ORBIFOLDS OF NONPOSITIVE CURVATURE AND THEIR LOOP SPACES

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance of a thesis entitled "**Orbifolds of nonpositive curvature and their loop space**" by **George Dragomir** in partial fulfillment of the requirements for the degree of **Master of Science**.

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Chapter 1

Introduction

The concept of orbifold (or V-manifold) was introduced by Satake as a natural generalization of manifold. While manifolds are locally modeled on $\mathbb{R}^n$, orbifolds are locally modeled on $\mathbb{R}^n$ modulo the action by a finite group. For example, the quotient M/Γ of a manifold by a discrete group Γ acting properly discontinuously admits an orbifold structure, and any orbifold Q that can be realized as a global quotient $Q = M/\Gamma$ in this way is called *good* or *developable.* Note that not all orbifolds are good: the $\mathbb{Z}_n$-teardrop from Section 2.6 is an example of a bad orbifold.

The study of orbifolds was initiated by Satake [Sa1, Sa2] in the 1950s and was furthered by Kawasaki [Ka] and Thurston, who was used them in his study of the hyperbolization problem for 3-manifolds [T1, Sc]. This traditional approach to orbifolds is presented in Chapter 2.

Orbifolds occur in many other branches of mathematics, such as string theory and algebraic geometry. For example, the Deligne-Mumford compactification $\mathscr{M}_g$ of the moduli space of stable curves of genus g is naturally an orbifold. From this perspective, an orbifold is a special kind of groupoid (or stack). This groupoid approach is the basis for the modern perspective on orbifolds presented in Chapter 3.

What is a groupoid? The formal definition (a small category in which all arrows are invertible) is not so intuitive. It is perhaps better to think of a groupoid as something that generalizes both the concept of a group and that of an equivalence relation. A groupoid $\mathscr{G}$ over a space X consists of a collection of arrows $g \in \mathscr{G}$ with source $x \in X$ and target $y \in X$. Here, the arrow g is viewed as identifying source x and target y, hence a groupoid gives rise to an equivalence relation on X. The advantage of using groupoids is that in addition to telling us when two elements are equivalent, they also keep track of the different ways in which two elements can be equivalent. This is the crucial property of groupoids which Grothendieck utilized in his construction of fine moduli stacks and which Connes used to give a unified framework for the study of operator algebras, foliations, and index theory that occur in noncommutative geometry.

In Chapter 3, we use the pseudogroup of local homeomorphisms to explain how to go from the traditional notion of an orbifold to a groupoid and back. From this perspective, every orbifold Q can be represented by a proper, étale groupoid $(\mathscr{G}, X)$ well defined up to Morita equivalence. Many intrinsic properties of the orbifold only become apparent by passing to a representative groupoid $(\mathscr{G}, X)$. This observation is the basis for several important developments, including the orbifold cohomology and orbifold Gromov-Witten theories recently introduced by Chen and Ruan [CR].

The main result of this thesis, which is presented in Chapter 4, is a new proof of a theorem of Gromov stating that any complete Riemannian orbifold of nonpositive curvature is developable. This result is closely related to the classical result (proved for instance in [E]) that a complete, connected n-dimensional manifold of constant curvature κ has universal cover isometric to spherical, Euclidean, or hyperbolic space depending on whether $\kappa > 0, \kappa = 0$, or $\kappa < 0$, respectively. Extending these ideas, Thurston proved developability of orbifolds with geometric structures (including those with metrics of constant curvature) in [T1], and Gromov's developability theorem is a deep generalization of Thurston's.

In [BH] Bridson and Haefliger prove Gromov's theorem by proving a more general developability result for groupoids of local isometries. In this thesis, we give a proof that is quite different from that in [BH] in that we work entirely within the context of Riemannian orbifolds.

Here is an outline our proof, which is explained fully in Chapter 4. We first prove an orbifold analogue of the Hopf-Rinow theorem that Q is complete if and only if it is geodesically complete. We then introduce the notion of Jacobi fields, conjugate points, and poles for Q, and we relate these to the curvature condition by proving that any point in Q is a pole provided the metric on Q has nonpositive curvature. The final step in the proof is to show that the exponential map is a covering map.

We close the paper with a brief discussion on loop groupoids of orbifolds, paying particular attention to good orbifolds, and we also present some interesting problems on orbifold Morse theory and existence of closed geodesics of positive length.

Chapter 2

Orbifolds

In this chapter we describe the category of orbifolds in the traditional approach. Our first definition for orbifolds is essentially the one introduced by Satake in [Sa1]. We distinguish a special class of orbifold maps called *good maps* under which bundles pull back nicely. These maps were introduced by Chen and Ruan in [CR], and they correspond to the morphisms in the category of proper étale groupoids which will be introduced in section 3.2.

In the first section we recall some definitions and immediate proprieties of properly discontinuous group actions on topological (or smooth) manifolds. We prove there (in the smooth case see Proposition 2.1.6) that if a finite group acts (effectively) by diffeomorphisms of a connected manifold, then the set of points with trivial isotropy group is open and dense. In section 2.2 we give the classical definition of orbifolds and an important class of orbifolds, developable orbifolds, i.e. orbifolds which are global quotients of manifolds by groups acting properly discontinuously (Proposition 2.2.3). We also define maps between orbifolds, as well as the tangent space to an orbifold. We see that as in the manifold case, any differentiable orbifold with paracompact base space admits a Riemannian metric (Proposition 2.6.1) and we describe a way to define on Riemannian orbifolds objects familiar from Riemannian geometry of manifolds. At the end of this chapter we give a description of the universal cover and of the orbifold fundamental group for 2-dimensional orbifolds and we introduce the Euler number of a compact orbifold.

The material in this chapter is well-known and there are many excellent sources. Besides the original work of Satake [Sa1] and Thurston [T1], other good references are chapter 6 in [K], section 2.4 in [MM], the appendix of [CR] (for a somewhat different approach), and the article of Scott [Sc] which gives a detailed presentation of the geometric structure of 2-dimensional orbifolds.

2.1 Group actions

In this section we briefly recall some facts from the theory of group actions on a topological spaces and smooth manifold and we will mainly focus on the *properly discontinuous* actions (see below and also Proposition 2.1.2). A useful special case of a discontinuous action is the action of a finite group on a Hausdorff topological space and as we will see in the next section this will play an important role in understanding local properties of orbifolds. For a more detailed introduction to actions of discrete groups the reader may consult [B] and [T2].

We will begin with some formal definitions. Suppose Γ is a group and M is a topological space or a smooth manifold.

Definition 2.1.1. *An* action *of* Γ *on* M *is a map* $\Gamma \times M \to M$, $(\gamma, x) \mapsto \gamma.x$ *such that, for all* $x \in M$

(i) $(\gamma \cdot \delta).x = \gamma.(\delta.x)$ *for all* $\gamma, \delta \in \Gamma$, *and*

(ii) $1.x = x$, *where* $1 \in \Gamma$ *is the identity element.*

The first rule says that two elements of Γ, acting successively, act as the product of two elements, and the second says that the identity of the group acts as identity.

There are some standard notions associated with such an action. For $x \in M$, the set $\Gamma(x) = \{\gamma.x |\ \gamma \in \Gamma\} \subseteq M$ is called the *orbit* of x. We can introduce a relation on M by $x \sim y$ if and only if x and y are on the same orbit. It is easy to check that this is an equivalence relation. The space of equivalence classes of points of M will be called the *space of orbits* and will be denoted M/Γ (we agree to denote it like this even if the action here is considered from the left).

The elements of Γ which leave an element $x \in M$ fixed form a subgroup $\Gamma_x = \{\gamma \in \Gamma |\ \gamma.x = x\}$ called the *isotropy group* at x. It is easy to see that if x and y are on the same orbit, say $y = \gamma.x$, then their isotropy groups are conjugate, $\Gamma_y = \gamma.\Gamma_x.\gamma^{-1}$, and in fact any conjugate subgroup to Γ_x occurs as an isotropy group Γ_y to some element $y \in \Gamma(x)$. If $\Gamma_x = \Gamma$, then x is said to be a *fixed point* of the action. The set of fixed points of the action is often denoted M^Γ. A subset $N \subset M$ is called Γ-*invariant* if it is left invariant by the action of Γ, i.e. $\gamma.N = N$ for every $\gamma \in \Gamma$.

An action can be thought of as a homomorphism from Γ into $S(M)$, the symmetric group of M (i.e. the group of bijections from M to itself). We say that Γ acts by *homeomorphisms* if there is a homomorphism $\rho : \Gamma \to \mathrm{Homeo}(M)$, where $\mathrm{Homeo}(M)$ denotes the group of homeomorphisms of M with the group law given by composition of maps. The action is said to be by *diffeomorphisms* if the above homomorphism is defined into $\mathrm{Diffeo}(M)$, the subgroup of diffeomorphisms of M. In what follows we will consider this kind of action.

Here are some basic properties of group actions:

(i) The action of Γ on M is called *effective* if no element of the group, besides the identity element, fixes all the elements of the space, i.e.

$$\bigcap_{x \in M} \Gamma_x = \{1\},$$

or equivalently if the $\Gamma \to \mathrm{Diffeo}(M)$ is a monomorphism. In this case we can regard Γ as a group of diffeomorphisms.

(ii) The action of Γ on M is called *free* if no point of M is fixed by an element of Γ other than the identity, or equivalently if the map

$$\Gamma \times M \to M \times M,\ \ (\gamma, x) \mapsto (\gamma.x, x) \text{ is injective}$$

in the sense that $\gamma.x = x$ implies $\gamma = 1$ for any γ and any x.

(iii) The action of Γ on M is called *discrete* if Γ is a discrete subgroup of the group of homeomorphisms (diffeomorphisms) with the compact-open topology.

(iv) The action of Γ on M is said to have *discrete orbits* if every $x \in M$ has a neighborhood U such that the set $\{\gamma \in \Gamma |\ \gamma.x \in U\}$ is finite.

(v) The action of Γ on M is called *discontinuous* if every $x \in M$ has a neighborhood U such that the set $\{\gamma \in \Gamma |\ \gamma.U \cap U \neq \varnothing\}$ is finite.

(vi) Assume now that M is locally compact. The action of Γ on M is called *properly discontinuous* if for any compact sets K_1, K_2 in M, the set $\{\gamma \in \Gamma |\ \gamma.K_1 \cap K_2 \neq \varnothing\}$ is finite, or equivalently if the map

$$\Gamma \times M \longrightarrow M \times M,\ (\gamma, x) \mapsto (\gamma.x, x) \text{ is proper.}$$

Recall that a map is proper if the preimages of compact sets are compact. Here Γ is assumed endowed with the discrete topology. Recall also that a proper map between locally compact Hausdorff spaces is closed.

Note that on a locally compact space any properly discontinuous action is discontinuous and any discontinuous action has discrete orbits, but the converses are not true in general.

The following characterization of properly discontinuously actions will be useful (see also section 3.5 in [T2]).

Proposition 2.1.2. *The action of a group Γ on a locally compact space X is properly discontinuous if and only if all of the following hold:*

(i) the space of orbits M/Γ is Hausdorff with the quotient topology;

(ii) each $x \in M$ has finite isotropy group;

(iii) each $x \in M$ has a Γ_x-invariant neighborhood U such that $\Gamma_x = \{\gamma \in \Gamma |\ \gamma.U \cap U \neq \varnothing\}$.

Proof. ($\Rightarrow$) (i) Let x and x' be points in M with distinct orbits of Γ and let K be a compact neighborhood of x. Since, by hypothesis, the set $\{\gamma \in \Gamma |\ \gamma.x' \in K\}$ is finite we can find a neighborhood U of x which is disjoint from the orbit of x'. Then $\bigcup_{\gamma\in\Gamma} \gamma.U$ is a Γ-invariant neighborhood of x which does not contain x' (so it does not intersect the orbit through x'). Similarly we can find a Γ-invariant neighborhood of x' which does not contain x, hence M/Γ is Hausdorff with the quotient topology.
(ii) is immediate from the definition by considering $K_1 = K_2 = \{x\}$.
(iii) Let $K_1 = \{x\}$ and K_2 be a compact neighborhood of x. Since the action is properly discontinuous, the set $\{\gamma \in \Gamma |\ \gamma.x \in K_2\}$ is finite and contains Γ_x. Thus we can find a compact neighborhood of x, say K_2' such that $\{\gamma \in \Gamma |\ \gamma.x \in K_2'\} = \Gamma_x$. Consider now the set $K = \bigcup_{\gamma\in\Gamma_x} \gamma.K_2'$ which is a compact and Γ_x-invariant neighborhood of x. Applying the definition for $K_1 = K_2 = K$, the set $\{\gamma \in \Gamma |\ \gamma.K \cap K \neq \varnothing\}$ is finite and contains Γ_x. Then we can find a neighborhood of x in K, say U', such that $\{\gamma \in \Gamma |\ \gamma.U' \cap U' \neq \varnothing\} = \Gamma_x$ and by taking $U = \bigcup_{\gamma\in\Gamma_x} \gamma.U'$ we obtain the Γ_x-invariant satisfying (iii).
($\Leftarrow$) By (i), for any $x, x' \in M$ such that x and x' are not on the same orbit there are neighborhoods U and U' such that $\gamma.U \cap U' = \varnothing$ for any $\gamma \in \Gamma$. If x and x' are on the same orbit, then we can find neighborhoods of x and x', say U and U', such that the set $\{\gamma \in \Gamma |\ \gamma.U \cap U' \neq \varnothing\}$ is finite and in fact that it has the cardinality equal to the order of the isotropy group of x (or x'). Indeed, assume that $x = \delta.x'$ for some $\delta \notin \Gamma_x$, and consider U to be a neighborhood of x as given by (iii) and U' to be $\delta.U$. Then $\{\gamma \in \Gamma |\ \gamma.U \cap U' \neq \varnothing\} = \{\gamma \in \Gamma |\ \gamma.U \cap \delta.U \neq \varnothing\} = \{\gamma \in \Gamma |\ U \cap (\gamma^{-1} \cdot \delta).U \neq \varnothing\} = \delta.\Gamma_x$ which by (ii) is finite. Thus, for any $x, x' \in M$ we can find neighborhoods U and U' such that $\{\gamma |\ \Gamma.U \cap U' \neq \varnothing\}$ is at most finite. Let now K be any compact in M. Then $K \times K$ is compact in $M \times M$ and so has a finite cover with sets of the form $U \times U'$ where $\{\gamma |\ \Gamma.U \cap U' \neq \varnothing\}$ is finite. Therefore the set $\{\gamma \in \Gamma |\ \Gamma.K \cap K \neq \varnothing\}$ is finite, i.e. the action is properly discontinuous. ☐

Remark 2.1.3. In the above proposition, if the action is free and the space M is a smooth manifold, then M/Γ is also a smooth manifold and the map $M \to M/\Gamma$ is a covering.

Consider now a smooth manifold M and a group Γ acting by diffeomorphisms on it. Note that the action of Γ on M induces an action on the tangent bundle TM defined by $\gamma.v := (d\gamma)_x.v$, for any $\gamma \in \Gamma$, $v \in T_xM$, $x \in M$. It is well known that if the manifold M is connected and paracompact, it admits a Riemannian metric. In the case when the action of Γ on M is proper one can prove that there exists a Γ-invariant Riemannian metric on M. In the special case when Γ is a finite subgroup of Diffeo(M) we have the following.

Lemma 2.1.4. *Let M be a connected paracompact differentiable manifold and Γ be a finite subgroup of Diffeo(M) acting on M. Then there exists a Γ-invariant Riemannian metric on M.*

Proof. Choose a Riemannian metric g on M and define a new one by averaging g over Γ:

$$\rho_x(v, w) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} g_{\gamma.x}(\gamma.v, \gamma.w)$$

for any $v, w \in T_xM$ and any $x \in M$. Then ρ is a Riemannian metric on M and it is Γ-invariant. □

Recall that for $x \in M$ the isotropy group Γ_x is the subgroup of Γ of the elements which leave x fixed. If Γ_x is non trivial we say that x is a singular point. The collection of all singular points in M is denoted Σ_Γ and is called the singular set of M. Let $\gamma \in \Gamma$ and denote $\Sigma_\gamma = \{x \in M |\ \gamma.x = x\}$ the subset of M which is fixed by γ. Then the singular set is

$$\Sigma_\Gamma = \{x \in M |\ \Gamma_x \neq 1\} = \bigcup_{\gamma \in \Gamma, \gamma \neq 1} \Sigma_\gamma.$$

We say that a subset S of M is Γ-*stable* if it is connected and if for any $\gamma \in \Gamma$ we have either $\gamma.S = S$ or $\gamma.S \cap S = \varnothing$. The isotropy group of the Γ-stable set S is $\Gamma_S = \{\gamma \in \Gamma |\ \gamma.S = S\}$. Note that the Γ-stable subsets of M are exactly the connected components of Γ-invariant subsets of M and the following holds.

Proposition 2.1.5. *If Γ is finite, for any $x \in M$ there exists an arbitrarily small open Γ-stable neighborhood S of x such that $\Gamma_x = \Gamma_S$* (compare with (iii) in Proposition 2.1.2). *Hence, the open Γ-stable subsets of M form a basis for the topology of M.*

Proposition 2.1.6. *Let M be a connected, paracompact smooth manifold and Γ a finite subgroup of Diffeo(M). Then Σ_Γ is a closed set with empty interior. Moreover, the homomorphism $d_x : \Gamma_x \longrightarrow Aut(T_xM)$ is injective for every $x \in M$.*

Proof. The set

$$\Sigma_\Gamma = \bigcup_{\gamma\in\Gamma,\gamma\neq 1} \Sigma_\gamma = \{x \in M |\ \exists\gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.x = x\}$$

is obviously closed. It is the finite union of the sets of fixed points of a diffeomorphism, which are closed. An alternative way of seeing this is the following. Consider a sequence of points (x_n) in Σ_Γ which converges to a point x in M (we will prove that $x \in \Sigma_\Gamma$). Since each x_n is in Σ_Γ we can form a sequence of elements of Γ, γ_n such that $\gamma_n.x_n = x_n$ and $\gamma_n \neq 1$ for any n. But Γ is finite, so γ_n contains at least a constant subsequence which we will denote again γ_n and assume $\gamma_n = \gamma \neq 1$ for any n. The corresponding subsequence of x_n converges to x and moreover the sequence $\gamma_n.x_n = \gamma.x_n$ converges to $\gamma.x$. But $\gamma_n.x_n = x_n$ for any n so $\gamma_n.x_n$ converges to x also. Since the quotient space is Hausdorff we have that $\gamma.x = x$ and since $\gamma \neq 1$ we conclude that $x \in \Sigma_\Gamma$.

Let's prove now that it has empty interior. Consider the Γ-invariant Riemannian metric on M given by the above lemma, and consider the exponential map associated with this metric. Then, for each $x \in M$ there are $\varepsilon > 0$ and open neighborhood of x, U such that

$$\exp_x : B(0, \varepsilon) \longrightarrow U$$

is a diffeomorphism from the ε-ball centered at the origin in the tangent space T_xM to U. Since the metric is Γ-invariant, the action of the isotropy group Γ_x on the tangent space at x is orthogonal, i.e. $(d\gamma)_x$ is an orthogonal transformation of T_xM, and

$$\exp_x \circ (d\gamma)_x = \gamma \circ \exp_x,$$

for any $\gamma \in \Gamma_x$. In particular, if $(d\gamma)_x = \text{id}$ then the restriction $\gamma|_U = \text{id}$. Since M is connected this implies $\gamma = 1$. To see this, consider the set

$$A = \{y \in M |\ \gamma.y = y \text{ and } (d\gamma)_y = \text{id}\}.$$

Then, $A \neq \varnothing$ (since $x \in A$) and it is obviously closed. It is also open. If we assume that $y \in A$ then the condition $\gamma.y = y$ implies that $\gamma \in \Gamma_y$ and the condition that $(d\gamma)_y = \mathrm{id}$ implies that the restriction of γ to an open neighborhood of y is 1. Hence the whole neighborhood of y is contained in A, i.e. A is open. The connectedness of M implies $A = M$ and so, $\gamma = 1$ on M.

This proves that $d_x : \Gamma_x \longrightarrow \mathrm{Aut}(T_xM)$ is injective, which in particular implies that Σ_γ has empty interior, for any $\gamma \neq 1$. Indeed, let $x \in \Sigma_\gamma$ and assume that there exists an open neighborhood U of x in Σ_γ. Then the restriction $\gamma|_U = 1$ and as we have seen this implies $\gamma = 1$ on M. Since Γ is finite, Σ_Γ has empty interior also and the proof is complete. □

Remark 2.1.7. The above proposition implies that the only diffeomorphism of finite order on a connected manifold which fixes an open set is the identity.

Proposition 2.1.8. *Let M be a connected, paracompact smooth manifold, Γ a finite subgroup of Diffeo(M) and φ the natural projection $M \longrightarrow M/\Gamma$. Let V be a nonempty, open, connected subset of M and $f : V \longrightarrow M$ a diffeomorphism onto its image such that $\varphi \circ f = \varphi|_V$. Then there exists a unique $\gamma \in \Gamma$ such that $f = \gamma|_V$.*

Proof. Consider on M a Γ-invariant Riemannian metric given by Lemma 2.1.4. By the previous result, the set $M_0 = M \setminus \Sigma_\Gamma$ is open and dense in M. For any $x \in V \cap M_0$, the condition $\varphi \circ f = \varphi|_V$ implies that there is a unique $\gamma \in \Gamma$ such that $f(x) = \gamma.x$ and on a sufficiently small connected neighborhood of x in $V \cap M_0$ we have $(df)_x = (d\gamma)_x$. Since the metric is Γ-invariant, $(df)_x$ preserves also the metric. So, the restriction of f to $V \cap M_0$ is a Riemannian isometry. Since $V \cap M_0$ is dense in V, by continuity f is a Riemannian isometry on V and $f = \gamma$ on a neighborhood of x in V. But V is connected, therefore the two isometries f and $\gamma|_V$ are equal. □

Remark 2.1.9. The last two results are still true in the case where M is assumed to be a connected topological manifold and Γ a finite subgroup of Homeo(M). This fact is a consequence of a result of Newman (see [T1] and the references there) which states that a nontrivial homeomorphism of a manifold which fixes an open set cannot have finite order (compare with Remark 2.1.7).

2.2 Traditional approach to orbifolds

In the following sections we present the definitions and the basic properties of orbifolds from a traditional approach. The point of view here is that an orbifold structure generalizes the manifold topological (differentiable) structure by allowing mild singularities. They are locally modeled by open sets in $\mathbb{R}^n$ modulo an action of some finite group of homeomorphisms or diffeomorphisms. Moreover the group is not fixed and can be changed as we pass from one point of the orbifold to another. An isomorphism of coordinate neighborhoods corresponds to equivariant actions of the same group on $\mathbb{R}^n$.

We begin with the formal definition of an orbifold as given in [T1].

Definition 2.2.1. *A topological (differentiable) n-dimensional* orbifold Q *consists of a Hausdorff space, denoted* $|Q|$ *and called the underlying space of* Q*, together with an additional structure given by the following*

(i) a countable basis of open sets $\{U_i\}_{i\in I}$ *which is closed under finite intersections and such that* $|Q| = \bigcup_{i\in I} U_i$*;*

(ii) to each U_i *is associated a finite group* Γ_i*, an action of* Γ_i *on some open subset* $\widetilde{U}_i$ *of* $\mathbb{R}^n$ *by homeomorphisms (diffeomorphisms) and a homeomorphism* $\varphi_i : \widetilde{U}_i/\Gamma_i \to U_i$*;*

(iii) whenever $U_i \subseteq U_j$*, there is a monomorphism*

$$\lambda_{ij} : \Gamma_i \to \Gamma_j$$

which induces an isomorphism between $\{\gamma \in \Gamma_i |\, \gamma.\widetilde{U}_i = \widetilde{U}_i\}$ *and* $\{\gamma \in \Gamma_j |\, \gamma.\widetilde{U}_j = \widetilde{U}_j\}$*, and a* λ_{ij}*-equivariant (smooth) embedding*

$$\widetilde{\varphi}_{ij} : \widetilde{U}_i \to \widetilde{U}_j$$

(i.e., for any $\gamma \in \Gamma_i$*,* $\widetilde{\varphi}_{ij}(\gamma.\widetilde{x}) = \lambda_{ij}(\gamma).\widetilde{\varphi}_{ij}(\widetilde{x})$ *for all* $\widetilde{x} \in \widetilde{U}_i$*) such that the following diagram commutes*

$$
\begin{array}{ccc}
\widetilde{U}_i & \xrightarrow{\widetilde{\varphi}_{ij}} & \widetilde{U}_j \\
\downarrow & & \downarrow \\
\widetilde{U}_i/\Gamma_i & \xrightarrow{\varphi_{ij} = \widetilde{\varphi}_{ij}/\Gamma_i} & \widetilde{U}_j/\Gamma_i \\
 & & \downarrow \lambda_{ij} \\
\varphi_i \downarrow & & \widetilde{U}_j/\Gamma_j \\
 & & \downarrow \varphi_j \\
U_i & \xrightarrow{\subset} & U_j
\end{array}
$$

Note that the actions of the Γ_i's on $\widetilde{U}_i$'s can be always assumed to be effective. Indeed, the set of elements in the group which act trivially form a normal subgroup and the action of the quotient is effective with the same space of orbits. Then each Γ_i can be regarded as a finite subgroup of Homeo$(\widetilde{U}_i)$ in the topological case and of Diffeo$(\widetilde{U}_i)$ in the differentiable case. In literature, an orbifold for which each Γ_i acts effectively is referred as *a reduced orbifold.*

The triple $(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)$ as in (ii) is called an *orbifold coordinate chart over* U_i or *uniformizing system of* U_i and the pair $(\lambda_{ij}, \widetilde{\varphi}_{ij}) : (U_i, \widetilde{U}_i/\Gamma_i, \varphi_i) \to (U_j, \widetilde{U}_j/\Gamma_j, \varphi_j)$ as in (iii) an *injection* between charts.

Remark 2.2.2. It easy to see that the composition of two injections is again an injection. We regard the maps $\widetilde{\varphi}_{ij}$ as being defined up to compositions with elements of Γ_j and the maps λ_{ij} as being defined up to conjugation by elements of Γ_j in the sense that if γ is an element of Γ_j then $\gamma.\widetilde{\varphi}_{ij}$ and $\gamma.\lambda_{ij}.\gamma^{-1}$ also satisfy the requirements in (iii) of the definition above. This is a consequence of Proposition 2.1.8 in the smooth case and of Remark 2.1.9 in the topological case. Moreover if $(\lambda_{ij}, \widetilde{\varphi}_{ij})$ and $(\lambda'_{ij}, \widetilde{\varphi}'_{ij})$ denote two injections between the same orbifold charts, then there is a unique $\gamma \in \Gamma_j$ such that $\widetilde{\varphi}'_{ij} = \gamma.\widetilde{\varphi}_{ij}$ and in this case $\lambda'_{ij} = \gamma \cdot \lambda_{ij} \cdot \gamma^{-1}$. In particular if $i = j$, $\widetilde{\varphi}_{ii}$ is an element $\gamma_i \in \Gamma_i$ and then λ_{ii} is just conjugation with γ_i. In general it is not true that whenever $U_i \subset U_j \subset U_k$ we have $\widetilde{\varphi}_{jk} \circ \widetilde{\varphi}_{ij} = \widetilde{\varphi}_{ik}$, but there exists an element

$\gamma \in \Gamma_k$ such that $\widetilde{\varphi}_{jk} \circ \widetilde{\varphi}_{ij} = \gamma.\widetilde{\varphi}_{ik}$ and $\lambda_{jk} \circ \lambda_{ij} = \gamma \cdot \lambda_{ik} \cdot \gamma^{-1}$.

As in the manifold case, the covering $\{U_i\}_i$ is not an intrinsic part of the orbifold structure. Two coverings will give the same orbifold structure if they can be consistently combined to give a covering which still satisfies the properties (ii) and (iii). By an *orbifold* one should understand the orbifold with the structure given by a such maximal cover.

We begin with the simplest examples of orbifolds, more interesting examples are given in 2.6.4.

Proposition 2.2.3. *Let Γ be a group acting properly discontinuously on a manifold M. Then the quotient space M/Γ has a natural orbifold structure.*

Proof. We have seen already, in Proposition 2.1.2 (i), that under these assumptions the quotient space M/Γ is Hausdorff. We will construct an orbifold atlas for M/Γ, $\mathscr{U}$ satisfying the conditions (i)-(iii) in definition.

Let $\pi : M \to M/\Gamma$ denote the quotient map and let $x \in M/\Gamma$. Choose $\widetilde{x} \in M$ such that $\pi(\widetilde{x}) = x$ and let $\Gamma_{\widetilde{x}} = \{\gamma \in \Gamma |\ \gamma.\widetilde{x} = \widetilde{x}\}$ denote the isotropy group of $\widetilde{x}$. By Proposition 2.1.2 (iii), there exists an open connected neighborhood of $\widetilde{x}$, $\widetilde{U}_{\widetilde{x}}$ which is invariant to $\Gamma_{\widetilde{x}}$ and disjoint from its translates by elements of Γ not in $\Gamma_{\widetilde{x}}$. Then the restriction

$$\pi|_{\widetilde{U}_{\widetilde{x}}} : \widetilde{U}_{\widetilde{x}} \to U_x := \widetilde{U}_{\widetilde{x}}/\Gamma_{\widetilde{x}}$$

is a homeomorphism. Let $\widetilde{\mathscr{U}}$ be a maximal atlas on M. By eventually shrinking $\widetilde{U}_{\widetilde{x}}$, we can assume that $\widetilde{U}_{\widetilde{x}} \in \widetilde{\mathscr{U}}$ and so there is a homeomorphism $\varphi : \widetilde{U}_{\widetilde{x}} \to \varphi(\widetilde{U}_{\widetilde{x}}) \subset \mathbb{R}^n$. Then the composition

$$\varphi_x := (\varphi/\Gamma_{\widetilde{x}})^{-1} \circ \pi|_{\widetilde{U}_{\widetilde{x}}} : \varphi(\widetilde{U}_{\widetilde{x}})/\Gamma_{\widetilde{x}} \to U_x$$

is again a homeomorphism.

Hence $\{U_x |\ x \in M/\Gamma\}$ forms an open cover for M/Γ and each U_x has an uniformizing system by $(U_x, \varphi(\widetilde{U}_{\widetilde{x}})/\Gamma_{\widetilde{x}}, \varphi_x)$. In order to get a suitable cover of M/Γ we should augment the above cover by adjoining finite intersections. Let now $x_1, x_2, \dots\ x_k \in M/\Gamma$ such that the corresponding sets $U_{x_1}, U_{x_2}, \cdots\ U_{x_k}$ as above satisfy $U_{x_1} \cap U_{x_2} \cap$

$\cdots \cap U_{x_k} \neq \varnothing$. Then, since Γ acts by permutations on the set of connected components of $\pi^{-1}(U_{x_1} \cap U_{x_2} \cap \cdots \cap U_{x_k})$, there exist $\gamma_1, \gamma_2, \ldots \gamma_k \in \Gamma$ such that $\gamma_1.\widetilde{U}_{\widetilde{x}_1} \cap \gamma_2.\widetilde{U}_{\widetilde{x}_2} \cap \cdots \cap \gamma_k.\widetilde{U}_{\widetilde{x}_k} \neq \varnothing$, where as above $\widetilde{U}_{\widetilde{x}_i}$ denote neighborhoods of $\widetilde{x}_i \in \pi^{-1}(x_i)$ invariant by $\Gamma_{\widetilde{x}_i}$. Then this intersection may be taken to be

$$U_{x_1} \widetilde{\cap \cdots} \cap U_{x_k}$$

which is obviously invariant to the action of the finite subgroup

$$\gamma_1 \cdot \Gamma_{\widetilde{x}_1} \cdot \gamma_1^{-1} \cap \cdots \cap \gamma_k \cdot \Gamma_{\widetilde{x}_k} \cdot \gamma_k^{-1}.$$

In this way we obtain a cover $\mathscr{U}$ of M/Γ which satisfies (i) and (ii) of Definition 2.2.1.

We will show that the condition (iii) in the definition is also satisfied. Consider U and U' in $\mathscr{U}$ such that $U' \subset U$ and let $x \in U'$ and $\widetilde{x} \in M$ such that $\pi(\widetilde{x}) = x$. For x and U, consider $\widetilde{U}_{\widetilde{x}}$ and $\Gamma_{\widetilde{x}}$ as above and choose $\widetilde{U}'_{\widetilde{x}}$ (note that it should be the neighborhood for the same lift of x, $\widetilde{x}$). In order to prove that there is an embedding between the two charts, it suffices to prove that $\widetilde{U}'_{\widetilde{x}} \subset \widetilde{U}_{\widetilde{x}}$. To see this, assume it is not true and choose $\widetilde{y} \in \widetilde{U}'_{\widetilde{x}} \setminus \widetilde{U}_{\widetilde{x}}$. Then there should exist $\gamma \in \Gamma_{\widetilde{x}}$ such that $\gamma.\widetilde{y} \in \widetilde{U}' \cap \widetilde{U}$, since $\pi(\widetilde{y}) = y \in U' \subset U$. But both $\widetilde{U}'_{\widetilde{x}}$ and $\widetilde{U}_{\widetilde{x}}$ are $\Gamma_{\widetilde{x}}$-invariant and hence so is $\widetilde{U}'_{\widetilde{x}} \cap \widetilde{U}_{\widetilde{x}}$. This means that $\widetilde{y} \in \widetilde{U}'_{\widetilde{x}} \cap \widetilde{U}_{\widetilde{x}}$ which contradicts the fact that $\widetilde{y} \in \widetilde{U}'_{\widetilde{x}} \setminus \widetilde{U}_{\widetilde{x}}$, i.e. proves $\widetilde{U}'_{\widetilde{x}} \subset \widetilde{U}_{\widetilde{x}}$. □

Note that the orbifold structure on M/Γ is *natural* in the sense that it depends only on the action of the group Γ and not on the choice of the atlas $\widetilde{\mathscr{U}}$ on M. It is called *the orbifold quotient of M by the properly discontinuous action of* Γ.

Definition 2.2.4. *An orbifold is called* good *if it arises as the global quotient by a discrete group acting properly discontinuously on a manifold. If the group can be chosen to be finite, then the orbifold is called* very good.

Remark 2.2.5. Sometimes, we refer to good orbifolds as being *developable.* In fact, another approach is to define orbifolds using charts for open sets U_i of the form (X_i, Γ_i), where each X_i is a manifold and Γ_i is a discrete group acting properly discontinuously on X_i with X_i/Γ_i homeomorphic to U_i. From this point of view, an orbifold Q is good (or developable) if it admits such an atlas with a single chart. As in [BH], we call the pair (X_i, q_i) a *uniformizing chart* of U_i, where $q_i : X_i \to U_i$ is a continuous map which induces a homeomorphism from X_i/Γ_i onto U_i. Then the compatibility condition (iii) in Definition 2.2.1 becomes: for all $x_i \in X_i$ and $x_j \in X_j$ such that $q_i(x_i) = q_j(x_j)$, there is a homeomorphism (diffeomorphism) h from an open connected neighborhood W of x_i to a neighborhood of x_j, such that $q_j \circ h = q_i|_W$. This time, we will call h a *change of charts*, and as we already saw it is defined up to composition with elements of Γ_j and in particular, if $i = j$ then h is the restriction of an element of Γ_i.

Similarly, we define *orbifolds with boundary* by taking as uniformizing charts connected manifolds with boundary (or equivalently open sets in the upper half plane $\mathbb{R}^n_{\geq 0} = \{(x_1, x_2, \dots x_n) |\ x_n \geq 0\}$). The *boundary* of such an orbifold consists of points $x \in |Q|$ that correspond to the boundary of X_i (or to $\mathbb{R}^n_0 = \{(x_1, x_2, \dots, x_n) |\ x_n = 0\}$). As in the manifold case, the boundary of an orbifold is an orbifold without boundary. A compact orbifold without boundary is called *closed*.

2.3 The singular set

Let Q be an orbifold and let $x \in Q$. Within an orbifold chart $(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)$ we can associate to x a group $\Gamma^{(i)}_x$ well defined up to isomorphism in the following way. Consider $\widetilde{x}, \widetilde{x}' \in \widetilde{U}_i$ such that $\varphi_i(\widetilde{x}) = \varphi_i(\widetilde{x}') = x$. Then their isotropy groups $\Gamma^{(i)}_{\widetilde{x}}$ and $\Gamma^{(i)}_{\widetilde{x}'}$ are conjugate to each other, since $\widetilde{x}$ and $\widetilde{x}'$ are on the same orbit. Denote this subgroup by $\Gamma^{(i)}_x$ and since it is independent of the choice of the lifts of x we will refer to it as the isotropy group of x in U_i.

Consider now $(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)$ and $(U_j, \widetilde{U}_j/\Gamma_j, \varphi_j)$ two orbifold charts containing x assume that $U_i \subset U_j$. Let $(\lambda_{ij}, \widetilde{\varphi}_{ij})$ denote the injection between them. Note that if x has non-trivial isotropy in U_i (i.e. $\Gamma^{(i)}_x \neq 1$) then it has non-trivial isotropy in U_j. Indeed, since the embedding $\widetilde{\varphi}_{ij}$ is λ_{ij}-equivariant, $\Gamma^{(j)}_x$ contains the subgroup $\lambda_{ij}(\Gamma^{(i)}_x)$ which is not trivial since λ_{ij} is injective (the inclusion above is considered up to isomorphism, i.e. at least one of the isomorphic subgroups defining $\Gamma^{(j)}_x$ should contain $\lambda_{ij}(\Gamma_x^{(i)})$, and this is fine since both $\Gamma_x^{(j)}$ and λ_{ij} are defined up to conjugation with elements of Γ_j).

Define the *isotropy group* Γ_x *at* x to be the smallest isotropy group of x corresponding to an orbifold chart containing x, i.e. the set

$$\Gamma_x = \bigcap_{x \in U_i} \Gamma_i.$$

It is obvious that Γ_x is always finite. (Equivalently the isotropy group at x can be defined as the *germ* of the action of $\Gamma_x^{(i)}$ at x.)

A point $x \in Q$ is called a *singular point* if it has non-trivial isotropy i.e. if $\Gamma_x \neq \{1\}$, and it is called a *regular point* otherwise. Define the *singular set* of an orbifold to be

$$\Sigma_Q := \{x \in Q \mid \Gamma_x \neq \{1\}\}.$$

Then we say that an orbifold is a manifold if the singular set is empty. The following result concerning the singular set of an orbifold holds:

Proposition 2.3.1. *The singular set of an orbifold is closed and nowhere dense.*

Proof. Let $(U, \widetilde{U}/\Gamma, \varphi)$ be any orbifold chart that has nonempty intersection with the singular set. Then

$$\begin{aligned}
\Sigma_Q \cap U &= \{x \in U |\ \Gamma_x \neq 1\} \\
&= \{x \in U |\ \exists \gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.\widetilde{x} = \widetilde{x} \text{ for some } \widetilde{x} \in \varphi^{-1}(x)\} \\
&= \bigcup_{\widetilde{x} \in \widetilde{U}} \{\varphi(\widetilde{x}) |\ \exists \gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.\widetilde{x} = \widetilde{x}\} \\
&= \varphi(\bigcup_{\widetilde{x} \in \widetilde{U}} \{\widetilde{x} |\ \exists \gamma \in \Gamma, \gamma \neq 1 \text{ such that } \gamma.\widetilde{x} = \widetilde{x}\}) \\
&= \varphi(\Sigma_\Gamma).
\end{aligned}$$

Since Σ_Γ is closed with empty interior (see Proposition 2.1.6 and Remark 2.1.9) and φ is a homeomorphism $\Sigma_\Gamma \cap U$ is closed and has empty interior. Hence Σ_Q is closed and since $|Q|$ is locally compact and Hausdorff, $\Sigma_Q = \bigcup_i \Sigma_Q \cap U_i$ has empty interior. $\square$

Note that in general the singular set is not a manifold and it may have several connected components of different dimension. From the definition we can see that any orbifold is locally compact. If we assume that in any uniformizing system the elements of the group have fixed point set of codimension at least two (i.e. the singular set has the codimension at least two) then the regular set is also locally path connected. In this case an orbifold is connected if and only if its regular set is path connected.

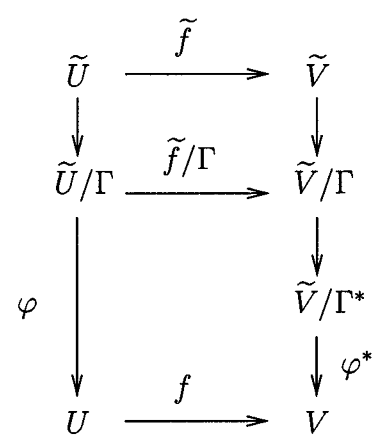
2.4 Maps between orbifolds

Consider two orbifolds Q and Q' and a continuous map $f : |Q| \to |Q'|$ between their underlying spaces.

Let $x \in |Q|$ and $y = f(x) \in |Q'|$ and let V be an open neighborhood of y and U an open neighborhood of x such that $f(U) \subset V$. Let $(V, \widetilde{V}/\Gamma^*, \varphi^*)$ be an uniformizing system over V and $(U, \widetilde{U}/\Gamma, \varphi)$ an uniformizing system over U. Corresponding to these uniformizing systems, a *continuous (resp. smooth) lifting* of $f|_U : U \to V$ is a continuous (resp. smooth) map

$$\widetilde{f} : \widetilde{U} \to \widetilde{V}$$

such that $\varphi^* \circ \widetilde{f} = f \circ \varphi$ and for any $\gamma \in \Gamma$ there exists $\gamma^* \in \Gamma^*$ satisfying $\gamma^*.\widetilde{f}(\widetilde{x}) = \widetilde{f}(\gamma.\widetilde{x})$ for any $\widetilde{x} \in \widetilde{U}$.



For a different choice of uniformizing systems $(V, \widetilde{V}'/\Gamma^{*'}, \varphi^{*'})$ over V, and $(U, \widetilde{U}'/\Gamma', \varphi')$ over U, we say that the lifting $\widetilde{f}' : \widetilde{U}' \to \widetilde{V}'$ is *isomorphic* to $\widetilde{f}$ if there are bijections

$$(\lambda, \widetilde{\varphi}) : (U, \widetilde{U}/\Gamma, \varphi) \to (U, \widetilde{U}'/\Gamma', \varphi')$$

and

$$(\lambda^*, \widetilde{\varphi}^*) : (V, \widetilde{V}/\Gamma^*, \varphi^*) \to (V, \widetilde{V}'/\Gamma^{*'}, \varphi^{*'})$$

such that $\lambda^* \circ \widetilde{f} = \widetilde{f}' \circ \lambda$.

Let now $x_0 \in U$ and $(U_0, \widetilde{U}_0/\Gamma_0, \varphi_0)$ be a uniformizing system over an open neighborhood $U_0 \subset U$ of x_0 and $(V_0, \widetilde{V}_0/\Gamma_0^*, \varphi_0^*)$ a uniformizing system over a neighborhood $V_0 \subset V$ of $f(x_0)$ such that $f(U_0) \subset V_0$. The lifting $\widetilde{f}$ will induce a lifting

$$\widetilde{f}_0 : \widetilde{U}_0 \to \widetilde{V}_0$$

of $f|_{U_0} : U_0 \to V_0$ in the following way. For any injection

$$(\lambda_0, \widetilde{\varphi}_0) : (U_0, \widetilde{U}_0/\Gamma_0, \varphi_0) \to (U, \widetilde{U}/\Gamma, \varphi),$$

consider the map $\widetilde{f} \circ \lambda_0 : \widetilde{U}_0 \to \widetilde{V}$ and note that $(\varphi^* \circ \widetilde{f} \circ \lambda_0)(\widetilde{U}_0) \subset V_0$ which implies $(\widetilde{f} \circ \lambda_0)(\widetilde{U}_0) \subset (\varphi^*)^{-1}(V_0)$. Therefore there is an injection

$$(\lambda_0^*, \widetilde{\varphi}_0^*) : (V_0, \widetilde{V}_0/\Gamma_0^*, \varphi_0^*) \to (V, \widetilde{V}/\Gamma^*, \varphi^*)$$

such that $(\widetilde{f} \circ \lambda_0)(\widetilde{U}_0) \subset \lambda_0^*(\widetilde{V}_0)$. Define now

$$\widetilde{f}_0 = (\lambda_0^*)^{-1} \circ \widetilde{f} \circ \lambda_0,$$

which is the induced lifting of $f|_{U_0} : U_0 \to V_0$. Note that different choices of injections $(\lambda_0, \widetilde{\varphi}_0)$ give isomorphic liftings. We say that two liftings are *equivalent* at a point x if they induce isomorphic liftings on a smaller neighborhood of x.

A *continuous (resp. smooth) lifting* of $f : |Q| \to |Q'|$ is the following: given any point $y \in |Q'|$ and any uniformizing chart $(V, \widetilde{V}/\Gamma^*, \varphi^*)$ at y there exists a uniformizing chart $(U, \widetilde{U}/\Gamma, \varphi)$ at $x \in f^{-1}(y)$ and a continuous (resp. smooth) lifting $\widetilde{f}_x : \widetilde{U} \to \widetilde{V}$ of $f|_U : U \to V$ such that for any $x' \in U$ and any uniformizing chart $(U', \widetilde{U}'/\Gamma', \varphi')$ at x' with $U' \subset U$, the lifting $\widetilde{f}_{x'} : \widetilde{U}' \to \widetilde{V}$ of $f|_{U'} : U' \to V$ is *isomorphic* with the *induced* one on $\widetilde{U}'$ from $\widetilde{f}_x$. We say that two liftings of $f : |Q| \to |Q'|$ are *equivalent* if their local liftings are equivalent at each point in $|Q|$.

Definition 2.4.1. *A continuous (resp. smooth)* orbifold map *between orbifolds $Q \to Q'$ is an equivalence class of continuous (resp. smooth) liftings of a continuous map between their underlying spaces $|Q| \to |Q'|$.*

We will denote by $\widetilde{f} : Q \to Q'$ an orbifold map whose underlying continuous map is $f : |Q| \to |Q'|$. Note that two non-isomorphic orbifold maps might have the same underlying continuous map (e.g. Example 4.1.6b in [CR])

We will define a particular kind of orbifold map called *a good map* (see [CR] Definition 4.4.1). The advantage of using these maps is that we can define the pull-back bundles (which cannot be done using general orbifold maps). We will further see that good maps correspond to morphisms in the category of groupoids (see section 3.2). In fact this notion of good maps was previously found by Moerdijk and Pronk in [MP], where they were called strict maps.

Let $\widetilde{f} : Q \to Q'$ be an orbifold map with underlying continuous function $f : |Q| \to |Q'|$. Suppose that there is an atlas $\mathscr{U}$ for Q and a collection of open subsets $\mathscr{U}'$ of Q' such that there is a one-to-one correspondence between the elements of $\mathscr{U}$ and $\mathscr{U}'$, say $U \leftrightarrow U'$, with $f(U) \subset U'$ and $U_1 \subset U_2$ implies $U_1' \subset U_2'$. Moreover, there is a collection of liftings of f such that $\widetilde{f}_{UU'} : \widetilde{U} \to \widetilde{U}'$ satisfies that for each injection

$$(\lambda, \widetilde{\varphi}) : (U_1, \widetilde{U}_1/\Gamma_1, \varphi_1) \to (U_2, \widetilde{U}_2/\Gamma_2, \varphi_2)$$

there is another injection associated to it

$$((\nu(\lambda), \nu(\widetilde{\varphi})) : (U_1', \widetilde{U}_1'/\Gamma_1', \varphi_1') \to (U_2', \widetilde{U}_2'/\Gamma_2', \varphi_2')$$

such that

$$\widetilde{f}_{U_1U_1'} \circ \lambda = \nu(\lambda) \circ \widetilde{f}_{U_2U_2'},$$

and for any composition of injections $\lambda' \circ \lambda$ we have $\nu(\lambda' \circ \lambda) = \nu(\lambda') \circ \nu(\lambda)$. The collection of liftings $\{\widetilde{f}_{UU'}, \nu\}$ defines a lifting of f. If this lifting is in the same equivalence class as $\widetilde{f}$, then the collection $\{\widetilde{f}_{UU'}, \nu\}$ is called a *compatible system* of $\widetilde{f}$.

Definition 2.4.2. *An orbifold map is called* good *if it admits a compatible system.*

The real line $\mathbb{R}$ as a smooth manifold is trivially an orbifold. The smooth orbifold maps $f : Q \to \mathbb{R}$ are called *smooth functions* on the orbifold Q. Note that an orbifold function is smooth if and only if the map $f \circ \varphi$ is smooth for any orbifold chart $(U, \widetilde{U}/\Gamma, \varphi)$ in an orbifold atlas of Q. A (smooth) map from $\mathbb{R}$ (or an interval I) into an orbifold Q is called a *(smooth) path* in Q.

Similarly we can define *immersions*, *submersions* and *embeddings* between orbifolds as differentiable maps between orbifolds that locally lift to immersions, submersions and embeddings, respectively. A *suborbifold* $Q' \subset Q$ is an orbifold Q' together with an orbifold embedding $i : Q' \hookrightarrow Q$.

2.5 The tangent space to an orbifold

Suppose now that $Q = M/\Gamma$ is a good orbifold. As we have seen in the previous section, we can extend the action of Γ on M to an action on TM by setting $\gamma.(\widetilde{x}, v) := (\gamma.\widetilde{x}, d(\gamma)_{\widetilde{x}} v)$, for all $\gamma \in \Gamma$ and $(\widetilde{x}, v) \in TM$. The quotient of TM by this action is the tangent bundle TQ of the orbifold Q. As in Proposition 2.2.3, it inherits a natural orbifold structure. For $x \in Q$, let $\widetilde{x} \in M$ denote one of its lifts. By taking the differentials $(d\gamma)_{\widetilde{x}}$ of the elements γ in the isotropy group Γ_x, we obtain a new group which acts on $T_{\widetilde{x}}M$. Since the group is independent of the choice of the lift, we will denote it by Γ_{x^*}. Hence the fiber in TQ above $x \in Q$ is $T_{\widetilde{x}}M/\Gamma_{x^*}$ and is denoted T_xQ. Because T_xQ will not be a vector space at the singular points, we will call it the *tangent cone to Q at x*.

Since any orbifold is locally good, the construction above gives a local way to work with tangent cones to orbifolds. So, if we consider an orbifold atlas $(X_i, q_i)_{i \in I}$

(see Remark 2.2.5) for the orbifold structure on Q, by making a quotient space of the tangent bundle TX_i over X_i by Γ_i as above, we obtain a $2n$-dimensional orbifold. We can easily patch these orbifolds together to obtain a $2n$-dimensional orbifold TQ with a projection map $p : TQ \to Q$ such that the inverse image of a point in the orbifold is a vector space modulo a finite group action (the tangent cone). A full description of the tangent bundle, as well as general bundles over orbifolds will be given in the next chapter.

2.6 Riemannian orbifolds

Let Q be a differentiable orbifold and let $\mathscr{U} = \{(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)\}_{i\in I}$ be the maximal orbifold atlas on Q.

Definition 2.6.1. *A* Riemannian metric *on the orbifold Q is a collection $\rho = (\rho_i)$, where ρ_i is a Riemannian metric on $\widetilde{U}_i$ such that any embedding $\widetilde{\varphi}_{ij}$ coming from an injection between orbifold charts $(U_{i,j}, \widetilde{U}_{i,j}/\Gamma_{i,j}, \varphi_{i,j})$ is an isometry as a map between $(\widetilde{U}_i, \rho_i)$ to $(\widetilde{U}_j, \rho_j)$. An orbifold with such a Riemannian metric is called a* Riemannian orbifold.

Remark 2.6.2. Note that the Riemannian metrics ρ_i on $\widetilde{U}_i$ are Γ_i-invariant, so locally Riemannian orbifolds look like the quotient of a Riemannian manifold by a finite group of isometries. By a suitable choice of coordinate charts it can be assumed that the local group actions are by finite subgroups of $O(n)$ for a general n-dimensional Riemannian orbifold, and by finite subgroups of $SO(n)$ for orientable Riemannian orbifolds.

As in the manifold case, the following proposition holds.

Proposition 2.6.3. *Any differentiable orbifold admits a Riemannian metric.*

Proof. Let Q be an orbifold and let $\{(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)\}_{i\in I}$ denote an orbifold atlas on it. Since $|Q|$ is paracompact we may assume that the cover $(U_i)_{i\in I}$ is locally finite. Then there is a smooth partition of unity $f_i : U_i \to \mathbb{R}$ subordinate to it. Indeed, we can choose on each $\widetilde{U}_i$ a Γ_i-invariant, non-negative smooth function $\widetilde{h}_i : \widetilde{U}_i \to \mathbb{R}$ such that $h_i = \widetilde{g}_i \circ \varphi_i$ can be extended over $|Q|$ by zeroes and such that $\{supp(h_i) \subset U_i : i \in I\}$ still covers $|Q|$. Denote $h(x) = \sum_{i \in I} h_i(x) \neq 0$ for all $x \in Q$ and consider $f_i := h_i/h$. Then $\{f_i : i \in I\}$ is a smooth partition of unity subordinate to the cover U_i.

Consider now an arbitrary Riemannian metric g_i on each $\widetilde{U}_i$. By Lemma 2.1.4 there exists a Γ_i-invariant Riemannian metric α_i on each $\widetilde{U}_i$ obtained from g_i by averaging over Γ_i. For any $i \in I$, define a new Riemannian metric ρ_i on $\widetilde{U}_i$ as follows:

$$(\rho_i)_{\widetilde{x}}(v, w) := \sum_{j \in I} f_j(\varphi_i(\widetilde{x}))(\alpha_j)_{\widetilde{\varphi}_{ij}(\widetilde{x})}\big(d(\widetilde{\varphi}_{ij})_{\widetilde{x}}(v), d(\widetilde{\varphi}_{ij})_{\widetilde{x}}(w)\big)$$

for any $\widetilde{x} \in \widetilde{U}_i$ and any $v, w \in T_{\widetilde{x}}\widetilde{U}_i$ and where $\widetilde{\varphi}_{ij}$ is an embedding coming from an injection between $(U_i, \widetilde{U}_i/\Gamma_i, \varphi_i)$ and any $(U_j, \widetilde{U}_j/\Gamma_j, \varphi_j)$, $j \in I$. Then, Lemma 2.1.4 together with the second part of the Remark guarantee that the Riemannian metric defined in this way is well defined, i.e. it is independent of the choice of the embedding between the uniformizing charts. It is also easy to check that each embedding is an isometry, hence the collection $\rho = (\rho_i)$ is defines a Riemannian metric in the sense of the definition above. □

Similar to Riemannian metric we can define general tensor fields for orbifolds. So, in an uniformizing system $(U_i, \widetilde{U}_i/\Gamma_i, \varphi)_i)$, for any tensor field $\widetilde{\omega}_i$ on $\widetilde{U}_i$ by pre-composing with an element $\gamma \in \Gamma_i$ we obtain a new tensor field $\widetilde{\omega}_i{}^{\gamma}$ on $\widetilde{U}_i$. Then, by averaging, we obtain a Γ_i-invariant tensor field on $\widetilde{U}_i$,

$$\widetilde{\omega}_i{}^{\Gamma_i} := \frac{1}{|\Gamma_i|} \sum_{\gamma \in \Gamma_i} \widetilde{\omega}_i{}^{\gamma}.$$

Such a Γ_i invariant tensor field on $\widetilde{U}_i$ gives a tensor field ω on U_i.

A *smooth tensor field* on an orbifold is one that lifts to smooth tensor fields of the same type in all uniformizing systems.

Using the Riemannian metrics on the local uniformizing systems we can define the objects familiar from the Riemannian geometry of manifolds.

Example 2.6.4. (1) (*The cone*) Let $M = \mathbb{R}^2$ and let $\Gamma = \mathbb{Z}_n$ acting by rotations on it. The quotient space is topologically $\mathbb{R}^2$ but metrically is a cone with cone angle $2\pi/n$. It is a Riemannian manifold except at the cone point where the metric has a singularity. In this case, the singular locus consists of a single point (the cone point) and the isotropy group at any point is trivial except the singular point where it is $\mathbb{Z}_n$.

(2) (*The $\mathbb{Z}_n$-football*) Let $M = \mathbb{S}^2 \subset \mathbb{R}^3$ and $\Gamma = Z_n$ acting by rotations around the z-axis by an angle of $2\pi/n$. The quotient space is topologically $\mathbb{S}^2$ but metrically there are two singular points: the north and the south pole (the points N and S in figure 2.1).

(3) (*The pillow case*) Let $M = \mathbb{T}^2$ and $\Gamma = \mathbb{Z}_2$ acting by rotations around one of its axis. Then the quotient space is an orbifold whose underlying space is S^2 and has four singular points with nontrivial isotropy $\mathbb{Z}_2$. The sphere inherits a Riemannian metric of curvature 0 in the complement of the singular locus, and has curvature π at each of the four points.

(4) Let $M = \mathbb{R}^3$ and $\Gamma = \mathbb{Z}_2$ acting by the antipodal map $x \mapsto -x$. Since topologically $\mathbb{R}^3/\mathbb{Z}_2$ is a cone over $\mathbb{RP}^2$, the underlying space of this orbifold is not (topologically) a manifold.

(5) (*The $\mathbb{Z}_n$-teardrop*) The underlying space is $\mathbb{S}^2$ and the singular locus is a single point with isotropy group $\mathbb{Z}_n$, $n > 1$ (see Figure 2.3).

(6) (*The $\mathbb{Z}_p$-$\mathbb{Z}_q$-football*) The underlying space is again $\mathbb{S}^2$ and the singular locus is a pair of cone points (N and S in Figure 2.4) with isotropy $\mathbb{Z}_p$ at N and $\mathbb{Z}_q$ at S with $p \neq q$.

Note that except for the last two examples, the orbifolds considered above are good (actually they are very good). The non-developability of the orbifolds in (5) and in (6) follows easily also from the next section.

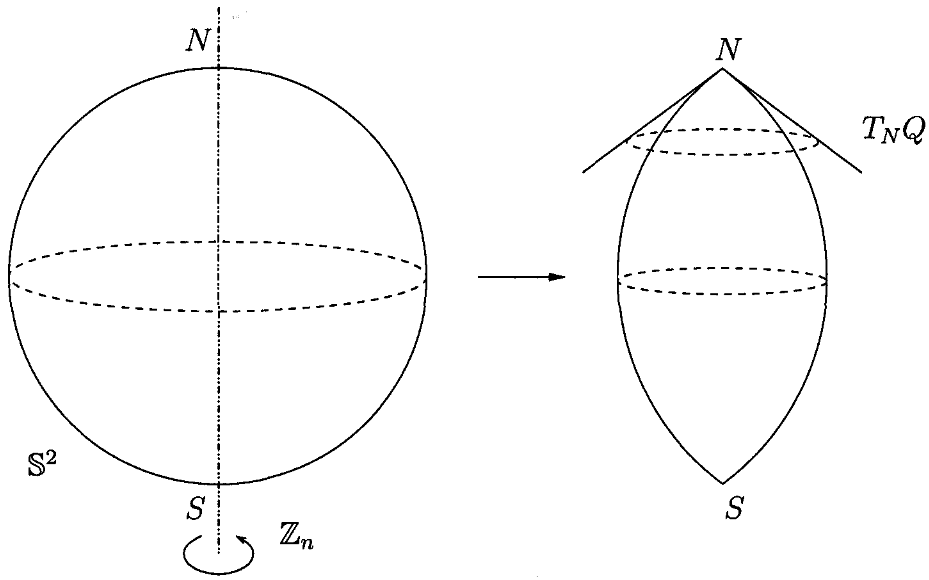


Figure 2.1: The $\mathbb{Z}_n$-football is covered by $\mathbb{S}^2$ and its fundamental group is $\mathbb{Z}_n$. The tangent cone at N is the cone $\mathbb{R}^2/\mathbb{Z}_n$ of angle $2\pi/n$.

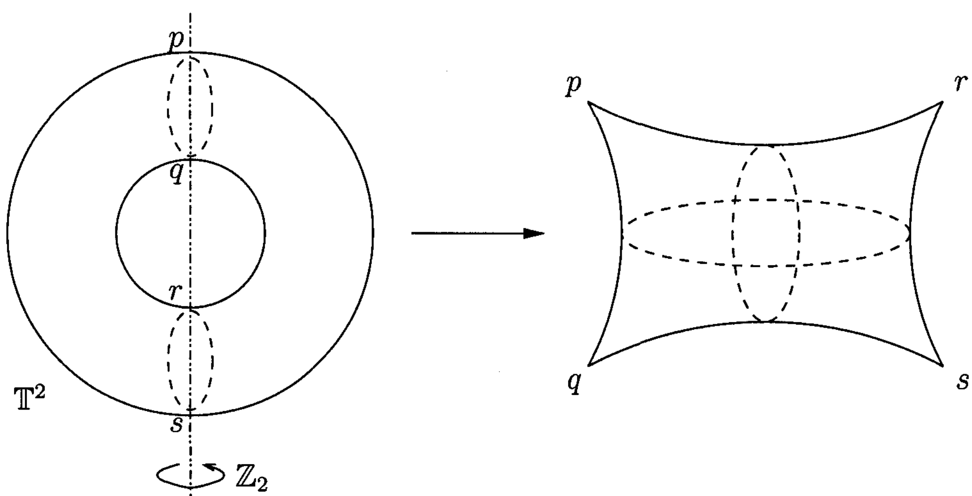


Figure 2.2: The pillowcase is covered by $\mathbb{T}^2$.

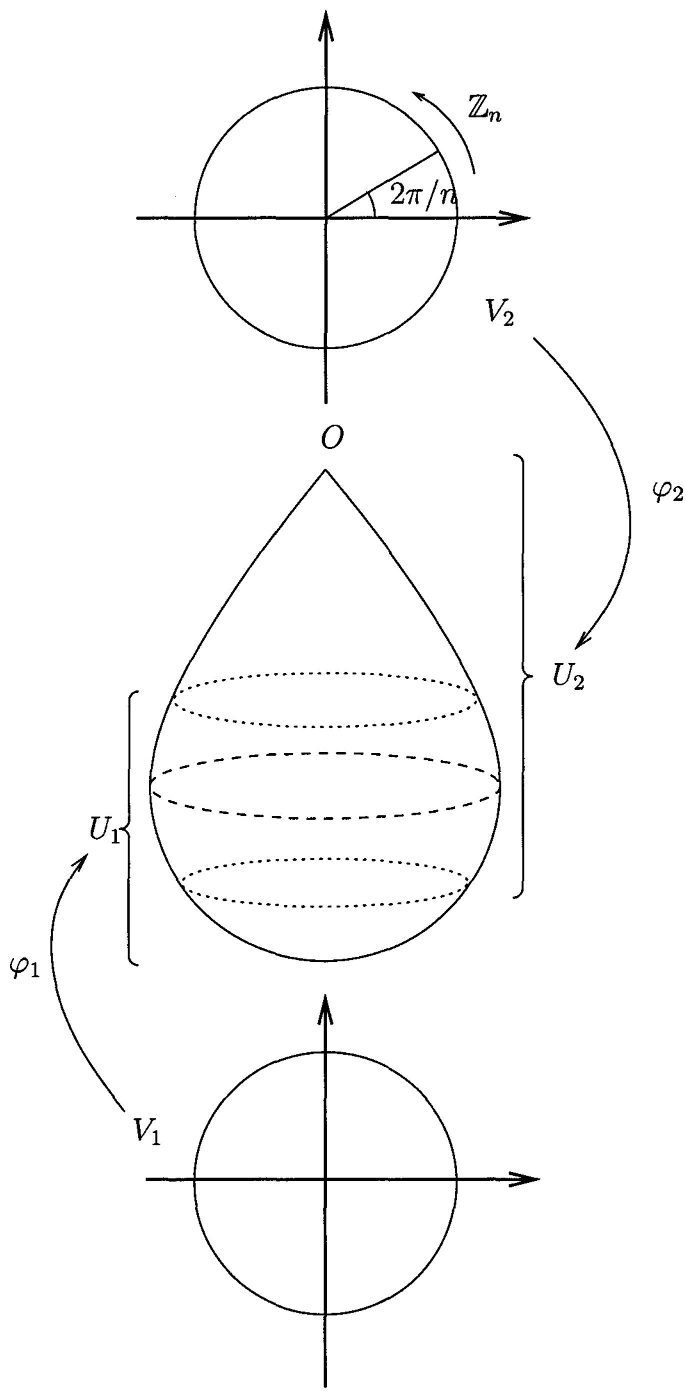


Figure 2.3: The $\mathbb{Z}_n$-teardrop is not developable. The orbifold atlas consists of two open sets U_1 and U_2 uniformized by $V_1 = D^2$ and $V_2 = D^2/\mathbb{Z}_n$ respectively.

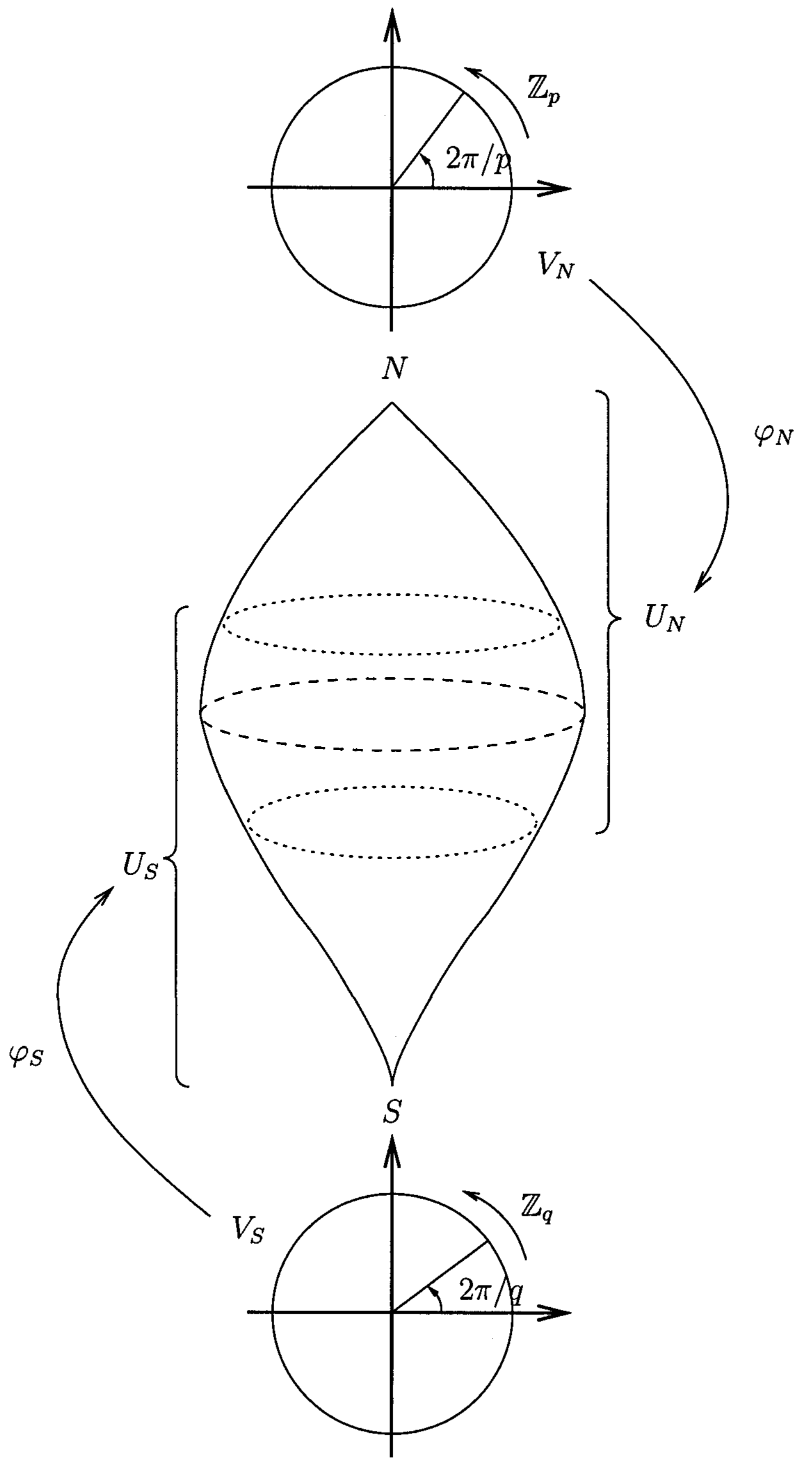
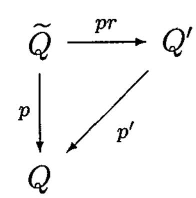


Figure 2.4: The $\mathbb{Z}_p$-$\mathbb{Z}_q$-football is not developable. The orbifold atlas consists of two open sets U_N and U_S uniformized by $V_N = D^2/\mathbb{Z}_p$ and by $V_S = D^2/\mathbb{Z}_q$ respectively.

2.7 Orbifold coverings and fundamental groups

We begin by briefly recalling the notion of coverings of topological spaces. A projection $\pi : \tilde{X} \to X$ is a covering map if every $x \in X$ has a neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of sets V_i for each of which the restriction $\pi|_{V_i} : V_i \to U$ is a homeomorphism.

Orbifold covering spaces are defined similarly. A projection $p : Q' \to Q$ between orbifolds is called a covering map if it satisfies the condition that, for each $x \in Q$, there exists a neighborhood U uniformized by $(\widetilde{U}, \Gamma)$ such that for each connected component U_i of ${p'}^{-1}(U)$ in Q', the uniformizing systems of U_i is $(\widetilde{U}, \Gamma_i)$ for some subgroup $\Gamma_i \leq \Gamma$. Note that the underlying space $|Q'|$ is not generally a covering space of $|Q|$. The *universal covering* of a connected orbifold Q is a connected covering orbifold $p : \widetilde{Q} \to Q$ such that for any connected covering orbifold $p' : Q' \to Q$, $\widetilde{Q}$ is a covering orbifold of Q' with a projection $pr : \widetilde{Q} \to Q'$ factoring $p : \widetilde{Q} \to Q$ through $p' : Q' \to Q$.



Thurston proved existence of universal orbifold covers (see Proposition 13.2.4 in [T1]) and used them to define the orbifold fundamental group in terms of deck transformations. We will present the details of this theory for the special case of 2-dimensional orbifolds, but before that we discuss the case of a global quotient $Q = M/\Gamma$. The quotient $M \to M/\Gamma$ can be regarded as an orbifold covering with Γ as the group of deck transformations. Similarly, any subgroup Γ' induces an intermediate orbifold covering $M/\Gamma' \to M/\Gamma$. On the other hand, any manifold covering $\tilde{M} \to M$ gives an orbifold covering by composing with the quotient map $M \to M/\Gamma$. In particular, the universal covering gives rise to a universal orbifold covering of Q, and the orbifold fundamental group belongs in a short exact sequence

$$1 \longrightarrow \pi_1 M \longrightarrow \pi_1^{orb} Q \longrightarrow \Gamma \longrightarrow 1.$$

For the remainder of this section, we take Q to be a 2-dimensional orbifold. For the purpose of proving the existence of universal covers, we can assume that Q has no boundary, for if it did, by doubling Q along its boundary, we get an orbifold without boundary that double covers Q and has the same universal cover as Q.

The singular locus of Q consists of cone points and corner reflectors, which are singularities modeled by $\mathbb{R}^2/D_n$ for the dihedral group of order $2n$. We can further assume Q does not contain any corner reflectors, for if it did, then by doubling Q along the reflector lines, we obtain an orbifold that covers Q with two cone points for each cone point in Q and one cone point for each corner reflector.

Denote by Σ the singular locus of Q and suppose $p : \hat{Q} \longrightarrow Q$ is an orbifold covering. Note that $\widehat{Q} - p^{-1}(\Sigma)$ is a manifold cover for the regular set $Q \setminus \Sigma$. Let now $x_i \in \Sigma$ be a singular point with cone angle $2\pi/n_i$. That is, a neighborhood of x_i is uniformized by $(\mathbb{R}^2, \mathbb{Z}_{n_i})$ with x_i corresponding to the cone point in $\mathbb{R}^2/\mathbb{Z}_{n_i}$. Any point in $\widehat{Q}$ above x_i will have cone angle $2\pi/m_i$, for some $m_i | n_i$. Denote by X the manifold obtained from Q by removing the interior of small cones centered at the singular points and denote by $\widehat{X} = p^{-1}(X) \subset \widehat{Q}$. Hence the regular set $Q \setminus \Sigma$ is just X with pointed discs attached to ∂X. The covering $p|_{\widehat{X}} : \widehat{X} \longrightarrow X$ induced by the orbifold covering $\widehat{Q} \longrightarrow Q$ has the property that a circle C_i of ∂X bounding a cone of angle $2\pi/n_i$ has the pre-image consisting of circles which projects with degree which divides n_i. Thus $\pi_1(\widehat{X})$ contains all conjugates of $\alpha_i^{n_i}$, where $\alpha_i \in \pi_1(X)$ which represents the circle C_i. Define G to be the quotient of $\pi_1(X)$ by adding the relations $\alpha_i^{n_i} = 1$ and let H be the kernel of the natural homomorphism $\pi_1(X) \longrightarrow G$. Clearly H is a subgroup of $\pi_1(\widehat{X})$. It follows that the covering $\widetilde{X}$ of X determined by H is universal among the covers of X which extend to a cover for Q. By adding to $\widetilde{X}$ the appropriate cones along $\partial\widetilde{X}$, we obtain an orbifold cover $\widetilde{Q} \longrightarrow Q$ and define $\pi_1^{orb}(Q)$ to be the group of deck transformation of $\widetilde{Q} \longrightarrow Q$.

Inspired by the construction above, we can easily give a presentation of the $\pi_1^{orb}(Q)$ in the case when the orbifold has only cone points as singular points. We start with the fundamental group of the manifold obtained from Q by removing small neighborhoods of the cone points and we add the relations $\alpha_i^{n_i} = 1$. Thus, if the underlying space $|Q|$ is a closed orientable surface of genus g and if Q has m cone points of order $n_1, \ldots, n_m$ then

$$\pi_1^{orb}(Q) = \langle a_1, b_1, \ldots, a_g, b_g, \alpha_1, \ldots, \alpha_m \mid \alpha_i^{n_i} = 1, \prod_{i=1}^{g}[a_i, b_i]\alpha_1 \ldots \alpha_m = 1\rangle.$$

Using this presentation of the orbifold fundamental group, we can see that the orbifold considered in example (5) above is not covered by a manifold (i.e. is bad). Indeed, since the underlying space is $\mathbb{S}^2$, the orbifold fundamental group for the $\mathbb{Z}_n$-teardrop is obtained from the fundamental group of $\mathbb{S}^2$ minus a point (which is trivial) by adding a relation, so it is obviously trivial. Hence the $\mathbb{Z}_n$-teardrop has no covering (other than itself).

We will introduce now the Euler number of a 2-dimensional orbifold Q. First note that in general if $\widetilde{X} \longrightarrow X$ is a k-fold covering space then $\chi(\widetilde{X}) = k\chi(X)$. As we have seen in the previous section every good compact 2-dimensional orbifold Q is finitely covered by a manifold N. Thus is natural to define the Euler number of Q by

$$\chi(Q) = \frac{1}{n}\chi(N),$$

where n is the degree of the covering $N \longrightarrow Q$. Note that the Euler number of an orbifold is not in general an integer, but is always a rational number. We will compute $\chi(Q)$ directly from the description of the orbifold.

Consider first the case when the orbifold Q is a good compact 2-dimensional orbifold which has r cone points of order n_i, $1 \leq i \leq r$ as singular points. Let X be the closure of the manifold obtained from $|Q|$ by removing r disjoint 2-discs $D_1, D_2, \ldots, D_r$ centered at the cone points. Thus $|Q| = X \cup \left(\bigcup_{i=1}^{r} D_i\right)$ and

$$\chi(|Q|) = \chi(X) + r$$

since $\chi(D_i) = 1$ and $\chi(\mathbb{S}^1) = 0$. If $p : N \to Q$ is a finite manifold cover of Q as above and $\widetilde{X} = p^{-1}(X)$, then $p|_{\widetilde{X}} : \widetilde{X} \to X$ is a manifold covering of the same degree n, hence $\chi(\widetilde{X}) = n\chi(X)$. The pre-image in N of the discs D_i by p are only n/n_i 2-discs, so we have

$$\chi(N) = n\chi(X) + \sum_{i=1}^{r} n/n_i.$$

The formula for the orbifold Euler number, known also as the Riemann-Hurwitz formula, follows immediately

$$\chi(Q) = \chi(|Q|) - \sum_{i=1}^{r} (1 - \frac{1}{n_i}).$$

Consider now the case when Q is a good compact 2-dimensional orbifold which has r cone points of order n_i, $1 \leq i \leq r$ and s corner reflectors of order m_j, $1 \leq j \leq s$. By doubling Q along the reflector curves, we obtain a 2-fold orbifold cover of Q, denoted DQ, which has r pairs of cone points with cone angle $2\pi/n_i$, $1 \leq i \leq r$ and s cone points with angle $2\pi/m_j$, $1 \leq j \leq s$. Then by above formula we have

$$\chi(DQ) = \chi(|DQ|) - 2\sum_{i=1}^{r} (1 - \frac{1}{n_i}) - \sum_{j=1}^{s} (1 - \frac{1}{m_j}).$$

Since $\chi(DQ) = 2\chi(Q)$ and $\chi(|DQ|) = 2\chi(|Q|)$, we obtain in this case the following formula for the orbifold Euler number

$$\chi(Q) = \chi(|Q|) - \sum_{i=1}^{r} (1 - \frac{1}{n_i}) - \frac{1}{2}\sum_{j=1}^{s} (1 - \frac{1}{m_j}).$$

This formula works also for the bad orbifolds. The $\mathbb{Z}_p - \mathbb{Z}_q$-football has the Euler number $2 - (1 - \frac{1}{p}) - (1 - \frac{1}{q}) = -\frac{p+q}{pq}$ which proves that the orbifold cannot be covered by a manifold, hence is bad.

If the orbifold Q is equipped with a Riemannian metric (see Definition 2.6.1), we can extend to orbifolds the Gauss-Bonnet theorem (see [Sa2])

$$\int_Q K dA = 2\pi\chi(Q),$$

where K denotes the sectional curvature of Q and dA denotes the area element. Note that an orbifold has a well defined area which has the same naturality property as the euler number, i.e. if $\widetilde{Q} \to Q$ is a finite orbifold covering of degree n, then $A(\widetilde{Q}) = nA(Q)$. The argument used in the proof is similar to the one used above to determine the orbifold Euler number : we consider the manifold with boundary obtained by removing disjoint small discs containing the singular points and we apply the usual Gauss-Bonnet theorem for manifolds with boundary. We say that the orbifold Q has an elliptic, parabolic or hyperbolic structure if $\chi(Q)$ is respectively positive, zero or negative. If the Q is elliptic or hyperbolic then the area $A(Q) = 2\pi|\chi(Q)|$.

Chapter 3

Groupoids

The purpose of this chapter is to describe orbifolds in terms of topological groupoids. As we will show in section 3.4, any orbifold structure can be represented in a natural way by an étale groupoid (the étale groupoid associated to the pseudogroup of change of charts of the orbifold). However, several different groupoids can represent the same orbifold structure. In section 3.2 we introduce an equivalence relation between groupoids, called *Morita equivalence* and which is a weak equivalence of categories. A theorem of Moerdijk and Pronk [MP] states that the category of orbifolds is equivalent to a quotient category of the proper étale groupoids after inverting Morita equivalence. Hence, whenever we consider an orbifold, we can choose, up to Morita equivalence a proper étale groupoid representing it. As we will see, the theory of groupoids provides a more convenient language for developing the foundation of the theory of orbifolds.

In the first section we begin by defining the groupoid and some standard notions associated to it. In section 3.3 we investigate the relation between étale groupoids and the pseudogroups of local homeomorphisms of a topological space. In section 3.2 we introduce the equivalence of étale groupoids and so, we have all the ingredients for describing orbifolds in terms of étale groupoids, which is done in section 3.4.

In section 3.5 we define the notion of $\mathscr{G}$-space and describe the groupoid associated to the action of a groupoid on a topological space. In section 3.6 we introduce morphisms from topological spaces to topological groupoids to be the groupoid homomorphism between the unit groupoid associated to the topological space and the topological groupoid. We also introduce the notion of relative morphisms, describe the homotopy relative to a given morphism of a subspace and define the homotopy groups of a topological groupoid.

In section 3.7 we investigate the particular case of morphisms from $\mathbb{R}$ to a topological groupoid. In this case a morphism is described by an equivalence class of $\mathscr{G}$-paths ($\mathscr{G}$-maps from $\mathbb{R}$ or the interval $I = [0,1]$ to the groupoid). This allows us to introduce the set of $\mathscr{G}$-paths between two points as well as the set of $\mathscr{G}$-loops based at a point. (Here by points we mean objects in the groupoid.) We give a description of the fundamental group of a groupoid based at a point and see that in the case of a connected or $\mathscr{G}$-connected groupoid, up to isomorphism, the fundamental group is independent of the base point.

We conclude this chapter with section 3.8 on the classifying space of a groupoid. We see there that an equivalence between groupoids induces a homotopy equivalence between their classifying spaces. This allows us to define the homotopy type of an orbifold in terms of the classifying space of a groupoid representing it. In this way we recover the orbifold homotopy groups defined in section 3.6. As we will see if Q is a Riemannian orbifold there is an explicit construction of the classifying space of the groupoid of germs of change of charts, which is independent of the particular atlas defining the orbifold structure.

3.1 Definitions

A groupoid can be thought of as a simultaneous generalization of a group, a manifold, and an equivalence relation. First as an equivalence relation, a groupoid has a set of relations that we will think of as arrows. It will be denoted here by $\mathscr{G}$. The arrows relate the objects, which are elements in a set X that we think of as points. Each arrow $g \in \mathscr{G}$ has a source $x = \alpha(g) \in X$ and a target $y = \omega(g) \in X$. Then we say $g : x \to y$, i.e. x is related to y. We want to have an equivalence relation. Then for the symmetry we need that each arrow to be invertible, for the reflexivity we need unit arrows 1_x which have both source and target equal to x, and for transitivity we need a way to compose arrows. We also require $\mathscr{G}$ and X to be more than just sets. We often require them to be locally Hausdorff, paracompact, locally compact topological spaces, or simply smooth manifolds.

Here is the formal definition of the notion of groupoid we will use in this framework.

Definition 3.1.1. *A* groupoid *is a small category in which each arrow is invertible.*

We will denote a groupoid by $(\mathscr{G}, X)$ and it consists of a set of *objects* X and a set of *arrows* $\mathscr{G}$, together with the following structure maps:

(i) the *source* and the *target* maps,

$$\alpha, \omega : \mathscr{G} \to X,$$

which assign to each arrow $g \in \mathscr{G}$ its initial object $\alpha(g)$, and its terminal object $\omega(g)$.

(ii) the *composition* map,

$$m : \mathscr{G} \times_X \mathscr{G} := \{(h, g) \in \mathscr{G} \times \mathscr{G} | \ \alpha(h) = \omega(g)\} \ \to \ \mathscr{G},$$

which assign to each pair of arrows (h, g) with $\alpha(h) = \omega(g)$ their composition hg with $\alpha(hg) = \alpha(g)$ and $\omega(hg) = \omega(h)$. This composition is required to be associative.

(iii) the *unit* map,

$$u : X \to \mathscr{G},$$

which identifies each object $x \in X$ with the unit arrow $1_x \in \mathscr{G}$. The unit arrow 1_x is a two-sided unit for the composition, i.e. $g1_x = g$ and $1_x h = h$ for any two arrows $g, h \in \mathscr{G}$ with $\alpha(g) = x = \omega(h)$.

(iv) the *inverse* map,

$$i : \mathscr{G} \to \mathscr{G},$$

which assigns to each arrow $g \in \mathscr{G}$ the inverse arrow $g^{-1} \in \mathscr{G}$ with $\alpha(g^{-1}) = \omega(g)$ and $\omega(g^{-1}) = \alpha(g)$. The inverse arrow g^{-1} is a two-sided inverse for the composition, i.e. $g^{-1}g = 1_{\omega(g)}$ and $gg^{-1} = 1_{\alpha(g)}$.

Let $(\mathscr{G}, X)$ be a groupoid and suppose $x \in X$. The set

$$\mathscr{G}_x := \{g \in \mathscr{G} |\ \alpha(g) = \omega(g) = x\}$$

is a group called the *isotropy group* of x. The subset

$$\mathscr{G}.x := \omega(\alpha^{-1}(x)) = \{y \in X |\ \exists g \in \mathscr{G}, \alpha(g) = x, \omega(g) = y\}$$

is called the *$\mathscr{G}$-orbit* of x. The $\mathscr{G}$-orbits form a partition of X and the sets of $\mathscr{G}$-orbits will be denoted by $X/\mathscr{G}$.

If the spaces $\mathscr{G}$ and X admit topologies for which the structure maps are continuous and the unit map is homeomorphism onto its image, then the groupoid $(\mathscr{G}, X)$ is called a *topological groupoid.* If further X and $\mathscr{G}$ admit smooth structures for which the structures maps are smooth, then the groupoid is called a *smooth groupoid.*

A topological groupoid is called *proper* if the map $(\alpha, \omega) : \mathscr{G} \to X \times X$ is a proper map. Note that in a proper groupoid every isotropy group is compact.

A *foliation groupoid* is a groupoid for which each isotropy group $\mathscr{G}_x$ is discrete.

An *étale groupoid* is a topological groupoid for which the source and target maps are étale maps, i.e. are local homeomorphisms. Note that any étale groupoid is a foliation groupoid and that any proper étale groupoid has finite isotropy groups.

A groupoid $(\mathscr{G}, X)$ of *local isometries* is an étale groupoid with a length metric on X that induces the given topology on X and is such that the elements of the associated pseudogroup are local isometries (see below Proposition 3.3.1).

A groupoid of local isometries $(\mathscr{G}, X)$ is *Hausdorff* if $\mathscr{G}$ is Hausdorff as a topological space and for every continuous map $c : [0, 1) \to \mathscr{G}$, if $\lim_{t\to 1} \alpha \circ c$ and $\lim_{t \to 1} \omega \circ c$ exist, then $\lim_{t\to 1} c(t)$ exists.

A groupoid $(\mathscr{G}, X)$ of local isometries of X is *complete* if X is locally complete (i.e. each point of X has a complete neighborhood) and if the space of orbits with the quotient pseudometric is complete.

3.2 Morita equivalence

In this section we introduce the equivalence of étale groupoids and an equivalence relation among étale groupoids, namely the Morita equivalence. The reader should be warned that the exposition in this section is very succinct. For a more general definition of Morita equivalent groupoids (not necessarily étale) and also for the properties of groupoids which are invariant under the Morita equivalence, a good reference is [MM].

A *homomorphism* $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ between two étale groupoids consists of a continuous functor $\psi : \mathscr{G} \to \mathscr{G}'$ inducing a continuous map $f : X \to X'$. We say that (ψ, f) is an *étale homomorphism* if f is an étale map.

We say that the homomorphism $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ is an *equivalence* if it is étale and the functor ψ is an equivalence, that is:

(i) for each $x \in X$, ψ induces an group isomorphism from $\mathscr{G}_x$ onto $\mathscr{G}'_{f(x)}$,

(ii) $f : X \to X'$ induces a bijection between the orbits sets $X/\mathscr{G}$ and $X'/\mathscr{G}'$.

We say that two étale groupoids $(\mathscr{G}_1, X_1)$ and $(\mathscr{G}_2, X_2)$ are *Morita equivalent* (or *weak equivalent*) if there exists a third étale groupoid $(\mathscr{G}, X)$ and equivalences $(\psi_1, f_1) : (\mathscr{G}, X) \to (\mathscr{G}_1, X_1)$ and $(\psi_2, f_2) : (\mathscr{G}, X) \to (\mathscr{G}_2, X_2)$.

$$\begin{array}{ccc} & (\mathscr{G}, X) & \\ {\scriptstyle (\psi_1, f_1)} \swarrow & & \searrow {\scriptstyle (\psi_2, f_2)} \\ (\mathscr{G}_1, X_1) & & (\mathscr{G}_2, X_2). \end{array}$$

This defines an equivalence relation among étale groupoids. The reflexivity and the symmetry of this relation between étale groupoids are obvious from the above diagram. Before we check the transitivity, note the following: if $(\varphi_1, h_1) : (\mathscr{G}_1, X_1) \to (\mathscr{G}', X')$ and $(\varphi_1, h_1) : (\mathscr{G}_1, X_1) \to (\mathscr{G}', X')$ are two étale homomorphisms which are equivalences, then the *fiber product* $(\mathscr{G}, X) := (\mathscr{G}_1 \times_{\mathscr{G}'} \mathscr{G}_2, X_1 \times_{X'} X_2)$ is naturally an étale groupoid and the projections $(\mathscr{G}, X) \to (\mathscr{G}_1, X_1)$ and $(\mathscr{G}, X) \to (\mathscr{G}_1, X_1)$ are equivalences.

$$\begin{array}{ccc}
(\mathscr{G}_1 \times_{\mathscr{G}'} \mathscr{G}_2, X_1 \times_{X'} X_2) & \xrightarrow{pr_1} & (\mathscr{G}_1, X_1) \\
{\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle (\varphi_1, h_1)} \\
(\mathscr{G}_2, X_2) & \xrightarrow[(\varphi_2, h_2)]{} & (\mathscr{G}', X').
\end{array}$$

Then the transitivity follows easily from the diagram

$$\begin{array}{ccccccc}
 & & & (\mathscr{G}, X) & & & \\
 & & {\scriptstyle pr_1}\swarrow & & \searrow{\scriptstyle pr_2} & & \\
 & (\mathscr{G}_1, X_1) & & & & (\mathscr{G}_2, X_2) & \\
{\scriptstyle \varphi_1'}\swarrow & & \searrow{\scriptstyle \varphi_2'} & & {\scriptstyle \varphi_2}\swarrow & & \searrow{\scriptstyle \varphi_1} \\
(\mathscr{G}_1', X_1') & & & (\mathscr{G}', X') & & & (\mathscr{G}_2', X_2').
\end{array}$$

If $(\mathscr{G}, X)$ and $(\mathscr{G}', X')$ are differentiable étale groupoids, then a homomorphism $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ is called a *differentiable equivalence* if it is an equivalence and f is locally diffeomorphism.

In the case of groupoids of local isometries, the equivalence is generated by étale homomorphisms of groupoids $(\varphi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ which are equivalences and such that $f : X \to X'$ is locally an isometry.

Example 3.2.1. (1) Any discrete group Γ can be viewed as a groupoid $(\Gamma, *)$, where the set of objects is the one-point space and the set of arrows is the group itself. In this case the composition of the groupoid is just the multiplication of the group. This groupoid is étale.

(2) Any manifold M can be viewed as an étale groupoid in two ways. One way is to consider the groupoid (M, M) where all the arrows are units (or equivalently

the unit map is a bijection). This is called the *unit groupoid* on M or the *trivial groupoid* M. The other way is to consider the groupoid $(M \times M, M)$ which is the groupoid with exactly one arrow $(x, y) : x \longrightarrow y$ from any object x to any other object y. This is called the *pair groupoid* of M. Note that the pair groupoid $(M \times M, M)$ is Morita equivalent to the trivial one point groupoid consisting of one object and one arrow.

(3) Let M be a (smooth) manifold and let $\mathscr{U} = \{U_i\}_{i \in I}$ be an open cover. To the pair $(M, \mathscr{U})$ we associate the groupoid $(\mathscr{G}, X)$ where $X = \{(x, i) \mid x \in U_i\} \subset M \times I$ and $\mathscr{G} = \{(x, i, j) \mid x \in U_i \cap U_j\} \subset M \times I \times I$. We denote $U_{ij} = U_i \cap U_j \times \{i\} \times \{j\}$. Note that in this notation $U_{ij} \neq U_{ji}$. We endow the space of objects with the topology of $\coprod_{i \in I} U_i \times \{i\}$ and the space of arrows with that of $\coprod_{ij} U_{ij}$. The structure maps are defined to be the natural maps: $\alpha|_{U_{ij}} : U_{ij} \longrightarrow U_i \times \{i\}$ the source, $\omega|_{U_{ij}} : U_{ij} \longrightarrow U_j \times \{j\}$ the target, $u|_{U_i \times \{i\}} : U_i \times \{i\} \longrightarrow U_i \times \{i\} \times \{i\} = U_{ii}$ the unit map, $i|_{U_{ij}} : U_{ij} \longrightarrow U_{ji}$ the inverse map and $m|_{U_{ijk}} : U_{ijk} \longrightarrow U_{ik}$ the multiplication map. The groupoid above is called the *covering groupoid* and is denoted $M_{\mathscr{U}}$.

(4) Let $p : N \longrightarrow M$ be a smooth map between manifolds. It induces a homomorphism of pair groupoids $(p \times p, p) : (N \times N, N) \longrightarrow (M \times M, M)$ in the obvious way. Furthermore, if p is a submersion, we can define a subgroupoid of $(N \times N, N)$ called the *kernel groupoid of p over N* to be the groupoid $(N \times_M N, N)$. If p is also surjective, then it induces an equivalence $(N \times_M N, N) \longrightarrow (M, M)$. An interesting situation is when $N = \coprod_i U_i$ is the disjoint union of an open cover $\mathscr{U} = \{U_i\}_{i \in I}$ of M and p is the natural map. In this case the kernel groupoid of p over N is the covering groupoid $M_{\mathscr{U}}$ and as above there is an equivalence from $M_{\mathscr{U}}$ to the unit groupoid (M, M). This implies that the covering groupoid $M_{\mathscr{U}}$ is Morita equivalent to the unit groupoid (M, M) and obviously that the covering groupoids $M_{\mathscr{U}}$ and $M_{\mathscr{U}'}$ associated to any two open covers $\mathscr{U}$ and $\mathscr{U}'$ of M, are also Morita equivalent.

(4) Let X be a topological space and Γ a group acting by homeomorphisms on it from the left. One can define the *étale groupoid associated to this action*

$(\Gamma \ltimes X, X)$ to be the topological small category whose space of objects is the space X and whose space of arrows $\mathscr{G} := \Gamma \times X$, where Γ is endowed with the discrete topology. The source and the target map are defined to be: $\alpha(\gamma, x) = x$ the projection, and $\omega(\gamma, x) = \gamma.x$ the action. The composition is defined from the multiplication of the group by $(\gamma, x)(\gamma', x') = (\gamma \cdot \gamma', x')$, whenever $x = \gamma'.x'$, and the inverse of (γ, x) is $(\gamma^{-1}, \gamma.x)$. The groupoid $(\Gamma \ltimes X, X)$ is also called the *translation groupoid* or *action groupoid* associated to the action of Γ on X. For right actions there is a similar groupoid denoted $(X \rtimes \Gamma, X)$ whose arrows $\gamma : x \to y$ are $\gamma \in \Gamma$ with $x = y.\gamma$. The semi-direct product symbol is used because this construction is a special case of the translation groupoid associated to the action of a groupoid on a topological space defined in section 3.5.

(5) An example of a groupoid of local isometries which is also Hausdorff, is the translation groupoid $(\Gamma \ltimes X, X)$ associated to an action by isometries of a group Γ on a length space X. This groupoid is complete if and only if X is complete as metric space.

We conclude this section with the definition of developability of groupoids.

Definition 3.2.2. *An étale groupoid* $(\mathscr{G}, X)$ *is* developable *if it is Morita equivalent to the translation groupoid* $(\Gamma \ltimes \widetilde{X}, \widetilde{X})$ *associated to an action of a group* Γ *by homeomorphisms of a space* $\widetilde{X}$.

3.3 Pseudogroups of local homeomorphisms

Of a particular interest is the correspondence between the étale groupoids $(\mathscr{G}, X)$ and the pseudogroup of local homeomorphisms of X.

First recall that a *pseudogroup* $\mathscr{H}$ of local homeomorphisms of a topological space X is a collection $\mathscr{H}$ of homeomorphisms $h : U \to V$ of open sets of X such that:

(i) the inverse and the composition of elements in $\mathscr{H}$ (whenever possible) are in $\mathscr{H}$;

(ii) the restriction of an element of $\mathscr{H}$ to any open subset of X belongs to $\mathscr{H}$;

(iii) the identity of X belongs to $\mathscr{H}$;

(iv) $\mathscr{H}$ is closed under union of elements in $\mathscr{H}$.

Proposition 3.3.1. *To each étale groupoid $(\mathscr{G}, X)$ there is associated a pseudogroup of local homeomorphisms of X.*

Indeed, if $(\mathscr{G}, X)$ is an étale groupoid, then each arrow $g \in \mathscr{G}$ with $\alpha(g) = x$ and $\omega(g) = y$, induces a well defined germ of a homeomorphism $\widetilde{g} : U_x \longrightarrow V_y$ of the form $\widetilde{g} = \omega \circ s$, where $s : U_x \longrightarrow \mathscr{G}$ is a continuous section of α over the (sufficiently small) open neighborhood U_x of x such that $s(x) = g$. It is easy to see now that the collection of all such local homeomorphisms induced by the arrows of $\mathscr{G}$ form a pseudogroup.

The groupoid $(\mathscr{G}, X)$ is called *effective* (or *reduced*) if the assignment $g \mapsto \widetilde{g}$ is faithful, i.e. for each point $x \in X$ the map $g \mapsto \widetilde{g}$ induces an injective group homomorphism between $\mathscr{G}_x$ the isotropy group of x, and the group $Hom_x(X)$ of homeomorphisms of X which fixes x. If X is a differentiable manifold and if the elements of the pseudogroup associated are diffeomorphisms, then the groupoid $(\mathscr{G}, X)$ is also a differentiable étale groupoid.

Proposition 3.3.2. *To each pseudogroup of local homeomorphisms we can associate an étale groupoid.*

Let $\mathscr{H}$ be a pseudogroup of local homeomorphisms of a topological space X. Let $f : U \longrightarrow V$ be an element of $\mathscr{H}$ and consider (f, x), its germ at some $x \in U$. That is, the equivalence class of pairs (f, x), given by the equivalence relation $(f, x) \sim (f', x')$, if and only if $x = x'$ and f is equal to f' on some neighborhood of x. The point x is called the *source* and the point $f(x)$ is called the *target* of the germ of f at x.

Denote by $\mathscr{M}_{\mathscr{H}}(X)$ the space of germs of the local homeomorphisms of $\mathscr{H}$. This is an open subspace of the space $\mathscr{M}(X)$ of germs of continuous maps from open sets of X to X, endowed with the germ topology (a basis of which consists of the subsets

U_f which are the unions of germs of continuous maps $f : U \to X$ at various points of U).

Define the source and target maps

$$\alpha, \omega : \mathscr{M}_{\mathscr{H}}(X) \to X$$

associating to each germ its source and its target respectively. With respect to the subspace topology on $\mathscr{M}_{\mathscr{H}}(X)$, the maps α and ω are étale maps, i.e. they are continuous, open maps and their restriction to any sufficiently small open set is a homeomorphism into its image. We can also define a continuous composition map

$$m : \mathscr{M}_{\mathscr{H}}(X) \times_X \mathscr{M}_{\mathscr{H}}(X) \to \mathscr{M}_{\mathscr{H}}(X),$$

by associating to any two germs (f, x) and $(f', f(x))$ their (well defined) composition, the germ $(f' \circ f, x)$. The unit map is the natural inclusion of X in $\mathscr{M}_{\mathscr{H}}(X)$ which associates to each $x \in X$ the germ (id_X, x) and the inverse map $i : \mathscr{M}_{\mathscr{H}}(X) \to \mathscr{M}_{\mathscr{H}}(X)$ associates to each germ (f, x) the well defined germ $(f^{-1}, f(x))$.

Hence, we obtained an étale groupoid $(\mathscr{G}, X)$, with $\mathscr{G} = \mathscr{M}_{\mathscr{H}}(X)$, of all the germs of the elements of $\mathscr{H}$, called the étale groupoid associated to $\mathscr{H}$.

Remark 3.3.3. From the étale groupoid $(\mathscr{G}, X)$ associated to a pseudogroup of local homeomorphisms $\mathscr{H}$ one can reconstruct $\mathscr{H}$ by considering the pseudogroup associated to $(\mathscr{G}, X)$. In general, the converse is not true. If $\mathscr{H}$ is the pseudogroup associated to an étale groupoid $(\mathscr{G}, X)$, then the étale groupoid associated to $\mathscr{H}$ is a quotient of the original groupoid $(\mathscr{G}, X)$. For instance one can consider the étale groupoid in Example 3.2.1 (1). In this case, the elements of the associated pseudogroup are the elements of the group and the space of germs of local homeomorphisms has only one element. However, if the groupoid $(\mathscr{G}, X)$ is effective and $\mathscr{H}$ is its associated pseudogroup, then it turns out that the étale groupoid associated to $\mathscr{H}$ is again $(\mathscr{G}, X)$.

3.4 Groupoid approach to orbifolds

In this section we will show how orbifolds can be completely characterize by a particular type of groupoids, namely the proper étale groupoids. We first begin with a differentiable orbifold Q and an orbifold atlas representing it. To this orbifold atlas we show that it can be associated a proper étale groupoid. We will see then that if we consider two atlases defining the same orbifold structure, the associated proper étale groupoids are Morita equivalent. This means that any orbifold structure can be represented up to Morita equivalence by a proper étale groupoid. Moreover, we can show that any such groupoid represents in a natural way an orbifold structure. It follows then that the category of orbifolds is the same as the category of proper étale groupoids after inverting Morita equivalence.

We will consider in this section the category of differentiable orbifolds, although everything applies equally to the category of topological ones. Recall the definition of a differentiable orbifold given in Remark 2.2.5. Assume that the orbifold structure Q is given by an atlas of uniformizing charts $\{(X_i, q_i)\}_{i\in I}$. Let $X = \coprod_{i\in I} X_i$. We identify each X_i to a connected component of the differentiable manifold X and denote by $q : X \to |Q|$ the union of maps q_i, that is $q(x) = q_i(x)$ whenever $x \in X_i$. Any diffeomorphism h from an open subset U of X to another open subset of X such that $q \circ h = q|_U$ will be called a change of charts.

The collection of change of charts form a pseudogroup $\mathscr{H}$ of local diffeomorphisms of X, called the pseudogroup of change of charts of the orbifold with respect to the atlas $\{(X_i, q_i)\}_{i\in I}$. We have seen that if $h : U \to V$ is a change of charts such that U and V are contained in the same X_i and U is connected, then h is the restriction to U of an element of Γ_i, hence $\mathscr{H}$ contains in particular all the elements of the groups Γ_i. We say that two points $x, x' \in X$ are on the same orbit of $\mathscr{H}$ if there is an element $h \in \mathscr{H}$ such that $h(x) = x'$. This defines an equivalent relation on X whose classes are called the orbits of $\mathscr{H}$. We will denote $X/\mathscr{H}$ the set of orbits endowed with the quotient topology. The map $q : X \to |Q|$ induces a homeomorphism from $X/\mathscr{H}$ to $|Q|$.

To the pseudogroup $\mathscr{H}$ of change of charts of an atlas of uniformizing charts $\{(X_i, q_i)\}_{i\in I}$ defining an orbifold Q , we can associate the étale groupoid $(\mathscr{G}, X)$ of all the germs of change of charts, with the topology of germs. Then $q : X \to |Q|$ induces an homeomorphism between the space of orbits $X/\mathscr{G}$ to $|Q|$.

Since $(\mathscr{G}, X)$ is the étale groupoid associated to a pseudogroup of change of charts of an orbifold, it is always effective (see section 3.3). Note that this groupoid is also proper. Indeed, let $(x_i, x_j) \in X_i \times X_j \subset X \times X$. If x_i and x_j are not in the same orbit, using the fact that the base space $|Q|$ is Hausdorff and locally compact, we can find a compact neighborhood $K = K_i \times K_j \subset X \times X$ such that $(\alpha, \omega)^{-1}(K) = \varnothing$ and hence compact. Suppose now that $q_i(x_i) = q_j(x_j) = x$, i.e. the points x_i and x_j are in the same orbit above $x \in |Q|$. From the compatibility condition between the orbifolds charts (X_i, q_i) and (X_j, q_j) there exists a diffeomorphism h from a connected open neighborhood W of x_i to a neighborhood of x_j such that $q_j \circ h = q_i|_W$. Moreover, we can assume that $W \subset X_i$ is Γ_i-stable and that the isotropy group of W is $(\Gamma_i)_{x_i}$ (see Proposition 2.1.5). This implies that $h(W) \subset X_j$ is Γ_j-stable and its isotropy group is $(\Gamma_j)_{x_j}$. By Proposition 2.1.8 we have

$$(\alpha, \omega)^{-1}(W \times h(W)) = \{(\gamma \circ h, y) \mid \gamma \in (\Gamma_j)_{x_j}, y \in W\} \cong (\Gamma_j)_{x_j} \times W.$$

Since the isotropy groups are finite, the map $(\alpha, \omega) : \mathscr{G} \to X \times X$ is proper, i.e. $(\mathscr{G}, X)$ is proper.

We would like to see now the relation between the proper étale groupoids associated (as above) to different atlases of uniformizing charts defining the same orbifold structure. First we have to introduce the notion of equivalent pseudogroups of local diffeomorphisms.

We say that two pseudogroups $\mathscr{H}_1$ and $\mathscr{H}_2$ of local diffeomorphisms of differentiable manifolds X_1 and X_2 respectively, are *equivalent* if there is a pseudogroup $\mathscr{H}$ of local diffeomorphisms of the disjoint union $X = X_1 \coprod X_2$ whose restriction to X_1 and X_2 is equal to $\mathscr{H}_1$ and $\mathscr{H}_2$, and such that the inclusions $X_1 \to X$ and $X_2 \to X$ induces homeomorphisms between the orbit spaces $X_1/\mathscr{H}_1 \to X/\mathscr{H}$ and $X_2/\mathscr{H}_2 \to X/\mathscr{H}$, respectively. If $\mathscr{H}_1$ and $\mathscr{H}_2$ are equivalent, their associated étale groupoids $(\mathscr{G}_1, X_1)$ and $(\mathscr{G}_2, X_2)$ are Morita equivalent. As we have seen in Remark 3.3.3, from the étale groupoids $(\mathscr{G}_1, X_1)$ and $(\mathscr{G}_2, X_2)$ associated to pseudogroups of local homeomorphisms $\mathscr{H}_1$ and $\mathscr{H}_2$, we can recover the pseudogroups and it turns out that if $(\mathscr{G}_1, X_1)$ is Morita equivalent to $(\mathscr{G}_2, X_2)$ then $\mathscr{H}_1$ and $\mathscr{H}_2$ are equivalent. Hence, two pseudogroups are equivalent if and only if their associated étale groupoids are Morita equivalent.

If two atlases of uniformizing charts $\{(X_i', q_i')\}_{i\in I}$ and $\{(X_j'', q_j'')\}_{j\in J}$ define the same orbifold structure on $|Q|$, their pseudogroups of change of charts $\mathscr{H}'$ and $\mathscr{H}''$ are equivalent and the associated proper étale groupoids are Morita equivalent. Therefore, an orbifold structure on a topological space Q could be defined as the Morita equivalence class of (differentiable) proper étale groupoids $(\mathscr{G}, X)$ together with a homeomorphism from $X/\mathscr{G}$ to Q, such that :

(i) $X/\mathscr{G}$ is Hausdorff, and

(ii) each point of X has a neighborhood U such that the restriction of $(\mathscr{G}, X)$ to U is the groupoid associated to an effective action of a finite group on U.

We will use the notation $Q = X/\mathscr{G}$ to denote the orbifold whose pseudogroup of change of charts is equivalent to the pseudogroup associated to the proper étale groupoid $(\mathscr{G}, X)$.

In fact, any proper étale groupoid $(\mathscr{G}, X)$ comes from an orbifold in this way. To see this, note first that any isotropy group of a proper étale groupoid is finite. Furthermore, any proper étale groupoid locally looks like the translation groupoid with respect to an action of the isotropy group. That is, for any $x \in X$ there exists an open neighborhood $U \subset X$ with an action of the isotropy group $\mathscr{G}_x$ such that the restriction $(\mathscr{G}|_U, U)$ is isomorphic to $(\mathscr{G}_x \ltimes U, U)$ (see Proposition 5.30 in [MM]). This implies that any proper étale groupoid defines a natural orbifold structure on its space of orbits $X/\mathscr{G}$. Recall that the space of orbits $X/\mathscr{G}$ of a groupoid $(\mathscr{G}, X)$ is the quotient of the space X in which two points are identified if there is an arrow in $\mathscr{G}$ between them (see section 3.1). Equivalently, $X/\mathscr{G}$ is the space of orbits of the canonical $\mathscr{G}$ action on X as defined in the next section. If we assume that the groupoid is effective then the orbifold structure on $X/\mathscr{G}$ is such that the associated proper étale groupoid is Morita equivalent to $(\mathscr{G}, X)$. However, in the non reduced case the associated proper étale groupoid to the orbifold structure on $X/\mathscr{G}$ is Morita equivalent to a quotient of the original groupoid $(\mathscr{G}, X)$ (compare with Remark 3.3.3).

Under the above correspondence it is obvious that the étale groupoid of change of charts of an orbifold is developable if and only if the orbifold structure is developable.

3.5 Structures over groupoids

Recall that if G is a topological group then topological space Y is called a (left) G-*space* if there is a continuous left action $G \times Y \to Y$, written $(g, y) = g.y$, satisfying $g'.(g.y) = (g'g)y$ and $1_G.y = y$ for any $g, g' \in G$ and $y \in Y$.

Let now $(\mathscr{G}, X)$ denote a topological groupoid with source and target projections $\alpha, \omega : \mathscr{G} \to X$. A (left) $\mathscr{G}$-*space* is a topological manifold E together with a continuous map $p_E : E \to X$ and a continuous left action of $\mathscr{G}$ on E with respect to the map p_E. That is, a continuous map from

$$\mathscr{G} \times_X E := \{(g, e) |\ p_E(e) = \alpha(g)\}$$

to E, written $(g, e) = g.e$, and such that $\omega(g) = p_E(g.e)$, $g'.(g.e) = (g'g).e$ and $1_x.e = e$. Note that for any groupoid $(\mathscr{G}, X)$, the space X is trivially a $\mathscr{G}$-space. In a similar way one can define a smooth left action of a smooth groupoid $(\mathscr{G}, X)$ on a smooth manifold E.

For two such $\mathscr{G}$-spaces (E, p_E) and $(E', p_{E'})$, a map of $\mathscr{G}$-spaces is a continuous (smooth) map $h : E \to E'$ which commutes with the structure, i.e. $p_{E'} \circ h = p_E$ and $h(g.e) = g.h(e)$ (here each action is to be considered in the corresponding $\mathscr{G}$-space). This defines a category $(\mathscr{G} - spaces)$. If

$$(\psi, f) : (\mathscr{G}', X') \to (\mathscr{G}, X)$$

is a homomorphism of groupoids, then there is a functor

$$\psi^* : (\mathscr{G} - spaces) \longrightarrow (\mathscr{G}' - spaces)$$

which associates to each $\mathscr{G}$-space E the pullback $X' \times_X E$ with the induced action. If (ψ, f) is an equivalence of groupoids, then ψ^* is an equivalence of categories. Thus, up to equivalence of categories, the category $(\mathscr{G} - spaces)$ depends only on the Morita equivalence of the groupoid $\mathscr{G}$.

Similar to the group case, if E is a $\mathscr{G}$-space, one can define the groupoid associated to the action of $\mathscr{G}$ on E to be the groupoid $(\mathscr{G} \times_X E, E)$. Its arrows $g : e \longrightarrow e'$ are arrows $g : p_E(e) \longrightarrow p_E(e')$ in $\mathscr{G}$ with $g.e = e'$, and its source and target

$$\alpha_E, \omega_E : \mathscr{G} \times_X E \longrightarrow E$$

are given by $\alpha_E(e, g) = e$ the projection and $\omega_E(e, g) = e'$ the action. We will denote this groupoid by $(\mathscr{G} \ltimes E, E)$, its dependence of the space X is to be understood.

There is an obvious homomorphism of groupoids

$$(\pi, p_E) : (\mathscr{G} \ltimes E, E) \longrightarrow (\mathscr{G}, X)$$

and the fiber over $x \in X$ is $p_E^{-1}(x) \subset E$. Note that if $(\psi, f) : (\mathscr{G}', X') \longrightarrow (\mathscr{G}, X)$ is a homomorphism of groupoids, then the diagram

$$\begin{array}{ccc} (\mathscr{G}' \ltimes \psi^*(E), \psi^*(E)) & \longrightarrow & (\mathscr{G} \ltimes E, E) \\ \downarrow & & \downarrow {\scriptstyle (\pi, p_E)} \\ (\mathscr{G}', X') & \xrightarrow[(\psi, f)]{} & (\mathscr{G}, X). \end{array}$$

is a weak pullback up to Morita equivalence.

We will define the quotient $E/\mathscr{G}$ to be the space of orbits of the groupoid $(\mathscr{G} \ltimes E, E)$. This in general is not a manifold and note that at the level of orbit spaces $E/\mathscr{G} \longrightarrow X/\mathscr{G}$ the fiber above x is $p_E^{-1}(x)/\mathscr{G}_x$ while at the level of groupoids the fiber of $(\mathscr{G} \ltimes E, E) \longrightarrow (\mathscr{G}, X)$ the fiber over $x \in X$ is the fiber $p_E^{-1}(x)$.

Remark 3.5.1. It can be proven that if $(\mathscr{G}, X)$ is an étale or a foliation groupoid, then so is $(\mathscr{G} \ltimes E, E)$. Moreover, if E is Hausdorff and $(\mathscr{G}, X)$ is proper, then $(\mathscr{G} \ltimes E, E)$ is also proper. In particular, if the groupoid $(\mathscr{G}, X)$ represents an orbifold $Q = X/\mathscr{G}$, then any Hausdorff $\mathscr{G}$-space E represents an orbifold $E/\mathscr{G} \to X/\mathscr{G}$ over Q.

There is also a notion of a (left) action of a groupoid $(\mathscr{G}, X)$ on another groupoid $(\mathscr{H}, Y)$ (see [MM]). AIt is defined by two left continuous actions of $\mathscr{G}$ over $\mathscr{H}$ and Y over continuous maps $p_{\mathscr{H}} : \mathscr{H} \to X$ and $p_Y : Y \to X$ respectively. That is, a continuous map

$$\mathscr{G} \times_X \mathscr{H} := \{(g, h) \mid p_{\mathscr{H}}(h) = \alpha(g)\} \;\to\; \mathscr{H},\; (g, h) \mapsto g \cdot h$$

such that $\omega(g) = p_{\mathscr{H}}(g \cdot h)$, $g' \cdot (g \cdot h) = g'g \cdot h$ and $1_x \cdot h = h$, and a continuous map

$$\mathscr{G} \times_X Y := \{(g, y) \mid p_Y(y) = \alpha(g)\} \;\to\; Y,\; (g, y) \mapsto g \cdot y$$

such that $\omega(g) = p_Y(g \cdot y)$, $g' \cdot (g \cdot y) = g'g \cdot y$ and $1_x \cdot y = y$. Assume moreover that the groupoid structure maps of $(\mathscr{H}, Y)$ are compatible with the action. In particular, for the source and the target map, this implies that $p_Y(\omega(h)) = p_{\mathscr{H}}(h) = p_Y(\alpha(h))$ for every $h \in \mathscr{H}$ and for multiplication map that $g \cdot (hh') = (g \cdot h)(g \cdot h')$ whenever $\alpha(h) = \omega(h')$. For every $x \in X$ the fiber $(\mathscr{H}_x, Y_x) = (p_{\mathscr{H}}^{-1}(x), p_Y^{-1}(x))$ above $x \in X$ is a full subgroupoid of $(\mathscr{H}, Y)$. This gives a family of subgroupoids indexed by the points $x \in X$ on which $\mathscr{G}$ "acts" by a groupoid isomorphism $(\mathscr{H}_x, Y_x) \to (\mathscr{H}_{x'}, Y_{x'})$ for each arrow $x \to x'$.

We can associate to a such action of $\mathscr{G}$ on $(\mathscr{H}, Y)$ a groupoid $(\mathscr{G} \ltimes \mathscr{H}, Y)$ where the space of arrows is

$$\mathscr{G} \ltimes \mathscr{H} = (\mathscr{G} \times_X Y) \times_Y \mathscr{H} := \{(g, y, h) \mid p_Y(y) = \alpha(g), g \cdot y = \alpha(h)\}.$$

If (g, y, h) is an arrow from y to $\omega(h)$ and (g', y', h') is an arrow from $y' = \omega(h)$ to $\omega(h')$, then their multiplication

$$(g, y, h)(g', y', h') = (gg', y, (g \cdot h)h')$$

is the arrow from y to $\omega(h')$.

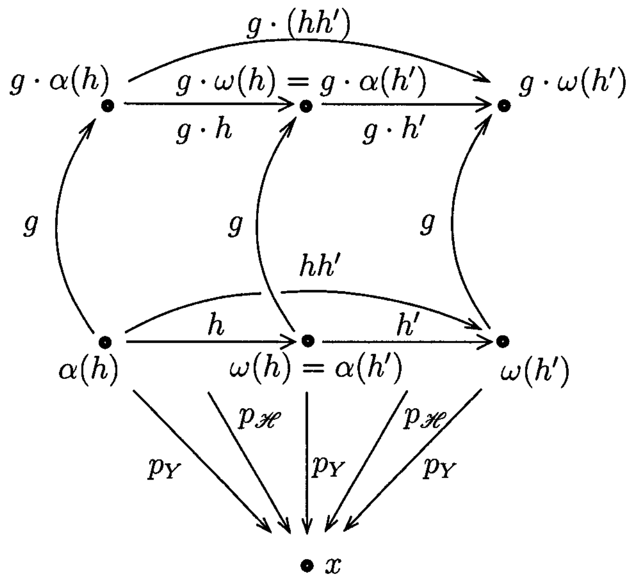


Figure 3.1: Compatibility with the $\mathscr{G}$-action in $(\mathscr{H}, Y)$.

Remark 3.5.2. In the particular case of a unit groupoid (E, E), this definition agrees with the one for $(\mathscr{G} \ltimes E, E)$ up to the obvious isomorphism.

In what follows we emphasize how the language of groupoids leads to a uniform definition of structures over orbifolds, like vector bundles, principal bundles and covering spaces. A more detailed presentation of the results there is contained in [MM].

A *vector bundle* over a groupoid $(\mathscr{G}, X)$ is a $\mathscr{G}$-space E for which the projection map $p : E \to X$ is a vector bundle over the space of objects and the action of $\mathscr{G}$ on E is fiberwise linear. In particular each fiber E_x is a linear representation of the isotropy group $\mathscr{G}_x$. We denote $Vect(\mathscr{G})$ for the category of vector bundles over $(\mathscr{G}, X)$. If the groupoid $(\mathscr{G}, X)$ is Morita equivalent to another groupoid $(\mathscr{G}', X')$ then the category $Vect(\mathscr{G})$ is equivalent to $Vect(\mathscr{G}')$. In particular, if $(\mathscr{G}, X)$ represents an orbifold Q then up to equivalence of categories there is a well defined category of vector bundles over Q denoted $Vect(Q)$. It is easy to see that the tangent bundle TX of the space of objects of a groupoid $(\mathscr{G}, X)$ has a natural structure of a vector bundle over $(\mathscr{G}, X)$. A metric on such a vector bundle is a metric in the usual sense which is preserved by the action of $\mathscr{G}$.

Let G be a topological group. A *principal G-bundle* over a groupoid $(\mathscr{G}, X)$ is a $\mathscr{G}$-space P with a left action $G \times P \to P$ which makes the projection $p : P \to X$ into a principal G-bundle over X and is compatible with the groupoid action in the following sense: for any $e \in P$, $\gamma \in G$ and $g \in \mathscr{G}$ such that $\omega(g) = p(e)$ we have $(\gamma e).g = \gamma(e.g)$.

A *covering space* over a groupoid $(\mathscr{G}, X)$ is a $\mathscr{G}$-space $\widetilde{X}$ for which the map $p : \widetilde{X} \to X$ is a covering map. The covering spaces of a groupoid form a full subcategory of $(\mathscr{G} - spaces)$ denoted $Cov(\mathscr{G})$. If $(\psi, f) : (\mathscr{G}', X') \to (\mathscr{G}, X)$ is a groupoid homomorphism, the functor $\psi^* : (\mathscr{G} - spaces) \to (\mathscr{G}' - spaces)$ restricts to a functor from $Cov(\mathscr{G})$ to $Cov(\mathscr{G}')$ and this is an equivalence of categories whenever (ψ, f) is an equivalence of groupoids.

Let Γ be a group acting by homeomorphisms on a simply connected topological space X. Let Γ_0 be a subgroup of Γ. Let $\widetilde{X} := \Gamma/\Gamma_0 \times X$, where Γ/Γ_0 has the discrete topology. The group Γ acts naturally on $\widetilde{X}$ by the rule $\gamma.(\gamma'\Gamma_0, x) = (\gamma.x, \gamma\gamma'\Gamma_0)$. Let $p : \Gamma/\Gamma_0 \times X \to X$ be the natural projection. The functor $\pi : \Gamma \ltimes \widetilde{X} \to \Gamma \ltimes X$ mapping $(\gamma, \widetilde{x}) \mapsto (\gamma, p(\widetilde{x}))$ gives the morphism $(\pi, p) : (\Gamma \ltimes \widetilde{X}, \widetilde{X}) \to (\Gamma \ltimes X, X)$ which can be considered as a covering. The natural inclusion $\Gamma_0 \times X \to \Gamma \times \widetilde{X}$ sending $(\gamma_0, x) \mapsto (\gamma_0, (\Gamma_0, x))$ defines an étale homomorphism $(\Gamma_0 \ltimes X, X) \to (\Gamma \ltimes \widetilde{X}, \widetilde{X})$ which is an equivalence. In fact the equivalence classes of connected coverings of the groupoid $(\Gamma \ltimes X, X)$ are in bijective correspondence with the conjugacy classes of subgroups of Γ.

In particular, for an orbifold Q, up to equivalence of categories, there is a well defined category $Cov(Q)$. In this case, if $(\mathscr{G}, X)$ is a proper étale groupoid associated to the orbifold structure on Q, then any covering $(\widetilde{X} \rtimes \mathscr{G}, \widetilde{X})$ of $(\mathscr{G}, X)$ is again proper and étale (see Remark 3.5.1) and its space of orbits $\widetilde{Q} := \widetilde{X}/\mathscr{G}$ is a covering orbifold for Q. To see this, note that the local charts on the orbit spaces are of the form $\widetilde{U}/\widetilde{\Gamma}$ and U/Γ where $\Gamma = \mathscr{G}_x$ and $\widetilde{\Gamma} \leq \Gamma$ is a subgroup.

3.6 Homotopy groups of groupoids

Let K be a topological space and $(\mathscr{G}, X)$ be a topological groupoid with source and target projections $\alpha, \omega : \mathscr{G} \to X$. A *morphism* from K to $(\mathscr{G}, X)$ is a homomorphism between the unit groupoid (K, K) (see Example 3.2.1 (2)) to the groupoid $(\mathscr{G}, X)$. A more direct definition of a morphism from a topological space to a groupoid can be given using $\mathscr{G}$-maps (or cocycles, see [GH]).

Let $\mathscr{U} := \{U_i\}_{i \in I}$ be an open cover of the topological space K. A $\mathscr{G}$*-map* from K to $(\mathscr{G}, X)$ over the cover $\mathscr{U}$ is a collection of continuous maps $f_i : U_i \to X$ such that whenever $U_i \cap U_j \neq \varnothing$ there is a continuous map

$$f_{ij} : U_i \cap U_j \to \mathscr{G} \text{ with } \alpha(f_{ij}(x)) = f_i(x) \text{ and } \omega(f_{ij}(x)) = f_j(x)$$

and which satisfies the cocycle condition

$$f_{ik}(x) = f_{ij}(x) f_{jk}(x),$$

for any $x \in U_i \cap U_j \cap U_k$. Note that the cocycle condition implies in particular that each $f_{ii}(x)$ is a unit of $\mathscr{G}$ and also that $f_{ij}^{-1} = f_{ji}$. Moreover, since the maps f_i can be identified with the maps f_{ii} via the natural inclusion $X \to \mathscr{G}$, the $\mathscr{G}$-map over $\mathscr{U}$ is completely characterized by the 1-cocycle f_{ij} over $\mathscr{U}$.

Two $\mathscr{G}$-maps over two open covers of K with value in $\mathscr{G}$ are equivalent if there is a $\mathscr{G}$-map with value in $\mathscr{G}$ on the disjoint union of those two covers extending the given ones on each of them. An equivalence class of $\mathscr{G}$-map is called a *morphism* from K to $\mathscr{G}$ (or when Q is an orbifold $X/\mathscr{G}$ a "continuous map" from K to Q). The set of equivalence classes of $\mathscr{G}$-maps on K with value in $\mathscr{G}$ is denoted, according Haefliger, by $H^1(K, \mathscr{G})$. Any morphism from K to $\mathscr{G}$ projects, via $q : X \to X/\mathscr{G} = |Q|$, to a continuous map from K to $|Q|$; note that two distinct morphisms may have the same projection.

If $\mathscr{G}$ and $\mathscr{G}'$ are the groupoids of germs of the changes of charts of two atlases defining the same orbifold structure on a space $|Q|$, then there is a natural bijection between the sets $H^1(K,\mathscr{G})$ and $H^1(K,\mathscr{G}')$. Any continuous map between topological spaces $f : K' \to K$ induces a map

$$f^* : H^1(K,\mathscr{G}) \to H^1(K',\mathscr{G}).$$

Two morphisms from K to $\mathscr{G}$ are homotopic if there is a morphism from $K \times [0,1]$ to $\mathscr{G}$ such that the morphisms from K to $\mathscr{G}$ induced by the natural inclusions $k \mapsto (k,i)$, $i = 0,1$, from K to $K \times [0,1]$ are the given morphisms.

Another description of morphisms from K to $\mathscr{G}$ can be given in terms of isomorphism classes of principal $\mathscr{G}$-bundles over K (see [GH]).

A *principal $\mathscr{G}$-bundle* over K is a topological space E together with a surjective continuous map $p : E \to K$ and a continuous action $(e,g) \mapsto e.g$ of $\mathscr{G}$ on E with respect to a continuous map $q_E : E \to X$ such that $p(e.g) = p(e)$. Moreover we assume that the action is simply transitive on the fibers of p in the following sense. Each point of K has an open neighborhood U with a continuous section $s : U \to E$ with respect to p such that the map $U \times_X \mathscr{G} \to p^{-1}(U)$ mapping pairs $(u,g) \in U \times \mathscr{G}$ with $\omega(g) = q_E s(u)$ to $s(u).g$ is a homeomorphism. It follows that if $\mathscr{U} = (U_i)_{i\in I}$ is an open cover of K and if $s_i : U_i \to E$ is a local continuous section of p above U_i for each $i \in I$, then there are unique continuous maps $f_{ij} : U_i \cap U_j \to \mathscr{G}$ such that $s_i(u) = s_j(u) f_{ji}(u)$ for each $u \in U_i \cap U_j$. Thus $f = (f_{ij})$ is a 1-cocycle over $\mathscr{U}$ with value in $\mathscr{G}$.

If $f = (f_{ij})$ is a 1-cocycle over an open cover $\mathscr{U} = (U_i)_{i\in I}$ of K with value in $\mathscr{G}$, then we can construct a principal $\mathscr{G}$-bundle E over K by identifying in the disjoint union of the fiber products

$$U_i \times_X \mathscr{G} = \{(u,g) \in U_i \times \mathscr{G} : \omega(g) = f_{ii}(u)\}$$

the point $(u,g) \in U_i \times_X \mathscr{G}$, $u \in U_i \cap U_j$, with the point $(u, f_{ji}(u)g) \in U_j \times_X \mathscr{G}$. The projections $p : E \to K$ and $q_E : E \to X$ map the equivalence class of $(u,g) \in U_i \times_X \mathscr{G}$ to u and $\alpha(g)$ resp. and the action of g' on the class of (u,g) is the class of (u, gg'). A principal $\mathscr{G}$-bundle obtained in this way by using an equivalent cocycle is isomorphic to the preceding one, i.e. there is a homeomorphism between them projecting to the identity of K and commuting with the action of $\mathscr{G}$. This isomorphism is determined uniquely by a cocycle extending the two given cocycles.

Therefore we see that *there is a natural bijection between the set $H^1(K, \mathscr{G})$ and the set of isomorphism classes of principal $\mathscr{G}$-bundles over K*. This correspondence is functorial via pull back: if E is a principal $\mathscr{G}$-bundle over K and if $f : K' \to K$ is a continuous map, then the pull back f^*E of E by f (or the bundle induced from E by f) is the bundle $K' \times_K E$ whose elements are the pairs $(k', e) \in K' \times E$ such that $f(k')$ is the projection of e.

$\mathscr{G}$ itself can be considered as a principal $\mathscr{G}$-bundle over X with respect to the projection $\omega : \mathscr{G} \to X$, the map $q_{\mathscr{G}} : \mathscr{G} \to X$ being the source projection. Any continuous map $f : K \to X$ induces the principal $\mathscr{G}$-bundle $f^*\mathscr{G} = K \times_X \mathscr{G}$ over K.

Let K be a topological space, $L \subseteq K$ be a subspace and F be a principal $\mathscr{G}$-bundle over L. A morphism from K to $\mathscr{G}$ relative to F is represented by a pair (E, φ) where E is a principal $\mathscr{G}$-bundles E over K and φ is an isomorphism from F to the restriction $E_{|L}$ of E above L. Two such pairs (E, φ) and (E', φ') represent the same morphism from K to $\mathscr{G}$ relative to F if there is an isomorphism $\Phi : E \to E'$ such that $\varphi' = \Phi \circ \varphi$.

Two morphisms represented by (E_0, φ_0) and (E_1, φ_1) from K to $\mathscr{G}$ relative to F are *homotopic (relative to F)* if there is a bundle E over $K \times I$ and an isomorphism from $E_{|(K \times \partial I) \cup (L \times I)}$ to the bundle obtained by gluing $F \times I$ to $E_0 \times \{0\}$ and $E_1 \times \{1\}$ using the isomorphism φ_0 and φ_1.

Let $I^n = [0, 1]^n$ be the n-cube, and let ∂I^n be its boundary. Fix a base point x in X. Let F be the bundle over ∂I^n induced from the bundle $\mathscr{G}$ by the constant map $\partial I^n \to X$ onto the point x. We define $\pi_n((\mathscr{G}, X), x)$ as the set of homotopy classes of principal $\mathscr{G}$-bundle over I^n relative to F. Similar to the case of topological spaces one proves that this set has a natural group structure, called the n^{th}- homotopy group of $(\mathscr{G}, X)$ based at x. In the case where $(\mathscr{G}, X)$ represents a connected orbifold Q, this group is called the n-th homotopy group of Q, and for $n = 1$ the (orbifold) fundamental group of Q.

3.7 Paths and loops on orbifolds

Let $(\mathscr{G}, X)$ be an étale topological groupoid representing an orbifold Q. A $\mathscr{G}$-path on Q is a morphism from $\mathbb{R}$ to $(\mathscr{G}, X)$. If x and y are two points in X a $\mathscr{G}$-path between them parametrized by $[0,1]$ is just a morphism from $[0,1]$ as topological space to $(\mathscr{G}, X)$. Using the compactness of $[0,1]$, such a morphism can be represented over a finite subdivision of the unit interval.

Definition 3.7.1. *Let x and y be two points of X. A* $\mathscr{G}$-path *from x to y over a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ of the interval $[0,1]$ is a sequence $c = (g_0, c_1, g_1, \ldots c_k, g_k)$ where:*

(i) each $c_i : [t_{i-1}, t_i] \to X$ is a continuous map,

(ii) $g_0, \ldots, g_k$ are arrows such that $\alpha(g_i) = c_i(t_i)$ for $i = 1, 2, \ldots k$, $\omega(g_i) = c_{i+1}(t_i)$ for $i = 0, 1, \ldots k-1$ and $\alpha(g_0) = x$, $\omega(g_k) = y$.

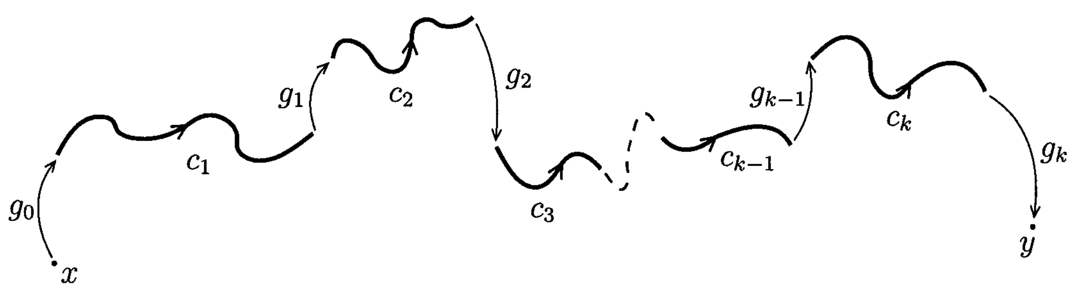


Figure 3.2: A $\mathscr{G}$-path.

If $(\mathscr{G}, X)$ is a groupoid of local isometries (for example if $(\mathscr{G}, X)$ represents a Riemannian orbifold Q) then we can define the *length* of the $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots c_k, g_k)$, $l(c)$ as being the sum of the length of the paths c_i. The pseudodistance on the space of orbits $X/\mathscr{G}$ between the orbits of two points x and y is the infimum of the length of the paths joining x to y.

Among $\mathscr{G}$-paths from x to y parametrized by $[0,1]$ we can define an equivalence relation given by the following operations:

(i) Given a $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ over the subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$, we can add a new point $t' \in (t_{i-1}, t_i)$ together with the unit element $g' = 1_{c_i(t')}$ to get a new sequence, replacing c_i in c by c_i', g', c_i'', where c_i' and c_i'' are the restriction of c_i to the intervals $[t_{i-1}, t']$ and $[t', t_i]$ respectively.

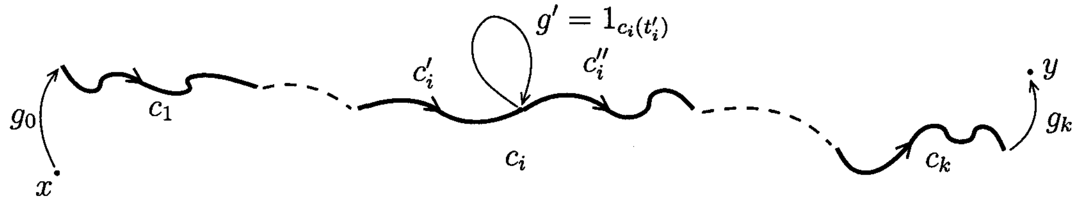


Figure 3.3: Equivalent $\mathscr{G}$-path over a subdivision with a new point $t' \in (t_{i-1}, t_i)$.

(ii) Replace the $\mathscr{G}$-path c by a new one $c' = (g_0', c_1', g_1', \ldots, c_k', g_k')$ over the same subdivision as follows: for each $i = 1, \ldots, k$ choose continuous maps $h_i : [t_{i-1}, t_i] \to \mathscr{G}$ such that $\alpha(h_i(t)) = c_i(t)$, and define $c_i'(t) := \omega(h_i(t))$, $g_i' := h_{i+1}(t_i) g_i (h_i(t_i))^{-1}$ for $i = 1, \ldots, k-1$ and $g_0' := h_1(0) g_0$, $g_k' := g_k (h_k(1))^{-1}$.

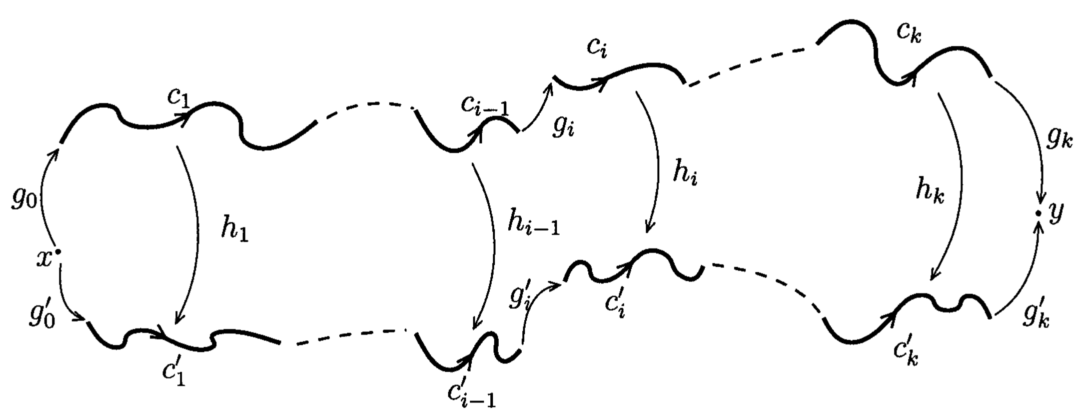


Figure 3.4: Equivalent $\mathscr{G}$-paths over the same subdivision.

Remark 3.7.2. (a) if two $\mathscr{G}$-paths on different subdivisions are equivalent, then we can pass from one to another first by considering their equivalent paths by (i) on a suitable common subdivision, and then by an operation similar to (ii).

(b) note that two equivalent $\mathscr{G}$-paths have the same initial and terminal point.

(c) for any $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ from x to y, we can find equivalent paths $c' = (g'_0, c'_1, g'_1, \ldots, c'_l, g'_l)$ such that $g'_0 = 1_x$ or $g'_l = 1_y$. For this reason, by abuse of notation we will denote sometimes the initial point of c by $c(0)$ and the terminal point by $c(1)$, even if g_0 or g_k may not be units.

(d) since the groupoid $(\mathscr{G}, X)$ is Hausdorff and étale, then the continuous maps h_i in (ii) above are uniquely defined by c and c'.

(e) if a $\mathscr{G}$-path c from x to y is such that all the c_i's are constant, then the equivalence class of c is completely characterized by an element $c \in \mathscr{G}$ with $\alpha(g) = x$ and $\omega(g) = y$.

The equivalence class of a $\mathscr{G}$-path c from x to y will be denoted by $[c]_{x,y}$, and the set of such equivalence classes will be denoted by $\Omega_{x,y}(\mathscr{G})$, or simply by $\Omega_{x,y}$.

As we have seen in the previous section, the set $\Omega_{x,y}$ corresponds bijectively to the set of isomorphism classes of principal $\mathscr{G}$-bundles E over $I = [0,1]$ relative to the bundle F over ∂I induced from $\mathscr{G}$ by the map $\partial I \to X$ sending 0 to x and 1 to y. The bundle E is obtained from c as the quotient of the union of the bundles $c_i^*(\mathscr{G})$ by the equivalence relation identifying $(t_i, g_i g) \in c_i^*(\mathscr{G})$ to $(t_i, g) \in c_{i+1}^*(\mathscr{G})$ for $i = 1, \ldots, k-1$. The isomorphism from $E|_{\partial I}$ to F maps $(0, g) \in c_1^*(\mathscr{G})$ to $(0, g_0 g)$ and $(1, g) \in c_k^*(\mathscr{G})$ to $(1, g_k^{-1} g)$.

If $x = y$, set $\Omega_x = \Omega_{xx}$ and then $c \in \Omega_x$ is called a closed $\mathscr{G}$-path with base point x. Its equivalence class is called a $\mathscr{G}$-loop based at x and is denoted by $[c]_x$. The set of $\mathscr{G}$-loops based at x is denoted by $\Omega_x(\mathscr{G})$ or simply Ω_x. The set $\bigcup_{x \in X} \Omega_x$ of based loops will be denoted by Ω_X.

The set Ω_x is in bijective correspondence with the set of isomorphisms classes of principal $\mathscr{G}$-bundle over the circle $\mathbb{S}^1$ relative to the bundle F over $1 \in \mathbb{S}^1$ induced from $\mathscr{G}$ by the map sending 1 to x. The relative bundle $E_{[c]_x}$ corresponding to c is constructed as follows. For $j = 1, \ldots, k$, let $\mathbb{S}^1_j$ be the image of the interval $[t_{j-1}, t_j]$ by the map $t \mapsto e^{2i\pi t}$; let E_j be the pull back of the principal $\mathscr{G}$-bundle $\mathscr{G}$ by the map

$\mathbb{S}^1_j \to X$ sending $e^{2i\pi t}$ to $c_j(t)$. The bundle $E_{[c]_x}$ is the quotient of the disjoint union of the E_j by the equivalence relation identifying $(e^{2i\pi t_j}, g) \in E_{j+1}$ to $(e^{2i\pi t_j}, g_j g) \in E_j$ for $j < k$ and $(1, g) \in E_1$ to $(1, g_k g_0 g) \in E_k$. The isomorphism from the restriction of $E_{[c]_x}$ over $\{1\}$ to F maps the equivalence class of $(1, g) \in E_1$ to $(1, g_0 g) \in F$.

Remark 3.7.3. If $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ and $c' = (g'_0, c'_1, g'_1, \dots c'_k, g'_k)$ are two equivalent $\mathscr{G}$-paths from x to y (or closed $\mathscr{G}$-loops at x) over the same subdivision, then the maps h_i in (ii) above induces an isomorphism from the relative principal $\mathscr{G}$-bundle associated to c to the one associated to c'. By Remark 3.7.2 (d) this isomorphism is unique.

Let $x, y \in X$ and $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ be a $\mathscr{G}$-path from x to y, defined over the subdivision $0 = t_0 \le t_1 \le \cdots \le t_k = 1$. We can define the *inverse of* c to be the $\mathscr{G}$-path from y to x given by $c^{-1} = (g'_0, c'_1, g'_1, \dots, c'_k, g'_k)$ defined over the subdivision $0 = t'_0 < t'_1 < \cdots < t'_k = 1$, where for each $i = 0, \dots, k$ we have $t'_i = 1 - t_{k-i}$, $g'_i = g^{-1}_{k-i}$ and $c'_i(t) = c_{k-i}(1-t)$ for $t \in [t'_{i-1}, t'_i]$ and $i = 1, \dots, k$. It is easy to see that the inverses of equivalent $\mathscr{G}$-paths are equivalent.

If we are given two $\mathscr{G}$-paths $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ over a subdivision $0 = t_0 < t_1 \le \cdots \le t_k = 1$ and $c' = (g'_0, c'_1, g'_1, \dots, c'_{k'}, g'_{k'})$ over $0 = t'_0 \le t'_1 \le \cdots \le t'_{k'} = 1$ such that the initial point of c' is the terminal point of c, we can define their *composition*(or concatenation) $c * c'$ to be the $\mathscr{G}$-path $c'' = (g''_0, c''_1, g''_1, \dots, c''_k, g''_k)$ over a subdivision $0 = t''_0 \le t''_1 \le \cdots \le t''_{k+k'} = 1$, where

$$t''_i = t_i/2, \text{ for } i = 0, \dots, k \text{ and } t''_i = 1/2 + t'_{i-k}/2, \text{ for } i = k, \dots, k+k' \;;$$

$$c''_i(t) = c_i(t/2), \text{ for } i = 1, \dots, k \text{ and } c''_i(t) = c'_{i-k}(2t-1), \text{ for } i = k+1, \dots, k+k' \;;$$

$$\text{and } g''_i = g_i, \text{ for } 0, \dots, k-i, \quad g''_k = g'_0 g_k, \quad g''_i = g'_{i-k}, \text{ for } i = k+1, \dots, k+k'.$$

Again, if c is equivalent to $\overline{c}$ and c' is equivalent to $\overline{c}'$, then the composition $c * c'$ is equivalent to $\overline{c} * \overline{c}'$.

An *elementary homotopy* between two $\mathscr{G}$-paths $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ over $0 = t_0 \le t_1 \le \cdots \le t_k = 1$ and $c' = (g'_0, c'_1, g'_1, \dots, c'_k, g'_k)$ over $0 = t'_0 \le t'_1 \le \cdots \le t'_k = 1$, with the same end points, is a family of $\mathscr{G}$-paths parametrized by $s \in [s_0, s_1]$, $c^s = (g_0^s, c_1^s, g_1^s, \ldots, c_k^s, g_k^s)$ over $0 = t_0^s \leq t_1^s \leq \cdots \leq t_k^s = 1$ where t_i^s, c_i^s and g_i^s depend continuously on the parameter s, g_0^s and g_k^s are independent of s and $c^{s_0} = c$, $c^{s_1} = c$.

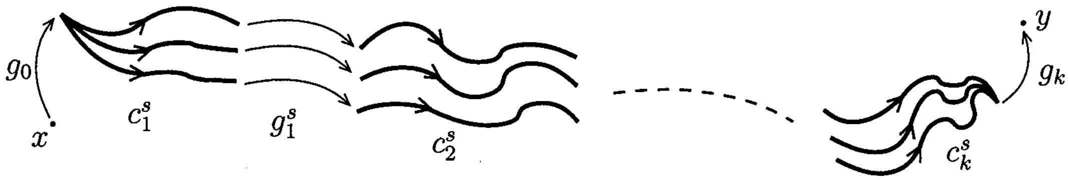


Figure 3.5: An elementary homotopy of $\mathscr{G}$-paths.

We say that two paths are *homotopic (relative to their end points)* if one can be obtained from the other by a finite sequence of the following operations:

(i) equivalence of $\mathscr{G}$-paths,

(ii) elementary homotopies.

The homotopy class of a $\mathscr{G}$-path c will be denoted by $[c]$. If c and c' are two composable $\mathscr{G}$-paths, the homotopy class of $c * c'$ depends only on the homotopy classes of c and c' and will be denoted $[c * c'] = [c] * [c']$. If c, c' and c'' are composable $\mathscr{G}$-paths, then $[c * c'] * [c''] = [c] * [c' * c''] = [c] * [c'] * [c'']$.

With the operation of composition of $\mathscr{G}$-paths, the set of homotopy classes of $\mathscr{G}$-loops based at a point x_0 form a group called *the fundamental group* $\pi_1((\mathscr{G}, X), x_0)$ *of* $(\mathscr{G}, X)$ *based at* x_0. A continuous homomorphism of étale groupoids $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ induces a homomorphism $\pi_1((\mathscr{G}, X), x_0) \to \pi_1((\mathscr{G}', X'), f(x_0))$.

As we can see in the following proposition in the case of a $\mathscr{G}$-connected étale groupoid, up to isomorphism, the fundamental group $\pi_1((\mathscr{G}, X), x_0)$ is independent of the choice of the base point (see [BH]).

Proposition 3.7.4. *Let $(\mathscr{G}, X)$ be an étale groupoid and $x_0 \in X$ be a base point and let a be a $\mathscr{G}$-path joining x_0 to $x_1 \in X$. Then the map that associates to each*

$\mathscr{G}$-loop based at x_0 the $\mathscr{G}$-loop $a^{-1} * c * a$ based at x_1 induces an isomorphism from $\pi_1((\mathscr{G}, X), x_0)$ to $\pi_1((\mathscr{G}, X), x_1)$.

If $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ is an equivalence of étale groupoids, then the induced homomorphism on the fundamental groups is an isomorphism.

A groupoid is called *simply connected* if it is $\mathscr{G}$-connected and its fundamental group is trivial.

In the case when the groupoid $(\mathscr{G}, X)$ represents a connected orbifold Q the fundamental group $\pi_1((\mathscr{G}, X), *)$ is denoted $\pi_1^{orb}(Q)$ and coincides with the previous definition of the orbifold fundamental group given in termas of covering spaces (see 2.7).

Let Q be a topological orbifold and $(\mathscr{G}, X)$ the étale groupoid associated to the pseudogroup of change of uniformizing charts. The set of free loops on Q is the set of equivalence classes of morphisms from the circle $\mathbb{S}^1$ to $(\mathscr{G}, X)$. Such a morphism can be represented over a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ by a closed $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ based at some point $x \in X$. This time the equivalence relation is generated by i) and ii) together with

(iii) for any element $g \in \mathscr{G}$ such that $\alpha(g) = x$, then $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ is equivalent to $c.g := (g^{-1}g_0, c_1, g_1, \ldots, c_k, g_k g)$.

The class of c under this equivalence relation is noted $[c]$ and is called a free loop on Q. It depends only of the orbifold structure Q and not on a particular atlas defining it.

The groupoid $\mathscr{G}$ acts naturally on the right on the set Ω_X of based $\mathscr{G}$-loops with respect to the projection $p : \Omega_X \to X$ associating to a based $\mathscr{G}$-loop its base point. If g is an element of $\mathscr{G}$ with source x and target y, then $([c]_x).g \in \Omega_y$ is the $\mathscr{G}$-loop based at y represented by $c.g$. The action of $\mathscr{G}$ on Ω_X with respect p is continuous. The quotient of Ω_X by this action is by definition the "space" of (continuous) free loops $|\Lambda(\mathscr{G})| = |\Lambda Q|$ on Q.

Under the projection $q : X \to |Q|$, every free $\mathscr{G}$-loop is mapped to a free loop on $|Q|$. Therefore if $\Lambda|Q|$ is the space of free loops on the topological space $|Q|$, we have a map

$$|\Lambda Q| \to \Lambda |Q|.$$

We denote $|\Lambda^0 Q|$ the subset of $|\Lambda Q|$ formed by the free loops on Q projecting to a constant loop. An element of this subset is represented by a closed $\mathscr{G}$-path $c = (g_0, c_1, g_1)$, where g_0 is a unit 1_x, c_1 is the constant map from $[0,1]$ to x and g_1 is an element of the subgroup $\mathscr{G}_x = \{\{g \in \mathscr{G} : \alpha(g) = \omega(g) = x\}$. The equivalence class $[c]$ of c corresponds to the conjugacy class of g_1 in $\mathscr{G}_x$.

In the developable case, if $Q = X/\Gamma$ is a global quotient of a connected manifold X by a properly discontinuous action of a discrete subgroup Γ of its group of diffeomorphisms, then the free loops based at $x \in X$ are in bijective correspondence with pairs (c, γ), where $c : [0,1] \to X$ is a continuous path with $c(0) = x$ and γ is an element of Γ mapping $c(1)$ to x. The free loops on Q are represented by classes of pairs (c, γ) as above, with (c, γ) equivalent to $(c \circ \delta, \delta^{-1}\gamma\delta)$, for all $\delta \in \Gamma$. Assuming X is simply connected, the set of homotopy classes of elements of $|\Lambda Q|$ is in bijective correspondence with the set of conjugacy classes in Γ.

For instance let $X = \mathbb{R}^2$ and $\Gamma = \mathbb{Z}_n$ be the group generated by a rotation ρ fixing 0 and of angle $2\pi/n$. Let $\mathscr{G}$ be the groupoid associated to the action of Γ on X. The orbifold $Q = X/\mathscr{G}$ is a cone. Consider the closed free $\mathscr{G}$-loop represented by the pair (c, ρ^k), where c is the constant path at 0. If we deform this loop slightly so that it avoids the origin, its projection to the cone $|Q|$ will be a curve going around the vertex a number of times congruent to k modulo n ; in particular, when $k = n$, it could also be a constant loop.

On free loop space, we have two operations, the first one defined by a change of parameter, and the second one by m-times iteration.

Since $|\Lambda Q| = H^1(\mathbb{S}^1, \mathscr{G})$, a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ induces a bijection $h^* : H^1(\mathbb{S}^1, \mathscr{G}) \to H^1(\mathbb{S}^1, \mathscr{G})$. This gives a continuous action on $|\Lambda Q|$ of the group of homeomorphisms of $\mathbb{S}^1$ leaving invariant the subspace $|\Lambda^0 Q|$ of free loops of length 0. By restriction we get an action of the group $O(2)$ of isometries of $\mathbb{S}^1$ on $|\Lambda Q|$. The fixed point set of the action of $O(2)$ (or $SO(2)$) on $|\Lambda Q|$ is the subspace $|\Lambda^0 Q|$ of free loops of length zero.

For a positive integer m, the map $e^{2i\pi t} \mapsto e^{2i\pi mt}$ from $\mathbb{S}^1$ to $\mathbb{S}^1$ induces a map $H^1(\mathbb{S}^1, \mathscr{G}) \to H^1(\mathbb{S}^1, \mathscr{G})$. If $c = (g_0, c_1, \ldots, c_k, g_k)$ is a closed $\mathscr{G}$-path then the image of $[c]$ by this map is equal to $[c^m]$, where c^m is the m-th iterate of c.

3.8 Classifying space

Given a groupoid $\mathscr{G}$ a very important construction is that of its *classifying space* $B\mathscr{G}$, the base space of a principal $\mathscr{G}$-bundle $E\mathscr{G} \to \mathscr{G}$. In the general case one possible construction is the geometric realization of the nerve of the groupoid representing the orbifold. Our description follows Segal's fat realization construction. An alternative method is Milnor's infinite join construction.

Let $(\mathscr{G}, X)$ denote a topological groupoid with source and target maps $\alpha, \omega : \mathscr{G} \to X$. Let $\mathscr{G}^{(n)}$ denote the iterated fiber product

$$\mathscr{G}^{(n)} := \{(g_1, g_2, \ldots, g_n) \mid g_i \in \mathscr{G}, \alpha(g_i) = \omega(g_{i+1}) \textit{ for } i = 1, \ldots, n-1\}$$

and let $\mathscr{G}^{(0)} = X$ denote the set of objects. It is worth thinking of $\mathscr{G}^{(n)}$ as the manifold of composable strings of arrows in $\mathscr{G}$:

$$x_0 \xleftarrow{g_1} x_1 \xleftarrow{g_2} x_2 \longleftarrow \cdots \xleftarrow{g_n} x_n$$

With this data we can form a simplicial set (see [Se]).

Definition 3.8.1. *A* semi-simplicial set (resp. group, space, manifold) $A_\bullet$ *is a sequence of sets (resp. groups, spaces, manifolds)* $\{A_n\}_{n \in \mathbb{N}}$ *together with maps*

$$A_0 \rightleftarrows A_1 \rightleftarrows A_2 \rightleftarrows \cdots \rightleftarrows A_m \rightleftarrows \cdots$$

$$\partial_i : A_m \to A_{m-1}, \quad s_j : A_m \to A_{m+1}, \quad 0 \leqslant i, j \leqslant m$$

called boundary *and* degeneracy *maps, satisfying*

$$\begin{aligned}
\partial_i\partial_j &= \partial_{j-1}\partial_i \quad \textit{if } i<j \\
s_i s_j &= s_{j+1}s_i \quad \textit{if } i<j \\
\partial_i s_j &= \begin{cases} s_{j-1}\partial_i & \textit{if } i<j \\ 1 & \textit{if } i=j,j+1 \\ s_j\partial_{i-1} & \textit{if } i>j+1 \end{cases}
\end{aligned}$$

The *nerve* of a category $\mathscr{C}$ (see [Se]) is a semi-simplicial set $N\mathscr{C}$ where the objects of $\mathscr{C}$ are vertices, the morphisms of $\mathscr{C}$ are the 1-simplexes, the triangular commutative diagrams are the 2-simplexes, and so on.

For a groupoid $(\mathscr{G}, X)$, the corresponding simplicial object $N\mathscr{G}$ is defined by $N\mathscr{G}_n = A_n = \mathscr{G}^{(n)}$ and the boundary maps $\partial_i : \mathscr{G}^{(n)} \to \mathscr{G}^{(n-1)}$:

$$\partial_i(g_1,\ldots,g_n) = \begin{cases} (g_2,\ldots,g_n) & \text{if } i=0 \\ (g_1,\ldots,m(g_i,g_{i+1}),\ldots,g_n) & \text{if } 1\le i\le n-1 \\ (g_1,\ldots,g_{n-1}) & \text{if } i=n \end{cases}$$

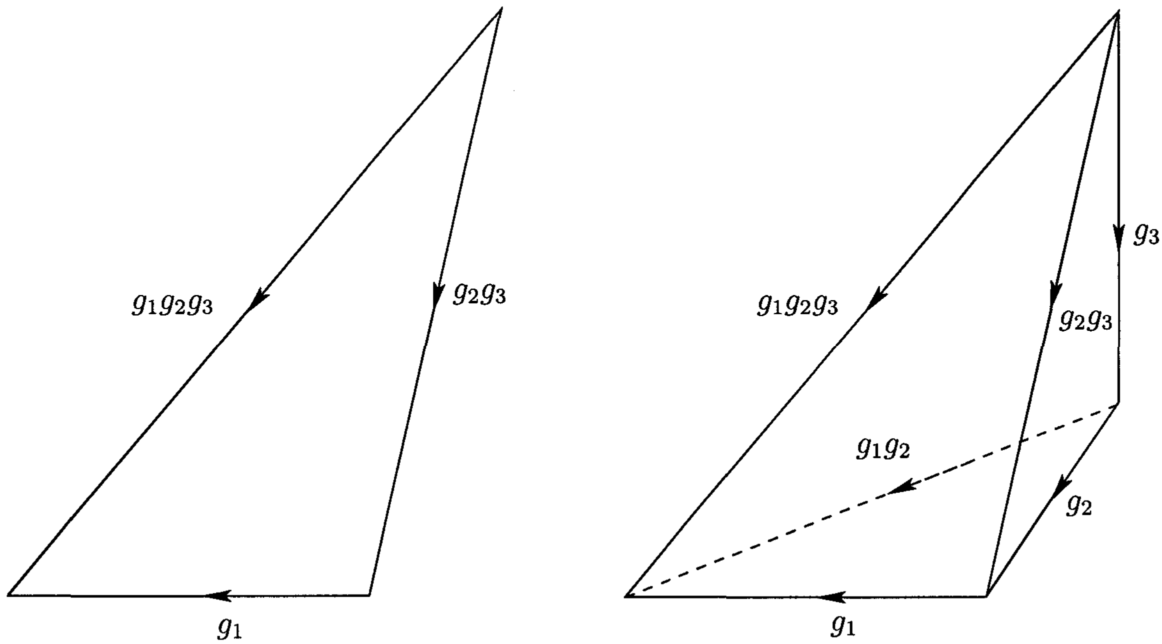
and the degeneracy maps $s_j : \mathscr{G}^{(n)} \to \mathscr{G}^{(n+1)}$:

$$s_j(g_1,\ldots,g_n) = \begin{cases} (u(\alpha(g_1)),g_1,\ldots,g_n) & \text{for } j=0 \\ (g_1,\ldots,g_j,u(\omega(g_j)),g_{j+1},\ldots,g_n) & \text{for } j\ge 1 \end{cases}$$

Denote by Δ^n the standard n-simplex in $\mathbb{R}^n$. Let $\delta_i\colon \Delta^{n-1} \to \Delta^n$ be the linear embedding of Δ^{n-1} into Δ^n as the i-th face, and let $\sigma_j\colon \Delta^{n+1} \to \Delta^n$ be the linear projection of Δ^{n+1} onto its j-th face. Define the spaces $\Delta^n \times \mathscr{G}^{(n)}$, where the points of $\Delta^n \times (g_1, g_2, \ldots, g_n)$ are understood as the points in the simplex with longest sequence of edges being named as $(g_1, g_2, \ldots, g_n)$.

By gluing these spaces together along the simplicial operators we obtain the so called *geometric realisation* of the simplicial object $A_\bullet$.

Definition 3.8.2. *The* geometric realization $|A_\bullet|$ *of the simplicial object* $A_\bullet$ *is the space*



$\Delta^2 \times \partial_2(g_1, g_2, g_3)$
$= \Delta^2 \times (g_1, g_2g_3)$

$\Delta^3 \times (g_1, g_2, g_3)$

Figure 3.6: A description of $\Delta^3 \times (g_1, g_2, g_3)$ and its boundary $\Delta^2 \times (g_1, g_2g_3)$.

$$|A_\bullet| = \left(\coprod_{n\in\mathbb{N}} \Delta^n \times A_n\right) \Big/ \begin{array}{l} (z, \partial_i(x)) \sim (\delta_i(z), x) \\ (z, s_j(x)) \sim (\sigma_j(z), x) \end{array}$$

The semi-simplicial object $N\mathscr{C}$ determines $\mathscr{C}$ and its topological realization $B\mathscr{C}$ is called the *classifying space of the category.* Here $\mathscr{C}$ is a *topological category* in Segal's sense [Se].

For a groupoid $(\mathscr{G}, X)$ we will call $B(\mathscr{G}, X) = B\mathscr{G} = |N\mathscr{G}|$ the *classifying space of the groupoid.* Note that $B\mathscr{G}$ is an infinite dimensional space and the topology of $B\mathscr{G}$ is the quotient topology induced from the topology of $\coprod \Delta^n \times \mathscr{G}^{(n)}$ (hence the topologies of $\mathscr{G}^{(n)}$'s are relevant here).

An important basic property of the classifying space construction is that an equivalence between groupoids $(\psi, f) : (\mathscr{G}, X) \to (\mathscr{G}', X')$ induces a homotopy equivalence between their classifying spaces (see [Se], [M])

$$B\psi : B\mathscr{G} \longrightarrow B\mathscr{G}'.$$

This means that for any point $x \in X$, the equivalence (ψ, f) induces an isomorphism of homotopy groups

$$\pi_n(B\mathscr{G}, x) \approx \pi_n(B\mathscr{G}', f(x)).$$

Thus, if Q is an orbifold whose orbifold structure is given by the groupoid $(\mathscr{G}, X)$ we can define its *homotopy type* as being that of the classifying space $B\mathscr{G}$:

$$\pi_n^{orb}(Q, x) = \pi_n(B\mathscr{G}, \tilde{x})$$

the definition being independent of the choice of the groupoid representing the orbifold and of the base point $x \in Q$ and of the lift $\tilde{x} \in X$ for which $q(\tilde{x}) \in |Q|$ is mapped to x by the induced homeomorphism $X/\mathscr{G} \longrightarrow |Q|$.

Example 3.8.3. (1) For the groupoid $(*, G)$ the classifying space coincides with the classifying space BG of the group G. This space classifies the principal G-bundles.

(2) If $(M, \mathscr{U})$ is a manifold then the classifying space of the groupoid $M_{\mathscr{U}}$ is homotopy equivalent to M: $BM_{\mathscr{U}} \simeq M$. (see [Se])

(3) For the groupoid $(G \ltimes M, M)$ associated to an action of a group G on a topological space M, the classifying space is the homotopy quotient

$$B(G \ltimes M) \simeq M_G = (EG \times M)/G$$

where the action of G is given by $g.(e, x) = (eg^{-1}, gx)$.

When $\mathscr{G}$ is the groupoid of germs of changes of charts of an atlas of uniformizing charts for a Riemannian orbifold Q of dimension n, there is an explicit construction of $B\mathscr{G}$ which is independent of the particular atlas defining Q and which will be therefore be denoted BQ (see [H] and also [GH]). It uses the fact that any reduced orbifold Q is isomorphic to the global quotient M/G of a manifold by an effective, smooth and almost free action of a topological group. An explicit construction of M is the so-called orthonormal frame bundle P_Q over Q and G is the orthogonal group $O(n)$. Then one considers the universal $O(n)$ bundle $EO(n) \to BO(n)$ and defines BQ to be the quotient of $EO(n) \times P_Q$ by the diagonal action of $O(n)$. This gives a natural projection $p : BQ \to Q$ with generic fiber the contractible space $EO(n)$ and Haefliger defined the cohomology, homology and homotopy groups by

$$H^i_{orb}(Q, \mathbb{Z}) = H^i(BQ, \mathbb{Z}), \quad H^{orb}_i(Q, \mathbb{Z}) = H_i(BQ, \mathbb{Z}), \quad \pi^{orb}_i(Q) = \pi_i(BQ).$$

The definition of π_1^{orb} is equivalent to that of Thurston ([T1]) in terms of deck transformations, and when Q is a smooth manifold these (orbifold) groups coincide with the usual groups. It is worth noting here that, in general, these groups are not topological invariants, but invariants of the orbifold structure only. However, rationally the (orbifold) groups are the same as the usual groups, hence they are topological invariants.

Chapter 4

Developability of orbifolds of nonpositive curvature

In this chapter we give a proof of developability for complete Riemannian orbifolds of nonpositive curvature, a result first proved by Gromov. The chapter is structured as follows. In first section we introduce the notion of $\mathscr{G}$-geodesic path on a connected Riemannian orbifold, where $(\mathscr{G}, X)$ is the groupoid associated to the germs of change of charts of the orbifold structure. This is possible since in this case $(\mathscr{G}, X)$ is a groupoid of local isometries which is Hausdorff and is $\mathscr{G}$-connected. It is also complete if the orbifold is assumed to be complete. In section 4.2 we introduce an analogue of the exponential map from the manifold case, as being the base map of an uniquely defined morphism from the tangent space T_xX into X. We see there that $\exp_x$ is continuous and that the action of $\mathscr{G}$ on T_xX with respect to this map induces a groupoid structure on the tangent space which is Morita equivalent to $(T_xX \rtimes \mathscr{G}_x, T_xX)$ and hence it is developable. This time, the induced groupoid homomorphism is the same as the good C^∞ orbifold map *Exp* considered by Ruan and Chen in [CR].

In section 4.3 we prove an orbifold analogue of the Hopf-Rinow Theorem for manifolds, namely that a connected Riemannian orbifold is complete if and only if it is geodesically complete. We also prove any two points in a complete connected Riemannian orbifold can be joined by a minimal geodesic $\mathscr{G}$-path.

In section 4.4 we introduce Jacobi fields and define the notion of conjugate points for orbifolds. We relate conjugate points along a geodesic $\mathscr{G}$-path to the critical points of the exponential map.

In the last section we consider the case of complete Riemannian orbifolds of non-positive curvature. Applying the curvature condition to Jacobi fields we prove that there are never conjugate points along any geodesic $\mathscr{G}$-path and using the results in section 4.4 we conclude that the exponential map is étale. Then the morphism from the developable (by Lemma 4.2.2) groupoid associated to the action of $\mathscr{G}$ on T_xX into $(\mathscr{G}, X)$ whose base map is the exponential map is a covering. This proves that $(\mathscr{G}, X)$ is developable, i.e. the orbifold Q is developable. The same result holds for complete Riemannian orbifolds Q which contain a pole.

4.1 Geodesic $\mathscr{G}$-paths

Let Q be a Riemannian orbifold. Recall that an orbifold structure is said to be Riemannian if each uniformizing chart X_i is a Riemannian manifold and if the change of charts are local isometries. On $X = \coprod X_i$ a Riemannian metric is defined as the union of the riemannian metrics on each X_i and this Riemannian structure induces a length metric whose quotient gives a pseudometric on the space of orbits $|Q|$. In this case, the étale groupoid of germs of change of uniformizing charts is a group of local isometries. The fact that the base space $|Q|$ is Hausdorff implies that the pseudometric is always a metric and induces the given topology on $|Q|$. Moreover the groupoid $(\mathscr{G}, X)$ associated to the pseudogroup of germs of change of charts is always Hausdorff and it is complete if and only if Q is complete.

In the case when Q is connected note that X as a disjoint union is not connected but it is *$\mathscr{G}$-path connected* in that, for any two points $x, y \in X$, there is a $\mathscr{G}$-path $c = (1_x, c_1, g_2, \dots, c_k, 1_y)$ defined over some subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ connecting x to y. We will assume in what follows that Q is connected.

Definition 4.1.1. *A* geodesic $\mathscr{G}$-path *from x to y in a Riemannian orbifold is a $\mathscr{G}$-path $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ from x to y such that:*

(i) each $c_i : [t_{i-1}, t_i] \to X$ is a geodesic segment with constant speed $\dot{c}_i$;

(ii) the differential dg_i of a representative of g_i at $c_i(t_i)$, maps the velocity vector $\dot{c}_i(t_i)$ to the velocity vector $\dot{c}_{i+1}(t_i)$.

We say that a geodesic $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ is normalized if the all the geodesic segments c_i are normalized geodesics, i.e. they are parametrized by the arc-length. In what follows we will consider normalized geodesic $\mathscr{G}$-paths.

Note that the projection $q : X \longrightarrow |Q|$ associates uniquely to each equivalence class of geodesic $\mathscr{G}$-paths $[c]$ parametrized over $[0,1]$ a continuous path $[0,1] \mapsto |Q|$. This is the base map of a smooth orbifold map which is also good in the sense of the Definition 2.4.2. Equivalently we could define a geodesic on an orbifold to be a good C^∞ orbifold map from $[0,1]$ to Q which locally lifts to geodesics. When there is no place of confusion, we will refer to the underlying map as of the geodesic path on the orbifold. Sometimes is convenient to consider the geodesic $\mathscr{G}$-paths issuing at a point which can be extended indefinitely. Then, one should consider (admissible) covers of $\mathbb{R}$ (or $[0,\infty)$) and the associated (possible infinite) sequence of elements g_i and geodesic paths c_i as in Definition 4.1.1.

It is easy to see that if c is a geodesic $\mathscr{G}$-path from x to y, then the vector $d(g_0)^{-1}\dot{c}_1(0)$ at x is an invariant of the equivalence class $[c]_{x,y}$ and is called the initial vector of the $\mathscr{G}$-geodesic c.

If c is a closed $\mathscr{G}$-path, it represents a closed geodesic $[c]$ (or a geodesic loop) on Q if moreover the differential of $g_0 g_k$ maps the velocity vector $\dot{c}_k(1)$ to the vector $\dot{c}_1(0)$. A free loop of length 0 is always a closed geodesic.

In the developable case, if the orbifold Q is the quotient of a connected Riemannian manifold X by a discrete subgroup Γ of its group of isometries, then any closed geodesic on Q is represented by a pair (c, γ), where $c : [0,1] \to X$ is a geodesic and γ an element of Γ such that the differential of γ at $c(1)$ maps the velocity vector $\dot{c}(1)$ to $\dot{c}(0)$; another such pair (c', γ') represents the same geodesic if and only if there is an element $\delta \in \Gamma$ such that $c' = \delta.c$ and $\gamma' = \delta^{-1}\gamma\delta$.

As an example (see [GH]), consider the orbifold Q which is the quotient of the round 2-sphere S^2 by a rotation ρ of angle π fixing the north pole N and the south pole S. The quotient space $|Q|$ is a sphere with two conical points $[N]$ and $[S]$, images of N and S. There are two homotopy classes of free loops on Q. Closed geodesics homotopic to a constant loop are represented by a closed geodesic on S^2 (their length is an integral multiple of 2π). If they have positive length, their image in $|Q|$ is either the equator, a figure eight or a meridian. Closed geodesics in the other homotopy class are represented by a pair (c, ρ), where c is either the constant map to N or S, or a geodesic arc on the equator of length an integral odd multiple of π.

4.2 Exponential map

We would like to introduce on X an analogue of the exponential map such that the geodesic $\mathscr{G}$-paths could be the image of a line segment containing the origin in T_xX. We will have to overcome the following difficulty. For any point $x \in X$ using the Riemannian structure on the connected component X_i which contains x, there is a uniquely defined exponential map $\exp_x^i : T_xX \to X_i$ on a sufficiently small neighborhood of the origin in T_xX. In this case the geodesic obtained as the image by the exponential map $\exp_x^i$ of a line segment containing the origin in the tangent space T_xX would lie in the connected component X_i. Hence it would be just a particular type of $\mathscr{G}$-geodesic, namely one that can be represented over the entire interval. However, this is not enough since the geodesic $\mathscr{G}$-paths can have their terminal point in other connected components of X and not every geodesic $\mathscr{G}$-path can be represented in its equivalence class by one defined on the whole interval. Using the local existence of the exponential map and the groupoid structure we will overcome this aspect by constructing a well defined "exponential $\mathscr{G}$-map" which, at a point $x \in X$, gives a morphism from T_xX to $(\mathscr{G}, X)$ as in 3.6.

We begin by recalling some facts from the Riemannian geometry of manifolds, which of course apply to any connected component of X. Recall first that the existence of geodesics $c_i : I \to X_i$ depends on the solutions of a certain system of differential equations obtained by writing in local coordinates the equation for geodesics $\frac{D}{dt}\frac{dc_i}{dt} = 0$. By the existence and uniqueness of solutions of differential equations applied to this system we obtain that for every $x_0 \in X_i$ there exist a neighborhood $U \subset X_i$ of x_0 and $\epsilon > 0$ such that for each $x \in U$ and each tangent vector $v \in T_x X$ with length less then ϵ there is a unique geodesic $c_i : (-2, 2) \to X_i$ satisfying the conditions $c_i(0) = x$ and $\dot{c}_i(0) = v$. In this case there is a uniquely defined map denoted $\exp_x^i$, given by $\exp_x^i(v) = c_i(1)$ and the geodesic $c_i : [0, 1] \to X_i$ can be described by the formula $c_i(t) = \exp_x^i(tv)$. Thus for every i and $x \in X_i$ the exponential map $\exp_x^i$ is defined throughout a neighborhood of $(x, 0)$ in $T_x X$ and it is differentiable. Moreover there exists a neighborhood V of the origin in $T_x X$ such that $\exp_x^i|_V$ is a diffeomorphism and the image $\exp_x^i(V) = U \subset X_i$ is called a geodesic neighborhood of x. If the open ball $B(0, \delta) \subset T_x X$ is such that $\overline{B}(0, \delta) \subset V$ then we call the image $\exp_x^i(B(0, \delta)) = B(x, \delta)$ the geodesic ball centered at x and of radius δ. The boundary of the geodesic ball is a submanifold of codimension one in X_i which is orthogonal to the geodesics rays issuing from x. Furthermore for any point $x \in X_i$ there exists a neighborhood W of x which is a geodesic neighborhood of each point $y \in W$. That is, a neighborhood $W \subset X_i$ of x together with a $\delta > 0$, such that for every $y \in W$, the exponential map $\exp_y^i$ is a diffeomorphism on $B(0, \delta) \subset T_y X$ and the geodesic ball $B(y, \delta)$ contains W. Such of pair (W, δ) is called a totally geodesic neighborhood for x. Note that any two points of W can be joined by a unique geodesic in X_i of length less than δ.

The following lemma (compare to Proposition 2.1.8) will allow us to extend the image of the exponential map to other connected components X_j of X.

Lemma 4.2.1. (Lemma 2.1.5 in [GH]) *Let $Q = X/\mathcal{G}$ be a Riemannian orbifold. Let $g \in \mathcal{G}$ with $x = \alpha(g)$ and $y = \omega(g)$. Let $\epsilon > 0$ be such that $B(x, \epsilon)$ and $B(y, \epsilon)$ are convex geodesic balls at x and y respectively. Then there is an element h of the pseudogroup of change of charts of Q which is an isometry from $B(x, \epsilon)$ and $B(y, \epsilon)$.*

First, the above lemma allows us to consider the following construction. Let $v \in T_x X$ such that the $\exp_x^i(v)$ is defined. Since the length of the geodesic $\exp_x^i(tv)$

is finite there exists an $\epsilon > 0$ and a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ of the interval $[0,1]$ such that $B(\exp^i_x(t_j v), \epsilon)$ are convex geodesic balls for all $j = 1, \ldots, k$. By applying inductively the above lemma for $j = 1, \ldots, k$ it is easy to see that we obtain a geodesic $\mathscr{G}$-path $c = (g_0, c_1, g_1, c_2, \ldots, c_k, g_k)$ equivalent with the one which is obtained as the image $\exp^i_x(tv)$, $t \in [0,1]$. Note that the g_j's may be units and that each $c_{j+1}(t_j)$ as well as $c_j(t_j)$ are in the orbit of $\exp^i_x(t_j v)$. In particular, the "end point" $\omega(g_k)$ of the geodesic $\mathscr{G}$-path c is in the same orbit of $\exp^i_x(v)$. Then the geodesic $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ is the image of the map defined on $[0,1]$,

$$t \mapsto (g_0, \exp^{i_1}_{x_{i_1}}(t(t_1 - t_0)v_{i_1}), g_1, \ldots, \exp^{i_k}_{x_{i_k}}(t(t_k - t_{k-1})v_{i_k}), g_k),$$

where for all $1 \leq j \leq k$

$$x_{i_j} = g_{j-1}(\exp^{i_{j-1}}_{x_{i_{j-1}}}((t_{j-1} - t_{j-2})v_{i_{j-1}})) \text{ and } v_{i_j} = A_{i_j}(v)$$

and where $x_{i_0} = x$ and the maps

$$A_{i_j} : T_x X \longrightarrow T_{x_{i_j}} X_{i_j}$$

is the composition of dg_j's and parallel transport maps along geodesics c_j.

The above construction works for any geodesic $\mathscr{G}$-path and not only for the geodesic $\mathscr{G}$-paths which are equivalent to one that can be represented over the whole interval $[0,1]$. Note now that if two points $x \in X_i$ and $y \in X_j$ ($i = j$ is allowed) are joined by a geodesic $\mathscr{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ then the terminal point $y = \omega(g_k)$ depends continuously on the initial vector $v = d(g_0)^{-1}\dot{c}(0)$. Consider any geodesic ball $B_y \in X_j$ centered at y and of sufficiently small radius such that there exists a section of the target map which is equal to g_k at y. Then there is a neighborhood $U_j \in T_x X$ of v such that for any $y' \in B_y$ there exist $v' \in U_j$ and a geodesic $\mathscr{G}$-path c' from x to y' with initial vector at x equal to v'. Moreover, the geodesic $\mathscr{G}$-path $c' = (g'_0, c'_1, g'_1, \ldots, c'_k, g'_k)$ can be defined over the same subdivision as c and such that each c'_l is in the same connected component of X as c_l.

Fix now $x \in X$ and consider the set $Y \subset X$ of all points $y \in X$ which can be joined to x by $\mathscr{G}$-geodescics. For each $y \in Y$ consider the initial vector $v(y)$ of a such

$\mathscr{G}$-geodesic. The set of all such vectors

$$K := \{v(y) \mid y \in Y\} \subset T_x X$$

is a connected open subset of $T_x X$. Moreover it is star-shaped and clearly contains the origin. It is important to note here that given a point $y \in Y$ the vector $v(y)$ is not unique. There might be more than one equivalence class of geodesic $\mathscr{G}$-paths issuing at x and having the terminal point y and to each equivalence class we can associate a such vector. Moreover, if the isotropy group $\mathscr{G}_x$ at x is not trivial, then any vector $(dg)(v)$ with $g \in \mathscr{G}_x$ is the initial vector of a geodesic $\mathscr{G}$-path c' connecting x to y which is not equivalent to c. For instance, in the case when $v = 0 \in T_x X$ every $\mathscr{G}$-path $(1_x, c, g)$ where $c : [0,1] \to X$ is the constant map at x and $g \in \mathscr{G}_x$ represents an equivalence class of geodesic $\mathscr{G}$-paths, and for distinct $g, g' \in \mathscr{G}_x$ the constant geodesic $\mathscr{G}$-paths $(1_x, c, g)$ and $(1_x, c, g')$ are not equivalent.

We will define an open cover of K in the following way. For any $v \in K$ choose a terminal point of a geodesic $\mathscr{G}$-path issuing at x with initial vector v, say $y \in X_j$, and consider as above the open neighborhood U_j. Let $\mathscr{U}$ be the collection of all such open neighborhoods.

We will construct now the $\mathscr{G}$-map $\mathscr{G}\text{-}\exp_x : K \subset T_x X \to (\mathscr{G}, X)$ over the cover $\mathscr{U}$ (see 3.6). For every $U_j \in \mathscr{U}$ define

$$\exp_x^j : U_j \to X, \;\; v \mapsto y.$$

Then $\exp_x^j$ is well defined for all j and it is continuous. Let now v_j, v_k be vectors in K and let U_j and U_k be open neighborhoods given by geodesic balls B_{y_j} and B_{y_k} of the terminal points y_j and y_k, respectively. Assume that $U_j \cap U_k \neq \varnothing$. Then, for every $v \in U_j \cap U_k$ there are geodesic $\mathscr{G}$-paths $c_k = (g_0^k, c_1^k, g_1^k, \ldots, c_m^k, g_m^k)$, and $c^j = (g_0^j, c_1^j, g_1^j, \ldots, c_l^j, g_l^j)$ issuing at x with initial vector v and such that $\omega(g_l^j) \in B_{y_j}$ and $\omega(g_m^k) \in B_{y_k}$. We define

$$\exp_x^{jk} : U_j \cap U_k \to \mathscr{G}$$

to be the composition $h_k^{-1} \circ h_j$ of sections as in Lemma 4.2.1 such that $h_k^{-1} \circ h_j(\omega(g_l^j)) = \omega(g_m^k)$. Then the maps $\exp_x^{jk}$ are continuous and

$$\alpha(\exp_x^{jk}(v)) = \alpha(h_j(\exp_x^j(v))) = \exp_x^j(v)$$

and

$$\omega(\exp_x^{jk}(v)) = \omega(h_k^{-1}(\exp_x^k(v))) = \exp_x^k(v),$$

for all $v \in U_j \cap U_k$. From the way we constructed the maps $\exp_x^{jk}$ we can see that it defines a 1-cocycle over $\mathscr{U}$ which completely characterize $\mathscr{G}$-$\exp_x$.

As in the section 3.6, two $\mathscr{G}$-maps over two open covers of K are equivalent if there is a $\mathscr{G}$-map over the union of the two covers and which extends the given ones on each of them. It is important to note that from the way we constructed the covers $\mathscr{U}$ any two $\mathscr{G}$-maps are equivalent. Therefore there is a uniquely defined morphism

$$\widetilde{\exp}_x : K \to (\mathscr{G}, X), \tag{4.1}$$

which associates to any vector $v \in K$ the terminal point of an equivalence class of geodesic $\mathscr{G}$-paths whose initial vector is v. Note that this morphism can be actually regarded as a groupoid homomorphism from the unit groupoid (K, K) to $(\mathscr{G}, X)$.

We will denote the base map of this homomorphism by $\exp_x : K \to X$ and we will refer to it as of exponential map at x. It associates to any vector $v \in K$ the terminal point of the unique equivalence class of geodesic $\mathscr{G}$-paths issuing at x with initial vector v.It is a well defined continuous map. We will say that X is $\mathscr{G}$-geodesically complete if in 4.1 $K = T_xX$ (as we will see in section 4.3 this is equivalent to the completeness of Q).

Assume now that X is $\mathscr{G}$-geodesically complete. With respect to the map $\exp_x : T_xX \to X$ we can define a natural continuous (left) action of the groupoid $\mathscr{G}$ on T_xX (see section 3.5) in the following way. If $v \in T_xX$ and $g \in \mathscr{G}$ such that $\alpha(g) = \exp_x(v)$, we define $g.v \in T_xX$ to be such that $\exp_x(g.v) = \omega(g)$. As usual we denote by $(\mathscr{G} \ltimes T_xX, T_xX)$ the groupoid associated to this action. Furthermore, if we denote by $\pi : \mathscr{G} \ltimes T_xX \to \mathscr{G}$ the map $(g, v) \mapsto g$, then $\exp_x$ induces a homomorphism

$$(\pi, \exp_x) : (\mathscr{G} \ltimes T_xX, T_xX) \to (\mathscr{G}, X) \tag{4.2}$$

In the following lemma we will show that the above groupoid associated to the action of $\mathscr{G}$ on T_xX is developable.

Lemma 4.2.2. *The groupoid* $(\mathscr{G} \ltimes T_xX, T_xX)$ *is Morita equivalent to the translation groupoid* $(\mathscr{G}_x \ltimes T_xX, T_xX)$.

Proof. We will prove that the isotropy group of any vector of positive length $v \in T_xX$ is trivial and that the isotropy at $0 \in T_xX$ is $\mathscr{G}_x$.

Let $v \in T_xX$ be a nonzero vector and let $c = (g_0, c_1, \ldots, c_k, g_k)$ be the geodesic $\mathscr{G}$-path connecting x to $\exp_x(v)$. For simplicity write $\mathscr{G}' = \{(g, v) \mid \alpha(g) = \exp_x(v)\}$ for the space of arrows of the groupoid $(\mathscr{G} \ltimes T_xX, T_xX)$ and recall that the source and the target map in this groupoid are $\alpha'(g, v) = v$, the projection and $\omega'(g, v) = g.v$, the action. The isotropy group $\mathscr{G}'_v$ of v consists of those arrows in $\mathscr{G}'$ with $\alpha'(g, v) = \omega'(g, v)$, i.e. of those $g \in \mathscr{G}$ such that $\alpha(g) = \exp_x(v)$ and $g.v = v$. Then $\exp_x(g.v) = \exp_x(v)$ which implies that for every element $(g, v) \in \mathscr{G}'_v$ the arrow g has to be an element of $\mathscr{G}_{\exp_x(v)}$.

Note now that if $g \in \mathscr{G}_{\exp_x(v)}$ is not a unit and if $\alpha(g) = \omega(g_k)$ is the terminal point of the c then the geodesic $\mathscr{G}$-path $c' = (g_0, c_1, \ldots, c_k, g_k g)$ connecting x to $\exp_x(g.v)$ is not in the same equivalence class as c. This follows from the fact that $\mathscr{G}$ is Hausdorff together with the Remark 3.7.2 (d). Indeed, since the geodesic $\mathscr{G}$-paths c and c' are represented over the same subdivision they are equivalent if there exist continuous maps $h_i : [t_{i-1}, t_i] \to \mathscr{G}$ with $\alpha(h_i(t)) = c_i(t)$ and $\omega(h_i(t)) = c'_i(t)$ and such that $g'_i h_i(t_i) = h_{i+1}(t_i) g_i$ for all $1 \leq i < k$ and $g'_0 = h_1(0) g_0$ and $g_k = g'_k h_k(1)$ (see Figure 3.4). Moreover since $\mathscr{G}$ is Hausdorff the maps h_i are unique. However, in our case $h_i(t)$ is a unit for all i and t and then the condition $g_k g = g_k h_k(1)$ implies $g = 1_\omega g_k$, i.e. g is a unit. Therefore the isotropy group $\mathscr{G}'_v = (1_{\exp_x(v)}, v)$ is trivial for any nonzero $v \in T_xX$.

At $0 \in T_xX$ the isotropy group $\mathscr{G}'_0$ is isomorphic to $\mathscr{G}_x$ as for every element $g \in \mathscr{G}_x$ the $\mathscr{G}$-paths $(1_x, c, g)$ are geodesic $\mathscr{G}$-paths of length zero and $\exp_x(g.0) = \exp_x(0) = x$.

To finally prove that $(\mathscr{G} \ltimes T_xX, T_xX)$ is Morita equivalent to the groupoid $(\mathscr{G}_x \ltimes T_xX, T_xX)$ we make use of Remark 3.5.1. Since $(\mathscr{G}, X)$ is proper and étale (as it represents the orbifold $Q = X/\mathscr{G}$) and the tangent space T_xX is Hausdorff, then the groupoid $(\mathscr{G} \ltimes T_xX, T_xX)$ represents an orbifold $T_xX/\mathscr{G}$ over Q. However, in the space of orbits $X/\mathscr{G}$ every element has trivial isotropy group except the zero vector whose isotropy group can be identified with the group $\mathscr{G}_x$. Thus orbifold $T_xX/\mathscr{G}$ can be represented by the translation groupoid groupoid $(\mathscr{G}_x \ltimes T_xX, T_xX)$ and the conclusion follows. □

At the level of spaces of orbits, the homomorphism in 4.2 gives a well defined orbifold map which we will denote $\mathrm{Exp}_{\underline{x}} : T_{\underline{x}}Q = T_xX/\mathscr{G}_x \to Q = X/\mathscr{G}$ where $q(x) = \underline{x} \in |Q|$, and whose base map $|\mathrm{Exp}_{\underline{x}}|$ associates to any "short" $\xi \in |T_{\underline{x}}Q|$, the projection $q(\exp_x(v))$ where $v \in T_xX$ such that $v = dg_0(\xi)$ where g_0 is the element of $\Gamma_{\underline{x}}$ which lifts $\underline{x}$ to x. Note that this map is also good and that it coincides with the one considered by Ruan and Chen in [CR].

4.3 Geodesically complete orbifolds

Note that the construction above works for "short" vectors $v \in T_xX$ and we would like to investigate whether a such of orbifold exponential map can be defined for vectors of any length. Similar to the manifold case, we will call an orbifold *geodesically complete* if the orbifold exponential map $\mathrm{Exp}_x : T_xQ \to Q$ is defined for all T_xQ. This is equivalent to the condition that for every $x \in X$ the exponential map $\exp_x$ is defined on all T_xX, i.e. any geodesic $\mathscr{G}$-path starting at x can be extended infinitely. By analogy, we will say in this situation that X is $\mathscr{G}$-geodesically complete. We will see that this is always the case when the orbifold Q is complete.

In what follows, using the connectedness of Q (or equivalently the $\mathscr{G}$-connectedness of X) we will introduce a pseudodistance topology on X which locally agrees the induced topology on X by the Riemannian structure (this could be easily avoided, it just make the analogy with the manifold case simpler). Define the $\mathscr{G}$-distance between the two points by

$$d_{\mathscr{G}}(x,y) = \inf\{\sum_{i=1}^{k} d(x_i, y_i)\},$$

where the infimum is considered on all possible sequences $(x_1, y_1, \dots, x_k, y_k)$ and for all k and where $d(\cdot, \cdot)$ denotes the distance induced by the Riemannian metric on each connected component in X. Note that $d_{\mathscr{G}}$ defines a *pseudodistance* on X (it is not a distance since any two distinct points in the same orbit have $d_{\mathscr{G}}$ distance zero) and that its restriction to each connected component $X_i \subset X$ is a distance which agrees with the one induced by the Riemannian structure. This implies that the topology on each component is the same as the one induced (as the subspace topology) by the pseudodistance topology given by $d_{\mathscr{G}}$ on X. Therefore, with respect to this pseudodistance topology, the exponential map $\exp_x : K \longrightarrow X$ is continuous. Note that the pseudodistance topology on X is not Hausdorff (it is not even Kolgomorov (T_0)) but as we mentioned before it is locally Hausdorff. The natural projection $q : X \longrightarrow |Q|$ is an isometry. Furthermore, we say that a geodesic $\mathscr{G}$-path between two points is minimal if it realizes the $\mathscr{G}$-distance between the two points.

We will say that a sequence $\{x^{(n)}\}_n$ of points in X is *$\mathscr{G}$-convergent* to a point $x \in X$ if it is convergent in the pseudodistance topology, i.e. if for every $\epsilon > 0$ there is $n_{(\epsilon)}$ such that for any $n \geq n_{(\epsilon)}$ we have $d_{\mathscr{G}}(x^{(n)}, x) < \epsilon$. As expected the limit point of the sequence $x^{(n)}$ is not unique and obviously any other point in the orbit of x will also be the limit point. In a similar way we say that the sequence $x^{(n)}$ of points in X is *$\mathscr{G}$-fundamental* if for every $\epsilon > 0$ there is $n_{(\epsilon)}$ such that for any $n, m \geq n_{(\epsilon)}$ we have $d_{\mathscr{G}}(x^{(n)}, x^{(m)}) < \epsilon$. It is easy to see that the completeness of $|Q|$ is equivalent with the $\mathscr{G}$-completeness of X via the projection $q : X \longrightarrow |Q|$.

The following proposition is an important property of geodesically complete orbifolds.

Proposition 4.3.1. *If a connected Riemannian orbifold $Q = X/\mathscr{G}$ is geodesically complete then any two points can be joined by a minimal geodesic $\mathscr{G}$-path.*

Proof. Let x and y be two points in X and let r denote the $\mathscr{G}$-distance between them. Consider a geodesic ball $B(x, \delta)$ at x (note that this ball is contained in the connected component of X containing x). The boundary of this ball is quasi-compact in the pseudometric topology and since the function

$$d_{\mathscr{G}}(y,\cdot):X\longrightarrow\mathbb{R}$$

is continuous, there exist a point in $x_0 \in \partial B(x,\delta)$ such that $d_{\mathscr{G}}(y,\partial B(x,\delta))$ attains its minimum. Then $x_0 = \exp_x(\delta v)$ for some $v \in T_xX$ with norm one.

Since X is $\mathscr{G}$-geodesically complete, there is an equivalence class of geodesic $\mathscr{G}$-paths issuing from x with velocity v and which can be extended indefinitely. We will show that these geodesic $\mathscr{G}$ paths intersect the orbit through y and therefore there exists a geodesic $\mathscr{G}$-path in this equivalence class which contain y. Moreover, we well prove that the length of this geodesic is equal to r.

Similar to the manifold case we will prove that a point which moves along one of the geodesic $\mathscr{G}$-paths as above must get closer (with respect to $d_{\mathscr{G}}$) to y. Let c be a such geodesic $\mathscr{G}$-path which is represented over an admissible cover of $[0,\infty)$ by a sequence $c=(g_0,c_1,g_1,\ldots,g_{k-1},c_k,\ldots)$. For convenience we will denote $c(t)$ the point $c_i(t)$ for $t\in[t_{i-1},t_i]$.

Consider the equation

$$d_{\mathscr{G}}(c(t),y)=r-t$$

and let A be the set of points in $[0,r]$ for which the equation holds.

A is not empty since the above equation is satisfied for $t=0$ and it is clearly closed. Let t^* be a point in A. We will show that if $t^*<r$ then the equation holds for $t^*+\delta'$ for some sufficiently small $\delta'>0$. This would imply that $\sup A=r$ and since A is closed we can conclude that $r\in A$, i.e. $c(r)$ is in the orbit of y. Denote $x^*=c(t^*)\in X$ and consider a geodesic ball $B(x^*,\delta')$ at x^*. Let $x_0'\in\partial B(x^*,\delta')$ be the point which minimizes the $\mathscr{G}$-distance $d_{\mathscr{G}}(y,\partial B(x^*,\delta'))$. Note that the Lemma 4.2.1 allows us to assume without loss of generality that $[t^*-\delta',t^*+\delta']\subset(t_{i-1},t_i)$ for some i. Then it suffices to show that the point x_0' equals $c(t^*+\delta')$.

Note first that

$$d_{\mathscr{G}}(c(t^*),y)=\inf_{z\in\partial B(x^*,\delta')}\big(d_{\mathscr{G}}(x^*,z)+d_{\mathscr{G}}(z,y)\big)=\delta'+d_{\mathscr{G}}(x_0',y)$$

and since $d_{\mathscr{G}}(c(t^*),y)=r-t^*$, it follows that

$$r-t^*=\delta'+d_{\mathscr{G}}(x_0',y)=d_{\mathscr{G}}(c(t^*+\delta'),y),$$

i.e.

$$d_{\mathscr{G}}(c(t^* + \delta'), y) = r - (t^* + \delta').$$

To prove now that $x_0' = c(t^* + \delta')$, note that $r - t^* = \delta' + d_{\mathscr{G}}(x_0', y)$ implies

$$d_{\mathscr{G}}(x, x_0') \geq d_{\mathscr{G}}(x, y) - d_{\mathscr{G}}(y, x_0') = r - (r - t^* - \delta') = t^* + \delta'.$$

On the other hand, the $\mathscr{G}$-distance between x and x_0' measured along the geodesic $\mathscr{G}$-path c up to x^* and then along the geodesic ray joining x^* to x_0' gives

$$d_{\mathscr{G}}(x, x_0') \leq d_{\mathscr{G}}(x, x^*) + d_{\mathscr{G}}(x^*, x_0') = t^* + \delta'.$$

Hence $d_{\mathscr{G}}(x, x_0') = t^* + \delta'$. Consider now $c(t_{i-1})$ which is a point in the same connected component of X as x^*. Clearly $d(c(t_{i-1}), x_0') \leq d(c(t_{i-1}), x^*) + d(x^*, x_0')$. We claim that we have equality in the inequality above. Indeed, if we assume that $d(c(t_{i-1}), x_0') < d(c(t_{i-1}), x^*) + d(x^*, x_0')$ then we have

$$\begin{aligned} t^* + \delta' &= d_{\mathscr{G}}(x, x_0') \leq d_{\mathscr{G}}(x, c(t_{i-1})) + d(c(t_{i-1}), x_0') \\ &< d_{\mathscr{G}}(x, c(t_{i-1})) + d(c(t_{i-1}), x^*) + d(x^*, x_0') \\ &= d_{\mathscr{G}}(x, x^*) + d_{\mathscr{G}}(x^*, x_0') \\ &= t^* + \delta' \end{aligned}$$

which cannot be true. This means that the broken path obtained from the geodesic segment $c_i|_{[t_{i-1}, t^*]}$ and the minimal geodesic joining $x^* = c_i(t^*)$ to x_0', is distance minimizing in the connected component of X containing c_i, and so it has to be an (unbroken) geodesic, i.e. it coincides with c_i. This proves that $x_0' = c(t^* + \delta')$ and completes the proof. □

The following proposition relates the concepts of $\mathscr{G}$-completeness and $\mathscr{G}$-geodesically completeness of X. It is the similar of the Hopf-Rinow theorem in the manifold case.

Theorem 4.3.2. *Let $Q = X/\mathscr{G}$ be a connected Riemannian orbifold. Then X is $\mathscr{G}$-complete if and only if it is $\mathscr{G}$-geodesically complete.*

Thus, a connected Riemannian orbifold Q is complete if and only if it is geodesically complete.

Proof. Assume that X is $\mathcal{G}$-complete and suppose that it is not $\mathcal{G}$-geodesically complete. That means that there exist a point $x \in X$ and a (normalized) geodesic $\mathcal{G}$-path issuing from x which is defined for $t < t^*$ and is not defined for t^*. Let $0 = t_0 < t_1 < \cdots < t_{k-1} < t^*$ be a subdivision of $[0, t^*]$ and $c = (g_0, c_1, g_1, \ldots, c_k)$ be a representative of this geodesic $\mathcal{G}$-path over it such that c_k is defined for $t \in [t_{k-1}, t*)$ and not for t^*. Consider a sequence $(t^n)_n$ in $[0, t^*)$ which converges at t^*. We can assume without loss of generality that $t^n \in [t_{k-1}, t^*)$. Note that t^m is Cauchy and therefore for every $\epsilon > 0$ there exists n_ϵ such that if $n, m > n_\epsilon$ then $|t^n - t^m| < \epsilon$. Since c is a normalized $\mathcal{G}$-geodesic, this implies that

$$d_{\mathcal{G}}(c_k(t^n), c_k(t^m)) \leq |t^n - t^m| < \epsilon,$$

i.e. the sequence $c_k(t^n)$ is $\mathcal{G}$-fundamental. Since X is $\mathcal{G}$-complete, this sequence is $\mathcal{G}$-convergent to a point $y \in X$ (actually it is $\mathcal{G}$-convergent to any point in the orbit of y). Let (W, δ) be a totally geodesic neighborhood of y. For $\delta > 0$ there exists n_δ such that for $n \geq n_\delta$ we have

$$d_{\mathcal{G}}(c_k(t^n), c_k(t^m)) < \delta.$$

We can choose n_δ large enough such that (eventually passing to a subsequence) the orbits through $c_k(t^n)$ have nonempty intersection with W. Denote these intersections with x_n. It follows that for any $n, m \geq n_\delta$ we have

$$d(x_n, x_m) = d_{\mathcal{G}}(x_n, x_m) = d_{\mathcal{G}}(c_k(t^n), c_k(t^m)) < \delta.$$

Then, there exists a unique geodesic connecting x_n to x_m of length smaller than δ. Denote c' this geodesic. It is clear that this geodesic is equivalent to c_k whenever this is defined. Since $\exp_{x_n}$ is a diffeomorphism on $B(0, \delta) \subset T_{x_n}X$ and $W \subset \exp_{x_n}(B(0, \delta))$, the geodesic c' can be extended over t^*. By replacing in $c = (g_0, c_1, \ldots, c_k)$ the geodesic c_k with c' we obtain a $\mathcal{G}$-geodesic path issuing at x which is defined beyond t^* and whose restriction to $[0, t^*)$ is equivalent to c, i.e. c can be extended in its equivalence class. This leads to a contradiction and proves the claim.

Conversely, assume that X is $\mathcal{G}$-geodesically complete. Consider a $\mathcal{G}$-fundamental sequence $(x^n)_n$ in X. It is clearly that the set $A = \{x^n \mid n\}$ is bounded in the $\mathcal{G}$-distance. Thus there exist a ball $B_{\mathcal{G}}$ centered at a point $x \in X$ which contains its closure. By Proposition 4.3.1, there exists a ball $B(0,r) \subset T_xX$ such that $B_{\mathscr{G}} \subset \exp_x(\overline{B}(0,r))$. As the map $\exp_x$ is continuous in the pseudodistance topology and $\overline{B}(0,r)$ is compact in T_xX, the image $\exp_x(\overline{B}(0,r))$ is quasi-compact in X. Then the closure of A is also quasi-compact and so the sequence (x^n) contains a $\mathscr{G}$-convergent subsequence and being $\mathscr{G}$-fundamental, it converges. This proves that X is also $\mathscr{G}$-complete. □

4.4 Jacobi fields

Let $x, y \in X$ and $c = (g_0, c_1, \ldots, c_k)$ be a geodesic $\mathscr{G}$-path connecting them over a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$.

Definition 4.4.1. *A* Jacobi field *along c is a sequence $J = (J_1, \ldots, J_k)$ of vector fields J_i along c_i and such that :*

(i) dg_i maps $(J_i, \frac{D}{dt}J_i(t_i)) \mapsto (J_{i+1}, \frac{D}{dt}J_{i+1}(t_i))$, for $i = 1, \ldots, k-1$

(ii) each J_i is a Jacobi field along c_i, i.e. satisfies the Jacobi equation:

$$\frac{D^2 J_i}{dt^2}(t) + R(\dot{c}_i(t), J_i(t))\dot{c}_i(t) = 0, \tag{4.3}$$

for any $t \in [t_{i-1}, t_i]$ and $i = 1, \ldots, k$.

If c is a closed geodesic, a periodic Jacobi field *is a Jacobi field such that the differential $d(g_0 g_k)$ maps $(J_k(1), \frac{DJ_k}{dt}(1))$ to $(J_1(0), \frac{DJ_1}{dt}(0))$.*

Note that for each i the Jacobi equation 4.3, as a second order differential equation, has $2n$ linearly independent smooth solutions. Since each J_i along the geodesic c_i is uniquely determined by its initial conditions $(J_i(t_{i-1}), \frac{DJ_i}{dt}(t_{i-1}))$, by condition (i) in the Definition 4.4.1 we see that a Jacobi field J along a geodesic $\mathscr{G}$-path c is uniquely determined by $(J_1(0), \frac{DJ_1}{dt}(0))$. So there are $2n$ linearly independent Jacobi fields, each of which can be defined throughout c. Moreover, note that the vector fields $\dot{c}(t)$ and $t\dot{c}(t)$ are Jacobi fields along c. The first one has the derivative zero and vanishes nowhere, and the second one is zero if and only if $t = 0$. Note also that the Jacobi fields do not depend on a particular choice of c in its equivalence class. That is, if $c' = (g'_0, c'_1, g'_1, c'_2, \dots, c'_k, g'_k)$ is another representative of the geodesic $\mathcal{G}$-path c over the same subdivision then $J' = (J'_1, J'_2, \dots, J'_k)$ given by $J'_i(t) = \big(dh_i(t)\big)_{c_i(t)}(J_i(t))$ is a Jacobi field along c' which is uniquely determined since the h_i's are unique by Remark 3.7.2 (d).

As in the manifold case, one can prove that every Jacobi field along a geodesic $\mathcal{G}$-path c may be obtained by a one-parameter variation of c through geodesics. That is, a sequence $\nu = (\tau_0, \nu_1, \tau_1, \nu_2, \dots, \tau_{k-1}, \nu_k)$ associated to the division $0 = t_0 \leq t_1 \leq \dots \leq t_k = 1$, where

(i) for each $i = 1, \dots, k$, $\nu_i : (-\varepsilon, \varepsilon) \times [t_{i-1}, t_i] \to X$ are differentiable functions such that $\nu_i(0,t) = c_i(t)$ and each $\overline{\nu}_i(s)$, $s \in (-\varepsilon, \varepsilon)$ given by $\overline{\nu}_i(s)(t) = \nu_i(s,t)$ is a geodesic, and

(ii) each $\tau_i : (-\varepsilon, \varepsilon) \to \mathcal{G}$, $i = 0, \dots, k$ are $\mathcal{G}$-valued differentiable functions such that $\tau_i(0) = g_i$ and such that $\alpha(\tau_i(s)) = \nu_i(s, t_i)$ and $\omega(\tau_i(s)) = \nu_{i+1}(s, t_i)$ for any $s \in (-\varepsilon, \varepsilon)$ and any $i = 0, \dots, k-1$.

We will give a construction of Jacobi fields along a geodesic $\mathcal{G}$-path using the exponential map $\exp_x : K \to X$, the base map of the morphism in 4.1. Here we will consider only Jacobi fields which satisfy $J(0) = 0$, but analogous constructions can be obtain in the general case.

Let c be a geodesic $\mathcal{G}$-path and J a Jacobi field along c with $J(0) = 0$. Denote $v = \dot{c}(0) \in T_{c(0)}X$ and $w = \frac{DJ}{dt}(0) \in T_v(T_{c(0)}X)$ and construct a path $v(s)$ in $T_{c(0)}X$ with $v(0) = v$ and $\dot{v}(0) = w$. Put

$$\nu(s,t) = \exp_{c(0)}(tv(s))$$

and define the Jacobi field $\overline{J}$ along c by

$$\overline{J}(t) = \frac{\partial \nu}{\partial s}(0,t).$$

Note that at $s = 0$ we have

$$\frac{D}{dt}\frac{\partial \nu}{\partial s} = \frac{D}{\partial t}((d\exp_{c(0)})_{tv}(tw)) = \frac{D}{\partial t}(t(d\exp_{c(0)})_{tv}(w))$$
$$= (d\exp_{c(0)})_{tv}(w) + t\frac{D}{\partial t}((d\exp_{c(0)})_{tv}(w))$$

Therefore, for $t = 0$

$$\frac{D\overline{J}}{dt}(0) = \frac{D}{\partial t}\frac{\partial \nu}{\partial s}(0,0) = (d\exp_{c(0)})_0(w) = w.$$

Since the initial conditions are $J(0) = \overline{J}(0) = 0$ and $\frac{DJ}{dt}(0) = \frac{D\overline{J}}{dt}(0) = w$, from the uniqueness of the Jacobi fields, we conclude that $J = \overline{J}$.

We have just proved that if c be a geodesic $\mathscr{G}$-path, then a Jacobi field J along c with $J(0) = 0$ is given by

$$J(t) = (dexp_{c(0)})_{t\dot{c}(0)}(t\frac{DJ}{dt}(0)), \ t \in [0,1]. \tag{4.4}$$

Definition 4.4.2. *A point $c(t') \in X$ is* conjugate to $c(0) \in X$ *along the geodesic $\mathscr{G}$-path c if there exists a non-zero Jacobi field along c such that $J(0) = 0 = J(t')$. The maximum number of such linearly independent Jacobi fields is called the* multiplicity *of $c(0)$ and $c(t')$ as conjugate points.*

Note that the multiplicity of two conjugate points never exceeds $n - 1$. Indeed, the dimension of the vector space consisting of all Jacobi fields which vanish at $t = 0$ has dimension at most n and the non-zero Jacobi field $J(t) = t\dot{c}(t)$ never vanishes for $t \neq 0$.

Note that this gives on the space of orbits $|Q|$ a well defined notion of conjugate points along a geodesic path and of their multiplicity. Thus two points $x, x' \in |Q|$ on a geodesic path $[c]$ are conjugate if their lifts to X are conjugate along a representative of $[c]$.

The relation between the conjugate points and the critical points of the exponential map is given by the following proposition.

Proposition 4.4.3. *Let c be a geodesic $\mathcal{G}$-path. Then $c(t')$ is conjugate to $c(0)$ along c if and only if the vector $t'\dot{c}(0)$ is a critical point for $\exp_{c(0)}$. Moreover, the multiplicity of $c(t')$ as conjugate point to $c(0)$ is equal to the dimension of the kernel of the linear map $(d\exp_{c(0)})_{t'\dot{c}(0)}$.*

Proof. Let J be a non-zero Jacobi field along c which vanishes at 0 and t'. Denote $v = \dot{c}(0)$ and $w = J'(0)$. Then, from 4.4 $J(t) = (d\exp_{c(0)})_{tv}(tw)$, $t \in [0,1]$ and since $(d\exp_{c(0)})_{tv}$ is linear, we have that $w \neq 0$, as J is not identically zero. But $0 = J(t') = (d\exp_{c(0)})_{t'v}(t'w)$ for $t' \neq 0$ and $w \neq 0$, which is possible if and only if $t'v$ is a critical point of the exponential map at $c(0)$.

For the second part of the proposition note that the Jacobi fields $J^{(1)}, J^{(2)}, \ldots, J^{(m)}$ along c which are zero at $t = 0$ are linear independent if and only if the vectors $(\frac{D}{dt}J^{(1)})(0)$, $\frac{D}{dt}(J^{(2)})(0), \ldots, \frac{D}{dt}(J^{(m)})(0)$ are linear independent in $T_{c(0)}X$. Then, from the construction above, we can see that the maximal number of linear independent Jacobi fields along c which vanish also at $t = t'$, is equal to the maximal number of linear independent vectors such that the linear map $(d\exp_{c(0)})_{t'\dot{c}(0)}$ is zero, i.e. the dimension of its kernel. □ □

We say that x is a *pole* if there are no conjugate points to it on any geodesic $\mathcal{G}$-path starting at x (or actually on any geodesic $\mathcal{G}$-path containing x). Note that if x is a pole, there are no Jacobi fields along any geodesic $\mathcal{G}$-path starting at x which vanish anywhere else than at x. In this case, by the proposition above, we see that the exponential map $\exp_x$ has no critical points, hence it is étale. In the next section we will prove that every point in an orbifold of nonpositive curvature is a pole.

4.5 Orbifolds of nonpositive curvature

Proposition 4.5.1. *Let Q be a Riemannian orbifold with nonpositive sectional curvature. Then any point of Q is a pole.*

Proof. Let $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ be an arbitrary geodesic $\mathcal{G}$-path on X, over a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$, and $J = (J_1, J_2, \ldots J_k)$ a non-zero Jacobi filed along it which vanishes at zero. Then, for each $i = 1, \dots, k$

$$\frac{D^2 J_i}{dt^2}(t) + R(\dot{c}_i(t), J_i(t))\dot{c}_i(t) = 0,$$

holds for every $t \in [t_{i-1}, t_i]$, and then

$$< \frac{D^2 J_i}{dt^2}(t), J_i > + < R(\dot{c}_i(t), J_i(t))\dot{c}_i(t), J_i > = 0.$$

Hence

$$< \frac{D^2 J_i}{dt^2}(t), J_i > = - < R(\dot{c}_i(t), J_i(t))\dot{c}_i(t), J_i > \geq 0,$$

for any $t \in [t_{i-1}, t_i]$. Therefore

$$\frac{d}{dt} < \frac{DJ_i}{dt}, J_i > = < \frac{D^2 J_i}{dt^2}, J_i > + \| \frac{DJ_i}{dt} \|^2 \geq 0$$

i.e. each function $< \frac{DJ_i}{dt}, J_i >$ is monotonically increasing on each $[t_{i-1}, t_i]$ and strictly increasing if $\frac{DJ_i}{dt} \neq 0$ on $[t_{i-1}, t_i]$.

Note that the condition (i) in the Definition 4.4.1 together with the fact that $\mathscr{G}$ is a groupoid of local isometries imply that $< \frac{DJ_i}{dt}(t_i), J_i(t_i) > = < \frac{DJ_{i-1}}{dt}(t_i), J_{i-1}(t_i) >$, for any $i = 1, \dots, k-1$. This defines a continuous function on the interval $[0, 1]$, which we will denote by $< \frac{DJ}{dt}, J >$. Moreover, this function is monotonically increasing on $[0, 1]$ and strictly increasing if $\frac{DJ}{dt} \neq 0$ on $[0, 1]$.

Suppose now that $c(0)$ has a conjugate points along c. Choose the first one. Then, there exists a Jacobi field as above and $t' \in (0, 1]$ such that $J_i(t') = 0$ for some $i \in \{1, 2, \dots, k\}$, and so $< \frac{DJ_i}{dt}, J_i >$ vanishes at t'. Therefore $< \frac{DJ}{dt}, J >$ vanishes at both 0 and t', hence it has to vanish identically on $[0, t']$. This implies that $\frac{DJ_1}{dt}(0) = 0$ and since $J_1(0) = 0$, we have that $J \equiv 0$, which contradicts the fact that J is a non-zero Jacobi field. Hence there are no conjugate points to $c(0)$ along c, and since c is arbitrary, $c(0)$ is a pole and of course any point of Q is a pole. □

In particular, for an orbifold $Q = X/\mathscr{G}$ of nonpositive curvature, the exponential map $\exp_x : T_x X \to X$ has no critical points, hence is étale. In what follows we will show that if the orbifold is also complete the exponential map is also a covering map. This will give a proof of Gromov's developability theorem.

Proposition 4.5.2. *Let $Q = X/\mathscr{G}$ be a complete Riemannian orbifold with nonpositive sectional curvature. Then for any $x \in X$ the exponential map $exp_x : T_xX \to X$ has the path lifting property. In particular, exp_x is a covering map.*

Proof. Since the orbifold is complete, the exponential map $\exp_x : T_xX \to X$ is defined for all the points $x \in X$ and is surjective. Since Q has nonpositive curvature, it is also étale.

This allows us to introduce a Riemannian metric on the tangent space T_xX such that $\exp_x : T_xX \to X$ is a local isometry. Indeed, consider $u \in T_xX$ and put for any $v, w \in T_u(T_xX) \cong T_xX$

$$\mu_u(v, w) := \rho_{\exp_x(u)}(d(\exp_x)_u(v), d(\exp_x)_u(w)),$$

where $\rho_{\exp_x(u)}$ denotes the inner product metric on the tangent space at $\exp_x(u)$ to X induced by the Riemannian metric on the connected component of X containing $\exp_x(u)$. Note that since the Riemannian metric is invariant, the definition is good in the sense that it does not depend on a particular choice in the fiber above $\exp_x(u)$. Then μ_u defines an inner product in each tangent space $T_u(T_xX) \cong T_xX$ which clearly varies smoothly with respect to $u \in T_xX$, i.e. defines a metric on T_xX and the exponential map is a local isometry. Note that, by Proposition 4.3.2, this metric is also complete, since the geodesics in T_xX through the origin are straight lines.

We will show now that $\exp_x$ has the path lifting property. Let $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ be an arbitrarily rectifiable $\mathscr{G}$-path in X, over a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$. Assume that $c(0) = x \in X$ and consider the origin $0 \in T_xX$. Then $\exp_x(0) = x$ and since $\exp_x$ is a local diffeomorphism at 0 we can lift c in a small neighborhood of x. That is, there exists $\varepsilon > 0$ such that we can define $\widehat{c} : [0, \varepsilon] \to T_xX$ with $\widehat{c} = 0$ and $\exp_x(\widehat{c}) = c|_{[0,\varepsilon]}$. Denote by $A \subset [0, 1]$ the set of values for which the $\mathscr{G}$-path c can be lifted to a path starting from $0 \in T_xX$. Then A is nonempty and since $\exp_x$ is a local diffeomorphism on all T_xX, A is open and connected in $[0, 1]$. That is, $A = [0, t')$. If we show that t' is also in A, then A would be also closed and it will follow that $A = [0, 1]$. This means that the $\mathscr{G}$-path c can be lifted throughout the whole interval $[0, 1]$, i.e. $\exp_x$ has the path lifting property.

To show that $t' \in A$, let $t^{(m)}, m = 1, \ldots$ be an increasing sequence in A such that $\lim t^{(m)} = t'$. Since $t' \in (0,1]$, there is $i \in 1, 2, \ldots, k$ such that $t_{i-1} < t' \leq t_i$, and without loss of generality we can assume that $t_{i-1} < t^{(m)} \leq t' \leq t_i$ for all m. Note now that the set $\{\widehat{c}(t^{(m)})\}$ is contained in a compact subset $K \subset T_xX$. Indeed, otherwise, since T_xX is complete the set $\{\widehat{c}(t^{(m)})\}$ would be unbounded, and so the distance between $\widehat{c}(t^{(m)})$ and $\widehat{c}(t_{i-1})$ could be made arbitrarily large. However this is not possible since

$$d(\widehat{c}(t^{(m)}), \widehat{c}(t_{i-1})) \leq L|_{t_{i-1}}^{t^{(m)}}(\widehat{c}) = \int_{t_{i-1}}^{t^{(m)}} |\frac{d\widehat{c}}{dt}|dt = \int_{t_{i-1}}^{t^{(m)}} |d(\exp_x)_{\widehat{c}(t)}(\frac{d\widehat{c}}{dt})|dt$$
$$= \int_{t_{i-1}}^{t^{(m)}} |\frac{dc_i}{dt}|dt = L|_{t_{i-1}}^{t^{(m)}}(c_i)$$

and the length of c_i is finite.

The completeness of T_xX and the fact that $\{\widehat{c}(t^{(m)})\} \subset K$ imply that there is an accumulation point $v \in T_xX$ of $\{\widehat{c}(t^{(m)})\}$. Let V be a neighborhood of v such that $\exp_x|_V$ is a diffeomorphism onto an open neighborhood of $\exp_x(v) \in X_i$. Then $c_i(t') \in \exp_x(V)$ and by continuity of c_i there is a subinterval $I \subset [t_{i-1}, t_i]$ containing t' such that $c_i(I) \subset \exp_x(V)$. Since $\exp_x|_V$ is a diffeomorphism there is a lift of $c_i|_I$ through v, say $\bar{c}$. But $v \in V$ is an accumulation point for $\{\widehat{c}(t^{(m)})\}$, so there exists an index m such that $\widehat{c}(t^{(m)}) \in V$. Since $e_x|_V$ is bijective the lifts $\widehat{c}$ and $\bar{c}$ coincide on the interval $[t_{i-1}, t^{(m)}) \cap I$. Hence $\bar{c}$ is an extension of $\widehat{c}$ to I and so $\widehat{c}$ is defined at t', i.e. $t' \in A$. This completes the proof that $\exp_x$ has the path lifting property. Note that equivalent $\mathscr{G}$-paths in X lift to the same path in T_xX. In particular $\exp_x$ maps the terminal point of the lift $\widehat{c}$ to the terminal point of c. Since $\exp_x$ is étale and surjective, it is a covering map. □

Theorem 4.5.3. (Gromov) *Every connected complete Riemannian orbifold with non-positive curvature is developable.*

Proof. Let $Q = X/\mathscr{G}$ be a such orbifold. By the previous proposition the homomorphism $(\pi, \exp_x) : (\mathscr{G} \ltimes T_xX, T_xX) \to (\mathscr{G}, X)$ (see 4.2) is a covering. By Lemma 4.2.2 the groupoid $(\mathscr{G} \ltimes T_xX, T_xX)$ developable. This implies that $(\mathscr{G}, X)$ is developable and thus Q is developable. □

Remark 4.5.4. The same proof works for the case when $Q = X/\mathscr{G}$ is a complete Riemannian orbifold which contains a pole by using the exponential map at the pole. In the similar way we can prove that the exponential map is a covering map (it is étale, surjective and has the path lifting property). Then the homomorphism given by 4.2 is a covering one which implies the developability of $(\mathscr{G}, X)$, i.e. that of Q.

4.6 Loop spaces for orbifolds of nonpositive curvature

The study of loop spaces of manifolds has proved very important in geometry, topology, representation theory, and more recently, string theory. If we regard orbifolds as a generalization of manifolds, it is a natural question to ask how much of the classical theory for loop spaces can be generalized to the orbifold setting. In this section we combine some known results with some new ideas and present a few open problems for further study.

For an orbifold $Q = X/\mathscr{G}$ we have introduced in section 3.7 the set $\Omega_{x,y}$ of equivalence classes of continuous $\mathscr{G}$-paths connecting x to y, and Ω_X the set of continuous based $\mathscr{G}$-loops to be the union of sets $\Omega_x = \Omega_{x,x}$ of closed $\mathscr{G}$-paths based at x. As in the manifold case (see [K1]) one can prove that these spaces have a natural structure of Banach manifolds (see [GH]). We have also seen that there is a natural continuous (right) action of $\mathscr{G}$ on the space of based loops over the projection $p : \Omega_X \to X$, which associates to any equivalence class $[c]_x \in \Omega_X$ its base point $x \in X$. Given an element $[c]_x \in \Omega_X$ represented by a $\mathscr{G}$-loop $c = (g_0, g_1, \ldots, c_k, g_k)$ based at x and an element $g \in \mathscr{G}$ such that $\alpha(g) = x$, we define $([c]_x).g$ to be the equivalence class of based $\mathscr{G}$-loops at $\omega(g)$ represented by $c.g = (g^{-1}g_0, c_1, \ldots, c_k, g_k g)$. The groupoid $(\Omega_X \rtimes \mathscr{G}, \Omega_X)$ associated to this action (see section 3.5) is called the *loop groupoid* of $(\mathscr{G}, X)$. An important property is that the Morita equivalence class of this loop groupoid depends only on the Morita equivalence class of $(\mathscr{G}, X)$. In the case when $(\mathscr{G}, X)$ represents an orbifold structure Q, the loop groupoid represents an infinite dimensional (Banach) orbifold structure $\Lambda Q = \Omega_X/\mathscr{G}$ over Q (see Remark 3.5.1). The base space $|\Lambda Q|$ of this infinite dimensional orbifold is called the "space" of continuous free orbifold loops ($\mathscr{G}$-loops) on Q. The projection $q : X \to |Q|$ induces a map $|\Lambda Q| \to \Lambda|Q|$ which associates to every free orbifold loop on Q a free loop on the topological space $|Q|$. The subset $|\Lambda^0 Q| \subset |\Lambda Q|$ of free orbifold loops projecting to constant loops on $|Q|$ inherits a natural orbifold structure. It is denoted $\Lambda^0 Q$ and called the *inertia orbifold.* It can also be viewed as the suborbifold of ΛQ invariant under the natural action of $\mathbb{S}^1$. In particular if Q is a smooth manifold, this gives the well known construction of the path and the loop spaces. For manifolds the fixed point set of the $\mathbb{S}^1$ action can be identified with the manifold itself, but for orbifolds the fixed point set (i.e. the inertia orbifold) is a suborbifold of ΛQ lying above Q.

$$\begin{array}{ccc} \Lambda^0 Q & \xrightarrow{\ \subset\ } & \Lambda Q \\ {\scriptstyle p}\big\downarrow & & \\ Q & & \end{array}$$

As in the section 3.8, we can associate classifying spaces to both ΛQ and $\Lambda^0 Q$, which we denote $B\Lambda Q$ and $B\Lambda^0 Q$. Theorem 3.2.2 of [GH] shows that the functor B "commutes" with the construction of loop spaces, in the sense that there is a weak homotopy equivalence

$$B\Lambda Q \simeq \Lambda BQ,$$

where ΛBQ denotes the loop space of the classifying space of Q. Therefore ΛBQ can be thought of as the classifying space for the orbifold ΛQ. Theorem 3.2.2 [GH] further shows that the space Ω_x has the same weak homotopy type as $\Omega_z BQ$, where z is a point in BQ projecting to the same point in $|Q|$ as $x \in X$. An important consequence of this is the fact that the homotopy type of Ω_x is independent of the base point x. If the orbifold is connected the same is true for the path space $\Omega_{x,y}$.

Similar results hold for the inertia orbifold. The *ghost loop space* $\Lambda^0 BQ \subset \Lambda BQ$ is defined as the subspace of the loop space of BQ whose elements project under

$\pi : BQ \longrightarrow |Q|$ to constant loops, and it is shown in [LU3] that

$$B\Lambda^0 Q \simeq \Lambda^0 BQ.$$

For a manifold M, there is a powerful relationship between the algebraic topology of the loop space $\mathscr{L}M$ and that of M. The standard loop fibration shows that $\pi_{i-1}(\mathscr{L}M) = \pi_i(M)$. It turns out that there is an extension of this to the orbifold (and even orbispace) setting. This approach is used in [C] to give an alternative way of defining the orbifold homotopy groups. Recall that these homotopy groups reflect properties of the underlying orbifold structure of Q and are not simply topological invariants of the base space $|Q|$.

Another interesting aspect is the connection between the geometric structure of an orbifold and its orbifold structure. An example in this sense is provided by the previous section, where we have seen that the assumption of nonpositive curvature (a geometric condition) gives developability.

For manifolds, Morse theory provides a powerful tool that connects the algebraic topology of the loop space $\mathscr{L}M$ to the geometry of M. This involves studying the energy functional on the path space [Mi]. It would be interesting to develop an analogue of this for the orbifolds. It would also be interesting to develop a Morse theoretic model for orbifold cohomology, which was introduced by Ruan and Chen in [CR] and is a new homology theory for orbifolds that incorporates the twisted sectors.

As the energy function is not defined on the spaces of continuous orbifold paths (loops), we have to consider certain subsets of orbifold paths (equivalence classes of $\mathscr{G}$-paths), namely those of class H^1. If $Q = X/\mathscr{G}$ is a complete Riemannian orbifold, such an orbifold path is represented by a $\mathscr{G}$-path $c = (g_0, c_1, \ldots, c_k, g_k)$ defined over a subdivision $0 = t_0 \leq t_1 \leq \ldots \leq t_k = 1$ but with the additional requirement that each c_i is of class H^1, i.e. that c_i is absolutely continuous with velocity function $t \mapsto |\dot{c}_i(t)|$ square integrable.

As before we define $\Omega'_{x,y}$ to be the completion of set of orbifold paths of class H^1 connecting x to y, and $\Omega'_X = \bigcup_{x\in X} \Omega'_x$ the set of based orbifold loops of class H^1.

Similar to the case of continuous orbifold paths one can prove that these sets have a natural structure of Riemannian Hilbert manifolds. Moreover, the natural action of $\mathscr{G}$ on Ω_X restricts to an action by isometries on Ω'_X. Therefore on the quotient space $\Omega'_{x,y}/\mathscr{G}$ we get a structure of Riemannian Hilbert orbifold which we denote as $\Lambda' Q$. The set $|\Lambda'^0 Q|$ of fixed points of the natural $\mathbb{S}^1$ action on the space of free orbifold loops of class H^1 has a suborbifold structure denoted $\Lambda'^0 Q$. In Proposition 3.3.5 of [GH] the authors prove that the natural inclusions

$$\Omega'_X \to \Omega_X, \quad \Omega'_{x,y} \to \Omega_{x,y}$$

are continuous and are homotopy equivalences. In particular, if Q is connected, $\Omega'_{x,y}$ has the same weak homotopy type as the space of loops on BQ based at a fixed point. The induced inclusion $B\Lambda' Q \to B\Lambda Q$ is also a homotopy equivalence.

The energy function is defined on all these spaces and it is invariant under the action of $\mathscr{G}$, thus it gives a well defined function on the quotient $|\Lambda' Q|$. The critical points $[c]_{x,y}$ of E on $\Omega'_{x,y}$ are the geodesics $\mathscr{G}$-paths from x to y and the critical points $[c]_x$ of E on Ω'_X are either points in Ω'^0_X (i.e. based $\mathscr{G}$-loops of length zero) or closed geodesic $\mathscr{G}$-paths at x.

Recall that for orbifolds that are global quotients (e.g. complete orbifolds of nonpositive curvature), the path and loop spaces admit a more concrete description as follows.The space $\Omega_{x,y}$ of equivalences classes of continuous orbifold paths from x to y is in bijective correspondence with the set of pairs (c, γ), where $c : [0, 1] \to X$ is a continuous path in X starting at x and γ is an element of Γ which maps $c(1)$ to y. In particular, any equivalence class $[c]_x$ of orbifold loops based at a point x is uniquely described by a pair (c, γ) where this time γ is an element of Γ mapping $c(1)$ to x. If moreover the manifold X is assumed to be simply connected then the set of homotopy classes of free orbifold loops is in bijective correspondence with the set of conjugacy classes in Γ. Furthermore, if $c : [0, 1] \to X$ is just the constant map at x, then the pair (c, γ) with $\gamma \in \Gamma_x$ defines an element of $|\Lambda^0 Q|$. At the level of groupoids, if $(\Gamma \ltimes X, X)$ is the translation groupoid associated to the action of Γ on X (it is also the proper étale groupoid describing the orbifold structure $Q = X/\Gamma$) then the loop groupoid is also developable. In fact it is Morita equivalent to a translation groupoid, defined in the following way (see [LU1]). The set of objects is given by the disjoint union of sets $\mathscr{P}_\gamma$ of all pairs (c,γ), where $\gamma \in \Gamma$ and $c : [0,1] \to X$ is a continuous path such that $c(0) = \omega(g)$ and $c(1) = \alpha(g)$. On this set there is a natural Γ left action by Γ by translation in the first component and conjugation in the second one: $\delta.(c,\gamma) \mapsto (\delta.c, \delta\gamma\delta^{-1})$. Then the loop groupoid is Morita equivalent to the translation groupoid associated to this action

$$\Big(\Gamma \times \coprod_{\gamma\in\Gamma} \mathscr{P}_\gamma \,, \coprod_{\gamma\in\Gamma} \mathscr{P}_\gamma\Big).$$

The elements in $\mathscr{P}_\gamma$ which are invariant under the natural $\mathbb{S}^1$ action can be identified with the set pairs (x,γ) where $\gamma \in \Gamma$ and x belongs to the set X^γ of points of X fixed by the action of γ. Then the inertia groupoid is Morita equivalent to the translation groupoid

$$\Big(\Gamma \times \coprod_{\gamma\in\Gamma} X^\gamma \times \{\gamma\}, \coprod_{\gamma\in\Gamma} X^\gamma \times \{\gamma\}\Big).$$

In the case when $Q = \widetilde{Q}/\Gamma$ is a complete orbifold of nonpositive curvature, there is an exact description of the space of geodesics connecting two points x and y in the orbifold. Since Q is complete and has nonpositive curvature, the manifold $\widetilde{Q}$ is also complete and also has nonpositive curvature. It follows that for any two points $\widetilde{x}$ and $\widetilde{y}$ in $\widetilde{Q}$ which project to x and y respectively, there is precisely one geodesic connecting them. This gives a unique geodesic orbifold path for each element $\gamma \in \Gamma$. Note that we can identify Γ with $\pi_1^{orb}(Q)$. If we assume that the fundamental theorem for Morse theory holds, then the homotopy type of the path space $\Omega_{x,y}$ (or equivalently $\Omega'_{x,y}$) is that of a 0-dimensional CW complex with one cell for each homotopy class in $\pi_1^{orb}(Q)$.

Another interesting problem is that of the existence of closed geodesics of positive length on compact Riemannian orbifolds. In [GH] this question is answered in the affirmative for bad orbifolds. For two dimensional orbifolds, one can also prove the existence of closed geodesics of positive length as follows. It follows mainly from the fact that in dimension $d = 2$ every good orbifold is actually very good, i.e. it admits a finite cover by a manifold (see Theorem 2.5 in [Sc]). Morse theory for compact manifolds provides a closed geodesic on the finite cover, and this pushes down to a closed geodesic on the orbifold.

For orbifolds of dimension $d > 2$ the problem of existence of closed geodesics of positive length is still open. The best results on this problem are contained in Theorem 4.2.1 of [GH].

The only remaining case is that of a compact Riemannian orbifold which is a global quotient M/Γ of a Riemannian manifold M by an infinite group Γ acting by isometries properly discontinuously such that all of the elements of Γ have finite order (see Remark 4.2.2. in [GH]). Of particular interest here is the case when Q is a compact orbifold of nonpositive curvature. Of course the answer follows if one shows Q is very good, and perhaps every compact Riemannian orbifold of nonpositive curvature is very good. On the other hand, even if they are not all very good, perhaps every compact Riemannian orbifold of nonpositive curvature admits a manifold cover $\widehat{Q} \to Q$ with $\pi_1(\widehat{Q})$ nontrivial. (In this case $\widetilde{Q} \to \widehat{Q}$ is a universal cover and $\pi_i(\widehat{Q}) = \pi_i(\widetilde{Q}) = 0$ for all $i \geq 2$.) Morse theory for manifolds shows that $\widehat{Q}$ contains a nontrivial closed geodesic since $\pi_1 \neq 1$, and this geodesic pushes down to a closed geodesics of positive length on the orbifold Q.

Bibliography

[BGS] W. Ballmann, M. Gromov, V. Schröder, *Manifolds of nonpositive curvature*, Progress in Mathematics, **61**, Birkhäuser Boston, Inc., Boston, MA, 1985.

[B] A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, **91**, Springer-Verlag, New York, 1995.

[BH] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften, **319**. Spinger-Verlag, Berlin, 1999.

[BL] J. E. Borzellino and B. G. Lorica, *The closed geodesic problem for compact Riemannian 2-orbifolds*, Pacific J. Math. **175** (1996), No. **1**, 39–46.

[C] W. Chen *A homotopy theory of orbispaces*, `math.AT/0102020 v3`.

[CR] W. Chen and Y. Ruan, *Orbifold Gromov-Witten theory*, Orbifolds in mathematics and physics (Madison, WI, 2001), 25–85, Contemp. Math. Soc., **310**, AMS, Providence, RI, 2002.

[dC] M. P. do Carmo, *Riemannian geometry*, Mathematics: Theory and Aplications, Birkhäuser Boston, Inc., Boston, MA, 1992.

[E] C. Ehresmann, *Sructures locales,* Annali de Matematica, Serie IV, **36** (1954), 133–142.

[GH] K. Guruprasad and A. Haefliger, *Closed geodesics on orbifolds*, `math.DG/0306238`.

[H] A. Haefliger, *Groupoids d'holonomie et classifiants*, Structures transverses des feiulletages (Touluose 1982), Astérisque **116** (1984), 70–97.

[K] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics, **183**, Birkhäuser Boston, Inc., Boston, MA, 2001.

[Ka] T. Kawasaki, *The signature theorem for V-manifolds*, Topology, **17** (1978), 75–83.

[K1] W. Klingenberg, *Closed geodesics on Riemannian manifolds*, CBMS Regional Conference Series in Mathematics, No. **53**, AMS, Providence, RI, 1982.

[K2] W. Klingenberg, *Lectures on closed geodesics*, Grundlehren der mathematischen Wissenschaften, **230**. Spinger-Verlag, Berlin, 1978.

[LU1] E. Lupercio and B. Uribe, *Gerbes over orbifolds and twisted K-theory*, Comm. Math. Phys. **245** (2004), No. 3, 449–489.

[LU2] E. Lupercio and B. Uribe, *Loop groupoids, gerbes, and twisted sectors on orbifolds*, Orbifolds in mathematics and physics (Madison, WI, 2001), 163–184, Contemp. Math., **310**, AMS, Providence, RI, 2002.

[LU3] E. Lupercio and B. Uribe, *Inertia orbifolds, configuration spaces and the ghost loop space.* Q.J. Math. **55**, (2004), No. 2, 185–201.

[Mi] J. Milnor, *Morse theory.* Annals of Mathematics Studies, No. **51**, Princeton University Press, Princeton, N.J. 1963.

[M] I. Moerdijk, *Orbifolds as groupoids: an introduction*, Orbifolds in mathematics and physics (Madison, WI, 2001), 205–222, Contemp. Math., **310**, AMS, Providence, RI, 2002.

[MM] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids.* Cambridge Studies in Advanced Mathematics, **91**. Cambridge University Press, Cambridge, 2003.

[MP] I. Moerdijk and D. Pronk, *Orbifolds, sheaves and groupoids*, K-Theory, **12** (1997), 3–21.

[Ra] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, **149**, Springer-Verlag, New York, 1994.

[Sa1] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363.

[Sa2] I. Satake, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan **9** (1957), 464–492.

[Sc] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1993), No. **5**, 401–487.

[Se] G. Segal, *Classifying spaces and spectral sequences*, Publications mathématiques de l'I.H.É.S., **34**, (1968), 105–112.

[T1] W. P. Thurston, *The geometry and topology of 3-manifolds*, Chapter 13. Princeton University, 1978-1979.

[T2] W. P. Thurston, *Three-dimensional geometry and topology*, Volume 1, Princeton Mathematical Series, **35**, Princeton University Press, Princeton, NJ, 1997.