PRECISION MEASUREMENT

OF

HIGH DIRECT VOLTAGE

1840

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HIGH DIRECT VOLTAGE

By

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SCOPE AND CONTENTS:

This thesis describes a new method to measure high direct voltage of the range of 100 KV to 200 KV with very high accuracy. The principle, based on the capacitive divider method, is presented in the first part.

This thesis is, however, principally concerned with the design of the low voltage capacitor of the divider and its accuracy. The investigations made lead to an instrument having the accuracy required, which is 0.1 partsper million.

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Introduction

This thesis describes a new method of measuring high direct voltage of the 100 KV to 200 KV range with an accuracy of about 2 parts per million.

The AECL/McMaster Research contract is related to the absolute calibration of the β-ray spectrometer of AECL.

A study of the method has been done and led to a division of the research into well defined parts:

Firstly the investigation of the limit of the possibility of detecting charges.

Secondly the switching device which presented some important problems.

Lastly the capacitive divider itself has been studied. This led to detailed investigations of the design of the high voltage capacitor as well as the low voltage one.

After a short survey, done in chapter I, of several known methods of measuring high direct voltage, the capacitive divider method is presented in chapter II.

The main part of the thesis is concerned with the design of the low voltage capacitor. Calculations and results are presented in chapter III.

Chapter I

Methods of Measurements of High Direct Voltage

For the determination of high direct voltage, five methods are of practical interest:

a) by means of gas gaps;

b) by measurement of electrostatic force in a voltage balance;

c) by measurement of the shortest wave length of radiation obtained from a discharge tube connected to this voltage;

d) by measurement of current in a known resistor;

e) by measurement of charge in a known capacitor.

These five methods of measurement can be divided into two main groups. The measurement by means of gas gaps is clearly the simplest one and does not require any elaborate equipment. The four other methods yield a much higher accuracy in measurement of the voltage but require more complicated pieces of equipment.

Voltage measurements by means of gas gaps.

The principle of the method is known: breakdown voltage between electrodes of a given shape in a gaseous atmosphere is a function of electrode spacing.

The breakdown voltage is a function of many parameters and generally a non-linear relationship to spacing exists. Because of simplicity, measurements are usually carried out in air.

Basically there are three different types of gaps:

1) uniform and quasi-uniform field gap

2) non-uniform field gap

3) geometrically non-uniform field gap.

Results are affected by humidity and air density. As a consequence, all measurements are normally referred to the so-called normal air condition: 20°C; 760 mm Hg; 11g m^{-3} H_2O.

Uniform field gap

Although not internationally accepted, the uniform field gap is the most accurate of all methods involving gaps. The measuring electrodes consist of parallel planes of circular shape with rounded edges according to a Rogowski profile. Even between parallel planes the function of breakdown voltage versus gap spacing is not linear, it is slightly concave to the axis of abscissa.

With proper precaution, the absolute accuracy of measurement is 0.3% to 0.5% depending on gap spacing. The method is sensitive to deviations of parallelism of the plates and to surface smoothness and flatness. For measurements of impulse voltage and a.c. voltage a major disadvantage of the uniform field gap is its increase of capacitance to infinity approaching zero spacing.

Quasi-uniform field gap

This type of field is represented by the sphere gap. This is the only internationally approved method involving gas gaps.

The absolute accuracy is always better than 3%, and when taking all necessary precautions often better than 1%.

Small values of spread of individual measurements call for small ratio of spacing s to sphere diameter D. In practice the ratio $\frac{s}{D} \leq 0.75$ is used for measurement of direct voltage.

When unsymmetrical voltages are applied, spark-over voltage for a given spacing will be independent of polarity up to a certain critical spacing, beyond which voltage measurement is dependent upon polarity. This critical point is called Toepler point.

For the quasi-uniform field gap, the function of breakdown voltage versus gap spacing is also non-linear and concave to the axis of abscissa.

Non-uniform field gap

Rod gaps are sometimes used for voltage measurements. The influence of humidity is very pronounced and is also a function of gap spacing.

Needle gaps were formerly used. The results were reasonably accurate. A disadvantage is the need of exchanging the needles for each single measurement.

The lower part of the characteristic breakdown voltage versus gap spacing, for needle gaps is again non-linear but is, however, independent of humidity. The upper part is linear, but is affected very much by humidity.

Geometrically non-uniform field gap

The two representatives of this type of field arrangement are the crossed wire gap and the wire-plane gap. Both types of gaps are suitable for measurement of direct voltage. The curves are strictly linear and the spread of individual measurements is smaller than 1%. A disadvantage is the rather heavy corona currents immediately preceeding breakdown, which may make the method unsuitable if the source of supply is not sufficiently powerful.

Although the gas gap method is very simple and reliable, it will not satisfy the research requirements since its accuracy is short of at least 3 orders of magnitude.

b) Measurement of electrostatic force in a voltage balance

Mechanical force exerted on a plane of a parallel plate system (uniform field) may also be used for measurement of voltage. In order to obtain very high accuracy, one plate, at the grounded side, may be connected to a balance and the force acutally "weighed". According to the National Bureau of Standards, the maximum accuracies obtained at voltage levels of 250 KV are a few parts in 10^4.

The accuracy obtained is much better than that of the first method, but it is still deficient by almost two orders of magnitude.

c) Determination of high direct voltage by measurement of the shortest wave length of radiation emitted in a discharge tube connected to this voltage.

If a highly evacuated X-ray tube is connected to a constant source of voltage, then a continuous spectrum will be emitted from the anti-cathode of the tube and there is a definite limit of shortest emitted wave length.

The tube contains three electrodes: the cathode, the anode and the anti-cathode. The tube is evacuated to a pressure of the order of 10^{-4} mm Hg for the hot cathode type, so that the mean free path is greater than the dimensions of the tube.

When sufficiently high a voltage is applied to the tube, a discharge will develop between cathode and anode. At such pressures this discharge consists of cathode rays. They will collide with the anti-cathode. The energy with which they strike the target corresponds to the voltage applied to the tube by which they were accelerated. The deceleration process in striking the anti-cathode gives rise to emission of X-rays with a continuous spectrum.

Unlike the spectrum of light, this X-ray spectrum (Brems spectrum) shows a sharp lower limit of wave length of radiation, which is a function of the voltage applied. The energy of the X-rays is identical to the energy of the electrons having been accelerated by the voltage applied to the tube. This was discovered by Duane and Hunt. They found that the product of accelerating voltage and wave length of the lowest limit is constant and equal to $12.2\ 10^3$ VÅ.

This method is being used to measure high direct voltage with excellent precision. The difficulty consists of determining where the intensity is really zero. The establishment of this wave length limit is possible to an accuracy of a few parts in 10^5 at present.

This accuracy is very good. It is thought, however, that other methods will lead to slightly better accuracies.

d) Determination of high direct voltage by measurement of current in a known resistor.

If probability of breakdown in the high voltage circuit can be ignored, then stray capacitance effect need not be considered and relatively high resistances carrying only small currents may be used. A suitable order of magnitude of current is 10^{-4} to 10^{-3} A. Nevertheless the heat dissipation in such resistors may well be of the order of 100 W or 1 KW.

In order to measure accurately, the current entering the high voltage end must be the same as the current flowing through the measuring instrument on the grounded end; moreover, the value of the resistance must stay constant. The first requirement is therefore negligible corona loss and leakage current loss on the resistor. The second requirement calls for the use of reliable resistors with small temperature coefficient of resistivity, small temperature rise and negligible voltage coefficient.

Corona can be practically avoided if the resistor is subdivided into units with a voltage drop of only 1 to 5 KV each. These units are enclosed in metal cans, connected to the mid-tap of each unit.

For resistors of very high accuracy, the shielding cans are connected to appropriate taps of an auxiliary resistor network, which supplies the potential of the cans and encloses the actual system of measuring resistors.

For accuracy requirements, only special wire-wound resistors may be used. The resistor network is then laid out as a voltage divider and the voltage between tap point and ground is determined with the voltage of a reference (standard cell) by means of a d.c. comparator.

An accuracy of 10 ppm for voltages of the order of 100 KV is possible.

Such an accuracy is very close to what is needed for the development of a method of calibrating high direct voltage. Therefore detailed consideration has been given to the design of a shielded resistive divider, which would be made of individual wire-wound units supplied by Julie Research Corporation.

It has been found that an absolute accuracy of less than 10 ppm can be expected from such an instrument, providing individual units are assorted and matched in pairs so that individual thermal coefficientsof resistivity almost cancel one another.

An auxiliary resistor network would supply the potential to several points of the shield in order to avoid any corona loss.

A special cooling system would be provided to minimize as much as possible variations in temperature.

The measurement of high direct voltage achieved by measurement of a charge in a known capacitor, method which is discussed in chapter II, has however, been preferred to the resistive divider method.

This choice is mostly due to the fact that the measurement of high direct voltage with the resistive divider method demands that the accuracy of all parts of the instrument be stretched to the utmost limit. Therefore any further improvement seems unlikely to be achievable.

Chapter II

Determination of high direct voltage by measurement of a charge on a known capacitor.

Principle of the method

The principle of the capacitor method consists of determining a high direct voltage by applying it to a capacitor of known value, and then to neutralize its charge with a known inverse charge. The basic circuit is shown on figure 1.

The circuit is operated in the following way:

At a certain time instant, switches S_1 and S_2 are put into position "a" and capacitors C_1 and C_2 are charged from sources G_1 and G_2 with voltages of inverse polarity.

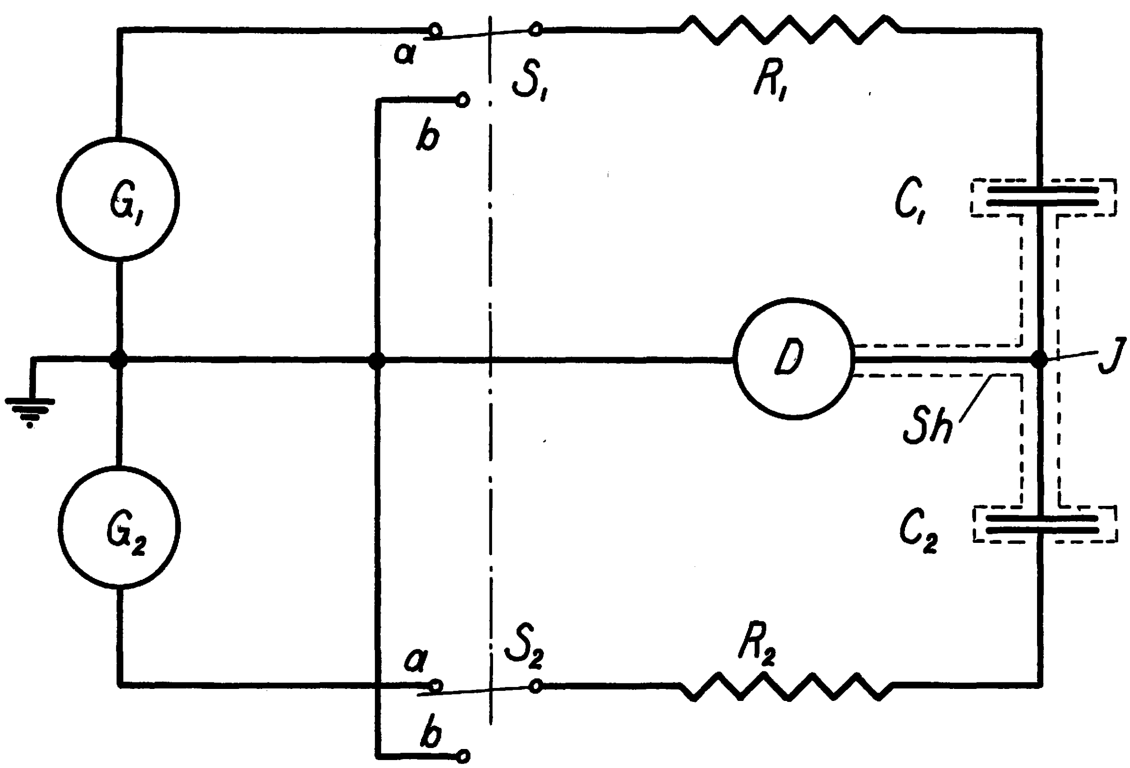
If at the end of the charging period, charges on capacitors C_1 and C_2 are inverse and of identical values, the potential of the junction J will remain unchanged and the total charge on the shielded portion of the circuit will be zero. If the time constants R_1C_1 and R_2C_2 are identical, then the charge will be zero at all times and no voltage will be detected by detector D.

The basic equations are:

$$Q_1 = C_1V_1 \quad \text{and} \quad Q_2 = C_2V_2$$

$$Q = Q_1 + Q_2$$

$$T = R_1C_1 \quad \text{and} \quad T_2 = R_2C_2$$



G_1 unknown, stabilized high direct voltage source

G_2 known, stabilized low direct voltage source

S_1 ; S_2 high and low voltage switches

C_1 high voltage capacitor of known value

C_2 low voltage capacitor of known value

R_1 ; R_2 high and low voltage resistors

D residual charge detector

J junction

Sh shield

Fig. 1

$$T = T_1 = T_2$$

The letter V is used for voltages, C for capacitances, Q for charges, T for time constants and R for resistances. Index 1 indicates the high voltage circuit and index 2 indicates the low voltage one.

The determination of the unknown voltage is done with the expression:

$$V_1 = -V_2 \frac{C_2}{C_1}$$

Satisfying the equation $Q_1 + Q_2 = 0$ is best achieved by adjusting voltage V_2 rather than the capacitance ratio $\frac{C_2}{C_1}$. In this case the values of C_1, C_2, R_1 and R_2 are constant.

Voltage V_2 may be expressed in terms of voltage V_o of a standard cell using direct voltage comparator techniques, and the ratio $\frac{C_2}{C_1}$ may be determined in an a.c. bridge by measurement of the ratio of currents with the aid of a current comparator.

From the measurement with switches in position "a", essentially the charge stored in capacitor C_1 has been determined. When S_1 and S_2 are thrown into position "b", capacitors C_1 and C_2 will be discharged again with identical time constants and thus no change in potential should occur at the junction.

Limits of accuracy

The final limitation of possible accuracy is set by the accuracy to which the reference voltage V_o of the standard cell is known. The total error of measurement, when assuming stable conditions of the circuit components during one measuring cycle, consists of the sum of four main errors:

ε_1 : error of determination of the reference voltage V_o

ε_2 : error of measurement of the ratio $\frac{V_2}{V_o}$

ε_3 : error of null detection of charge Q

ε_4 : error of measurement of the ratio $\frac{C_2}{C_1}$

The total error, when considering ideal circuit components, is:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$$

When taking into account non-ideal components, half a dozen of individual errors have to be added. Therefore the total error reads:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{10}$$

where

ε_5 : error introduced by capacitor C_1

ε_6 : error introduced by capacitor C_2

It seems realistic to expect ε not to exceed 2 parts per million.

<u>Order of magnitude of ratios and of components.</u>

The ratio of the order of magnitude of voltage V_1 to voltage V_o is 10^4, if we assume 10 standard cells connected in series. In order to obtain minimum error, it is best to make the capacitance ratio $\frac{C_2}{C_1}$ large, for example 10^3, and the voltage ratio $\frac{V_2}{V_o}$ small, say 10, since the accuracy in the determination of $\frac{C_2}{C_1}$ is much higher than in the measurement of $\frac{V_2}{V_o}$. The ratio $\frac{C_2}{C_1}$ may be determined under a.c. with the Schering -

Kusters-Petersons Bridge. The voltage ratio is obtained by d.c. comparator techniques. It may be assumed that with most modern techniques, the errors ε_2 and ε_4 become about equal if the following ratios are chosen:

$$\frac{C_2}{C_1} = 1000 \qquad \frac{V_2}{V_o} = 10$$

Then ε_2 and ε_4 should not exceed 0.1 ppm.

If the condition of unbalance at the junction is measured in terms of charge rather than voltage, it does not matter how the ratio 10^4 is split. The total charge, however, is important. It must be taken so large, that the error ε_3 will not exceed the order of 0.1 ppm. The most suitable method to detect the residual charge is by means of a vibrating reed electrometer. It will permit the detection of 0.1 ppm of the total charge on a 10 pF capacitor charged to 100 KV.

To choose the optimum values for ratios $\frac{C_2}{C_1}$ and $\frac{V_2}{V_1}$, and the magnitude of C_1 and C_2, two more considerations are important.

It must be physically realistic to make the capacitor C_2. When C_1 = 10 pF and $\frac{C_2}{C_1}$ = 1000 the value of C_2 is equal to 10 nF, which is quite suitable. Of course it is also possible to design C_1 = 10 pF.

The value of the low voltage V_2 should be chosen from considerations of simplicity of the comparator, of the design of C_2 and of the low voltage source G_2. This voltage, however, should not be too low, because thermal emfs and contact potentials would become important in the low voltage circuit. With the ratio $\frac{V_2}{V_o}$ = 10 the value V_2 is equal to 100 V which is convenient.

Therefore the following values have been chosen:

$$\frac{C_2}{C_1} = 1000 \qquad C_1 = 10\,pF \qquad C_2 = 10\,nF$$

$$\frac{V_2}{V_o} = 10 \qquad V_1 = 150\ KV \qquad V_2 = 150\ V$$

The capacitive divider

General considerations on high precision capacitors

As seen earlier, the capacitive divider has a ratio of 1000 and consists of a high voltage capacitor $C_1 = 10\,pF$ rated for a voltage of 150 KV and of a low voltage capacitor $C_2 = 10\,nF$ rated for 150 V.

In general a capacitor will have three types of errors:

ε' : error of value due to maladjustment

ε'' : error caused by temperature variations

ε''' : error which is a function of applied voltage

These three errors will be discussed in more detail in chapter III **when treating** the design of capacitor C_2. It can already be mentioned that ε''' has not to be considered in capacitor C_2.

Errors of the two components will tend to cancel one another in a divider if C_1 and C_2 have properties in common.

It is clear that capacitors of high quality may not contain solid or liquid dielectrics which would invariably give rise to losses and dielectric absorption. Therefore the measuring electrode system must not see solid dielectric, which means that no line of force originating from any point of the measuring electrode must be permitted to touch or to end on solid dielectric. In order to check this, the entire field map must be known.

Since solid or liquid dielectric cannot be used, the only suitable dielectrics are either vacuum or a gas. Vacuum is the purest dielectric; it is loss-free and its dielectric constant is invariable and well known. It has, however, some disadvantages as compared with some gases:

Heat transfer is restricted to radiation alone. Vacuum of the order of 10^{-5} mm Hg will be required, and it is difficult to realize this practically in a capacitor of large geometrical dimensions.

All vacuum capacitors contain residual gases. This is unimportant at any pressure, as long as voltage applied to their electrode system is smaller than the Paschen minimum value. If this is not the case, instabilities in field configuration will appear and these will become important in poor vacuum. Compressed gas is therefore chosen as being more suitable.

Surface smoothness as obtained by honing and machine lapping is sufficient to avoid losses up to breakdown in the ten-atmosphere-pressure range compressed gas capacitor.

A sealed-off capacitor system requires a container. Mechanical complications are associated with it, but its complexity is not very sharply influenced by the pressure for which it is designed, as long as pressures remain in the 10 atmosphere range.

Here are some advantages of compressed gas capacitors:

The relative dielectric constant of a sealed-off volume of an ideal gas is invariant, since it depends only on density. Pressure changes caused by variations in temperature have, therefore, no influence.

A sealed-off atmosphere will also reduce temperature gradients, the source of these variations being external.

Onset or breakdown stress of suitable gases increases almost proportionally to pressure up to 15 or 20 atmospheres. Thus geometrical dimensions will be reduced inversely proportional to pressure.

It has been decided to choose a pressure of 14 atmospheres for both high and low voltage capacitors for reasons of similarity of performance.

There are many factors influencing the choice of the gas. A careful survey of all the necessary properties, which are known, has led to the conclusion that the best gas is pure nitrogen. This gas at 14 atmospheres can still be considered as an ideal gas.

Experience has shown that the last step of gas purification is obtained inside the capacitor by electro-deposition of impurities. It is claimed that this method is of permanent nature. It is done by subjecting the capacitor to a voltage, which will cause internal breakdown or flash-over. In order to prevent damage to the surface of the measuring electrode or the corresponding portion of the high voltage electrode, internal breakdown stress must be slightly higher there, than between other internal parts. Furthermore, it is clear that the value of external flash-over must be higher than the internal value.

Chapter III

General consideration on the capacitor C_2

In the previous chapters the position of the capacitor in the electrical circuit and its functions have been explained. Perhaps it would be convenient to recall that its capacitance is 10nF or 10,000pF and its accuracy $\pm 10^{-7}$ or speaking in the language of parts per million (p.p.m.) 0.1 p.p.m.

As far as the design is concerned, the word accuracy in the past sentence, bears two different meanings. Firstly it means that once the parts of the capacitor have been machined and assembled, its capacitance must be 10nF $\pm$0.1 p.p.m. Secondly, this value must not vary by more than 0.1 p.p.m. for a temperature change of $1^{o}C$.

Therefore the specifications the design has to meet are:

1) capacitance 10nF
2) accuracy $\pm$ 0.1 p.p.m.

The errors ε', ε'', ε'''

The errors encountered in designing high precision gas filled capacitors can be divided into three main categories:

an error ε' in the value of the capacitance (or maladjustment)

an error ε'' due to variation of temperature

an error ε''' which is a function of the voltage applied and due to the action of electrostatic forces.

The error ε''' does not appear in the design of the low voltage capacitor C_2, the dielectric stress being much too low. However in the high voltage capacitor ε''' must be taken into account.

Later it will be shown how the errors ε' and ε'' can be minimized in order to meet the specifications.

The coaxial design

Though the coaxial design has never been used for low voltage capacitors of high accuracy, it is thought that the requirements can be more easily reached with it than with the parallel plate system.

The capacitance of two coaxial cylinders of radii R and r is given by the well known formula:

$$C = \frac{2\pi\varepsilon_o\varepsilon}{\ln\frac{R}{r}} \text{ pF cm.}^{-1}$$

ε_o = 0,08854 pF cm.$^{-1}$ ε : relative dielectric constant

For all further calculations the capacitance will be given per unit length of the cylinders.

The use of only two cylinders would have involved geometrical dimensions of much too large values.

It was decided to use seven cylinders. Four of them, bearing numbers 1, 3, 5, 7 are called voltage cylinders and the three others are the measuring cylinders, as shown on Figure 2. Their length is approximately 36 cm. The exact value will be found later.

The radii are:

Cylinder	R = inner radius [cm.]	r = outer radius [cm
1	9.85	10.35
2	10.50	11.00
3	11.15	11.65
4	11.80	12.30
5	12.45	12.95
6	13.10	13.60
7	13.75	14.25

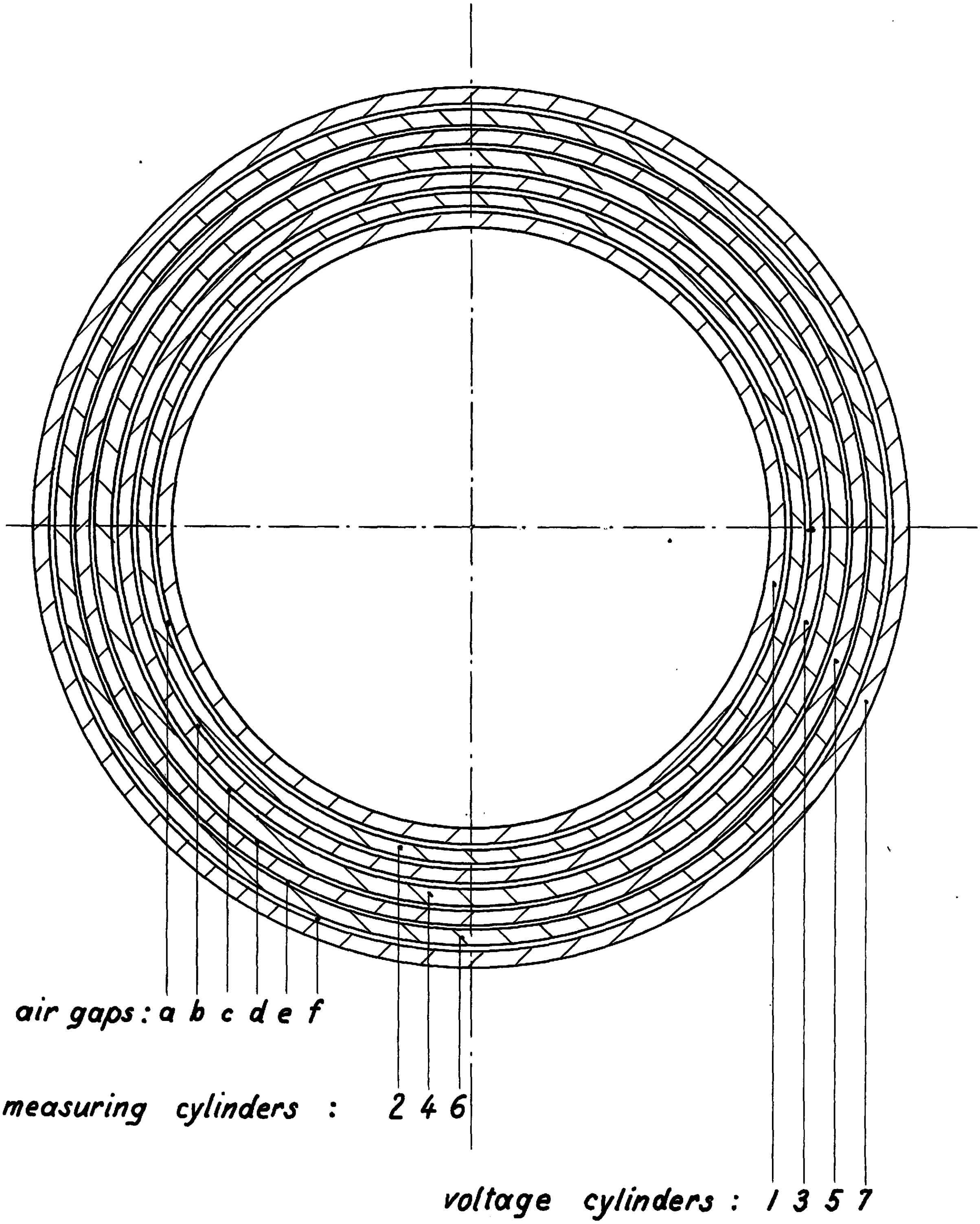

air gaps : a b c d e f
measuring cylinders : 2 4 6
voltage cylinders : 1 3 5 7

Fig. 2

From these figures it appears that the thickness of the wall of the cylinder is 5 mm, and that the air gap is 1.5 mm wide. One might think that this last value is relatively large. However it is the lowest one which could be chosen for reasons of machining tolerances.

As can be seen on the general assembly drawing (Figure A), the measuring cylinders are supported by guard rings at the bottom end. Their purpose is to keep the field as even as possible in the vicinity of the end of the electrode. It was first decided to put guard rings at both ends of the electrodes but then the problem of bringing the connections of the top ones out had to be solved by using a lead inside a hole along the wall of the measuring cylinders. This involved great difficulties in machining. Moreover the insulation level of the measuring electrode to shield must be kept very high. The insertion of an insulated wire in the wall of the cylinder would bring down this level by several orders of magnitude. For this reason the idea of a second guard ring was dropped.

The error ε'

A close examination of the influence of the machining tolerances has to be taken in order to find out what is the deviation in capacitance after assembly of all the different parts of the capacitor.

A very small field of tolerances would bring the capacitance near the desired value of 10nF. Only a fine and simple means of adjustment would be needed. This would involve great difficulties in machining. It is thought that the smallest value of tolerances which can be chosen involving only great care in machining but without big difficulties is $\pm$ 6 µm in diameter.

Let $\Delta_R = \pm 3\,\mu m$ be the tolerances in the radius.

The capacitance per unit length is:

$$C = K \frac{1}{\ln \frac{R}{r}} \ \text{pF cm}^{-1}$$

$K = 2\pi\ \varepsilon_o \varepsilon$ pF cm $^{-1}$

R = inner radius of cylinder

r = outer radius of next smaller cylinder

R-r = air gap

The biggest incremental change in capacitance is found for the worst set of tolerances corresponding to the smallest value of the air gap: $\beta_1 = 1.5\,mm - 6\,\mu m$. The other extreme case is $\beta_2 = 1.5\,mm + 6\,\mu m$ and will be assumed to produce the same change but with negative sign. This is theoretically incorrect, however when β is so small compared to the radius, the variations of capacitance can be assumed to follow a linear law within such an interval.

Case 1: air gap $\beta_1 = 1.5\,mm - 6\,\mu m$

$$R_\Delta = R - \Delta = R(1 - \frac{\Delta}{R}) = R(1 - \delta)$$

$$r_\Delta = r + \Delta = r(1 + \frac{\Delta}{r}) = r(1 + \delta)$$

assume $\frac{\Delta}{R} = \frac{\Delta}{r} = \delta$ since $\Delta \ll R$ and r

The capacitance after variation Δ is:

$$C_\Delta = K \frac{1}{\ln \frac{R_\Delta}{r_\Delta}} = K \frac{1}{\ln R(1-\delta) - \ln r(1+\delta)}$$

$$\Delta_C = \frac{C_\Delta - C}{C} = \frac{C_\Delta}{C} - 1 = \frac{\ln R - \ln r}{\ln R(1-\delta) - \ln r(1+\delta)} - 1$$

$$= \frac{\ln R - \ln r}{\ln R - \ln r + \ln\frac{1-\delta}{1+\delta}} - 1$$

Let

$$M = \ln\frac{1-\delta}{1+\delta} = -2(\delta + \frac{\delta^3}{3} + \frac{\delta^5}{5} + \ldots.)$$

$$\Delta_C = \frac{\ln R - \ln r}{\ln R - \ln r - M} - 1 = \frac{1}{1 - \frac{M}{\ln R - \ln r}} - 1$$

$$= 1 + \frac{M}{\ln R - \ln r} + \frac{M^2}{(\ln R - \ln r)^2} + \ldots. - 1$$

$$\Delta_C = \frac{2(\delta + \frac{\delta^3}{3} + \frac{\delta^5}{5} + \ldots.)}{\ln R - \ln r} + \frac{\left[2(\delta + \frac{\delta^3}{3} + \frac{\delta^5}{5} + \ldots.)\right]^2}{\ln R - \ln r} + \ldots.$$

$\delta \ll$, therefore:

$$\Delta_C = + \frac{2\delta}{\ln R - \ln r}$$

When taking the other case $\beta_2 = 1.5\,mm + 6\,\mu m$, that is to say

$$R_\Delta = R + \Delta \quad \text{and} \quad r_\Delta = r - \Delta$$

and with the same method, Δ_C is

$$\Delta_C = - \frac{2\delta}{\ln R - \ln r}$$

Thus, corresponding to the field of tolerances $\mp 6\,\mu m$:

$$\Delta_C = \pm \frac{2\delta}{\ln R - \ln r}$$

Instead of speaking of the capacitance of cylinders 1 and 2, it is more convenient to call it capacitance of air gap a.

Δ_C must be found for each of the six air gaps:

air gap	$\delta = \frac{\Delta}{R} \cong \frac{\Delta}{r}$	$\frac{1}{\ln R - \ln r}$	$\Delta_C = \pm \frac{2\delta}{\ln R - \ln r}$
a	2.857 142 10^{-5}	69.498 79	3.97 10^{-3}
b	2.690 383 10^{-5}	73.832 20	3.97 10^{-3}
c	2.542 373 10^{-5}	78.165 60	3.97 10^{-3}
d	2.409 639 10^{-5}	82.498 99	3.97 10^{-3}
e	2.290 076 10^{-5}	86.832 37	3.97 10^{-3}
f	2.181 818 10^{-5}	91.165 75	3.97 10^{-3}

For the whole capacitor:

$$\Delta_{CT} = \sum_{i=a}^{f} \Delta_{ci} = 2.38.10^{-2} = \pm 2.38\%$$

+ 2.38% and - 2.38% are the upper and lower limits of the error which can be expected when keeping within the field of tolerance for the diameters of the cylinders.

The influence of the field of tolerances on the length of the measuring electrodes is also to be calculated. As will be found later the length of each measuring cylinder is approximately 35.6 cm. They can easily be machined to an accuracy of 0.1 mm in length. The error is

$$\Delta_C = \frac{0.1}{356} = 2.8.10^{-4} = 0.028\%$$

If all geometrical dimensions are within both fields of tolerances the capacitance after assembly of all the parts is **within:**

$$C = 10nF \pm 2.38\% \pm 0.028\% = 10nF \pm 2.41\%$$

Adjustment of the capacitance

A means of adjustment to bring the value of the capacitor from $\pm$ 2.41% to within 0.1 p.p.m. has to be found. To achieve this, use will be made of the property of eccentric cylinders.

The capacitance of two eccentric cylinders of radii R and r is given by the relation:

$$C^* = \frac{2\pi\ \varepsilon_0 \varepsilon}{\ln \frac{R}{r} \cdot \frac{R^2 - r^2 - D^2 + \sqrt{(R^2 - r^2 + D^2)^2 - 4R^2D^2}}{R^2 - r^2 + D^2 + \sqrt{(R^2 - r^2 + D^2)^2 - 4R^2D^2}}} \text{ pF cm}^{-1}$$

D: eccentricity as shown on figure 3.

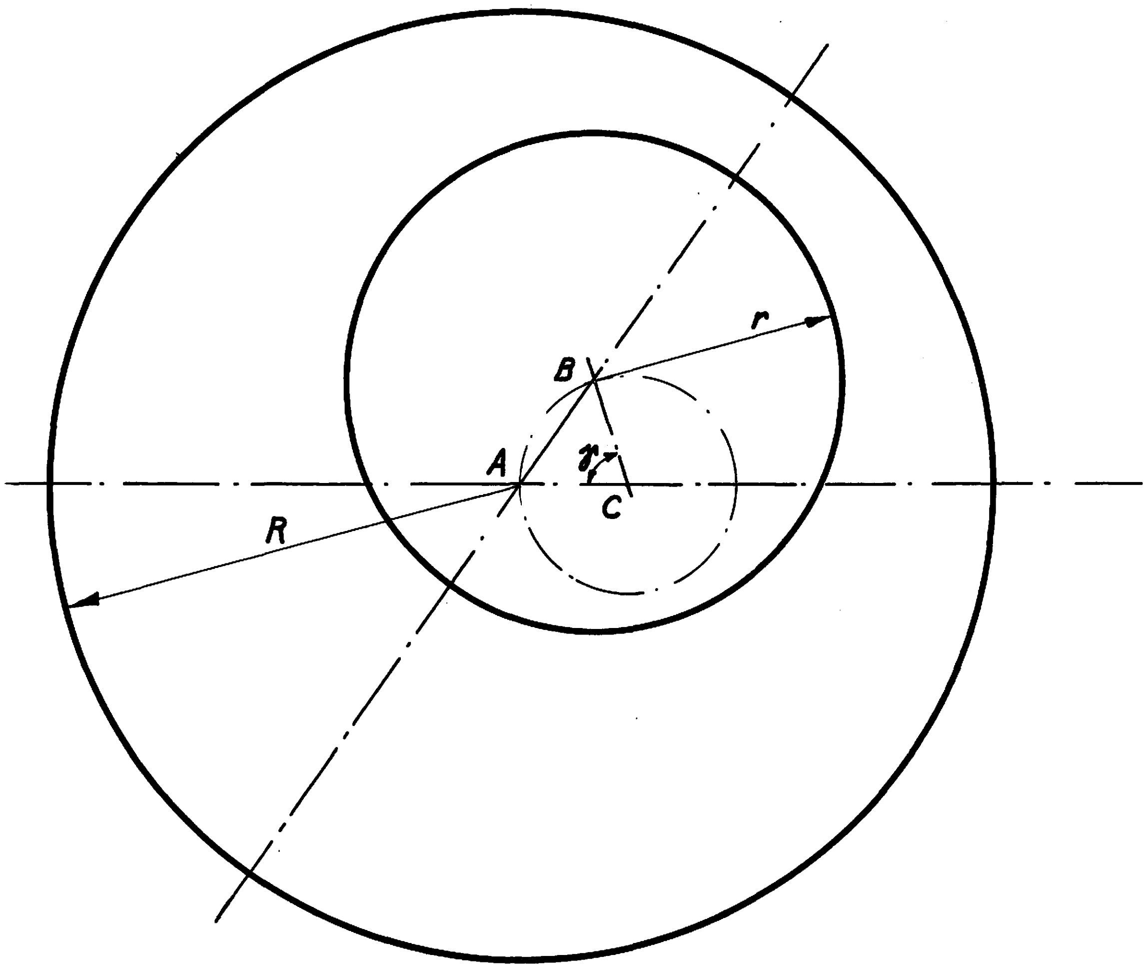
This formula shows that C* is a minimum when D vanishes, that is to say, in the case of coaxial cylinders.

The change in capacitance is found in the usual manner

$$\Delta^* = \frac{C^* - C}{C}$$

To have an idea of how Δ* varies as a function of the eccentricity D, only the air gap d will be considered. This will represent almost the mean variation. The eccentricity shall take four different values:

D_1 = 0.15 mm. D_2 = 0.3 mm. D_3 = 0.4 mm. D_4 = 0.5 mm



A center of outer cylinder

B center of inner cylinder

$\overline{AB} = D$ eccentricity due to rotation of B around C

$D_t = 2\ \overline{AC}$ maximum deviation

Fig. 3

Let

$$m = \frac{R^2 - r^2 - D^2 + \sqrt{(R^2 - r^2 + D^2)^2 - 4R^2D^2}}{R^2 - r^2 + D^2 + \sqrt{(R^2 - r^2 + D^2)^2 - 4R^2D^2}}$$

Δ^* will thus become:

$$\Delta^* = \frac{C_d^*}{C_d} - 1 = \frac{\ln d}{\ln d + \ln m} - 1$$

where $\ln d = \ln \frac{R_d}{r_d}$

R_d : outer diameter of air gap d

r_d : inner diameter of air gap d

Calculation of Δ^* using the four different values of D for the air gap d.

$R^2 = 15500.25$ $r^2 = 15129.00$ $\ln d = 0.012121360$

D	0.15	0.30
D^2	0.0225	0.09
$M = R^2 - r^2 - D^2$	371.2275	371.16
$N = R^2 - r^2 + D^2$	371.2725	371.34
N^2	137843.27	137893.40
$4D^2R^2$	1395.0225	5580.09
$O^2 = N^2 - 4D^2R^2$	136448.25	132313.31
O	369.399	363.749
$M + O$	740.6165	734.909
$N + O$	740.6665	735.029
$m = \frac{M + O}{N + O}$	0.99993924	0.99975510

ln m	-0.0000607637	-0.0002449579
ln d + ln m	0.012060596	0.011876400
$\Delta^* = \frac{\ln d}{\ln d + \ln m} - 1$	0.00504	0.02062
Δ^* in %	0.50	2.06

And for the last values of D:

D	0.40	0.50
D^2	0.16	0.25
M	371.09	371.00
N	371.41	371.50
N^2	137945.39	138012.25
$4D^2R^2$	9920.16	15500.25
O^2	128025.25	122512.00
O	357.810	350.017
M + O	728.90	721.017
N + O	729.22	721.617
$m = \frac{M + O}{N + O}$	0.99956117	0.9993071
ln m	-0.0004390226	-0.0006933801
ln d + ln m	0.011682337	0.011427980
$\Delta^* = \frac{\ln d}{\ln d + \ln m} - 1$	0.03758	0.06067
Δ^* in %	3.76	6.07

Figure 4 shows the variation of Δ* in % versus the eccentricity D in mm. From this curve it appears that an eccentricity of slightly less than 0.5 mm will suffice to correct an error of the order of ± 2.5%. The adjustability must, however, also include the tolerance of the high voltage capacitor, which is ± 0.018%.

As a safety measure the value of eccentricity which will be written down on the shop drawings shall be 0.5 mm (with tolerances 0 and -0.02 mm.), which corresponds to a total adjustment of 6%.

It will be observed that the length of the capacitor has to be given such, that the obtained value of capacitance is 10 nF - 3% for D = 0, that is to say in the case of coaxial cylinders.

The eccentricity is obtained by rotating one set of cylinders (the four voltage cylinders) with respect to the others. An angle of rotation of 180° is thus available to go from zero to maximum eccentricity D = 0.5 mm.

As can be seen from figure 3 the eccentricity D varies sinusoidally with respect to the angle of rotation γ:

$$\frac{D}{D_t} = \sin\frac{\gamma}{2} \qquad D_t = 0.5 \text{ mm.}$$

angle of rotation γ	$\frac{D}{D_t}$	D [mm.]	Δ* [%]
0	0	0	0
30	0.258	0.129	0.36
60	0.5	0.250	1.36
90	0.707	0.354	2.8
120	0.866	0.433	4.35
150	0.966	0.483	5.5
180	1.0	0.500	6.0

Variatons of capacitance caused by eccentricity.

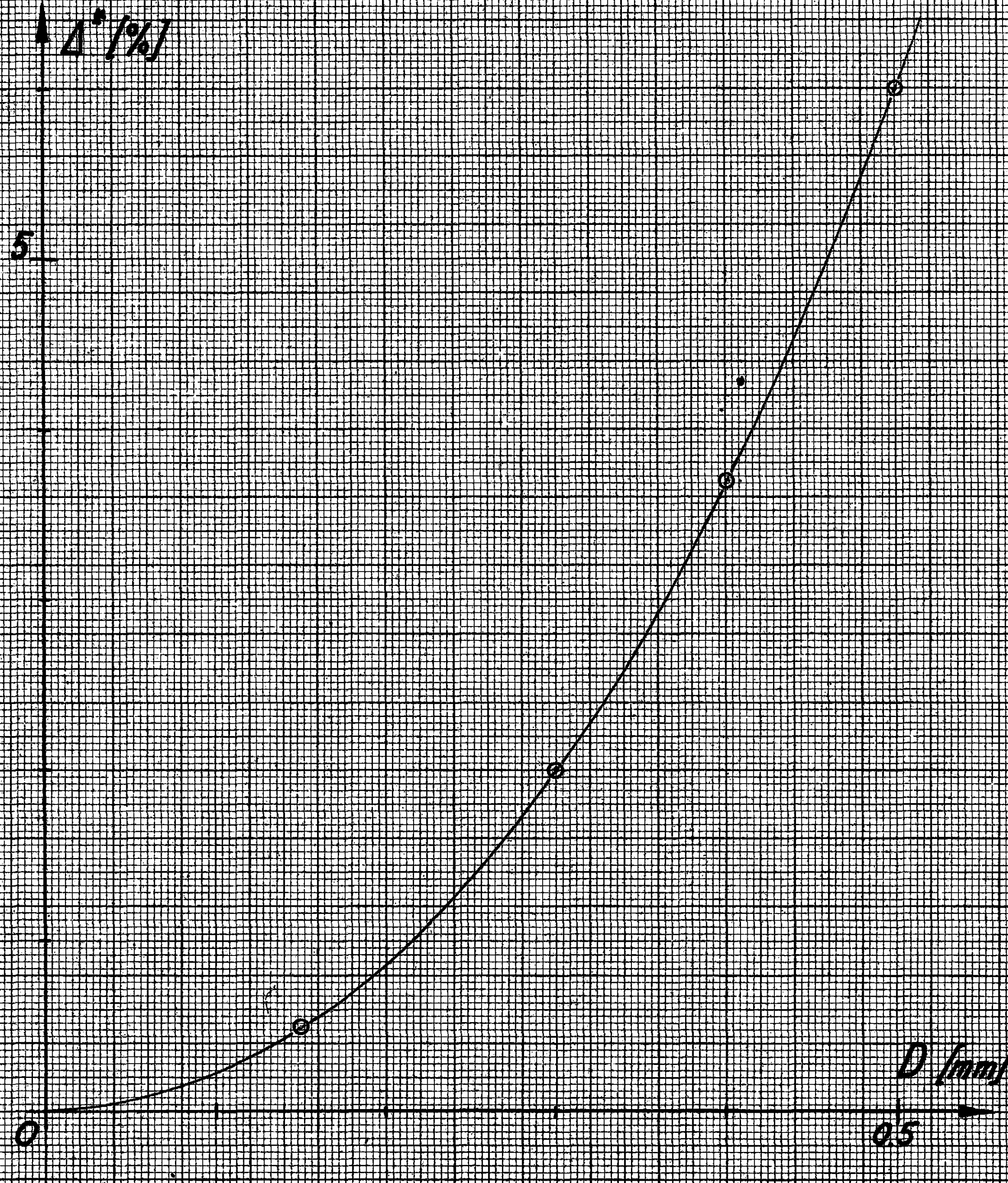


Fig. 4

Figure 5 shows Δ^* as a function of the angle of rotation γ. A rather good symmetry with respect to the middle point P (Δ = 3%) is evident. The calculation of the length of the capacitor can also be made in order to have 10nF at point P.

The exact value of capacitance at that point is calculated taking into account the seven cylinders. The angle of rotation at P is slightly more than 93° which corresponds to an eccentricity D = 0.363.

A slightly smaller eccentricity is used to obtain Δ^* = 3% with all the cylinders, since the calculations were made earlier on the fourth air gap. The effect of eccentricity is most noticeable in air gap a and decreases for the others.

Let

$$D = 0.358$$

$$D^2 = 0.1282$$

air gap	a	b	c
R^2	11025.0	12432.25	13924.0
r^2	10712.25	12100.0	13572.25
M	312 622	332 122	351 622
N	312 878	332 378	351 878
N^2	97892.643	110475.15	123818.13
$4D^2R^2$	5653.62	6375.28	7140.23
O^2	92239.023	104099.87	116677.90
O	303 709	322 644	341 582

Variation of capacitance due to eccentric rotation of one cylinder.

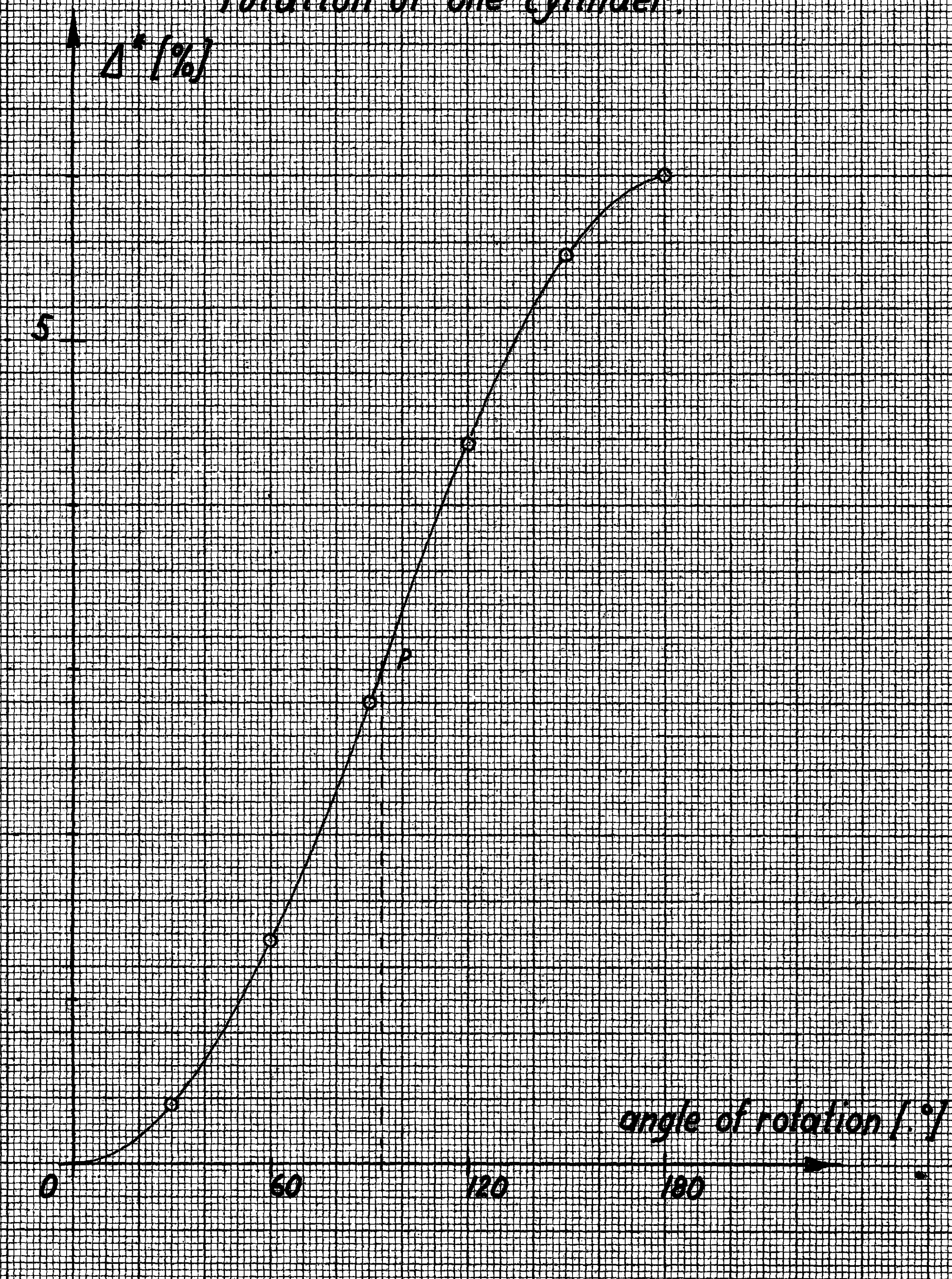


Fig. 5

air gap	a	b	c
M + O	616.331	654.766	693.204
N + O	616.587	655.622	693.460
m	0.99958481	0.99960917	0.99963083
ln m	-0.00041527619	-0.00039090637	-0.00036923814
$\ln \frac{R}{r}$	0.014388738	0.013544225	0.012793351
$\ln \frac{R}{r} + \ln m$	0.0139734618	0.0131533186	0.0124240629
$[\ln \frac{R}{r}]^{-1}$	69.49879	73.83220	78.16560
$[\ln \frac{R}{r} + \ln m]^{-1}$	71.56442	76.02643	80.48896

air gap	d	e	f
R^2	15500.25	17161.0	18906.25
r^2	15129.0	16770.25	18496.0
M	371.122	390.622	410.122
N	371.378	390.878	410.378
N^2	137921.62	152785.610	168410.10
$4D^2R^2$	7948.53	8800.16	9695.15
O^2	129973.09	143985.50	158714.97
O	360.518	379.454	398.391
M + O	731.640	770.076	808.514
N + O	731.896	770.332	808.769
m	0.99965022	0.99966767	0.99968346
ln m	-0.00034984117	-0.00033238522	-0.00031659010
$\ln \frac{R}{r}$	0.012121360	0.011516442	0.010969631
$\ln \frac{R}{r} + \ln m$	0.011771519	0.0111840568	0.0106530409
$[\ln \frac{R}{r}]^{-1}$	82.49899	86.83237	91.16575
$[\ln \frac{R}{r} + \ln m]^{-1}$	84.95080	89.41299	93.86991

For coaxial cylinders (D = 0)

$$\sum \frac{1}{\ln \frac{R}{r}} = 481.993\ 70$$

with an eccentricity D = 0.358

$$\sum \frac{1}{\ln \frac{R}{r} + \ln m} = 496.313\ 31$$

The change in capacitance is:

$$\Delta_T^* = \frac{496.313\ 31 - 481.993\ 70}{481.993\ 70} = 0.02\ 9709$$

$$\Delta_T^* = 2.9709\%$$

The exact value for the capacitance per unit length is given by

$$C = 2\pi\ \varepsilon_o \varepsilon\ \sigma \text{ pF cm.}^{-1}$$

$$\sigma = \sum \frac{1}{\ln \frac{R}{r} + \ln m} = 496.313\ 31$$

<u>The auxiliary electrode</u>

To bring the accuracy of the capacitor from ± 2.5% to ± 0.1 ppm by means of one single adjustment is not easily feasible. It has been decided to use the rotation of the main cylinders as a coarse adjustment to bring the accuracy from ± 2.5% to around 10 ppm. To complete the adjustment use of an auxiliary electrode is made. Its capacitance must be quite small because the incremental change in its capacitance due to thermal expansion must be negligible. With the aid of the property of eccentric electrodes it is possible to obtain the fine adjustment required. It takes over the trimming from 10 ppm to 0.1 ppm. The variation in capacitance of this electrode ΔC_f must be equal to 10 ppm at 10nF.

$$\Delta C_f = 10.10^{-6}.10^4 = 0.1\text{pF}$$

The auxiliary electrode with an external diameter of $2r = 2{:}9.35$ cm is inside the smallest cylinder which has an inner diameter $2R = 2{:}9.85$ cm (figure A).

An eccentricity $D_f = 0.5$ mm. is chosen in order to calculate the corresponding incremental change in capacitance Δ_f^*.

The capacitance of coaxial electrodes is

$$C_f = K \frac{1}{\ln \frac{R}{r}} \text{ pF cm}^{-1}$$

The capacitance when introducing the eccentricity D_f is:

$$C_f = K \frac{1}{\ln \frac{R}{r} + \ln \frac{L - D_f^2}{L + D_f^2}}$$

where $$L = R^2 - r^2 + \sqrt{(R^2 - r^2 + D_f^2)^2 - 4D_f^2R^2}$$

let $$\varphi = \frac{D_f^2}{L}$$

$$C_f = K \frac{1}{\ln \frac{R}{r} + \ln \frac{1-\varphi}{1+\varphi}}$$

$$\Delta_f^* = \frac{C_f^* - C_f}{C} = \frac{C_f^*}{C} - 1 = \frac{\ln \frac{R}{r}}{\ln \frac{R}{r} + \ln \frac{1-\varphi}{1+\varphi}} - 1$$

$$\Delta_f^* = \frac{\ln R - \ln r}{\ln R - \ln r - 2(\varphi + \frac{\varphi^3}{3} + \frac{\varphi^5}{5} + \ldots)} - 1$$

$$\Delta_f^* = \frac{2}{\ln R - \ln r}(\varphi + \frac{\varphi^3}{3} + \frac{\varphi^5}{5} + \ldots) + \left[\frac{2}{\ln R - \ln r}(\varphi + \frac{\varphi^3}{3} + \ldots)\right]^2$$

$D_f = 0.05$ cm $\qquad D_f^2 = 25.10^{-4}$ cm.

$$L = R^2 - r^2 + \sqrt{(R^2 - r^2 + D_f^2)^2 - 4D_f^2R^2}$$

$$= 97.0025 - 87.4225 + \sqrt{(97.0025 - 87.4225 + 25\ 10^{-4})^2 - 10^{-2}\ 97.0025}$$

$$= 9.58 + \sqrt{91.87 - 0.97} = 19{,}11$$

$$\varphi = \frac{D_f^2}{L} = \frac{25\ 10^{-4}}{19.11} = 1.31\ 10^{-4}$$

Therefore the terms φ^3 and of higher orders can be neglected.

The incremental change becomes:

$$\Delta_f^* = \frac{2}{0.0521}\ 1.31\ 10^{-4} = 5.03\ 10^{-3} \simeq 0.5\%$$

But $\Delta C_f = 0.1$pF has been found earlier.

The capacitance C_f between auxiliary electrode and cylinder no. 1 is given by:

$$C_f = \frac{\Delta C_f}{\Delta_f^*} = \frac{0.1}{5.03\ 10^{-3}} \simeq 20.0\ \text{pF}$$

The length of this electrode is

$$\ell_f = \frac{C_f \ln \frac{R}{r}}{2\pi\varepsilon_o\varepsilon} \text{ cm}$$

$$\varepsilon_o = 0.0885 \text{ pF cm}^{-1}$$

$$\ln R - \ln r = 0.0529$$

ε is assumed to be 1 since C_f need not be obtained with great precision.

$$\ell_f = \frac{20.0}{2\pi} \frac{0.0529}{0.0885} = 1.92 \text{ cm.}$$

$$\ell_f = 19.2 \text{ mm.}$$

This auxiliary electrode has two guard rings in order to avoid any capacitance effects from the ends which would be difficult to calculate.

The error ε''

Influence of the coefficient of thermal expansion

As mentioned earlier the error ε'' stands for the deviation in capacitance due to thermal expansion of the electrodes.

The calculations will usually be made assuming an incremental change in temperature of 1^oC. In the relation to calculate changes in capacitance due to thermal expansion, the factor $\alpha\Delta t$ (here written $\alpha\,\delta$), where α is the coefficient of thermal expansion and Δt ($=\delta$) the variation in temperature, will simply be written α since $\Delta t = 1$.

To obtain the smallest deviation in capacitance, use is made of the metal having the lowest coefficient of thermal expansion. Invar, manufactured by Carpenter, has a coefficient α lying between 1.196 $10^{-6}\,{}^{o}C^{-1}$ and 1.525 $10^{-6}\,{}^{o}C^{-1}$.

The capacitance of two coaxial cylinders at t°C is:

$$C = K \frac{1}{\ln \frac{R}{r}} \text{ pF cm}^{-1}$$

The capacitance of the same cylinders at t + 1°C is:

$$C_\Delta = K \frac{1 + \alpha}{\ln \frac{R(1 + \alpha)}{r(1 + \alpha)}} = K \frac{1 + \alpha}{\ln \frac{R}{r}}$$

The incremental change is:

$$\Delta = \frac{C_\Delta - C}{C} = \alpha \cong 1.5 \; 10^{-6}$$

This is a variation of 1.5 ppm whereas the accuracy has to reach 0.1 ppm or better.

Differential expansion

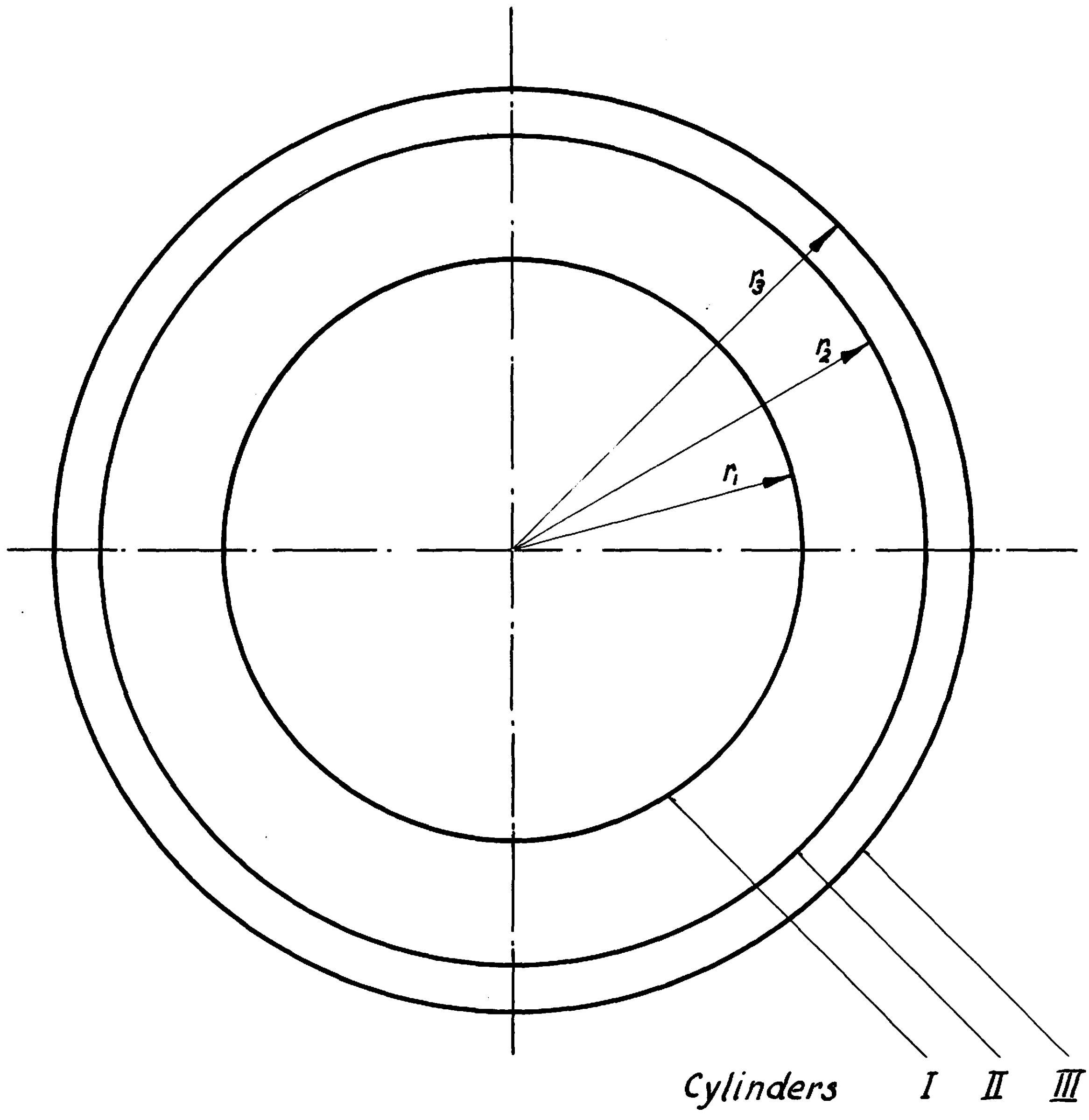
The design of the three coaxial cylinders of figure 6 is made in such a way that the measuring length, here equal to one, is taken on cylinder II. The walls are infinitely thin.

At temperature t°C the capacitance is

$$C = C_1 + C_2$$

where

$$C_1 = K \frac{1}{\ln \frac{r_2}{r_1}} \quad \text{and} \quad C_2 = K \frac{1}{\ln \frac{r_3}{r_2}}$$



Cylinder I made of Invar with coefficient of thermal expansion α_1

Cylinders II-III made of Invar with coefficient of thermal expansion α_2

Fig. 6

At temperature $t + \delta$ °C the capacitance becomes:

$$C' = C_1' + C_2'$$

where

$$C_1' = K \frac{1 + \alpha_2\delta}{\ln \frac{r_2(1 + \alpha_2\delta)}{r_1(1 + \alpha_1\delta)}} \qquad C_2' = K \frac{1 + \alpha_2\delta}{\ln \frac{r_3(1 + \alpha_2\delta)}{r_2(1 + \alpha_2\delta)}} = K \frac{1 + \alpha_2\delta}{\ln \frac{r_3}{r_2}}$$

The value of C_2' is definitively larger than C_2 since α_2 is larger than zero.

Looking at C_1' it is noticed that a suitable choice for the coefficient α_1 and α_2 can yield either a negative or a positive deviation of C_1. Thus α_1 and α_2 must be found in order to have:

$$C = C'$$

$$K \frac{1}{\ln \frac{r_2}{r_1}} + K \frac{1}{\ln \frac{r_3}{r_2}} = K \frac{1 + \alpha_2\delta}{\ln \frac{r_2(1 + \alpha_2\delta)}{r_1(1 + \alpha_1\delta)}} + K \frac{1 + \alpha_2\delta}{\ln \frac{r_3}{r_2}}$$

Let $\quad \ln \frac{r_2}{r_1} = a \qquad \ln \frac{r_3}{r_2} = b \qquad \ln \frac{r_2(1 + \alpha_2\delta)}{r_1(1 + \alpha_1\delta)} = c$

$$\frac{1}{a} + \frac{1}{b} = \frac{1 + \alpha_2\delta}{c} + \frac{1 + \alpha_2\delta}{b}$$

$$c = \frac{1 + \alpha_2\delta}{\frac{1}{a} - \frac{\alpha_2}{b}} = a + \ln(1 + \alpha_2\delta) - \ln(1 + \alpha_1\delta)$$

$$\ln(1 + \alpha_1\delta) = \ln(1 + \alpha_2\delta) + a - \frac{1 + \alpha_2\delta}{\frac{1}{a} - \frac{\alpha_2\delta}{b}}$$

$$\alpha_1\delta \ll 1 \quad \text{and} \quad \alpha_2\delta \ll 1$$

$$\alpha_1\delta = \alpha_2\delta + \ln\frac{r_2}{r_1} - \frac{1+\alpha_2\delta}{\dfrac{1}{\ln\dfrac{r_2}{r_1}} - \dfrac{\alpha_2\delta}{\ln\dfrac{r_3}{r_2}}}$$

If the correlatives of the radii r_1, r_2 and r_3 of the cylinders I, II and II can be determined, then the above result can be applied to the more complicated case of seven cylinders.

It is evident that:

$$r_1 = r_a \quad \text{and} \quad r_2 = R_a$$

where

r_a is the inner radius of air gap a or the outer radius of cylinder 1.

R_a is the outer radius of air gap a or the inner radius of cylinder 2.

thus

$$C_1 = C_a$$

where

$$C_1 = K\frac{1}{\ln\dfrac{r_2}{r_1}} \qquad C_a = K\frac{1}{\ln\dfrac{R_a}{r_a}}$$

For the five other air gaps the capacitance for each of them is:

$$C_i = K\frac{1}{\ln\dfrac{R_i}{r_i}}\ \text{pF cm}^{-1} \qquad i = b;\ c;\ d;\ e;\ f.$$

The capacitance per unit length of gaps b to f is:

$$C_5 = K\sum_{i=b}^{f}\frac{1}{\ln\dfrac{R_i}{r_i}}$$

Let $$\sum_{i=b}^{f} \frac{1}{\ln \frac{R_i}{r_i}} = \frac{1}{\ln \frac{r_3}{r_2}}$$

The capcitance of cylinders II and III was:

$$C_2 = K \frac{1}{\ln \frac{r_3}{r_2}}$$

Thus $$C_5 = C_2$$

The seven cylinders used in the design can be replaced by the three fictitious cylinders I, II and III. Cylinder II is a measuring cylinder to which corresponds cylinder 2, which is also a measuring one.

As far as the property of the materials is concerned the seven cylinders can be split into two groups. On one side cylinder 1 is made of Invar having a coefficient of thermal expansion α_1 and on the other side the cylinders 2 to 7 are made of Invar with a coefficient α_2.

Calculation of the ratio of coefficient of thermal expansion $\Psi = \frac{\alpha_1}{\alpha_2}$

It has been found earlier:

$$\alpha_1 \delta = \alpha_2 \delta + \ln\left(\frac{r_2}{r_1}\right) - \frac{1 + \alpha_2 \delta}{\frac{1}{\ln \frac{r_2}{r_1}} - \frac{\alpha_2 \delta}{\ln \frac{r_3}{r_2}}}$$

$$\ln \frac{r_2}{r_1} = \ln \frac{R_a}{r_a} = 0.014388738$$

$$\text{and } \frac{1}{\ln \frac{r_2}{r_1}} = 69.498798$$

$$\frac{1}{\ln \frac{r_3}{r_2}} = \sum_{i=b}^{f} \frac{1}{\ln \frac{R_i}{r_i}} = 412.49491$$

Let $\alpha_2\delta$ take five different values:

$\alpha_2\delta$	$\frac{\alpha_2\delta}{\ln \frac{r_3}{r_2}}$	$p = \frac{1}{\ln \frac{r_2}{r_1}} - \frac{\alpha_2\delta}{\ln \frac{r_3}{r_2}}$	$q = \frac{1 + \alpha_2\delta}{p}$
$1\ 10^{-6}$	$0.41249491\ 10^{-3}$	69.4983855	0.014388838
$2\ 10^{-6}$	$0.82498982\ 10^{-3}$	69.4979730	0.014388938
$5\ 10^{-6}$	$2.06247455\ 10^{-3}$	69.4967355	0.014389237
$10\ 10^{-6}$	$4.1249491\ 10^{-3}$	69.4946731	0.014389736
$20\ 10^{-6}$	$8.2498982\ 10^{-3}$	69.4905481	0.014390734

$\alpha_2\delta$	$\ln \frac{r_2}{r_1}$	$\alpha_1\delta$	$\frac{\alpha_1\delta}{\alpha_2\delta}$
$1\ 10^{-6}$	$-0.1\ 10^{-6}$	$0.9\ 10^{-6}$	0.9000
$2\ 10^{-6}$	$-0.2\ 10^{-6}$	$1.8\ 10^{-6}$	0.9000
$5\ 10^{-6}$	$-0.499\ 10^{-6}$	$4.501\ 10^{-6}$	0.9002
$10\ 10^{-6}$	$-0.998\ 10^{-6}$	$9.002\ 10^{-6}$	0.9002
$20\ 10^{-6}$	$-1.996\ 10^{-6}$	$18.004\ 10^{-6}$	0.9002

These results lead to the plotting of the curve of figure 7.

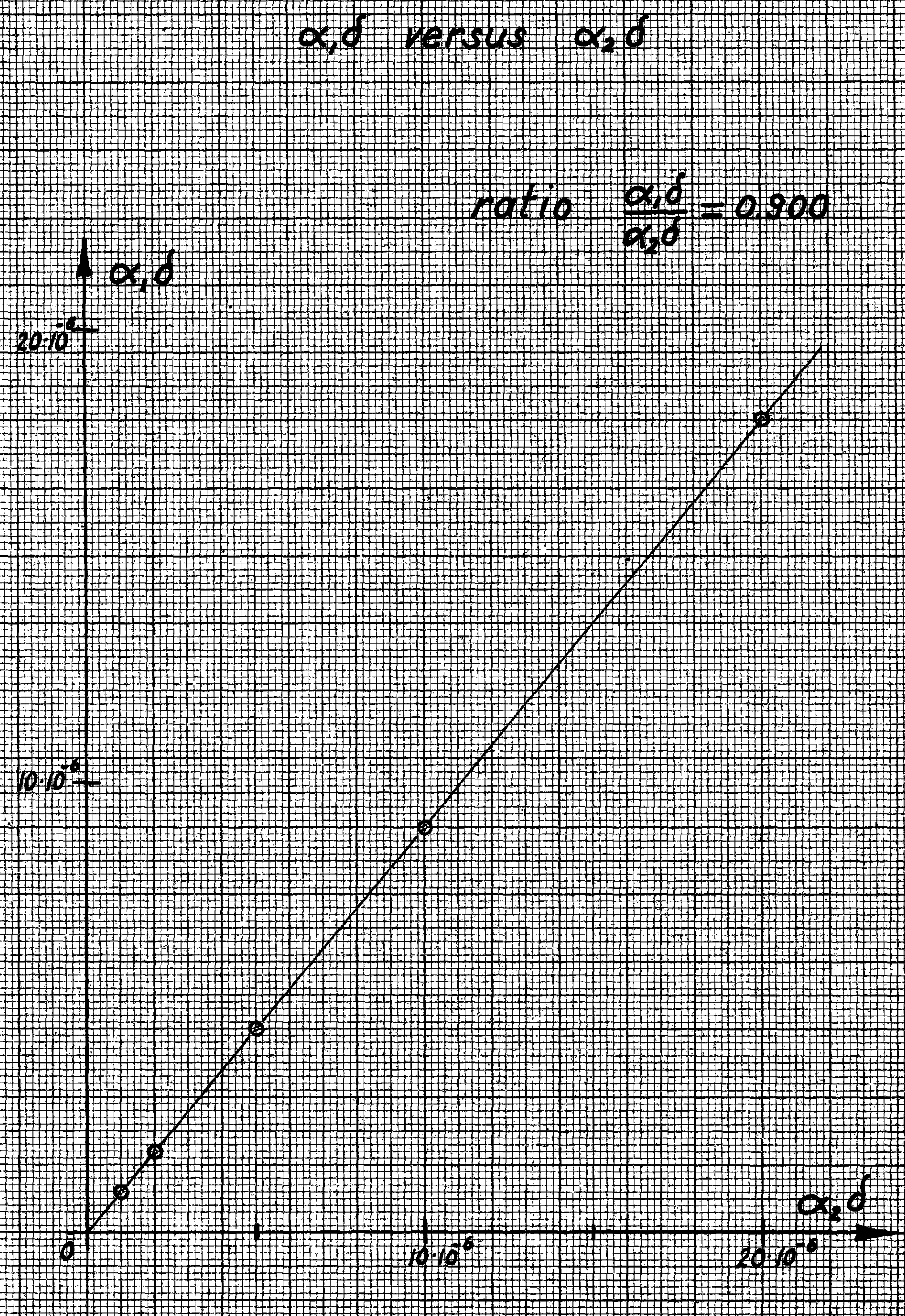

α₁δ versus α₂δ
ratio α₁δ/α₂δ = 0.900
α₁δ
20·10⁻⁶
10·10⁻⁶
0
10·10⁻⁶
20·10⁻⁶
α₂δ

Fig. 7

The two sets of electrodes, which, under steady state conditions, have the same temperature, are inside a pressure vessel.

Let $\alpha_2 = 1.4\ 10^{-6}\ {}^oC^{-1}$. Therefore α_1 is

$$\alpha_1 = \alpha_2\, 0.9 = 1.26\ 10^{-6}\ {}^oC^{-1}$$

These two figures are well within the data given by the manufacturer of Invar.

From Figure 7 it is evident that a variation of 14°C in temperature produces no change in the ratio $\Psi = \frac{\alpha_1}{\alpha_2}$. Actually this ratio Ψ remains constant even for larger values of δ.

The temperature changes will not have any influence on the capacitor as long as the two sets of cylinders are at the same temperature, and are made of Invar taken from two different yields so that the two coefficients of thermal expansion satisfy the ratio:

$$\Psi = \frac{\alpha_1}{\alpha_2} = 0.9$$

<u>Influence of deviation of Ψ from the value 0.9</u>.

Investigation is made of the influence of small changes of ratio Ψ.

In order to find the variation in capacitance due to a deviation of 1.5% in α_1, let $\alpha_2 = 1.4\ 10^{-6}\ {}^oC^{-1}$ and let α_1 take different values.

Capacitance of the set of 6 cylinders at temperature t:

$$C_2 = K \frac{1}{\ln \frac{r_3}{r_2}}$$

C_2 at temperature t + 1°C

$$C_2' = K \frac{1 + \alpha_2}{\ln \frac{r_3}{r_2}}$$

$$\Delta C_2 = C_2' - C_2 = K \left(\frac{1 + \alpha_2}{\ln \frac{r_3}{r_2}} - \frac{1}{\ln \frac{r_3}{r_2}} \right)$$

$$\frac{\Delta C_2}{K} = \frac{1}{\ln \frac{r_3}{r_2}} \alpha_2 = 0.000\ 577$$

The capacitance of the first two cylinders at temperature t is:

$$C_1 = K \frac{1}{\ln \frac{r_2}{r_1}}$$

C_1 at temperature t + 1°C

$$C_1' = K \frac{1 + \alpha_2}{\ln \frac{r_2(1 + \alpha_2)}{r_1(1 + \alpha_1)}}$$

$$\Delta C_1 = C_1' - C_1 = K \left(\frac{1 + \alpha_2}{\ln \frac{r_2(1 + \alpha_2)}{r_1(1 + \alpha_1)}} - \frac{1}{\ln \frac{r_2}{r_1}} \right)$$

$$\frac{\Delta C_1}{K} = \frac{1 + \alpha_2}{\ln \frac{r_2(1 + \alpha_2)}{r_1(1 + \alpha_1)}} - \frac{1}{\ln \frac{r_2}{r_1}}$$

Since $\alpha_1 \ll 1$

$$\ln \frac{r_2(1 + \alpha_2)}{r_1(1 + \alpha_1)} = \ln \frac{r_2}{r_1} + \alpha_2 - \alpha_1$$

The variation in capacitance is given by:

$$\Delta C = \Delta C_1 + \Delta C_2$$

or

$$\frac{\Delta C}{K} = \frac{\Delta C_1}{K} + \frac{\Delta C_2}{K}$$

Calculation of $\frac{\Delta C_1}{K}$

α_1	$\alpha_2 - \alpha_1$	$\ln \frac{r_2}{r_1} + \alpha_2 - \alpha_1$	$\frac{1 + \alpha_2}{\ln \frac{r_2}{r_1} + \alpha_2 - \alpha_1}$	$\frac{\Delta C_1}{K}$
$1.22\ 10^{-6}$	$0.18\ 10^{-6}$	0.014 388 918	69.498 027	-0.000 771
$1.24\ 10^{-6}$	$0.16\ 10^{-6}$	0.014 388 898	69.498 123	-0.000 675
$1.26\ 10^{-6}$	$0.14\ 10^{-6}$	0.014 388 878	69.498 219	-0.000 579
$1.28\ 10^{-6}$	$0.12\ 10^{-6}$	0.014 388 858	69.498 316	-0.000 482
$1.30\ 10^{-6}$	$0.10\ 10^{-6}$	0.014 388 838	69.498 413	-0.000 385

It is now possible to find the variation of the capacitance in p.p.m. as a function of the deviation of α_1 in % from its value given by Ψ. (figure 8)

$$C = K \sum_{i=b}^{f} \frac{1}{\ln \frac{r_3}{r_2}} = K\ 1109.831\ 3$$

$$\frac{C}{K} = 1109.8313$$

Variation of capacitance due to deviation of coefficient α_1.

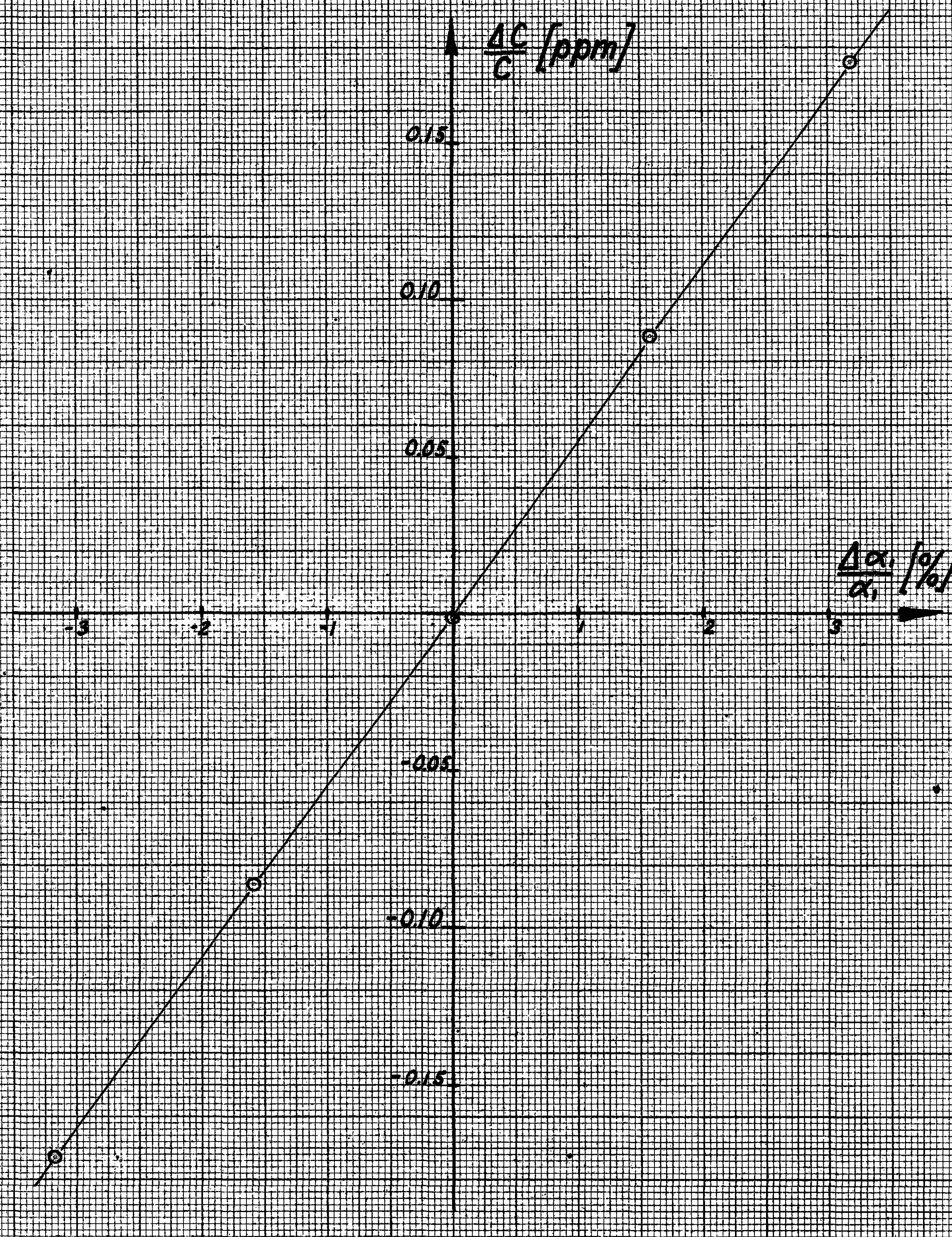


Fig. 8

α_1	$\frac{\Delta C}{K}$	$\frac{\Delta\alpha_1}{\alpha_1}$ [%]	$\frac{\Delta C}{C}$ [ppm]
$1.22\ 10^{-6}$	-0.000 194	-3.17	-0.1748
$1.24\ 10^{-6}$	-0.000 098	-1.58	-0.0883
$1.26\ 10^{-6}$	-0.000 002	0.00	-0.0018
$1.28\ 10^{-6}$	0.000 095	+1.58	0.0856
$1.30\ 10^{-6}$	0.000 192	+3.17	0.1730

As long as the ratio Ψ is 0.9 ± 1.5%, it is now evident that the variations of capacitance per unit length due to thermal expansion of the electrodes can be minimized to within ± 0.0812 ppm per °C.

The thermal expansion of the auxiliary electrode.

This auxiliary electrode, sitting inside the cylinder no. 1, as shown on figure A, is made of Invar having the same coefficient of thermal expansion α_1 as the cylinder 1.

Capacitance of a ring of unit length at temperature t°C:

$$C_r = K \frac{1}{\ln \frac{R_r}{r_r}} \text{ pF cm}^{-1}$$

R_r inner radius of cylinder 1

r_r outer radius of auxiliary electrode

C_r at temperature t + 1°C is:

$$C_r' = K \frac{1 + \alpha_1}{\ln \frac{R_r(1 + \alpha_1)}{r_r(1 + \alpha_1)}} = K \frac{1 + \alpha_1}{\ln \frac{R_r}{r_r}} = C_r(1 + \alpha_1) \text{ pF cm}^{-1}$$

$$\Delta C_r = C_r' - C_r = C_r \alpha_1 = K \frac{\alpha_1}{\ln \frac{R_r}{r_r}}$$

$$K = 2\pi\varepsilon_o\varepsilon$$

ε is assumed to be equal to 1

$$\Delta C_r = \frac{2\pi\ 0.0885\ \ 1.8\ 10^{-6}}{\ln 9.85 - \ln 9.35} = 18.9\ 10^{-6}\ pF\ cm^{-1}$$

The length of the auxiliary electrode has been found earlier:

$$\ell = 1.92\ cm$$

$$\Delta C_r = 18.9\ 1.92\ 10^{-6} = 36.3\ 10^{-6}\ pF$$

This deviation compared to the capacitance C_2 = 10nF is very small:

$$\frac{\Delta C_r}{C} = \frac{36.3\ 10^{-6}}{10\ 10^3} = 3.63\ 10^{-9} = 0.00363\ ppm.$$

This result shows that the capacitance of the auxiliary electrode is so low compared to the whole capacitor that its variation due to thermal expansion will not be noticeable.

Capacitance of the ends of the measuring electrode.

A very close investigation is now made of the influence of thermal expansion at both ends of the measuring electrode.

Area no. 1 designates the bottom end of the electrode where an insulator connects it mechanically to the guard ring.

Area no. 2 designates the top end of the measuring electrode.

Field maps of these two areas were plotted. All field pictures and calculations made from them would only apply to the two dimensional case, because they correspond to a parallel plane system of electrodesand not to coaxial cylinders. The air gap, however, is very small compared to the radii of curvature of the cylinders so that the two dimensional representation need not be transposed into a three dimensional one. In fact, when the air gap is much smaller than the radii the capacitance of two cylinders can be calculated using the relation for parallel plate system.

It is assumed $R-r = a \ll R;r$

$$C = \frac{2\pi\ \varepsilon_o \varepsilon \ell}{\ln \frac{R}{r}} = \frac{2\pi\ \varepsilon_o \varepsilon \ell}{\ln \frac{r+a}{r}} = \frac{2\pi\ \varepsilon_o \varepsilon \ell}{\ln \frac{1+\frac{a}{r}}{1}} = \frac{2\pi\ \varepsilon_o \varepsilon \ell}{\ln(1+\frac{a}{r})}$$

$$C = \frac{2\pi\ \varepsilon_o \varepsilon \ell}{\frac{a}{r} - \frac{1}{2}(\frac{a}{r})^2 + \ldots} = \frac{2\pi\ \varepsilon_o \varepsilon \ell r}{a} = \frac{\varepsilon_o \varepsilon S}{a}$$

Thus

$$C = \frac{2\pi\ \varepsilon_o \varepsilon \ell}{\ln \frac{R}{r}} = \frac{\varepsilon_o \varepsilon S}{a} \quad \text{if } \frac{a}{r} \ll 1$$

$S = 2\pi \ell r$ area of cylinder

To plot the field maps, conducting paper has been used, and a voltage of unity was applied on one electrode. The other electrode was at zero potential.

Area No. 1

Figure 9 shows where this area is situated. It must be found where the lines of force end, which start from the voltage electrode between the point A and B . The line passing through point C is the axis of symmetry of the spacing between the measuring electrode and the guard ring.

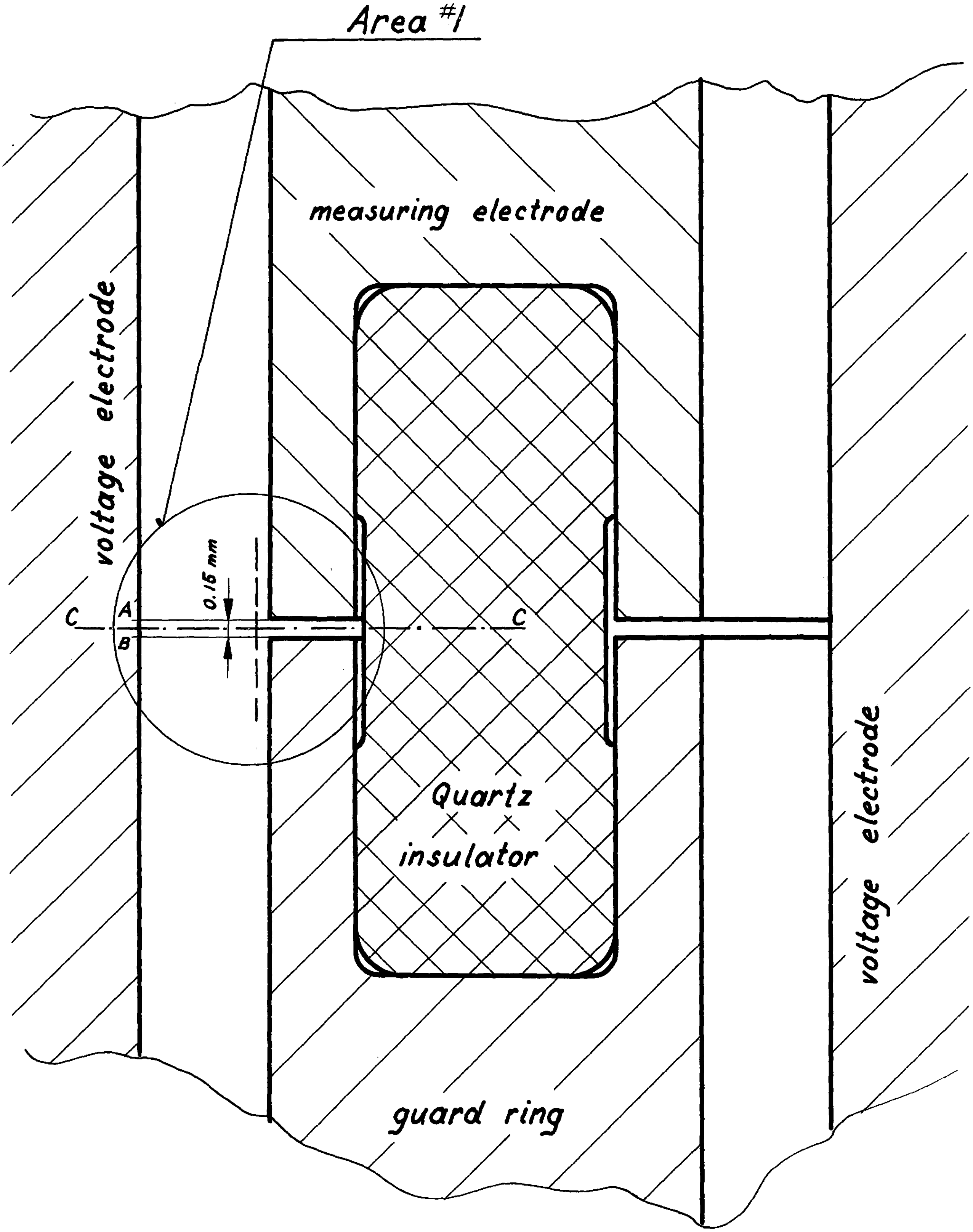

Area #1
measuring electrode
voltage electrode
0.15 mm
C
A
B
C
Quartz
insulator
voltage electrode
guard ring

Fig. 9

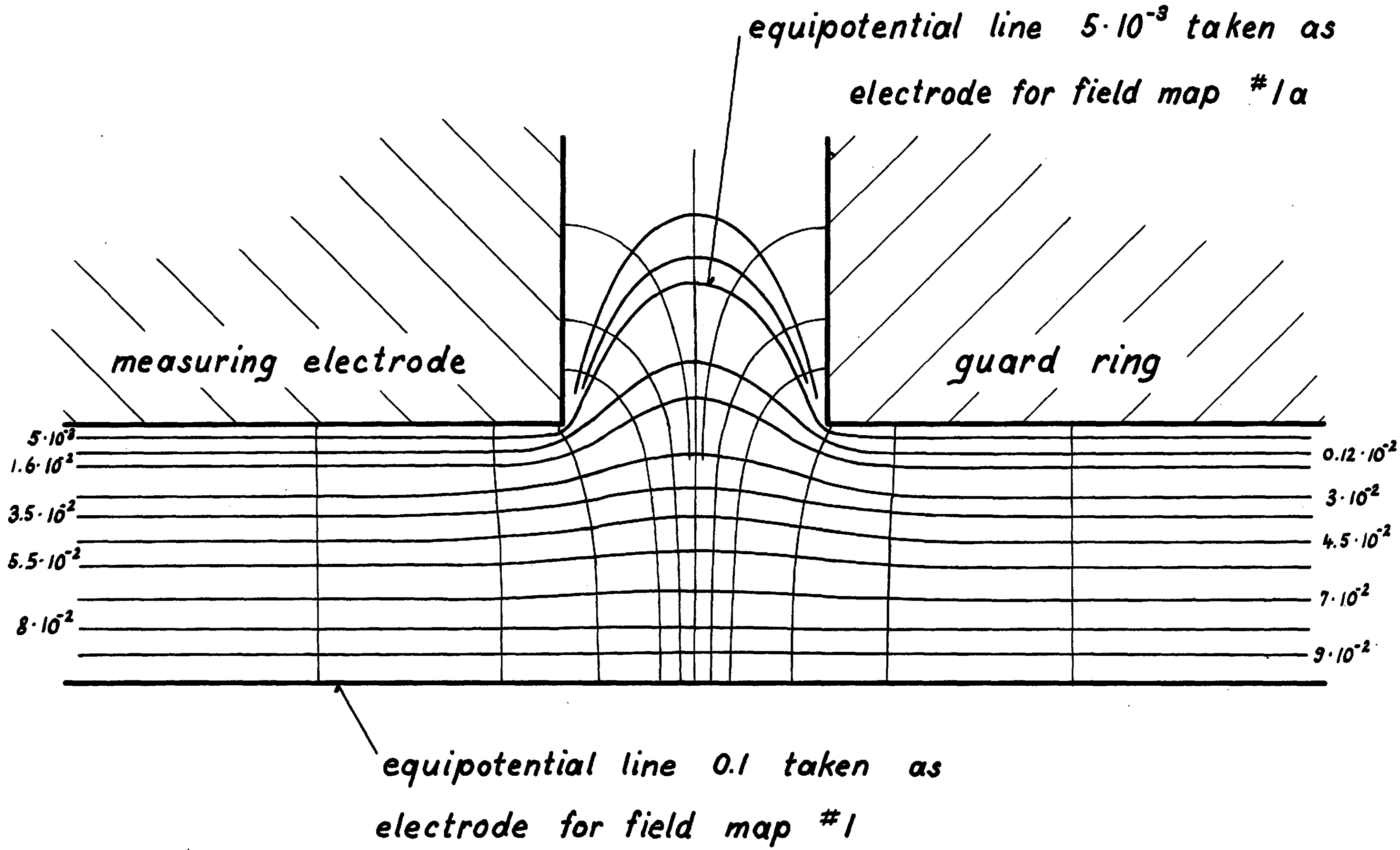
The insulator maintaining the spacing is made of quartz. Both ends of the quartz ring are coated with a thin layer of metal in order to permit soldering to the measuring electrode as well as to the guard ring.

Several field maps of area no. 1 have been made, and they have shown that it can be assumed that the 0.1 equipotential line (on figure 9, the line ----) lying between the measuring electrode and the guard ring is a straight line. Field map no. 1 is made with this assumption. From it, it is still difficult to obtain a good picture of the field in the spacing AB. One of the equipotential lines is again chosen as an electrode in order to plot field map no. 1a, showing half the length of the spacing. This time, however, it is a curve corresponding to the $5\ 10^{-3}$ equipotential.

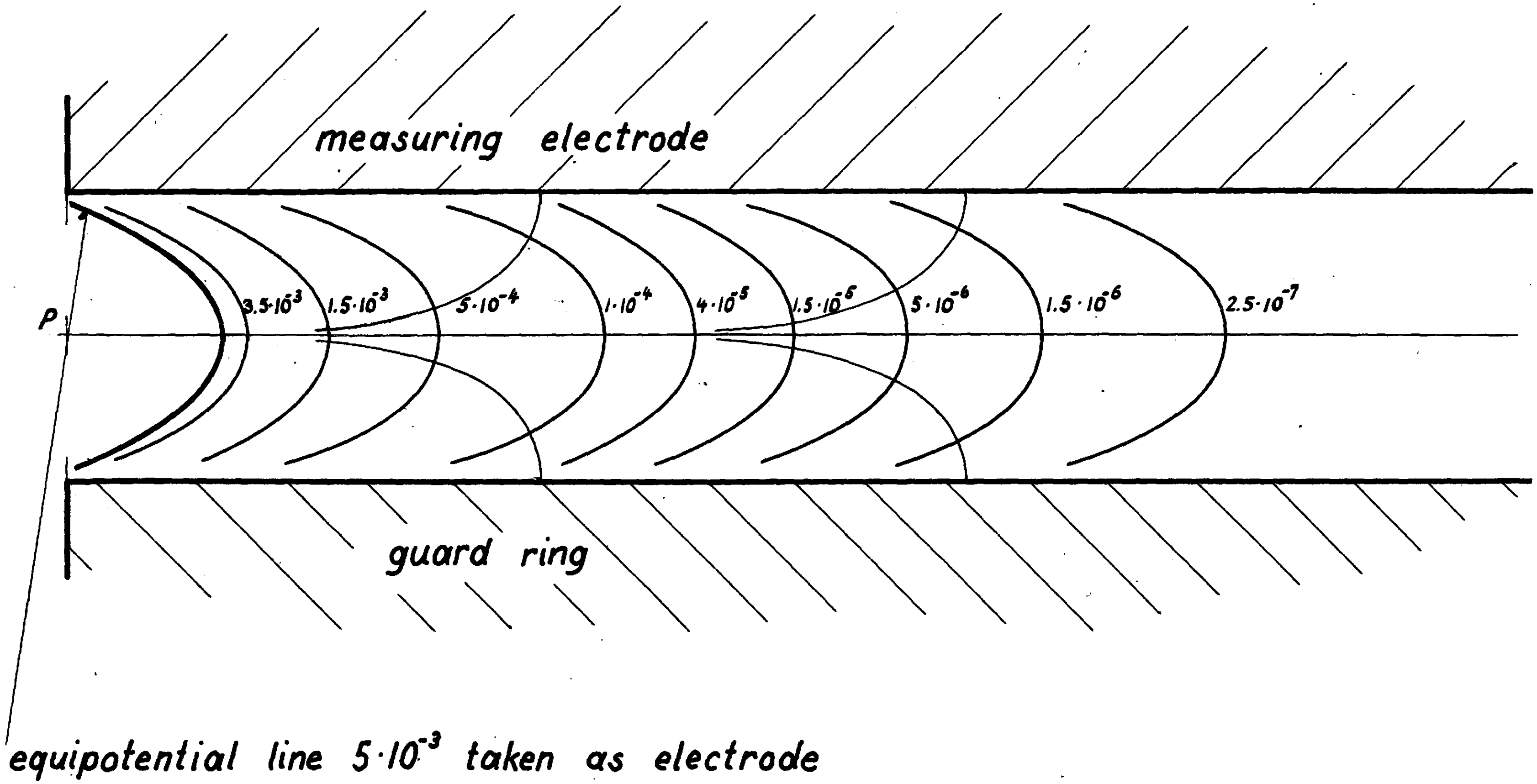
On figure 10 the variation of the voltage inside the spacing is plotted versus the distance d measured from point P to the intersection of the equipotential lines and the axis of symmetry. This variation follows an exponential law. The total length of the spacing is 1 mm. The last equipotential line, $2.5\ 10^{-7}$ shown on the field picture is situated at 60% of the length of the air gap. By extrapolation it is found that the equipotential line touching the insulator at distance d = 1 mm. from P is of the order of 10^{-10}. Thus the requirement to have no lines of forces reaching the insulating material is met.

Furthermore, all lines of force above the axis of symmetry reach the measuring electrode, whereas the lines leaving the voltage electrode underneath the axis touch the guard ring.

To calculate the deviation in capacitance it is assumed that the capacity per unit length is constant up to the axis C-C.


equipotential line $5\cdot10^{-3}$ taken as electrode for field map #1a
measuring electrode
guard ring
$5\cdot10^{-3}$
$1.6\cdot10^{-2}$
$3.5\cdot10^{-2}$
$5.5\cdot10^{-2}$
$8\cdot10^{-2}$
$0.12\cdot10^{-2}$
$3\cdot10^{-2}$
$4.5\cdot10^{-2}$
$7\cdot10^{-2}$
$9\cdot10^{-2}$
equipotential line 0.1 taken as electrode for field map #1

Field map # 1.


measuring electrode
P
3.5·10-3
1.5·10-3
5·10-4
1·10-4
4·10-5
1.5·10-5
5·10-6
1.5·10-6
2.5·10-7
guard ring
equipotential line 5·10-3 taken as electrode

Field map # 1a.

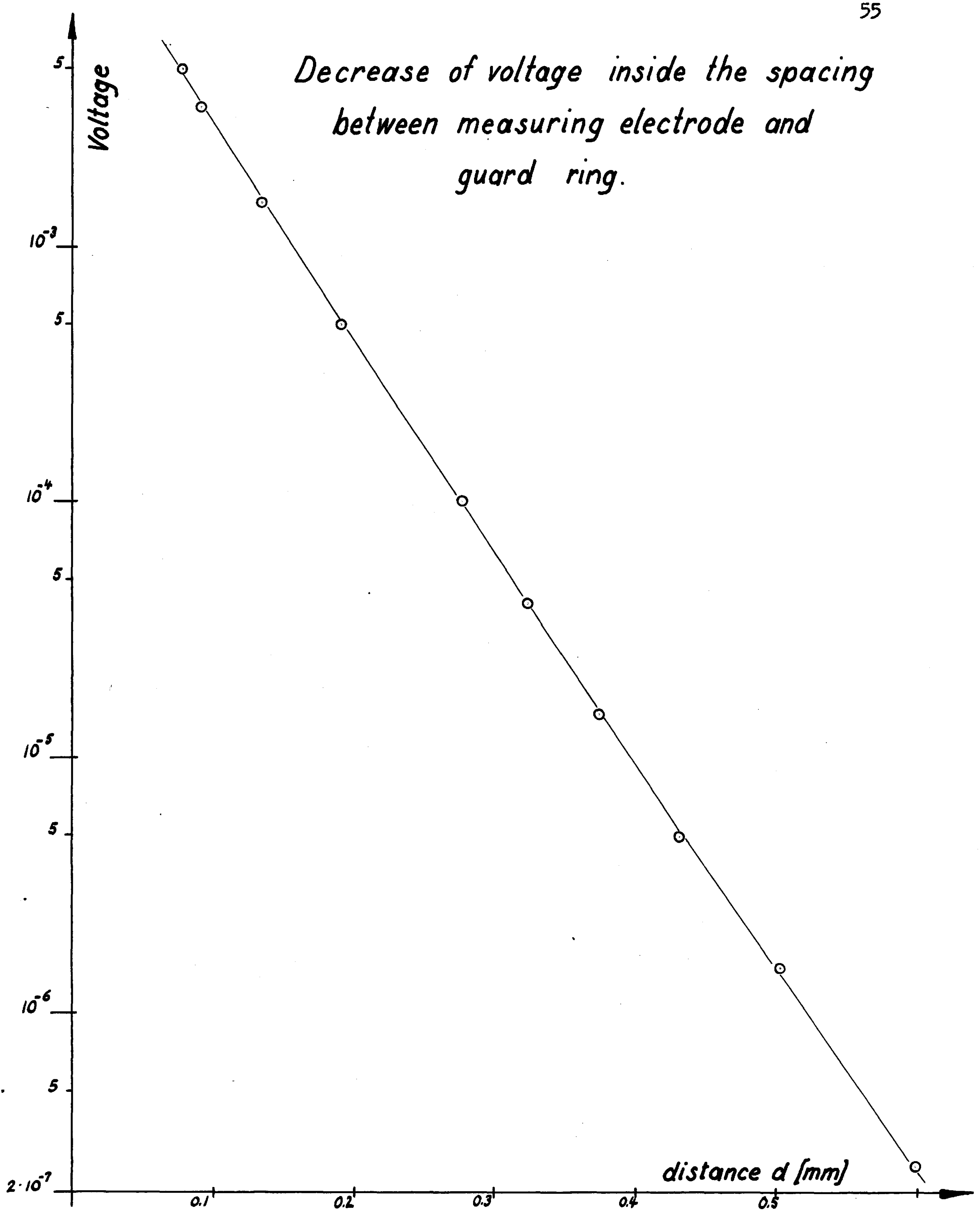

Decrease of voltage inside the spacing between measuring electrode and guard ring.
Voltage
5
10^{-3}
5
10^{-4}
5
10^{-5}
5
10^{-6}
5
$2 \cdot 10^{-7}$
distance d [mm]
0.1
0.2
0.3
0.4
0.5

Fig. 10

Theoretically it is not the case, since the length of the tubes of flux increases when approaching C-C. The deviation which is found is therefore, slightly larger than the actual one.

The variation of the spacing is due to the thermal expansion of the insulator made of quartz (or Vycor) as well as the thermal expansion of the corresponding part of the measuring electrode. The height of the insulating ring is 8mm. The difference of the two coefficients of expansion is:

$$\alpha_r = \alpha_{invar} - \alpha_{vycor} = (1.3 - 0.8)\ 10^{-6}\ {}^{o}C^{-1} = 0.5\ 10^{-6}\ {}^{o}C^{-1}$$

The distance measured between the axis C-C and the measuring electrode decreases by:

$$\Delta\ell_q = \frac{1}{2}\ 0.8\ 0.5\ 10^{-6} = 2\ 10^{-7}\ \text{cm}\ {}^{o}C^{-1}$$

The length of the measuring electrode is approximately 35.6 cm. The deviation in capacitance of area no. 1 caused by thermal expansion is:

$$\Delta_1 = \frac{2\ 10^{-7}}{35.6} = 5.61\ 10^{-9} = 0.00561\ \text{ppm.}$$

<u>Area No. 2</u>

The location of this area is shown on figure 11. The capacitance of the top of the measuring electrode to the voltage electrode must be known. The latter is composed of two sides being the voltage cylinders and of the top plate. The variation of capacitance due to thermal expansion must be calculated.

For reasons of experimental convenience the 1 V potential used to plot the field map is applied to the measuring electrode. The voltage electrode system is now held at zero potential. The two sets of electrodes will still be called by the original name.

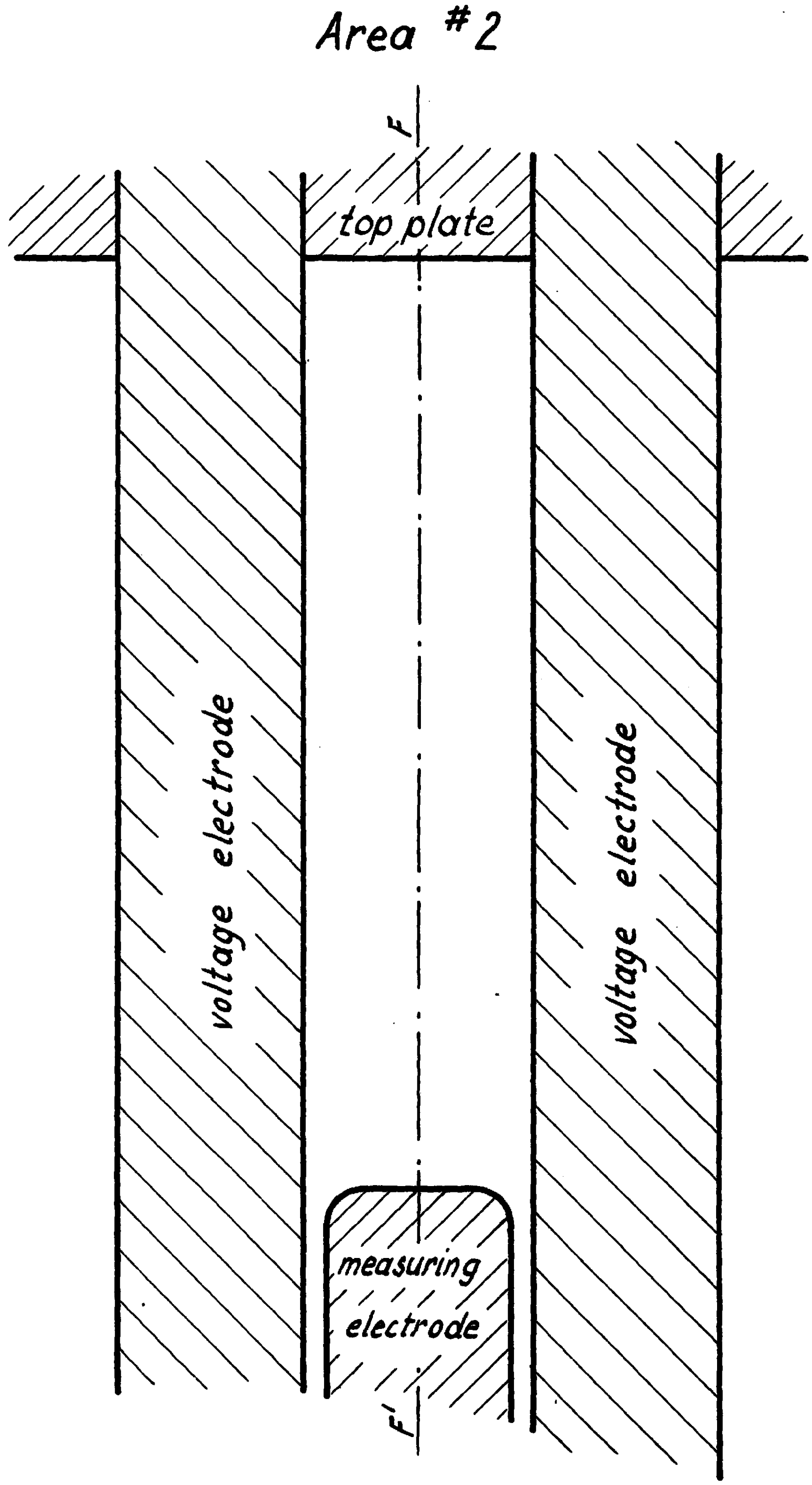

Area #2
F
top plate
voltage electrode
voltage electrode
measuring
electrode
F'

Fig. 11

The field picture no. 2 shows the distribution of the equipotential lines as well as some lines of force. Here again, the voltage decreases exponentially along the axis of symmetry as it did in area no. 1. However, owing to the fact that the top of the spacing is now part of the voltage electrode, this exponential law does not hold until the very end. The voltage must be zero at the top plate. Up to approximately 85% of the distance the distribution of voltage is the one represented on figure 13. From the exponential line $8\ 10^{-5}$ on, the decrease of voltage is shown in figure 12.

Variation of capacitance of area no. 2

In case of temperature rise, the top plate to which the voltage cylinders are fastened, moves up. The measuring cylinders also move up. The central support, made of steel having a coefficient of thermal expansion $\alpha_S = 12\ 10^{-6}\ {}^{o}C^{-1}$, holds the top plate.

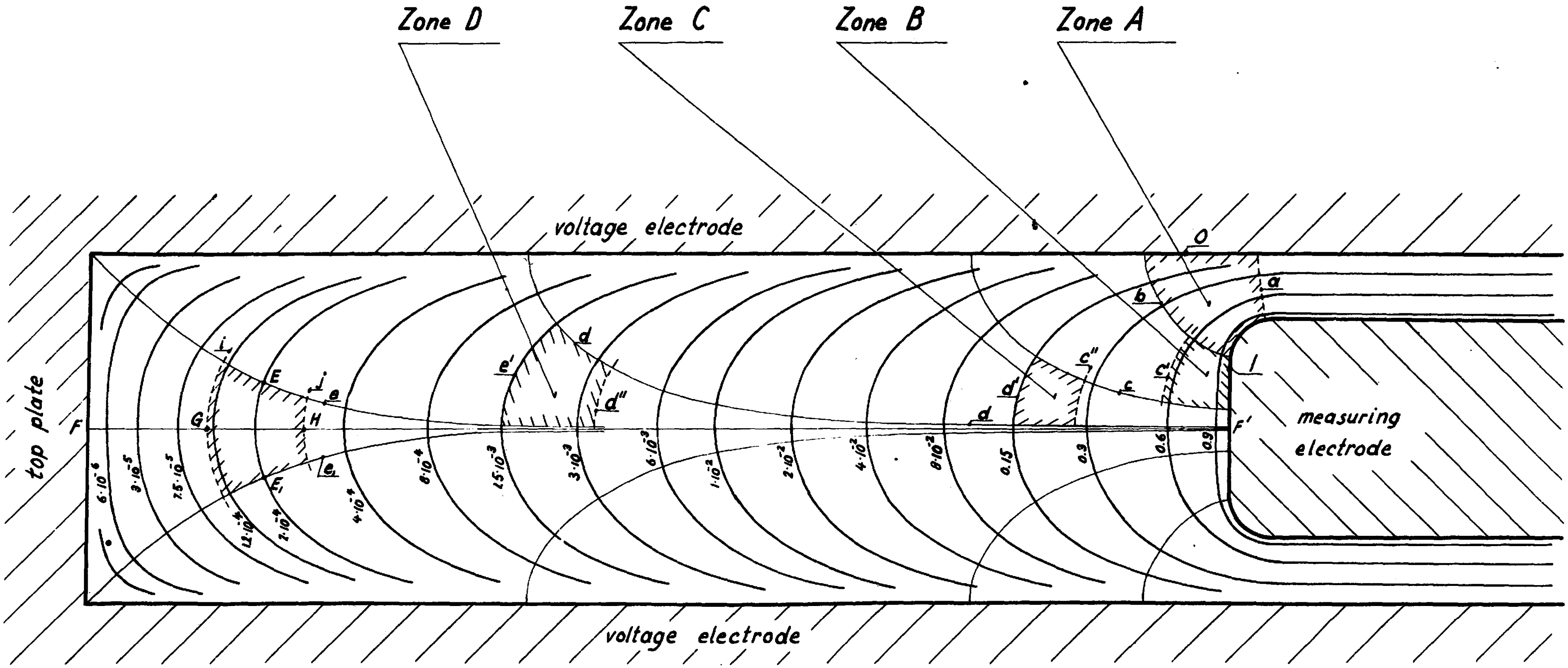
Δe represents the variation of the spacing e measured on axis F F' which is due to a change in temperature of $1^{o}C$.

To calculate this differential expansion use is made of an approximate length $\ell = 38$ cm.

$$\Delta e \ = \ \ell\ (\alpha_S\ \ \alpha_i) \ = \ 38\ (12 - 1.4) \ = \ 40.4\ 10^{-5} cm.$$

The deviation in capacitance caused by this expansion can be split into two parts. The variation of the capacity composed of the top plate and the top of the measuring cylinder, and the variation of the capacity composed of the two sides of the voltage cylinders and the measuring electrode. The first change is negative whereas the second one is positive. The positive variation is so much smaller than the negative one that it can be ignored.

Field map # 2


Zone D
Zone C
Zone B
Zone A
voltage electrode
voltage electrode
top plate
measuring electrode
F
F'
O
a
b
c
c'
c"
d
d'
d"
e
e'
e_1
E
E_1
G
H
i
l
$6\cdot10^{-6}$
$3\cdot10^{-5}$
$7.5\cdot10^{-5}$
$1.2\cdot10^{-4}$
$2\cdot10^{-4}$
$4\cdot10^{-4}$
$8\cdot10^{-4}$
$1.5\cdot10^{-3}$
$3\cdot10^{-3}$
$6\cdot10^{-3}$
$1\cdot10^{-2}$
$2\cdot10^{-2}$
$4\cdot10^{-2}$
$8\cdot10^{-2}$
0.15
0.3
0.6
0.9

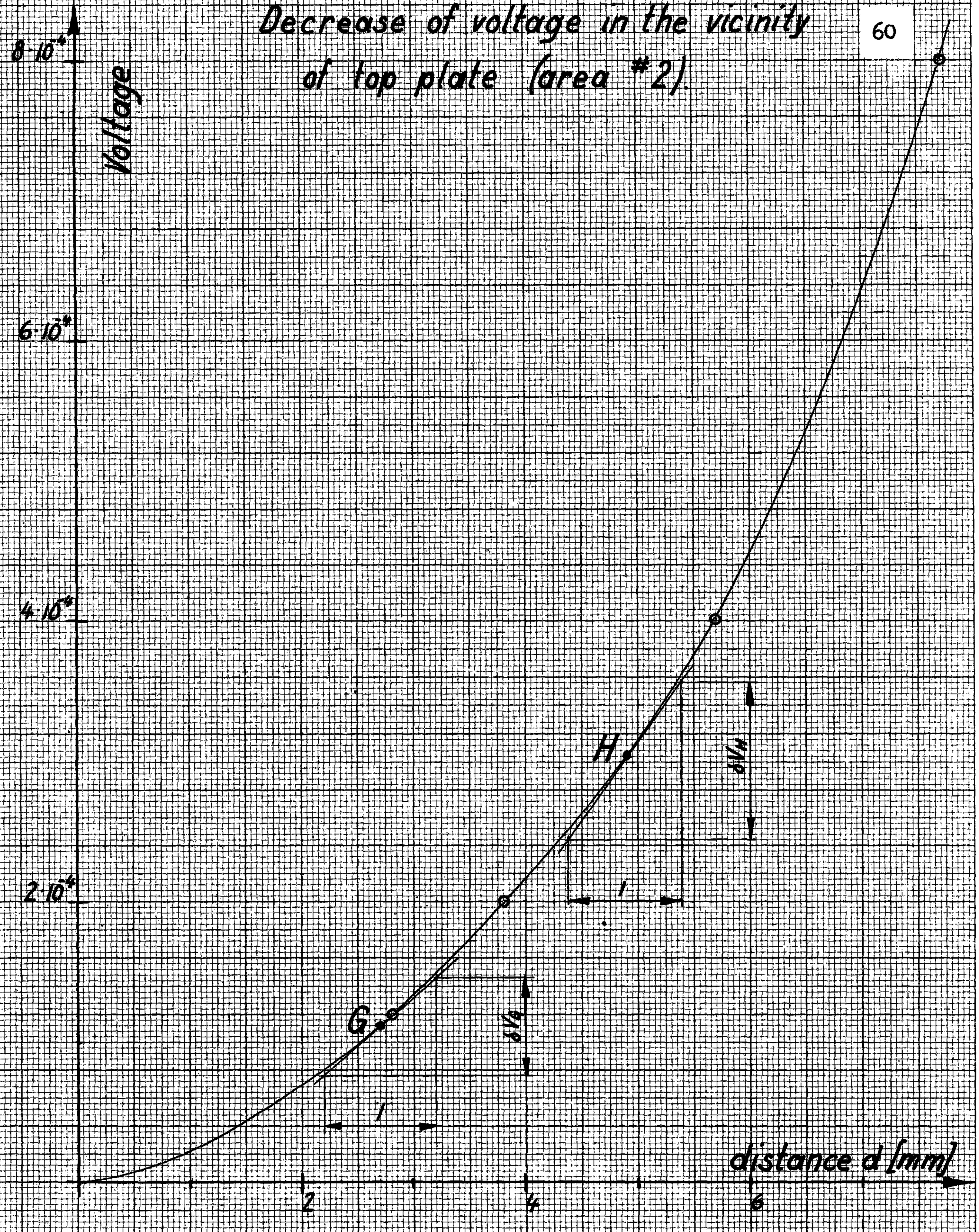

Decrease of voltage in the vicinity
of top plate (area #2).
Voltage
8·10⁻⁴
6·10⁻⁴
4·10⁻⁴
2·10⁻⁴
H
δV_H
1
G
δV_G
1
distance d [mm]
2
4
6

Fig. 12

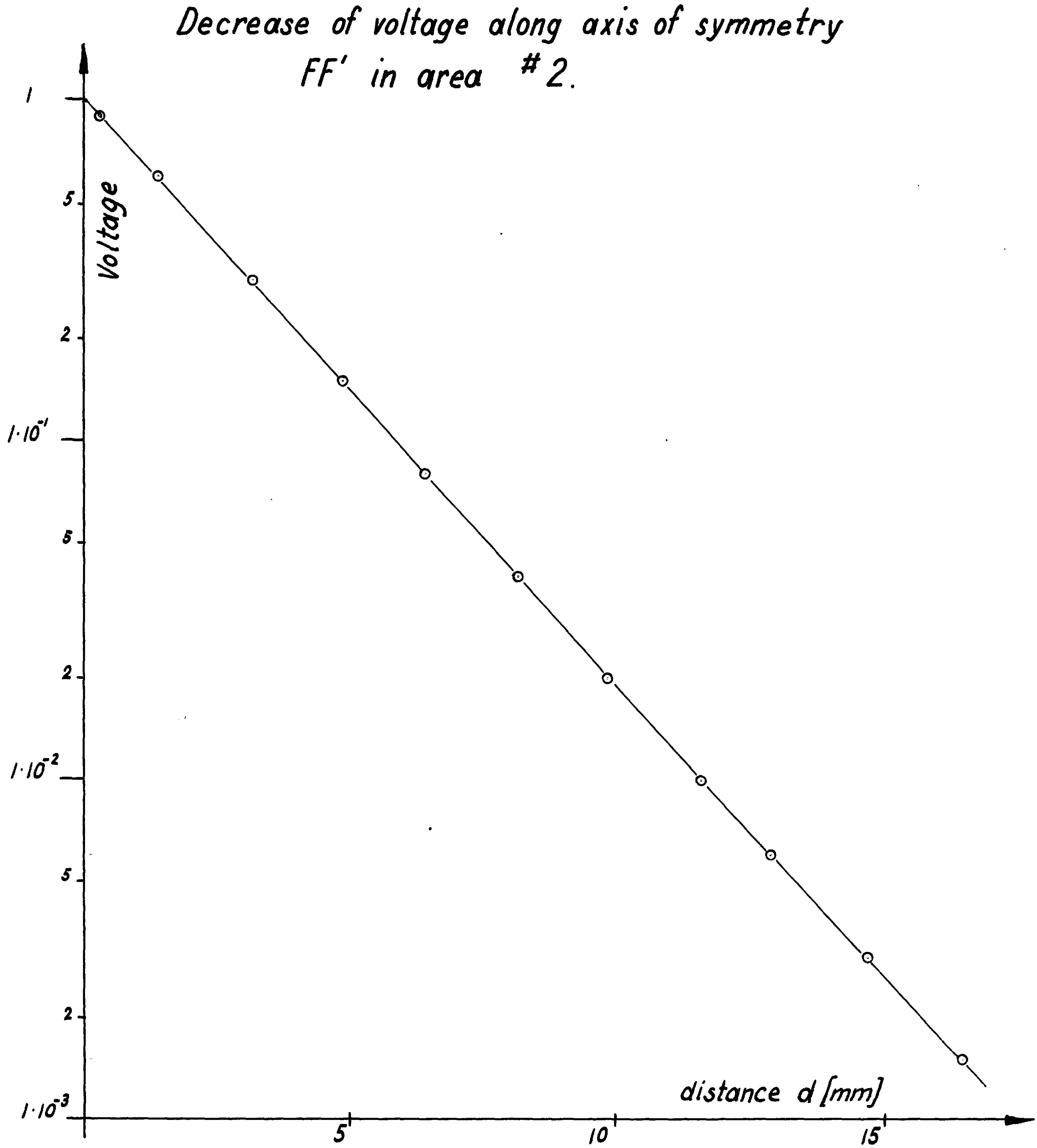

Decrease of voltage along axis of symmetry
FF' in area #2.
Voltage
1
5
2
1·10⁻¹
5
2
1·10⁻²
5
2
1·10⁻³
distance d [mm]
5
10
15

Fig. 13

The capacitance of the top plate and the measuring electrode is equal to the capacity of the tube of force lying between the two lines e and e_1. Lehmann's method is used to calculate it.

In the "curved" square, lying between equipotential lines i and j and lines of force e and e_1 shown in field map no. 2, the length $\overline{EE_1}$, measured on the equipotential at equal distance from i and j, is the same as the length $\overline{GH}$, measured on axis of symmetry FF'.

The capacity of this "curved" square is equal to the capacity of a square lying between the lines i and j, but located inside the air gap where, as shown on figure 13, the equipotential lines are parallel. Moreover, a square filling the whole width of the air gap has also the same capacity. It is:

$$C_{sq} = 0.15 \frac{2\pi\varepsilon_o\varepsilon}{6} \sum_{i=a}^{f} \frac{1}{\ln \frac{R_i}{r_i}} = 6.96 \text{ pF}$$

By interpolation on figure 12, the potential of the lines i and j is found to be:

$$i = 1.12 \ 10^{-4} \ V$$

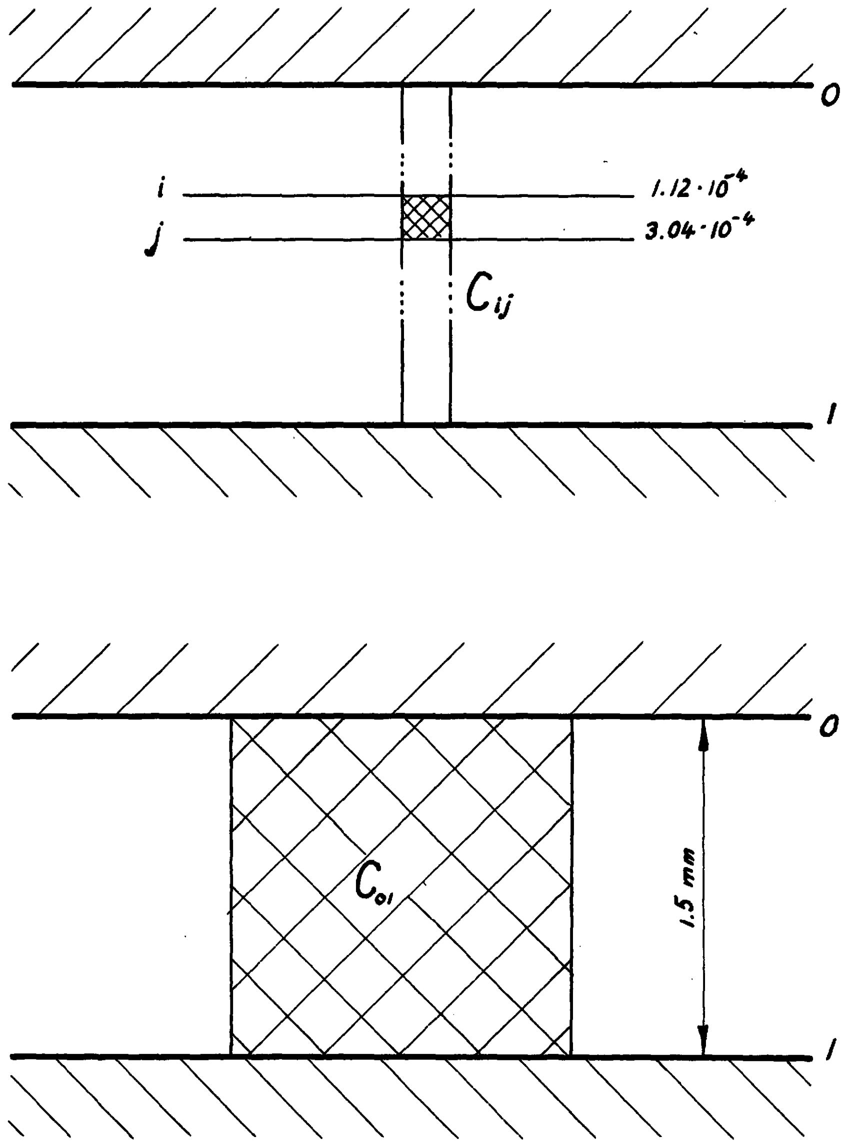
$$j = 3.04 \ 10^{-4} \ V$$

The side of the square is $\Delta V = 1.92 \ 10^{-4}$ V.

The length of the tube of force expressed in terms of potential is 1 V. Therefore the number of squares of equal capacitance is:

$$N = \frac{1}{1.92 \ 10^{-4}}$$

The capacitance of the top plate and the measuring electrode is



According to Lehmann's method, the capacity of the square C_{ij} is equal to the capacity of square C_{01} :

$$C_{sq} = C_{ij} = C_{01}$$

Fig. 14

equal to the capacity C_{tu} of the tube of force having N squares in series:

$$C_{tu} = \frac{C_{sq}}{N} = C_{sq}\ 1.92\ 10^{-4}\ \text{pF}$$

Due to thermal expansion it has been found earlier that the spacing increases from e to e + Δe, where :

$$\Delta e = 40.4\ 10^{-5}\ \text{cm.}$$

The field picture is assumed to remain the same. The same "curved" square will be used after thermal expansion, but the equipotential lines will not have the same values. To find their new values a linear interpolation will be made around the points G and H. Furthermore it will be assumed that Δe is distributed linearly along the axis of symmetry FF'. This will give a larger value of deviation than the actual one.

The deviation of G along axis FF' due to Δe is:

$$\Delta e_G = 2.7\ \frac{40.4\ 10^{-5}}{25.8} = 4.22\ 10^{-5}\ \text{cm.}$$

The deviation of H along axis FF' due to Δe is:

$$\Delta e_H = 4.9\ \frac{40.4\ 10^{-5}}{25.8} = 7.68\ 10^{-5}\ \text{cm.}$$

The linear variation of voltage is calculated from the slope of the tangents at points G and H:

$$\delta V_G = 7\ 10^{-5}\ \text{V cm}^{-1}$$

$$\delta V_H = 11.8\ 10^{-5}\ \text{V cm}^{-1}$$

The new potentials of lines i and j are:

$$i' = i - 4.22\ 10^{-5}\ 7\ 10^{-5} = i - 2.95\ 10^{-9}\ V$$

$$j' = j - 7.68\ 10^{-5}\ 11.8\ 10^{-5} = j - 9.05\ 10^{-9}\ V$$

The side of the new "curved" square is:

$$\Delta V = j' - i' = j - i - (9.05 - 2.95)\ 10^{-9}\ V$$

$$= 1.92\ 10^{-4} - 6.10\ 10^{-9}\ V$$

The number of squares in the tube of force is:

$$N' = \frac{1}{1.92\ 10^{-4} - 6.10\ 10^{-9}}$$

Therefore the new capacitance becomes:

$$C_{tu}' = \frac{C_{sq}}{N} = C_{sq}\ (1.92\ 10^{-4} - 6.10\ 10^{-9})$$

The deviation in capacitance is:

$$\Delta C_{tu} = C_{tu} - C_{tu}' = C_{sq}\ 6.10\ 10^{-9}\ pF$$

Only one measuring cylinder has been considered up to now. However there are three. The total deviation in capacitance is

$$\Delta C_{3tu} = 3\ C_{sq}\ 6.10\ 10^{-9}\ pF$$

$$= 3\ 6.96\ 6.10\ 10^{-9} = 1.28\ 10^{-7}\ pF$$

The deviation conpared to the value of the capacitor will give the variation in ppm due to thermal expansion of area no. 2:

$$\Delta_2 = \frac{\Delta C_{3tu}}{C} = \frac{1.28\ 10^{-7}}{10^4} = 1.28\ 10^{-11}$$

$$\Delta_2 = 0.0000128\ ppm.$$

Capacitance of area no. 2.

The capacitance of the top is known. To calculate the capacitance of the whole area, the side is divided into 4 tubes of force. In each of them a zone is chosen in order to calculate the capacitance.

According to field map no. 2:

tube I includes zone A

tube II includes zone B

tube III includes zone C

tube IV includes zone D

Each zone has the shape of a "curved" square and is found according to the Lehmann method. It lies between two lines of force and two equipotential lines which are

zone	line of force	equipotential line
A	a and b	O and 1
B	b and c	1 and c'
C	c and d	c" and d'
D	d and e	d" and e'

Capacitance of tube I.

Zone A is limited by equipotential lines O and 1, its side is thus unity. The capacity of tube I is:

$$C_A = C_{sq}$$

Capacity of tube II.

The potential of one of the lines is found by interpolation on figure 13, whereas the other one is already known to be 1.

The potential of line c' is 0.575 V.

The side of zone B is 1 - 0.572 = 0.425 V.

The capacity of tube II is:

$$C_B = C_{sq}\, 0.425$$

Capacity of tube III.

The potential of the lines c" and d' of zone C are found in the same way as for zone B:

potential of line c" : 0.26 V

potential of line d' : 0.15 V.

The side of zone C is 0.26 - 0.15 = 0.11 V. Thus the capacity of tube III is:

$$C_C = C_{sq}\, 0.11$$

Capacity of tube IV.

By interpolation the potential of lines d" and e' are found to be:

potential of line d" : 0.00355

potential of line e' : 0.0015

The side of zone D is 0.00355 - 0.0015 = 0.00205. The capacity of tube IV is:

$$C_D = C_{sq}\, 0.00205$$

The sum of the capacity of the 4 tubes of force I, II, III and IV is equal to the capacitance between the side of the voltage electrode and the top of the measuring electrode:

$$C_{si} = C_A + C_B + C_C + C_D = C_{sq}\,(1 + 0.425 + 0.11 + 0.00205)$$

$$C_{si} = C_{sq}\, 1.53705$$

The capacitor has 6 air gaps, therefore 6 such sides. The capacitance of the three ends of the measuring electrode is given by:

$$C_T = 3\ C_{tu} + 6\ C_{si}$$

$$= (3\ 1.92\ 10^{-4} + 6\ 1.53705)\ C_{sq}$$

$$C_T = C_{sq}(9.2223 + 0.000576)$$

$$C_T = 6.96\ 9.2229 = 64.1\,pF$$

The area corresponding to the capacitance C_T starts at line a which is 0.8 mm. under the top of the measuring electrode. This has to be remembered when calculating the length of the measuring electrode.

Determination of the dielectric constant ε of nitrogen.

The dielectric constant of nitrogen varies with pressure and with temperature, in other words with the density of the gas.

The pressure vessel, in which the capacitor is, will be filled with purified nitrogen at a pressure of 14 atmospheres. The temperature is assumed to be 20°C.

The dielectric constant ε is given by:

$$\varepsilon - 1 = 3\ c_1 d_1 + 3\ c_1^2\ d_1^2 + \ldots.$$

When only the first two order terms are considered, ε becomes:

$$\varepsilon - 1 = 3\ c_1 d_1\ (1 + c_1 d_1)$$

where

$$c_1 = 193.3\ 10^{-6}$$

and $d_1 = \dfrac{p}{1 + \frac{t}{273}}$ with $p = 15$ ata

$t = 20°C$

$$\varepsilon - 1 = 3\ 1.933\ 10^{-4} \frac{15}{1.0733} (1 + 1.933\ 10^{-4} \frac{15}{1.0733})$$

$$\varepsilon - 1 = 81.26\ 10^{-4}$$

The dielectric constant of N_2 at 14 atmospheres and 20^oC is:

$$\varepsilon = 1.008126$$

ε is very sensitive to the pressure. A variation in ε of 14 ppm is due to a deviation of only 0.1% in pressure. However, the adjustment provided by means of eccentricity of one set of cylinders with respect to the other is large enough to correct an error due to a deviation of the pressure much larger than 0.1%.

Determination of the length of the measuring electrode.

It is now possible to calculate the length of the measuring electrode in order to obtain, after having assembled the different parts, a value of capacitance equal to 10 nF.

The length must be calculated in such a way that at half eccentricity the capacitance reads 10 nF.

This capacitance is given by:

$$C = 2\pi\varepsilon_o \varepsilon \ell \sigma$$

where

ε_o = 0.08854 pF cm^{-1}

ε = 1.008126

ℓ = length of electrode

$$\sigma = \sum_{i=a}^{f} \left[\ln \frac{R_i}{r_i} \cdot \frac{R_i^2 - r_i^2 - D^2 + \sqrt{(R_i^2 - r_i^2 + D^2)^2 - 4R_i^2 D^2}}{R_i^2 - r_i^2 + D^2 + \sqrt{(R_i^2 - r_i^2 + D^2)^2 - 4R_i^2 D^2}} \right]^{-1}$$

σ = 496.31331 as found on page 33

$C = 278.3495 \cdot \ell$

It is not correct to use 10 nF for the value of C, since there are two additional capacitances, which are:

1) Capacitance of the top of the measuring electrode:

$$C_T = 64.1 \text{ pF}$$

2) Capacitance of auxiliary electrode

$$C_f = 20.0 \text{ pF}$$

Therefore the value of C is:

$$C = 10\,000 - 64.1 - 20.0 = 9915.9 \text{ pF}$$

The length is:

$$\ell = \frac{9915.9}{278.3495} = 35.6239 \text{ cm} = 356.24 \text{ mm}$$

According to the calculations of the capacitance of area no. 1 and area no. 2, the length ℓ is measured from the middle of the spacing between measuring electrode and guard ring (C-C) to a distance of 0.8 mm from the top of the measuring electrode.

The length L measured between both ends of the measuring cylinder becomes:

$$L = \ell - \frac{0.15}{2} + 0.8$$

$$L = 356.239 - 0.075 + 0.8 = 356.96 \text{ mm.}$$

Design recommendations

Plating of the cylinders with Rhodium

The dielectric absorption on the surface of the electrodes must be as low as possible. Investigations made at N.P.L. indicate that the minimum value is obtained with rhodium-plated surfaces.

It is not possible to purchase Invar pipe from which the cylinder could be machined. The electrodes will be made from rolled sheets. The welded seam must be made according to the recommendations of the manufacturer of Invar.

In order to ensure good dimensional stability, electrodes will be annealed prior to finished machining.

Lastly a thin film of rhodium will go on the cylinders to minimize the dielectric absorption.

It has been seen that the value of capacitance is highly sensitive to deviations from the geometrical dimensions, especially on the radii. The cylinders should be machined keeping in mind that a thin film of rhodium goes on the surface. Rhodium has a coefficient of thermal expansion which is much larger than that of Invar. The film must be thin so that the thermal expansion of the cylinders will not be disturbed.

The eccentric drives

Coarse adjustment

The set of four voltage cylinders rotates in order to adjust its eccentricity with respect to the measuring electrode system.

A leakproof means of connecting the inside of the pressure vessel to the outside is called the "cats tail" drive and is shown on the general assembly drawing. The shaft is connected to an external gear drive composed of one spur gear of one hundred teeth driven by a worm gear. The drive must be made with a spring loaded worm gear in order to avoid any play.

Fine adjustment

The "cats tail" drive is also used here. Unlike the coarse adjustment, the drive of the fine adjustment is internal. As shown on the general assembly drawing, spur gears are used. The number of teeth satisfies the ratio 1 to 2.

When the coarse adjustment is in operation, the auxiliary electrode may or may not rotate, depending on the fits of the fine adjustment. But, when the fine adjustment takes over the trimming, it is clear that the coarse adjustment cannot rotate since it is driven with a worm gear.

Dimensioning of the pressure vessel

The vessel containing the capacitor is filled with nitrogen at a pressure of 14 atmospheres, which produces mechanical forces on the walls.

The dimensions of the top and bottom plates as well as the cylindrical part of the vessel are calculated in order to stay within suitable limits of stress and deformation.

The top and bottom plates of the vessel

The inner diameter of the vessel is 31 cm; however, to calculate the force produced by the pressure on the plates, the inner diameter of the pressure gasket, which is 32 cm., has to be used.

The thickness of the plate is not given from the maximum permissible stress but rather from the maximum deflection δ_{max} which can be tolerated to avoid any change in the position of the electrodes.

The theory of flexure of plates yields two relations to calculate the maximum stress σ_{max} and the maximum deflection δ_{max} for a fixed-edge circular plate under uniform load. They are called Grashof's formulas.

The maximum deflection δ_{max} is given by:

$$\delta_{max} = \left[1 + 5.72 \left(\frac{t}{r}\right)^2\right] \frac{3}{16} (1 - \mu^2) \frac{pr^4}{E\,t^3}$$

where

$t = 3.8$ cm : thickness of the plate

$r = 16$ cm : inner radius of pressure gasket

$\mu = 0.29$: Poisson's ratio

$p = 14$ kg cm^{-2}: evenly distributed load

$E = 2.1\ 10^{-6}$ kg cm^{-2} : modulus of elasticity of steel

$$\delta_{max} = \left[1 + 5.72 \frac{14.44}{256}\right] \frac{3}{16} (1 - 0.084) \frac{14\ 256^2}{2.1\ 10^{-6}\ 55}$$

$$= 1.323\ 1.365\ 10^{-3} = 1.81\ 10^{-3} \text{cm}$$

$$\delta_{max} = 1.81\ 10^{-2} \text{ mm}$$

This value of δ_{max} is acceptable. Actually this deflection will be smaller because the thickness t will be at some places larger in order to allow the machining of the grooves into which the guard rings are inserted.

The maximum stress for a fixed-edge circular plate is at the edges. It is given by the relation:

$$\sigma_{max} = \frac{3}{4}\, p \frac{r^2}{t^2} = \frac{3}{4}\, 14 \frac{256}{14.44} = 186 \text{ kg cm}^{-2}$$

The maximum stress is rather low. The stress will, however, be larger around the clamping bolts because of the holes.

<u>The wall of the vessel</u>

As shown on figure 15, a small element of the wall is strained in three different directions.

The theory of the thick wall cylinder gives two formulas to calculate σ_t and σ_r:

$$\sigma_t = p \frac{r_1^2}{r_2^2 - r_1^2} \left(\frac{r_2^2}{\rho^2} + 1 \right)$$

$$\sigma_r = p \frac{r_1^2}{r_2^2 - r_1^2} \left(\frac{r_2^2}{\rho^2} - 1 \right)$$

The tensile stress σ_ℓ, due to the fact that the cylinder is not infinitely long but has two end plates, is given by:

$$\sigma_\ell = \frac{p r_c}{2t}$$

Stresses in the wall of the vessel.

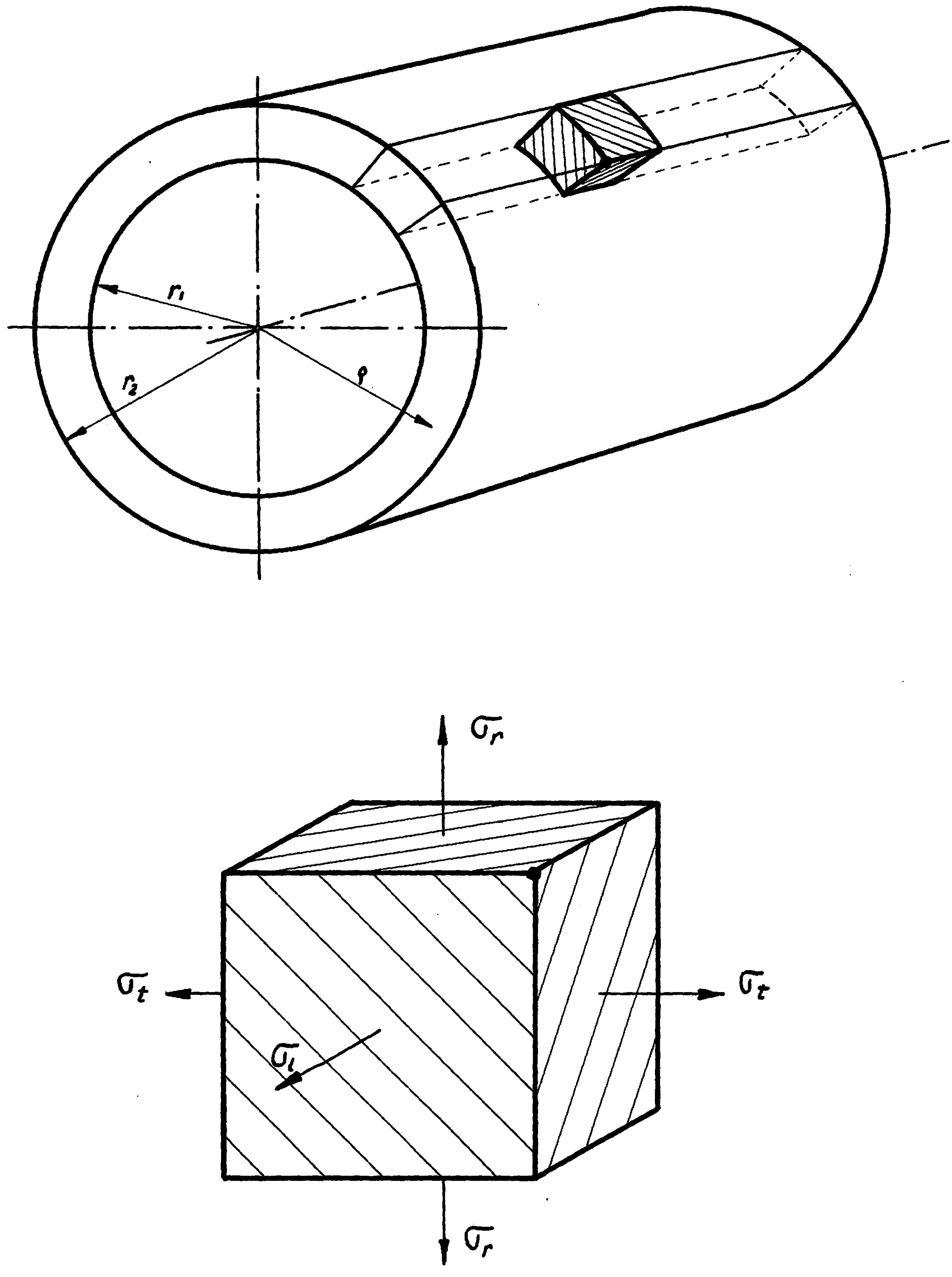


Fig. 15

σ_t and σ_r are maximum for $\rho = r_1$:

$$\sigma_r = p$$

$$\sigma_t = p \frac{r_2^2 + r_1^2}{r_2^2 - r_1^2}$$

The thickness of the wall $t_w = r_2 - r_1$ is assumed to be 1 cm. The pressure is 14 kg cm^{-2}. The radii are $r_1 = 15.5$ cm and $r_2 = 16.5$ cm.

$$\sigma_r = 14 \text{ kg cm}^{-2}$$

$$\sigma_t = 14 \frac{272.25 + 240.25}{272.25 - 240.25} = 224 \text{ kg cm}^{-2}$$

$$\sigma_\ell = \frac{14.160}{2.1} = 112 \text{ kg cm}^{-2}$$

These stresses are very low. Even their combination within the plane of maximum stress is fairly low.

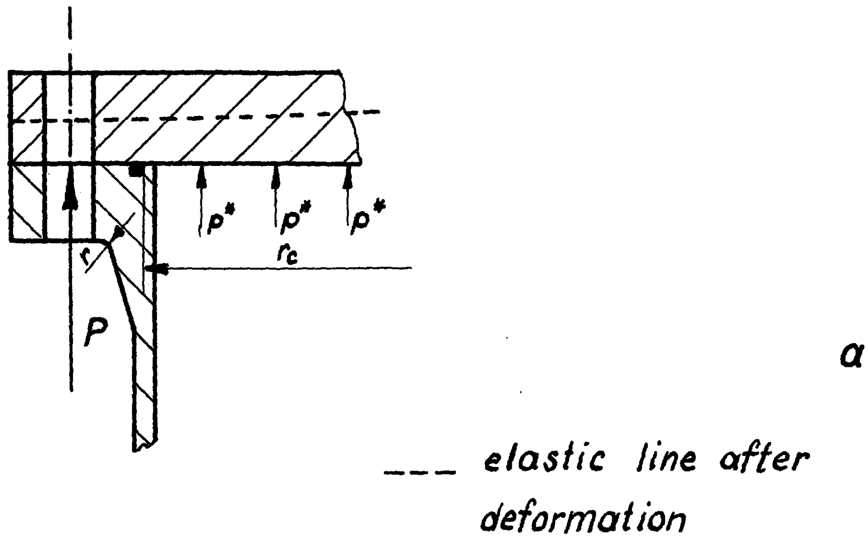
So much for the cylindrical portion of the vessel. The ends of the cylinder still remain to be calculated, in order to hold the top and the bottom plates.

It is assumed, as shown in figure 16, that the force caused by the pressure p is transmitted to the cylinder at several points P. This is true when the elastic line of the plate follows the line --- (figure 16 a); that is to say, is flat as far as the axis of the bolt.

The cylinder is considered to be developed and put flat on a plane as represented in figure 16 b.

To clamp the plate to the cylinder, twelve bolts are used. The force in each bolt, caused by the pressure is called P.

Development of the wall on a plane.



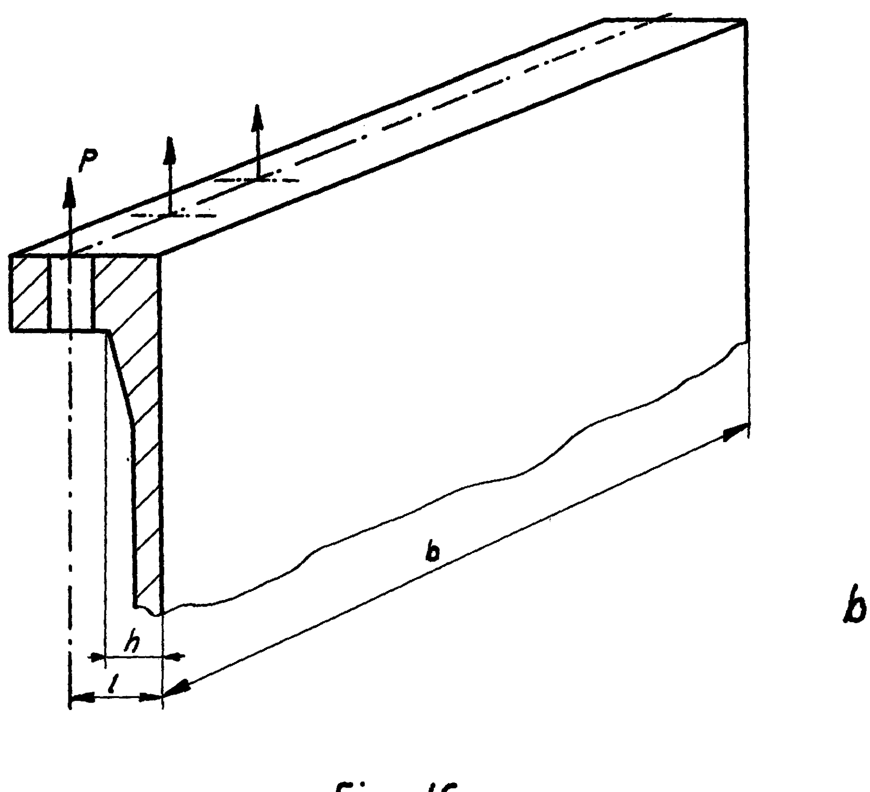


Fig. 16

$$18\ P = \sum p^* = p\pi r_c^{\ 2}$$

$$P = \frac{14\pi\ 16^2}{18} = 630\ \text{kg.}$$

The cross section of bolts M 16, measured at the inside of the thread is:

$$s = 1.37\ \text{cm}^2$$

Consequently, the tensile stress in the bolts is

$$\sigma_{max} = \frac{P}{s} = \frac{630}{1.37} = 460\ \text{kg cm}^{-2}$$

According to Bach (Hutte II, page 13) the tensile stress should not exceed 800 kg cm^{-2} for steel Ac.60.11 when neither stress concentration factor nor stress due to tightening of the bolts are taken into consideration. The bolts are made of steel Ac. 60.11

Thickness of the wall of the cylinder at the flange

The stress will be calculated using the standard relation:

$$\sigma_{fl} = \frac{M_{fl}}{W} = \frac{\ell \sum p^*}{\frac{bh^2}{6}} = \frac{6 \sum p^*}{b} \cdot \frac{\ell}{h^2}$$

where

b = $2\pi r$ developed circumference of the inside of the cylinder.

ℓ = 1.6 + h as shown on figure 16 b.

The figure 1.6 cm represents half the diameter of the washer for bolts M 16.

Let h = 1.9 cm.

$$\sigma_{fl} = \frac{6\ 14\ \pi\ 15.5^2\ 3.5}{2\pi\ 15.5\ \ \ 3.61} = 630\ \text{kg cm}^{-2}$$

The tensile stress found earlier $\sigma_{\ell} = 112$ kg cm^{-2} must be combined with σ_{fl}.

σ_s of steel Ac60.11 is 3000 kg cm^{-2}. The combination of σ_{ℓ} and σ_{fl} gives a stress which is less than 30% of σ_s of steel Ac.60.11.

These calculations give the main dimensions of the pressure vessel represented on the general assembly drawing.

σ_s is the stress corresponding to the elastic limit of the material.

CONCLUSION

In chapter II the method has been presented. General considerations on the design of the high voltage and the low voltage capacitors, composing the capacitive divider, were given.

Although no word has been said about the switching device, it has already been studied far enough to be sure that it will not present unsolvable problems. It will, however, require special techniques and careful examination will be made which will lead to the final design.

The high voltage capacitor has been designed and it is now certain that its accuracy will be within the desired limits.

It is shown in chapter III that the error, ε_6 is only composed of the error ε' and ε''. Furthermore, it is proved that ε' can be sufficiently minimized by means of eccentric adjustment to become negligible.

The error ε_6 is therefore equal to ε''. Figure 17 shows the error ε_6 in ppm as a function of temperature variations. The parameter is $\frac{\Delta\alpha_1}{\alpha_1}$ which represents the deviation of the value α_1 according to the ratio $\frac{\alpha_1}{\alpha_2}$.

It is clear that the requirement for capacitor C_2 are fulfilled.

Figure A represents the general assembly drawing of the capacitor C_2.

Total error ε_6 versus temperature variations.

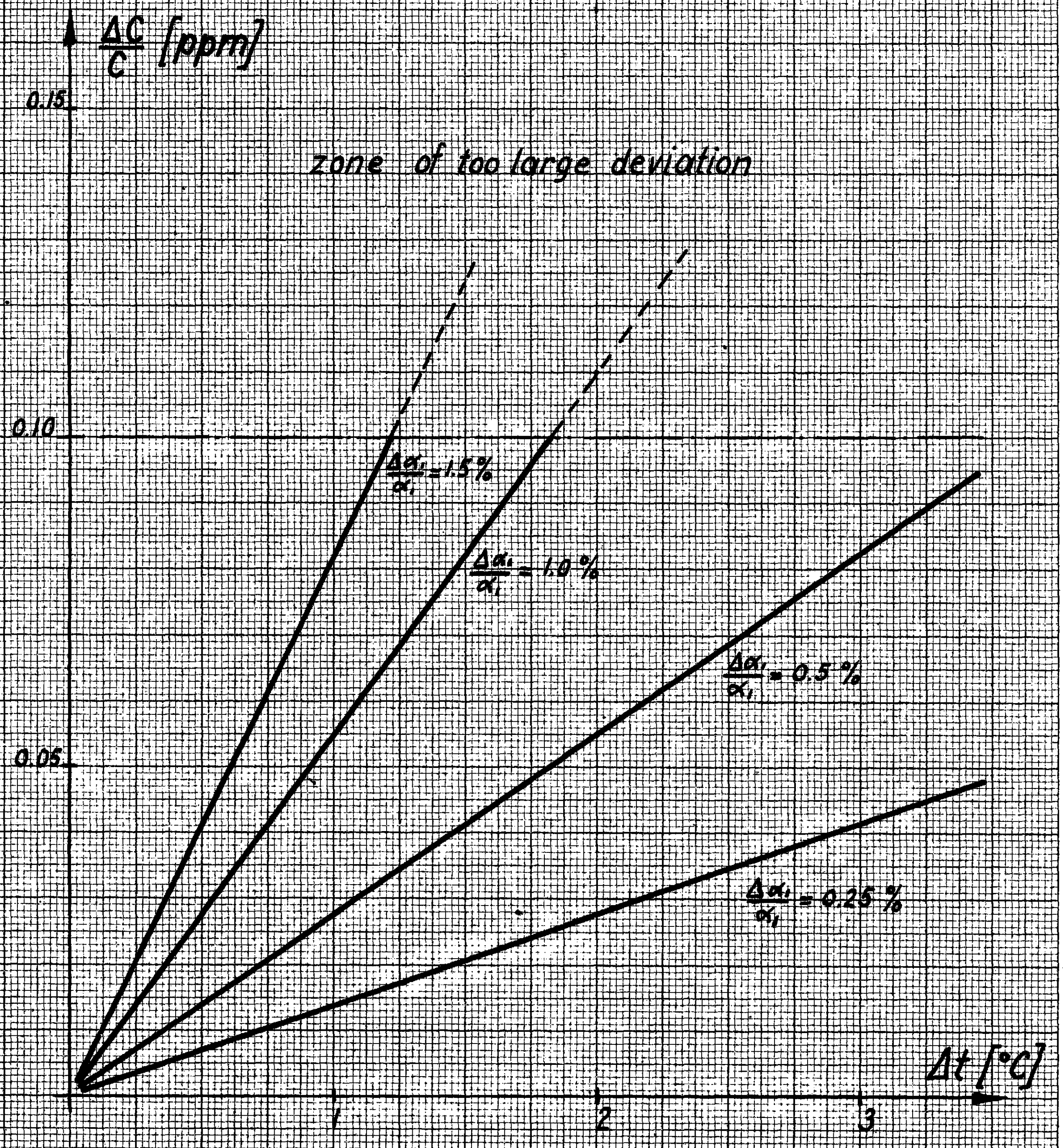


Fig. 17

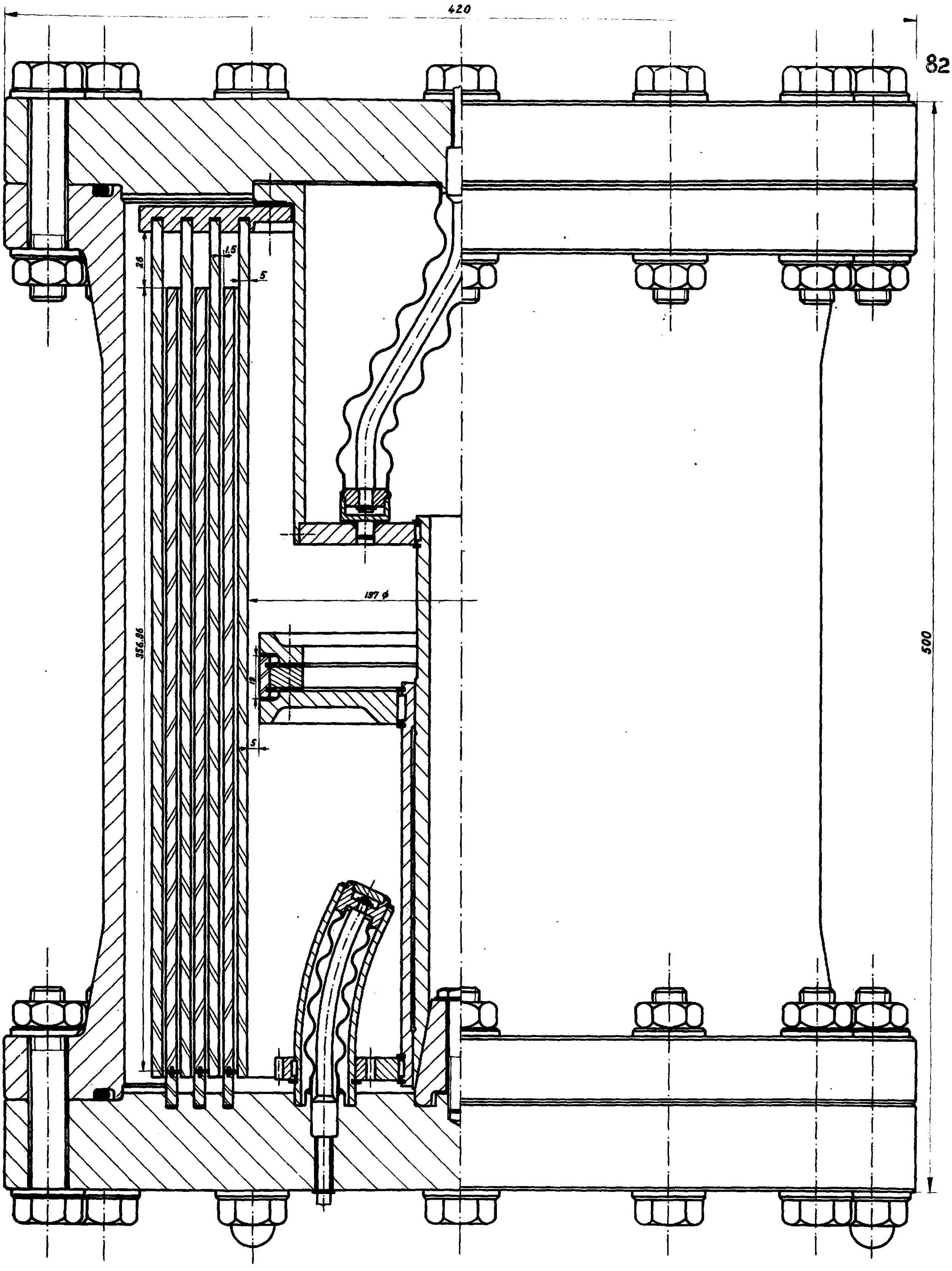


Fig. A

No. Required		Title	Item No.	Material	Weight	Comments
2	1	Revisions			Scale	1:1
		Project: *Absolute calibration of Beta-Ray Spectrometer*			By: [illegible]	Date: 29.9.62
		Assembly: *Low Voltage Capacitor C_2*			No: AE-2-62	
		Detail			No:	
		McMASTER UNIVERSITY Electrical Engineering Department			Acc. No: B-20-225	

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