ALTERNATE DUALS OF GABOR SUBSPACE FRAMES

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By

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENT FOR THE DEGREE OF
MASTER OF SCIENCE
AT
MCMASTER UNIVERSITY
HAMILTON, ONTARIO
AUGUST 2005

McMASTER UNIVERSITY

DEGREE:	Master of Science, 2005
DEPARTMENT:	Mathematics and Statistics, Hamilton, Ontario
UNIVERSITY:	McMaster University
TITLE:	Alternate Duals of Gabor Subspace Frames
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SUPERVISOR:	Jean-Pierre Gabardo
PAGES:	vii, 74

McMASTER UNIVERSITY

DEPARTMENT OF

MATHEMATICS AND STATISTICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a project entitled "**Alternate duals of Gabor subspace frames**" by **Mehmet Ali Akinlar** in partial fulfillment of the requirement for the degree of **Master of Science**.

Dated: August 2005

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To Mom and Dad

for your love and encouragement

Contents

Abstract

In this thesis we mainly give a characterization of dual frames of Gabor subspace frames. We give necessary and sufficient conditions for the existence and the uniqueness of a function h (called window) in the closed linear span of a Gabor subspace frame $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ such that the Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ serves as the dual frame of the original frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. We solve the problem for three cases, first $ab = 1$, second $ab = p \in \mathbf{N}$, and third $ab = p/q$, $\gcd(p, q) = 1$. In each case, we first find the conditions for upper frame bound (known as Bessel collection). Secondly, we characterize the functions which are orthogonal to $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ in terms of the Zak transform, and then obtain necessary and sufficient conditions for lower frame bound. Here we state obtained conditions for normalized tight frame as a corollary. Finally, using all this information we solve the duality problem.

Keywords: Bessel collection, Frames, Gabor Frames, Alternate dual frames, The Zak Transform

Acknowledgements

I would love to thank my supervisor, Dr. Jean-Pierre Gabardo, for his instructions and guidance. He taught me most of the Fourier analysis that I know, and inspired me to first conceptualize a problem by working for special cases, and then extending it to the general case. I deeply acknowledge for his overwhelming generosity with his time, suggestions and insightful comments. I would also like to acknowledge that this thesis is the outcome of my graduate studies in mathematics, and it was conducted at and financed by the Department of Mathematics and Statistics, McMaster University.

My sincere thanks go to Dr. Slim Ibrahim for very useful conversations and helpful explanations. As I move forward in the pursuit of my Ph.D in Software Engineering, I would like to extend my gratitude to Dr. Martin Von Mohrenschildt for accepting me as his future Ph.D student, whose support and interest in my work kept me going. I look forward to working together with him very soon.

I also would like to thank you very much to Dr. Walter Craig and Dr. Bartosz Protas for accepting to be in my committee. As for the preparation of the thesis, most of the manuscript was typeset by Debbie Iscoe and, who I thank for her patience and diligence, and for doing a wonderful job. Finally, I would like to extend my thanks to my parents, my brother and my sisters, and my friends for being extremely supportive and committed to my career.

Chapter 1

Introduction

The theory of frames is a relatively new and fast developing subject in mathematics. It is a very useful tool for Functional Analysis and Fourier Analysis as well as wavelets. Although some of the topics in frame theory have only been recently formalized, frames have deep roots in physics and engineering. They play an extremely useful role in applications such as signal processing, image processing, data compression and sampling theory.

The general concept of frames was first introduced by R. J. Duffin and A. C. Schaeffer in connection with non-harmonic Fourier series [DS]. The Hilbert space under consideration here will be $L^2(\mathbf{R})$, the space of all finite-energy signals on the real line $\mathbf{R}$.

This thesis provides an introduction to Bessel sequences and to the theory of Gabor subspace frames and their dual frames and gives a characterization for the existence and uniqueness of a window function h in the linear span of a Gabor subspace frame $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ such that the Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ serves as a dual frame of the original Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$.

In the thesis some of the results are due to others and a more detailed and complete treatment to the frame theory and Gabor subspace frames can be found in the expository papers by [Ca] and [DS], [HW], [CE]. O. Christensen and Y. C.

Eldar presented an article titled "Oblique dual frames and Shift-invariant spaces" in 2004. (see [CE]). In this paper, their main focus was on shift-invariant frame sequences of the form $\{\phi(\cdot - k)\}_{k\in\mathbf{Z}}$ in subspaces of $L^2(\mathbf{R})$ where $\phi(\cdot - k) = T_k\phi(\cdot)$ for $k \in \mathbf{Z}$ is a translation operator on $\mathbf{R}$ that we define in the equation (1.1.7) (below). For such frame sequences they characterized the set of shift-invariant oblique dual Bessel sequences. They characterized the oblique dual frame on a closed subspace $\mathcal{V}$ of a Hilbert space $\mathcal{H}$ in the following theorem.

Theorem 1.0.1 *([CE]) Let $\{f_k\}_{k=1}^{\infty}$ be a frame for a subspace $\mathcal{W} \subseteq \mathcal{H}$, and let $\mathcal{V}$ be a closed subspace such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$. Then the oblique dual frames of $\{f_k\}_{k=1}^{\infty}$ on $\mathcal{V}$ are precisely families*

$$\{g_k\}_{k=1}^{\infty} = \left\{ E_{\mathcal{V}\mathcal{W}^{\perp}} S^{-1} f_k + h_k - \sum_{j=1}^{\infty} \langle S^{-1} f_k, f_j \rangle h_j \right\}_{k=1}^{\infty}$$

where $\{h_k\}_{k=1}^{\infty} \subset \mathcal{V}$ is a Bessel collection, and S^{-1} stands for the inverse of the frame operator S, which is a bounded, invertible and positive mapping of $\mathcal{V}$ onto itself (defined by the equation 1.1.4 below), and $E_{\mathcal{V}\mathcal{W}^{\perp}}$ denotes the oblique projection of $\mathcal{H}$ on $\mathcal{V}$ along $\mathcal{W}^{\perp}$ and defined by $E_{\mathcal{V}\mathcal{W}^{\perp}}(v + w^{\perp}) = v$ for $v \in \mathcal{V}$, $w^{\perp} \in \mathcal{W}^{\perp}$.

In the same paper these authors also showed how to find an oblique dual frame of $\{f_k\}_{k=1}^{\infty}$ on an arbitrary closed subspace $\mathcal{U}$ for which $\mathcal{H} = \mathcal{W} \oplus \mathcal{U}^{\perp}$ for a subspace $\mathcal{W} \subseteq \mathcal{H}$.

Theorem 1.0.2 *([CE]) Assume that $\{f_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are Bessel sequences in $\mathcal{H}$, and that*

$$f = \sum_{k=1}^{\infty} \langle f, h_k \rangle f_k$$

for every $f \in \mathcal{W}$. Let $\mathcal{U}$ be any closed subspace of $\mathcal{H}$ for which $\mathcal{H} = \mathcal{W} \oplus \mathcal{U}^{\perp}$. Then $\{E_{\mathcal{U}\mathcal{W}^{\perp}} h_k\}_{k\in\mathbb{Z}}$ is an oblique dual frame of $\{f_k\}_{k\in\mathbb{Z}}$ on $\mathcal{U}$.

Given frame sequences $\{\phi(\cdot - k)\}_{k\in\mathbf{Z}}$ and $\{\phi_1(\cdot - k)\}_{k\in\mathbf{Z}}$, they gave a condition implying that $\overline{\text{span}}\{\phi_1(\cdot - k)\}_{k\in\mathbf{Z}}$ contains a generator for a shift-invariant dual of $\{\phi(\cdot - k)\}_{k\in\mathbf{Z}}$ in the following way. For $\phi_1 \in L^2(\mathbf{R})$, let $\mathcal{W} = \overline{span}\{T_k\phi_1\}_{k\in Z}$, and denote the orthogonal projection of $L^2(\mathbf{R})$ onto $\mathcal{W}$ by $P_{\mathcal{W}}$. Given two Bessel sequences $\{T_k\phi_1\}_{k\in Z}$ and $\{T_k\phi_2\}_{k\in Z}$, they provided a necessary condition on the generators such that $\{T_k\phi_2\}_{k\in Z}$ is a dual frame of $\{T_k\phi_1\}_{k\in Z}$ in the following theorem:

Theorem 1.0.3 *Let ϕ_1 and $\phi_2 \in L^2(\mathbf{R})$, and assume that $\{T_k\phi_1\}_{k\in Z}$ and $\{T_k\phi_2\}_{k\in Z}$ are Bessel sequences. Then the following are equivalent:*

i-) $f = \sum_{k\in\mathbf{Z}}\langle f, T_k\phi_2\rangle T_k\phi_1$, for every $f \in \mathcal{W}$;

ii-) $\sum_{k\in\mathbf{Z}} \hat{\phi}_1(\gamma + k)\overline{\hat{\phi}_2(\gamma + k)} = 1$ almost everywhere on $\{\gamma : \phi_1(\gamma) \neq 0\}$,

where $\hat{\phi}$ is the Fourier transform of ϕ defined by $\hat{\phi}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}dx$. (As usual, the Fourier transform is extended to a unitary operator on $L^2(\mathbf{R})$.)

If the conditions are satisfied, then $\{T_k\phi_1\}_{k\in Z}$ and $\{P_{\mathcal{W}}T_k\phi_2\}_{k\in Z}$ are dual frames for $\overline{span}\{T_k\phi_1\}_{k\in Z}$.

Finally, assuming that $\{T_k\phi_2\}_{k\in\mathbb{Z}}$ is a frame sequence, they obtained the following conditions on a function ϕ_1 which imply that the subspace

$$\mathcal{V} := \overline{span}\{T_k\phi_1\}_{k\in\mathbf{Z}}$$

contains a function ϕ_2 generating an oblique dual frame $\{T_k\phi\}_{k\in\mathbf{Z}}$ of $\{T_k\phi_2\}_{k\in\mathbf{Z}}$ in the following theorem:

Theorem 1.0.4 *Let ϕ_1 and $\phi_2 \in L^2(\mathbf{R})$, and assume that $\{T_k\phi_2\}_{k\in Z}$ and $\{T_k\phi_1\}_{k\in Z}$ are frame sequences. If there exists a constant $A > 0$ such that*

$$\left|\sum_{k\in\mathbf{Z}} \hat{\phi}(\gamma + k)\overline{\hat{\phi}_1(\gamma + k)}\right| \geq A$$

almost everywhere on $\{\gamma : \phi(\gamma) \neq 0\}$, then the following holds:

(i) There exists a function $\phi \in \mathcal{V} = \overline{span}\{T_k\phi_1\}_{k\in\mathbf{Z}}$ *such that*

$$f = \sum_{k\in\mathbf{Z}} < f, T_k\phi > T_k\phi_2, \quad for \quad every \quad f \in \overline{span}\{T_k\phi\}_{k\in\mathbf{Z}}; \qquad (1.0.1)$$

(ii) One choice of $\phi \in \mathcal{V}$ *satisfying 1.0.1 is given in the Fourier domain by*

$$\hat{\phi}(\gamma) = \begin{cases} \frac{\hat{\phi_1}(\gamma)}{\sum_{k\in\mathbf{Z}} \hat{\phi_2}(\gamma+k)\overline{\hat{\phi_1}(\gamma+k)}}, & on \ \{\gamma : \phi(\gamma) \neq 0\}, \\ 0, & on \ \{\gamma : \phi(\gamma) = 0\}, \end{cases}$$

(iii) There is a unique function $\phi \in \mathcal{V}$ *such that 1.0.1 is satisfied if and only if*

$$\mathcal{N}(\phi_2) = \mathcal{N}(\phi_1);$$

if this condition is satisfied, then $\{T_k\phi\}_{k\in\mathbf{Z}}$ *is a frame for* $\mathcal{V}$ *and an oblique dual of* $\{T_k\phi_2\}_{k\in\mathbf{Z}}$ *on* $\mathcal{V}$*, where* $\mathcal{N}(\cdot)$ *denotes the null space of the function.*

The interested reader can have a look at the paper [CE] by O. Christensen and Y. C. Eldar and make a detailed comparison of each result for Bessel collection, frame condition, and alternate dual frame condition etc. with the results in this thesis. In our thesis, we will consider a modified version of the problem above involving both translation and modulation operators.

J. -P. Gabardo and D. Han presented an article titled "Balian-Low phenomenon for subspace Gabor frames" in August 2004, (See [GH1].) In this work, they extended the Balian-Low theorem to Gabor frames for subspaces, and more particularly, they pointed out the relationship of Balian-Low phenomenon with the unique Gabor dual property for subspace Gabor frames. For the classical situation where $g = k$ for the uniqueness problem in this thesis, they proved that a necessary and sufficient condition for the uniqueness of the Gabor dual belonging to the original subspace is that

$$rank\{G^i\}_{i=0}^{q-1} = \{0, q\},$$

where G^i is a vector valued function defined as $G^i = (G_0^i, \ldots, G_{q-1}^i)$ and $G_k^i(x, w) = Zg^i\,(x, w + k/p)$ for $i = 0, \ldots, q-1$ and $k = 0, \ldots, p-1$ and $g^i(x) = g(x-ip/q)$ and

g is the window function of the original Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. In order to make a comparison with the results in the paper [GH1] of Gabardo and Han by the interested readers, here we want to point out our results for conditions of the uniqueness of the (alternate) dual frame. The dual frame of the original Gabor subspace frame uniquely exists if and only if

(i) $\Omega_1 = \bigcup_i \{(x,w) : G^i(x,w) \neq 0\} = \Omega_2 = \bigcup_i \{(x,w) : K^i(x,w) \neq 0\}$

(ii) $rank\{G^i\}_{i=0}^{q-1} = q$ (i.e, $G^0, \ldots, G^{q-1}$ are linearly independent).

(Note that we can define the vector valued function K^i as we defined the G^i in the previous paragraph.)

The purpose of this thesis is to give a detailed description of the alternate dual frames of Gabor subspace frames and give a solution to the existence and uniqueness of these alternate duals.

In the thesis,

- We give all the background material needed throughout the thesis, including results from Harmonic Analysis, Functional Analysis and Hilbert space theory in the first chapter.

- We give a detailed introduction to the Zak transformation, and point out its relations to Bessel collections and Gabor subspace frames.

- We obtain some necessary and sufficient conditions for the existence of a Bessel collection and lower frame bound for a Gabor subspace frame in terms of the Zak transformation in the so-called rational case.

- We give a simple result for the existence of a dual frame as a corollary for the classical situation where the dual belongs to the original space (i.e., when $g = k$) after we solve the problem.

- In particular, we state our results for the normalized Gabor subspace frames and normalized tight Gabor subspace frames as simple conclusions.

- We conclude the thesis by pointing out some open problems regarding Gabor frames and some recent developments in the frame theory.

The main contribution of the thesis to the subject is:

- Construction of a solution of the problem of finding a necessary and sufficient condition for the existence and uniqueness of a window function in one of the given two Gabor subspace frames such that the Bessel collection generated by that window function serves as the dual frame of the original Gabor subspace frame.

1.1 Frames in Hilbert spaces and Introduction to Gabor subspace frames

In this chapter we introduce some key concepts such as Bessel collection, frame, Gabor subspace frame and the Zak transform as well as some results related with them.

In this section we will describe some of the basic properties of frames in Hilbert spaces, showing that they are useful generalizations of orthonormal bases. By a **Hilbert space**, we mean a vector space, H over $\mathbb{C}$, which possesses an **inner product** $\langle x, y\rangle$ and which is **complete** in the **norm** $\|x\| = \langle x, x\rangle^{1/2}$.

The only Hilbert space that actually we will be using in this thesis is $L^2(\mathbf{R})$, the space of all complex-valued signals f defined on the real line, $\mathbf{R}$, which have finite energy, i.e., for which

$$\|f\| = \left(\int_{\mathbf{R}} |f(t)|^2 dt\right)^{1/2} < \infty.$$

The inner product in this Hilbert space is

$$\langle f, g\rangle = \int_{\mathbf{R}} f(t)\overline{g(t)}dt,$$

where the bar indicates complex conjugation.

Frames were first introduced in 1952 by R. J. Duffin and A. C. Schaeffer in the paper [DS], in connection with nonharmonic Fourier series. Their first use in connection with wavelets was in 1986 in the paper [DGY] by I. Daubechies, A. Grossmann, and Y. Meyer.

We need the following definitions from the theory of frames. A countable sequence $\{g_i\}_{i\in\mathcal{I}}$ of elements in a subspace $\mathcal{W}$ of a separable Hilbert space H is a **Bessel sequence** if there exists a constant $B > 0$ (called **Bessel bound**) such that

$$\sum_{i\in\mathcal{I}} |\langle f, g_i\rangle|^2 \leq B\|f\|^2, \quad \text{for all} \quad f \in \mathcal{W}.$$

If, in addition, $\mathcal{W}$ is a closed subset of H, and there is a constant $0 < A < B$ such that

$$A\|f\|^2 \leq \sum_{i \in \mathcal{I}} |\langle f, g_i \rangle|^2 \leq B\|f\|^2, \quad \text{for all} \quad f \in \mathcal{W},$$

then $\{g_i\}_{i \in \mathcal{I}}$ is called a **frame** for $\mathcal{W}$. The numbers A, B are called the **lower** and **upper frame bounds**, respectively. The frame is **tight** if A and B can be chosen so that $A = B$, and is a **Parseval frame (or normalized tight frame)** if $A = B = 1$.

It is well-known that given any Hilbert space H, there always exists an **orthonormal basis**, i.e., a set of vectors $\{e_n\}_n$ such that

(1) $\langle e_m, e_n \rangle = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{if } m \neq n; \end{cases}$

(2) $\sum_{n \in \mathbf{Z}} |\langle x, e_n \rangle|^2 = \|x\|^2 \quad$ for all $x \in H$.

Every orthonormal basis is clearly a frame with $A = B = 1$. A fundamental property of orthonormal bases is that every element $x \in H$ can be written in terms of the orthonormal basis in a unique way as $x = \sum_{n \in \mathbf{Z}} \langle x, e_n \rangle e_n$. We will see that frames also give representations of elements of the Hilbert space, although these representations need not be unique. However, they are still computable and under good control.

As a trivial example of a frame which is not an orthonormal basis, consider the following. Let $\{e_1, e_2, \ldots\}$ and $\{e'_1, e'_2, \ldots\}$ be two different orthonormal bases for a Hilbert space H. Both of them are surely then a frame for H with bounds $A = B = 1$. However, the set

$$\{e_1/\sqrt{2}, e'_1/\sqrt{2}, e_2/\sqrt{2}, e'_2/\sqrt{2}, \ldots\}$$

is also a frame for H with bounds $A = B = 1$, but is not an orthonormal basis.

Example 1.1.1 An orthonormal basis $(e_n)_{n\in\mathbf{Z}}$ for a Hilbert space H is a normalized tight frame for H. The collection

$$\{e_1, 0, e_2, 0, e_3, 0, \ldots\}$$

is also a normalized tight frame for H. But for the orthonormal basis $(e_n)_{n\in\mathbf{Z}}$ for H, $(\frac{e_n}{n})_{n\in\mathbf{Z}}$ and (ne_n) are not frames since $(\frac{e_n}{n})_{n\in\mathbf{Z}}$ does not have a finite lower frame bound and similarly, (ne_n) fails to have a finite upper frame bound.

Example 1.1.2 Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H.

1. Define $\{x_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, \ldots\}$. Then $\{x_k\}$ is a tight frame with frame bound $A = B = 2$.

2. Define $\{x_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_3, \ldots\}$. Then $\{x_k\}$ is a frame with $A = 1$, $B = 2$.

3. Define $\{x_k\}_{k=1}^{\infty} = \left\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots\right\}$. Then $\{x_k\}$ is a normalized tight frame with $A = B = 1$.

Notation: From now on, $\ell^2(\mathbf{N})$ will denote the Hilbert space of square summable sequences. Elements in $\ell^2(\mathbf{N})$ will be denoted by $c = (c_k),\ d = (d_k), \ldots$ etc.

Lemma 1.1.3 ([El]) *Let $\{f_k\}_{k=1}^{\infty} \subset H$ be a sequence that $\sum_{k\in\mathbf{N}} c_k f_k \in H$ for each $c = (c_k) \in \ell^2(\mathbf{N})$. The operator $T : \ell^2(\mathbf{N}) \to H$ defined by*

$$Tc = \sum_{k=1}^{\infty} c_k f_k$$

is linear and bounded. Its adjoint $T^* : H \to \ell^2(\mathbf{N})$ is defined by

$$T^*f = (\langle f, f_k\rangle).$$

Moreover,

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq \|T\|^2\|f\|^2. \tag{1.1.2}$$

Proof: Define $T_n : \ell^2(\mathbf{N}) \to H$ by

$$T_n c = \sum_{k=1}^{n} c_k f_k.$$

T_n is bounded and $T_n c \to Tc$ for each $c \in \ell^2(\mathbf{N})$. By the uniform boundedness principle, T is linear and bounded. To compute T^* observe first that

$$\langle Tc, f \rangle = \left\langle \sum_{k=1}^{\infty} c_k f_k, f \right\rangle = \sum_{k=1}^{\infty} c_k \langle f_k, f \rangle.$$

Hence, $\sum_{k=1}^{\infty} c_k \langle f_k, f \rangle$ converges for each $c \in \ell^2(\mathbf{N})$. Define the linear functionals $l_n : \ell^2(\mathbf{N}) \to \mathbb{C}$ by

$$l_n c = \sum_{k=1}^{n} c_k \langle f_k, f \rangle.$$

Then $l_n c \to lc$. Therefore, l is a bounded linear functional. By the Riesz Representation Theorem,

$$lc = \sum_{k=1}^{\infty} c_k \bar{a}_k$$

for some $a \in \ell^2(\mathbf{N})$. Thus

$$\sum_{k=1}^{\infty} c_k \langle f_k, f \rangle = \sum_{k=1}^{\infty} c_k \bar{a}_k.$$

Taking c to be the i^{th} standard basis element $\epsilon_i \in \ell^2(\mathbf{N})$, we get

$$\langle f_i, f \rangle = \bar{a}_i \quad \forall\, i \in \mathbf{N}.$$

therefore,

$$a = (\langle f, f_i \rangle) \in \ell^2(\mathbf{N}).$$

Now

$$\begin{aligned} \langle c, T^* f \rangle &= \langle Tc, f \rangle = \sum_{k=1}^{\infty} c_k \langle f_k, f \rangle \\ &= \langle c, (\langle f, f_i \rangle) \rangle_{\ell^2(\mathbf{N})}. \end{aligned}$$

Therefore

$$T^* f = (\langle f, f_i \rangle).$$

Also,

$$\begin{aligned}\sum_{k=1}^{\infty}|\langle f, f_k\rangle|^2 &= \|T^*f\|^2 \le \|T^*\|^2\|f\|^2 \\ &= \|T\|^2\|f\|^2.\end{aligned}$$

□

Now we will give another useful theorem regarding Bessel sequences and Hilbert space of square summable sequences.

Theorem 1.1.4 ([El]) *$\{f_k\}_{k=1}^{\infty} \subset H$ is a Bessel sequence with Bessel bound B if and only if the mapping*

$$c \mapsto \sum_{k=1}^{\infty} c_k f_k$$

defines a linear bounded operator T on $\ell^2(\mathbf{N})$ into H with $\|T\| \le \sqrt{B}$.

Proof: Suppose $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with Bessel bound B. We want to show that the mapping $c \mapsto \sum_{k=1}^{\infty} c_k f_k$ defines a bounded linear operator T from $\ell^2(\mathbf{N})$ into H. This will follow from Lemma 1.1.3 if we can show that $\sum_{k=1}^{\infty} c_k f_k \in H$ for every $c \in \ell^2(\mathbf{N})$. This can be done by showing that $\sum_{k=1}^{n} c_k f_k$ is a Cauchy sequence. Let $m > n$.

$$\begin{aligned}\left\|\sum_{k=1}^{n} c_k f_k - \sum_{k=1}^{m} c_k f_k\right\| &= \left\|\sum_{k=m+1}^{n} c_k f_k\right\| \\ &= \sup_{\|g\|=1}\left|\left\langle \sum_{k=m+1}^{n} c_k f_k, g\right\rangle\right| \\ &= \sup_{\|g\|=1}\left|\sum_{k=m+1}^{n} c_k\langle f_k, g\rangle\right| \\ &\le \sup_{\|g\|=1}\left(\sum_{k=m+1}^{n}|c_k|^2\right)^{1/2}\left(\sum_{k=m+1}^{n}|\langle f_k, g\rangle|^2\right)^{1/2} \\ &= \sqrt{B}\left(\sum_{k=m+1}^{n}|c_k|^2\right)^{1/2}.\end{aligned}$$

This establishes what we want since $\sum_{k=1}^{\infty}|c_k|^2 < \infty$. To check the inequality

$\|T\| \le \sqrt{B}$, we use Lemma 1.1.3 again to get

$$\|T^* f\|^2 = \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \le B\|f\|^2.$$

Hence, $\|T\| = \|T^*\| \le \sqrt{B}$.

$\Leftarrow$: This is the content of Lemma 1.1.3 □

Corollary 1.1.5 *It thus follows that to check that a sequence $\{f_k\}_{k=1}^{\infty} \subset H$ is a Bessel sequence, we only need to check that the operator T is well defined.*

Now we want to give another lemma about Bessel sequences and their applications to frames.

Lemma 1.1.6 ([HW]) *Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be Bessel sequences in a Hilbert space H, and assume that*

$$f_j = \sum_{k=1}^{\infty} \langle f_j, g_k \rangle f_k, \quad \forall\, j \in \mathbf{N}.$$

Then $\{f_k\}_{k=1}^{\infty}$ is a frame for $\overline{span}\{f_k\}_{k=1}^{\infty}$, and

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \quad \forall f \in \overline{span}\{f_k\}_{k=1}^{\infty}. \tag{1.1.3}$$

Proof: The assumptions immediately imply that (1.1.3) holds for $f \in \text{span}\{f_k\}_{k=1}^{\infty}$. By the Bessel assumption, the operator $f \mapsto \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k$ is continuous, and therefore (1.1.3) actually holds for all $f \in \overline{\text{span}}\{f_k\}_{k=1}^{\infty}$. Finally, Cauchy-Schwarz' inequality applied to

$$\|f\|^2 = \sum_{k=1}^{\infty} \langle f, g_k \rangle \langle f_k, f \rangle, f \in \overline{\text{span}}\{f_k\}_{k=1}^{\infty},$$

yields the announced frame property. □

Definition 1.1.7 Let H be a Hilbert space, and suppose that $(x_n)_{n \in \mathbf{Z}}$ is a frame a subspace $W \subseteq H$. A Bessel sequence $(y_n)_{n \in \mathbf{Z}}$ for the subspace W is called an

alternate dual, or dual in short for $(x_n)_{n\in\mathbf{Z}}$ if

$$x = \sum_n \langle x, y_n \rangle x_n,$$

for every $x \in W$ and $n \in \mathbf{Z}$.

Now our goal is to give the definition of the standard dual frame of a subspace of a Hilbert space, and then give the relations between standard duals and alternate duals.

Given a frame $\{g_i\}_{i\in\mathcal{I}}$ of H with frame bounds α and β, **the frame operator** S, defined by $Sf = \sum_{i\in\mathcal{I}} \langle f, g_i \rangle g_i$ is a bounded, invertible and positive mapping of H onto itself. This provides the **frame decomposition:**

$$f = S(S^{-1}f) = \sum_{i\in\mathcal{I}} \langle S^{-1}f, g_i \rangle g_i = \sum_{i\in\mathcal{I}} \langle f, S^{-1}g_i \rangle g_i, \quad \text{for all} \quad f \in H, \qquad (1.1.4)$$

with convergence in H. The sequence $\{S^{-1}g_i\}_{i\in\mathcal{I}}$ is also a frame for H, called **the canonical dual frame** of $\{g_i\}_{i\in\mathcal{I}}$, and has upper and lower frame bounds β^{-1} and α^{-1}, respectively. If the frame is tight, then $S^{-1} = \alpha^{-1}I$, where I is the identity operator, and the frame decomposition becomes:

$$f = \frac{1}{\alpha} \sum_{i\in\mathcal{I}} \langle f, g_i \rangle g_i \quad \text{for all} \quad f \in H, \qquad (1.1.5)$$

with convergence in H. Equations (1.1.3) and (1.1.4) show that a frame provides a basis-like representation. In general, however, a frame need not be a basis, and the elements $\{g_i\}_{i\in\mathcal{I}}$ need not be linearly independent.

Note also that a frame may have many dual frames. For instance, let our frame be $\{e_1, e_1, e_2, e_2, e_3, e_3, \ldots\}$, where (e_n) is an orthonormal basis for the Hilbert space H. It is obvious that each of the following collections is an alternate dual frame for this frame:

$$\{e_1, 0, e_2, 0, \ldots\}$$

$$\{0, e_1, 0, e_2, \ldots\}$$

And also, the canonical dual frame for this frame is

$$\left\{\frac{e_1}{2}, \frac{e_1}{2}, \frac{e_2}{2}, \frac{e_2}{2}, \ldots\right\}.$$

Next we will give a theorem about the normalized tight frames.

Theorem 1.1.8 *Let $(x_n)_{n\in\mathbf{Z}}$ be a normalized tight frame for a Hilbert space H. Then, every $x \in H$ has a representation as*

$$x = \sum_{n\in\mathbf{Z}} \langle x, x_n\rangle x_n$$

Proof: We defined the frame operator as $S : H \longrightarrow H$ with

$$Sx = \sum_{n\in\mathbf{Z}} \langle x, x_n\rangle x_n$$

for a frame of a Hilbert space H. Therefore, we must prove that the frame operator is the identity map I of Hilbert space H onto itself if the frame is a normalized tight frame. In order to prove that $S = I$, we must show that $\langle Sx, y\rangle = \langle x, y\rangle$ for every $x, y \in H$

Using the definition of the frame operator, we can clearly see that $\langle Sx, x\rangle = \|x\|^2$ for every $x \in H$.

Now let us show that

$$\langle Sx, y\rangle = \langle x, y\rangle \quad \text{for any} \quad x, y \in H.$$

Let $x, y \in H$ and $\lambda \in \mathbb{C}$ be a constant. Suppose that

$$\langle S(x + \lambda y), (x + \lambda y)\rangle = \langle x + \lambda y, x + \lambda y\rangle$$

Then,

$$\langle Sx, x\rangle + \lambda\langle Sy, x\rangle + \bar{\lambda}\langle Sx, y\rangle + |\lambda|^2\langle Sy, y\rangle = \langle x, x\rangle + \lambda\langle y, x\rangle + \bar{\lambda}\langle x, y\rangle + |\lambda|^2\langle y, y\rangle.$$

Therefore, we have that

$$\|x\|^2 + |\lambda|^2\|y\|^2 + 2Re\bar{\lambda}\langle Sx, y\rangle = \|x\|^2 + |\lambda|^2\|y\|^2 + 2Re\bar{\lambda}\langle x, y\rangle,$$

this implies that

$$2Re\bar{\lambda}(\langle Sx, y\rangle - \langle x, y\rangle) = 0 \tag{1.1.6}$$

Now let

$$\beta := \langle Sx, y\rangle - \langle x, y\rangle \in \mathbb{C},$$

then the equality (1.1.6) is equal to $2Re\bar{\lambda}\beta = 0$ for any $\lambda \in \mathbb{C}$. Choosing $\beta = \lambda$, we see that $\beta = \langle Sx, y\rangle - \langle x, y\rangle$ has to be 0. Hence,

$$\langle Sx, y\rangle = \langle x, y\rangle$$

for every $x, y \in H$, therefore $S = I$ as claimed. □

Theorem 1.1.9 ([HW]) *Suppose that $(f_n)_{n\in\mathbf{Z}}$ is a normalized tight frame for a Hilbert space H. Then,*

1. *The norm of any element of $(f_n)_{n\in\mathbf{Z}}$ is less than or equal to 1.*
2. *If the norm of every element of $(f_n)_{n\in\mathbf{Z}}$ is equal to 1, then $(f_n)_{n\in\mathbf{Z}}$ is an orthonormal basis for H.*

Proof:

1. If $k \in \mathbf{Z}$ we have

$$\|f_k\|^2 = \sum_{n\in\mathbf{Z}} |\langle f_k, f_n\rangle|^2 = \|f_k\|^4 + \sum_{n\in\mathbf{Z}\setminus\{k\}} |\langle f_k, f_n\rangle|^2.$$

Hence, $\|f_k\|^2 - \|f_k\|^4 \geq 0$, which implies that $\|f_k\|^2(1 - \|f_k\|^2) \geq 0$, thus $\|f_k\| \leq 1$.

2. Suppose that $\|f_k\| = 1$ for every $k \in \mathbf{Z}$. Then by part (1), we have that

$$0 = \|f_k\|^2 - \|f_k\|^4 = \sum_{n\in\mathbf{Z}\setminus\{k\}} |\langle f_k, f_n\rangle|^2$$

Therefore, $\langle f_k, f_n\rangle = 0$ for every $n \in \mathbf{Z}\setminus\{k\}$. This means that $(f_n)_{n\in\mathbf{Z}}$ is an orthonormal basis for H. □

In this thesis we will be mainly interested in constructing the dual frames which have the same Gabor structure as the original Gabor frame of Gabor subspace frames (or Weyl-Heisenberg frames) for $L^2(\mathbf{R})$.

The Gabor subspace frame elements have a particularly simple form, for they are functions which are generated from a single fixed function (called window) by applications of the basic operations of translation, modulation, defined by:

$$\text{Translation:} \quad T_a f(x) = f(x-a), \quad \text{for} \quad a \in \mathbf{R};$$

$$\text{Modulation:} \quad E_a f(x) = e^{2\pi i a x} f(x), \quad \text{for} \quad a \in \mathbf{R}.$$

It is clear that the translation and modulation operators satisfy the equalities:

$$\begin{aligned} T_a E_b g(x) &= e^{2\pi i b(x-a)} g(x-a) \\ E_b T_a g(x) &= e^{2\pi i b x} g(x-a) \\ (E_{mb} T_{na} g)(x) &= e^{2\pi i b m x} g(x-na). \end{aligned} \tag{1.1.7}$$

A **Weyl-Heisenberg frame (or Gabor subspace frame)** for $L^2(\mathbf{R})$ is a frame consisting of the set of translates and modulates of a fixed function in $L^2(\mathbf{R})$. That is, a collection of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$, with $a, b > 0$, and $g \in L^2(\mathbf{R})$, and sometimes denoted by WH-frames.

WH-frames were introduced in 1946 by D. Gabor when he formulated a fundamental approach for signal decomposition in terms of elementary signals. Since then, a central question in this area has been to give necessary and sufficient conditions on g, a, b so that $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ forms a frame.

Although this question is still open, the necessary and sufficient conditions for this family to form a tight WH-frame is given by Casazza and Christensen in a paper (see [CCJ1]), and these authors hope that this will eventually lead to the solution to the general problem. In our thesis we are giving a characterization for alternate dual frames of Weyl-Heisenberg frames in terms of the Zak transform.

It can be shown that the frame operator S for a Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\mathbf{Z}}$ commutes with translation by a and modulation by b.

Next we show that the image of a Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\mathbf{Z}}$ under the frame operator S is another Gabor subspace frame.

Theorem 1.1.10 *Let $\{E_{mb}T_{na}g\}_{m,n\mathbf{Z}}$ be a Gabor subspace frame with the frame operator S. Then, a direct computation shows that*

$$S(E_{mb}T_{na}g) = E_{mb}T_{na}Sg,$$

and

$$S^{-1}(E_{mb}T_{na}g) = E_{mb}T_{na}S^{-1}g.$$

In particular, the canonical dual frame of a Gabor subspace frame is another Gabor subspace frame.

Next we want to give a proposition which gives a simple characterization for the functions which belong to $L^2(\mathbf{R})$ are orthogonal to Gabor subspace frames.

Proposition 1.1.11 ([HW]) *Let $g, h \in L^2(\mathbf{R})$ and $a, b \in \mathbf{R}$.*

1. *$h \perp E_{mb}g$, for all $m \neq 0$ if and only if there is a constant C so that*

$$\sum_{n\in\mathbf{Z}} h(x - n/b)\overline{g(x - n/b)} = C \quad \textit{almost everywhere}$$

2. *If $n \neq 0$, then $h \perp E_{mb}T_{na}g$, for all $m \in \mathbf{Z}$ if and only if*

$$\sum_{k} h(x - k/b)\overline{g(x - k/b - na)} = 0 \quad \textit{almost everywhere}$$

Proof:

1. We have that $h \perp E_{mb}g$, for all $m \neq 0$ if and only if,

$$\begin{aligned} 0 &= \langle h, E_{mb}g\rangle = \int_{\mathbf{R}} h(x)\overline{E_{mb}g(x)}\,dx = \int_{\mathbf{R}} h(x)\overline{g(x)}E_{-mb}\,dx \\ &= \int_0^{1/b} \sum_{n\in\mathbf{Z}} h(x - n/b)\overline{g(x - n/b)}E_{-mb}\,dx, \quad \text{for all} \quad m \in \mathbf{Z}. \end{aligned}$$

(1) now follows.

2. Similar to (1) we have that $h \perp E_{mb}T_{na}g$, for all $m \in \mathbf{Z}$ and $n \neq 0$ if and only if

$$\begin{aligned} 0 &= \langle h, E_{mb}T_{na}g\rangle = \int_{\mathbf{R}} h(x)\overline{E_{mb}g(x-na)}\,dx = \int_{\mathbf{R}} h(x)\overline{g(x-na)}E_{-mb}\,dx \\ &= \int_0^{1/b} \sum_{k\in\mathbf{Z}} h(x-k/b)\overline{g(x-k/b-na)}E_{-mb}\,dx, \quad \text{for all} \quad m \in \mathbf{Z}. \end{aligned} \tag{1.1.8}$$

Now, (1.1.8) is equivalent to

$$\sum_{k\in\mathbf{Z}} h(x-k/b)\overline{g(x-k/b-na)} = 0, \quad \text{almost everywhere.}$$ □

A crucial tool in the Gabor systems analysis is the **Zak transform**. This transform has been introduced independently by many groups in many different areas of pure and applied mathematics. J. Zak investigated it for quantum mechanical reasons beginning in the 1960s ([Za]); recent work includes that of A. J. E. M. Janssen ([Ja1], [Ja2]).

Definition 1.1.12 The **Zak transform** of a function $f \in L^2(\mathbf{R})$ is (formally)

$$Zf(x,w) = \sum_{k\in\mathbf{Z}} f(x-k)e^{2\pi ikw}$$

for every $x, w \in \mathbf{R}$.

The Zak transform is a mapping from $L^2(\mathbf{R})$ onto $L^2([0,1]\times[0,1])$ and another very useful version of it which will be the key tool for the second chapter is defined for $\alpha > 0$ as

$$Z_\alpha g(t,w) = \alpha^{-\frac{1}{2}} \sum_{k\in\mathbf{Z}} g\left(\frac{t-k}{\alpha}\right) e^{2\pi ikw}.$$

A detailed discussion of the Zak transform can be seen in ([Ja2], [Za]). For extensive examples using the Zak transform see ([CCJ1]). A nice application of it to Balian-Low phenomenon for subspace Gabor frames can be seen in [GH1].

Now we want to point out some useful properties of the Zak transform and its relation with Gabor subspace frames in the next lemma and theorems.

Theorem 1.1.13 ([Ca]) *The Zak transform is a unitary map from $L^2(\mathbf{R})$ onto $L^2(Q)$ where $Q = [0,1] \times [0,1]$.*

Proof. In order to prove that the Zak transform is a unitary map from $L^2(\mathbf{R})$ onto $L^2(Q)$, we will show that the image of an orthonormal basis of $L^2(\mathbf{R})$ under the Zak transform is an orthonormal basis for $L^2(Q)$.

It can be seen that the following collection, for $\alpha > 0$

$$\begin{aligned} \{\phi_{m,n}\}_{m,n\in\mathbf{Z}} &= \{E_{m\alpha}T_{n/\alpha}\chi_{[0,1/\alpha)}(t)\}_{m,n\in\mathbf{Z}} \\ &= \{e^{2\pi i m\alpha t}\chi_{[0,1/\alpha)}(t - n/\alpha)\}_{m,n\in\mathbf{Z}} \end{aligned}$$

is an orthonormal basis for $L^2(\mathbf{R})$.

$$Z_\alpha(\phi_{m,n})(t,w) = \alpha^{-1/2}\sum_{k\in\mathbf{Z}} e^{2\pi i m\alpha(\frac{t-k}{\alpha})}\chi_{[0,1/\alpha)}(\frac{t-k-n}{\alpha})e^{2\pi i k w}.$$

The only non zero term in this series occurs when $k = -n$, thus,

$$\begin{aligned} Z(\phi_{m,n})(t,w) &= e^{2\pi i m t}e^{-2\pi i n w} \\ &= E_{m,-n}(t,w). \end{aligned}$$

It is obvious that $\{E_{m,-n}\}_{m,n\mathbf{Z}}$ is an orthonormal basis for $L^2(Q)$. This completes the proof. □

We now give a characterization of the Zak transformation of the Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ for the case of $ab \in \mathbf{N}$.

Proposition 1.1.14 ([Ca]) *The Zak transform of $E_{mb}T_{na}g$ is $E_{m,-np}Z_b g$ for $ab = p \in \mathbf{N}$ and $g \in L^2(\mathbf{R})$.*

Proof: We have

$$E_{mb}T_{na}g = g_{mb,na} = g_{mb,\frac{np}{b}}$$

for $ab = p$. Thus, we have that

$$g_{mb,\frac{np}{b}}(x) = e^{2\pi i m b x}g\left(x - \frac{np}{b}\right).$$

By definition of Z_b, we have

$$
\begin{aligned}
Z_b(g_{mb,\frac{np}{b}}) &= b^{-\frac{1}{2}} \sum_{k\in\mathbf{Z}} e^{2\pi i m b \frac{t-k}{b}} g\left(\frac{t-k-pn}{b}\right) e^{2\pi i k w} \\
&= b^{-\frac{1}{2}} e^{2\pi i m t} \sum_{k\in\mathbf{Z}} g\left(\frac{t-k-pn}{b}\right) e^{2\pi i k w}, \text{ (let } k' := k+pn), \\
&= b^{-\frac{1}{2}} e^{2\pi i m t} \sum_{k'\in\mathbf{Z}} g\left(\frac{t-k'}{b}\right) e^{2\pi i k' w} e^{-2\pi i p n w} \\
&= b^{-\frac{1}{2}} e^{2\pi i m t} e^{-2\pi i p n w} \sum_{k'\in\mathbf{Z}} g\left(\frac{t-k'}{b}\right) e^{2\pi i k' w} \\
&= e^{2\pi i m t} e^{-2\pi i p n w} Z_b g(t,w),
\end{aligned}
$$

as claimed. □

1.2 The Zak Transformation of a Gabor Subspace Frame

In this section we will present some results showing how the Zak transform can be used to study Gabor subspace frames.

Theorem 1.2.1 ([Ca]) *Let $g \in L^2(\mathbf{R})$ and let $g_{m,n} = E_m T_n g$ for $m, n \in \mathbf{Z}$. Then,*

1. *$\{g_{m,n}\}_{m,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$ if and only if $|Zg| = 1$ almost everywhere.*

2. *$\{g_{m,n}\}_{m,n\in\mathbf{Z}}$ is a frame for $L^2(\mathbf{R})$ with frame bounds A and B if and only if $A \le |Zg|^2 \le B$ almost everywhere.*

Proof:

1. We proved in Theorem 1.1.13 that the Zak transform is a unitary map from $L^2(\mathbf{R})$ onto $L^2(Q)$, and by Theorem 1.1.14, we know that the Zak transform of $E_m T_n g$ is $E_{m,-n} Zg$. Therefore, $\{g_{m,n}\}_{m,n\in\mathbf{Z}} = \{E_m T_n g\}_{m,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$ if and only if $\{E_{m,-n} Zg\}_{m,n\in\mathbf{Z}}$ is an orthonormal

basis for $L^2([0,1] \times [0,1])$. By Parseval's theorem, we have that for each $f \in L^2(\mathbf{R})$ with $F = Zf \in L^\infty(Q)$, where $Q = [0,1] \times [0,1]$.

$$\begin{aligned} \|F\|_2^2 &= \sum_{m,n} |\langle F, E_{m,-n} Zg \rangle|^2 \\ &= \sum_{m,n} |\langle F\overline{Zg}, E_{m,-n} \rangle|^2 \\ &= \|F\overline{Zg}\|_2^2, \end{aligned}$$

since $E_{m,-n}$ is an orthonormal basis for $L^2(Q)$. Then using the definition of the norm, and unitarity of the Zak transform, we have the following identity:

$$\int_0^1 \int_0^1 |F(x,w)|^2 \, dx \, dw = \int_0^1 \int_0^1 |F(x,w)\overline{Zg(x,w)}|^2 \, dx \, dw$$

Therefore we have that

$$\int_0^1 \int_0^1 (|F(x,w)\overline{Zg(x,w)}|^2 - |F(x,w)|^2) dx dw = 0. \tag{1.2.9}$$

Since this equality is true for every bounded function F, we can choose F to be the characteristic function χ_S of S, where

$$S = \{(x,w) | \, |\overline{Zg(x,w)}|^2 > 1\}.$$

Hence equality (1.2.9) turns into the form

$$\int_S \int (|\overline{Zg(x,w)}|^2 - 1) dx dw = 0.$$

This implies that the measure of S must be 0. In a similar way, if we choose F to be $\chi_{S'}$ where

$$S' = \{(x,w) | \, |\overline{Zg(x,w)}|^2 < 1\},$$

the equality (1.2.9) turns into the form

$$\int_{S'} \int (|\overline{Zg(x,w)}|^2 - 1) \, dx \, dw = 0.$$

In the same way, this implies that S' has measure 0.

Therefore it follows that $|Zg| = 1$ almost everywhere on $[0,1] \times [0.1]$.

2. Since the Zak transform is a unitary map and $Z(E_mT_ng) = E_{m,-n}Zg$, we have that

$$\{g_{m,n}\}_{m,n\in\mathbf{Z}} = \{E_mT_ng\}_{m,n\in\mathbf{Z}}$$

is a frame for $L^2(\mathbf{R})$ with frame bounds A and B if and only if $\{E_{m,-n}Zg\}_{m,n\in\mathbf{Z}}$ is a frame for $L^2([0,1]\times[0,1])$ with the frame bounds A and B.

By part (1), we know that

$$\|F\overline{Zg}\|^2 = \sum_{m,n} |\langle F, E_{m,-n}Zg\rangle|^2$$

for $F \in L^2(Q)$ with F bounded.

Hence by the definition of a frame we have the following inequalities

$$A\|F\|_2^2 \;\le\; \|F\overline{Zg}\|_2^2 = \sum_{m,n} |\langle F, E_{m,-n}Zg\rangle|^2 \le B\|F\|_2^2.$$

Therefore, using the right hand side of this inequality, we have that

$$\int_0^1\int_0^1 \left(|F(x,w)\overline{Zg(x,w)}|^2 - B|F(x,w)|^2\right) dxdw \le 0$$

and using the left hand side of the same inequality, we have that

$$\int_0^1\int_0^1 \left(|F(x,w)\overline{Zg(x,w)}|^2 - A|F(x,w)|^2\right) dxdw \ge 0.$$

By defining the sets

$$S_1 = \{(x,w)|\ |Zg(x,w)|^2 - B > 0\}$$

and

$$S_2 = \{(x,w)|\ |Zg(x,w)|^2 - A < 0\},$$

and letting $F = \chi_{S_1}$ and χ_{S_2}, respectively, we obtain that S_1 and S_2 are measure 0 sets. Therefore we obtain that

$$A \le |Zg|^2 \le B \quad \text{almost everywhere}$$

on $[0,1]\times[0,1]$ as claimed. □

We can extend Theorem 1.2.1 to the Gabor subspace frames in the following way:

Theorem 1.2.2 ([Ca]) *Let* $g \in L^2(\mathbf{R})$ *and* $g_{m,n} = E_m T_n g$ *for* $m, n \in \mathbf{Z}$. *Then*

1. $\{g_{m,n}\}_{m,n\in\mathbf{Z}}$ *is a normalized tight Gabor subspace frame if and only if* $|Zg| = \chi_\Omega$ *almost everywhere, where* Ω *is a measurable subset of* $[0,1] \times [0,1]$.

2. $\{g_{m,n}\}_{m,n\in\mathbf{Z}}$ *is a Gabor subspace frame for* $L^2(\mathbf{R})$ *with frame bounds* A *and* B *if and only if there exists a measurable subset* Ω *of* Q *such that* $A \le |Zg|^2 \le B$ *almost everywhere on* Ω, *and* $Zg = 0$ *on* $Q\backslash\Omega$, *where* $Q = [0,1] \times [0,1]$.

Proof. The proof of both parts (1) and (2) will come as an easy corollary after we characterize the frame condition in terms of the Zak transform in the second chapter. □

Theorem 1.2.3 *Let* $\{E_m T_n g\}_{m,n\in\mathbf{Z}}$ *form a Bessel collection, where* $g \in L^2(\mathbf{R})$ *and define* $\mathcal{M}(g)$ *to be the closure of span*$\{E_m T_n g\}_{m,n\in\mathbf{Z}}$ *in* $L^2(\mathbf{R})$ *and*

$$Z(\mathcal{M}(g)) = \{Zf | f \in \mathcal{M}(g)\}.$$

Then

$$Z(\mathcal{M}(g)) = L^2(\Omega),$$

where

$$\Omega = \{(x,w) \in [0,1] \times [0,1] | Zg(x,w) \neq 0\}.$$

Proof: Since the Zak transform of a finite sum $\sum_{m,n} \alpha_{m,n} E_m T_n g$ has the form $H(x,w)Zg(x,w)$ where

$$H(x,w) = \sum_{m,n} \alpha_{m,n} E_{m,-n}(x,w),$$

it follows immediately that

$$Z(\mathcal{M}(g)) \subseteq L^2(\Omega).$$

To show the converse inclusion, consider a function $h \in L^2(\Omega)$ such that h is orthogonal to $Z(\mathcal{M}(g))$.

Using the Zak transform, it follows that

$$\begin{aligned} 0 &= \langle h(x,w), E_{m,-n}(x,w)Zg(x,w)\rangle \\ &= \langle h(x,w)\overline{Zg(x,w)}, E_{m,-n}(x,w)\rangle \end{aligned} \tag{1.2.10}$$

Since Zg is bounded, we have that

$$h\overline{Zg} \in L^2(Q) \quad \text{where} \quad Q = [0,1] \times [0,1].$$

Hence by the equation (1.2.10), we have that $h(x,w)\overline{Zg(x,w)} = 0$ since $E_{m,-n}(x,w)$ is an orthonormal basis for $L^2(Q)$, and, thus, $h = 0$ almost everywhere on Ω.

Thus,

$$L^2(\Omega) \subseteq Z(\mathcal{M}(g)) \quad \text{and, therefore,}$$

$$Z(\mathcal{M}(g)) = L^2(\Omega).$$

This completes the proof. □

Proposition 1.2.4 *Let ab be a rational number and $g \in L^2(\mathbf{R})$. Then the collection $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a Bessel collection if and only if the Zak transform Zg of g is bounded.*

Proof: The proof of the proposition is an easy conclusion of Lemma 2.3.5. □

Chapter 2

Main Problem

Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ and $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ be two Gabor subspace frames. We want to obtain a condition for the existence and uniqueness of a window function h in the Gabor subspace frame $\overline{span}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ which generates a Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ such that $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of the original Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$.

2.1 Solution for the case of $ab = 1$

Theorem 2.1.1 *Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ and $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ be two Gabor subspace frames. Then there exists a window function h in $\overline{span}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ such that the Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of the Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ if and only if $\overline{span}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ is a subset of $\overline{span}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$.*

Proof: First let us define the sets

$$\begin{aligned}\Omega_1 &= \{(x,w)|Zg(x,w)\neq 0\}\\ \Omega_2 &= \{(x,w)|Zk(x,w)\neq 0\}.\end{aligned}$$

Since the Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ will serve as a dual frame of the Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$, using the definition of the dual frame, we have the following equality:

$$f(t) = \sum_{m,n\in\mathbf{Z}} \langle f(t), E_{mb}T_{na}h(t)\rangle E_{mb}T_{na}g(t)$$

for every $f \in \operatorname{span}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$, which has a bounded Zak transform. Taking the Zak transform of both sides, we get that

$$\begin{aligned} Zf(x,w) &= \sum_{m,n\in\mathbf{Z}} \langle Zf(x,w), E_{m,-n}(x,w)Zh(x,w)\rangle E_{m,-n}(x,w)Zg(x,w) \\ &= Zg(x,w)\sum_{m,n\in\mathbf{Z}} \langle Zf(x,w)\overline{Zh(x,w)}, E_{m,-n}(x,w)\rangle E_{m,-n}(x,w) \\ &= Zg(x,w)Zf(x,w)\overline{Zh(x,w)} \quad \text{for every } (x,w)\in\Omega_1 \\ &\qquad\qquad \text{by the Parseval equality.} \end{aligned}$$

Thus, we have the equality $Zh(x,w) = \frac{1}{\overline{Zg(x,w)}}$ on the set Ω_1. It is clear that $Z(h) \in L^2(\Omega_2)$ and $Z(g) \in L^2(\Omega_1)$ since $h \in \overline{span}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ and g is the window function of $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$.

Hence, because of the equality

$$Zh(x,w) = \frac{1}{\overline{Zg(x,w)}} \neq 0 \quad \text{on } \Omega_1,$$

and $Zh(x,w) = 0$ for every $(x,w) \in Q - \Omega_2$, it is obvious that $\Omega_1 \subseteq \Omega_2$, and this implies that $L^2(\Omega_1) \subseteq L^2(\Omega_2)$.

Define now the sets

$$\begin{aligned} M_1 &= \overline{\operatorname{span}}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}} \\ M_2 &= \overline{\operatorname{span}}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}} \end{aligned}$$

We just showed that $L^2(\Omega_1) \subseteq L^2(\Omega_2)$ and using Theorem 1.2.3, we can say that

$$Z(M_1) = L^2(\Omega_1) \qquad \text{and} \qquad Z(M_2) = L^2(\Omega_2),$$

hence, we have that

$$Z(M_1) = L^2(\Omega_1) \subseteq Z(M_2) = L^2(\Omega_2).$$

Therefore we conclude that if $M_1 \subseteq M_2$, then always there exists a window function h, defined as $Zh(x,w) = \frac{1}{Zg(x,w)}\chi_{\Omega_1}(x,w)$, in span$\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ such that the Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of the Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ as claimed. □

Proposition 2.1.2 *The window function h satisfying the conditions of Theorem 2.1.1 exists uniquely if*

$$\Omega_1 = \{(x,w)|Zg(x,w) \neq 0\} = \Omega_2 = \{(x,w)|Zk(x,w) \neq 0\},$$

or

$$\overline{span}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}} = \overline{span}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}},$$

respectively.

Proof: Suppose that $\Omega_1 \neq \Omega_2$. Then Ω_1 must be strictly contained in Ω_2, since the condition for the existence of the dual is $\Omega_1 \subseteq \Omega_2$. Now since we assumed that $\Omega_1 \subset \Omega_2$, we can define two $L^2(\Omega_2)$ functions satisfying the equality

$$Zh(x,w) = \frac{1}{Zg(x,w)} \qquad on \qquad \Omega_1$$

in the following way.

Define

$$Zh_1(x,w) = \begin{cases} \frac{1}{Zg(x,w)}, & (x,w) \in \Omega_1 \\ 1, & (x,w) \in \Omega_2 \setminus \Omega_1 \\ 0, & (x,w) \in \Omega_2^c \end{cases} \tag{2.1.1}$$

$$Zh_2(x,w) = \begin{cases} \frac{1}{Zg(x,w)}, & (x,w) \in \Omega_1 \\ 2, & (x,w) \in \Omega_2 \setminus \Omega_1 \\ 0, & (x,w) \in \Omega_2^c \end{cases} \tag{2.1.2}$$

It is clear that Zh_1 and Zh_2 are both in $L^2(\Omega_2)$ and satisfies the equality $Zh(x,w) = \frac{1}{Zg(x,w)}$ on Ω_1, i.e. they satisfy the condition for the existence of the dual. Thus we conclude that if Ω_1 is strictly contained in Ω_2, we can define as many as window functions h we want such that the Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ serves as a dual frame for the original Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. Therefore the condition for the uniquely existence of the dual frame must be $\Omega_1 = \Omega_2$, (or $M_1 = M_2$, or $Z(M_1) = L^2(\Omega_1) = Z(M_2) = L^2(\Omega_2)$, respectively) as claimed. $\square$

2.2 Solution for the case of $ab = p \in \mathbf{N}$

Before we start to give our proofs and conclusions for the case of $ab \in \mathbf{N}$, we want to explain the scheme of the solution of our problem since we believe that it will be very useful in order to understand the steps of our proofs and we will follow the similar steps in the proof for the case of $ab = p/q$, $gcd(p,q) = 1$.

First, we will characterize the set of Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ in terms of the Zak transform. Secondly, we will characterize the functions in the closed linear span of the Bessel collection $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$, and then obtain a condition for the lower frame bound for $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ to hold and as a corollary we will obtain a condition for being a Gabor subspace frame of the Bessel collection $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. Finally, in order to obtain a condition characterizing the dual frame condition, we will characterize functions $h \in L^2(\mathbf{R})$ which generates a Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ that serves as a dual frame of the original Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$, and then generalize this characterization to our main duality problem.

Since we believe that mentioning the problem again will be helpful to follow up the proofs, here we state our problem one more time:

Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ and $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ be two Gabor subspace frames. We want to obtain a condition for the existence and uniqueness of a window function h in the Gabor subspace frame $\overline{span}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ which generates a Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ such that $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ serves as a dual frame of the original Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$.

It is very useful to remember that the Zak transform of $E_{mb}T_{na}g$ is $E_{m,-np}Z_b g$ for $ab = p \in \mathbf{N}$, since we will make our characterizations in terms of the Zak transform.

Now we will give a remark from calculus which will be very useful in our calculations.

Remark 2.2.1 From calculus, we know that the equality

$$\int_0^1 f(x)\,dx = \int_0^{\frac{1}{p}} \sum_{n=0}^{p-1} f\left(x+\frac{n}{p}\right)\,dx$$

holds for every integrable function f defined on the interval $[0,1]$.

Proposition 2.2.2 *Let $ab = p \in \mathbf{N}$. There exists a positive constant B such that the inequality*

$$\sum_{m,n\in\mathbf{Z}} |\langle f(t), E_{mb}T_{na}g(t)\rangle|^2 \le B\|f(t)\|^2 \tag{2.2.3}$$

holds for every $f \in \overline{span}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ if and only if $\|G(x,w)\|^2_{\mathbb{C}^p} \le Bp$ almost everywhere on $[0,1]\times[0,1/p]$, where

$$\begin{aligned} G &= (G_0,\dots,G_{p-1}) \in \mathbb{C}^p \\ G_i(x,w) &= Z_b g\left(x, w+\frac{i}{p}\right), \qquad i=0,\dots,p-1. \end{aligned}$$

Proof: Let us consider the left hand side of the inequality in (2.2.3) first, and assume that f or g has a bounded Zak transform.

$$\begin{aligned} &\sum_{m,n\in\mathbf{Z}} |\langle f(t), E_{mb}T_{na}g(t)\rangle|^2 \\ &= \sum_{m,n\in\mathbf{Z}} |\langle Z_b f(x,w), Z_b(E_{mb}T_{na}g)(x,w)\rangle|^2 \\ &\qquad\qquad \text{(Since the Zak transform is a unitary map.)} \\ &= \sum_{m,n\in\mathbf{Z}} |\langle Z_b f(x,w), E_{m,-np}(x,w) Z_b g(x,w)\rangle|^2 \qquad \text{(by Theorem 1.1.14)} \\ &= \sum_{m,n\in\mathbf{Z}} \left| \int_0^1 \int_0^1 Z_b f(x,w)\overline{Z_b g(x,w)} e^{-2\pi i m x} e^{2\pi i n p w}\,dx\,dw \right|^2 \\ &= \sum_{m,n\in\mathbf{Z}} \left| \int_0^1 \int_0^{1/p} \sum_{k=0}^{p-1} Z_b f(x, w+k/p)\overline{Z_b g(x,w+k/p)} e^{-2\pi i m x} e^{2\pi i n p w}\,dx\,dw \right|^2 \\ &= \frac{1}{p}\int_0^1 \int_0^{1/p} \left| \sum_{k=0}^{p-1} Z_b f(x,w+k/p)\overline{Z_b g(x,w+k/p)} \right|^2 dx\,dw \end{aligned}$$

where the last equality follows from the fact that $\{e^{-2\pi imx}e^{2\pi inpw}\}_{m,n\in\mathbf{Z}}$ is an orthogonal basis for $L^2([0,1]\times[0,1/p])$, and the fact that $Z_bf\overline{Z_bg}\in L^2(Q)$. Similarly,

$$\|f(x)\|^2 = \int_0^1\int_0^{1/p}\sum_{k=0}^{p-1}|Z_bf(x,w+k/p)|^2\ dx\ dw \tag{2.2.4}$$

Since $Z_bf(x,w+k/p)$ is an arbitrary function in $L^2([0,1]\times[0,1/p])$, we can define the functions

$$F_k(x,w) := Z_bf(x,w+k/p),\quad k=0,\dots,p-1 \tag{2.2.5}$$

$$\overline{G_k(x,w)} := \overline{Z_bg(x,w+k/p)},\quad k=0,\dots,p-1 \tag{2.2.6}$$

and the $\mathbb{C}^p$-valued functions $F=(F_0,\dots,F_{p-1})$ and $G=(G_0,\dots,G_{p-1})$. Therefore the inequality (2.2.3) can be written as

$$\begin{aligned}
&\int_0^1\int_0^{1/p}\left|\sum_{k=0}^{p-1}F_k(x,w)\overline{G_k(x,w)}\right|^2\ dx\ dw\\
&= \int_0^1\int_0^{1/p}|\langle F(x,w),G(x,w)\rangle_{\mathbb{C}^p}|^2\ dx\ dw\\
&\leq Bp\int_0^1\int_0^{1/p}\sum_{k=0}^{p-1}|F_k(x,w)|^2\ dx\ dw\\
&= Bp\int_0^1\int_0^{1/p}\|F(x,w)\|^2_{\mathbb{C}^p}\ dx\ dw
\end{aligned} \tag{2.2.7}$$

We can rewrite the inequality (2.2.7) as

$$\int_0^1\int_0^{1/p}(|\langle F(x,w),G(x,w)\rangle_{\mathbb{C}^p}|^2 - Bp\|F(x,w)\|^2_{\mathbb{C}^p})\ dx\ dw \leq 0. \tag{2.2.8}$$

Now, suppose that $\|G(x,w)\|^2_{\mathbb{C}^p}\leq Bp$ almost everywhere, then using the Cauchy-Schwartz inequality, it is clear that we have the inequality (2.2.8).

Conversely, suppose that we have the inequality (2.2.8), and let us define the set

$$S=\{(x,w):\|G(x,w)\|^2_{\mathbb{C}^p}-Bp>0\}.$$

If we choose $F(x,w) = \frac{G(x,w)}{\|G(x,w)\|}\chi_S(x,w)$, we see that each component of F belongs to $L^\infty(Q)$ and the inequality (2.2.8) turns into the form

$$\int_S\int (\|G(x,w)\|^2_{\mathbb{C}^p} - Bp)\ dx\ dw \leq 0$$

This implies that S has measure 0, and thus, that $\|G(x,w)\|^2_{\mathbb{C}^p} \leq Bp$ almost everywhere. This proves our claim. □

We will need the following proposition next.

Proposition 2.2.3 *Let $ab = p \in \mathbf{N}$. Let $g \in L^2(\mathbf{R})$ and define $G \in L^2([0,1] \times [0,1/p], \mathbb{C}^p)$ by $G = (G_0, \ldots, G_{p-1})$ where $G_i(x,w) = Z_b g(x, w+i/p)$, $i = 0, \ldots, p-1$. Assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ is a Bessel collection and let $\mathcal{M}(b,a,g)$ be the closure of the closed linear span of $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. Let $f \in L^2(\mathbf{R})$ and define the vector valued function $F \in L^2([0,1]\times[0,1/p], \mathbb{C}^p)$ by $F = (F_0, \ldots, F_{p-1})$ where $F_i(x,w) = Z_b f(x, w+i/p)$, $i = 0, \ldots, p-1$. Then $f \in \mathcal{M}(b,a,g)$ if and only if there exists a complex-valued measurable function L defined on $[0,1]\times[0,1/p]$ such that $F = LG$.*

Proof: Let $h \in L^2(\mathbf{R})$ and suppose that $h \perp \overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. Then, we have that

$$\begin{aligned}
0 &= \langle h(x), E_{mb}T_{na}g(x)\rangle \\
&= \langle Z_b h(x,w), E_{m,-np}(x,w) Z_b g(x,w)\rangle \\
&\quad \text{(since the Zak transform is a unitary map and by Theorem 1.1.14)} \\
&= \langle Z_b h(x,w)\overline{Z_b g(x,w)}, E_{m,-np}(x,w)\rangle \\
&= \int_0^1\int_0^1 Z_b h(x,w)\overline{Z_b g(x,w)} \quad e^{-2\pi imx}e^{2\pi inpw}\ dx\ dw \\
&= \int_0^1\int_0^{1/p}\sum_{k=0}^{p-1} Z_b h(x, w+k/p)\overline{Z_b g(x, w+k/p)} \quad e^{-2\pi imx}e^{2\pi inpw}\ dx\ dw \\
&= \int_0^1\int_0^{1/p} \langle H(x,w), G(x,w)\rangle_{\mathbb{C}^p} \quad e^{-2\pi imx}e^{2\pi inpw}\ dx\ dw
\end{aligned}$$

where G is defined as in the hypothesis of Proposition and H is defined similarly. This implies that $h \perp \overline{span}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ if and only if

$$\langle H(x,w), G(x,w)\rangle_{\mathbb{C}^p} = 0$$

for almost every $(x,w) \in [0,1] \times [0,1/p]$, since $\{e^{-2\pi imx}e^{2\pi inpw}\}_{m,n\in\mathbf{Z}}$ is an orthogonal basis for $L^2([0,1]\times[0,1/p])$. This implies, in particular, that if a measurable function $\mathcal{V}$ defined on Q is such that $\mathcal{V}G \in L^2(Q)$, then the function $y \in L^2(\mathbf{R})$ obtained by $Z_b y(x, w+i/p) = \mathcal{V}(x,w)G_i(x,w)$ belongs to $\mathcal{M}(b,a,g)$.
Indeed, if $h \perp \mathcal{M}(b,a,g)$, we have that

$$\int_{\mathbf{R}} y(t)\overline{h(t)}dt = \int_0^1 \int_0^{1/p} \mathcal{V}(x,w) < G(x,w), H(x,w) >_{\mathbb{C}^p} dxdw = 0.$$

In particular, if $F \in L^2(Q)$, then, we have

$$\frac{\langle F(x,w), G(x,w)\rangle_{\mathbb{C}^p}}{\|G(x,w)\|^2_{\mathbb{C}^p}} G(x,w) \in L^2(Q).$$

Since,

$$\left\| \frac{\langle F(x,w), G(x,w)\rangle_{\mathbb{C}^p}}{\|G(x,w)\|^2_{\mathbb{C}^p}} G(x,w) \right\|_{\mathbb{C}^p} \leq \|F(x,w)\|_{\mathbb{C}^p},$$

if $F \in L^2([0,1]\times[0,1/p], \mathbb{C}^p)$ is arbitrary, we can define

$$H := F - \frac{\langle F, G\rangle_{\mathbb{C}^p}}{\|G\|^2_{\mathbb{C}^p}} G, \tag{2.2.9}$$

where $\langle H, G\rangle_{\mathbb{C}^p} = 0$.

It follows from here that

$$\langle F, H\rangle_{\mathbb{C}^p} = 0 \quad \text{for} \quad f \in \mathcal{M}(b,a,g).$$

Therefore if $f \in \mathcal{M}(b,a,g)$, then by the definition of F in the equality (2.2.9), it is clear that

$$0 \;=\; < F, H >_{\mathbb{C}^p} = \langle \frac{\langle F, G\rangle_{\mathbb{C}^p}}{\|G\|^2_{\mathbb{C}^p}} G + H, H\rangle_{\mathbb{C}^p},$$

and this implies that

$$\|H\|^2_{\mathbb{C}^p} = 0, \qquad \text{i.e.} \quad H = 0.$$

Conversely, if $F = \frac{\langle F,G\rangle_{\mathbb{C}^p}}{\|G\|^2_{\mathbb{C}^p}} G$, then it is obvious that $f \in \mathcal{M}(b, a, g)$.

Therefore

$$F = LG \quad \text{where} \quad L = \frac{\langle F, G\rangle_{\mathbb{C}^p}}{\|G\|^2_{\mathbb{C}^p}} \tag{2.2.10}$$

for $f \in \overline{span}\{E_{mb}T_{na}g\}$ on the set $\Omega = \{(x, w)|G(x, w) \neq 0\}$. □

After we characterized the functions in the closure of the linear span of the Bessel sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ as in the equation (2.2.10), let us obtain a condition for lower frame bound of the Bessel collection $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$.

Proposition 2.2.4 *Let $ab = p \in \mathbf{N}$. Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ be a Bessel collection. Then, there exists a positive constant A such that the inequality*

$$A\|f(x)\|^2 \leq \sum_{m,n\in\mathbf{Z}} |\langle f(x), E_{mb}T_{na}g(x)\rangle|^2 \tag{2.2.11}$$

holds for every $f \in \overline{span}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ if and only if $Ap \leq \|G(x, w)\|^2_{\mathbb{C}^p}$ almost everywhere on the set $\Omega_1 = \{(x, w) : G(x, w) \neq 0\}$, where

$$\begin{aligned} G &= (G_0, \ldots, G_{p-1}) \in \mathbb{C}^p, \\ G_i(x, w) &= Z_b g(x, w + i/p), \quad i = 0, \ldots, p-1. \end{aligned}$$

Proof: We can rewrite the inequality (2.2.11) as we did in inequality (2.2.8).

$$\int_0^1 \int_0^{1/p} \left(Ap\|F(x, w)\|^2_{\mathbb{C}^p} - |\langle F(x, w), G(x, w)\rangle_{\mathbb{C}^p}|^2\right) \, dx \, dw \leq 0 \tag{2.2.12}$$

Since we are working for $f \in \overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$, by the equality $F = LG$ in (2.2.10), we can rewrite the inequality (2.2.12) as

$$\int_0^1 \int_0^{1/p} |L(x, w)|^2\|G(x, w)\|^2_{\mathbb{C}^p}(Ap - \|G(x, w)\|^2_{\mathbb{C}^p}) \, dx \, dw \leq 0 \tag{2.2.13}$$

It is clear that if $Ap \leq \|G(x, w)\|^2$ on Ω, then 2.2.13 holds.

Conversely, assume that we have the inequality 2.2.13. Define now the set

$$S := \{(x, w)|Ap - \|G(x, w)\|^2_{\mathbb{C}^p} > 0 \quad and \quad \|G(x, w)\| > 0\}.$$

Choosing L as the characteristic function of S, i.e., $L = \chi_S$, the inequality (2.2.13) becomes

$$\int_S \int \|G(x,w)\|^2_{\mathbb{C}^p}(Ap - \|G(x,w)\|^2_{\mathbb{C}^p})\, dx\, dw \leq 0,$$

which implies that S has zero measure. Hence $Ap - \|G(x,w)\|^2_{\mathbb{C}^p} \leq 0$ on the set $\Omega_1 = \{(x,w)|G(x,w) \neq 0\}$. This proves our claim. □

Therefore, combining the propositions 2.2.2 and 2.2.4, we get the following condition characterizing Gabor subspace frames.

Corollary 2.2.5 *Let $ab = p \in \mathbf{N}$. The collection $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ is a Gabor subspace frame if and only if there exist two constants A and B such that $Ap \leq \|G(x,w)\|^2_{\mathbb{C}^p} \leq Bp$ almost everywhere on the set $\Omega_1 = \{(x,w)|G(x,w) \neq 0\}$ where G is defined as $G = (G_0, \ldots, G_{p-1})$ where $G_i(x,w) = Z_b g(x, w + i/p)$, $i = 0, \ldots, p-1$.*

Now we are ready to obtain the condition for the dual frame. First we want to characterize the functions h (not necessarily in the linear span of $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$) such that the Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of the Gabor subspace $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$.

Proposition 2.2.6 *Let $ab = p \in \mathbf{N}$. Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ be a Gabor subspace frame. Then, the function $h \in L^2(\mathbf{R})$ generates a Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ such that*

$\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ if and only if

$p = \langle G(x,w), H(x,w)\rangle_{\mathbb{C}^p}$ almost everywhere on $\Omega_1 = \{(x,w) : G(x,w) \neq 0\}$ and where G and H are defined as $G = (G_0, \ldots, G_{p-1})$ and $H = (H_0, \ldots, H_{p-1})$ where $G_i(x,w) = Z_b g(x, w + i/p)$, and $H_i(x,w) = Z_b h(x, w + i/p)$, $i = 0, \ldots, p-1$.

Proof: Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ be a Gabor subspace frame and suppose that the Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. Using the definition of dual frame, we have the following equality

$$f(x) = \sum_{m,n\in\mathbf{Z}} \langle f(x), E_{mb}T_{na}h(x)\rangle E_{mb}T_{na}g(x)$$

for every $f \in \operatorname{span}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. Then,

$$\begin{aligned}
&Z_bf(x,w)\\
&= \sum_{m,n\in\mathbf{Z}} \langle Z_bf(x,w), Z_b(E_{mb}T_{na}h)(x,w)\rangle Z_b(E_{mb}T_{na}g)(x,w)\\
&\qquad\qquad \text{(since the Zak transform is a unitary map)}\\
&= \sum_{m,n\in\mathbf{Z}} \langle Z_bf(x,w), E_{m,-np}(x,w)Z_bh(x,w)\rangle E_{m,-np}(x,w)Z_bg(x,w)\\
&\qquad\qquad\qquad \text{(by the theorem 1.1.14)}\\
&= Z_bg(x,w)\sum_{m,n\in\mathbf{Z}} \langle Z_bf(x,w)\overline{Z_bh(x,w)}, E_{m,-np}(x,w)\rangle E_{m,-np}(x,w)\\
&= Z_bg(x,w)\sum_{m,n\in\mathbf{Z}} \left(\int_0^1\int_0^1 Z_bf(x,w)\overline{Z_bh(x,w)}e^{-2\pi imx}e^{2\pi inpw}\,dx\,dw\right)E_{m,-np}(x,w)\\
&= Z_bg(x,w)\sum_{m,n\in\mathbf{Z}} \left(\int_0^1\int_0^{1/p}\sum_{k=0}^{p-1} Z_bf(x,w+k/p)\overline{Z_bh(x,w+k/p)}.\right.\\
&\qquad\qquad\qquad \left. e^{-2\pi imx}e^{2\pi inpw}\,dx\,dw\right)E_{m,-np}(x,w)\\
&= \frac{1}{p}Z_bg(x,w)\sum_{k=0}^{p-1} Z_bf(x,w+k/p)\overline{Z_bh(x,w+k/p)}\\
&\qquad \text{(since } e^{-2\pi imx}e^{2\pi inpw} \text{ is an orthogonal basis for } [0,1]\times[0,1/p])\\
&= \frac{1}{p}Z_bg(x,w)\langle F(x,w), H(x,w)\rangle_{\mathbb{C}^p}
\end{aligned}$$

where F and H are defined as in the hypothesis of Proposition. Hence,

$$Z_bf(x,w) = \frac{1}{p}Z_bg(x,w)\langle F(x,w), H(x,w)\rangle_{\mathbb{C}^p}. \tag{2.2.14}$$

Since we are working with the functions $f \in \operatorname{span}\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ and we characterized these f's as $F = LG$ in the equality (2.2.10), the equality (2.2.14) becomes

$$
\begin{aligned}
L(x,w)G(x,w) &= \frac{1}{p}G(x,w)\langle L(x,w)G(x,w), H(x,w)\rangle_{\mathbb{C}^p} \\
&= \frac{1}{p}G(x,w)L(x,w)\langle G(x,w), H(x,w)\rangle_{\mathbb{C}^p}, \quad \text{(since } L \text{ is constant)}
\end{aligned}
$$

this implies that

$$p = \langle G(x,w), H(x,w)\rangle_{\mathbb{C}^p} \tag{2.2.15}$$

on the set $\Omega_1 = \{(x,w) | G(x,w) \neq 0\}$. This proves our claim. □

We can generalize this result to our main duality problem very easily.

Note that in Proposition 2.2.6, we worked with a window function h whose Zak transform was bounded but it was not necessary that h belonged to the linear span of the Gabor subspace frame $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$. Now if we choose h, in particular, from the closed linear span of the Gabor subspace frame $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$, then we can write H as

$$H = LK \tag{2.2.16}$$

as we did in equation (2.2.10), where L is a scalar-valued function and

$$
\begin{aligned}
K &= (K_0, \ldots, K_{p-1}) \in \mathbb{C}^p \quad \text{and} \\
K_i(x,w) &= Z_b k(x, w + i/p), \quad i = 0, \ldots, p-1.
\end{aligned}
$$

Next we obtain a condition for the main duality problem.

Theorem 2.2.7 *Let $ab = p \in \mathbf{N}$. Let $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ and $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ be two Gabor subspace frames. Then, there exists a function $h \in \overline{span}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ which generates a Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ such that $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of the Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ if and only if there exists a positive constant C such that inequality*

$$|\langle G(x,w), K(x,w)\rangle_{\mathbb{C}^p}| \geq C \tag{2.2.17}$$

holds almost everywhere on $\Omega_1 = \{(x,w) \in [0,1] \times [0, \frac{1}{p}] : G(x,w) \neq 0\}$, *where*

$$\begin{aligned} K &= (K_0, \dots, K_{p-1}) \in \mathbb{C}^p \quad \textit{and} \\ K_i(x,w) &= Z_b k(x, w + i/p), \quad i = 0, \dots, p-1, \end{aligned}$$

and

$$\begin{aligned} G &= (G_0, \dots, G_{p-1}) \in \mathbb{C}^p \quad \textit{and} \\ G_i(x,w) &= Z_b g(x, w + i/p), \quad i = 0, \dots, p-1. \end{aligned}$$

Proof: First let us prove the necessity part of the implication. We want to prove that if we have the dual frame, then inequality 2.2.17 holds almost everywhere. Note that we can write $H = LK$ for some scalar-valued function L by the equality 2.2.16 since $h \in \text{span}\{E_{mb}T_{na}k\}_{m,n \in \mathbf{Z}}$. By the previous proposition, we have that

$$\begin{aligned} p &= \langle G(x,w), H(x,w) \rangle_{\mathbb{C}^p} \quad \text{on the set} \quad \Omega_1 = \{(x,w) | G(x,w) \neq 0\} \\ &= \langle G(x,w), L(x,w) K(x,w) \rangle_{\mathbb{C}^p} \\ &= \overline{L(x,w)} \langle G(x,w), K(x,w) \rangle_{\mathbb{C}^p}. \end{aligned}$$

This implies that

$$\overline{L(x,w)} = \frac{p}{\langle G(x,w), K(x,w) \rangle_{\mathbb{C}^p}}$$

on Ω_1. Thus, we can rewrite $H = LK$ as $H(x,w) = \frac{pK(x,w)}{\langle G(x,w), K(x,w) \rangle_{\mathbb{C}^p}}$ on Ω_1.

Since the dual frame $\{E_{mb}T_{na}h\}_{m,n \in Z}$ is a Bessel collection, $\|H\|_{\mathbb{C}^p}$ is bounded. Hence,

$$\|H(x,w)\|_{\mathbb{C}^p} = \frac{\|pK(x,w)\|_{\mathbb{C}^p}}{\langle G(x,w), K(x,w) \rangle_{\mathbb{C}^p}} \leq C_1 \tag{2.2.18}$$

for some positive constant C_1 on Ω_1, where K can not be 0. Using the frame condition we see that K is bounded below on the set $\Omega_2 = \{(x,w) : K(x,w) \neq 0\}$, hence $\Omega_1 \subseteq \Omega_2$. Note that K is bounded on Ω_2, and thus on Ω_1. Thus, the equality 2.2.18 implies that $|\langle G(x,w), K(x,w) \rangle_{\mathbb{C}^p}| \geq C$ on $\Omega_1 = \{G(x,w) \neq 0\}$ for some positive constant C. Therefore, we conclude that if we have dual frame, then inequality $|\langle G(x,w), K(x,w)\rangle_{\mathbb{C}^p}| \geq C$ holds on $\Omega_1 = \{(x,w) : G(x,w) \neq 0\}$ for some positive constant C. Conversely, now we want to show that if inequality 2.2.17 holds, then we can construct a dual frame.

Define,

$$L = \begin{cases} \frac{p}{\langle G(x,w), K(x,w)\rangle_{\mathbb{C}^p}}, & \text{on } \Omega_1; \\ 0, & \textit{otherwise.} \end{cases}$$

By this definition of L, it is obvious that

$$\|H(x,w)\|_{\mathbb{C}^p} = \|L(x,w)K(x,w)\|_{\mathbb{C}^p} = \frac{\|pK(x,w)\|_{\mathbb{C}^p}}{\langle G(x,w), K(x,w)\rangle_{\mathbb{C}^p}} \leq \frac{\sqrt{B}p^{3/2}}{C},$$

since $\|K(x,w)\|^2_{\mathbb{C}^p} \leq Bp$. Therefore, we conclude that $H = LK$ is bounded function and then we can say that we have dual frame if inequality 2.2.17 holds. This completes the proof. □

Now let us obtain the condition for the uniqueness of the dual frame of the Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$.

Proposition 2.2.8 *Let $ab = p \in \mathbf{N}$. The window function h satisfying the conditions of the theorem 2.2.7 exists uniquely if and only if*

$$\{(x,w) \in [0,1]\times[0,1/p] : G(x,w) \neq 0\} = \{(x,w) \in [0,1]\times[0,1/p] : K(x,w) \neq 0\}.$$

Proof: First let us prove the necessity part of the implication.

Let $\Omega_1 := \{(x,w) \in [0,1] \times [0,1/p] : G(x,w) \neq 0\}$, and $\Omega_2 := \{(x,w) \in [0,1] \times [0,1/p] : K(x,w) \neq 0\}$. We know that the condition for the existence of dual frame implies that $\Omega_1 \subseteq \Omega_2$. If $\Omega_2 \backslash \Omega_1$ has a positive measure, then we can define the vector functions

$$H_1(x,w) = \begin{cases} \frac{pK}{\langle G(x,w), K(x,w)\rangle_{\mathbb{C}^p}}, & (x,w) \in \Omega_1 \\ K, & (x,w) \in \Omega_2 \setminus \Omega_1 \\ 0, & (x,w) \in \Omega_2^c \end{cases} \tag{2.2.19}$$

, and

$$H_2(x,w) = \begin{cases} \frac{pK}{\langle G(x,w),K(x,w)\rangle_{\mathbb{C}^p}}, & (x,w) \in \Omega_1 \\ 0, & (x,w) \in \Omega_2 \setminus \Omega_1 \\ 0, & (x,w) \in \Omega_2^c. \end{cases} \tag{2.2.20}$$

Since both of H_1 and H_2 satisfy the conditions of the theorem 2.2.7, we conclude that if $\Omega_2 \backslash \Omega_1$ has a positive measure, then dual frame can not be unique. Therefore, if we have a unique dual frame, then $\Omega_1 = \Omega_2$ almost everywhere. Conversely, if $\Omega_1 = \Omega_2$ up to a set of zero measure, the dual is clearly unique and is obtained by the equality: $H := \frac{pK}{<G,K>_{\mathbb{C}^p}} \chi_{\Omega_1}$. □

2.3 Solution for the case of $ab = p/q,\ gcd(p, g) = 1$

This chapter is the core of this thesis since we are giving the main theorem, Theorem 2.3.20, here together with a proof of it using matrix-valued functions. It is also very useful to point out here that in Theorem 2.3.20, we give a condition for the existence of the dual frame, and in this chapter we obtain a result characterizing the existence and uniqueness of the dual frame.

Before we give a characterization of Bessel sequence, dual frame, etc. in terms of the Zak transform, we will make some remarks related to the Zak transform of $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ for the case of $ab = p/q,\ gcd(p, g) = 1$.

Let us express the Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ in the following way:

$$E_{mb}T_{na}g = g_{mb,na} = g_{mb,\frac{np}{qb}} \quad (for \quad ab = p/q,\ gcd(p, g) = 1).$$

We know that every $n \in \mathbf{Z}$ can be written uniquely as $n = i + \ell q$ with $i \in \{0, 1, \ldots, q-1\}$ for some $\ell \in \mathbf{Z}$. Thus, we get that

$$\frac{np}{qb} = \frac{ip}{qb} + \frac{\ell p}{b}.$$

Therefore, we have that

$$E_{mb}T_{na}g(x) = g_{mb,\frac{np}{bq}}(x) = g_{mb,\frac{ip}{qb}+\frac{\ell p}{b}}(x) = e^{2\pi imbx}g\left(x - \frac{ip}{qb} - \frac{\ell p}{b}\right). \quad (2.3.21)$$

Now let us define the function $g^i(x) := g\left(x - \frac{ip}{qb}\right)$. Then the equality (2.3.21) turns into

$$e^{2\pi imbx}g^i\left(x - \frac{\ell p}{b}\right) = g^i_{mb,\frac{\ell p}{b}}(x).$$

In the theorem 1.1.14, we proved that the Zak transform with parameter $\alpha = b$ of $g_{mb,\frac{np}{b}}$ is

$$e^{2\pi imx}e^{-2\pi inpw}Z_b g(x, w).$$

Therefore,

$$Z_b(g^i_{mb,\frac{\ell p}{b}}) = e^{2\pi imx}e^{-2\pi i\ell pw}Z_b g^i(x, w) = E_{m,-\ell p}(x, w)Z_b g^i(x, w).$$

We will now define matrix-valued functions and discuss some of their properties which will be useful later on.

We will write as $A \geq 0$ if A is any positive-semidefinite matrix, and $A \geq B$ if the difference $A - B$ of square matrices A and B of the same size is a positive-semidefinite matrix.

Definition 2.3.1 Let (Ω, μ) be a measure space. Let $\mathcal{M}_{r \times s}$ be the set of complex matrices of size $r \times s$.

Let $\gamma : \Omega \to \mathcal{M}_{r \times s}$ be a map. Then γ is measurable (respectively in $L^p(\Omega)$ where $1 \leq p \leq \infty$) if and only if each entry of γ is measurable (respectively in $L^p(\Omega)$ where $1 \leq p \leq \infty$).

Lemma 2.3.2 *Let (Ω, μ) be a measure space where $\Omega = [0,1] \times [0, 1/p]$. Let $A : \Omega \longrightarrow \mathcal{M}_{r \times s}$ be a measurable matrix-valued function.*

If A is not positive-semidefinite almost everywhere, then there exists a vector $\xi \in \mathbb{C}^r$ such that $\langle A(x,w)\xi, \xi \rangle < 0$ for all (x,w) in a set B of positive measure.

Proof: Let $\{\xi_i\}_{i=1}^{\infty}$ be a countable dense subset of $\mathbb{C}^r$.

Note that a fixed matrix A of size $r \times r$ is positive-semidefinite if and only if $\langle A\xi_i, \xi_i \rangle \geq 0$ for every $i = 1, 2, \ldots$.

Define

$$B_i = \{(x,w) | \langle A(x,w)\xi_i, \xi_i \rangle < 0\}, \quad i = 1, 2, \ldots.$$

At least one of these B_i's has a positive measure, since, otherwise, for every i,

$$\langle A(x,w)\xi_i, \xi_i \rangle \geq 0 \tag{2.3.22}$$

on $[0,1] \times [0, 1/p] \backslash B_i$. Hence, the inequality 2.3.22 holds for every i on the set $[0,1] \times [0,1/p] \backslash \cup_{i=1}^{\infty} B_i$ where measure of the set $\cup_{i=1}^{\infty} B_i$ is 0. This implies that $\langle A(x,w)\xi_i, \xi_i \rangle \geq 0$ almost everywhere on $[0,1] \times [0,1/p]$ for every i.

This means that A is a positive-semidefinite matrix almost everywhere on $[0,1] \times [0,1/p]$, which is a clear contradiction with the hypothesis of the lemma. Hence, at least one of the B_i's has a positive-measure. Therefore, $\langle A(x,w)\xi, \xi\rangle < 0$ on a set B of positive measure as claimed if we let $\xi = \xi_i$ and $B = B_i$. □

Definition 2.3.3 Given window function g, we can associate g with a matrix valued function G defined on $[0,1] \times [0,1/p]$ as $G = (G^0, \ldots, G^{q-1})$, and $G^i = (G^i_0, \ldots, G^i_{p-1})$ for $i = 0, \ldots, q-1$ and $G^i_k(x,w) = Z_b g^i(x, w + k/p)$ for $k = 0, \ldots, p-1$.

Keeping these notations in mind, and letting $G_{i,j} = G^i_j$, we can define a new matrix valued function G^* as $G^*_{i,j} = \overline{G^j_i}$, for $i = 0, \ldots, q-1$ and $j = 0, \ldots, p-1$, where the bar indicates complex conjugation.

Note that for the window functions $f, k, h \in L^2(\mathbf{R})$, we can define the same type of matrix valued functions, but in this case, we will replace G with F, K and H respectively.

Notation: We will denote the closure of the closed linear span of $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ with $\mathcal{M}(b,a,g)$.

Next we obtain the condition for the Bessel collection.

Proposition 2.3.4 *Let $a, b > 0$ and $ab = p/q$ and $gcd(p,q) = 1$. Let G and G^* be defined as in the definition 2.3.3. Then the following are equivalent.*

(a) There exists a positive constant B such that the inequality

$$\sum_{m,n} |\langle f(x), E_{mb}T_{na}g(x)\rangle|^2 = \sum_{i=0}^{q-1}\sum_{m,\ell} \left|\langle f(x), g^i_{mb,\frac{\ell p}{b}}(x)\rangle\right|^2 \leq B\|f(x)\|^2 \tag{2.3.23}$$

holds for every $f \in \mathcal{M}(b,a,g)$.

(b) $G^*G \leq BpI$ almost everywhere.

(c) $GG^* \leq BpI$ almost everywhere.

Proof: First, let us prove the equivalence $(a) \iff (b)$. Let us consider the left-hand side of the inequality (2.3.23) first.

$$\begin{aligned}
&\sum_{m,n} |\langle f(x), E_{mb}T_{na}g(x)\rangle|^2 \\
&= \sum_{i=0}^{q-1} \sum_{m,\ell\in\mathbf{Z}} |\langle f(x), g^i_{mb,\frac{\ell p}{b}}(x)\rangle|^2 \\
&= \sum_{i=0}^{q-1} \sum_{m,\ell\in\mathbf{Z}} |\langle Z_b f(x,w), Z_b g^i_{mb,\frac{\ell p}{b}}(x,w)\rangle|^2 \\
&= \sum_{i=0}^{q-1} \sum_{m,\ell\in\mathbf{Z}} |\langle Z_b f(x,w)\overline{Z_b g^i(x,w)}, E_{m,-\ell p}(x,w)\rangle|^2 \\
&= \sum_{i=0}^{q-1} \sum_{m,\ell\in\mathbf{Z}} \left| \int_0^1 \int_0^1 Z_b f(x,w)\overline{Z_b g^i(x,w)} e^{-2\pi i m x} e^{2\pi i \ell p w} \, dx \, dw \right|^2 \\
&= \sum_{i=0}^{q-1} \sum_{m,\ell\in\mathbf{Z}} \left| \int_0^1 \int_0^{1/p} \sum_{k=0}^{p-1} Z_b f\left(x, w+\frac{k}{p}\right) \overline{Z_b g^i\left(x, w+\frac{k}{p}\right)} e^{-2\pi i m x} e^{2\pi i \ell p w} \, dx \, dw \right|^2 \\
&= \frac{1}{p}\sum_{i=0}^{q-1} \int_0^1 \int_0^{1/p} \left| \sum_{k=0}^{p-1} Z_b f\left(x, w+\frac{k}{p}\right) \overline{Z_b g^i\left(x, w+\frac{k}{p}\right)} \right|^2 \, dxdw. \qquad (2.3.24)
\end{aligned}$$

Define

$$F_k(x,w) = Z_b f\left(x, w+\frac{k}{p}\right), \qquad k = 0,\ldots,p-1$$

and

$$G^i_k(x,w) = Z_b g^i\left(x, w+\frac{k}{p}\right), \qquad k = 0,\ldots,p-1 \text{ and } i = 0,\ldots,q-1.$$

Consider the vector-valued functions $F = (F_0,\ldots,F_{p-1})$ and $G^i = (G^i_0,\ldots,G^i_{p-1})$. Since

$$|\langle F(x,w), G^i(x,w)\rangle_{\mathbb{C}^p}|^2 = \left| \sum_{k=0}^{p-1} F_k(x,w)\overline{G^i_k(x,w)} \right|^2,$$

The last term in the equation 2.3.24 becomes

$$\frac{1}{p}\int_0^1 \int_0^{1/p} \sum_{i=0}^{q-1} |\langle F(x,w), G^i(x,w)\rangle_{\mathbb{C}^p}|^2 \, dx \, dw$$

$$\begin{aligned}
&= \frac{1}{p}\int_0^1\int_0^{1/p}\sum_{i=0}^{q-1}\left|\sum_{k=0}^{p-1}F_k(x,w)\overline{G_k^i(x,w)}\right|^2\, dx\, dw \\
&= \frac{1}{p}\int_0^1\int_0^{1/p}\sum_{i=0}^{q-1}\left(\sum_{k=0}^{p-1}F_k(x,w)\overline{G_k^i(x,w)}\sum_{\ell=0}^{p-1}\overline{F_\ell(x,w)}G_\ell^i(x,w)\right)\, dx\, dw \\
&= \frac{1}{p}\int_0^1\int_0^{1/p}\left(\sum_{k,\ell=0}^{p-1}\sum_{i=0}^{q-1}\overline{G_k^i(x,w)}G_\ell^i(x,w)F_k(x,w)\overline{F_\ell(x,w)}\right)\, dx\, dw.
\end{aligned} \tag{2.3.25}$$

Define the matrices G and G^* of size $q\times p$ and $p\times q$ respectively as

$$G_{i,j} = G_j^i \quad \text{and} \quad G_{i,j}^* = \overline{G_i^j}, \quad i=0,\ldots,q-1 \quad \text{and} \quad j=0,\ldots,p-1. \tag{2.3.26}$$

Note that

$$(G^*G)_{i,j} = \sum_{k=0}^{q-1} G_{i,k}^* G_{k,j} = \sum_{k=0}^{q-1}\overline{G_i^k}G_j^k.$$

Since

$$\|f\|^2 = \int_0^1\int_0^{1/p}\|F(x,w)\|_{\mathbb{C}^p}^2\, dx\, dw,$$

we can rewrite the inequality 2.3.23 as

$$\int_0^1\int_0^{1/p}\left(\sum_{k,\ell=0}^{p-1}(G^*G)_{k,\ell}(x,w)F_k(x,w)\overline{F_\ell(x,w)} - Bp\|F(x,w)\|_{\mathbb{C}^p}^2\right)\, dx\, dw \le 0. \tag{2.3.27}$$

Now assume that (b) holds. In other words, suppose that $BpI-(G^*G)$ is a positive-semidefinite matrix-valued function almost everywhere. That is, suppose that the inequality

$$\langle BpI-(G^*G)(x,w)\xi,\ \xi\rangle_{\mathbb{C}^p}\ge 0$$

holds for almost every $(x,w)\in[0,1]\times[0,1/p]$ and $\xi\in\mathbb{C}^p$.

Thus,

$$\sum_{k,\ell=0}^{p-1}(G^*G)_{k,\ell}(x,w)\xi_k\bar{\xi}_\ell - Bp\|\xi\|_{\mathbb{C}^p}^2\le 0$$

for almost every $\xi \in \mathbb{C}^p$ and $(x, w) \in [0, 1] \times [0, 1/p]$. In this inequality if we replace ξ with $\overline{F} = (\overline{F_0}, \ldots, \overline{F_{p-1}})$, we immediately obtain the inequality (2.3.27). Hence, we conclude that (b) implies (a).

Assume now that (a) holds, or, equivalently, that equality (2.3.27) holds, and let us prove that $BpI - (G^*G)$ is a positive-semidefinite matrix-valued function almost everywhere.

We can rewrite inequality (2.3.27) as

$$\int_0^1 \int_0^{1/p} \langle A(x, w)F(x, w), F(x, w)\rangle_{\mathbb{C}^p} \; dx \; dw \geq 0 \tag{2.3.28}$$

where $A = BpI - (G^*G)$. Now suppose that inequality (2.3.28) holds but that A is not positive semidefinite almost everywhere.

Using lemma 2.3.2, we can then find a vector $\xi \in \mathbb{C}^p$ such that $\langle A(x, w)\xi, \xi\rangle < 0$ for every $(x, w) \in B$ where $B \subseteq [0, 1] \times [0, 1/p]$ is a measurable set with positive measure. Letting $F = \xi\chi_B$, we have

$$\int_0^1 \int_0^{1/p} \langle A(x, w)\xi\chi_B(x, w), \xi\chi_B(x, w)\rangle_{\mathbb{C}^p} \; dxdw$$
$$= \int_B \int \langle A(x, w)\xi, \xi\rangle_{\mathbb{C}^p} \; dx \; dw < 0$$

which contradicts the inequality 2.3.28. Thus, we conclude that (a) implies (b). Finally, we want to prove the equivalence of (b) and (c). Suppose that $G^*G - BpI \leq 0$, then we have that $G(G^*G - Bp)G^* \leq 0$ and then $(GG^*)^2 - Bp(GG^*) \leq 0$. Thus, if λ is an eigenvalue of GG^*, then we have $\lambda^2 \leq Bp\lambda$ or $\lambda \leq Bp$. Therefore, we conclude that $GG^* - BpI \leq 0$. The proof of the converse is similar. □

Now before we obtain a condition for lower frame bound, we want to characterize the functions which are orthogonal to $\mathcal{M}(b, a, g)$ in terms of the Zak transform.

Lemma 2.3.5 *Let $a, b > 0$ and $ab = p/q$ and $gcd(p, q) = 1$. Let $g \in L^2(\mathbf{R})$ and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ is a Bessel collection. Let $h \in L^2(\mathbf{R})$. Then h is orthogonal to $\mathcal{M}(b, a, g)$ if and only if $\langle H(x, w), G^i(x, w)\rangle_{\mathbb{C}^p} = 0$ for almost* *every* $(x,w) \in [0,1] \times [0,1/p]$, $i = 0,\dots,p-1$, *where* $H = (H_0,\dots,H_{p-1}) \in \mathbb{C}^p$, $H_i(x,w) = Z_b h(x, w+i/p)$, $G^i = (G_0^i,\dots,G_{p-1}^i)$ *and* $G_k^i(x,w) = Z_b g^i(x, w+k/p)$, $i = 0,\dots,p-1$.

Proof: Let us prove the necessity part of the implication first. Let $h \in L^2(\mathbf{R})$, and assume that $h \perp \mathcal{M}(b,a,g)$. Then, we have that

$$0 = \langle h(x), E_{mb}T_{na}g(x)\rangle = \langle Z_b h(x,w)\overline{Z_b g^i(x,w)}, E_{m,-\ell p}(x,w)\rangle$$
$$= \int_0^1 \int_0^{1/p} \sum_{k=0}^{p-1} Z_b h(x, w+\frac{k}{p}) \overline{Z_b g^i\left(x, w+\frac{k}{p}\right)} e^{-2\pi i m x} e^{2\pi i \ell p w}\, dx\, dw. \quad (2.3.29)$$

Define

$$H_k(x,w) = Z_b h\left(x, w+\frac{k}{p}\right), \quad G_k^i(x,w) = Z_b g^i\left(x, w+\frac{k}{p}\right), \quad k = 0,\dots,p-1 \quad (2.3.30)$$

$$H = (H_0,\dots,H_{p-1}), \quad G^i = (G_0^i,\dots,G_{p-1}^i) \quad G_{i,j} = G_j^i, \quad G_{i,j}^* = \overline{G_i^j}.$$

Note that

$$\begin{aligned}(G^*G)_{k,k}(x,w) &= \sum_{i=0}^{q-1} G_{k,i}^*(x,w) G_{i,k}(x,w) = \sum_{i=0}^{q-1} |G_{i,k}(x,w)|^2 \\ &= \sum_{i=0}^{q-1} \left| Z_b g^i\left(x, w+\frac{k}{p}\right)\right|^2 \le Bp. \end{aligned} \quad (2.3.31)$$

It is clear that G is a $q \times p$ matrix, and G^* is a $p \times q$ matrix and G^*G is a $p \times p$ matrix. After these definitions, it is obvious that the equality (2.3.29) becomes

$$\int_0^1 \int_0^{1/p} \langle H(x,w), G^i(x,w)\rangle_{\mathbb{C}^p} \quad e^{-2\pi i m x} e^{2\pi i l p w}\, dx\, dw = 0 \quad (2.3.32)$$

Now let us show that the function $\langle H(x,w), G^i(x,w)\rangle_{\mathbb{C}^p}$ is in $L^2([0,1]\times[0,1/p])$.

$$\begin{aligned}&\int_0^1 \int_0^{1/p} |\langle H(x,w), G^i(x,w)\rangle_{\mathbb{C}^p}|^2\, dx\, dw \\ &\le \int_0^1 \int_0^{1/p} \|H(x,w)\|_{\mathbb{C}^p}^2 \|G^i(x,w)\|_{\mathbb{C}^p}^2\, dx dw \\ &\le Bp \int_0^1 \int_0^{1/p} \|H(x,w)\|_{\mathbb{C}^p}^2\, dx\, dw \end{aligned} \quad (2.3.33)$$

by the inequality 2.3.31, since $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ is a Bessel collection and by the definition of G^i in the equation (2.3.26). The integral on the right hand side of inequality (2.3.35) is finite since H is an L^2 function. Hence, $\langle H(x,w), G^i(x,w)\rangle_{\mathbb{C}^p}$ is an L^2 function on $[0,1]\times[0,1/p]$.

Since $\{e^{2\pi imx}e^{-2\pi ilpw}\}_{m,l\in\mathbf{Z}}$ is an orthogonal basis for $L^2([0,1]\times[0,1/p])$, and $\langle H, G^i\rangle_{\mathbb{C}^p}$ is in $L^2([0,1]\times[0,1/p])$, the equality 2.3.32 implies that

$$\langle H(x,w), G^i(x,w)\rangle_{\mathbb{C}^p} = 0$$

almost everywhere in $[0,1]\times[0,1/p]$, for every $i=0,\ldots,q-1$. Since this argument can clearly be reversed, our proof is now completed. □

Now we want to characterize the functions in $\mathcal{M}(b,a,g)$, and this characterization of the functions will be very useful tool when solving the duality problem.

Proposition 2.3.6 *Let $a,b>0$ and $ab=p/q$ and $gcd(p,q)=1$. Let $g\in L^2(\mathbf{R})$ such that $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ forms a Bessel collection. Then, $f\in L^2(\mathbf{R})$ belongs to $\mathcal{M}(b,a,g)$ if and only if*

$$F=\sum_{i=0}^{q-1} a_i G^i \tag{2.3.34}$$

where $a_i:[0,1]\times[0,1/p]\longrightarrow\mathbb{C}$ is measurable function for each $i=0,\ldots,q-1$ such that

$$\int\int_{[0,1]\times[0,1/p]}\left|\sum_{i=0}^{q-1} a_i(x,w)G^i(x,w)\right|^2 dx\ dw<\infty$$

In particular, if $a_i\in L^2([0,1]\times[0,1/p])$, $i=0,\ldots,q-1$, then there exist a function f which belongs to $\mathcal{M}(b,a,g)$ such that

$$F=\sum_{i=0}^{q-1} a_i G^i.$$

Note that the relation between f and F is given in the definition 2.3.3.

Proof: First we want to prove that if $h \perp \mathcal{M}(b,a,g)$ and $F = \sum_{i=0}^{q-1} a_i G^i$, then $f \in \mathcal{M}(b,a,g)$. In order to prove this, we will show that $< f, h >= 0$.

$$\begin{aligned}
\langle h(t), f(t)\rangle &= \langle Z_b h(x,w), Z_b f(x,w)\rangle \\
&= \int_0^1 \int_0^1 Z_b h(x,w)\overline{Z_b f(x,w)}dxdw \\
&= \int_0^1 \int_0^{1/p} \sum_{k=0}^{p-1} Z_b h(x, w + k/p)\overline{Z_b f(x, w + k/p)}dxdw \\
&= \int_0^1 \int_0^{1/p} \langle H(x,w), F(x,w)\rangle dxdw \\
&= \int_0^1 \int_0^{1/p} \langle H(x,w), \sum_{i=0}^{q-1} a_i G^i(x,w)\rangle dxdw \\
&= \sum_{i=0}^{q-1} \int_0^1 \int_0^{1/p} a_i \langle H(x,w), G^i(x,w)\rangle dxdw \\
&= 0
\end{aligned} \tag{2.3.35}$$

by Lemma 2.3.5. Therefore, if $h \perp \mathcal{M}(b,a,g)$ and $F = \sum_{i=0}^{q-1} a_i G^i$, then $f \in \mathcal{M}(b,a,g)$.

Now, let us consider indices $i_1, \ldots, i_r$ with $0 \leq i_1 \leq i_2 \leq \ldots \leq i_r \leq q-1$ for $1 \leq r \leq q-1$.

Note that the set $E_{i_1,\ldots,i_r}$

$$E_{i_1,\ldots,i_r} = \{(x,w) \in [0,1] \times [0,1/p] | G^{i_1}(x,w), \ldots, G^{i_r}(x,w)$$

$$\textit{are linearly independent and}$$

$$\text{span}(G^{i_1}(x,w), \ldots, G^{i_r}(x,w)) = \text{span}(G^0(x,w), \ldots, G^{q-1}(x,w))\}$$

is a measurable set. Indeed, $E_{i_1,\ldots,i_r}$ can be expressed as the intersection of the measurable sets

$$\{\det(\langle G^k, G^l\rangle_{\mathbb{C}^p})_{k,l \in \{i_1,\ldots,i_r\}} \neq 0\}$$

and

$$\{\det(\langle G^k, G^l\rangle_{\mathbb{C}^p})_{k,l \in \{j,i_1,\ldots,i_r\}} = 0\},$$

where j varies over all the indices in $\{0,\ldots,q-1\}$ different from $i_1,\ldots,i_r$.

Let $K=\{(x,w)\in[0,1]\times[0,1/p]|G^i(x,w)=0, \quad \text{for every} \quad i=0,\ldots,q-1\}$. Then, we have

$$[0,1]\times[0,1/p]=K\bigcup\left[\bigcup_{r=1}^{q-1}\bigcup_{0\leq i_1\leq i_2\leq\ldots\leq i_r\leq q-1}E_{i_1,\ldots,i_r}\right] \tag{2.3.36}$$

Since the sets $E_{i_1,\ldots,i_r}$ are not necessarily disjoint, we can replace them with a pairwise disjoint collection of the sets $F_{i_1,\ldots,i_r}$ with $F_{i_1,\ldots,i_r}\subseteq E_{i_1,\ldots,i_r}$ such that the equality 2.3.36 holds.

Assume that the measure of the set $F_{i_1,\ldots,i_r}$ is not zero. Let $f\in L^2(\mathbf{R})$, and F is the corresponding matrix-valued function defined as in the definition 2.3.3. Note that we can write the equality

$$F=\sum_{i=0}^{q-1}a_iG^i+H$$

uniquely on the set $F_{i_1,\ldots,i_r}$, where $a_i=0$ for $i\notin\{i_1,\ldots,i_r\}$, and $<H,G^i>=0$ for $i=0,\ldots,q-1$.

Let $\sum_{i=0}^{q-1}a_iG^i=K$, then the corresponding function $k\in L^2(\mathbf{R})$ belongs to $\mathcal{M}(b,a,g)$ by the previous argument. Now suppose that $f\in\mathcal{M}(b,a,g)$, then $f-k\in\mathcal{M}(b,a,g)$ as well, and, thus,

$$H=F-K\bot\{G^0,\ldots,G^{q-1}\}.$$

Therefore the function h corresponding to H must be orthogonal to $\mathcal{M}(b,a,g)$. This implies that $h=0$, and therefore,

$$F=K=\sum_{i=0}^{q-1}a_iG^i.$$

Now we only need to prove that each a_i, $i=0,\ldots,q-1$, in the equation (2.3.34) is a measurable function. It is obvious that each a_i, $i=0,\ldots,q-1$ is a measurable function on the set $E_{i_1,\ldots,i_r}$ and $a_i=0$ if $i\neq i_1,\ldots,i_r$ and

$$\langle F,G^{i_k}\rangle_{\mathbb{C}^p}=\sum_{j=1}^{r}a_{i_j}\langle G^{i_j},G^{i_k}\rangle_{\mathbb{C}^p}$$

$$\begin{pmatrix} \langle F, G^{i_1} \rangle \\ \vdots \\ \langle F, G^{i_r} \rangle \end{pmatrix} = \begin{pmatrix} \langle G^{i_1}, G^{i_1} \rangle & \cdots & \langle G^{i_r}, G^{i_1} \rangle \\ \vdots & & \vdots \\ \langle G^{i_1}, G^{i_r} \rangle & \cdots & \langle G^{i_r}, G^{i_r} \rangle \end{pmatrix} \begin{pmatrix} a_{i_1} \\ \vdots \\ a_{i_r} \end{pmatrix}$$

and, let,

$$B := \begin{pmatrix} \langle G^{i_1}, G^{i_1} \rangle_{\mathbb{C}^p} & \cdots & \langle G^{i_r}, G^{i_1} \rangle_{\mathbb{C}^p} \\ \vdots & & \vdots \\ \langle G^{i_1}, G^{i_r} \rangle_{\mathbb{C}^p} & \cdots & \langle G^{i_r}, G^{i_r} \rangle_{\mathbb{C}^p} \end{pmatrix}.$$

We know that B is invertible on the set $E_{i_1,\ldots,i_r}$. Therefore, we can compute each a_i by taking the inverse of B if $i \in \{i_1, \ldots, i_r\}$. This completes the proof. □

Before we obtain a condition for the lower frame bound for the Bessel collection $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$, let us give a lemma which will be used in the proof of the next proposition.

Lemma 2.3.7 *Let F be a bounded matrix-valued function of size $p \times p$, and the inequality*

$$0 \leq \int_0^1 \int_0^1 < F(x,w)a(x,w), a(x,w) >_{\mathbb{C}^p} dxdw \qquad (2.3.37)$$

holds for every vector $a \in \mathbb{C}^p$ such that each component of a is an $L^2(Q)$ function, then F is positive semidefinite almost everywhere.

Proof: Suppose that F is not positive semi definite almost everywhere. Then, we can find a vector $\eta = (\eta_i)_{i=0}^{q-1}$ such that $< F\eta, \eta >< 0$ on a set S of positive measure. Since the inequality 2.3.37 is true for every $a \in \mathbb{C}^q$, if we let $a_i = \eta_i \chi_S$, we get a contradiction. Therefore, we conclude that if the inequality 2.3.37 holds, then F must be positive definite almost everywhere. □

Proposition 2.3.8 *Let $a, b > 0$ and $ab = p/q$ and $gcd(p,q) = 1$. Let $\{E_{mb}T_{na}g(x)\}$ be a Bessel sequence. Then, there exists a positive constant A such that the in-*

equality

$$A\|f(x)\|^2 \le \sum_{m,n} |\langle f(x), E_{mb}T_{na}g(x)\rangle|^2 = \sum_{i=0}^{q-1}\sum_{m,\ell} \left|\langle f(x), g_{mb,\frac{ip}{qb}+\frac{\ell p}{b}}(x)\rangle\right|^2 \tag{2.3.38}$$

holds for every $f \in \mathcal{M}(b,a,g)$ *if and only if* $Ap\xi \le \xi^2$*, where,* $\xi = GG^*$*, and* G *and* G^* *are matrix-valued functions defined as in the definition 2.3.3.*

Proof: We can express the inequality (2.3.38) as

$$\begin{aligned} A&\int_0^1\int_0^{1/p} \|F(x,w)\|^2_{\mathbb{C}^p}\, dx\, dw \\ &\le \frac{1}{p}\sum_{i=0}^{q-1}\int_0^1\int_0^{1/p} |\langle F(x,w), G^i(x,w)\rangle_{\mathbb{C}^p}|^2\, dx\, dw. \end{aligned} \tag{2.3.39}$$

We showed that $F = \sum_{i=0}^{q-1} a_i G^i$ for $f \in \mathcal{M}(b,a,g)$ in the previous proposition.

First, let us look at the sum on the right hand side of the inequality 2.3.39:

$$\begin{aligned} \sum_{i=0}^{q-1} |\langle F, G^i\rangle_{\mathbb{C}^p}|^2 &= \sum_{i=0}^{q-1}\left|\langle\sum_{k=0}^{q-1} a_k G^k, G^i\rangle_{\mathbb{C}^p}\right|^2 \\ &= \sum_{i=0}^{q-1}\left|\sum_{k=0}^{q-1} a_k\langle G^k, G^i\rangle_{\mathbb{C}^p}\right|^2 \\ &= \sum_{i=0}^{q-1}\sum_{k=0}^{q-1} a_k\langle G^k, G^i\rangle_{\mathbb{C}^p}\sum_{\ell=0}^{q-1}\overline{a_\ell\langle G^\ell, G^i\rangle_{\mathbb{C}^p}} \\ &= \sum_{i=0}^{q-1}\sum_{k=0}^{q-1} a_k\langle G^k, G^i\rangle_{\mathbb{C}^p}\sum_{\ell=0}^{q-1}\overline{a_\ell}\langle G^i, G^\ell\rangle_{\mathbb{C}^p} \\ &= \sum_{k,\ell=0}^{q-1}\left(\sum_{i=0}^{q-1}\langle G^k, G^i\rangle_{\mathbb{C}^p}\langle G^i, G^\ell\rangle_{\mathbb{C}^p}\right) a_k\overline{a_\ell}. \end{aligned}$$

Define $\langle G^k, G^i\rangle = \xi_{k,i}$. So by this definition, we have that

$$\sum_{k,\ell=0}^{q-1}\left(\sum_{i=0}^{q-1}\xi_{i,\ell}\xi_{k,i}\right) a_k\overline{a_\ell} = \sum_{k,\ell=0}^{q-1}\xi^2_{k,\ell}a_k\overline{a_\ell}.$$

Now, let us look at the integral on the left hand side of the inequality (2.3.39)

$$A\int_0^1\int_0^{1/p} \|F(x,w)\|^2_{\mathbb{C}^p}\, dx\, dw, \qquad F = \sum_{k=0}^{q-1} a_k G^k,$$

$$\|F\|^2_{\mathbb{C}^p} = \sum_{k,\ell=0}^{q-1} \langle G^k, G^\ell \rangle_{\mathbb{C}^p} a_k \overline{a_\ell} = \sum_{k,\ell=0}^{q-1} \xi_{k,\ell} a_k \overline{a_\ell}.$$

Then, the inequality (2.3.39) becomes

$$0 \leq \int_0^1 \int_0^{1/p} \sum_{k,\ell=0}^{q-1} \left(\frac{1}{p} \xi^2_{k,\ell}(x,w) - A\xi_{k,\ell}(x,w) \right) a_k(x,w) \quad \overline{a_\ell}(x,w) \ dx \ dw. \tag{2.3.40}$$

If the condition $Ap\xi \leq \xi^2$ is true, then it is obvious that the inequality 2.3.40 holds, and conversely if the inequality 2.3.40 holds, then using lemma 2.3.7, we conclude that

$$\frac{1}{p} \xi^2_{k,\ell} - A\xi_{k,\ell} \geq 0 \Rightarrow \xi^2_{k,\ell} \geq Ap\xi_{k,\ell}.$$

Hence, in general, the condition for the lower frame bound is $Ap\xi \leq \xi^2$. □

In proposition 2.3.4, we proved that the condition for the Bessel collection is equivalent to the matrix inequality $\xi^2 \leq Bp\xi$, where ξ is as in the definition 2.3.1. Therefore, combining the propositions 2.3.4 and 2.3.6, we obtain a condition for the collection $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ to be a Gabor subspace frame in the following corollary:

Corollary 2.3.9 *Let $a, b > 0$, and $ab = p/q$, $gcd(p,q) = 1$. The collection $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ is a Gabor subspace frame with frame bounds A and B if and only if $Ap\xi \leq \xi^2 \leq Bp\xi$, where $\xi = GG^*$, and $G = (G^0, \ldots, G^{q-1})$, $G^i = (G^i_0, \ldots, G^i_{p-1})$, $G^i_k(x,w) = Z_b g^i(x, w + k/p)$.*

Before we start to solve the duality problem, let us give a useful conclusion about Gabor subspace frames.

Corollary 2.3.10 *Let $\xi = GG^*$. The frame condition $Ap\xi \leq \xi^2 \leq Bp\xi$ reduces to $\xi^2 = p\xi$ for normalized tight frames since we have that $A = B = 1$ and $p\xi \leq \xi^2 \leq p\xi$. Furthermore, $\xi^2 = p\xi$ means that $\frac{1}{p}\xi$ is a self-adjoint projection operator.*

Next we will find a condition for duality problem.

Proposition 2.3.11 *Let $a, b > 0$ and $ab = p/q$ and $gcd(p, q) = 1$. Let $\{E_{mb}T_{na}g\}$ be a Bessel collection. Assume that a function $h \in L^2(\mathbf{R})$ generates a Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$. Then, $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of the Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ if and only if*

$$G^k(x, w) = \frac{1}{p}\sum_{i=0}^{q-1} G^i(x, w)\langle G^k(x, w), H^i(x, w)\rangle_{\mathbb{C}^p},$$

for almost every $(x, w) \in [0, 1] \times [0, \frac{1}{p}]$.

Proof: Remember that we can express the Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ in the following way:

$$E_{mb}T_{na}g(x) = g_{mb,\frac{np}{bq}}(x) = g_{mb,\frac{ip}{qb}+\frac{\ell p}{b}}(x) = e^{2\pi imbx}g\left(x - \frac{ip}{qb} - \frac{\ell p}{b}\right). \quad (2.3.41)$$

Defining the function $g^i(x) := g\left(x - \frac{ip}{qb}\right)$, the equality 2.3.41 turns into the form

$$e^{2\pi imbx}g^i\left(x - \frac{lp}{b}\right) = g^i_{mb,\frac{\ell p}{b}}(x).$$

Suppose $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a Bessel collection, and further assume that $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame for $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. Let $f \in \mathcal{M}(b, a, g)$. Then,

$$\begin{aligned}
f(x) &= \sum_{m,n}\langle f(x), h_{mb,\frac{np}{q}}(x)\rangle g_{mb,\frac{np}{q}}(x)\\
&= \sum_{m,\ell}\sum_{i=0}^{q-1}\langle f(x), h^i_{mb,\ell p/b}(x)\rangle g^i_{mb,\ell p/b}(x)\\
Z_bf(x, w) &= \sum_{m,\ell}\sum_{i=0}^{q-1}\langle Z_bf(x, w), Z_bh^i_{mb,\ell p/b}(x, w)\rangle Z_bg^i_{mb,\ell p/b}(x, w)\\
&= \sum_{i=0}^{q-1} Z_bg^i(x, w)\sum_{m,\ell}\langle Z_bf(x, w)\overline{Z_bh^i(x, w)}, E_{m,-lp}(x, w)\rangle E_{m,-lp}(x, w)\\
&= \sum_{i=0}^{q-1} Z_bg^i(x, w)\sum_{m,\ell}\left(\int_0^1\int_0^1 Z_bf(x, w)\overline{Z_bh^i(x, w)}e^{-2\pi imx}e^{2\pi i\ell pw}\, dx\, dw\right)
\end{aligned}$$

$$\times\ e^{-2\pi imx}e^{2\pi i\ell pw}$$
$$= \sum_{i=0}^{q-1} Z_b g^i(x,w) \sum_{m,\ell} \Big(\int_0^1 \int_0^{1/p} \sum_{k=0}^{p-1} Z_b f\left(x, w+\frac{k}{p}\right) \overline{Z_b h^i\left(x, w+\frac{k}{p}\right)}$$
$$\times\ e^{-2\pi imx}e^{2\pi i\ell pw}\ dx\ dw \Big) e^{-2\pi imx}e^{2\pi i\ell pw} \tag{2.3.42}$$

Define

$$\begin{aligned} F_k(x,w) &= Z_b f\left(x, w+\frac{k}{p}\right), \qquad F = (F_0, \ldots, F_{p-1}) \\ H_k^i(x,w) &= Z_b h^i\left(x, w+\frac{k}{p}\right), \qquad H^i = (H_0^i, \ldots, H_{p-1}^i). \end{aligned}$$

Then equality (2.3.42) turns into

$$\sum_{i=0}^{q-1} Z_b g^i(x,w) \sum_{m,\ell} \left(\int_0^1 \int_0^{1/p} \langle F(x,w), H^i(x,w)\rangle_{\mathbb{C}^p} \quad e^{-2\pi imx}e^{2\pi i\ell pw}\ dx\ dw \right)$$
$$\times e^{-2\pi imx}e^{2\pi i\ell pw}$$
$$= \frac{1}{p}\sum_{i=0}^{q-1} Z_b g^i(x,w)\langle F(x,w), H^i(x,w)\rangle_{\mathbb{C}^p}.$$

Then we get that, for each $k = 0, \ldots, q-1$,

$$Z_b f(x, w+k/p) = \frac{1}{p}\sum_{i=0}^{q-1} Z_b g^i\left(x, w+\frac{k}{p}\right)\langle F(x,w), H^i(x,w)\rangle_{\mathbb{C}^p}.$$

Thus, we obtain that

$$F(x,w) = \frac{1}{p}\sum_{i=0}^{q-1} G^i(x,w)\langle F(x,w), H^i(x,w)\rangle_{\mathbb{C}^p}. \tag{2.3.43}$$

This implies that

$$G^k(x,w) = \frac{1}{p}\sum_{i=0}^{q-1} G^i(x,w)\langle G^k(x,w), H^i(x,w)\rangle_{\mathbb{C}^p}, k = 0, \ldots, q-1. \tag{2.3.44}$$

Conversely, if the equality 2.3.44 holds, then so does the equality 2.3.43, since $F = \sum_{k=0}^{q-1} a_k G^k$ by the proposition 2.3.6. This completes our proof. □

Since we have to choose the window function h from the Gabor subspace frame $\overline{span}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ in our main duality problem, we can modify the duality problem as in the following way:

Question: Given $G^0,\dots,G^{q-1}\in\mathbb{C}^p$ and $K^0,\dots,K^{q-1}\in\mathbb{C}^p$ when does there exist $H^i\in span\{K^0,\dots,K^{q-1}\}$, $i=0,\dots,q-1$ such that the equality $F=\frac{1}{p}\sum_{i=0}^{q-1}G^i\langle F,H^i\rangle_{\mathbb{C}^p}$ holds for every $F\in span\{G^0,\dots,G^{q-1}\}$ or such that the equality

$$G^j=\frac{1}{p}\sum_{i=0}^{q-1}G^i\langle G^j,H^i\rangle_{\mathbb{C}^p} \tag{2.3.45}$$

holds for every $j=0,\dots,q-1$?

Since $H^i\in \text{span}\{K^0,\dots,K^{q-1}\}$, we can write H^i as

$$H^i=\sum_{k=0}^{q-1}L_{i,k}K^k$$

for some constants $L_{i,k}$. Hence, we can rewrite the sum $\sum_{i=0}^{q-1}\langle G^j,H^i\rangle$ as

$$\sum_{i=0}^{q-1}\langle G^j,H^i\rangle_{\mathbb{C}^p}=\sum_{i=0}^{q-1}\langle G^j,\sum_{k=0}^{q-1}L_{i,k}K^k\rangle_{\mathbb{C}^p}=\sum_{i=0}^{q-1}\sum_{k=0}^{q-1}\overline{L_{i,k}}\langle G^j,K^k\rangle_{\mathbb{C}^p}. \tag{2.3.46}$$

Therefore, the equality 2.3.45 becomes

$$G^j=\frac{1}{p}\sum_{i=0}^{q-1}\sum_{k=0}^{q-1}\overline{L_{i,k}}\langle G^j,K^k\rangle_{\mathbb{C}^p}G^i. \tag{2.3.47}$$

By taking the inner product of both sides of the equality 2.3.47 with G^ℓ, we obtain that

$$\langle G^\ell,G^j\rangle_{\mathbb{C}^p}=\frac{1}{p}\sum_{i=0}^{q-1}\sum_{k=0}^{q-1}L_{i,k}\langle K^k,G^j\rangle_{\mathbb{C}^p}\langle G^\ell,G^i\rangle_{\mathbb{C}^p}. \tag{2.3.48}$$

Let

$$\langle G^\ell,G^j\rangle_{\mathbb{C}^p}=\xi_{\ell,j}\quad and\quad \langle K^k,G^j\rangle_{\mathbb{C}^p}=\eta_{k,j}. \tag{2.3.49}$$

Therefore, using the notations defined in the equality 2.3.49, we can rewrite the equality 2.3.48 in the following way:

$$\xi_{\ell,j} = \frac{1}{p}\sum_{i=0}^{q-1}\sum_{k=0}^{q-1} L_{i,k}\eta_{k,j}\xi_{\ell,i} = \frac{1}{p}\sum_{i=0}^{q-1}(L\eta)_{i,j}\xi_{\ell,i} = \frac{1}{p}(\xi L\eta)_{\ell,j}. \qquad (2.3.50)$$

Consequently, we can rewrite the equality 2.3.50 as

$$\xi = \frac{1}{p}(\xi L\eta).$$

Now we will give a series of lemmas which will be used in the proof of the duality theorem.

Lemma 2.3.12 *Let A be an invertible positive-semidefinite matrix of size $d \times d$ for a positive constant d, and suppose that I is the $d \times d$ identity matrix. If the difference $A - I$ is positive-semidefinite, then the matrix $I - A^{-1}$ is also positive-semidefinite.*

Proof: Since $A \geq 0$, there exits an invertible positive-semidefinite matrix B with $B^2 = A$. It is clear that we have that $B^{-1}(A-I)B^{-1} \geq 0$, or $B^{-1}AB^{-1} - (B^{-1})^2 \geq 0$. This implies that $I - A^{-1} \geq 0$ since $B^2 = A$. This proves our claim. □

Lemma 2.3.13 *Let A be a positive-semidefinite matrix. Then the inequality*

$$|A_{i,j}| \leq (A_{i,i})^{1/2}(A_{j,j})^{1/2}$$

holds for every i, j.

Proof: Since A is positive semi-definite matrix, there exists a matrix B such that $A = B^*B$. Therefore,

$$A_{i,j} = \sum_k B^*_{i,k}B_{k,j} = \sum_k B_{k,j}\overline{B_{k,i}}.$$

Using the Cauchy-Schwarz inequality, we have that

$$|A_{i,j}| \leq \left(\sum_k B^2_{k,j}\right)^{1/2}\left(\sum_k B^2_{k,i}\right)^{1/2}.$$

This implies that

$$|A_{i,j}| \leq (A_{j,j})^{1/2} (A_{i,i})^{1/2},$$

which completes the proof. □

Lemma 2.3.14 *Let (Ω, μ) be a measure space. Let A be an invertible positive semi-definite matrix-valued function. Then, each entry of $A^{-1}(\cdot)$ is an L^∞ function if and only if there exists a positive constant C such that $A(\cdot) \geq CI$, where I is the identity matrix of the same size with A.*

Proof: $\Leftarrow$: Let $\{e_n\}$ be the standard-orthogonal basis of $\mathbb{C}^q$. It is clear that

$$\langle Ax, y \rangle = \sum_i \left(\sum_j A_{ij} x_j \right) \overline{y_i}$$

for $x = (x_1, \ldots, x_q) \in \mathbb{C}^q$ and $y = (y_1, \ldots, y_q) \in \mathbb{C}^q$, and it is also obvious that

$$A_{ij} = \langle Ae_j, e_i \rangle.$$

We want to prove that each entry of $A^{-1}(\cdot)$ is an L^∞ function.

Since $A(\cdot) \geq CI$ by hypothesis, we have that $A^{-1}(\cdot) \leq C^{-1}I$, by lemma 2.3.12. First let us show that A_{ii}^{-1} is in L^∞ for each i. We have that

$$A_{ii}^{-1} = \langle A^{-1} e_i, e_i \rangle \leq \langle C^{-1} I e_i, e_i \rangle = C^{-1} \langle I e_i, e_i \rangle = C^{-1}.$$

Thus, A_{ii}^{-1} is an L^∞ function for each i. Finally, using the lemma 2.3.13, we have that

$$|A_{ij}^{-1}| \leq (A_{ii}^{-1})^{1/2} (A_{jj}^{-1})^{1/2} \leq C^{-1},$$

showing that A_{ij}^{-1} is an L^∞ function for each i, j. Therefore, each entry of $A^{-1}(\cdot)$ is an L^∞ function as claimed.

$\Rightarrow$: Define $\lambda(\cdot)$ to be the smallest eigenvalue of A. Then $\lambda(\cdot)$ is measurable since

$$\begin{aligned}
\lambda(\cdot) &= \inf_i (\langle A(\cdot) x_i, x_i \rangle) \\
&= \inf_i (\langle A^{-1}(\cdot) x_i, x_i \rangle)^{-1} = [\sup_i (\langle A^{-1}(\cdot) x_i, x_i \rangle)]^{-1}
\end{aligned}$$

where $\{x_i\}_{i=1}^{\infty}$ is a countable dense set in the unit ball of $\mathbb{C}^q$.

It is obvious that $\langle A^{-1}(\cdot)x_i, x_i\rangle$ is measurable, hence $\langle A^{-1}(\cdot)x_i, x_i\rangle^{-1}$ is measurable and, therefore, $\lambda(\cdot)$ is measurable.

It is clear that $A(\cdot) \geq \lambda(\cdot)I$, and $\lambda^{-1}(\cdot) = \sup_i \langle A^{-1}(\cdot)x_i, x_i\rangle$ is an L^∞ function. Indeed,

$$\begin{aligned}\langle A^{-1}(\cdot)x_i, x_i\rangle = \left|\sum_{k,l} A^{-1}_{k,l} x^i_l \overline{x^i_k}\right| &\leq \sum_{k,l} |A^{-1}_{k,l}| \; \|x^i\|^2 \\ &= \sum_{k,l} |A^{-1}_{k,l}| < D,\end{aligned}$$

for a positive constant D, since $\|x^i\|^2 = 1$. By hypothesis, each entry of $A^{-1}(\cdot)$ is an L^∞ function. We can thus choose $C = \|\lambda\|_\infty^{-1}$, which completes the proof. □

Lemma 2.3.15 *Let η be a measurable invertible matrix-valued function. If each entry of η^{-1} is an L^∞ function, then each entry of $(\eta^*\eta)^{-1}$ is an L^∞ function as well.*

Proof: Since, $(\eta^*\eta)^{-1} = \eta^{-1}(\eta^*)^{-1} = \eta^{-1}(\eta^{-1})^*$,

$$(\eta^{-1}(\eta^{-1})^*)_{i,j} = \sum_k \eta^{-1}_{i,k}(\eta^{-1}_{k,j})^* = \sum_k \eta^{-1}_{i,k}\overline{\eta^{-1}_{j,k}},$$

we conclude that the entries of $(\eta^*\eta)^{-1}$ are in L^∞ if those of η^{-1} are. □

Lemma 2.3.16 *Let ξ and η be two measurable matrix-valued functions defined as in the equation 2.3.49. Then $C\xi - \eta^*\eta$ is positive-semidefinite if the positive constant C is large enough.*

Proof: Let C be a positive constant large enough. We have to show that $0 \leq \langle (C\xi - \eta^*\eta)x, x\rangle_{\mathbb{C}^q}$ for every $x \in \mathbb{C}^q$. Note that

$$\begin{aligned}\langle (C\xi - \eta^*\eta)x, x\rangle_{\mathbb{C}^q} &= C\langle \xi x, x\rangle_{\mathbb{C}^q} - \langle \eta^*\eta x, x\rangle_{\mathbb{C}^q} \\ &= C\langle \xi x, x\rangle_{\mathbb{C}^q} - \langle \eta x, \eta x\rangle_{\mathbb{C}^q}, \qquad (2.3.51)\end{aligned}$$

and,

$$\begin{aligned}
\|\eta x\|^2_{\mathbb{C}^q} &= \sum_j \left|\sum_i \langle K^j, G^i\rangle_{\mathbb{C}^p} x_i\right|^2 \\
&= \sum_j \left|\left\langle K^j, \sum_i G^i \overline{x_i}\right\rangle_{\mathbb{C}^p}\right|^2 \\
&\leq \sum_j \|K^j\|^2_{\mathbb{C}^p} \left\|\sum_i G^i\overline{x_i}\right\|^2_{\mathbb{C}^p},
\end{aligned}$$

by the Cauchy-Schwarz inequality. Furthermore, since $(\xi x)_j = \sum_i \langle G^j, G^i\rangle_{\mathbb{C}^p} x_i$, we have

$$\begin{aligned}
\langle \xi x, x\rangle &= \sum_j \left(\sum_i \langle G^j, G^i\rangle_{\mathbb{C}^p} x_i\right)\overline{x_j} \\
&= \langle \sum_j G^j \overline{x_j}, \sum_i G^i\overline{x_i}\rangle_{\mathbb{C}^p} \\
&= \left\|\sum_i G^i \overline{x_i}\right\|^2_{\mathbb{C}^p}
\end{aligned}$$

Therefore, it is clear that $C\xi - \eta^*\eta \geq 0$ if

$$\left\|\sum_i G^i\,\overline{x_i}\right\|^2_{\mathbb{C}^q} \sum_j \|K^j\|^2_{\mathbb{C}^q} \leq C \left\|\sum_i G^i\,\overline{x_i}\right\|^2_{\mathbb{C}^q}.$$

It suffices to choose the constant C such that

$$\sum_j \|K^j\|^2_{\mathbb{C}^q} \leq C,$$

which is possible since $\sum_j \|K^j\|^2_{\mathbb{C}^q}$ satisfies the Bessel condition. Thus, $C\xi - \eta^*\eta$ is a positive-semidefinite matrix, or, equivalently, $\eta^*\eta \leq C\xi$ on $\mathbb{C}^q$ if C is large enough. □

Lemma 2.3.17 *Let η and ξ be a measurable matrix-valued functions defined as in the equation 2.3.49. Then, the restriction of η to $\ker \xi(\cdot)$ is $\{0\}$, i.e. $\eta|_{\ker \xi(\cdot)} = \{0\}$.*

Proof: By Lemma 2.3.16, we know that $\eta^*\eta \leq C\xi$ for a large enough positive constant C. Therefore, for $x \in \ker\xi(\cdot)$, we have that

$$0 \leq \langle \eta x, \eta x\rangle_{\mathbb{C}^q} \;=\; \langle \eta^*\eta x, x\rangle_{\mathbb{C}^q} \leq C\langle \xi x, x\rangle_{\mathbb{C}^q} = 0.$$

This implies that

$$\langle \eta x, \eta x\rangle_{\mathbb{C}^q} = \|\eta x\|^2_{\mathbb{C}^q} = 0,$$

hence $\eta x = 0$ for every $x \in \ker\xi(\cdot)$. This proves our claim. □

Lemma 2.3.18 *Let ξ be a measurable matrix-valued function defined as in the equation 2.3.49. Then, the restriction of ξ to $(\ker\xi)^\perp$ is $1-1$, invertible and $\xi((\ker\xi)^\perp) \subseteq (\ker\xi)^\perp$.*

Proof: First let us show that ξ is $1-1$ on $(\ker\xi)^\perp$. Take $y \in (\ker\xi)^\perp$, and assume that $\xi y = 0$. This implies that $y \in \ker\xi$, hence y has to be 0, since $y \in (\ker\xi)^\perp$. This means that ξ is $1-1$ on $(\ker\xi)^\perp$.

Since we proved that ξ is $1-1$ on $(\ker\xi)^\perp$, we only need to show that $\xi((\ker\xi)^\perp) \subseteq (\ker\xi)^\perp$ in order to show that ξ is invertible on $(\ker\xi)^\perp$.

Let $y \in (\ker\xi)^\perp$ and $x \in \ker\xi$.

$$\langle \xi y, x\rangle = \langle y, \xi x\rangle = 0$$

since $\xi x = 0$. This implies that $\xi((\ker\xi)^\perp) \subseteq (\ker\xi)^\perp$.
This completes the proof. □

Lemma 2.3.19 *Let ξ, L and η be measurable matrix-valued functions defined as in the equation 2.3.49, assume also that the equality $\xi = \frac{1}{p}(\xi L\eta)$ holds almost everywhere on $[0,1]\times[0,\frac{1}{p}]$. Let $Q : \mathbb{C}^q \longrightarrow (\ker\xi(\cdot))^\perp$ be the projection map. Then, $I = \frac{1}{p}(QL\eta)$ on the subspace $(\ker\xi(\cdot))^\perp$, where I is the identity matrix.*

Proof: Let $Q : \mathbb{C}^q \longrightarrow (\ker\xi(\cdot))^\perp$ be the projection map. For any $x, y \in (\ker\xi)^\perp$, we have that

$$\langle x, \xi y\rangle_{\mathbb{C}^q} = \langle \xi x, y\rangle_{\mathbb{C}^q} = \frac{1}{p}\langle \xi L\eta x, y\rangle_{\mathbb{C}^q} = \frac{1}{p}\langle L\eta x, \xi y\rangle = \frac{1}{p}\langle QL\eta x, \xi y\rangle_{\mathbb{C}^q}.$$

Since $\xi y \in (\ker \xi(\cdot))^{\perp}$ by Lemma 2.3.18, using the previous equality, we obtain that

$$\langle x, \xi y\rangle_{\mathbb{C}^q} - \frac{1}{p}\langle QL\eta x, \xi y\rangle_{\mathbb{C}^q} = 0.$$

This implies that

$$\langle x - \frac{1}{p}QL\eta x, \xi y\rangle_{\mathbb{C}^q} = 0,$$

or,

$$\langle \xi(x - \frac{1}{p}QL\eta x), y\rangle_{\mathbb{C}^q} = 0.$$

Hence,

$$\xi(x - \frac{1}{p}QL\eta x) = 0,$$

since $\xi(\mathbb{C}^q) \subseteq (\ker \xi(\cdot))^{\perp}$ and y is an arbitrary element of the subspace $(\ker \xi(\cdot))^{\perp}$.

Therefore, $x - \frac{1}{p}QL\eta x = 0$, since $x - \frac{1}{p}QL\eta x \in (\ker \xi(\cdot))^{\perp}$.

This means that

$$I = \frac{1}{p}QL\eta \quad \text{on} \quad (\ker \xi(\cdot))^{\perp}.$$

(In order to see that the projection map $Q : \mathbb{C}^q \longrightarrow (\ker \xi(\cdot))^{\perp}$ is measurable, see [Da1], pp. 978). □

Now we give the key theorem of the thesis in which we provide a necessary and sufficient condition for the existence of a dual frame.

Theorem 2.3.20 *Let $a, b > 0$ with $ab = p/q$ and $gcd(p, q) = 1$. Let A and B be two positive constants and suppose that ξ and η are measurable matrix-valued functions defined in the equation 2.3.49. Suppose also that the frame condition*

$$Ap\xi \leq \xi^2 \leq Bp\xi \tag{2.3.52}$$

holds almost everywhere. Then, there exists a measurable $q \times q$ matrix-valued function L whose each entry is an L^∞ function such that the equality

$$\xi = \frac{1}{p}(\xi L \eta) \tag{2.3.53}$$

holds almost everywhere if and only if there exists a positive constant C such that the inequality

$$C\xi \leq \eta^* \eta \tag{2.3.54}$$

holds almost everywhere.

Proof: First let us prove the sufficiency part of the equivalence. Suppose that the frame condition $Ap\xi \leq \xi^2 \leq Bp\xi$ holds almost everywhere. Assume also that there exists a positive constant C such that the inequality 2.3.54 holds almost everywhere.

Using the restriction of the matrix-valued function η to the subspace $(\ker \xi)^\perp$, we will define the matrix valued function L as in the equality 2.3.57 below.

Let $\eta|_{(\ker \xi)^\perp} = \eta_1$, where η_1 is considered is a linear map from $(\ker \xi)^\perp$ to $\mathbb{C}^q$ and let us show that $\eta_1^* \eta_1$ is $1-1$ and invertible matrix valued function on the subspace $(\ker \xi)^\perp$. In order to prove that $\eta_1^* \eta_1$ is $1-1$ on $(\ker \xi)^\perp$, suppose that $(\eta_1^* \eta_1)(y) = 0$ for some $y \in (\ker \xi)^\perp$. We must then prove that $y = 0$.

Using inequality (2.3.54), we have that

$$\begin{aligned} 0 &\leq \langle (\eta_1^* \eta_1 - C\xi) y, y \rangle_{\mathbb{C}^q} \\ &= \langle (\eta_1^* \eta_1) y, y \rangle_{\mathbb{C}^q} - C \langle \xi y, y \rangle_{\mathbb{C}^q} = -C \langle \xi y, y \rangle_{\mathbb{C}^q} \end{aligned}$$

This implies that

$$\langle \xi y, y \rangle_{\mathbb{C}^q} \leq 0 \tag{2.3.55}$$

since C is a positive constant. By the Bessel condition $\xi^2 \leq Bp\xi$, we also have that

$$0 \leq \langle \xi y, \xi y \rangle_{\mathbb{C}^q} \leq Bp \langle \xi y, y \rangle_{\mathbb{C}^q}, \tag{2.3.56}$$

which implies that

$$\langle \xi y, \xi y\rangle_{\mathbb{C}^q} = \|\xi y\|^2_{\mathbb{C}^q} = 0$$

Thus, $\xi y = 0$. Thus, $\eta_1^*\eta_1$ is $1-1$ on $(\ker \xi)^\perp$ as claimed. Since $\eta_1^*\eta_1$ maps $(\ker \xi)^\perp$ to itself, it must be invertible.

Since $\eta_1^*\eta_1$ is $1-1$ and invertible matrix-valued function on the subspace $(\ker \xi)^\perp$, it follows that

$$I = (\eta_1^*\eta_1)^{-1}(\eta_1^*\eta_1)$$

on the subspace $(\ker \xi)^\perp$, where I is the identity matrix on the subspace $(\ker \xi)^\perp$. Let

$$L = p(\eta_1^*\eta_1)^{-1}\eta_1^* \tag{2.3.57}$$

on $\mathbb{C}^q$ where Q is the projection map from $\mathbb{C}^q$ onto the subspace $(\ker \xi)^\perp$ defined as in the lemma 2.3.19. Note that we can write every $x \in \mathbb{C}^q$ as

$$Qx = \eta_1 u \tag{2.3.58}$$

for some $u \in (\ker \xi)^\perp$, and $\eta_1 = \eta|_{(\ker \xi)^\perp}$. Note also that the lower frame condition $Ap\xi \le \xi^2$ reduces to $ApI \le \xi$ on $(\ker \xi)^\perp$. Therefore,

$$\begin{aligned} ApC\|u\|^2_{\mathbb{C}^p} &\le C\langle \xi u, u\rangle_{\mathbb{C}^q} \\ &\le \langle \eta^*\eta u, u\rangle_{\mathbb{C}^q} \quad \text{(by the inequality (2.3.54))} \\ &= \langle \eta u, \eta u\rangle_{\mathbb{C}^q} = \|\eta u\|^2_{\mathbb{C}^q} = \|Qx\|^2_{\mathbb{C}^q} \quad \text{by the equality (2.3.58).} \end{aligned} \tag{2.3.59}$$

Hence, rearranging the inequality (2.3.59), we have that

$$\|u\|_{\mathbb{C}^q} \le \frac{1}{\sqrt{ApC}}\|Qx\|_{\mathbb{C}^q} \le \frac{1}{\sqrt{ApC}}\|x\|_{\mathbb{C}^q} \tag{2.3.60}$$

where C is a positive constant . Now we want to prove that all of the entries of L defined in (2.3.57) are L^∞ functions. In order to do this, we must show that

$\langle Lx, y\rangle$ is in L^∞ for every $x, y \in \mathbb{C}^q$. We can write every $y \in \mathbb{C}^q$ as $y = y_1 + y_2$, where $y_1 \in (\ker\xi)^\perp$ and $y_2 \in \ker\xi$, and Qx as in the equation 2.3.58.

$$\begin{aligned}
|\langle Lx, y\rangle| &= |\langle p(\eta_1^*\eta_1^*)^{-1}\eta_1^* x, y\rangle_{\mathbb{C}^q}| \\
&= p|\langle(\eta_1^*\eta_1)^{-1}\eta_1^*\eta_1 u, (y_1 + y_2)\rangle_{\mathbb{C}^q}| \\
&= p|\langle(\eta_1^*\eta_1)^{-1}\eta_1^*\eta_1 u, y_1\rangle_{\mathbb{C}^q}| \\
&= p|\langle u, y_1\rangle_{\mathbb{C}^q}| \\
&\le p\|u\|_{\mathbb{C}^q}\|y_1\|_{\mathbb{C}^q} \\
&\le \sqrt{\frac{p}{AC}}\|x\|_{\mathbb{C}^q}\|y_1\|_{\mathbb{C}^q} \quad \text{by the inequality (2.3.60).}
\end{aligned}$$

Thus, $\langle Lx, y\rangle_{\mathbb{C}^q}$ is in L^∞ for every $x, y \in \mathbb{C}^q$. Furthermore,

$$\begin{aligned}
\frac{1}{p}(\xi L\eta) &= \frac{1}{p}(\xi p(\eta_1^*\eta_1)^{-1}\eta_1^*\eta) \\
&= \frac{1}{p}(\xi p(\eta_1^*\eta_1)^{-1}\eta_1^*\eta_1) \quad \text{on} \quad (\ker\xi)^\perp \\
&= \xi \quad \text{on} \quad (\ker\xi)^\perp.
\end{aligned}$$

On the other hand, for any $x \in \ker\xi$,

$$\xi x = 0 = \eta x = \frac{1}{p}(\xi L\eta)x.$$

It follows that $\xi = \frac{1}{p}(\xi L\eta)$ on $\ker\xi$ as well.

Since we proved that $\xi = \frac{1}{p}(\xi L\eta)$ on both of the subspaces $(\ker\xi)^\perp$ and $\ker\xi$, $\xi = \frac{1}{p}(\xi L\eta)$ as claimed.

Suppose that the frame condition $Ap\xi \le \xi^2 \le Bp\xi$ holds almost everywhere on $[0,1]\times[0,\frac{1}{p}]$. Assume also that there exists a matrix-valued function L whose each entry is an L^∞ function such that the equality $\xi = \frac{1}{p}(\xi L\eta)$ holds almost everywhere on $[0,1]\times[0,\frac{1}{p}]$.

We want to show that there exists a positive constant C such that the inequality $C\xi \le \eta^*\eta$ holds almost everywhere on $[0,1]\times[0,\frac{1}{p}]$. Let $\widehat{L} := \frac{L}{p}$. Since each entry of L is an L^∞ function, $\|\widehat{L}\| \le C_1$ for a positive constant C_1, where $\|\widehat{L}\|$ is the operator norm of $\widehat{L}$. Now we can write the following identities:

$$\begin{aligned}
\langle \xi x, \xi x\rangle &= \langle \xi \widehat{L}\eta x, \xi \widehat{L}\eta x\rangle_{\mathbb{C}^q} \\
&= \langle \eta x, \widehat{L}^* \xi^2 \widehat{L}\eta x\rangle_{\mathbb{C}^q} \\
&\leq \|\eta x\|_{\mathbb{C}^q} \|\widehat{L}^* \xi^2 \widehat{L}\eta x\|_{\mathbb{C}^q} \\
&\leq \|\eta x\|_{\mathbb{C}^q} (C_1 Bp)^2 \|\eta x\|_{\mathbb{C}^q} \quad (\text{Since } \|\widehat{L}\| \leq C_1 \quad \text{and by frame condition}) \\
&= \|\eta x\|^2_{\mathbb{C}^q} D \quad (\text{for letting} \quad D := (C_1 Bp)^2 > 0)
\end{aligned}$$

Hence, rearranging the previous inequality, we obtain that:

$$\langle \eta^* \eta x, x\rangle_{\mathbb{C}^q} = \|\eta x\|^2_{\mathbb{C}^q} \geq \frac{1}{D}\langle \xi x, \xi x\rangle_{\mathbb{C}^q} \geq \frac{1}{D} Ap \langle \xi x, x\rangle_{\mathbb{C}^q}$$

by the lower frame condition. Therefore, letting $C := \frac{1}{D}Ap$, we obtain the desired equality:

$$\langle \eta^* \eta x, x\rangle_{\mathbb{C}^q} \geq C\langle \xi x, x\rangle_{\mathbb{C}^q},$$

for every $x \in \mathbb{C}^q$. In other words, we obtain that $\eta^*\eta \geq C\xi$ for a positive constant C. □

We now state the following corollary which provides a bridge between duality problem and the main theorem, Theorem 2.3.20, of the thesis.

Corollary 2.3.21 *Let $a, b > 0$ and $ab = p/q$ with $gcd(p, q) = 1$. For $g, h \in L^2(\mathbf{R})$, let $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ and $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ be Gabor subspace frames. Then, there exists a window function $h \in \overline{span}\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ which generates a Bessel collection $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ such that $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ if and only if the inequality $\eta^*\eta \geq C\xi$ holds almost everywhere on $[0,1]\times[0,1/p]$ for a positive constant C, where ξ and η are the measurable matrix-valued functions defined in the equation 2.3.49.*

Remark 2.3.22 Note that we have a unique dual frame if and only if any vector-valued function $H^i \in span\{K^0, \ldots, K^{q-1}\}$ and in L^∞ satisfying the equality

$$0 = \sum_j \langle G^i, H^j \rangle G^j \tag{2.3.61}$$

are 0 for each $i \in \{0, \ldots, q-1\}$.

Now we are ready to obtain the conditions for the uniqueness theorem.

Theorem 2.3.23 *The dual frame of the original Gabor subspace frame satisfying the conditions of Theorem 2.3.20 is unique if and only if the following conditions hold:*

(i) $\Omega_1 = \bigcup_i \{(x, w) : G^i(x, w) \neq 0\} = \Omega_2 = \bigcup_i \{(x, w) : K^i(x, w) \neq 0\}$

(ii) $rank\{G^i\}_{i=0}^{q-1} = q$ *(i.e,* $G^0, \ldots, G^{q-1}$ *are linearly independent) on* Ω_1*.*

Proof: First let us prove the necessity part of the implication. We want to show that the first condition holds if the dual frame exists uniquely. By the condition of the existence of the dual frame, we know that $\Omega_1 \subseteq \Omega_2$. If $\Omega_2 \setminus \Omega_1$ has positive measure, then on the set $\Omega_2 \setminus \Omega_1$, we can define arbitrarily many H satisfying the equality 2.3.61 as in the following way:

$$H^i = \begin{cases} 0, & \text{on } \Omega_1; \\ K^\ell \chi_S, & \text{for each} \quad i \in \{0, \ldots, q-1\}; \end{cases}$$

where $S \subseteq \Omega_2 \setminus \Omega_1$ has a positive Lebesgue measure.

Therefore, we conclude that $\Omega_2 = \Omega_1$ if the dual frame is unique.

In order to prove that the second condition holds, we argue by contradiction. In order words, let us prove that if second condition does not hold, then the dual frame can not be unique. Now assume that $G^0, \ldots, G^{q-1}$ are linearly dependent on a subset $\Omega \subseteq \Omega_1$ with positive Lebesgue measure.

Let $e_0, \ldots, e_{q-1}$ be the standard orthogonal basis of $\mathbb{C}^q$. Then, there must be at least one e_i such that $P_{ker\xi}(e_i) \neq 0$ for a subset $\Omega' \subset \Omega$ with positive Lebesgue measure. Let c

$$c := P_{ker\xi}(e_i)\chi_{\Omega'}. \tag{2.3.62}$$

There exists at least one $K^j \neq 0$ for some $j \in \{0, \ldots, q-1\}$ on a subset $\Omega" \subset \Omega'$ with positive Lebesgue measure. Let $R := K^j$ and define $H^j := \overline{c_j}R$ for $c = (c_0, \ldots, c_{q-1}) \in C^q$, defined as in the equation 2.3.62. It is obvious that H^j is bounded. Note that $(\xi c)_j = 0$ since $c \in \ker \xi$ by the equation 2.3.62. Therefore,

$$(\xi c)_i = \sum_j \langle G^i, G^j \rangle_{\mathbb{C}^p} c_j = \langle G^i, \sum_j G^j \overline{c_j} \rangle_{\mathbb{C}^p} = 0. \tag{2.3.63}$$

for every i on Ω_1. Hence, the equality 2.3.63 implies that $\sum_j G^j \overline{c_j} = 0$ on the subspace Ω_1.

$$\sum_j \langle G^i, H^j \rangle_{\mathbb{C}^p} G^j = \sum_j \langle G^i, \overline{c_j} R \rangle G^j = \langle G^i, R \rangle_{\mathbb{C}^p} \sum_j c_j G^j \quad = \quad 0.$$

This contradicts with the uniqueness of the dual frame. Therefore, we conclude that the rank of the set $\{G^0, \ldots, G^{q-1}\}$ must be q on Ω_1 if the dual frame is unique.

Now let us prove the sufficiency part of the implication. Suppose that both conditions (i) and (ii) hold. We must then prove that the dual frame exists uniquely, that is, we must show that H^i is 0 for $i \in \{0, \ldots, q-1\}$ if the equation 2.3.61 holds for $i \in \{0, \ldots, q-1\}$.

Clearly, by (i), $H^i = 0$ on Ω_1^c for each $i \in \{0, \ldots, q-1\}$. We know that $rank\{G^i\}_{i=0}^{q-1} = q$ on Ω_1. The fact that

$$\sum_j \langle G^i, H^j \rangle_{\mathbb{C}^p} G^j = 0$$

implies that $\langle G^i, H^j \rangle_{\mathbb{C}^q} = 0$ for each $i, j \in \{0, \ldots, q-1\}$. Writing

$$H^j = \sum_k \overline{L_{j,k}} K^k \quad \text{for} \quad j \in \{0, \ldots, q-1\},$$

we have $\langle G^i, \sum_k \overline{L_{j,k}} K^k \rangle = 0$, or $\sum_k L_{j,k} \langle G^i, K^k \rangle_{\mathbb{C}^p} = 0$. Since $\eta_{k,i} = \langle K^k, G^i \rangle_{\mathbb{C}^p}$, we get that $\sum_k L_{j,k} \overline{\eta_{k,i}} = 0$, or, $L\overline{\eta} = 0$. We know that $\eta^* \eta \geq C\xi$. Therefore, rank of the matrix valued function η is q. That is, η is invertible. Then $L\overline{\eta} = 0$ implies that $L = 0$. By the definition of H^j, we see that $H^j = 0$ on the Ω_1 for every $j \in \{0, \ldots, q-1\}$. Hence, we conclude that dual frame exists uniquely if both conditions hold in the hypothesis of the theorem. $\square$

Chapter 3

Summary and Open Problems

In this thesis, we solved an open problem regarding with the dual frames of the Gabor subspace frames.
We characterized a condition for the existence and uniqueness of the window function h in the Gabor subspace frame $\{E_{mb}T_{na}k\}_{m,n\in\mathbf{Z}}$ such that $\{E_{mb}T_{na}h\}_{m,n\in\mathbf{Z}}$ is a dual frame of the original Gabor subspace frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$. Consequently, we stated our results for the classic situations, for instance $g = k$, and for the normalized Gabor subspace frames.

This thesis can be used as a friendly-user reference for the introduction of the Bessel collections, Gabor frames, the Zak transformation, and relations amongst them, the dual frames of Gabor subspace frames with the two dimensional modulation and translation operators. A long standing question about the Gabor frames is to find all $a, b \in \mathbf{R}$ and $g \in L^2(\mathbf{R})$ such that $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ forms a frame for $L^2(\mathbf{R})$. Although some special cases of this problem have been solved, there are many variations of it which are still open, and another very interesting problem related to the Gabor frames is to identify all those $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}}$ which have finite upper frame bounds. Some other important open problems concerning the Gabor frames can be listed in the following way and more detailed version of them with historical explanations can be seen in the introductory paper by P.Cazassa, (See [Ca]).

Problem: ([Ca]) Given $g \in L^2(\mathbf{R})$ with $g \neq 0$, and any finite set $A \subset \mathbf{R} \times \mathbf{R}$, is the set $\{E_{mb}T_{na}g\}_{(a,b)\in A}$ linearly independent?

Because this characterization uses the dual frame generator, it is difficult to apply until one answers the following important question:

Problem: Given a Gabor subspace frame $\{E_{m/b}T_{n/a}g\}_{m,n\in\mathbf{Z}}$, give an explicit representation of the dual frame generator $S^{-1}g$.

Pete Cazassa presented an article titled "Every frame is a sum of three (but not two) orthonormal bases-and other frame representations" in 1998 (See[Ca1]). In this paper he gave a theorem stating that "Every frame is a sum of three orthonormal bases." The problem with this is that it uses strong results from operator theory, and hence in practice is often not usable, but the next question appeared related with this:

Problem: ([Ca]) For a Gabor subspace frame $\{E_{m/b}T_{n/a}g\}_{m,n\in\mathbf{Z}}$, give an explicit representation of this frame as a sum of three orthonormal bases.

Problem:([Ca]) Find all those g, a, b so that $\{E_{m/b}T_{n/a}g\}_{m,n\in\mathbf{Z}}$ is complete in $L^2(\mathbf{R})$.

Problem:([Ca]) Give an explicit representation of all functions $g \in L^2(\mathbf{R})$ and ab irrational so that $\{E_{m/b}T_{n/a}g\}_{m,n\in\mathbf{Z}}$ generates a normalized tight Gabor subspace frame.

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