is $\frac{2(b+\beta_L w_1)\lambda_2+2\beta w_1\lambda_1}{4(b+\beta_L w_1)(b+\beta_G w_1)-\beta^2 w_1^2}$. We know the gap is wider if:

$$\frac{2(b+\beta_G w_1)\lambda_1+2\beta w_1\lambda_2}{4(b+\beta_L w_1)(b+\beta_G w_1)-\beta^2 w_1^2} < \frac{2(b+\beta_L w_1)\lambda_2+2\beta w_1\lambda_1}{4(b+\beta_L w_1)(b+\beta_G w_1)-\beta^2 w_1^2}$$

as the term on the right is subtracted from p_2 and the term on the left is subtracted from p_1. After some algebraic manipulations this yields

$$\frac{(b-\beta_L w_1)}{(b-\beta_G w_1)} < \frac{\lambda_2}{\lambda_1}.$$

■

We assume that the direct price effects (b) will always be larger than the reference effects (β_L, β_G) and since $0 < w_1 \leq 1$, it follows that the denominator of the left side term will never be zero. We can easily see that both the numerator and denominator on both sides are greater than one (recall $\beta_G > \beta_L$) so we cannot make a definitive statement about a wider or narrower price gap. We can see that, although it is possible, it is unlikely that the price gap will be the same as that it would be in the absence of stochasticity.

We can see then, that consideration must be given to stochasticity in reference price models. While it would have to be evaluated computationally or by determining the different possible combinations mathematically, we would conjecture that regardless of cycle length, the total price spread will be wider with a stochastic component of demand than without it. At the very least, the $M = 2$ results suggest the need to at least consider the results will change given stochastic demand and reference pricing.

In the case where $M > 1$, we also need to give consideration within the price cycles to the case where $x_t > q_t^*$. In an $M = 2$ cycle where $p_1 > p_2$ this can only happen for q_1 because $q_1 < q_2$. In this case we have an issue similar to the one at the start of a planning horizon and need to make a decision on pricing based on the marginal impact on profit of clearing inventory versus the marginal cost of holding additional inventory for one period. In a given time period if $x_t > q_t^*$, Proposition 5.4.1 would hold here as well. A steady state plan would exist with an M stage pricing cycle. In a specific time period, t, if $x_t > q_t^*$, we would determine the optimal plan starting at that specific time period that would get us back to the optimal steady state price with a maximum expected profit.

5.5 Computations

We implemented the same dynamic program in MatLab as with the threshold model. The linear demand function uses parameters $a = 1$, $b = 0.2$. We assume $c = 0.5, 2.5$ to evaluate both high and low margin products. In the absence of reference effects, the optimal price is $p^* = \frac{a+bc}{2b} = 2.75$ for $c = 0.5$ and $p^* = 3.75$ for $c = 2.5$. This range of costs has a clear impact on z and provides some perspective on the relative impacts for low and high margin items. We note that a higher level of c will not only increase the numerator for the calculation of z (see Equation 5.4.9) but will also likely decrease the shortage cost s. The shortage cost is sometimes reflective of lost margin. We evaluate three levels of shortage cost, s ($s = 2.25, 1.25, 0.5$). We use the first two for $c = 0.5$ and $c = 2.5$ respectively. This reflects the smaller margins (the cost of lost sales) for low and high costs. The value of $s = 0.5$ represents a lower cost of shortage and could reflect replacement of the product for a shortage. For disposal costs we evaluate at $h = -0.49, -2.48, 0.05$. This reflects a small holding cost (i.e. a disposal cost that in negative but slightly smaller than $-c$) and suggests that we can use the inventory in the next period if it carries over. The last value, $h = 0.05$, reflects a small disposal cost for the item and allows us to evaluate two different types of products.

For the loss seeking case, in which a price cycle is optimal, we evaluate the impact of changes in gain and loss reference parameters (β_G, β_L) and memory parameter (α). We evaluate at $\alpha = 0.2$ and 0.35, the range which Mazumdar et al (2000) reported as being found in the research. Finally we evaluate values of 0.25, 0.2 and 0.1 for β_L and 0.2, 0.1 and 0.05 for β_G.

In order to undertake the computations we need to specify a distribution for the stochastic component of demand. We use the uniform and the triangular for these computations. The uniform distribution has been used previously (Zabel [1970], Gerchak et al. [2002], Zhou [2007], and Urban [2008]) and the triangular distribution can be used to approximate a normal distribution (if it is specified as symmetric). Both distributions also have a lower bound which allows us to maintain our non-negative demand constraint. These two seem suitable to evaluate the characteristics of the reference price model with a stochastic component on demand. We need to derive the key elements required for the expected profit function which we will use in the recursion.

For the uniform distribution in which the lower the lower limit, $G = 0$, the upper limit is H. We recall that the stocking factor is:

$$1 - F(z) = 1 - \frac{z}{H}$$

thus

$$z^* = H\left[1 - \frac{c+h}{p+s+h}\right].$$

We also determine:

$$\int_0^z (z-u)f(u)du = \int_0^z (z-u)\frac{u}{H}du = \frac{z^3}{6H}$$

and

$$\int_z^H (u-z)f(u)du = \int_z^H (u-z)\frac{u}{H}du = \frac{(H-z)^2}{2H}.$$

Similarly for the triangular distribution (G=0, upper limit =H and mode=I) we can show that :for:

$$z^* = \begin{cases} \sqrt{HI\left[1 - \frac{c+h}{p+s+h}\right]} \text{ if } 0 < z \leq I \\ H - \sqrt{H(H-I)\left[\frac{c+h}{p+s+h}\right]} \text{ if } I < z \leq H \end{cases}$$

and

$$\int_0^z (z-u)f(u)du = \begin{cases} \frac{z^3}{3HI} \text{ if } 0 < z \leq I \\ \frac{3H^2 - z^3 - 3zHI}{3H(H-I)} \text{ if } I < z \leq H \end{cases}$$

and

$$\int_z^H (u-z)f(u) = \begin{cases} \frac{IH^2 - 3zHI + I^2H + z^3}{3HI} \text{ if } 0 < z \leq I \\ \frac{H^3 - z^3 - 3zH^2 + 3Hz^2}{3H(H-I)} \text{ if } I < z \leq H \end{cases}$$

Given these we can now compute the expected profit function and complete the recursions. These are presented below. We evaluate three uniform distributions within the recursions. All have a lower limit $G = 0$. We consider $H = 0.045, 0.09, 0.225$. These were selected as proportions of the optimal quantity in the absence of uncertainty. The larger the value of H, the larger the contribution of uncertainty to total demand. We evaluate nine different triangular distributions. The upper and lower limits are the same as for the uniform ones. A symmetric triangular distribution ($I = 0.0225, 0.045, .1125$) approximates a normal distribution. We also evaluate a left skewed triangular ($I = 0$) and a right skewed triangular distribution where $I = H$. We assume a starting inventory $x_1 = 0$.

5.5.1 Loss Averse Case

We know in the loss averse case that a single price will be optimal in the steady state. This is the price we are interested in and evaluate. The complete set of results is available in Appendix B. We highlight again that we are evaluating the steady state price and do not consider the adjustment periods at the beginning and the end of the time horizon. We also note that because the optimal policy is one in which the price does not change (EDLP), we do not need to consider alternate levels of the memory parameter, α, of the reference loss or gain parameters, β_L and β_G.

Table 5.1 provides an overview of some of the results. The results shown are for the case where $c = 2.5$ which reflects a lower margin product. For the high margin product, $c = 0.5$, whether in the holding cost case ($h = -0.49$) or the disposal cost case ($h = 0.05$) and for the full margin shortage cost ($s = 2.25$) or the product switching shortage cost ($s = 0.5$), the optimal price is very close to optimal price p^* in the absence of a stochastic component. We do not vary shortage cost based on price. While one may suggest that shortage cost is a cost of lost margin, it is often true that stores will provide a "rain cheque" for low cost (or special) items if they are stocked out and thus the lost margin is actually the lost full cost margin. This is true for both the uniformly distributed error term and for the error term which has any of the nine specified triangular distributions. Recall that in all cases the lower limit of both the triangular and uniform distributions is zero. The mean of the distribution affects p^*, the riskless price. This requires consideration when making any comparison but allows us to insure that demand is never negative.

We see that when there is just a holding cost (i.e. excess inventory can be carried into the next period) there is no price discount due to the stochastic element regardless of the distribution chosen for the stochastic demand. This means that there is no impact on expected profit (i.e. the riskless profit is the same as the expected profit with the stochasticity included. When the cost of lost sales is lower and there is a positive disposal cost there is some effect on the price for the greater degree of variation, particularly for the right skewed and symmetric triangular distribution. In this case the expected profit with stochasticity is lower than the riskless profit. This not unexpected. If we consider a grocery store context, there are goods (canned goods for example) that do not deteriorate from one week to the next. The cost of "disposal" is merely the cost of holding that product. If the product is one that deteriorates or expires (produce for example), the cost of disposal is actually, at least in part, a disposal cost. This might include the cost of sorting the good from the bad and actual disposal costs. This can be clearly seen through the value of z relative to the parameters of the probability distributions. In the case of the uniform distribution with holding cost and high cost of lost sales the value of z is close to the upper limit of the distribution. This is what we would expect. The stocking factor covers almost all of the potential sales as the cost of extra inventory is low and the cost of lost sales is high. In the case of the left skewed triangular (mode = 0, the lower bound), the z values are lower when cost of lost sales is lower and the disposal cost is positive.

The difference between riskless and stochastic profits is affected by two factors. As the cost of disposal increases, the stochastic profits decrease relative to the riskless profits. The degree to which this happens depends on the both the distribution of the stochastic component and the size of the stochastic component relative to the certain component of demand. Disposal cost can be negative ($h \geq -c$) which infers that there is at least some salvage. As h approaches $-c$ it infers that there are only holding costs from one period to the next. In the situation where there are merely holding costs, we would expect to price to capture all potential sales as the reduction in profit for carrying excess inventory would be small.

The second factor which affects the size of the reduction of expected profit with stochastic effects relative to riskless profits is the size of shortage costs (s). As shortage costs increase, expected profits with stochastic effects decrease creating a bigger gap to riskless profits. It appears that the impact of the shortage cost is less significant than that for salvage costs with the parameters as specified here.

Table 5.1: Loss Averse Reference Model with Stochastic Component ($c = 2.5$)

Distribution	Upper Limit	Mode	s	h	z	p^*	π^*	p^{opt}	π^{opt}	% diff in π
Uniform	0.045	n/a	2.25	-2.48	0.0449	3.81	0.341	3.81	0.341	0
Uniform	0.225	n/a	0.50	0.05	0.1904	4.03	0.469	3.92	0.395	16
Triangular	0.045	0.0225	2.25	-2.48	0.0433	3.81	0.341	3.81	0.341	0
Triangular	0.090	0.0450	0.50	0.05	0.0644	3.86	0.371	3.84	0.339	9
Triangular	0.225	0.1125	0.50	0.05	0.1626	4.03	0.469	3.97	0.385	18
Triangular	0.045	0	2.25	-2.48	0.0426	3.79	0.332	3.79	0.331	0
Triangular	0.090	0	0.50	0.05	0.0534	3.83	0.351	3.79	0.350	0
Triangular	0.225	0	0.50	0.05	0.1337	3.94	0.413	3.86	0.405	2
Triangular	0.045	0.045	2.25	-2.48	0.0449	3.83	0.351	3.83	0.351	0
Triangular	0.090	0.090	0.50	0.05	0.0825	3.90	0.392	3.88	0.353	10
Triangular	0.225	0.225	0.50	0.05	0.2075	4.13	0.528	4.07	0.425	20

5.5.2 Loss Seeking Case

We evaluated the loss seeking case in a similar fashion. In this case, we found that all of the models specified resulted in an $M = 2$ price cycle. Tables 5.2 and 5.3 provide an overview of some of the key results. All of the results are tabulated in Appendix B. There are a number of points worth highlighting in the results.

- The reductions in optimal prices (both p_1 and p_2) for the $M = 2$ loss seeking case is greater for similar distributions than it was in the loss averse case. The price cycles interact with the stochasticity causing the greater price reductions.

- For this linear specification with the parameters used and the distributions chosen the price gap was wider than it would have been in the absence of stochasticity for all cases. In the case of holding costs ($h = -2.48$) the changes in the gap were not sufficient to cross the one cent price barrier and so the difference was not manifest in actual prices. It is easily seen that the price gaps widen in other cases.

- A decrease in the loss parameter (β_L) widens the basic price spread between p_1 and p_2 and also widens the spread that occurs due to stochasticity. That is, stochasticity has a bigger effect on the price spread when loss parameter is lower.

- We see that the value of z behaves similarly to the loss averse case.

It is worth noting that the impact on stochastic profit relative to riskless profit of salvage value (h) and shortage costs (s) are the same as in the loss averse case. The relative impact, however, is bigger for loss seeking consumers. This is due to the moderating impact of the stochasticity on the price gaps under these parameters and distributions. We know from Lemma 5.4.1 that the price gap can be bigger, smaller or the same with stochasticity than without. In this case the price gap is smaller and thus some of the reference gains are lost in addition to the lower prices charged.

Table 5.3: Loss Seeking Reference Model with Stochastic Component - ($\beta_L = 0.075$)

Dist.	Up.Lim.	Mode	s	h	z_1	z_2	p_1^*	p_2^*	π^{*1}	p_1^{opt}	p_2^{opt}	π^{opt2}	$\% \downarrow \pi$
Unif.	0.045	n/a	2.25	-2.48	0.0449	0.0448	4.23	2.57	0.169	4.23	2.57	0.169	0
Unif.	0.225	n/a	0.50	0.05	0.1027	0.0369	4.37	2.71	0.263	4.18	2.49	0.094	64
Tri.	0.045	0.0225	2.25	-2.48	0.0318	0.0429	4.23	2.56	0.169	4.23	2.57	0.169	0
Tri.	0.090	0.0450	0.50	0.05	0.0433	0.0313	4.26	2.60	0.228	4.21	2.54	0.142	38
Tri.	0.225	0.1125	0.50	0.05	0.1086	0.0790	4.37	2.71	0.418	4.25	2.56	0.192	54
Tri.	0.045	0	2.25	-2.48	0.0427	0.0420	4.22	2.56	0.150	4.22	2.56	0.149	0
Tri.	0.090	0	0.50	0.05	0.0238	0.0065	4.24	4.16	0.188	4.18	2.51	0.112	40
Tri.	0.225	0	0.50	0.05	0.0587	0.0147	4.31	4.12	0.309	4.16	2.47	0.110	64
Tri.	0.045	0.045	2.25	-2.48	0.0049	0.0449	4.24	2.58	0.189	4.24	2.58	0.189	27
Tri.	0.090	0.090	0.50	0.05	0.0614	0.0355	4.29	2.63	0.268	4.23	2.56	0.163	39
Tri.	0.225	0.225	0.50	0.05	0.1546	0.0911	4.43	2.77	0.532	4.29	2.60	0.253	52

$c = 2.5, \beta_L = 0.75, \beta_G = 0.2$

Table 5.2: Loss Seeking Reference Model with Stochastic Component ($\beta_L = 0.1$, $\beta_G = 0.2$, $c = 2.5$)

Dist.	Up.Lim.	Mode	s	h	z_1	z_2	p_1^*	p_2^*	π^*	p_1^{opt}	p_2^{opt}	π^{opt}	$\% \downarrow \pi$
Unif.	0.045	n/a	2.25	-2.48	0.0449	0.0448	4.26	2.66	0.179	4.26	2.66	0.179	0
Unif.	0.225	n/a	0.50	0.05	0.1121	0.0085	4.40	2.81	0.440	4.19	2.55	0.084	80
Tri.	0.045	0.0225	2.25	-2.48	0.0434	0.0430	4.26	2.66	0.179	4.26	2.66	0.179	0
Tri.	0.090	0.0450	0.50	0.05	0.0436	0.0287	4.29	2.70	0.238	4.25	2.65	0.161	32
Tri.	0.225	0.1125	0.50	0.05	0.1091	0.0219	4.40	2.81	0.426	4.29	2.69	0.225	47
Tri.	0.045	0	2.25	-2.48	0.0421	0.0421	4.25	2.65	0.160	4.25	2.65	0.160	0
Tri.	0.090	0	0.50	0.05	0.0091	0.0092	4.26	2.68	0.198	4.21	2.61	0.129	35
Tri.	0.225	0	0.50	0.05	0.0219	0.0219	4.34	2.75	0.135	4.20	2.59	0.319	58
Tri.	0.045	0.045	2.25	-2.48	0.0449	0.0449	4.27	2.68	0.199	4.27	2.68	0.198	0
Tri.	0.090	0.090	0.50	0.05	0.0620	0.0411	4.32	2.72	0.249	4.27	2.67	0.184	26
Tri.	0.225	0.225	0.50	0.05	0.1575	0.1067	4.46	2.87	0.462	4.35	2.73	0.295	36

$c = 2.5, \beta_L = 0.1, \beta_G = 0.2$

5.6 Summary and Conclusion

In this chapter we develop a reference demand model with a stochastic component of demand. We find analytical solutions for the loss averse case (where a single price is optimal) and for the special case of loss seeking demand in which $M = 2$. These solutions provide insight into the steady state case. The loss averse case is similar to the non-reference demand in the steady state. There will be differences in the adjustment periods that could arise due to cost or other parameter changes or at the end of a product life cycle or season. In the loss seeking case, we find that the stochasticity in both price periods affects the choice of price in each individual period. We also find that as shortage costs decrease. This is a significant insight. We also find that the introduction of a stochastic component in demand can change the size of the price spread between the period prices. We show that the price spread can be wider or narrowed depending on the specific parameters of the model and the stochastic component of demand. We do computational experiments to evaluate optimal pricing strategies for both loss averse and loss seeking models. We show that as disposal cost increases, expected profit decreases relative to riskless profit and that increasing shortage costs also decrease expected profits relative to riskless profit but to a lesser degree. We showed that the price gap for loss seeking consumers can either increase, decrease or stay the same.

We also introduce a discussion of results for multi-period models in which price cycles. In steady state, for the loss averse model, the price does not cycle so the ending inventory will never exceed the stocking factor, z for the next period. When prices cycle, the period in which a price increases may result in a starting inventory that exceeds the stocking factor. Historically researchers have assumed expensive disposal (i.e. inventory cannot be cost effectively carried forward). There may be cases in which a small holding cost on the expected ending inventory may be more cost effective than the reduction in expected profit incurred by lowering price such that the inventory is expected to be cleared. In the reference case, the impact on future periods also needs to be considered.

The incorporation of a stochastic component of demand to reference models provides additional insight for both reference research and research into stochastic demand.

Chapter 6

The Model with Setup and Holding Costs

6.1 Introduction

The dynamic lot sizing problem, where production or ordering decisions are made based on setup and holding costs, has been well studied in the literature. The order or production quantity can, depending on the relative holding and setup costs, create inventory that carries over several periods in the planning horizon. The incorporation of reference prices will add a second inter-temporal effect that creates an issue for consideration in the production/order planning decision. The inter-temporal pricing implications clearly interact with the lot sizing problem. In the case of loss averse consumers, the implications of increasing prices which leads to reference losses in future periods need to be considered. The work of Wagner and Whitin [1958] and Thomas [1970] lay a good foundation for addressing this complex computational problem. We address the problem with the incorporation of reference price models. Our focus is to analyze the problem in the context of steady state pricing and lot sizing strategies. That is, we have stationary demand over the planning horizon and we evaluate the optimal repeating strategy.

This chapter is laid out as follows. We introduce the general lot sizing problem and some of the results that facilitate the solution of the problem. We then discuss the incorporation of reference prices into the demand function and the modifications to the solution approach that the reference price considerations require. We finish with some computational experiments to illustrate the behaviour of the model that we have used to date.

6.2 The Dynamic Lot Sizing Problem with Pricing

The dynamic lot sizing problem with endogenous price is well established (Yano and Gilbert [2005], Bhattacharjee and Ramesh [2000]). We present it here to develop the complete notation within the context of our problem.

$$\underset{x,p,\delta}{Max} \sum_{t=1}^{T} \{\theta^{t-1}(p_t D_t(p_t) - \delta_t K - cx_t - hI_t\} \tag{6.2.1}$$

subject to

$$\begin{aligned} I_{t-1} + x_t - D_t(p_t) - I_t &= 0 \quad t = 1, ..., T \\ \delta_t &\in \{0,1\} \quad \delta_t = 1 \text{ if } x_t > 0, \text{ and } 0 \text{ otherwise} \\ x_t &\geq 0 \quad t = 1, ..., T \\ I_t &\geq 0 \quad t = 1, ..., T \\ x_t &\leq CP \end{aligned}$$

We have defined some of the notation before but in the interest of completeness we outline it again. The discount factor, θ, provides a present value of future streams. In our case, we assume that $\theta = 1$ so that we determine a repeating steady state pricing and ordering strategy. As was the case before, the price in the current period, p_t, is an endogenous choice variable. Demand in each period, $D_t(p_t)$, is a function of price. The specific form and parameters of demand can change between periods (reflecting, for example, seasonal differences in demand). In our case, we assume that demand is consistent between periods for the purposes of determining the optimal pricing strategy with dynamic lot sizing and reference price effects. As was the case previously, the variable production or ordering cost is represented by c and is assumed to be fixed across the planning horizon, although this need not always be the case. The production quantity in period t, x_t, incurs variable cost c, and fixed cost K. The binary variable, δ_t, ensures that the fixed cost is only incurred in periods in which there is actual production (the second constraint). The inventory at the end of the period, I_t, is charged holding cost, h, which is also fixed for all periods. The first constraint is the inventory balance equation. It ensures that starting inventory, I_{t-1}, plus production in period t, minus the demand equals to the ending inventory. The final constraint is a capacity constraint such that production in any period can not exceed the capacity, CP.

The problem is to determine jointly an optimal price path and an optimal production or ordering plan. Bhattacharjee and Ramesh [2000] study the production optimization problem with a discrete set of price choices and a finite life for the product. The finite life of the product means that the product will only last a given number of periods, k, so that the maximum production cycle is k periods. While having a maximum cycle length, k, clearly simplifies the problem, the combined problem of pricing and production creates a complex problem and even short horizon problems can be very difficult to solve.

The single period expected profit is:

$$\pi_t(r_t, p_t, I_t, x_t) = p_t D_t(p_t) - \delta_t K - cx_t - hI_t$$

where

$$D_t = \begin{cases} a - bp_t + \beta_G(r_t - p_t); \text{ if } r_t > p_t \\ a - bp_t + \beta_L(r_t - p_t); \text{ if } r_t \leq p_t \end{cases}$$

and the same balance constraints between periods hold as was the case in Equation 6.2.1.

The dynamic programming recursion stated in the form used in earlier chapters and including the reference price is:

$$f_n(r_t) = \max_{p_t \in J_t, x_t, I_t} [\theta \pi_t(r_t, p_t, I_t, x_t) + f_{t-1}(r_{t-1})]$$

where

$t = 2, 3, ..., T$;

$r_t = \alpha r_{t-1} + (1-\alpha)p_{t-1}$, $t = 2, ..., T$, and r_1 is given;

$f_1 = 0$;

J_t is a vector of discrete prices which goes from c to $\frac{a+\beta_L r_t}{b+\beta_L}$ in increment δ;

p_t is the discrete price period t from vector J_t;

π_t is the profit function based on the reference demand function with loss averse reference parameters; and

θ is the discount factor, $0 < \theta \leq 1$.

We highlight again that we use $\theta = 1$ which maximizes the average profit and gives insight into the optimal price and production/order cycle given consistent reference price demand. We have omitted capacity to simplify the discussion and have not included a maximum production/order cycle length due to product degradation or expiry.

This establishes the model to be solved for the optimal decision. We now introduce a solution approach for this problem.

6.3 A General Solution Approach

The results in Wagner and Whitin [1958] and Thomas [1970] provide a good foundation for solving this problem. Wagner and Whitin [1958] proved four key theorems with price as exogenous upon which Thomas built to include price as an endogenous decision variable. The four key results presented by Thomas [1970] as Lemmas 1 through 4 are:

1. An optimal program exists in which each period has either production or a starting inventory equal to zero but not both: $I_t x_t = 0$.

2. Analogous to the first Lemma, an optimal program exists in which the starting inventory is either zero (in which case production occurs) or is exactly equal to $\sum_{t=1}^{k} D_t(p_t)$ for some $k, t \leq k \leq T$. This means that production covers either all or none of a subsequent period's demand.

3. An optimal program exists such that if demand in period t is met by production in period $t^*, t^* < t$, then all production in the intermediate periods $(t^*, ..., t)$ is also met by production in period t^*. Stated differently, if the demand in period t is met by production, x_{t^*}, the demand in all of the intervening periods is also met by this production.

4. In the optimal program where $I_t = 0$, for some period t, it is optimal to consider periods 1 through $t-1$ alone. This means that each individual production cycle is independent of each other one.

These results prove very powerful in developing a dynamic programming algorithm for solving the production problem. Results 1 through 3 would be expected to hold regardless of the form of the demand function. Result 4 needs to be considered carefully. It states that when starting inventory is zero for a period we can optimize within the previous periods separately. This is based on the assumption that demand in the different periods is independent. In the reference demand case this is not true. The reference price links demand in period t to the pricing decisions in previous periods. We must optimize prices in periods prior to a production period while taking into account the demand impacts post production. This takes away one of the key results that would simplify the computational process in Thomas' original price setting algorithm. We discuss this in greater detail later.

The planning horizon results also require consideration. Lemmas 6 and 7 in Thomas [1970] provide lower and upper bounds for price consideration. Let us start with Lemma 6. It states that if the optimal last setup time is in the current period, no price lower than p_t^* is ever optimal for that subsequent production cycle. We recall from the previous numerical analysis (for thresholds and stochastic demand) that there were adjustment periods at the beginning and the end of the planning horizon depending on the starting reference price. We can reduce the adjustment period at the beginning of the planning horizon by beginning with a reference price equal to the optimal reference price in the absence of setup/ordering and holding costs.

Thomas [1970] uses Lemma 6 to set a lower bound on the optimal price within a planning horizon. We first revisit and extend this result for the special case of stationary demand, to build on it in the reference context. Lemma 6.3.1 builds on the result in Thomas but parts b and c provide more detail and further reduce the search space. Lemmas 6.3.1 and 6.3.2 do not include reference impacts but extend the results of Thomas [1970] and set a foundation for subsequent results specific to reference models.

Lemma 6.3.1 *If demand is linear and stationary, and if production occurs in period t,*

a) p_t^* will set a lower bound for the price for $m = t, ..., k-1$ where k is the time of the next production run.

b) p_t^* will be the maximizer of $g(p_t)$, the profit function without reference effects.

c) for any period $m, t \leq m < k-1$, p_m sets a lower bound for all subsequent prices, $p_{m+1}, ..., p_{k-1}$.

Proof. Consider any production cycle of length M without reference prices. It is straight forward to establish the concavity of this cycle profit function. In that cycle, there is, by definition, only a single production run in the first period. We know the profit from that production cycle equals:

$$\begin{aligned}\pi_M &= \sum_{j=1}^{M} p_j(a-bp_j) - K - c\left(\sum_{j=1}^{M}(a-bp_j)\right) - h(a-bp_2+\ldots \\ &+a-bp_M) - 2h(a-bp_3+\ldots+a-bp_M)) - \ldots(M-1)h(a-bp_M)\end{aligned} \tag{6.3.2}$$

If we differentiate with respect to individual prices we get:

$$\begin{aligned}\frac{\partial \pi_M}{\partial p_1} &= a - 2bp_1 + cb \\ \frac{\partial \pi_M}{\partial p_2} &= a - 2bp_2 + cb + hb \\ &\vdots \\ \frac{\partial \pi_M}{\partial p_M} &= a - 2bp_M + cb + hb\sum_{j=1}^{i-1} j\end{aligned}$$

We can then solve for optimal values of each price:

$$\begin{aligned}p_1 &= \frac{a+cb}{2b} \\ p_2 &= \frac{a+cb+hb}{2b} \\ p_{M-1} &= \frac{a+cb+hb\sum_{j=1}^{M-2} j}{2b} \\ p_M &= \frac{a+cb+hb\sum_{j=1}^{M-1} j}{2b}\end{aligned}$$

It is clear that p_1 sets a lower bound for all of the prices in the cycle and that each price is incrementally higher through the production cycle. ■

While we still need to solve the problem, we have significantly reduced the search space. We can easily see that this extends to the case where demand is not stationary.

Lemma 6.3.2 *If $g_t(p_t)$, the profit function, is based on downward sloping demand, concave, continuous and differentiable for each period and if production occurs in* *period t, p_t^* will set a lower bound for the price for $t = i, ..., k-1$ where k is the time of the next production run and $p_1^{opt} = p_1^*$, where p_1^{opt} is the optimal first price in the production cycle and p_1^* is the maximizer of the the profit function. The maximizer of $g_t(p_t)$ (the non-reference component of the profit function), p_t^*, will set a lower bound for each period t and all subsequent prices.*

Proof. Given demand is downward sloping we know:

$$\frac{\partial D_t}{\partial p_t} < 0$$

which says that demand is decreasing in price. The profit function for any M period cycle is:

$$\pi_M = \sum_{j=1}^{M} p_j D(p_j) - K - c\left(\sum_{j=1}^{M} D(p_j)\right) - h(\sum_{j=2}^{M} D(p_j) - 2h(\sum_{j=3}^{M} D(p_j)) - ... - (M-1)hD(p_M) \quad (6.3.3)$$

As we did in Lemma 6.3.1, we can easily solve for the optimal price in each period by differentiating with respect to p_i:

$$\begin{aligned}
\frac{\partial \pi_M}{\partial p_1} &= \frac{\partial (p_1 D(p_1)}{\partial p_1} - \frac{c \partial D(p_1)}{\partial p_1} \\
\frac{\partial \pi_M}{\partial p_t} &= \frac{\partial (p_t D(p_t))}{\partial p_t} - c\frac{\partial D(p_t)}{\partial p_t} - \sum_{j=1}^{i-1} h \frac{\partial D(p_t)}{\partial p_t} \; \forall t = 2, ...M
\end{aligned}$$

We can see easily that $p_1^{opt} = p_1^*$. We can also see that the last term relative to p_t is negative so it increases the derivative (because it is subtracted). Therefore, $p_t^{opt} > p_t^*$. ∎

We highlight that this does reduce the search space when determining a cycle. It still requires the evaluation in each case as to whether the cycle is the appropriate length. This result allows us to reduce the range of prices we evaluate in any given cycle length.

We would also benefit from an upper limit for prices under consideration. We offer first the upper bound set by Thomas [1970] as there was no proof published. The proof in Thomas [1970], says that if $g(p_t)$, the profit function, is based on downward sloping demand, concave, continuous and differentiable for each period, t, and if production occurs in period t, no price p_t, where $g(p^*) - g(p_t) > K$, is ever optimal.

Lemma 6.3.3 *If $g_t(p_t)$, the profit function, is based on downward sloping demand, concave, continuous and differentiable for each period and if production occurs in period i, no price p_t, where $(t-i)g(p^*) - g(p_{i+1}) - \ldots - g(p_t) > K$, is ever optimal.*

Proof. The proof is straight forward. If the profit contribution lost due to charging lower than optimal costs exceeds the setup cost, K, then it would be more profitable to produce with cost K in period i. ■

This lemma provides a tightening upper bound as M, the cycle length, grows. We now explore the case where we have loss averse reference demand We first consider the price sequences between production periods similar to Lemma 6.3.1 but with reference effects. The result is similar to the previous one.

Lemma 6.3.4 *If demand is linear and stationary with loss averse reference effects, and if production occurs in period t,*
a) p_t^* will set a lower bound for the price for $m = t, \ldots, k-1$ where k is the time of the next production run.
b) for any period $m, t \leq m < k-1$, p_m sets a lower bound for all subsequent prices, $p_{m+1}, \ldots, p_{k-1}$.

Proof. The proof is analogous to that for Lemma 6.3.1. We illustrate the approach for a two period cycle but it is easy to extend to a cycle of any length M. We also consider a risk neutral scenario, i.e. $\beta_L = \beta_G$ but we label the two parameters β_1 and β_2 which are associated with p_1 and p_2 respectively. In the $M = 2$ case. the two period profit function is

$$\begin{aligned}\pi_2 &= p_1(a - bp_1 + \beta_1(r_1 - p_1)) + p_2(a - bp_2 + \beta_2(r_2 - p_2)) - k \\ &\quad -c(a - bp_1 + \beta_1(r_1 - p_1) + a - bp_2 + \beta_2(r_2 - p_2) - h(a - bp_2 + \beta_2(r_2 - p_2)).\end{aligned}$$

For $b > \beta_1, \beta_2$ it is straight forward to establish concavity (see Proposition 3.4.1 on page 47). We can use the first order conditions to solve for p_1^* and p_2^* yielding

$$p_1^* = \frac{a + (b + w_1\beta_1 - w_2\beta_2)c - h\beta_2 w_1 + w_1(\beta_1 + \beta_2)p_2}{2(b - w_1\beta_1)}$$

and

$$p_2^* = \frac{a + (b + w_1\beta_2 - w_2\beta_1)c - h\beta_2 w_1 + hb + w_1(\beta_1 + \beta_2)p_1}{2(b - w_1\beta_2)}$$

where

$$w_1 = (1 - \alpha) + (\alpha^2 - \alpha^3) + (\alpha^4 - \alpha^5) + ...$$

and

$$w_2 = (\alpha - \alpha^2) + (\alpha^3 - \alpha^4) + (\alpha^5 - \alpha^6) + ...$$

It is easy to show that $p_1^* < p_2^*$ even when $\beta_1 = \beta_2$. As $\beta_1 = \beta_2$, we did not impose either a reference loss or reference gain within the cycle so we did not assume that $p_1^* < p_2^*$. We have now shown that in an $M = 2$ case with risk neutral consumers the second price in the cycle is higher than the first. We know from Proposition 3.3.1 that in the case of loss averse consumers without setup and holding costs, a single price is optimal. It is easily shown that the same is true for loss neutral consumers. We know from above that for loss neutral consumers, $p_2^* > p_1^*$. If $\beta_2 > \beta_1$ (i.e. $\beta_L > \beta_G$), it is still true that $p_2^* > p_1^*$ because there is no value in a lower price from a reference perspective. This proves the case for $M = 2$. The approach for any M is analogous and yields similar results. ■

This allows us to set a tightening lower bound on price as we evaluate cycle length and pricing computationally. We now consider the length of the cycle. Let M_{NR} be the optimal cycle length in the absence of reference effects and M_R be the optimal cycle length with loss averse reference effects.

Lemma 6.3.5 *If demand is linear and stationary, with an additive loss averse reference component, the optimal cycle length will be less than or equal to the optimal cycle length given the same linear demand function and no reference effects. The optimal number of periods, $M_R \leq M_{NR}$, in the optimal steady state pricing program*

Proof. Consider any M period cycle. In the absence of reference effects, the total cost of production and inventory is:

$$TC_{NR} = K + cx_t + \sum_{t=1}^{M-1} hI_t$$

We know from Proposition 3.3.1 that in the absence of setup/production and holding costs, a single price would be optimal. We also know that the sum of all of the reference losses equals the sum of all the reference gains in a price cycle. The total cost of production and inventory in the case of reference effects equals:

$$TC_R = K + cx_t + \sum_{t=1}^{M-1} [hI_i + (\beta_L - \beta_G)(r_{t+1} - p_{t+1})]$$

It is clear that the optimal cycle length in non-decreasing in K and non-increasing in c and h. We know from Lemma 6.3 that prices increase as M increases which means that r_M increases with M. Since $\beta_L > \beta_G$, $\sum_{t=1}^{M-1}(\beta_L - \beta_G)(r_M - p_1) > 0$. There are net reference losses that accrue due to each additional period. The marginal cost of inventory is higher, $\sum_{t=1}^{M-1} [hI_i + (\beta_L - \beta_G)(r_M - p_1)] > \sum_{t=1}^{M-1} hI$ for loss averse consumers so with all other parameters being the same, additive reference losses will be less than or equal to the cycle length than is the case without loss averse reference effects (i.e. $M_R \leq M_{NR}$). ■

This lemma allows us to set an upper limit on the length of the optimal cycle for risk averse consumers. While our focus is on loss averse consumers in this chapter, we offer the following corollary as a straight forward follow-up to Lemma 6.3.5.

Corollary 6.3.1 *If $g_t(p_t)$, the non-reference component of the profit function, is based on down sloping demand, concave, continuous and differentiable for each period t, with additive loss seeking reference effects, the optimal number of periods, $M_R \geq M_{NR}$, in the optimal steady state pricing program*

Proof. The proof follows directly from that for Lemma 6.3.5. We revisit

$$TC = K + cx_t + \sum_{i=1}^{M-1} hI_i + (\beta_L - \beta_G)(r_M - p_1)$$

In the case where $\beta_L < \beta_G$, it is clear that cost decreases as r_M increases. Adding loss seeking reference effects will reduce the cost of cycle of length M, so, therefore, M_{NR} sets a lower bound for M_R. ■

We have established a maximum cycle and we now require bounds for the loss averse case. We first revisit the upper bounds for the problem. In this case we can offer a tighter bound than was the case with loss averse reference demand.

Lemma 6.3.6 *If $g_t(p_t)$, the profit function, is based on downward sloping demand, concave continuous and differentiable for each period, with an additive loss averse reference component and if production occurs in period t, no price p_t, where $g(p^*) - g(p_t) - (\beta_L - \beta_G)(w_1 p_i + w_2 p_t - p_t) > K$, is ever optimal. Where*

$$w_1 = (1-\alpha) + (\alpha^2 - \alpha^3) + (\alpha^4 - \alpha^5) + ...$$

and

$$w_2 = (\alpha - \alpha^2) + (\alpha^3 - \alpha^4) + (\alpha^5 - \alpha^6) + ...$$

Proof. *This proof follows directly from that of Lemma 6.3.3. In this case we not only take into account the difference in the direct profit but the net reference effect of charging the higher price.* ■

We need to consider the adjustment periods for the beginning and end of the planning horizon so we get a stable steady state result in the middle. In order to establish an effective starting and finishing adjustment period we use the model without setup and holding costs and use the adjustment periods required for those. We also know from Lemma 6.3.5 that the maximum $M_R \leq M_{NR}$. We develop a simple algorithm to establish the cycle length with no reference effects. First we require an additional lemma.

Lemma 6.3.7 *If $g_t(p_t)$, the profit function, is based on stationary downward sloping demand, concave, continuous and differentiable for each period, and if production occurs in period t, the optimal cycle length, $M = T$, occurs at the point where $\pi_{T-1} < \pi_T > \pi_{T+1}$.*

Proof. The shortest possible cycle occurs at $M = 1$. We know demand is stationary, i.e. that demand parameters do not change. If $\frac{\pi_2}{2} > \pi_1$ then the reduction in fixed cost per unit outweighs the holding costs for the inventory (at the higher price) and increases profits. As M increases, fixed cost per unit decreases and inventory cost increases. Once inventory cost increases exceed fixed cost per unit decreases, there is no case in which profit will increase again. ■

We now set the algorithm for the non-reference stationary demand case. This is similar to that of Thomas [1970] and sets the context for the reference model algorithm. Algorithm 6.3.1 does use tighter bounds than was the case in Thomas (which did not require stationary demand) and also has a stopping mechanism when the optimal cycle length is determined.

Algorithm 6.3.1 *Initiation: Set $M = 2$, $p_1 = p^*$, calculate original upper bound,. $\pi^{opt} = p^*D(p^*) - K - cD(p^*)$.*

Step 1

Solve the M period problem using the restricted decision space as specified in Lemmas 6.3.1 to 6.3.7. Record optimal prices for each of the M periods and π_M.

Step 2

If $\frac{\pi_M}{N} > \pi^{opt}$, $\pi^{opt} = \frac{\pi_M}{M}, M = M + 1$, record $p_1, ..., p_m$, go back to Step 1

Else go to Step 3

Step 3

$M = M^{opt}, p_1, ..., p_M$ are optimal. Production quantities can be computed.

The entire planning horizon, T, for the reference pricing problem is established by adding the adjustment period at the beginning of the enumeration. In the case where we are interested in the transition (which is beyond the scope of this chapter) we would need to evaluate that period separately. We will understand the adjustment period required at the end and add that to T. We choose to incorporate $2M^{opt}$ in the middle of the T periods to ensure that we get a stable and consistent result. The reference price algorithm is then:

Algorithm 6.3.2 *Initiation: Set* $M = 2$. *Establish* T *as outlined above, calculate the upper bound, initialize* $\pi^{opt} = p^* D(p^*) - K - cD(p^*)$. *Set two* M *period cycles in* T.

Step 1

Solve the M *period problem using the restricted decision space as specified in the Lemmas 6.3.1 to 6.3.7.*

If there are at least two stable M *period cycles, record optimal prices for each of the* M *periods and compute* π_M. *Go to Step 2*

Else add another M *period cycle to the middle of the horizon,* $T = T + M$. *Repeat Step 1.*

Step 2

If $\frac{\pi_M}{M} > \pi^{opt}$, $\pi^{opt} = \frac{\pi_M}{M}$, *go to Step 3*

Else go to Step 4

Step 3

If $M = M^{opt}$ *from Lemma 6.3.1, Record* $p_1, ..., p_M$,

Else Record $p_1, ..., p_M$, $M = M + 1$, *reset* T *as outlined above,go back to Step 1*

Else go to Step 4

Step 4

$M = M^{opt}, p_1, ..., p_M$ *are optimal. Production quantities can be computed.*

We have now established a solution approach for the dynamic lot sizing problem with reference prices.

6.4 Computations

We perform some computations using the demand formulation used in earlier chapters. The problem is computationally intensive but we have narrowed the search space for the problem with the lemmas outlined in the previous section. As was the case in previous chapters, we implement the dynamic programming formulation in MatLab with the addition of the inventory balance equations, fixed setup/ordering and holding costs. We choose arbitrary levels of the costs based on the parameters of demand in order.

The results of the computations are presented in Table 6.1. The results from this model as specified perform as we would expect. We see that, depending on the difference between β_L and β_G, with the given parameters, the reference effects tend to overwhelm the production/ordering and holding cost effects. Only when the difference between β_L and $\beta_G$was very small did a multiple period strategy with pricing actions make sense. We note that the first line of Table 6.1 is a baseline evaluation. In the absence of reference effects, ordering/setup and holding costs and with stationary demand, we would expect the optimal strategy to be a constant price equal to p^*.

We can review the individual results in detail to isolate these results. The first row in Table 6.1 is the case in which there are no reference effects and no set up or holding costs. In this case the lack of a holding cost means that we order every cycle. We know that given the lack of reference effects a constant price is optimal. It is also worth noting that if ordering cost is positive and holding cost is zero then the production/ordering cycle would be infinite. The next four rows in the table represent the baseline cycles with different levels of setup and holding costs. We know the non-reference models set the upper bound on cycle length for loss averse models (Lemma 6.3.5) but they also serve as a benchmark for comparison for the reference model computations. We note that cycle length increases as holding cost decreases. As expected, price increases over the course of the cycle. The price increases are more significant for the higher holding costs.

Table 6.1: Computational Results for Dynamic Lot Sizing Problem

β_L	β_G	K	h	M	p
0	0	0	0	1	2.75
0	0	0.05	0.0250	3	2.75, 2.76, 2.78
0	0	0.05	0.0125	4	2.75, 2.75, 2.76, 2.77
0	0	0.10	0.0500	3	2.75, 2.78, 2.80
0	0	0.10	0.0250	4	2.75, 2.76, 2.78, 2.79
0.10	0.05	0.05	0.0250	1	2.75
0.10	0.05	0.05	0.0125	1	2.75
0.10	0.05	0.10	0.0500	1	2.75
0.10	0.05	0.10	0.0250	1	2.75
0.05	0.25	0.10	0.0250	1	2.75
0.05	0.05	0.10	0.0100	3	2.75. 2.76, 2.77
0.06	0.05	0.05	0.0250	3	2.75. 2.76, 2.77

In the loss averse reference model case, we see that the inclusion of loss averse reference effects moderate both cycle length and the highest price charged. In the first five cases (lines 6-10 in Table 6.1) the reference effects completely dominate the setup/ordering cost effects and a single period cycle is optimal. That means production/ordering occurs in each period. In the final two lines in the table, there are three period cycles. We note that in this first one the consumers are assumed to be loss neutral (that is $\beta_L = \beta_G$). This case also uses the largest setup cost considered and the smallest holding cost. The resulting three period cycle has smaller price increases than was the case with no reference effects and a higher carrying cost. The final example shown has slightly loss averse consumers and setup and holding costs identical to those in line 2 of the table. This highlights that, in this case the cycle is shorter but the price increases are moderated.

These results have implications for managers that go beyond the specific parameters evaluated in these computations. The consideration of inter-temporal reference effects will shorten order cycles and moderate the price impacts in the presence of order and holding costs. Given the strong empirical evidence for loss averse reference effects, there is real value in these results and further work would have merit. In the absence of consideration of reference effects, order/production cycles may be too long and price adjustments may be too high during the cycles resulting in lower profits.

6.5 Summary and Conclusions

We evaluate the dynamic lot sizing problem with reference prices. The dynamic lot sizing problem with endogenous price is already recognized as a difficult problem (Thomas [1970], Bhattacharjee and Ramesh [2000]), the incorporation of reference prices into the specification increases the complexity of the problem as the impacts of pricing decisions extend beyond the current period and this needs to be considered in undertaking the computations.

We outline the basic dynamic lot sizing model, which involves production planning and pricing and includes setup/ordering and holding costs. We develop steady state pricing and production cycles that maximize profit. We introduce a number of lemmas which set bounds for the dynamic programming model when finding optimal solutions. This involves both upper and lower bounds for pricing as well as a the development of an upper bound for the length of the steady state loss averse production planning and pricing cycle. We then present an algorithm for determining the optimal steady state production planning and pricing model for stationary demand in the absence of reference prices. This serves as an input into the algorithm for developing the model for inventory and pricing planning with reference prices. We undertake some computational experiments to show how the model performs.

Chapter 7

Summary and Conclusion

7.1 Summary and Overview

In this dissertation we have introduced three significant extensions to the reference price models to provide insight into not only optimal pricing decisions, but also inventory stocking and production or ordering decisions. The existence of a reference price, or expected price based on previous purchases, has strong empirical support and merits consideration when developing pricing strategies for retailers. We establish models that extend the scope of the reference price models studied to date and give richer insight into the pricing strategies.

We present a comprehensive review of the literature into reference prices from the operations and marketing perspectives. We also provide an overview of research with demand models that include a stochastic component, focusing on additive error. Finally we introduce the research into dynamic lot sizing. Neither of these last two include any reference price work but establish the context for the reference models presented later.

Chapter 3 introduces the basic reference model which serves as a starting point for the extension in the subsequent chapters. It provides an introduction to the notation and establishes key basic results that are used and extended later in the context of the other models. We introduce the loss averse and loss seeking cases. We also discuss the inter-temporal pricing effects and the importance of considering multiple periods even when a price for a single period is needed. We cannot consider any single period in isolation because the demand in future periods is affected by the current price.

The threshold model presented in Chapter 4 is the first significant extension to the previous reference price work. The model is based on the empirical work that suggests there are "zones of insensitivity" or threshold around reference price within which there is no reference response. This is interesting as it provides for a rigourous model with the flexibility to yield either single price (EDLP) strategies or cyclical pricing strategies depending on the parameters of the model. We specify the model and present some results and key findings. We also develop a set of results that reduce the search space for the dynamic programming enumeration. We conduct substantial computational experiments and provide a discussion of the results. This chapter concludes with the introduction of a nonlinear integer programming model for optimizing the threshold model. It provides optimal results relatively quickly and makes the implementation of threshold analysis more practical. This threshold model offers valuable insight and merits extension.

We next introduce the reference model with an additive stochastic component. We analyze both the loss averse and loss seeking cases to determine the impact of stochastic demand on the optimal pricing strategy. In the case of the loss averse consumer, we find that the steady state pricing strategy is the same as the non-reference demand as there is a single pricing strategy which precludes reference effects. The adjustment process, with stochastic demand merits attention and the loss averse case with stochastic demand continues to merit attention. In the loss seeking case, in which price cycles, we find that the introduction of a stochastic component can either increase or decrease the spread between the prices in the cycle. This is clearly an important result. We offer some computational analysis with uniform and triangular distributions.

The final model is the dynamic lot sizing model which includes setup/ordering and holding costs. When prices and production or order quantities are determined simultaneously, the interactions need to be considered to optimize profits. The incorporation of loss averse reference effects add further sophistication. We analyze the problem in detail and offer a series of results which reduce the search space. We then present an algorithm for solving the production/order and pricing problems simultaneously in the presence of loss averse reference effects. The inclusion of reference effects does change the optimal decision.

7.2 Further Research

The extension of reference price models offered here sets the foundation for considerable additional research going forward. Reference prices are well established in the empirical marketing literature. Despite the complexity inherent in the models there is clear value in expanding the application of models that include reference prices into a broader range of supply chain and operations models. The reference models have potential to yield interesting results and meaningful managerial insight.

The work of Mazumdar et al. [2005] highlighted the need for more reference price research to provide managers with insight into pricing and promotional strategies. The previous models at the aggregate level (see for example Kopalle et al. [1996], Fibich et al. [2007]) have been constrained to either a constant price (loss averse consumers) or a cyclical price (loss seeking consumers). This constrains the analysis of pricing strategies. The consideration of dynamic lot sizing or threshold models allows there to be both cyclical and single price strategies. These more robust models can be applied in a variety of contexts to generate insight into both pricing and production/ordering strategies.

This initial analysis was done in a monopolistic context. This can provide significant insight, particularly in the context of frequently purchased items such as groceries where switching costs are high and there are inter store price differences. There would clearly be potential additional insight from consideration of a multi-seller oligopolistic market. The analysis of the pricing strategy arising from a threshold model in a competitive context is a logical extension of the work here.

This initial work also focused on individual consumer segments as was the case in all of previous firm level reference price modelling (see for example Anderson et al. [2005], Greenleaf [1995]). The analysis of multiple consumer segments with different demand, reference price formation, threshold, gain and loss parameters could also provide valuable managerial insight. The existence of price cycles (or an EDLP) strategy may be affected by different consumer characteristics.

There has been some suggestion that the product category and variability within the category is one of the factors that contributes to the formation of thresholds. There is also evidence in the literature that different products and retail stores have different degrees of sensitivity to pricing actions. The consideration of the impact of complements and/or substitutes could also provide additional rigour and insight into this pricing analysis. This would take the analysis beyond anything published to date.

The three models in this work were each analyzed independently and showed some promise. It would be interesting to integrate the models to assess the impact on pricing strategy. Including a stochastic element of demand into a threshold reference model, for example, could yield interesting insight into stocking and pricing decisions. It is worth noting that the stochastic model results indicated that the size of the price spreads are likely different with a stochastic component to demand. This would suggest that there might be interesting insight for pricing strategy and stocking decisions in a threshold context.

The real value in these reference price models is to provide insight to managers into optimal pricing strategies. This work has focused on the role of internal reference price in affecting pricing strategies. That is, the focus has been on the impact of previous purchase experiences on the current purchase decision. External reference price, in which a retailer provides a signal of the regular price or the degree of discount can also play a role. It may play a role in moderating the impact of an external reference price on the internal reference price due to a promotional or sale price. There is some evidence in the marketing literature to this effect. If an external reference price might modulate the downward revision of reference price, particularly in a threshold model, the pricing strategy might be affected. In fact, it may allow for cyclical pricing in a loss averse reference model. There is merit in considering this potential in future work.

The formulation of the threshold model presented here did not include any measure of the cost of changing prices. While it may be argued that the cost of price change is very small with the current technology, there is also some indication in the literature that consumers begin to adapt their purchasing patterns (their demand per period) to reflect the predictable promotional schedule. It would be interesting to develop a pricing strategy based on a differing time line which may not be optimal generally but be better than a single price and preclude the adjustment of consumers. This could also include provision for two prices, a regular price and a sale price, and a time varying promotional pattern.

Our focus in this work has been on developing optimal steady state pricing strategies. We built explicit "adjustment periods" into the beginning and end of the models we analyzed to make allowance for the adjustment to the steady state and the price reductions at the end of the planning horizon. It is important for managers to understand the steady state pricing strategies. There would also be value in understanding the approach to price adjustments in cases such as cost changes, which would change the steady state strategy and may merit a period of adjustment rather than a one period change to the new price. It is also clear that products come to the end of the product lifecycle or selling season and the optimal strategy with reference prices merits research attention. There may also be interesting insight into the product rollover problem as price changes from one product to the next. Seasonal demand or time varying demand parameters would also merit consideration in both steady state and transitional contexts.

Greenleaf [1995] is the only one, to our knowledge, who has to date estimated an aggregate reference demand function. The empirical implementation of, in particular, the threshold model could provide more specific validation for the approach and an explicit insight into optimal pricing strategies. It would be interesting to estimate a threshold model with some grocery data.

It is clear that the foundation laid by the analysis presented here has the potential to yield considerable additional research opportunities.

Appendix A

Summary of Notation

The following is a comprehensive synthesis of the notation used in this dissertation.

A.1 General Notation

D_t = demand in period t

a = intercept in linear demand function

b = slope parameter in linear demand function - the direct price effect

p_t = retail price in period t

r_t = is the reference price in period t

β = reference effect parameter. This is the effect of a "good deal" or "bad deal" represented by the difference between reference price and actual retail price (the reference gap).

β_L = reference loss parameter in the case where reference effects are asymmetric

β_G = reference gain parameter in the case where reference effects are asymmetric

α = the memory parameter in reference price formation

$T =$ the number of periods in the planning horizon

$\pi_t =$ profit in period t

$c =$ cost which is assumed to be constant over time

$g_t(p) =$ non reference component of the profit function

$p_t^* =$ the optimizer of $g(p)$

$f(t) =$ value of recursion in time t

$\theta =$ is a discount factor

$J =$ vector of discrete prices

$M =$ cycle length in a repeating pricing cycle

$p_{c,m} =$ price in the m^{th} period in an M period cycle

$p_{c,\min} =$ the minimum price in an M period cycle

$p_{c,\max} =$ the maximum price in an M period cycle

A.2 Threshold Model

$\tau =$ absolute gain threshold, $\tau \geq 0$

$\rho =$ absolute loss threshold, $\rho \leq 0$

$\omega =$ percentage gain threshold

$\psi =$ percentage loss threshold

$$A = \sum_{m=1}^{M} (p_{c,m} - c)\beta_G(r_{c,m} - \tau - p_{c,m})^+,$$

$$B = \sum_{m=1}^{M} (g(p^*) - g(p_{c,m})), \text{ and}$$

$$C = \sum_{m=1}^{M} (p_{c,m} - c)\beta_L(r_{c,m} + \rho - p_{c,m})^-.$$

A.2.1 Math Programming Model

$YG_t =$ binary variable equals 1 if gain occurs in period t

$YL_t =$ binary variable equals 1 if loss occurs in period t

$w_i =$ weight for reference price approximation

$G_t =$ reference gain in period t

L_t = reference loss in period t

A.3 Stochastic Model

ϵ = random component of demand

ϕ = distribution function of ϵ

Φ = cumulative distribution function of ϵ

O = lower limit on ϵ

H = upper limit on ϵ

μ = mean of ϵ

z = stocking factor

u_t = realized demand in period t

h = disposal cost

SH_t = shortage cost in period t

EX_t = excess inventory in period t

A.4 Inventory Model

I_t = opening inventory in period t

x_t = binary variable equals 1 if production occurs in period t and 0 otherwise.

CP = capacity

h = holding cost

K = fixed production cost

c = production/ordering cost

Appendix B

Threshold Computational Results

Table B.1: Threshold Results ($\beta_L = 0.25$, $\beta_G = 0.2$ $\alpha = 0.35$)

ρ	τ	M	Prices	Reference Prices[1]	Average Profit
0.1	0	4	2.55, 2.76, 2.82, 2.89	2.86, 2.66, 2.72, 2.79	1.0410
0.2	0	3	2.54, 2.88, 3.01	2.94, 2,68, 2.81	1.0583
0.3	0	3	2.44, 2.95, 3.14	3.04, 2.65, 2.85	1.0709
0.4	0	2	2.45, 3.08	2.92, 2.61	1.0755
0.5	0	2	2.45, 3.08	2.92, 2.61	1.0843
0.2	0.1	4	2.40, 2.81, 2.94, 3.07	3.00, 2.61, 2.74, 2.87	1.0468
0.3	0.1	3	2.45, 2.95, 3.14	3.04, 2.65, 2.85	1.0580
0.5	0.1	2	2.42, 3.10	2.93, 2.60	1.0681
0.2	0.2	5	2.27, 2.75, 2.88, 3.01, 3.14	3.07, 2.55, 2.68, 2.81, 2.94	1.0363
0.3	0.2	3	2.42, 2.93, 3.12	3.02, 2.63, 2.83	1.0452
0.5	0.2	3	2.31, 3.06, 3.41	3.23, 2.64, 2.91	1.0510
0.2	0.3	4	2.41, 2.82, 2.95, 3.08	3.01, 2.62, 2.75, 2.88	1.0277
0.3	0.3	3	2.37, 2.91, 3.11	3.00, 2.59, 2.80	1.0312
0.5	0.3	3	2.25, 2.97, 3.33	3.16, 2.57, 2.83	1.0413

Table B.2: Threshold Results ($\beta_L = 0.2$, $\beta_G = 0.1$, $\alpha = 0.2$)

ρ	τ	M	Prices	Reference Prices	Average Profit
0.1	0	3	2.63, 2.77, 2.85	2.83, 2.67, 2.75	1.0250
0.2	0	3	2.54, 2.82, 2.98	2.94, 2.62, 2.78	1.0329
0.3	0	2	2.60, 2.96	2.90, 2.66	1.0373
0.4	0	2	2.57, 3.05	2.97, 2.65	1.0417
0.5	0	2	2.54, 3.14	3.04, 2.64	1.0439
0.2	0.1	3	2.53, 2.82, 2.98	2.94, 2.62, 2.78	1.0264
0.3	0.1	3	2.45, 2.87, 3.11	3.05, 2.57, 2.81	1.0294
0.5	0.1	3	2.42, 2.83, 3.14	3.07, 2.55, 2.77	1.0297
0.2	0.2	5	2.45, 2.73, 2.79, 2.87, 3.05	3.01, 2.53, 2.69, 2.77, 2.85	1.0187
0.3	0.2	3	2.45, 2.87, 3.11	3.05, 2.57, 2.81	1.0230
0.5	0.2	3	2.41, 2.83, 3.16	3.03, 2.55, 2.79	1.0233
0.2	0.3	5	2.50, 2.75, 2.79, 2.91, 3.08	3.05, 2.61, 2.72, 2.76, 2.88	1.0146
0.3	0.3	3	2.50, 2.83, 3.00	2.96, 2.59, 2.78	1.0147
0.5	0.3	3	2.50, 2.83, 3.00	2.96, 2.59, 2.78	1.0147

Table B.3: Threshold Results ($\beta_L = 0.2$, $\beta_G = 0.1$, $\alpha = 0.35$)

ρ	τ	M	Prices	Reference Prices	Average Profit
0.1	0	3	2.64, 2.81, 2.87	2.84, 2.71, 2.78	1.0245
0.2	0	3	2.51, 2.85, 2.98	2.91, 2.65, 2.78	1.0313
0.3	0	2	2.58, 2.99	2.89, 2.69	1.0356
0.4	0	2	2.53, 3.07	2.93, 2.67	1.0380
0.5	0	2	2.51, 3.14	2.98, 2.68	1.0405
0.2	0.1	3	2.55, 2.82, 2.94	2.88, 2.66, 2.77	1.0225
0.3	0.1	3	2.43, 2.90, 3.10	3.00, 2.62, 2.80	1.0259
0.5	0.1	3	2.43, 2.90, 3.10	3.00, 2.62, 2.80	1.0259
0.2	0.2	5	2.38, 2.76, 2.79, 2.87, 3.03	2.96, 2.59, 2.70, 2.76, 2.83	1.0175
0.3	0.2	3	2.42, 2.90, 3.10	3.00, 2.62, 2.80	1.0196
0.5	0.2	3	2.41, 2.87, 3.11	3.00, 2.62, 2.78	1.0200
0.2	0.3	5	2.40, 2.76, 2.79, 2.87, 3.03	2.96, 2.59, 2.70, 2.76, 2.83	1.0137
0.3	0.3	5	2.36, 2.76, 2.79, 2.86, 3.05	2.97, 2.57, 2.69, 2.76, 2.82	1.0138
0.5	0.3	5	2.36, 2.76, 2.79, 2.86, 3.05	2.97, 2.57, 2.69, 2.76, 2.82	1.0138

Table B.4: Threshold Results ($\beta_L = 0.1$, $\beta_G = 0.05$, $\alpha = 0.2$)

ρ	τ	M	Prices	Reference Prices	Average Profit
0.1	0	3	2.63, 2.77, 2.85	2.83, 2.67, 2.75	1.0179
0.2	0	3	2.55, 2.81, 2.96	2.93, 2.62, 2.77	1.0194
0.3	0	2	2.58, 2.97	2.90, 2.64	1.0216
0.4	0	2	2.58, 2.97	2.90, 2.64	1.0216
0.5	0	2	2.58, 2.97	2.90, 2.64	1.0216
0.2	0.1	3	2.54, 2.80, 2.96	2.92, 2.62, 2.73	1.0160
0.3	0.1	3	2.54, 2.80, 2.96	2.92, 2.62, 2.73	1.0160
0.5	0.1	3	2.54, 2.80, 2.96	2.92, 2.62, 2.73	1.0160
0.2	0.2	4	2.52, 2.76, 2.79, 2.95	2.91, 2.60, 2.73, 2.78	1.0127
0.3	0.2	4	2.52, 2.76, 2.79, 2.95	2.91, 2.60, 2.73, 2.78	1.0127
0.5	0.2	4	2.52, 2.76, 2.79, 2.95	2.91, 2.60, 2.73, 2.78	1.0127
0.2	0.3	1	2.75	2.75	1.0125
0.3	0.3	1	2.75	2.75	1.0125
0.5	0.3	1	2.75	2.75	1.0125

Table B.5: Threshold Results ($\beta_L = 0.1$, $\beta_G = 0.05$, $\alpha = 0.35$)

ρ	τ	M	Prices	Reference Prices	Average Profit
0.1	0	3	2.64, 2.80, 2.86	2.83, 2.71, 2.77	1.0174
0.2	0	2	2.64, 2.91	2.84, 2.71	1.0194
0.3	0	2	2.59, 2.95	2.86, 2.69	1.0199
0.4	0	2	2.59, 2.95	2.86, 2.69	1.0199
0.5	0	2	2.59, 2.95	2.86, 2.69	1.0199
0.2	0.1	3	2.55, 2.82, 2.94	2.88, 2.66, 2.77	1.0149
0.3	0.1	3	2.55, 2.82, 2.94	2.88, 2.66, 2.77	1.0149
0.5	0.1	3	2.55, 2.82, 2.94	2.88, 2.66, 2.77	1.0149
0.2	0.2	1	2.75	2.75	1.0125
0.3	0.2	1	2.75	2.75	1.0125
0.5	0.2	1	2.75	2.75	1.0125
0.2	0.3	1	2.75	2.75	1.0125
0.3	0.3	1	2.75	2.75	1.0125
0.5	0.3	1	2.75	2.75	1.0125

Table B.6: Threshold Results ($\beta_L = 0.2$, $\beta_G = 0.05$, $\alpha = 0.2$)

ρ	τ	M	Prices	Reference Prices	Average Profit
0.1	0	3	2.63, 2.77, 2.85	2.83, 2.67, 2.75	1.0179
0.2	0	3	2.55, 2.81, 2.96	2.93, 2.62, 2.77	1.0194
0.3	0	2	2.58, 2.97	2.90, 2.64	1.0216
0.4	0	2	2.58, 2.97	2.90, 2.64	1.0216
0.5	0	2	2.58, 2.97	2.90, 2.64	1.0216
0.2	0.1	3	2.54, 2.80, 2.96	2.92, 2.62, 2.73	1.0160
0.3	0.1	3	2.54, 2.80, 2.96	2.92, 2.62, 2.73	1.0160
0.5	0.1	3	2.54, 2.80, 2.96	2.92, 2.62, 2.73	1.0160
0.2	0.2	4	2.52, 2.76, 2.79, 2.95	2.91, 2.60, 2.73, 2.78	1.0127
0.3	0.2	4	2.52, 2.76, 2.79, 2.95	2.91, 2.60, 2.73, 2.78	1.0127
0.5	0.2	4	2.52, 2.76, 2.79, 2.95	2.91, 2.60, 2.73, 2.78	1.0127
0.2	0.3	1	2.75	2.75	1.0125
0.3	0.3	1	2.75	2.75	1.0125
0.5	0.3	1	2.75	2.75	1.0125

Table B.7: Threshold Results ($\beta_L = 0.2$, $\beta_G = 0.05$,$\alpha = 0.35$)

ρ	τ	M	Prices	Reference Prices	Average Profit
0.1	0	3	2.64, 2.80, 2.86	2.83, 2.71, 2.77	1.0174
0.2	0	2	2.64, 2.91	2.84, 2.71	1.0194
0.3	0	2	2.59, 2.95	2.86, 2.69	1.0199
0.4	0	2	2.59, 2.95	2.86, 2.69	1.0199
0.5	0	2	2.59, 2.95	2.86, 2.69	1.0199
0.2	0.1	3	2.55, 2.82, 2.94	2.88, 2.66, 2.77	1.0149
0.3	0.1	3	2.55, 2.82, 2.94	2.88, 2.66, 2.77	1.0149
0.5	0.1	3	2.55, 2.82, 2.94	2.88, 2.66, 2.77	1.0149
0.2	0.2	1	2.75	2.75	1.0125
0.3	0.2	1	2.75	2.75	1.0125
0.5	0.2	1	2.75	2.75	1.0125
0.2	0.3	1	2.75	2.75	1.0125
0.3	0.3	1	2.75	2.75	1.0125
0.5	0.3	1	2.75	2.75	1.0125

Appendix C

Stochastic Model Computational Results

Table C.1: Loss Averse Stochastic Results (Uniform Distribution, $c = 0.5$)

Upper Limit	s	h	z	p^*	Riskless π	p^{opt}	π^{opt}	% diff in π
0.045	2.25	-0.49	0.045	2.81	1.064	2.81	1.064	0
0.090	2.25	-0.49	0.090	2.86	1.116	2.86	1.116	0
0.225	2.25	-0.49	0.225	3.03	1.280	3.03	1.280	0
0.045	2.25	0.05	0.040	2.81	1.064	2.81	1.053	0
0.090	2.25	0.05	0.080	2.86	1.116	2.86	1.094	0
0.225	2.25	0.05	0.202	3.03	1.281	3.03	1.226	1
0.045	0.50	-0.49	0.045	2.81	1.064	2.81	1.064	0
0.090	0.50	-0.49	0.090	2.86	1.116	2.86	1.116	0
0.225	0.50	-0.49	0.224	3.03	1.281	3.03	1.280	0
0.045	0.50	0.05	0.038	2.81	1.064	2.81	1.060	0
0.090	0.50	0.05	0.075	2.86	1.116	2.86	1.112	0
0.225	0.50	0.05	0.190	3.03	1.281	3.03	1.271	1

Table C.2: Loss Averse Stochastic Results (Triangular Distribution, Symmetric, $c = 0.5$)

Upper Limit	Mode	s	h	z	p^*	Riskless π	p^{opt}	π^{opt}	% diff in π
0.045	0.0225	2.25	-0.49	0.0433	2.81	1.064	2.81	1.064	0
0.090	0.0450	2.25	-0.49	0.0867	2.86	1.116	2.86	1.116	0
0.225	0.1125	2.25	-0.49	0.2168	3.03	1.281	3.03	1.281	0
0.045	0.0225	2.25	0.05	0.0333	2.81	1.064	2.81	1.064	1
0.090	0.0450	2.25	0.05	0.0669	2.86	1.116	2.86	1.116	1
0.225	0.1125	2.25	0.05	0.1682	3.03	1.281	3.03	1.281	3
0.045	0.0225	0.50	-0.49	0.0431	2.81	1.064	2.81	1.281	0
0.090	0.0450	0.50	-0.49	0.0862	2.86	1.116	2.86	1.064	0
0.225	0.1125	0.50	-0.49	0.2159	3.03	1.281	3.03	1.116	0
0.045	0.0225	0.50	0.05	0.0321	2.81	1.064	2.81	1.062	1
0.090	0.0450	0.50	0.05	0.0644	2.86	1.116	2.86	1.064	1
0.225	0.1125	0.50	0.05	0.1626	3.03	1.281	3.03	1.116	1

Table C.3: Loss Averse Stochastic Results (Triangular Distribution, Left Skewed, $c = 0.5$)

Upper Limit	Mode	s	h	z	p^*	Riskless π	p^{opt}	π^{opt}	% diff in π
0.045	0	2.25	-0.49	0.0246	2.79	1.047	2.79	1.046	0
0.090	0	2.25	-0.49	0.0852	2.83	1.081	2.82	1.081	0
0.225	0	2.25	-0.49	0.2130	2.94	1.188	2.94	1.187	0
0.045	0	2.25	0.05	0.0284	2.79	1.047	2.79	1.036	1
0.090	0	2.25	0.05	0.0569	2.83	1.081	2.82	1.061	2
0.225	0	2.25	0.05	0.1425	2.94	1.188	2.93	1.138	4
0.045	0	0.50	-0.49	0.0423	2.79	1.047	2.79	1.046	0
0.090	0	0.50	-0.49	0.0846	2.83	1.081	2.82	1.081	0
0.225	0	0.50	-0.49	0.2115	2.94	1.188	2.94	1.187	0
0.045	0	0.50	0.05	0.0267	2.79	1.047	2.78	1.037	1
0.090	0	0.50	0.05	0.0534	2.83	1.081	2.82	1.063	2
0.225	0	0.50	0.05	0.1337	2.94	1.188	2.92	1.141	4

Table C.4: Loss Averse Stochastic Results (Triangular Distribution, Right Skewed, $c = 0.5$)

Upper Limit	Mode	s	h	z	p^*	Riskless π	p^{opt}	π^{opt}	% diff in π
0.045	0	2.25	-0.49	0.0246	2.79	1.047	2.79	1.046	0
0.090	0	2.25	-0.49	0.0852	2.83	1.081	2.82	1.081	0
0.225	0	2.25	-0.49	0.2130	2.94	1.188	2.94	1.187	0
0.045	0	2.25	0.05	0.0284	2.79	1.047	2.79	1.036	1
0.090	0	2.25	0.05	0.0569	2.83	1.081	2.82	1.061	2
0.225	0	2.25	0.05	0.1425	2.94	1.188	2.93	1.138	4
0.045	0	0.50	-0.49	0.0423	2.79	1.047	2.79	1.046	0
0.090	0	0.50	-0.49	0.0846	2.83	1.081	2.82	1.081	0
0.225	0	0.50	-0.49	0.2115	2.94	1.188	2.94	1.187	0
0.045	0	0.50	0.05	0.0267	2.79	1.047	2.78	1.037	1
0.090	0	0.50	0.05	0.0534	2.83	1.081	2.82	1.063	2
0.225	0	0.50	0.05	0.1337	2.94	1.188	2.92	1.141	4

Table C.5: Loss Averse Stochastic Results (Uniform Distribution, $c = 2.5$)

Upper Limit	s	h	z	p^*	Riskless π	p^{opt}	π^{opt}	% diff in π
0.045	2.25	-0.49	0.0446	3.81	0.341	3.81	0.341	0
0.090	2.25	-0.49	0.0893	3.86	0.371	3.86	0.370	0
0.225	2.25	-0.49	0.2234	4.03	0.469	4.03	0.467	0
0.045	2.25	0.05	0.0225	3.81	0.341	3.79	0.327	4
0.090	2.25	0.05	0.0450	3.86	0.371	3.83	0.342	8
0.225	2.25	0.05	0.1093	4.03	0.469	3.96	0.394	16
0.045	0.50	-0.49	0.0445	3.81	0.341	3.81	0.341	0
0.090	0.50	-0.49	0.0890	3.86	0.371	3.86	0.370	0
0.225	0.50	-0.49	0.2228	4.03	0.469	4.03	0.467	0
0.045	0.50	0.05	0.0186	3.81	0.341	3.79	0.327	4
0.090	0.50	0.05	0.0370	3.86	0.371	3.82	0.343	8
0.225	0.50	0.05	0.0823	4.03	0.469	3.92	0.395	16

Table C.6: Loss Averse Stochastic Results (Triangular Distribution, Symmetric, $c = 2.5$)

Upper Limit	Mode	s	h	z	p^*	Riskless π	p^{opt}	π^{opt}	% diff in π
0.045	0.0225	2.25	-0.49	0.0422	3.81	0.341	3.81	0.341	0
0.090	0.0450	2.25	-0.49	0.0845	3.86	0.371	3.86	0.370	0
0.225	0.1125	2.25	-0.49	0.2116	4.03	0.469	4.03	0.467	0
0.045	0.0225	2.25	0.05	0.0225	3.81	0.341	3.80	0.322	6
0.090	0.0450	2.25	0.05	0.0452	3.86	0.371	3.84	0.352	6
0.225	0.1125	2.25	0.05	0.1145	4.03	0.469	3.99	0.415	12
0.045	0.0225	0.50	-0.49	0.0417	3.81	0.341	3.81	0.341	0
0.090	0.0450	0.50	-0.49	0.834	3.86	0.371	3.86	0.370	0
0.225	0.1125	0.50	-0.49	0.2093	4.03	0.469	4.03	0.467	0
0.045	0.0225	0.50	0.05	0.0205	3.81	0.341	3.79	0.325	5
0.090	0.0450	0.50	0.05	0.0413	3.86	0.371	3.84	0.339	9
0.225	0.1125	0.50	0.05	0.1053	4.03	0.469	3.97	0.386	18

Table C.7: Loss Averse Stochastic Results (Triangular Distribution, Left Skewed, $c = 2.5$)

Upper Limit	Mode	s	h	z	p^*	Riskless π	p^{opt}	π^{opt}	% diff in π
0.045	0.0225	2.25	-0.49	0.0410	3.79	0.332	3.79	0.331	0
0.090	0.0450	2.25	-0.49	0.0821	3.83	0.351	3.82	0.350	0
0.225	0.1125	2.25	-0.49	0.2057	3.94	0.413	3.94	0.411	1
0.045	0.0225	2.25	0.05	0.0131	3.79	0.332	3.77	0.323	3
0.090	0.0450	2.25	0.05	0.0264	3.83	0.351	3.8	0.333	5
0.225	0.1125	2.25	0.05	0.0670	3.94	0.413	3.87	0.365	12
0.045	0.0225	0.50	-0.49	0.4030	3.79	0.332	3.79	0.331	0
0.090	0.0450	0.50	-0.49	0.0806	3.83	0.351	3.82	0.350	0
0.225	0.1125	0.50	-0.49	0.2022	3.94	0.413	3.94	0.411	1
0.045	0.0225	0.50	0.05	0.0104	3.79	0.332	3.77	0.331	0
0.090	0.0450	0.50	0.05	0.0210	3.83	0.351	3.79	0.350	0
0.225	0.1125	0.50	0.05	0.0539	3.94	0.413	3.86	0.405	2

Table C.8: Loss Averse Stochastic Results (Triangular Distribution, Right Skewed, $c = 2.5$)

Upper Limit	Mode	s	h	z	p^*	Riskless π	p^{opt}	π^{opt}	% diff in π
0.045	0.0225	2.25	-0.49	0.448	3.83	0.351	3.82	0.351	0
0.090	0.0450	2.25	-0.49	0.897	3.90	0.392	3.90	0.391	0
0.225	0.1125	2.25	-0.49	0.2242	4.13	0.528	4.12	0.527	0
0.045	0.0225	2.25	0.05	0.319	3.83	0.351	3.82	0.329	6
0.090	0.0450	2.25	0.05	0.0641	3.90	0.392	3.88	0.347	12
0.225	0.1125	2.25	0.05	0.1632	4.13	0.528	4.09	0.411	22
0.045	0.0225	0.50	-0.49	0.0448	3.83	0.351	3.82	0.351	0
0.090	0.0450	0.50	-0.49	0.0895	3.90	0.392	3.90	0.391	0
0.225	0.1125	0.50	-0.49	0.2239	4.13	0.528	4.12	0.527	0
0.045	0.0225	0.50	0.05	0.0290	3.83	0.351	3.81	0.332	6
0.090	0.0450	0.50	0.05	0.0586	3.90	0.392	3.88	0.353	10
0.225	0.1125	0.50	0.05	0.1506	4.13	0.528	4.07	0.425	20

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