Dihomotopy and Concurrent Computing

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree
Master of Science

McMaster University

MASTER OF SCIENCE (2005) McMaster University
(Mathematics) Hamilton, Ontario

TITLE:	Dihomotopy and Concurrent Computing
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SUPERVISOR:	Dr. Andrew Nicas
NUMBER OF PAGES:	vii, 42

TO MY PARENTS

Acknowledgements

This Thesis would not have been possible without the support and encouragement of many people. First, I would like to thank my advisor Dr. Andrew Nicas for generously supporting me. His willingness to meet with me, patience and flexibility helped me greatly. I thank my committee members Dr. Hans Boden and Dr. Ian Hambleton for agreeing to the task, and for taking the time to read my thesis. Dr. Boden was also very helpful to me in his role as Graduate Student Advisor, and Dr. Ian Hambleton literally changed the way I learned mathematics.

I am very grateful to the Department of Mathematics at McMaster University for supporting me and for giving me the opportunity to pursue my graduate studies.

I am also very grateful to Dr. Aaron Abrams, for the time and opportunity he gave me at the University of Georgia. Though not directly connected with my thesis, my time spent in Georgia was instrumental in helping me decide on a topic for my thesis.

My thanks also go out to Mehmetcik Pamuk, Semra Pamuk and George Dragomir for studying topology with me, as well as Brad Wagner and Jens Corsman, who were good friends. Special thanks to Jens for waking up at 6 a.m. every Monday morning, to travel to the Fields Institute with me.

I would like to thank my parents for their boundless love and support.

Finally, I would like to thank my girlfriend Cecilia, for her kindness.

Abstract

Concurrent Computing has certain interesting links with Algebraic Topology. There are various geometric models for concurrent computing. We examine one geometric approach to modeling concurrency, via the notion of a locally partially ordered space. We examine a notion analogous to that of homotopy, called di-homotopy, that is compatible with a local partial order. In the category of locally partially ordered spaces and di-maps we examine the isomorphisms, which are called di-homeomorphisms. We classify all di-homeomorphic embeddings of the unit square into the Euclidean plane.

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Introduction

Concurrent programming is a form of computer programming in which a complex programming task is broken up into a series of independent sequential programs that run relatively independently of each other, but need to co-operate towards a common goal. A simple example of this is a database server being accessed by two different users simultaneously. From the perspective of the database server, each user is a separate process, but their actions need to be managed in a way that maintains the integrity of the database. For example, they cannot both be allowed to write to the same record at the same time.

Concurrency has nothing to do with the number of processors available, in fact, most often, the number of processes exceeds the number of processors available. For example, the database server may be running on a single computer, but it may be accessed by any number of users simultaneously, each user running as a separate process. Concurrency Theory deals with the study of concurrent programming and the various problems that concurrent programming faces. Concurrency Theory is concerned with the interaction of multiple processes with respect to the resources that they share.

There are many mathematical models for concurrent systems, see [6], however a geometric model for concurrency called a Progress Graph first seems to have appeared in [5], where it is attributed to Dijkstra. Progress Graphs are formalized and used in [3] to prove the existence of deadlocks in certain types of concurrent systems. Progress Graphs are the geometric model that we shall be concerned with when modeling concurrency and which provide some of the motivation for dihomotopy theory.

The aim of dihomotopy theory is to set the general background for a homotopy theory of concurrency. Such a theory is suggested by Gunawardena in [7], where he gives a conceptually simple and homotopy flavored proof of an old result in Database Theory. One aspect of dihomotopy theory concerns itself with the category of locally-partially-ordered spaces and dimaps between them. We outline some of the basic constructions in dihomotopy from this perspective. We then examine the notion of a dihomeomorphism, and classify the possible dihomeomorphisms of the square.

Dijkstra introduced the concept of a "semaphore" in [4]. An "n-semaphore" is a resource that can be used by at most n processes simultaneously. An example of a 1-semaphore is a record in a database that only one process should be allowed to write to at a given time.

We shall use Dijkstra's notation to indicate the status of a semaphore, P to indicate that the semaphore is locked, and V to indicate that it has been unlocked. So for example, given a semaphore "a" we will indicate that it is locked by writing "Pa" and we will indicate that it is unlocked by writing "Va."

We are concerned with the way that processes interact with each other through the resources that they share, i.e. the semaphores. Thus from our point of view a process is characterized by its actions of locking or unlocking of the various semaphores. Dijkstra's notation allows us to write down the actions of a single process which locks and unlocks semaphores a, b and c, as a string of the form PaPbVaPcVbVc. For our purposes, we shall consider that a particular process accesses a particular semaphore at most once.

We follow [3] and define a Progress Graph to be a Cartesian graph in which the progress of n independent processes is measured against an independent time axis. The time axis of a particular process is labelled with the various actions of the process in the order in which they occur. The time between two successive actions of a process is considered to be greater than zero. Each point of a progress graph represents a state of the concurrent system, i.e. it represents the progress of each of the n concurrent processes. Within a progress graph there are regions whose states are impossible.

These regions are called Forbidden Regions. Given two states p_1 and p_2, a transition from p_1 to p_2 is represented by a vector based at p_1 directed towards p_2. An execution trace is an ordered set of states that defines a particular execution sequence of the system of concurrent processes.

Consider n concurrent processes $T_1, ..., T_n$. Each process is considered to start at time 0 and end at time 1. Let the string of P's and V's associated to process T_i be represented by a sequence of numbers in the interval $(0, 1)$ on the i'th co-ordinate axis. An execution trace is modeled by a continuous path in the unit n-cube I^n. Such a continuous path cannot pass through the point $(..., Pa, ..., Pa)$, if a is a 1-semaphore. Thus the locking of k-semaphores by k processes determines a k-dimensional forbidden region in I^n. In a digraph this is modeled by deleting an open k-dimensional cube from the interior of I^n. This has to be done for each semaphore. So the resulting forbidden region is a union of cubes of various dimensions.

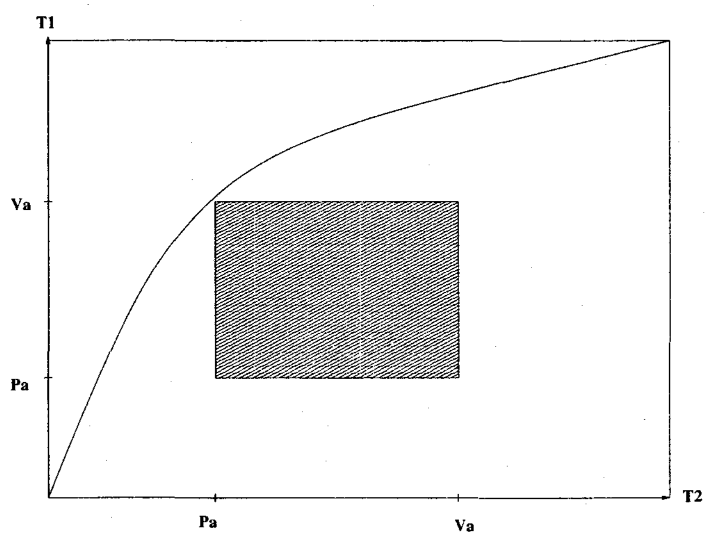

Figure 1: Progress Graph

Figure 1 shows us the progress graph of a system of two processes T_1 and T_2, which share a single 1-semaphore denoted as a. Process T_1 first locks semaphore a and then unlocks it. Thus we can represent its actions by the string PaVa. Process

T_2 performs the same actions on semaphore a, thus its actions are also represented by the string PaVa. The axis labelled T1 indicates the actions of process T_1 as well as the times at which they were performed. Similarly the axis labelled T2 indicates the actions of process T_2 as well as the times at which they were performed. The shaded block is called the forbidden region. It represents impossible states of the system, namely, states in which processes T_1 and T_2 have both locked semaphore a. Such a state is clearly impossible as a is a 1-semaphore. The curved line which avoids the forbidden region indicates an execution trace.

A question that one can ask at this point is why a continuous geometric model is being applied to something discrete, namely a collection of processes running on a computer. There do exist discrete geometric models for concurrent processes. Transition Systems are an example of such a model.

In the Transition System model the execution of a set of n processes can be modeled as a directed graph in which each vertex represents a state of the system and each directed edge represents an action of a process.

Definition 0.1. (Transition System) *A Transition System is a quadruple $(S, i, L,$ Tran) where S is a set of states with a distinguished initial state i, L is a set of labels and Tran $\subset S \times L \times S$ is a relation called the transition relation.*

In a concurrent system of processes modeled by a Transition System, the set of states is taken to represent all the possible states of the concurrent systems. Each label in the set L is taken to represent a possible action that some process can perform to take the system from one state to another. The relation Trans encodes how the various process actions take the system from one state to another, and we may think of it as a directed graph.

Consider an element $(s_i, L_k, s_j) \in$ Trans, this tells us that when the system is in state s_i the process action L_k takes the system to state s_j. We would thus represent the element (s_i, L_k, s_j) by a directed edge labelled L_k with s_i being the vertex at the tail and s_j being the vertex at the head. By thinking of every element of Trans in this way, we build up a directed graph.

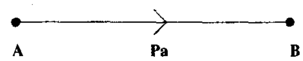

Figure 2: Single Process Transition System

To begin with, let us model a one process system, with only two possible states and a single possible transition between the two states. For example consider a one process system in which all our process does is lock a semaphore "a" by performing the action Pa. Thus the two states of the system would be, A in which the semaphore is unlocked and B in which the semaphore is locked. Thus the set of states is $S = \{A, B\}$. Thus the set of labels is $L = \{\text{Pa}\}$. The set of transitions is Trans $= \{(A,\ \text{Pa},\ B)\}$. This transition system would be represented by a directed graph with one edge labelled Pa, from vertex A to vertex B. Thus the directed edge Pa can be thought of as process performing action Pa to take the system from state A to state B.

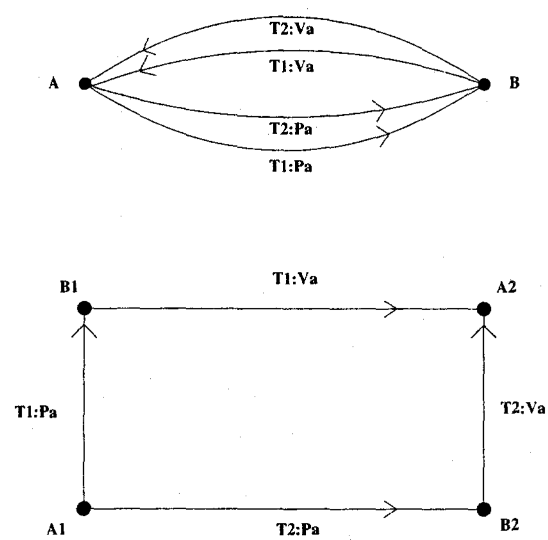

Figure 3: Two processes sharing a 1-semaphore

Consider a system with two concurrent processes T_1 and T_2. Let T_1 perform the actions PaVa and let T_2 perform the same actions, PaVa. Here "a" is a 1-semaphore;

which means that only one process can lock it at a time. Thus the system has two possible states, one in which semaphore "a" is locked, and one in which it is unlocked. We shall denote these two states by A and B respectively. Thus in our Transition System the set of states $S = \{A, B\}$. Our set of labels is $L = \{T_1$:Pa, T_1:Va, T_2:Pa, T_2:Va$\}$. Here T_1:Pa indicates the action of process T_1 performing Pa. The set of transitions is Trans $= \{(A,\ T_1\text{:Pa},\ B), (B,\ T_1\text{:Va},\ A), (A,\ T_2\text{:Pa},\ B), (B,\ T_1\text{:Va},\ A)\}$. In order to aid the reader in understanding this example better, we "open up" the directed graph of this example, by splitting vertex A into vertices A_1 and A_2. The vertex A_1 represents the unlocked state of semaphore "a" before the processes have performed any actions, and the vertex A_2 represents the unlocked state of semaphore "a" after the processes have performed their last action. We also split vertex B into vertices B_1 and B_2. The vertex B_1 represents the state in which semaphore "a" is locked by process T_1 and the vertex B_2 represents the state in which semaphore "a" is locked by process T_2. In Figure 3 we have drawn both the actual Transition System, and the "opened up" one below it.

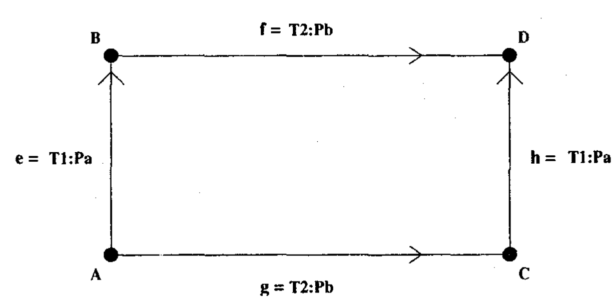

Figure 4: Two processes locking different semaphores

Consider again a system of two concurrent processes T_1 and T_2. Let T_1 perform the action Pa and T_2 perform the action Pb on semaphores "a" and "b" respectively. This system would be modeled by a directed graph with four vertices and four edges. Let vertex A be the state of the system in which semaphores a and b are free. Let vertex B be the state of the system in which semaphore "a" is locked by process

T_1 and semaphore "b" is free. Let vertex C be the state of the system in which semaphore "b" is locked by process T_2 and semaphore "a" is free. Let D be the state of the system in which semaphores "a" and "b" are locked by processes T_1 and T_2 respectively. In the directed graph representing this system, each edge represents a transition between states. However, in this example we are going to label each edge by the action that caused the transition. Thus different edges will be labelled the same, as there may be different transitions that are caused by the same action. Refer to Figure 4.

An execution trace is a directed path in the directed graph of a transition system, that agrees with the direction of each edge, and which starts at the initial state. It represents a possible sequence of actions by a process or processes. If we are dealing with a concurrent system of processes, concurrency is captured by the possibility of interleaving the actions of the various processes. Note that the underlying "direction" in the directed graph captures the causal order in which the various states occur.

If one is given more complicated processes, the basic idea of constructing the directed graph that represents the system remains the same. In a directed graph representing a particular system, there is a distinguished initial state, which is the state that the system is in before any of the processes have performed any actions. If one were to follow a directed path along a series of directed edges from the initial state to some other state of the system, then the sequence of edges that were followed would represent an interleaving of the various actions of the various processes that the edges represented. Such a directed path would represent one possible execution trace that our concurrent system could follow in order to get from the initial state to the resultant state.

Given two execution traces, we say that they "commute" with each other if it is possible for both of them to be followed simultaneously on the same system without interfering with each other. Consider figure 4. The execution trace corresponding to the path along edge "e" and then "f", and the execution trace along edge "g" and then "h," do not interfere with each other as they are locking different semaphores.

So in a sense they can be run "simultaneously." And so we say that they commute.

On the other hand if the action of process T_1 is PaVa, and the action of process T_2 is PaVa. We get a three state system, and hence a three vertex graph. Vertex A is the state of the system in which semaphore a is free. Vertex B is the state of the system in which semaphore a is locked by process T_1. Vertex C is the state of the system in which semaphore a is locked by process T_2. Edge e from A to B is the action of locking semaphore a by process T_1. Edge f from A to C is the action of locking semaphore a by process T_2. Edge g from B to A is the action of unlocking semaphore a by process T_1. Edge h from C to A is the action of unlocking semaphore a by process T_2. It is clear that the execution traces specified by the paths eg and fh cannot be performed simultaneously, as they both involve locking semaphore a. Thus these execution traces do not commute.

The notion of commutativity discussed above leads us to consider filling in the "hole" between commuting execution traces of n different processes by an n-cell, and letting the homotopy between the commuting execution traces speak for the fact that they commute. This points toward the notion of a digraph, which is an old geometric model for concurrency that was invented by Dijkstra.

Finally we note that the type of directed topology discussed in this paper may be relevant to a program for the quantization of gravity being pursued Sorkin [11]. This is noted in the conclusion of [12].

Chapter 1

Po-Spaces and Local Po-Spaces

We impose a partial order on a topological space, which gives us a preferred global direction in our space. However, as this is too rigid a constraint, and because we need our preferred direction to be more local than global, we are led on to define a local partial order on a topological space. We then start to define the basic tools of a directed homotopy theory, maps between locally partially ordered spaces that are compatible with the local partial order, paths that travel only in the "forward" direction at each point and homotopies through such paths.

Topological spaces with partial orders on them are not new to Computer Science. Most notably Dana Scott [10], applied such sets to certain aspects of the Lambda Calculus, and invented what is now called Domain Theory.

Definition 1.1. [1] **Po-Space** *A po-space is a pair* $(X, \leq)$ *formed by a topological space* X *together with a partial order* $\leq$ *such that* $\leq$ *is closed as a subset of* $X \times X$ *with the product topology.*

The only link that our partial order has with the topology of X is that it is a closed subset of $X \times X$. However, this is a strong enough constraint to force X to be a Hausdorff space, a fact that is noted in [2]. If X is a partially ordered space then the partial order relation $R \subset X \times X$ is a closed subset in the product topology. Define a homeomorphism $h : X \times X \to X \times X$ by setting $h(x, y) = (y, x)$. So $h^{-1}(R)$

is closed, and so $h^{-1}(R) \bigcap R$ is closed as it is the intersection of two closed sets. But $\Delta = h^{-1}(R) \bigcap R$. So the diagonal Δ is closed. So the space is Hausdorff.

Example 1.1. *Consider Euclidean space $\mathbb{R}^n$ with the product partial order given by, $(x_1, ..., x_n) \leq (y_1, ..., y_n) \Leftrightarrow x_1 \leq y_1, ..., x_n \leq y_n$. To see that this is a partial order one checks reflexivity, anti-commutativity and transitivity. This is easy to do as it entails proving those properties for the Euclidean total order for each variable. Our partial order relation must be a closed set of the product space $\mathbb{R}^n \times \mathbb{R}^n$. This is true because the Euclidean total order as a relation, is a closed subset R of $\mathbb{R} \times \mathbb{R}$. The relation that we define on $\mathbb{R}^n$, as a subset of $\mathbb{R}^n \times \mathbb{R}^n$ is precisely the set R^n. Thus it is a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$. This proves that Euclidean space with the product partial order is a po-space.*

Example 1.2. *Consider $I \times I$, with the partial order specified by $(a, b) \leq (c, d) \Leftrightarrow a \leq c$ and $b = d$.*

Example 1.3. *Consider a tree Γ. Identify each edge of Γ with I, and assign it the order corresponding to I. Do this arbitrarily. For example, consider the edge E with vertices a and b. We can identify it with I and assign it an order in two different ways, so that $a \leq_E b$ or so that $b \leq_E a$, either is valid. When considering a tree, we get many possible combinations based on our choices for the individual edges. When considering such a tree with ordered edges, for an edge E, we shall denote its vertices as a_E and b_E where $a_E \leq_E b_E$. Now define a relation $\leq_v$ on the vertices of Γ, so that given any two vertices (not necessarily on the same edge) $c, d \in \Gamma$ we define $c \leq d \Leftrightarrow \exists$ edges $E_1, ..., E_n$ such that c is a vertex of E_1 and d is a vertex of E_n and $c \leq_{E_1} b_{E_1} \leq_{E_2} b_{E_2} \leq_{E_3} \cdots \leq_{E_{(n-1)}} b_{E_{(n-1)}} = a_{E_n} \leq_{E_n} d$. Now define a relation $\leq$ on Γ so that for any points $x, y \in \Gamma$, with x on edge E and y on edge F, we have $x \leq y \Leftrightarrow \exists$ vertices $b_E, a_F \in \Gamma$ such that $x \leq_E b_E \leq_v a_F \leq_F y$. The relation $\leq$ is a partial order on Γ. We get that $\leq$ is reflexive as it is a total order when restricted to an edge. For transitivity consider $x, y, z \in \Gamma$, on edges E, F and H respectively such that $x \leq y$ and $y \leq z$. Thus there exist vertices c, de and f such that*

$x \leq_E c \leq_v d \leq_F y$ and $y \leq_F e \leq_v f \leq_H z$. Thus e and d are vertices of the same edge F, and we have $d \leq_F y \leq_F e$ and so $d \leq_F e$, which implies $d \leq_v e$. So we have $x \leq_E c \leq_v d \leq_v e \leq_v f \leq_H z$ which implies $x \leq z$. If antisymmetry were not true, it would imply that there were cyclic paths in Γ, this is impossible as Γ is a tree, so we must have antisymmetry. Now consider the cell complex $\Gamma \times \Gamma$. If $E_1, ..., E_n$ are the edges of Γ, then each cell in $\Gamma \times \Gamma$ is of the form $E_i \times E_j$ where i and j take values from 1 to n. On the unit interval I, the total order is a closed subset of $I \times I$, and so the partial order intersects $E_i \times E_i$ in a closed subset of $E_i \times E_i$ for $i = 1, ..., n$. If $i \neq j$ then either we have $x \leq y$ or $x \nleq y$ for all $x \in E_i$ and $y \in E_j$. This tells us that the partial order contains $E_i \times E_j$ as a subset, or is completely disjoint from $E_i \times E_j$. In either case, the partial order intersects $E_i \times E_j$ in a closed subset of $E_i \times E_j$. As the partial order intersects each cell of $\Gamma \times \Gamma$ in a closed subset of that cell, it implies that the partial order is a closed subset of $\Gamma \times \Gamma$. Thus the tree Γ is a po-space.

We would like to model an entire concurrent system by a geometric object that has a preferred time direction at each point. However, to put a partial order on our object would be too strong a constraint, as it would not allow loops in the execution traces of our system. Thus, if we had some way of designating a preferred time direction in a more local way, that would tell us the "right direction" at a point, rather than in the space as a whole, we could conceivably have a path with loops in it, as long as it travels in the "right direction" at each point that it passes through. With this in mind, we define a local partial order on a topological space X.

Definition 1.2. [1] **Local Partial Order** *Let X be a topological space. A collection $\mathcal{U}$ of po-spaces $(U, \leq_U)$ that are open subsets of X and which cover X is called a local partial order on X if for all $x \in X$ there exists a non-empty open neighborhood of x, W, such that the restriction of the partial orders $\leq_U$ to W coincide for all $U \in \mathcal{U}$.*

A local partial order on a topological space X is an additional structure on that space, and there are many ways that one can put such a structure on a space. This is analogous to what happens when one puts a smooth structure on a manifold, there are many possible smooth structures that one can put on it, however we define two smooth structures to be equivalent if their union is a smooth structure. We do the same for a local partial order.

Definition 1.3. [1] **Local Po-Space** *Two local partial orders on a space are called equivalent if their union is a local partial order on X. A local po-space consists of a topological space X with an equivalence class of local partial orders on it.*

A Local Po-Space may also be specified by a topological space with a "maximal local partial order" on it, which we define to be the union of all local partial orders that are equivalent.

Example 1.4. *As an example of a local po-space consider the unit circle S^1. Let N and S be the north and south poles respectively. Let $U = S^1 \setminus \{N\}$ and $V = S^1 \setminus \{S\}$. U and V form an open cover of S^1. Let θ_U and θ_V be local co-ordinates in U and V respectively. The co-ordinates are chosen so that they represent the angle from the X axis. We now give U and V total orders, by ordering points within them according to their co-ordinates. Thus U and V are each totally ordered charts, which implies that they are each partially ordered charts. Also note that the respective partial orders are closed subsets of $U \times U$ and $V \times V$ respectively. Denote the order on U by $\leq_U$ and the order on V by $\leq_V$. For $x, y \in U \bigcap V$, we have $x \leq_U y \Leftrightarrow x \leq_V y$. Thus we have given S^1 the structure of a local po-space.*

So far we have just defined local po-spaces. However, we would like to turn this into a category. What we need is an appropriate class of maps between local po-spaces that are compatible with the local partial order. To this end, we define the notion of a dimap.

Definition 1.4. [1] **Dimap** *Let X and Y be two local po-spaces. A continuous function $f : X \to Y$ is called a dimap if for all $x \in X$ there exists an open neighborhood $(V, \leq_V)$ of $f(x)$ in Y and a neighborhood of x, $(U, \leq_U)$ in X, such that $f(U) \subset V$ and for all $x_1, x_2 \in U, x_1 \leq_U x_2 \Rightarrow f(x_1) \leq_V f(x_2)$.*

In this paper, if the domain of a dimap is $\mathbb{R}$ or some subset of it, then unless otherwise stated, the local partial ordering is taken to be the total ordering of $\mathbb{R}$. If the domain of a dimap is $\mathbb{R}^n$ or a product of subsets of $\mathbb{R}$, then the local partial ordering is taken to be the product partial order of $\mathbb{R}^n$. Refer to Example 1.1.

One of the first steps in classical homotopy theory is the notion of a path in a topological space. In our local po-spaces we want our paths to be compatible with the local partial order, i.e. we want our paths to point in the forward time direction. Thus we define a dipath.

Definition 1.5. [1] **Dipath** *Let X be a local po-space. Let $a, b \in X$. A dipath on X from a to b is a dimap $f : I \longrightarrow X$ such that $f(0) = a$ and $f(1) = b$.*

Our dipaths are intended to symbolize execution traces in a concurrent system, just like the execution traces in a progress graph. We would like to distinguish execution traces based on their history, and whether or not they "commute." Thus we define a dihomotopy between two dipaths, to be a homotopy between two dipaths, through a continuous family of dipaths.

Definition 1.6. [2] **Dihomotopy** *Let f and g be two dipaths on X from a to b. A dihomotopy between f and g is a continuous map, $H : I \times I \longrightarrow X$, such that $H_t : I \longrightarrow X$ is a di-path from a to b $\forall t \in I$, and $H_0 = f$, $H_1 = g$.*

The above definition, which is the one that we shall use, is equivalent to the following definition which appeared in [1].

Definition 1.7. [1] **Dihomotopy** *Let f and g be two dipaths on X from a to b. A dihomotopy between f and g is a dimap $H : I \times I \longrightarrow X$ such that for all $x \in I, H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ and for all $t \in I, H(0, t) = a$ and $H(1, t) = b$. Here, we consider $I \times I$ with the partial order of Example 1.2*

In the category of local po-spaces, isomorphisms turn out to be a very specific type of homeomorphism. If $h : X \longrightarrow Y$ is to be an isomorphism of local po-spaces, then on every partially ordered neighborhood $(U, \leq_U)$ in X, for $x_1, x_2 \in U$, we must have $x_1 \leq_U x_2 \Leftrightarrow h(x_1) \leq_{h(U)} h(x_2)$, where $h(x_1), h(x_2) \in h(U)$ the homeomorphic image of U in Y, and $\leq_{h(U)}$ is a partial order on $h(U)$. Thus h and its inverse h^{-1} need to be dimaps.

Definition 1.8. [2] **Di-homeomorphism** *If X and Y are local po-spaces, then $h : X \to Y$ is called a di-homeomorphism if h is a homeomorphism and $h : X \to Y$ and $h^{-1} : Y \to X$ are dimaps.*

Example 1.5. *Consider I with the Euclidean order. Consider $a \in Int(I)$.*

Define $h : I \to I$ to be,

$$h(x) = \begin{cases} \frac{1}{2a}x & 0 \leq x \leq a \\ \\ \frac{1}{2} + (x-a)\frac{1}{2(1-a)} & a \leq x \leq 1 \end{cases}$$

Define $h^{-1} : I \to I$ to be,

$$h^{-1}(x) = \begin{cases} 2ax & 0 \leq x \leq \frac{1}{2} \\ \\ a + (x - \frac{1}{2})2(1-a) & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Consider $x_1, x_2 \in I$ such that $x_1 \leq x_2$.

If $x_1, x_2 \leq a$ then $\frac{1}{2a}x_1 \leq \frac{1}{2a}x_2$ as a is positive. And so $h(x_1) \leq h(x_2)$.

If $a \leq x_1, x_2$ then $(x_1 - a) \leq (x_2 - a)$. And so $\frac{1}{2} + (x_1 - a) \leq \frac{1}{2} + (x_2 - a)$. And so $\frac{1}{2} + (x_1 - a)\frac{1}{2(1-a)} \leq \frac{1}{2} + (x_2 - a)\frac{1}{2(1-a)}$. So we have $h(x_1) \leq h(x_2)$.

If $x_1 \leq a \leq x_2$ then $h(x_1) \leq h(a)$ and $h(a) \leq h(x_2)$ therefore $h(x_1) \leq h(x_2)$.

Thus h is a dimap.

If $x_1, x_2 \leq 1/2$ then $2ax_1 \leq 2ax_2$. So $h^{-1}(x_1) \leq h^{-1}(x_2)$.

If $1/2 \leq x_1, x_2$ then $x_1 - 1/2 \leq x_2 - 1/2$. So $(x_1 - 1/2)2(1-a) \leq (x_2 - 1/2)2(1-a)$. And so $a + (x_1 - 1/2)2(1-a) \leq a + (x_2 - 1/2)2(1-a)$. Thus $h^{-1}(x_1) \leq h^{-1}(x_2)$.

If $x_1 \leq 1/2 \leq x_2$ then $h^{-1}(x_1) \leq h^{-1}(1/2)$ and $h^{-1}(1/2) \leq h^{-1}(x_2)$. So we have $h^{-1}(x_1) \leq h^{-1}(x_2)$.

Thus h^{-1} is also a dimap, and we conclude that $h : I \to I$ is a dihomeomorphism.

Note also that $h(a) = 1/2$. Thus for any point $a \in Int(I)$, there is a dihomeomorphism $h : I \longrightarrow I$ that takes a to $1/2$. From now on, we shall denote the dihomeomorphism defined above, that takes a to $1/2$ on the interval I by $h_a : I \longrightarrow I$.

Example 1.6. *Consider the unit cube I^n with the Euclidean product partial order. Consider $(a_1, ..., a_n) \in Int(I^n)$. Define $g : I^n \longrightarrow I^n$ to be $g(x_1, ..., x_n) = (h_{a_1}, ..., h_{a_n})$. Where the h_{a_i} are as defined in Example 1.5. Similarly define $g^{-1} : I^n \longrightarrow I^n$ to be $g^{-1}(x_1, ..., x_n) = (h_{a_1}^{-1}, ..., h_{a_n}^{-1})$. Clearly g is a homeomorphism and g^{-1} is its inverse, they are also dimaps because they are dimaps componentwise.*

Thus $g : I^n \longrightarrow I^n$ is a dihomeomorphism, and $g(a_1, ..., a_n) = (1/2, ..., 1/2)$.

Thus for every $(a_1, ..., a_n) \in Int(I^n)$, there is a dihomeomorphism $g : I^n \longrightarrow I^n$ that takes $(a_1, ..., a_n)$ to $(1/2, ..., 1/2)$.

Example 1.7. *Consider the unit cube I^n with the Euclidean product partial order. Let $C = [a_1, b_1] \times ... \times [a_n, b_n]$. Define $h_i : [a_i, b_i] \longrightarrow I$ by $h_i(x) = \frac{1}{(b_i - a_i)}(x - a_i)$. Now define $h : C \longrightarrow I$ by $h(x_1, ..., x_n) = (h_1(x_1), ..., h_n(x_n))$. The inverse of h_i, $h_i^{-1} : I \longrightarrow [a_i, b_i]$, is given by $h_i^{-1}(x) = (b_i - a_i)(x) + a_i$. And so $h^{-1} : I \longrightarrow C$ is given by $h^{-1}(x_1, ..., x_n) = (h_1^{-1}(x_1), ..., h_n^{-1}(x_n))$.*

Both h and h^{-1} are dimaps and so $h : C \longrightarrow I^n$ is a dihomeomorphism.

In the definition of a dihomeomorphism, $f : X \longrightarrow Y$ between local po-spaces, the requirement that the inverse map $g : Y \longrightarrow X$ be a dimap is necessary. In other words, if $f : X \longrightarrow Y$ is a homeomorphism and a dimap, it need not be a dihomeomorphism.

Example 1.8. *Consider the unit square I^2 with the Euclidean product partial order. Consider the region R in the Euclidean plane, enclosed by the lines $y = x, y = x - 1, y = 0, y = 1$, together with its boundary. Let R inherit the product partial order of the Euclidean plane. Thus R is a local po-space(in fact a po-space).*

Define $f : I^2 \longrightarrow R$ by $f(x, y) = (x + y, y)$, and define $g : R \longrightarrow X$ by $g(x, y) = (x - y, y)$.

Clearly f and g are continuous. Also, $g \circ f(x, y) = g(x + y, y) = (x + y - y, y) = (x, y)$. Thus, g is the inverse of f. Thus f is a homeomorphism, and R is homeomorphic to I^2.

Consider $(x_1, y_1) \leq (x_2, y_2)$ in I^2. Thus we have $x_1 \leq x_2$ and $y_1 \leq y_2$. This implies that $x_1+y_1 \leq x_2+y_2$. And so $(x_1+y_1, y_1) \leq (x_2+y_2, y_2)$. Thus $f(x_1, y_1) \leq f(x_2, y_2)$. Thus $f : I^2 \longrightarrow R$ is a dimap between po-spaces.

However, consider $(\frac{3}{4}, \frac{1}{4}) \leq (1, 1)$ in R. $g(\frac{3}{4}, \frac{1}{4}) = (\frac{1}{2}, \frac{1}{4})$. And $g(1, 1) = (0, 1)$. However $(\frac{1}{2}, \frac{1}{4}) \not\leq (0, 1)$. Thus $g(\frac{3}{4}, \frac{1}{4}) \not\leq g(1, 1)$. Thus $g : R \longrightarrow I^2$ is not a dimap.

Thus $f : I^2 \longrightarrow R$ is a homeomorphism and a dimap, but it is not a dihomeomorphism.

Theorem 1.1. *Let X be a local po-space, and let $p : \tilde{X} \longrightarrow X$ be a covering projection. Then there exists a unique local po-structure on $\tilde{X}$ such that $p : \tilde{X} \longrightarrow X$ is a local dihomeomorphism.*

Proof. Consider an arbitrary $x \in X$. It is contained in an evenly covered neighborhood W_x. Let V_x be a partially ordered neighborhood of x. Let $U_x = W_x \bigcap V_x$, thus U_x is an evenly covered partially ordered neighborhood of x. Thus $\{U_x\}_{x \in X}$ is an open cover of X, as x was an arbitrary point. Let $\{V_{x,\alpha}\}_{\alpha \in \Lambda}$ be the connected components of $p^{-1}(U_x)$. The collection $\{V_{x,\alpha}\}_{\alpha \in \Lambda, x \in X}$ forms an open cover of $\tilde{X}$ as the map p is surjective. Consider an arbitrary $V_{x,\alpha}$ in this cover and define a relation on it by setting $a \leq_{x,\alpha} b \Leftrightarrow p(a) \leq_{U_x} p(b)$ for all $a, b \in V_{x,\alpha}$. The map p restricts to a homeomorphism $p : V_{x,\alpha} \longrightarrow U_x$, and so the map $p \times p : V_{x,\alpha} \times V_{x,\alpha} \longrightarrow U_x \times U_x$ is a local homeomorphism. The relation in $U_x \times U_x$ the relation in $V_{x,\alpha} \times V_{x,\alpha}$ are images of each other. Thus the relation is a closed subset of $U_x \times U_x$. Thus $V_{x,\alpha}$ is a partially ordered space.

Now consider $V_{x,\alpha}$ and $V_{y,\beta}$. If $x = y$ then $V_{x,\alpha} \bigcap V_{x,\beta} = \emptyset$. When $x \neq y$ this intersection may be non-empty. If $V_{x,\alpha} \bigcap V_{x,\beta} \neq \emptyset$ then let $V = V_{x,\alpha} \bigcap V_{x,\beta}$. V is an open set. For $a, b \in V$, we have $a \leq_{x,\alpha} b \Leftrightarrow p(a) \leq_{U_x} p(b)$ and $a \leq_{y,\beta} b \Leftrightarrow p(a) \leq_{U_y} p(b)$. Now $p(a), p(b) \in U_x \bigcap U_y$, and X is a local po-space, so the partial orders of U_x and U_y agree on their intersection, i.e. $p(a) \leq_{U_x} p(b) \Leftrightarrow p(a) \leq_{U_y} p(b)$. So we have $a \leq_{x,\alpha} b \Leftrightarrow a \leq_{y,\beta} b$. Thus $\{V_{x,\alpha}, \leq_{x,\alpha}\}$ forms a local partial order on $\tilde{X}$, turning it into a local po-space.

This automatically turns p into a dimap.

Consider the restriction $p : V_{x,\alpha} \to U_x$ and let $s : U_x \to V_{x,\alpha}$ be the corresponding section. Both p and s are homeomorphisms. For any $a, b \in V_{x,\alpha}$, $a \leq_{x,\alpha} b \Rightarrow p(a) \leq_{U_x} p(b)$ and $p(a) \leq_{U_x} p(b) \Rightarrow a \leq_{x,\alpha} b$, i.e. $p(a) \leq_{U_x} p(b) \Rightarrow s \circ p(a) \leq_{x,\alpha} s \circ p(b)$. Thus p and s are dimaps, and so p is a local dihomeomorphism.

Now let $\{V_\gamma, \leq_{V_\gamma}\}$ be a local partial order on $\tilde{X}$, such that $p : \tilde{X} \to X$ is a local dihomeomorphism. For all $\tilde{x} \in \tilde{X}$ there is a neighborhood $W_{\tilde{x}}$ such that p is a dihomeomorphism on $W_{\tilde{x}}$. In other words, if P is the image of $W_{\tilde{x}}$ under p, then for $a, b \in W_{\tilde{x}}$ we have $a \leq_{W_{\tilde{x}}} b \Leftrightarrow p(a) \leq_P p(b)$. Now consider $W_{\tilde{x}}$ and $V_{y,\alpha}$ such that $W_{\tilde{x}} \bigcap V_{y,\alpha} \neq \emptyset$. Consider $a, b \in W_{\tilde{x}} \bigcap V_{y,\alpha}$. We have $a \leq_{W_{\tilde{x}}} b \Leftrightarrow p(a) \leq_P p(b)$ where P is the image of $W_{\tilde{x}}$ under p. And $a \leq_{y,\alpha} b \Leftrightarrow p(a) \leq_{U_y} p(b)$. Now X is a local po-space, so the partial orders on P and U_y restrict to the same partial order on their intersection. Thus $p(a) \leq_P p(b) \Leftrightarrow p(a) \leq_{U_y} p(b)$. This implies that $a \leq_{W_{\tilde{x}}} b \Leftrightarrow a \leq_{y,\alpha} b$. Thus the union of $\{V_\gamma, \leq_{V_\gamma}\}$ and $\{V_{x,\alpha}, \leq_{x,\alpha}\}$ is still a local partial order on $\tilde{X}$, and so they are equivalent. □

Chapter 2

The Fundamental Category

Given two concurrent systems, we would like ways of characterizing the differences between them. In classical homotopy theory, one of the ways to differentiate between two topological spaces is to find homotopy invariants by constructing functors from the category of topological spaces and continuous maps to the category of groups and group homomorphisms. One of the homotopy invariants found in this way is the fundamental group.

Thus, we would like to follow the example of classical homotopy theory and define a functor from the category of local po-spaces to a more algebraic category. However, if we would like to create something like the fundamental group we hit a problem. Given a dipath in a local po-space, its inverse need not be a dipath. In fact, unless the path is the constant path this is not the case.

However all is not lost, for there is another construction, called the fundamental groupoid of a topological space X. It is a category whose objects are the points of the topological space, with the morphisms between any pair of points being the homotopy classes of paths between those two points. The composition law is defined as follows, given points x, y and z, and morphisms $[f]$ from x to y, and $[g]$ from y to z, where $[f]$ and $[g]$ the homotopy classes of paths f and g from x to y and y to z respectively, define $[g] \circ [f]$ to be the homotopy class of the path obtained by concatenating the paths f and g. Clearly this is a morphism from x to z.

Following this example we define the fundamental category of a local po-space to be the category whose objects are the points in the space, with the morphisms between any pair of points being the dihomotopy classes of dipaths between those two points. We define the composition law in a similar manner.

Note that the fundamental category is not a groupoid, as the morphisms in the fundamental category are almost never isomorphisms, unless they are identity morphisms.

Given points x, y and z in a local po-space X, let $f : I \to X$ be a dipath from x to y and let $g : I \to X$ be a dipath from y to z. Define a dipath called the composition $(g \circ f) : I \to X$ from x to z to be,

$$(g \circ f)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

Note that the re-parametrizations of f and g are monotonically increasing. As f and g are dipaths, their monotonically increasing re-parametrizations must also be dipaths. Also, $f|_{[0,1/2]}$ is a dipath and $g|_{[1/2,1]}$ is a dipath. So we have neighborhoods V and W of the point $a = (g \circ f)(1/2) = f(1) = g(0)$ such that for $O = f^{-1}(V)$ and $U = g^{-1}(W)$, if $x_1, x_2 \in O, x_1 \leq_O x_2 \Rightarrow f(x_1) \leq_V f(x_2)$ and if $x_1, x_2 \in U, x_1 \leq_U x_2 \Rightarrow g(x_1) \leq_W g(x_2)$. So let $K = V \bigcap W$ and let $L = (g \circ f)^{-1}(K)$. So for $x_1, x_2 \in K$ if $x_1 \leq_K 1/2 \leq_L x_2$ then $f(x_1) \leq_K f(1)$ and $g(0) \leq_K g(x_2)$. So by transitivity we have $f(x_1) \leq_K g(x_2)$, which is the same as saying that $(g \circ f)(x_1) \leq_K (g \circ f)(x_2)$. So the path $(g \circ f)(x)$ is indeed a dipath.

Now let $[f], [g]$ and $[(g \circ f)]$ denote the dihomotopy classes of the dipaths f, g and $(g \circ f)$. Define the composition $[g] \circ [f]$ to be the dihomotopy class of the composition of the dipaths, $[(g \circ f)]$.

Proposition 2.1. [1] *The composition of their dihomotopy classes* $[g] \circ [f]$ *as defined above, is associative with identities.*

Proof. Let X be a local po-space. Let x, y, z and w be points in X. Let f, g and h be dipaths from x to y, y to z and z to w respectively.

We claim that $([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f])$.

Consider the homotopy $H : I \times I \to X$, given by,

$$H(t,s) = \begin{cases} f(\frac{4}{1+s}t) & 0 \leq t \leq \frac{1+s}{4} \\ g(4(t - \frac{1+s}{4})) & \frac{1+s}{4} < t \leq \frac{2+s}{4} \\ h(\frac{4}{2-s}(t - \frac{2+s}{4})) & \frac{2+s}{4} < t \leq 1 \end{cases}$$

We need only check that the above homotopy is a dihimotopy.

The functions $f(\frac{4}{1+s}t), g(4(t - \frac{2+s}{4}))$ and $h(\frac{4}{2-s}(t - \frac{2+s}{4}))$ are dipaths for every fixed s, as f, g and h are dipaths and each of their re-parametrizations are monotonic for every fixed s. The homotopy $H(t,s)$ is a dipath for every fixed s as it is just a concatenation of dipaths for a fixed s, and so by the argument for the concatenation operation above it must be a dipath for every fixed s.

□

Definition 2.1. [1] *Given a local po-space X, we define the Fundamental Category $\vec{\pi_1}(X)$ to be the category whose objects are the points of the space X and whose morphisms are the dihomotopy classes of dipaths between any pair of points.*

Theorem 2.2. [2] *The isomorphism class of the fundamental category is a di-homeomorphism invariant.*

Proof. Let X and Y be local po-spaces with fundamental categories $\vec{\pi_1}(X)$ and $\vec{\pi_1}(Y)$ respectively. Let $h : X \to Y$ be a di-homeomorphism.

The function h gives a bijective correspondence between the sets X and Y. Thus, the induced functor $h_* : \vec{\pi_1}(X) \to \vec{\pi_1}(Y)$ is a bijection on the objects of the categories. All this is simply because h is a homeomorphism and hence bijective.

Now consider objects $a, b \in \vec{\pi_1}(X)$ and their images $h_*(a), h_*(b) \in \vec{\pi_1}(Y)$. Let $[f]$ be a dihomotopy class of a dipath $f : I \to X$ from a to b. As h is a dimap the composition $h \circ f : \vec{I} \to Y$ is a dipath in Y from $h_*(a)$ to $h_*(b)$. And so there is a dihomotopy class $[h \circ f] \in \vec{\pi_1}(Y)$ from $h_*(a)$ to $h_*(b)$.

Similarly, given objects $c, d \in \vec{\pi_1}(Y)$ and their pre-images ${h^{-1}}_*(c), {h^{-1}}_*(d) \in \vec{\pi_1}(X)$. If $[g]$ is the dihomotopy class of a dipath $g : I \to Y$ from c to d. Then as h^-1 is a dimap, the composition $h^{-1} \circ g : I \to X$ is a dipath in X from ${h^{-1}}_*(c)$ to ${h^{-1}}_*(d)$. And so there is a dihomotopy class $[h^{-1} \circ g] \in \vec{\pi_1}(X)$ from ${h^{-1}}_*(c)$ to ${h^{-1}}_*(d)$.

Now consider $[f], [g] \in \vec{\pi_1}(X)$ be two dihomotopy of dipaths f and g from a to b in X. We have corresponding dihomotopy classes $[h \circ f], [h \circ g] \in \vec{\pi_1}(Y)$ from $h_*(a)$ to $h_*(b)$ in Y. If $[h \circ f] = [h \circ g]$, then there must be a dihomotopy $H : I \times I \to Y$ between the representative dipaths $h \circ f$ and $h \circ g$. As $h^{-1} : Y \to X$ is a dimap, the composition $h^{-1} \circ H : I \times I \to X$ is a dihomotopy between the dipaths $h^{-1} \circ h \circ f$ and $h^{-1} \circ h \circ g$ in X. This is none other than a dihomotopy between the paths f and g in X. Thus we have $[f] = [g]$.

Thus there is a bijective correspondence between the objects and the arrows in the categories $\vec{\pi_1}(X)$ and $\vec{\pi_1}(Y)$, and so they are isomorphic as categories.

Thus, the fundamental category is a dihomeomorphism invariant. □

We now carry out some basic calculations of the fundamental categories of some simple examples. One of the obstacles to these calculations is finding some elegant way of stating the result. When calculating the fundamental group of a space, the answer is usually stated in the form of a group presentation. However, when calculating the fundamental category, there is a lot less structure on which to rely when stating the result. We shall attempt to state the results of our calculations in terms of familiar families of categories, though this shall not always be possible. To this end, we shall begin by stating the definitions of a few familiar categories.

Definition 2.2. [8] *A category P in which, given objects p and p', there is at most one arrow $p \to p'$, is called a preorder. Given a partially ordered set Q, the associated preorder is the category whose objects are the elements of Q, such that for q and q' in Q, $q \to q'$ if and only if $q \leq q'$.*

Proposition 2.3. *The fundamental category of I^n with the product partial order, is the associated preorder.*

Proof. Let $f : I \to I^n$ be a dipath. We have $0 \leq 1$ and so we must have $f(0) \leq f(1)$ as I^n is a partially ordered space. Note that we would not have been able to say this if we were working in local po-space.

Now, if we write f as $f(t) = (f_1(t), ..., f_n(t))$ then $t_1 \leq t_2$ implies that $f(t_1) \leq f(t_2)$ i.e. that $(f_1(t_1), ..., f_n(t_1)) \leq (f_1(t_2), ..., f_n(t_2))$, which tells us that $f_1(t_1) \leq f_1(t_2)...f_n(t_1) \leq f_n(t_2)$. Thus, each of the component functions of f must be monotonic as functions from I to I.

Consider two points $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in I^n.

If $x \not\leq y$ then there can be no dipath from x to y.

Without loss of generality, if $x \leq y$ then consider the path g, from x to y defined by, $g(t) = x + t(y - x)$ or $g(t) = (x_1 + t(y_1 - x_1), ..., x_n + t(y_n - x_n))$. For $t_1 \leq t_2$ in I, $t_1(y_i - x_i) \leq t_2(y_i - x_i)$, and so we have $x_i + t_1(y_i - x_i) \leq x_i + t_2(y_i - x_i)$. Thus $g(t_1) \leq g(t_2)$. So g is a dipath from x to y. We claim that all other dipaths from x to y are dihomotopic to g, or in other words that $[g]$ is the only morphism from x to y in the fundamental category.

Let f be any dipath from x to y. Define a homotopy $H : I \times I \to I^n$ between f and g by setting $H(t, s) = (1-s)f(t) + sg(t)$. For $t_1 \leq t_2$ in I, and for any s in I, we have, on the component functions of f and g, the following inequalities. $f_i(t_1) \leq f_i(t_2)$ and $g_i(t_1) \leq g_i(t_2)$, and so we have $(1 - s)f_i(t_1) \leq (1 - s)f_i(t_2)$ and $sg_i(t_1) \leq sg_i(t_2)$, as s and $(1 - s)$ are greater than 0. Thus, $(1 - s)f_i(t_1) + sg_i(t_1) \leq (1 - s)f_i(t_2) + sg_i(t_2)$, i.e. $H(s, t_1) \leq H(s, t_2)$ for all s in I. Thus H is a homotopy through dipaths and we conclude that H is a dihomotopy between f and g. As f was an arbitrary dipath from x to y, we can conclude that in the fundamental category, there is one and only one morphism $[g]$ from x to y.

Thus, the fundamental category of I^n with the product partial order, is the associated preorder. □

Chapter 3

Configuration Spaces

We now define the notions of Homotopy History, and Homotopy Future, of a point in a local po-space. The Homotopy History of a point x in a local po-space X is taken to be the set of all points y from which there exist dipaths terminating in x. The Homotopy Future of a point x in a local po-space X is taken to be the set of all points y at which a dipath from x can terminate.

The notions of Homotopy History and Homotopy Future of points in a local po-space are used to define what initial and final points are. These are defined to be points with no Homotopy History other than themselves, and no Homotopy Future other than themselves, respectively. Thus a dipath that begins at an initial point cannot be extended "further back in the past" and a dipath that begins at a terminal point cannot be extended "further into the future." An Inextensible Dipath is a dipath that begins at an initial point and ends at a final point, and thus cannot be extended "into the past" or "into the future."

Definition 3.1. [1] **(Homotopy History)** *If X is a local po-space then for $x \in X$, the Homotopy History of x is defined to be the set $P(x)$, containing of all points $y \in X$ such that there exists a dipath $f : I \to X$ with $f(0) = y$ and $f(1) = x$.*

Definition 3.2. (Homotopy Future) *If X is a local po-space then for $x \in X$, the Homotopy Future of x is defined to be the set $F(x)$, containing all points $y \in X$ such* *that there exists a dipath $f : I \to X$ with $f(0) = x$ and $f(1) = y$.*

Definition 3.3. [2] **(Initial Point)** *In a local po-space X, an initial point $x \in X$ is a point such that $P(x) = \{x\}$.*

Definition 3.4. [2] **(Final Point)** *In a local po-space X, a final point $x \in X$ is a point such that $F(x) = \{x\}$.*

Dihomeomorphisms preserve dipaths, and thus they map initial points to initial points and final points to final points.

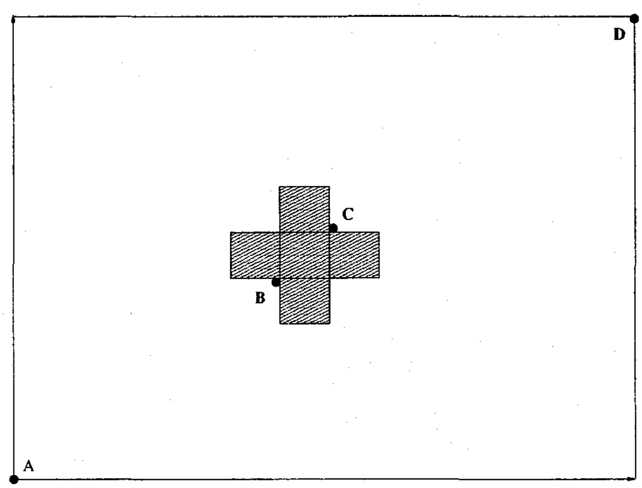

Figure 3.1: Initial and Final Points

Consider Figure 3.1, the partial order is the Euclidean Product partial order. The points A and C are initial points, and the points B and D are final points.

Definition 3.5. [2] **(Inextensible Dipath)** *In a local po-space X, an inextensible dipath is a dipath $f : I \to X$ such that $f(0)$ and $f(1)$ are initial and final points respectively. The set of inextensible dipaths in X is denoted by $\vec{P}_1(X)$.*

Consider Example 1.1. The Directed Cube I^n is a partially ordered space, and for an arbitrary point $(x_1, ..., x_n) \in I^n$, we have $(0, ..., 0) \leq (x_1, ..., x_n)$. Thus all dipaths that end at $(0, ..., 0)$ must also begin there. Thus the point $(0, ..., 0)$ is an initial point. It's Homotopy History is just itself, and its Homotopy Future is the entire cube I^n. Similarly, we can conclude that the point $(1, ..., 1)$ is a final point, its Homotopy History is the entire cube I^n, and its Homotopy Future is just itself. So in Example 1.1, any dipath from $(0, ..., 0)$ to $(1, ..., 1)$ is an inextensible dipath.

In Figure 3.1 a dipath starting at A and ending at B would be an example of an inextensible dipath.

As dihomeomorphisms preserve initial and final points, they must preserve inextensible dipaths.

Initial and final points don't always exist, and so inextensible paths don't always exist. Consider Example 1.4, the Directed Circle, which has no initial and final points and so does not admit any inextensible dipaths.

We now define an equivalence relation on the points in a local po-space called Homotopy History Equivalence. Two points are said to be Homotopy History Equivalent if the dihomotopy classes of all the Inextensible Dipaths passing through them are the same.

Definition 3.6. [2] *In a local po-space X, the Homotopy History of an Inextensible Dipath $f : I \to X$ is defined to be the set $h(f) = \{y \in X | \exists$ a dipath g through y such that g is dihomotopic to $f\}$*

Definition 3.7. [2] (Homotopy History Equivalence) *In a local po-space X, two points $x, y \in X$ are Homotopy History Equivalent if $x \in h(f) \Leftrightarrow y \in h(f)$ for all $f \in \vec{P}_1(X)$.*

Observe, that were we to consider the local po-space I^2 with n-points deleted, then the number of deformation equivalence classes would depend on the configuration of those n-points. In other words, we get a function from the configuration space of n-points in I^2 to the natural numbers $\mathbb{N}$. We now investigate the properties of this function.

Following Birman [9], we define the configuration space of n-points in I^2 as follows.

Definition 3.8. (Configuration Space) *The Configuration Space of n points in I^2 is the space $I^2 \times ... \times I^2 \setminus \Delta$, where $\Delta = \{(x_1, ..., x_n) \in (I^2)^n | x_i = x_j \text{ for some } i \neq j\}$. We denote this space by $F_{0,n}$.*

Given a point $x \in F_{0,n}$, where $x = (x_1, ..., x_n)$, we can form a corresponding partially ordered space from the unit square, with those points deleted, i.e. $I^2 \setminus \{x_1, ..., x_n\}$ with the Euclidean product partial order. We shall denote this partially ordered space by I_x^2.

Definition 3.9. (Homotopy History Equivalence Count) *Define the Homotopy History Equivalence Count to be the function $\mathbb{D} : F_{0,n} \to \mathbb{N}$ that counts the number of Homotopy History Equivalence Classes of I_x^2 for $x \in F_{0,n}$. In particular $\mathbb{D}(x) = \#(I_x^2)$, where $\#$ denotes the number of Homotopy History Equivalence Classes.*

Definition 3.10. (Generic Configuration) *A configuration of n points in I^2 is called Generic, if no two points lie on the same vertical or horizontal line, and all of the points lie in the interior of I^2.*

Proposition 3.1. *Consider $x, y \in F_{0,n}$, if the po-spaces I_x^2 and I_y^2 are dihomeomorphic, then $\mathbb{D}(x) = \mathbb{D}(y)$.*

Proof. Let $F : I_x^2 \to I_y^2$ be a dihomeomorphism.

Consider $a, b \in I_x^2$ such that a and b belong to the same Homotopy History Equivalence class. Thus $a \in h(f) \Leftrightarrow b \in h(f) \forall f \in \vec{P_1}(I_x^2)$.

Suppose that $F(a), F(b) \in I_y^2$ do not belong to the same Homotopy History Equivalence class. Thus there exists an inextensible dipath $s \in \vec{P_1}(I_y^2)$ such that $F(a) \in h(s)$ and $F(b) \notin h(s)$, Or $F(a) \notin h(s)$ and $F(b) \in h(s)$.

Consider the case in which $F(a) \in h(s)$ and $F(b) \notin h(s)$. Let z_s be a dipath in I_y^2 passing through $F(a)$ that is dihomotopic to s. Such a dipath can be picked by Definition 3.6 and Definition 3.7.

Now s is an inextensible dipath in I_y^2, so $F^{-1} \circ s$ is an inextensible dipath in I_x^2, call it t. Also $F^{-1} \circ z_s$ is a dipath in I_x^2 passing through a, call it u. As F is a dihomeomorphism we must have that u is dihomotopic to t, where $t \in \vec{P_1}(I_y^2)$. So we have that $a \in h(t)$. As a and b belong to the same Homotopy History Equivalence class, we must have that $b \in h(t)$. Let g_t be a dipath passing through b that is dihomotopic to t. Thus we have that u, g_t and t are all dihomotopic to each other, where u and g_t are dipaths passing through a and b respectively, and t is an inextensible dipath.

Let $v = F \circ g_t$ be the corresponding dipath in I_y^2, it passes through $F(b)$. Note also that $z_s = F \circ u$ is a dipath in I_y^2 that passes through $F(a)$.

As F is a dihomeomorphism, we have that z_s is dihomotopic to v. But z_s is dihomotopic to s, therefore v is dihomotopic to s and as v passes through $F(b)$ we have that $F(b) \in h(s)$.

Thus we have that $F(a) \in h(s)$ and $F(b) \in h(s)$. This is a contradiction.

The case in which $F(a) \notin h(s)$ and $F(b) \in h(s)$ is handled in a similar manner, and we obtain a contradiction here too. Thus our supposition that $F(a)$ and $F(b)$ do not belong to the same Homotopy History Equivalence class is false.

Thus, if $a, b \in I_x^2$ lie in the same Homotopy History Equivalence class and $F : I_x^2 \longrightarrow I_y^2$ is a dihomeomorphism, then $F(a), F(b) \in I_y^2$ must lie in the same Homotopy History Equivalence class.

Conversely, consider $c, d \in I_x^2$ such that c and d do not lie in the same Homotopy History Equivalence class.

Let if possible that $F(c), F(d) \in I_y^2$ belong to the same Homotopy History Equivalence class.

We have that $F^{-1} : I_y^2 \longrightarrow I_x^2$ is a dihomeomorphism, and so c and d must belong to the same Homotopy History Equivalence class, by our result above. This is a contradiction.

Thus, given $c, d \in I_x^2$ such that c and d do not belong to the same Homotopy History Equivalence class, and $F : I_x^2 \longrightarrow I_y^2$ a dihomeomorphism, then $F(c), F(d) \in I_y^2$ do not belong to the same Homotopy History Equivalence class.

Thus we can conclude that for arbitrary $p, q \in I_X^2$ we have that p and q belong to the same Homotopy History Equivalence class if and only if $F(p)$ and $F(q)$ belong to the same Homotopy History Equivalence class.

Thus a dihomeomorphism $F : I_x^2 \to I_y^2$ preserves Homotopy History Equivalence classes, so that $\mathbb{D}(x) = \mathbb{D}(y)$. □

Proposition 3.2. *Given a generic configuration $x = (x_1, ..., x_n) \in F_{0,n}$ with $\mathbb{D}(x) = k$, there exists an open neighborhood U_x around x such that for all $y \in U_x$, I_x^2 is dihomeomorphic to I_y^2 and consequently $\mathbb{D}(y) = \mathbb{D}(x) = k$.*

Proof. Let $p_1 : I^2 \to I$ and $p_2 : I^2 \to I$ be the projections on the X and Y axes respectively. As none of the x_i's lie on the same vertical or horizontal, $p_1(x_1), ..., p_1(x_n)$ are all distinct points. Similarly $p_2(x_1), ..., p_2(x_n)$ are all distinct points.

Let $a_0, ..., a_n$ be a partition of the interval I, such that $a_0 = 0$ and $a_n = 1$ and $p_1(x_i) \in (a_{i-1}, a_i)$.

Let $b_0, ..., b_n$ be a partition of the interval I, such that $b_0 = 0$ and $b_n = 1$ and $p_2(x_i) \in (b_{i-1}, b_i)$.

Let $U_i = (a_{i-1}, a_i) \times (b_{i-1}, b_i)$ be a "square shaped" neighborhood of x_i. By "square shaped" neighbourhood, we mean a rectangular neighbourhood which is the product of two open intervals.

Let $U = U_1 \times ... \times U_n$.

For $a \in U_i$ let the coordinate representation of $a$be (a^1, a^2) where $a^1 = p_1(a)$, $a^2 = p_2(a)$.

Consider a point $y = (y_1, ..., y_n) \in U$, where $y_i \in U_i$. Let (y_i^1, y_i^2) be its co-ordinate representation.

Define $h_{i_1} : I \to I$ to be,

$$h_{i_1}(x) = \begin{cases} x & 0 \leq x \leq a_{i-1} \\ a_{i-1} + (x - a_{i-1})\frac{(y_i^1 - a_{i-1})}{(x_i^1 - a_{i-1})} & a_{i-1} < x \leq x_i^1 \\ y_i^1 + (x - x_i^1)\frac{(a_i - y_i^1)}{(a_i - x_i^1)} & x_i^1 < x \leq a_i \\ x & a_i < x \leq 1 \end{cases}$$

Similarly, define $h_{i_2} : I \to I$ to be,

$$h_{i_2}(x) = \begin{cases} x & 0 \leq x \leq b_{i-1} \\ b_{i-1} + (x - b_{i-1})\frac{(y_i^2 - b_{i-1})}{(x_i^2 - b_{i-1})} & b_{i-1} < x \leq x_i^2 \\ y_i^2 + (x - x_i^2)\frac{(b_i - y_i^2)}{(b_i - x_i^2)} & x_i^2 < x \leq b_i \\ x & b_i < x \leq 1 \end{cases}$$

$H_{i_1}(r,s) = (h_{i_1}(r), s)$ is a dihomeomorphism that takes (x_i^1, x_i^2) to (y_i^1, x_i^2) and fixes all points outside $[a_{i-1}, a_i] \times I$.

$H_{i_2}(r,s) = (r, h_{i_2}(s))$ is a dihomeomorphism that takes (x_i^1, x_i^2) to (x_i^1, y_i^2) and fixes all points outside $I \times [b_{i-1}, b_i]$.

Thus the dihomeomorphism $H_{i_2} \circ H_{i_1}(r,s)$ takes (x_i^1, x_i^2) to (y_i^1, y_i^2), and keeps all points outside $[a_{i-1}, a_i] \times I \bigcup I \times [b_{i-1}, b_i]$ fixed.

Thus the dihomeomorphism $H_{1_2} \circ H_{1_1} \circ ... \circ H_{n_2} \circ H_{n_1} : I^2 \to I^2$ takes x to y. Thus I_x^2 is dihomeomorphic to I_y^2, by Proposition 3.1.

This implies that $\mathbb{D}(y) = \mathbb{D}(x) = k$.

And $y = (y_1, ..., y_n) \in U$ was arbitrary. Thus I_x^2 is dihomeomorphic to I_y^2 and $\mathbb{D}(y) = \mathbb{D}(x) = k$ for all $y \in U$.

□

Corollary 3.3. *Let G^c be the set of non-generic configurations. Then the function $\mathbb{D}$ is constant on the path components of $F_{0,n} \setminus G^c$.*

Proof. Let $a, b \in F_{0,n} \setminus G^c$ be two generic configurations that belong to the same path component. Let $f : I \to F_{0,n} \setminus G^c$ be a path from a to b.

Given a Generic Configuration $x \in F_{0,n}$, there exists a neighbourhood V_x in $F_{0,n}$ such that for all $y \in V_x$ we have that I_x^2 is dihomeomorphic to I_y^2. As every point on the image of the path f is a generic configuration, we can cover the path by such neighbourhoods. As the image of f is compact, there are only a finite number of such neighbourhoods, $V_{x_1}, ...V_{x_k}$, where $x_1 = a$ and $x_k = b$, and such that consecutive neighbourhoods V_{x_i} and $V_{x_{i+1}}$ have non-empty intersection. Thus we have $I_{x_i}^2$ dihomeomorphic to $I_{x_{i+1}}^2$. As dihomeomorphism is an equivalence relation, we have I_a^2 dihomeomorphic to I_b^2.

Thus by Proposition 3.1 we have $\mathbb{D}(a) = \mathbb{D}(b)$.

Thus the function $\mathbb{D}$ is constant on the path components of $F_{0,n} \setminus G^c$. □

Corollary 3.4. *Let $h : I^2 \to I^2$ be the map that sends (x, y) to (y, x). It induces a map $H : F_{0,n} \to F_{0,n}$, of the form $H = (h, ..., h)$. Then $\mathbb{D}(x) = \mathbb{D}(H(x))$.*

Proof. The map h is a dihomeomorphism, and so the result follows from Proposition 3.1. □

Chapter 4

Di-homeomorphisms of the Unit Square

We restrict our attention to the Euclidean product partial order. We look at di-homeomorphic embeddings of the unit square I^2, in the plane $\mathbb{R}^2$. We show that the di-homeomorphic image of I^2 must be of the form $[a, b] \times [c, d]$. We also classify all possible di-homeomorphisms that embed the unit square in the Euclidean plane. Namely, we show that all such di-homeomorphisms are, up to composition with automorphism of I^2 given by permuting the coordinates, products of independent one-dimensional di-homeomorphisms of intervals.

This result tells us that the isomorphisms in the category of local po-spaces are of a very restricted nature. Thus, if one is to deduce useful relations between different concurrent systems based on models formulated in the category of local po-spaces, a weaker notion of equivalence is required. In particular, this indicates the need to pass to an appropriate homotopy category.

In [13] various categories of fractions of the fundamental category are defined and in each case a component category of the category of fractions is also defined. Martin Raussen has suggested that an appropriate notion of equivalence is likely to be in the form of morphisms between local po-spaces that induce isomorphisms between components when one passes to their appropriate component categories.

Let us now return to our result, which states that in the 2-dimensional case, a dihomeomorphic embedding of the unit square in the plane must factor as a product of two 1-dimensional dihomeomorphisms of the unit interval, up-to a change in orientation.

It is not hard to intuit why a di-homeomorphic image of I^2 in the plane, must be of the form $[a, b] \times [c, d]$. However this intuition does not easily pass to a proof.

Consider the fundamental category of I^2. It's objects can be classified into two distinct sets, the points that form the interior of I^2 and those that form the boundary(in the manifold sense) of I^2. If one observes the di-homotopy classes of dipaths between these two distinct sets, one can subdivide the set of boundary points into four connected sets, $A = \{0\} \times [0, 1) \bigsqcup [0, 1) \times \{0\}$, $B = \{1\} \times (0, 1] \bigsqcup (0, 1] \times \{1\}$ and $C = \{(0, 1)\}, D = \{(1, 0)\}$. There are no dipaths from points in the interior of I^2 to points in A, there are no dipaths from points in B to points in the interior of I^2. There are no dipaths at all between points in the interior of I^2 and the sets C and D. As the fundamental category is a di-homeomorphism invariant, any di-homeomorphism of I^2 must preserve the properties of these sets in the image.

Applying this intuition to a simple example, we see that there are severe restrictions on what the di-homeomorphic image can be.

Consider a parallelogram P which is not a rectangle, in the plane, with the induced partial order (as a subspace). Consider its boundary (as a manifold). One can show that in " most" cases, the set of boundary points P does not have subsets that satisfy the dipath conditions of either A, B, C or D. And in the few cases where such subsets can be found, additional considerations on the set of boundary points, derived from the fundamental category can be shown to fail. Thus it can be shown that a parallelogram P in the plane, is not di-homeomorphic to I^2, in fact, such parallelograms may not even be di-homeomorphic to each other.

However, it is hard to turn this intuition into a proof that the di-homeomorphic image of I^2 in the plane must be of the form $[a, b] \times [c, d]$. This is because the homeomorphic image of the boundary of I^2 can be pretty wild, making it hard to distinguish its subsets based on dipath dihomotopy class properties. Instead, we show that the topological boundary of the partial order relation R as a subset of $\mathbb{R}^2 \times \mathbb{R}^2$, and as a subset of $I^2 \times I^2$, consists of pairs of points along lines parallel to the X or Y axes. We show that any di-homeomorphic embedding of I^2 in the plane, must map points on the topological boundary of the partial order relation in I^2 to those on the topological boundary of the relation in $\mathbb{R}^2$. We show that this forces all di-homeomorphisms that embed I^2 in the plane, to preserve vertical and horizontal lines parallel to the X and Y axes, up-to orientation reversal of the image of I^2. We then see how this forces the di-homeomorphic image of I^2 to be of the form $[a, b] \times [c, d]$.

First we shall characterize the properties of the topological boundary of the local partial order as a subset of $\mathbb{R}^2 \times \mathbb{R}^2$. We show that a pair of points in the partial order lie on its topological boundary if and only if they lie on the same vertical or horizontal line.

Proposition 4.1. (Po-Boundary in the Plane) *Consider the relation $R \subset \mathbb{R}^2 \times \mathbb{R}^2$ defined by the product partial order, $R = \{((a_1, a_2), (b_1, b_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$ with $a_1 \leq b_1$ and $a_2 \leq b_2\}$.*

The boundary of the relation R as a subset of $\mathbb{R}^2 \times \mathbb{R}^2$ is
$\mathrm{bd}(R) = \{((a_1, a_2), (b_1, b_2)) | a_1 = b_1 \text{ or } a_2 = b_2\}$

Proof. Let $A = \{((a_1, a_2), (b_1, b_2)) | a_1 < b_1 \text{ and } a_2 < b_2\}$ be the set of ordered pairs of points (a_1, a_2) and (b_1, b_2) where the co-ordinates a_1 and a_2 are strictly less than b_1 and b_2 respectively. A is a subset of R.

Note that $R \setminus A = \{((a_1, a_2), (b_1, b_2)) | a_1 = b_1 \text{ or } a_2 = b_2\}$.

We will show that $A = \mathrm{Int}(R)$ which would imply that $\mathrm{bd}(R) = R \setminus A$ and the result would follow.

Define $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ by $F((a_1, a_2), (b_1, b_2)) = (b_1 - a_1, b_2 - a_2)$. It is easy to check that F is continuous.

Now, note that $A = F^{-1}(\mathbb{R}^+ \times \mathbb{R}^+)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$.

$\mathbb{R}^+ \times \mathbb{R}^+$ is open, and so A is open.

Now pick arbitrary $((a_1, a_2), (b_1, b_2)) \in R \setminus A$ thus $a_1 = b_1$ or $a_2 = b_2$.

We will show that all neighbourhoods of $((a_1, a_2), (b_1, b_2))$ contain points that are not in R.

If $a_1 = b_1$. Let $B \subset \mathbb{R}^2 \times \mathbb{R}^2$ be an open neighborhood of $((a_1, a_2), (b_1, b_2))$. There exist open δ-balls $U_\delta((a_1, a_2))$ and $V_\delta((b_1, b_2))$, in $\mathbb{R}^2$, of (a_1, a_2) and (b_1, b_2) respectively, such that $U_\delta((a_1, a_2)) \times V_\delta((b_1, b_2)) \subset B$ as the products of such balls form a basis.

We now construct a point in in R^c that lies in $U_\delta((a_1, a_2)) \times V_\delta((b_1, b_2))$ and hence in B.

Pick ϵ, with $0 < \epsilon < \delta$. Let $(a_1 + \epsilon, a_2) \in U_\delta((a_1, a_2))$,
therefore $((a_1 + \epsilon, a_2), (b_1, b_2)) \in B$.

Now $a_1 + \epsilon = b_1 + \epsilon > b_1$ and $a_2 < b_2$.

Thus $(a_1+\epsilon, a_2) \not\leq (b_1, b_2)$ and so $((a_1+\epsilon, a_2), (b_1, b_2)) \in R^c$. As B was an arbitrary open neighborhood of (a, b), we conclude that for every neighborhood $B \subset \mathbb{R}^2 \times \mathbb{R}^2$ of (a, b) the intersection $B \bigcap R^c$ is non-empty.

If $a_2 = b_2$. The proof is similar to the previous case.

Thus, A is the largest open set contained in R. And so we can conclude that $\text{Int}(R) = A$.

Thus $\text{bd}(R) = R \setminus A$, i.e. $((x_1, x_2), (y_1, y_2)) \in \text{bd}(R) \Leftrightarrow x_1 = y_1$ or $x_2 = y_2$. □

We now recall how the topological boundary behaves when we restrict it to a subspace of the ambient space.

Lemma 4.2. (Topological Boundary in a Subspace) *Let X be a topological space, and $A \subset X$ a subset of X. Let $S \subset X$ be a subspace of X. Let $bd_S(A)$ denote the boundary of $S \bigcap A$ in the subspace S. Then* $\text{bd}_S(A \bigcap S) \subset \text{bd}_X(A)$.

Proof. $x \in \text{bd}_S(A \bigcap S)$ if and only if for every neighbourhood V_x of x in X we have, $S \bigcap V_x \bigcap A \neq \emptyset$ and $S \bigcap V_x \bigcap A^c \neq \emptyset$.

These conditions imply that $V_x \bigcap A \neq \emptyset$ and $V_x \bigcap A^c \neq \emptyset$, which tell us that $x \in \text{bd}_X(A)$. □

We now work out what the boundary of the partial order is in a disc in the plane. We need Proposition 4.1(Po-Boundary in the Plane) and Lemma 4.2(Topological Boundary in a Subspace) for this. We show that if two points in the partial order lie on the topological boundary, then they lie on the same vertical or horizontal line.

Corollary 4.3. (Po-Boundary in a Subspace) *Let S be any subspace of $\mathbb{R}^2$. Let $\leq_S$ denote the restriction of the Euclidean product partial order to S. Let $R|_S \subset S \times S$ be the set defined by $R|_S = \{(a, b) \in S \times S | a \leq_S b\}$.*

Then $R|_S = R \bigcap (S \times S)$ and is a closed subset of $S \times S$, making S a partially ordered space, and $((a_1, a_2), (b_1, b_2)) \in \text{bd}_{S\times S}(R|_S)$ implies $a_1 = b_1$ or $a_2 = b_2$.

Proof. By Lemma 4.2 (Topological Boundary in a Subspace) we have that $\text{bd}_{S\times S}(R|_S) \subset \text{bd}(R)$.

The conclusion then follows from Proposition 4.1 (Po-Boundary in the Plane). □

We get a more detailed picture than the above Corollary if we consider the subspace in the plane to be "square shaped," i.e. of the form $[a, b] \times [c, d]$. More precisely, we infer that the topological boundary of the partial order is made up of pairs of points that lie on the same vertical or horizontal line.

Proposition 4.4. (Po-Boundary in the Square)
For $(a_1, a_2), (b_1, b_2) \in P = [p, q] \times [r, s]$, $((a_1, a_2), (b_1, b_2)) \in \text{bd}_{P\times P}(R|_P)$ if and only if $a_1 = b_1$ or $a_2 = b_2$.

Proof. Let $P = [p, q] \times [r, s]$ and let $R \subset P \times P$ be the partial-order relation. Consider $((a_1, a_2), (b_1, b_2)) \in R$. Thus $(a_1, a_2) \leq (b_1, b_2)$ and we have $a_1 \leq b_1$ and $a_2 \leq b_2$.

By Corollary 4.3 (Po-Boundary in a Subspace), we have $((a_1, a_2), (b_1, b_2)) \in \text{bd}_{P\times P}(R|_P)$ implies that $a_1 = b_1$ or $a_2 = b_2$.

If $a_1 = b_1$ and $a_2 \neq b_2$ then $((a_1, a_2), (b_1, b_2)) \in \text{bd}_{P\times P}(R|_P)$.
Let $B \subset P \times P$ be an open neighborhood of (a, b). There exist δ-balls $U_\delta((a_1, a_2))$ and $V_\delta((b_1, b_2))$, in P, of (a_1, a_2) and (b_1, b_2) respectively, such that $U_\delta((a_1, a_2)) \times V_\delta((b_1, b_2)) \subset B$. Pick ϵ, with $0 < \epsilon < \delta$. Thus $(a_1 + \epsilon, a_2) \in U_\delta((a_1, a_2))$,

therefore $((a_1+\epsilon, a_2), (b_1, b_2)) \in B$. Now $a_1 + \epsilon = b_1 + \epsilon > b_1$ and $a_2 < b_2$. Thus $(a_1+\epsilon, a_2) \not\leq (b_1, b_2)$ and so $((a_1+\epsilon, a_2), (b_1, b_2)) \in R^c$. As B was an arbitrary open neighborhood of $((a_1, a_2), (b_1, b_2))$ and since $((a_1, a_2), (b_1, b_2)) \in R$, we conclude that $((a_1, a_2), (b_1, b_2)) \in \text{bd}(R)$.

If $a_1 \neq b_1$ and $a_2 = b_2$ then the proof is similar to the previous case.

Thus $((a_1, a_2), (b_1, b_2)) \in \text{bd}_{P\times P}(R|_P)$ if and only if $a_1 = b_1$ or $a_2 = b_2$. □

We now begin the process of proving that a dihomeomorphism of a square shaped region preserves rectangles. In the following Lemma and Corollary we prove that a dihomeomorphism of I^2 must map vertical and horizontal lines in the domain to vertical and horizontal lines in the image.

Proposition 4.5. (Po-Boundary Mapping) *Let $f : [p,q] \times [r,s] \to S$ be a di-homeomorphism from $[p,q] \times [r,s]$ to a S a subset of $\mathbb{R}^2$. Consider $a, b \in [p,q] \times [r,s]$, let $a = (a_1, a_2), b = (b_1, b_2)$. Let $c = f(a), d = f(b)$ with $c = (c_1, c_2)$ and $d = (d_1, d_2)$.*

If $a_1 = b_1$ or $a_2 = b_2$, then $c_1 = d_1$ or $c_2 = d_2$.

Proof. Let $P = [p,q] \times [r,s]$. $f : P \to S$ is a di-homeomorphism. Thus f is a homeomorphism, and so $(f \times f) : P \times P \to S \times S$ is also a homeomorphism.

Consider $k, l \in P$. As f is a di-homeomorphism, we have $f(k) \leq f(l)$ if and only if $k \leq l$. Thus $(f\times f)(R|_P) \subset R|_S$ and $(f\times f)^{-1}(R|_S) \subset R|_P$. As f is a homeomorphism we must conclude that $(f\times f)(R|_P) = R|_S$. $(f\times f)$ must map $R|_P$ homeomorphically onto $R|_S$. In particular, $(f \times f)$ must map $\text{bd}_{P\times P}(R|_P)$ homeomorphically onto $\text{bd}_{S\times S}(R|_S)$.

Consider $a, b \in P$ such that $a_1 = b_1$ or $a_2 = b_2$. We must have either $a \leq b$ or $b \leq a$. And by Proposition 4.4(Po-Boundary in the Square), $(a, b) \in \text{bd}(R|_P)$. Let $c = f(a), d = f(b)$, with $c = (c_1, c_2), d = (d_1, d_2)$. As $(f\times f)$ maps $\text{bd}_{P\times P}(R|_P)$ homeomorphically onto $\text{bd}_{S\times S}(R|_S)$, we must have $(c, d) \in \text{bd}(R|_S)$. Thus by Corollary 4.3(Po-Boundary in a Subspace) we must have $c_1 = d_1$ or $c_2 = d_2$. □

Corollary 4.6. *Let $f : I^2 \to S \subset \mathbb{R}^2$ be a di-homeomorphism. Consider the sets $I \times \{a_0\}$ and $\{b_0\} \times I$. The image of these sets under f must be of the form $[x, y] \times \{c_0\}$ or $\{d_0\} \times [z, w]$, where x, y, z, w, c_0 and d_0 are some real numbers.*

Proof. Consider the set $I \times \{a_0\}$ and the points $(0, a_0), (1, a_0)$ in this set.
Let $(r_1, r_2) = f(0, a_0), (s_1, s_2) = f(1, a_0)$.
By Prop 4.5(Po-Boundary Mapping), $r_1 = s_1$ or $r_2 = s_2$.

Consider an arbitrary point $u = (u_1, u_2) \in f(I \times \{a_0\})$.
We must have $(r_1, r_2) \leq (u_1, u_2) \leq (s_1, s_2)$ as f is a dimap. Thus $r_1 \leq u_1 \leq s_1$ and $r_2 \leq u_2 \leq s_2$.

If $r_1 = s_1 = c_0$ for some real number c_0, then $u_1 = r_1 = s_1 = c_0$, and if $r_2 = s_2 = d_0$ for some real number d_0 then $u_2 = r_2 = s_2 = d_0$.

As (u_1, u_2) was an arbitrary point of $f(I \times \{a_0\})$ and since $f(I \times \{a_0\})$ is homeomorphic to an interval, it must be of the form $[x, y] \times \{c_0\}$ or $\{d_0\} \times [z, w]$. Where x, y, z and w are some real numbers.

To show that $f(\{b_0\} \times I)$ is of the form $[x, y] \times \{c_0\}$ or $\{d_0\} \times [z, w]$, the proof is similar. □

The following Corollary is the same result as the previous Corollary, except for the fact that it applies to a general square shaped region rather than I^2 alone.

Corollary 4.7. (Grid Preservation Property) *Let $f : [p, q] \times [r, s] \to S \subset \mathbb{R}^2$ be a di-homeomorphism. Consider the sets $[p, q] \times \{a_0\}$, and $\{b_0\} \times [r, s]$. The image of these sets under f must be of the form $[x, y] \times \{c_0\}$ or $\{d_0\} \times [z, w]$.*

Proof. Define $h : I^2 \to [p, q] \times [r, s]$ by $h(x, y) = (x(q - p) + p, y(r - s) + s)$. The inverse, $h^{-1} : [p, q] \times [r, s] \to I^2$ is given by, $h^{-1}(v, w) = (\frac{v-p}{q-p}, \frac{w-r}{s-r})$. Both h and h^{-1} are dimaps, and so $h : I^2 \to [p, q] \times [r, s]$ is a di-homeomorphism.

Thus $f \circ h : I^2 \to S$ is a di-homeomorphism.

Subsets of $[p, q] \times [r, s]$ of the form $[p, q] \times \{a_0\}$ or $\{b_0\} \times [r, s]$, have inverse images in I^2 under h^{-1} of the form $I \times \{\frac{a_0 - r}{s-r}\}$, and $\{\frac{b_0 - p}{q-p}\} \times I$ respectively.

Thus by Corollary 4.6, $f \circ h(I \times \{\frac{a_0-r}{s-r}\})$ and $f \circ h(\{\frac{b_0-p}{q-p}\} \times I)$, are sets of the form $[x, y] \times \{c_0\}$ or $\{d_0\} \times [z, w]$ in S. However $f \circ h(I \times \{\frac{a_0-r}{s-r}\}) = f([p, q] \times \{a_0\})$ and $f \circ h(\{\frac{b_0-p}{q-p}\} \times I) = f(\{b_0\} \times [r, s])$.

Thus $f([p, q] \times \{a_0\})$ and $f(\{b_0\} \times [r, s])$ are of the form $[x, y] \times \{c_0\}$ or $\{d_0\} \times [z, w]$ in S. □

We now show that the dihomeomorphic image in the plane, of a square shaped region, must be a square shaped region.

Theorem 4.8. (Rectangles to Rectangles) *Let $f : [p, q] \times [r, s] \to S \subset \mathbb{R}^2$ be a di-homeomorphism.*

Then S must be of the form $[a, b] \times [c, d]$.

Proof. Consider the four distinct sets $[p, q] \times \{r\}, [p, q] \times \{s\}, \{p\} \times [r, s], \{q\} \times [r, s]$. The union of these four sets form the boundary of $[p, q] \times [r, s]$.

The images of these four sets under f must be four distinct sets, each of the form $[x, y] \times \{c_0\}$ or $\{d_0\} \times [z, w]$. This follows from Corollary 4.7(Grid Preservation Property).

The images of these four sets also form the boundary of S.

Also the union of the four sets is homeomorphic to a circle. Thus the image under f, of their union, must be homeomorphic to a circle. As the image under f of their union is made up of four distinct sets, each of the form $[x, y] \times \{c_0\}$ or $\{d_0\} \times [z, w]$, it must be a 4-sided polygon. Thus the boundary of S must be a rectangle. Since S must be homeomorphic to $[p, q] \times [r, s]$ and have a rectangular boundary formed from the union of sets of the form $[x, y] \times \{c_0\}$ or $\{d_0\} \times [z, w]$, it must be of the form $[a, b] \times [c, d]$. □

If a dihomeomorphism maps one vertical line to another vertical line, then it must map all vertical lines to vertical lines, and all horizontal lines to horizontal lines. If it mapped it to a horizontal line, then it must map all vertical lines to horizontal lines, and all horizontal lines to vertical lines. The proof of this is outlines in the following Lemma.

Lemma 4.9. (Grid Consistency) *Let $f : [p,q] \times [r,s] \rightarrow [a,b] \times [c,d]$ be a dihomeomorphism. Consider $\{x\} \times [r,s]$ and $[p,q] \times \{y\}$, subsets of $[p,q] \times [r,s]$.*

Then $f(\{x\} \times [r,s]) = [a,b] \times \{i\}$ and $f([p,q] \times \{y\}) = \{j\} \times [c,d]$ where $j \in [a,b], i \in [c,d]$ or $f(\{x\} \times [r,s]) = \{j\} \times [c,d]$ and $f([p,q] \times \{y\}) = [a,b] \times \{i\}$ where $j \in [a,b], i \in [c,d]$.

Proof. By Corollary 4.7(Grid Preservation Property) $f(\{x\} \times [r,s]) = \{j\} \times [c,d]$ or $f(\{x\} \times [r,s]) = [a,b] \times \{i\}$ and $f([p,q] \times \{y\}) = \{j\} \times [c,d]$ or $f([p,q] \times \{y\}) = [a,b] \times \{i\}$. Where $i \in [c,d]$ and $j \in [a,b]$.

The intersection of $\{x\} \times [r,s]$ and $[p,q] \times \{y\}$ is the point $(x,y) \in [p,q] \times [r,s]$. Thus $f(x,y) \in f(\{x\} \times [r,s]) \bigcap f(\{x\} \times [r,s])$.

Let if possible that $f(\{x\} \times [r,s]) = \{j\} \times [c,d]$ and $f([p,q] \times \{y\}) = \{j'\} \times [c,d]$ for some $j, j' \in [a,b]$ or that $f(\{x\} \times [r,s]) = [a,b] \times \{i\}$ and $f([p,q] \times \{y\}) = [a,b] \times \{i'\}$ for some $i, i' \in [c,d]$. However $f(\{x\} \times [r,s]) \bigcap f([p,q] \times \{y\})$ is nonempty as it contains at least the point $f(x,y)$. But this implies that $(\{j\} \times [c,d]) \bigcap (\{j'\} \times [c,d]) \neq \emptyset$ or $([a,b] \times \{i\}) \bigcap ([a,b] \times \{i'\}) \neq \emptyset$, and since $\{j\} \times [c,d], \{j'\} \times [c,d]$ are parallel, and $[a,b] \times \{i\}, [a,b] \times \{i'\}$ are parallel, we would have $\{j\} \times [c,d] = \{j'\} \times [c,d]$ in one case and $[a,b] \times \{i\} = [a,b] \times \{i'\}$ in the other case. Thus the map f would not be injective and would fail to be a homeomorphism. This is a contradiction.

Thus we must have $f(\{x\} \times [r,s]) = [a,b] \times \{i\}$ and $f([p,q] \times \{y\}) = \{j\} \times [c,d]$ or $f(\{x\} \times [r,s]) = \{j\} \times [c,d]$ and $f([p,q] \times \{y\}) = [a,b] \times \{i\}$.
Where $i \in [c,d], j \in [a,b]$ □

Dihomeomorphisms must respect vertical and horizontal lines, in the manner stated above. Thus, if one restricts the dihomeomorphism to a vertical or horizontal line, then the restriction is constant either in the X or Y co-ordinate. This allows us to factor the dihomeomorphism into a product of dihomeomorphisms up-to a change of orientation. Thus we are able to classify all the possible dihomeomorphic embeddings of a square shaped region into the plane.

Theorem 4.10. (Classification of Di-Homeomorphic Embeddings of a Square) *Given a di-homeomorphism $f : [a,b] \times [c,d] \to [m,n] \times [p,q]$, we can factor f as $f(x,y) = (X(x), Y(y))$ or $f(x,y) = (Y(y), X(x))$.*

Proof. Let $h : [a,b] \times [c,d] \to \mathbb{R}^2$ be the dimap defined by $h(x,y) = (y,x)$. It is also a dihomeomorphism onto its image.

Without loss of generality we can assume that f(up-to composition with h) maps sets of the form $[a,b] \times \{x\}$ to $[m,n] \times \{i\}$ and sets of the form $\{y\} \times [c,d]$ to $\{j\} \times [p,q]$, by Lemma 4.9(Grid Consistency).

Let f be of the form $f(x,y) = (f_1(x,y), f_2(x,y))$. Let the image of f be $[m,n] \times [p,q]$, the image is of this form by Theorem 4.8(Rectangles to Rectangles).

Define $X : [a,b] \to [m,n]$ by $X(x) = f_1(x,c)$. Define $Y : [c,d] \to [p,q]$ by $Y(y) = f_2(a,y)$.

Both X and Y are dihomeomorphisms.

Note that $X(x) = f_1(x,t) \forall t \in [c,d]$. If this were not true, there would exist $t_0 \in [c,d]$ such that $X(x) \neq f_1(x,t_0)$, i.e. $f_1(x,c) \neq f_1(x,t_0)$, which implies that $f([a,b] \times \{x\})$ is not of the form $[m,n] \times \{i\}$. This is a contradiction.

Similarly note that $Y(y) = f_2(s,y) \forall s \in [a,b]$.

Now define $k : [a,b] \times [c,d] \to \mathbb{R}^2$ by $k(x,y) = (X(x), Y(y))$.

For arbitrary $(x,y) \in [a,b] \times [c,d]$ we have,

$k(x,y) = (X(x), Y(y)) = (f_1(x,c), f_2(a,y)) = (f_1(x,y), f_2(x,y)) = f(x,y)$.

Thus $f(x,y)$ is of the form $(X(x), Y(y))$ where $X : [a,b] \to [m,n]$ and $Y : [c,d] \to [p,q]$ are dihomeomorphisms. If f needed to be composed with h then f is of the form $f(x,y) = (Y(y), X(x))$. □

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