BOUNDING THE DECAY OF P-ADIC OSCILLATORY INTEGRALS WITH A CONSTRUCTIBLE AMPLITUDE FUNCTION AND A SUBANALYTIC PHASE FUNCTION

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To my mother, father and brother

Abstract

We obtain an upper bound for oscillatory integrals of the form $\int_{R_v^m} f(x)\psi(y.\phi(x))|dx|$ where ψ is an additive character, $\phi : R_v^m \to K$ is an analytic map satisfying the hyperplane condition and $f \in C(R_v^m)$ is integrable. Igusa, Lichtin and Cluckers have proved that we can find the decay rate for such oscillatory integrals with certain conditions on f and ϕ. In this thesis we generalize those results by imposing the hyperplane condition on ϕ.

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Chapter 1

Introduction

In this thesis, we want to study a certain property of some specific p-adic integrals, namely the decay rate of p-adic oscillatory integrals of the form

$$\int_{R_v^m} f(x)\psi(\phi(x).y)|dx|$$

in which R_v is the valuation ring of a local p-adic field, $\phi : R_v^m \to K$ is an analytic function, $f(x)$ is a constructible function on R_v^m ($f \in C(R_v^m)$),$|dx|$ is the Haar measure and ψ is an additive character. We call f the amplitude and ϕ the phase function. The main theorem we prove in this thesis is:

Theorem 1.1. *Let $\phi : R_v^m \to K$ be an analytic map satisfying the hyperplane condition. Let $f \in C(R_v^m)$ be integrable and suppose ψ is an additive character. Let $\epsilon > 0$. Then there are real numbers $s < 0$ and $c > 0$ such that*

$$|\int_{R_v^m} f(x)\psi(y.\phi(x))|dx|| \leq c\min\{1, |y|^s\} + \epsilon$$

for all $y \in K^\times$. Moreover, s does not depend on ϵ while c does.

The Vinogradov symbol $\ll$ has its usual meaning, namely that for complex valued functions f and g with g taking non-negative real values $f \ll g$ means $|f| \leq cg$ for some constant c.

This theorem has a long history. In his book [14] of 1978, Igusa proves Theorem 1.1 in the case that $f(x) = 1$ for all x and $\phi : K^m \to K$ is a nonconstant homogeneous polynomial and then he used the results to give a nice description of the generalized Gaussian sum of the p-adic oscillatory integrals. He formulates the problem of generalizing this to the case of homogeneous polynomial maps $\phi : K^m \to K^r$ for $r > 1$. By a very careful analysis of embedded resolutions of f, Lichtin [17] is able to prove Igusa's version of Theorem 1.1 in the case that $\phi : K^m \to K^2$ is a dominant map (a map whose image is Zariski dense in the co-domain) whose coordinate maps are polynomial. By using cell decomposition, Cluckers [5] proves Theorem 1.1 in the case that $\phi : R_v^m \to K^r$ is a restricted power series such that $\phi(R_v^m)$ has nonempty interior in K^r, for arbitrary r. The goal of this thesis is to replace the dominancy condition by the hyperplane condition which is a more general case.

The hyperplane condition is defined as :

Definition 1.2. *Let $X \subseteq K^r$ be a subanalytic set. We say that a measurable function $\phi : X \times K^m \to K^n$ satisfies the hyperplane condition over X if for every $x \in X$ and every affine hyperplane H in K^n, the set $\{y \in K^m : \phi(x, y) \in H\}$ has measure zero.*

The hyperplane condition is an adaptation of a condition from Stein's book [23], to the context of analytic maps instead of C^∞ maps. Stein also proved decay results under this hyperplane condition (but with compactly supported smooth amplitudes). In [8], Cluckers and Miller prove Theorem 1.1 in the case of a real field by using real analytic tools such as the van der Corput lemma and its corollaries and a specific version of cell-decomposition. In this thesis, we prove a further modification of the version of Van Der Corput's lemma on the p-adic fields which is proved by Cluckers in [7].

In the first six chapters we discuss the requirements to proving the main theorem. In chapter two, we review some basic definitions and theorems regarding valued fields and subanalytic sets. We also discuss the model theoretic setting for studying p-adic integration. In chapter three, we discuss Haar measure and p-adic integration. In the fourth chapter, we state cell decomposition and we verify some properties of constructible functions. In chapter five, we review some properties of additive characters. In chapter six, we state Van Der Corput's lemma proved by Cluckers and we prove a generalized version of that lemma. In the last chapter, we restate the main theorem and we prove it.

Chapter 2

Valued fields and the model-theoretic setting

In this chapter, we present the classic definitions and theorems about valued fields and local fields. We also discuss the required model-theoretic setting for studying p-adic integration. We refer the reader to [15],[12],[25] and [10] for more details.

Let Γ be an ordered abelian group. We define a valuation v on a field K to be a surjective map $v : K \to \Gamma \cup \{\infty\}$ satisfying the following axioms:

- $v(x) = \infty \Leftrightarrow x = 0$;
- $v(xy) = v(x) + v(y)$;
- $v(x + y) \geq min\{v(x), v(y)\}$

for all $x, y \in K$. The set

$$R_v := \{x \in K | v(x) \geq 0\}$$

is a valuation ring of K, i.e., a subring of K such that for all $x \in K^\times$ either $x \in R_v$ or $x^{-1} \in R_v$. The set of non-units $M_v := \{x \in K | v(x) > 0\}$ forms a maximal ideal of R_v; in fact, the only such. We call the quotient $\overline{K_v} := R_v/M_v$ the residue field of v.

An example is given by the p-adic valuation, where p is any prime number. We define the

p-adic valuation $v : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ by $v(0) = \infty$ and

$$v(p^r \frac{m}{n}) = r$$

where $m, n \in \mathbb{Z} \setminus \{0\}$ are not divisible by p. Clearly, the valuation ring R_v is the localization $\mathbb{Z}_{(p)}$ of the ring $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ and the maximal ideal M_v is $p\mathbb{Z}_{(p)}$. Thus the residue field is isomorphic to the finite field $\mathbb{F}_p$. We define a metric on $\mathbb{Q}$ by using the p-adic norm $|x|_p = p^{-v(x)}$.

If $\Gamma \subseteq \mathbb{R}$, the valution v induces an ultrametric on K by $|x - y|_v = e^{-v(x-y)}$ that satisfies the ultrametric inequality

$$|x + y|_v \leq \max\{|x|_v, |y|_v\}.$$

We call the topology induced by this metric the v-topology and it has the following properties:

- For each $a \in K$ and $\gamma \in \Gamma$ we define

$$U_\gamma(a) = \{x \in K : v(x - a) > \gamma)\}.$$

  These sets form a basis of open neighborhoods of a.

- The sets $\{x \in K : v(x - a) \geq \gamma\}$, $\{x \in K : v(x - a) \leq \gamma\}$,$\{x \in K : v(x - a) = \gamma\}$ and $U_\gamma(a)$ are both open and closed.

- The field operations are continuous with respect to this metric topology.

A valuation v of K is called discrete if Γ is a discrete subgroup of $(\mathbb{R}, +)$; that is, if $\Gamma = \mathbb{Z}\beta$ for some real number $\beta \geq 0$. Since in this thesis we just need to deal with discrete valuations, from now on we assume that v is a discrete valuation on K.

A sequence of points, x_1 , x_2 , x_3 , . . . , in K converges to $x \in K$ in the v-topology , that is

$$\lim_{n\to\infty} x_n = x$$

if and only if $\lim_{n\to\infty} v(x_n - x) = \infty$. When this is so, then $\lim_{n\to\infty} v(x_n) = v(x)$. In fact, if $x \neq 0$, then $v(x_n) = v(x)$ for all sufficiently large n. A sequence x_1 , x_2 , x_3 , . . . , in K is called a

Cauchy sequence in the v-topology when

$$v(x_n - x_m) \to \infty, \text{ as } m, n \to \infty.$$

A convergent sequence is, of course, a Cauchy sequence, but the converse is not necessarily true. The valuation v is called complete if every Cauchy sequence in the v-topology converges to a point in K. If v is complete then the infinite sum

$$\sum_{n=1}^{\infty} x_n = \lim_{i\to\infty} \sum_{n=1}^{i} x_n$$

converges in K if and only if $v(x_n) \to \infty$, as $n \to \infty$.

Let K' be an algebraic extension field of K, and v' a valuation of K'. Let $v'|_K$ denote the function on K, obtained from v' by restricting its domain to the subfield K. Then $v'|_K$ is a valuation of K, and we call it the restriction of v' to the subfield K. On the other hand, if v is a valuation of K, any valuation v' on K' such that $v'|_K = v$ is called an extension of v to K'.

Let (K', v') be an extension of (K, v). It is easy to see that the residue field of v is naturally embedded in the residue field of v'. On the other hand, $v'|_K = v$ also implies that Γ is a subgroup of Γ'. Let

$$e = [\Gamma' : \Gamma], \quad f = [\bar{K'}_{v'} : \bar{K}_v]$$

where $[\Gamma' : \Gamma]$ is the group index and $[\bar{K'}_{v'} : \bar{K}_v]$ is the degree of the extension $\bar{K'}_{v'}/\bar{K}_v$. The integers e and f are called the ramification index and the residue degree of the extension v'/v, respectively. The following proposition is a fundamental result on the extension of valuations:

Theorem 2.1. *Let v be a complete valuation of K and let K' be an algebraic extension of K. Then v can be uniquely extended to a valuation v' of K'. If in particular, K'/K is a finite extension, then v' is also complete, and*

$$v'(x') = \frac{1}{n} v(N_{K'/K}(x'))$$

for all $x' \in K'$, where $n = [K' : K]$ is the degree and $N_{K'/K}$ is the norm of the extension K'/K.

Proof. We refer the reader to van der Waerden, [25]. □

Let v be a valuation of K, not necessarily complete. It is well known that there exists an extension field K' of K and an extension v' of v on K' such that v' is complete and K is dense in K' in the v'-topology of K'. Such a field K' is called a completion of K with respect to the valuation v. By the definition, each $x' \in K'$ is the limit of a sequence of points, x_1 , x_2 , x_3 , . . . , in K in the v'-topology:

$$x' = \lim_{n\to\infty} x_n.$$

Then $v'(x') = \lim_{n\to\infty} v(x_n)$ and hence if $x' \neq 0$, then $v'(x') = v(x_n)$ for all sufficiently large n. It follows that the valued group does not change.

Let $\pi_0 \in K$ be an element with least positive valuation. Any such element π_0 is called a prime element of K. Let

$$M_v^n = (\pi_0^n) = R_v\pi_0^n = \{x \in K : v(x) \geq nv(\pi_0)\}$$

be the ideal of R_v generated by π_0^n for $n \in \mathbb{Z}$. Fix A a complete set of representatives of the residue field of K. The following theorem can be found in [15].

Theorem 2.2. *Each nonzero $x \in K$ can be uniquely expressed in the form*

$$x = \sum_{n=i}^{\infty} a_n\pi_0^n$$

where $a_n \in A$ for all n and $v(x) = i$. We call $x = \sum_{n=i}^{\infty} a_n\pi_0^n$ the p-adic expansion of x. We can obtain the p-adic expansion of $0 \in K$ by choosing all coefficients to be zero.

Proof. The uniqueness is easy to verify. Without loss of generality, we assume $x \neq 0$, $v(x) = i < \infty$. Now, by the definition of A,

$$R_v = A + M_v = \{a + M_v : a \in A\}.$$

Since $M_v^n = \{x \in K : v(x) \geq nv(\pi_0)\}$ for $n \in \mathbb{Z}$, it follows that

$$M_v^n = A\pi_0^n + M_v^{n+1} = A\pi_0^n + A\pi_0^{n+1} + ... + A\pi_0^m + M_v^{m+1}$$

for all $m \geq n$. As $x \in M_v^i$, we see that there exists a sequence of elements in A, $a_i, a_{i+1}, ...$, such that

$$x \equiv \sum_{n=i}^{j} a_n \pi_0^n \pmod{M_v^{j+1}}$$

for any $j \geq i$. It then follows that

$$x = \lim_{j \to \infty} \sum_{n=i}^{j} a_n \pi_0^n = \sum_{n=i}^{\infty} a_n \pi_0^n.$$

□

Moreover every such series $\sum_{n=i}^{\infty} a_n \pi_0^n$ converges if v is a complete valuation on K since $v(a_n \pi_0^n) \to \infty$.

Let A^∞ denote the set of all sequences $(a_0, a_1, a_2, ...)$, where a_n are taken arbitrarily from the set A defined above. Thus A^∞ is the set-theoretical direct product of the sets $A_n = A$ for all $n \geq 0$:

$$A^\infty = \prod_{n=0}^{\infty} A_n.$$

Introduce a topology on A^∞ as the direct product of discrete spaces A_n, $n \geq 0$.

Corollary 2.3. *The map*

$$(a_0, a_1, a_2, ...) \to \sum_{n=0}^{\infty} a_n \pi_0^n,$$

defines a homeomorphism of A^∞ onto the valuation ring R_v of (K, v).

Proof. Let $x = \sum a_n \pi_0^n$ and $y = \sum b_n \pi_0^n$ where $a_n, b_n \in A$. Then it is easy to verify that for any integer i,

$$v(x - y) \geq i \Longleftrightarrow a_n = b_n \ for \ all \ n < i.$$

This fact shows that the map is bijective and Theorem 2.2 implies that it is a homeomorphism. □

Corollary 2.4. *K is a locally compact field in its v-topology.*

Proof. Since A^∞ is a compact space, by Corollary 2.3, R_v is compact and hence K is locally compact. □

Next, we want to review some important properties of the valued field extensions. Let (K', v') be a complete extension of the complete valued field (K, v). We discussed before that $\bar{K}_v$ can be naturally embedded into $\bar{K}'_v$. Let $w_1, ..., w_s$ be any finite number of elements in $\bar{K}'_v$, which are linearly independent over $\bar{K}_v$, and for each i, $1 \leq i \leq s$, choose an element ξ_i in $R'_{v'}$ that belongs to the residue class w_i in $\bar{K}'_v$. Fix a prime element $\pi'_0 \in K'$ and let

$$\eta_{ij} = \xi_i \pi'^j_o,$$

for $1 \leq i \leq s$ and $1 \leq j \leq e$ where e is the ramification index of the extension K'/K.

Theorem 2.5. *With the notation from previous paragraph*

1. *Let $x' = \sum x_{ij}\eta_{ij}$ with $x_{ij} \in K$. Then*

$$v(x') = \min\{ev(x_{ij}) + j : 1 \leq i \leq s \text{ and } 0 \leq j < e\},$$

and the elements η_{ij} are linearly independent over K.

2. *If the residue degree is finite then the elements η_{ij} form a basis of K' over K and*

$$[K' : K] = ef.$$

Proof. We refer the reader to [15]. □

The following corollary can be easily proved by using Theorem 2.5:

Corollary 2.6. *Let (K', v') be a complete extension of the complete valued field (K, v) with $[K' : K] = n$. Suppose $\{z_1, ..., z_n\}$ is basis of K' over K. Then the map*

$$(x_1, ..., x_n) \longmapsto \sum_i^n x_i z_i$$

defined from K^n into K' is a topological isomorphism.

The following lemma is well known as Hensels lemma and it plays an important role in this thesis. From now on, we assume the K is a complete valued field.

Lemma 2.7. *Let $g \in R_v[x]$ be a polynomial and let $a_0 \in R_v$ be such that $v(g(a_0)) > 2v(g'(a_0))$. Then there exists some $a \in R_v$ with $g(a) = 0$ and $v(a - a_0) > v(g'(a_0))$.*

Proof. We refer the reader to [12]. □

If $g(x) = c_0 + c_1x + ... + c_nx^n \in R_v[x]$ then by $\bar{g}(x)$ we mean $\bar{c}_0 + \bar{c}_1x + ... + \bar{c}_nx^n$ where $\bar{c}_i$ is the residue class corresponding to c_i. The next corollary is an easy consequence of Lemma 2.7.

Corollary 2.8. *Suppose $g \in R_v[x]$ such that $\bar{g}$ has a simple root $\bar{a}_0$ in the residue class $\bar{K}_v$. Then g has a zero $a \in R_v$ such that $\bar{a} = \bar{a}_0$.*

We write $P_n = \{y^n : y \in K^\times\}$ for the collection of n-th powers in $K^\times = K \setminus \{0\}$. By Corollary 2.8, each P_n has finite index when we consider it as a subgroup of the multiplicative group $K^\times$. The following lemma shows a relation between being an element of P_n and the way we can express that element as a convergent series (Theorem 2.2). By using the following lemma we can easily prove that P_n is an open subset of K for all n.

Lemma 2.9. *Let A be a complete set of representatives of the residue field of K. Let $x = \sum_{n\geq i} a_n\pi_0^n \in K$ where $a_n \in A$ for all n and $v(x) = i \in \mathbb{Z}$. For $m \in \mathbb{N}$, $x \in P_m$ if and only if $a_i = b^m$ for some $b \in K$ and $m|v(x) = i$.*

Proof. Suppose $x \in P_m$. Then there is $y \in K$ such that $x = y^m$ and hence $v(x) = mv(y)$. Thus

$$m|v(x).$$

Let $y = \sum_{n\geq j} b_n\pi_0^n$ where $b_n \in A$ for all n and $v(y) = j \in \mathbb{Z}$. Then $a_i\pi_0^i = b_j^m\pi_0^{mj}$ and hence $a_i = b_j^m$. Now suppose $a_i = b^m$ for some $b \in K$ and $m|v(x) = i$. We want to find $y = \sum_{n\geq j} b_n\pi_0^n \in K$ such that $x = y^m$. Let $j = \frac{v(x)}{m}$ and let $b_j = b$. We can then find all subsequent b_ns for $n > j$ recursively. □

Now we want to discuss the model-theoretic setting required for our purpose. Let L_a (a for "algebraic") be the language consisting of the binary operation symbols $+$ and $.$, the unary operation symbol $-$, the constant symbols 0 and 1, and the relation symbols P_n for all $n > 1$. We consider K as an L_a–structure using the natural interpretations of the symbols of L_a. In [19], Macintyre proved that K has elimination of quantifiers in L_a.

For $x = (x_1, ..., x_m)$, let $K\{x\}$ be the ring of restricted power series over K in the variables x; it is the ring of power series $\sum a_i x^i$ in $K[[x]]$ such that $|a_i|$ tends to 0 as $|i| \to \infty$. Equivalently, $\sum a_i x^i \in K\{x\}$ if $\sum a_i x^i$ is convergent on R_v^m. Here we use the multi-index notation where $i = (i_1, ..., i_m)$, $|i| = i_1 + ... + i_m$ and $x^i = x_1^{i_1} ... x_m^{i_m}$. For $x_0 \in R_v^m$ and $f = \sum a_i x^i \in K\{x\}$ the series $\sum a_i x_0^i$ converges to a limit in K, thus, one can associate to f a restricted analytic function given by

$$f : K^m \to K : x \mapsto \begin{cases} \sum a_i x_0^i, & \text{if } x \in R_v^m; \\ 0, & \text{else.} \end{cases}$$

We let L_{an} be the first order language consisting of the symbols of L_a together with an extra function symbol f for each restricted analytic function associated to restricted power series in $\bigcup_m K\{x_1, ..., x_m\}$ and $^{-1}$, the inverse operator on K with the convention $0^{-1} = 0$. We consider K as an L_{an}-structure using the natural interpretations of the symbols of L_{an}. In [10], Denef and van den Dries proved that Z_p admits quantifier elimination in this language and thus it follows that K admits quantifier elimination in this language too. To describe the definable subsets of K in this language we need to introduce the concept of D-function. Cluckers [6] gives the following definition of D-functions:

Definition 2.10. *A D-function is a function $K^m \to K$ for some $m \geq 0$, obtained by repeated application of the following rules:*

1. *for each $f \in K\{x_1, ..., x_m\}$, the associated restricted analytic function $x \mapsto f(x)$ is a D-function;*

2. *for each polynomial $f \in K[x_1, ..., x_m]$, the associated polynomial function $f(x)$ is a D-function;*

3. *the function $x \mapsto x^{-1}$, where $0^{-1} = 0$ by convention, is a D-function;*

4. *for each D-function f in n variables and each D-functions $g_1, ..., g_n$ in m variables, the function $f(g_1, ..., g_n)$ is a D-function.*

Now it is easy to see what the definable subsets of K look like:

Definition 2.11. *$X \subseteq K^m$ is a subanalytic set if X is definable in L_{an}. For $A \subseteq K^m$, we call $f : A \to K^n$ a subanalytic map if its graph is subanalytic.*

By using quantifier elimination and an inductive construction we can see that X is subanalytic if X is a boolean combination of the sets $\{x \in K^m \mid f(x) = 0\}$ or $\{x \in K^m \mid g(x) \in P_n\}$, where the functions g and f are D-functions and $n > 0$. We refer to [10] for more details on the inductive construction of subanalytic sets.

Chapter 3

Haar measure and p-adic integration

In this chapter, we state the theorem related to the existence of Haar measure for an arbitrary locally compact topological group. Next, we state and prove some properties of Haar measure which we need to prove the main theorem in this thesis. Moreover, we define the p-adic integration and prove some properties related to calculating p-adic integrals. We refer the reader to [21],[20],[16] and [22] for more details.

We first recall a sequence of fundamental definitions from analysis that culminate in the definition of a Haar measure. A collection $\mathbb{M}$ of subsets of a set X is called a σ-algebra if it satisfies the following conditions:

- $X \in \mathbb{M}$.
- If $A \in \mathbb{M}$, then $A^c \in \mathbb{M}$, where A^c denotes the complement of A in X.
- Suppose that $A_n \in \mathbb{M}(n \geq 1)$, and let $A = \cup A_n$. Then $A \in \mathbb{M}$.

It follows from these axioms that the empty set is in $\mathbb{M}$ and that $\mathbb{M}$ is closed under finite and countably infinite intersections. A set X together with a σ-algebra of subsets $\mathbb{M}$ is called a measurable space. If X is moreover a topological space, we may consider the smallest σ-algebra $\mathbb{B}$ containing all of the open sets of X. The elements of $\mathbb{B}$ are called the *Borel* subsets of X.

A positive measure μ on an arbitrary measurable space $(X, \mathbb{M})$ is a function $\mu : \mathbb{M} \to \mathbb{R}^{\geq 0} \cup \{\infty\}$

that is countably additive; that is,

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

for any family $\{A_n\}$ of disjoint sets in $\mathbb{M}$. In particular, a measure defined on the Borel sets of X is called a Borel measure.

Let μ be a Borel measure on a locally compact Hausdorff space X, and let E be a Borel subset of X. We say that μ is outer regular on E if

$$\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}.$$

We say that μ is inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}.$$

A Radon measure on X is a Borel measure that is finite on compact sets, outer regular on all Borel sets, and inner regular on all open sets. One can show that a Radon measure is, moreover, inner regular on σ-finite sets (that is, countable unions of μ-measurable sets of finite measure).

Let G be a locally compact Hausdorff topological abelian group and let μ be a Borel measure on G. We say that μ is translation invariant if for all Borel subsets E of G,

$$\mu(sE) = \mu(Es) = \mu(E)$$

for all $s \in G$.

Definition 3.1. *Let G be a locally compact Hausdorff topological abelian group. Then a Haar measure on G is a nonzero Radon measure μ on G that is translation-invariant.*

Theorem 3.2. *Let G be a locally compact Hausdorff topological abelian group. Then G admits a Haar measure. Moreover, this measure is unique up to a scalar multiple.*

Proof. For a thorough proof of existence and uniqueness, see [21]. $\square$

Proposition 3.3. *Let G be a locally compact topological abelian group with a nonzero Haar measure μ. Then:*

1. *μ is positive on all nonempty open subsets of G.*

2. *$\mu(G)$ is finite if and only if G is compact.*

Proof. 1. Since μ is not identically zero, by inner regularity there is a compact set K such that $\mu(K)$ is positive. Let U be any nonempty open subset of G. Then from the inclusion

$$K \subseteq \bigcup_{s \in G} sU$$

we deduce that K is covered by a finite set of translates of U, all of which must have equal measure. Since $\mu(K) > 0$, $\mu(U) > 0$.

2. If G is compact, then certainly $\mu(G)$ is finite by definition of a Haar measure. To establish the converse, assume that G is not compact. Let K be a compact set whose interior contains the identity element, e (there is such a K since G is locally compact). Then no finite set of translates of K covers G (which would otherwise be compact), and there must exist an infinite sequence $\{s_j\}$ in G such that

$$s_n \notin \bigcup_{j<n} s_j K.$$

Now suppose $W \subseteq K$ is an open neighborhood of e. Since $. : G \times G \to G$ (. is the group operation) is continuous, there is an open neighborhood of e, $U \subseteq W$, such that $U = U^{-1}$ and $UU \subseteq W$.

We claim that the translates $s_j U (j \geq 1)$ are disjoint, from which it follows at once from (1) that $\mu(G)$ is infinite. To prove the claim, suppose for $i < j$ we have $s_i u = s_j v$ where $u, v \in U$. Then $s_j = s_i u v^{-1}$, since U is symmetric and $UU \subseteq K$. But this contradicts the fact that $s_j \notin \bigcup_{i<j} s_i K$.

□

The following theorem states the property of Haar measure which is called continuity.

Theorem 3.4. *Let G be a locally compact topological abelian group with a Haar measure μ. Suppose $(A_1, A_2, ...)$ is a sequence of Borel subsets of G. Then*

- *If the sequence is increasing then*

$$\mu(\cup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n).$$

- *If the sequence is decreasing and $\mu(A_1) < \infty$ then*

$$\mu(\cap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n).$$

Now we want to define Haar integration. Let G be a locally compact topological abelian group with a Haar measure μ. Let $S \subseteq G$ be a Borel set and let χ_S be its characteristic function. We define

$$\int_G \chi_S d\mu := \mu(S).$$

Now suppose $s = \sum_{k=1}^{k=n} a_k \chi_{S_k}$ where S_k is a Borel subset of G and a_ks are real numbers for all k. We call s a simple function. Suppose s is a non-negative simple function. We define

$$\int_G s d\mu := \sum_{k=1}^{k=n} a_k \mu(S_k).$$

Now suppose $f : G \to [0, +\infty]$ is a measurable function. Let $T_f = \{s : 0 \leq s \leq f \text{ and } s \text{ is simple}\}$ The integral of f over G is defined as:

$$\int_G f d\mu := \sup_{s\in T_f} \int_G s d\mu.$$

If f is any measurable real-valued function on G we define:

$$\int_G f d\mu = \int_G f^+ d\mu - \int_G f^- d\mu$$

where f^+ and f^- are measurable and represent the positive and negative part of f, respectively:

$$f^+(x) = \max(+f(x), 0) \qquad f^-(x) = \max(-f(x), 0).$$

If E is a Borel subset of G then we define:

$$\int_E f d\mu := \int_G \chi_E f d\mu$$

A nonnegative measurable function f is called integrable if its integral $\int_G f d\mu$ is finite. An arbitrary measurable function is integrable if f^+ and f^- are each integrable.

Now suppose $h : G \to \mathbb{C}$ is a complex-valued function. We define:

$$\int_G h d\mu = \int_G Re(h) d\mu + i \int_G Im(h) d\mu$$

where $Re(h)$ and $Im(h)$ are the real and imaginary part of h respectively.

The next theorem is the analogue to the Lebesgue integrability of continuous functions with compact support in the real case:

Theorem 3.5. *Any continuous function $f : G \to \mathbb{C}$ with compact support is μ-integrable.*

Let G be a locally compact topological abelian group with a Haar measure μ. The next proposition combines the Haar integration with being translate-invariant of Haar measure:

Proposition 3.6. *Let G be a locally compact topological abelian group with a Haar measure μ. Let $f : G \to \mathbb{C}$ be a μ-integrable function. Then for every $g \in G$*

$$\int_G f(x) d\mu = \int_G f(gx) d\mu.$$

Proof. The statement is clear for characteristic and simple functions by using translate-invariance of μ. For a nonnegative measurable function the statement is obvious by using the integral definition and the fact that the claim is true for simple functions. Now the statement is clear for any μ-integrable function by using the previous fact for f^+ and f^-. $\square$

Next, we state the countable additivity of Haar integral:

Theorem 3.7. *Let G be a locally compact topological abelian group with a Haar measure μ. Let $E \subseteq G$ be a Borel set and $f : G \to \mathbb{C}$ be a μ-integrable function on E. Suppose $\{E_n\}$ is a disjoint countable family of Borel sets such that $E = \cup_i E_i$. Then*

$$\int_E f d\mu = \sum_{n=1}^{\infty} \int_{E_n} f d\mu$$

Now let K be any finite extension of $\mathbb{Q}_p$ with the valuation ring R_v and prime element π_0. By Corollary 2.4, $(K, +)$ is a locally compact topological group. Let μ be the induced Haar measure on $(K, +)$. Since R_v is compact and μ is unique up to a scalar multiple, we can assume $\mu(R_v) = 1$.

Suppose $q \in \mathbb{N}$ is the cardinality of residue field of K. Then

Lemma 3.8. *For $m \in \mathbb{N}$, $\mu(\pi_0^m R_v) = 1/q^m$.*

Proof. Since $|R_v/\pi_0^m R_v| = q^m$, with a set of representatives $a_0, a_1, ..., a_{q^m-1}$ in which $a_0 = 0$, we have a disjoint union decomposition

$$R_v = \pi_0^m R_v \cup (\pi_0^m R_v + a_1) \cup ... \cup (\pi_0^m R_v + a_{q^m-1}).$$

By translation invariance, all of the sets on the right have the same measure, and since $\mu(R_v) = 1$, this immediately gives the result. □

The Haar integral induced by μ on K is called the p-adic integral. Calculating p-adic integrals is difficult and complicated in general. Sometimes we only need to calculate the p-adic integral of an integrable function with a countable image. In this case, we can use the additivity of Haar integral mentioned in Theorem 3.7. For example:

Example 3.9. *Let $s \geq 0$ be a real number, and $d \geq 0$ an integer. Then*

$$\int_{R_v} |x^d|^s d\mu = \frac{q-1}{q - q^{-ds}}.$$

Proof. We take advantage of the fact that in this context the function we are integrating is the analogue of a step function, as in the comment above. We clearly have:

- $|x^d|^s = 1$ for $x \in R_v \setminus \pi_0 R_v$.

- $|x^d|^s = 1/q^{ds}$ for $x \in \pi_0 R_v \setminus \pi_0^2 R_v$.
- $|x^d|^s = 1/q^{2ds}$ for $x \in \pi_0^2 R_v \setminus \pi_0^3 R_v$

and so on. Since these sets partition R_v we get

$$\int_{R_v} |x^d|^s d\mu = 1.\mu(R_v \setminus \pi_0 R_v) + 1/q^{ds}.\mu(\pi_0 R_v \setminus \pi_0^2 R_v) + 1/q^{2ds}.\mu(\pi_0^2 R_v \setminus \pi_0^3 R_v) + ...$$

Using Lemma 3.8, this sum is equal to:

$$1.(1 - 1/q) + 1/q^{ds}.(1/q - 1/q^2) + 1/q^{2ds}.(1/q^2 - 1/q^3) + ... =$$

$$(1 - 1/q)(\frac{1}{1 - q^{-ds-1}}) = \frac{q-1}{q - q^{-ds}}.$$

□

For any $r \geq 1$, we can also consider the Haar measure on K^r with the product topology, normalized such that $\mu(R_v^r) = 1$. This is the same as the product measure. We can easily generalize Lemma 3.8 and Example 3.9 for K^r:

- For any non-negative integers $k_1, ..., k_r$, one has

$$\mu(\pi_0^{k_1} R_v \times ... \times \pi_0^{k_r} R_v) = \frac{1}{q^{k_1+...+k_r}}.$$

- For any non-negative integers $k_1, ..., k_r$,

$$\int_{R_v^r} |x_1^{k_1} \cdot ... \cdot x_r^{k_r}|^s d\mu = \prod_{i=1}^{r} \frac{q-1}{q - q^{-k_i s}}.$$

Remark 3.10. *Suppose $f : K^r \to K$ is an integrable function. Let $(x_1, ..., x_r) \in K^r$ and for $s \leq r$ let $x_s = (x_1, ..., x_s)$. Then by*

$$\int_{K^s} f|dx_s|$$

we mean the p-adic integral with regard to the Haar measure on K^s.

The following is the p-adic analogue of the change of variable theorem.

Theorem 3.11. *Let U be an open subset of K^n and consider analytic functions $f_1, ..., f_n$ on U. Assume $f = (f_1, ..., f_n) : U \to K^n$ is an analytic (f_i is locally given by a convergent power series for all i) isomorphism between U and an open subset V of K^n. Then, for every integrable function ϕ on V ,*

$$\int_V \phi d\mu = \int_U (\phi \circ f)|\partial(f_1, ..., f_n)/\partial(x_1, ..., x_n)|d\mu$$

where $|\partial(f_1, ..., f_n)/\partial(x_1, ..., x_n)|$ is the determinant of the jacobian matrix of f.

Finally, we notice that since $K = \cup_{n\in\mathbb{N}}\{x : v(x) \geq n\}$ and $\{x : v(x) \geq n\}$ is compact for all n, μ is σ-finite. Thus we have Fubini's theorem :

Theorem 3.12. *Suppose $f : K^n \to K^r$ is an integrable function and let l be an arbitrary positive integer less than n. Let $x_l = (x_1, ..., x_l)$ and $x'_{n-l} = (x_{l+1}, ..., x_n)$. Then*

$$\int_{K^n} f d\mu = \int_{K^l}\int_{K^{n-l}} f|dx'_{n-l}||dx_l| = \int_{K^{n-l}}\int_{K^l} f|dx_l||dx'_{n-l}|$$

In particular, if $E \subseteq K^2$ and $\mu(E) < \infty$ then

$$\mu(E) = \int_K \mu(E_y)d\mu = \int_K \mu(E_x)d\mu$$

in which $E_x = \{y : (x, y) \in E\}$ and $E_y = \{x : (x, y) \in E\}$.

If $f : K^n \to K^r$ is a continuous function then the graph of f is a closed subset of $K^n \times K^r$ and thus it is measurable. By Fubini's theorem, we can easily prove that the measure of the graph of f is zero.

Chapter 4

Cell decomposition and constructible functions

In this chapter, we state the definition of analytic cells and we give the cell decomposition theorem. Moreover, we discuss a modified version of the cell decomposition theorem for constructible functions. We refer the reader to [5] and [6] for more details.

Let K be a finite extension of $\mathbb{Q}_p$ with valuation ring R_v and prime element π_0. Suppose the cardinality of residue field is q. We denote the p-adic norm by $|.|_p$ as in chapter 2. For $n \in \mathbb{N}$ let $P_n = \{y^n : y \in K\}$. Let

$$L_{an} = \{+, ., -, ^{-1}, \{P_n\}_{n\in\mathbb{N}}, \{\text{a function symbol for each restricted analytic function}\}\}.$$

We consider K as an L_{an}-structure and we call the definable subsets of K subanalytic sets as we discussed in chapter 2.

Definition 4.1. *An analytic cell $A \subseteq K$ is a (nonempty) set of the form*

$$\{t \in K : \ |\alpha|_p \square_1 |t-c|_p \square_2 |\beta|_p, \ t-c \in \lambda P_n\},$$

with constants $n > 0$, $\lambda, c \in K$, $\alpha, \beta \in K^\times$ and $\square_i$ either $<$ or no condition. An analytic cell

$A \subseteq K^{m+1}$, $m \geq 0$, is a set of the form

$$\{(x,t) \in K^{m+1} : \ x \in D, \ |\alpha(x)|_p \square_1 |t - c(x)|_p \square_2 |\beta(x)|_p, \ t - c(x) \in \lambda P_n\},$$

with $(x,t) = (x_1, ..., x_m, t)$, $n > 0$, $\lambda \in K$, $D = \pi_m(A)$ a cell where π_m is the projection $K^{m+1} \to K^m$, subanalytic functions $\alpha, \beta : K^m \to K^\times$ and $c : K^m \to K$ and $\square_i$ either $<$ or no condition such that the functions α, β and c are analytic on D. We call c the center of the cell A and λP_n the coset of A.

Note that a cell is either the graph of an analytic function defined on D (namely if $\lambda = 0$) and thus of measure zero, or for each $x \in D$, the fiber $A_x = \{t : (x,t) \in A\}$ is a nonempty open (if $\lambda \neq 0$).

Theorem 4.2 below is a subanalytic analogue of the semialgebraic cell decomposition (see [9] and [4]):

Theorem 4.2. *[6, Theorem 2.8] Let $X \subseteq K^{m+1}$ be a subanalytic set and $f_j : X \to K$ subanalytic functions for $j = 1, 2, ..., r$. Then there exists a finite partition of X into cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that*

$$|f_j(x,t)|_p = |\delta_{ij}(x)|_p \cdot |(t - c_i(x))^{a_{ij}} \lambda_i^{-a_{ij}}|_p^{1/n_i},$$

for each $(x,t) \in A_i$, with $(x,t) = (x_1, ..., x_m, t)$, integers a_{ij}, and $\delta_{ij} : K^m \to K$ subanalytic functions, analytic on $\pi_m(A_i)$ for $j = 1, 2, ..., r$. If $\lambda_i = 0$ we use the convention that $a_{ij} = 0$.

Proof. We refer the reader to R. Cluckers, [6], Theorem 2.8. □

Remark 4.3. *Theorem 4.2 can be seen as a p-adic analogue of the preparation theorem [18] for real subanalytic functions, or as an analogue of cell decomposition for real subanalytic sets (see e.g. [11]).*

Certain algebras of functions from K^m to the rational numbers $\mathbb{Q}$ are closed under p-adic integration. These functions are called subanalytic constructible functions and they come up naturally when one calculates parametrized p-adic integrals.

Definition 4.4. *For each subanalytic set $X \subseteq K^m$, we let $C(X)$ be the $\mathbb{Q}$-algebra generated by the functions $|h|_p$ and $v(h)$ for all subanalytic functions $h : X \to K^\times$. We call $f \in C(X)$ a subanalytic constructible function on X.*

For $x = (x_1, ..., x_m)$ an m-tuple of variables, we will write $|dx|$ to denote the Haar measure on K^m, so normalized that R_v^m has measure 1. To any function f in $C(K^{m+n})$, $m, n \geq 0$, we associate a function $I_m(f) : K^m \to \mathbb{Q}$ by putting

$$I_m(f)(x) = \int_{K^n} f(x, y)|dy|$$

if the function $y \mapsto f(x, y)$ is absolutely integrable for all $x \in K^m$, and by putting $I_m(f)(x) = 0$ otherwise. The next theorem indicates that the set of constructible functions is closed under p-adic integration.

Theorem 4.5. *([6, Theorem 4.2]) For any function f in $C(K^{m+n})$, the function $I_m(f)$ is in $C(K^m)$.*

Proof. We refer the reader to R. Cluckers, [6], Theorem 4.2. □

In [5], Cluckers states the cell decomposition theorem for constructible functions as follows.

Lemma 4.6. *Let $X \subseteq K^{m+1}$ be a subanalytic set and let g_j be functions in $C(X)$ in the variables $(x_1, ..., x_m, t)$ for $j = 1, ..., r$. Then there exists a finite partition of X into cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that each restriction $g_j|_{A_i}$ is a finite sum of functions of the form*

$$|(t - c_i(x))^a \lambda^{-a}|_p^{1/n_i} v(t - c_i(x))^s h(x),$$

where $h : K^m \to \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers.

Proof. Without loss of generality we can assume $j = 1$. First, suppose $g = |f|_p$ for some subanalytic functions $f : X \to K^\times$. By cell-decomposition theorem, there exists a finite partition of X into cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that

$$|f(x, t)|_p = |\delta_i(x)|_p \cdot |(t - c_i(x))^{a_i} \lambda_i^{-a_i}|_p^{1/n_i},$$

for $(x,t) \in A_i$, with $(x,t) = (x_1, ..., x_m, t)$, integers a_i, and $\delta_i : K^m \to K$ subanalytic functions, analytic on $\pi_m(A_i)$. Let $h_i(x) = |\delta_i(x)|_p$.Then we have the desired property for g.

Now let $g = v(f)$ for some subanalytic functions $f : X \to K^\times$. By using $|x|_p = p^{-v(x)}$ we have

$$v(f) = v(\delta_i) + \frac{a_i}{n_i} v(t - c_i(x)) + v(\lambda_i^{a_i})^{1/n_i}$$

Now it is clear that g has the desired property.

Finally suppose

$$g = a_1 g_{11} g_{12} ... g_{1n_1} + ... + a_m g_{m1} g_{m2} ... g_{mn_m},$$

where $a_i \in \mathbb{Q}$ for $i \in [m]$ and $g_{ij} : X \to K^\times$ are constructible subanalytic functions. Then by using the results of the previous two paragraphs we can easily see that g satisfies the claim of the theorem. $\square$

In [5], Cluckers states the following corollary which he then uses to prove a modified version of cell decomposition theorem. Since there is not proof in [5], we prove it here.

Corollary 4.7. *For any function $g \in C(K^{m+1})$ there exists a closed subanalytic set $A \subseteq K^{m+1}$ of measure zero such that g is locally constant on $K^{m+1} \setminus A$.*

Proof. By Lemma 4.6, there exists a finite partition of K^{m+1} into cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that for each i, $g|_{A_i}$ is a finite sum of functions of the form

$$H(x,t) = |(t - c_i(x))^a \lambda_i^{-a}|_p^{1/n_i} v(t - c_i(x))^s h(x),$$

where $h : K^m \to \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers. It is enough to prove the theorem for each $H(x,t)$. Then the general case follows immediately. We proceed by induction on m.

First, suppose $m = 0$. Then each A_i is of the form

$$\{t \in K : \ |\alpha_i|_p \square_1 |t - c_i|_p \square_2 |\beta_i|_p, \ t - c_i \in \lambda_i P_{n_i}\},$$

with constants $n_i > 0$,$\lambda_i, c_i \in K$,$\alpha_i, \beta_i \in K^\times$, and $\square_i$ either $<$ or no condition. Moreover,

$$H(t) = r|(t - c_i)^a \lambda^{-a}|_p^{1/n_i} v(t - c_i)^s$$

where $r \in \mathbb{Q}$, $s \geq 0$ and a are integers. If $\lambda_i = 0$ then the measure of A_i is zero. Suppose $\lambda_i \neq 0$. Let $t_0 \in A_i$. Let $A_{t_0} = \{t : |t - t_0|_p < |t_0 - c_i|_p\}$. Then A_{t_0} is an open set. Since A_i is an open set too, $A_{t_0} \cap A_i$ is an open set. Moreover, $t_0 \in A_{t_0} \cap A_i$. Now let $t_1 \in A_{t_0} \cap A_i$. Then

$$|t_1 - c_i|_p = |t_1 - t_0 + t_0 - c_i|_p = |t_0 - c_i|_p$$

and thus $H(t)$ is constant on $A_{t_0} \cap A_i$. Hence H is locally constant on A_i.

Now suppose the claim is true for $m = k$. Suppose A_i is of the form

$$\{(x, t) \in K^{m+1} : \ x \in D, \ |\alpha(x)|_p \square_1 |t - c(x)|_p \square_2 |\beta(x)|_p, \ t - c(x) \in \lambda P_n\},$$

with $(x, t) = (x_1, ..., x_m, t)$,$n > 0$,$\lambda \in K$, $D = \pi_m(A_i)$ a cell, subanalytic functions $\alpha, \beta : K^m \to K^\times$ and $c : K^m \to K$ and $\square_i$ either $<$ or no condition. Moreover, suppose

$$H(x, t) = |(t - c_i(x))^a \lambda^{-a}|_p^{1/n_i} v(t - c_i(x))^s h(x)$$

where $h : K^m \to \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers. If $\lambda = 0$ in the definition of any $\pi_k(A_i)$ for $k = 1, ..., m$ then the measure of A_i is zero. So suppose $\lambda \neq 0$ for all $\pi_1(A_i), \ \pi_2(A_i), ..., \ \pi_m(A_i)$. Let $(x_0, t_0) \in A_i$.

Since both $h(x)$ and $|t_0 - c(x)|_p$ are constructible functions on K^m, by induction hypothesis there is a neighborhood of x_0 in $\pi_m(A_i)$, U_{x_0}, on which both $h(x)$ and $|t_0 - c(x)|_p$ are constant (for simplicity we assume that x_0 is not in the closed set of measure zero for both functions). Suppose B is the projection of A_i on the last coordinate. Let

$$A_{t_0} = \{t \in B : |t - t_0|_p < |t_0 - c(x_0)|\}.$$

Let $(x,t) \in U_{x_0} \times A_{t_0}$. Then we have

$$|t - c(x)|_p = |t - c(x_0)|_p = |t - t_0 + t_0 - c(x_0)|_p = |t_0 - c(x_0)|_p.$$

Thus $H(x,t)$ is a constant function on $U_{x_0} \times A_{t_0}$. □

In [5], Cluckers states the following corollary which plays an important role in proving the main theorem. Recall that the Vinogradov symbol $\ll$ here means that for complex valued functions f and g with g taking non-negative real values $f \ll g$ means $|f| \leq cg$ for some constant c.

Corollary 4.8. *Let g be in $C(K)$. Suppose that as $|t|_p$ tends to ∞ then $g(t)$ converges to zero. Then there exists a real number $\alpha < 0$ such that $g(t) \ll \min\{1, |t|^\alpha\}$.*

Proof. We refer the reader to R. Cluckers, [5], Corollary 2.6. □

Now we are ready to state the following modified version of cell decomposition for constructible functions that that Cluckers proves as part of the proof for Theorem 4.1. The modification gives more information on integrability of $H(x,t)$.

Theorem 4.9. *Let $X \subseteq K^{m+1}$ be a subanalytic set and let g be a function in $C(X)$ in the variables $(x_1, ..., x_m, t)$ such that g is integrable for almost all $(x,t) \in K^{m+1}$. Then there exists a finite partition of X into cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that each restriction $g|_{A_i}$ is a finite sum of functions of the form*

$$H_{ij}(x,t) = |(t - c_i(x))^a \lambda_i^{-a}|_p^{1/n_i} v(t - c_i(x))^s h_{ij}(x),$$

where $h_{ij} : K^m \to \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers and $j = 1, ..., k_i$ for some $k \in \mathbb{N}$.

Moreover, after refining the partition, we can assure that for each A_i either the projection $\pi_m(A_i) \subseteq K^m$ has zero measure, or we can write $g|_{A_i}$ as a sum of terms H_{ij} of the above form such that H_{ij} is integrable over A_i and does not change its sign on A_i.

Proof. We refer the reader to R. Cluckers [5], Lemma 2.5 and the proof of Theorem 4.1. □

The following theorem is result of applying Theorem 4.9 recursively.

Theorem 4.10. *Let $X \subseteq K^m$ be a subanalytic set and $f \in C(X)$ be a function in the variables $x = (x_1, ..., x_m)$. Then there exists a finite partition of X into cells A_i with centers c_{i_j} and cosets $\lambda_{i_j} P_{n_j}$ for $j \in \{1, 2, ..., m\}$ such that each restriction $f|_{A_i}$ is a finite sum of functions of the form*

$$H(x) = r.(\prod_{j=1}^{m} |(x_j - c_{i_j}(x_1, ..., x_{j-1}))^{a_j} \lambda_{i_j}^{a_j}|^{\frac{1}{n_{i_j}}})(\prod_{j=1}^{m} v(x_j - c_{i_j}(x_1, ..., x_{j-1}))^{s_j})$$

where $r = |t|$ or $r = v(t)$ for some $t \in K$ and $s_j \geq 0$ and a_j are integers. Moreover, after refining the partition, we can assure that for each A_i either the projection $A_i' := \pi_{m-1}(A_i) \subseteq K^r$ has zero measure, or we can write $f|_{A_i}$ as a sum of terms H of the above form such that H is integrable over A_i and does not change its sign on A_i.

Proof. We prove the theorem by induction on m. For $m = 1$, the theorem is an immediate consequence of Theorem 4.9. Now assume that the theorem is true for all positive integers less than m. To prove the theorem for m, first we use Theorem 4.9 to decompose X into finitely many cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that each restriction $f|_{A_i}$ is a finite sum of functions of the form

$$H(x_1, x_2, ..., x_m) = |(x_m - c_i(x_1, x_2, ..., x_{m-1}))^a \lambda_i^{-a}|_p^{1/n_i} v(x_m - c_i(x_1, x_2, ..., x_{m-1}))^s h(x_1, x_2, ..., x_{m-1}),$$

where $h : K^{m-1} \to \mathbb{Q}$ is a subanalytic constructible function, and $s \geq 0$ and a are integers. Moreover, after refining the partition, we can assure that for each A_i either the projection $\pi_{m-1}(A_i) \subseteq K^{m-1}$ has zero measure, or we can write $f|_{A_i}$ as a sum of terms H of the above form such that H is integrable over A_i and does not change its sign on A_i.

By applying induction assumption on $h(x_1, ..., x_{m-1})$ and $\pi_{m-1}(A_i)$, we obtain the desired form and the proof is complete. □

To conclude this section, we state an important theorem from [10]. First, we need two new definitions:

Definition 4.11. *For any open set $U \subseteq K^r$, a K-analytic function $f : U \to K$ is a function which is locally around any point in U given by a convergent power series. We call $f = (f_1, ..., f_m) : U \to K^m$ a K-analytic map if all f_i are K-analytic functions.*

Definition 4.12. *Let X be a Haussdorff topological space, and $n \geq 0$ an integer. A chart of X is a pair (U, ϕ_U) consisting of an open subset of X together with a homeomorphism $\phi_U : U \to V$ onto an open set $V \subseteq K^n$. An analytic atlas is a family of charts $\{(U, \phi_U)\}$ such that for every U_1, U_2 with $U_1 \cap U_2 = \emptyset$ the composition*

$$\phi_{U_2} \circ \phi_{U_1}^{-1} : \phi_{U_1}(U_1 \cap U_2) \to \phi_{U_2}(U_1 \cap U_2)$$

is bi-analytic. Two atlases are equivalent if their union is also an atlas. Finally, X together with an equivalence class of atlases as above is called a K-analytic manifold of dimension n.

The next theorem explains the connection between subanalytic functions and analytic functions.

Theorem 4.13. *Let $X \subseteq K^n$ be a subanalytic set and $f : X \to K$ a subanalytic function. Then there exists a finite partition of X into p-adic submanifolds A_j of K^n such that the restriction of f to each A_j is analytic and such that each A_j is subanalytic.*

Proof. We refer the reader to J. Denef and L. van den Dries, [10], Proposition 3.29. □

Chapter 5

Additive characters

In this chapter, we discuss the main characteristic of additive characters on p-adic fields. The additive characters on a p-adic field have a nice representation form that we aim to exploit to prove the main theorem in this thesis. We refer the reader to [2],[13] and [24]. for more details. The results of this chapter come from [24]. However, since the proofs in those references are incomplete, we present the full detailed proofs.

Throughout this chapter, we fix K as a finite extension of $\mathbb{Q}_p$ as we did in the previous chapters. R_v is the valuation ring of K, q is the cardinality of the residue field and π_0 is the prime element of K. We also fix A a complete set of representatives of the residue fields of K. As we observed in the Chapter 2, each nonzero $x \in K$ can be uniquely expressed in the form

$$x = \sum_{n=i}^{\infty} a_n \pi_0^n$$

where $a_n \in A$ for all n and $v(x) = i$. In the case where $x = 0$, we can take all the coefficients to be zero. We need the notion of fractional part of the elements in K:

Definition 5.1. *Using the notation from the previous paragraph, let $x = \sum_{n=i}^{\infty} a_n \pi_0^n \in K$. We define $\{x\}_p$, the fractional part of x, as:*

$$\{x\}_p = \begin{cases} 0, & \text{if } v(x) \geq 0 \text{ or } x = 0, \\ \sum_{n=i}^{-1} a_n \pi_0^n, & \text{if } v(x) < 0. \end{cases}$$

Fractional parts are not closed under addition. Hence, if $x, y \in \mathbb{Q}_p$ then $\{x+y\}_p$ is not necessarily equal to $\{x\}_p + \{y\}_p$. For example if $x = (\frac{p-1}{p} + ...)$ and $y = (\frac{p-1}{p} + ...)$ then

$$\{x\}_p = \frac{p-1}{p} \qquad \{y\}_p = \frac{p-1}{p}.$$

Since $x + y = (2\frac{p-1}{p} + ...)$ and $\frac{2p-2}{p} = 2 + \frac{p-2}{p}$

$$\{x+y\}_p = \frac{p-2}{p}.$$

However

$$\{x\}_p + \{y\}_p = \frac{p-2}{p} + 2.$$

The good point is that $\{x\}_p + \{y\}_p - \{x+y\}_p$ is always an integer as we prove here.

Lemma 5.2. *Suppose $a, b \in \mathbb{Q}_p$. Then*

$$(\{a\}_p + \{b\}_p) - \{a+b\}_p \in \mathbb{Z}.$$

Proof. Let $a, b \in \mathbb{Q}_p$ with $v(a) = i < 0$ and $v(b) = j < 0$. Suppose $a = \sum_{i \leq n} a_n p^n$ and $b = \sum_{j \leq m} b_m p^m$ where a_n and b_m are in $\{0, 1, ..., p-1\}$ for all n and m. Then $\{a\}_p = \sum_{i \leq n}^{-1} a_n p^n$ and $\{b\}_p = \sum_{j \leq m}^{-1} b_m p^m$. Suppose $i < j$. Then

$$\{a\}_p + \{b\}_p = \sum_{i \leq n}^{j-1} a_n p^n + \sum_{j \leq k}^{-1} (a_k + b_k) p^k$$

Now for $j \leq k \leq -1$ let $c_k \in \{0, 1, ..., p-1\}$ be such that

$$\{a+b\}_p = \sum_{i \leq n}^{j-1} a_n p^n + \sum_{j \leq k}^{-1} c_k p^k$$

Thus

$$(\{a\}_p + \{b\}_p) - \{a+b\}_p = \sum_{j \leq k}^{-1} (a_k + b_k - c_k) p^k$$

On the other hand we know that $a - \{a\}_p$, $b - \{b\}_p$ and $a + b - \{a + b\}_p$ are in $\mathbb{Z}_p$. Thus

$$(\{a\}_p + \{b\}_p) - \{a + b\}_p = \sum_{j \leq k}^{-1} (a_k + b_k - c_k) p^k$$

is in $\mathbb{Z}_p$. We can write

$$\sum_{j \leq k}^{-1} (a_k + b_k - c_k) p^k = \{\sum_{j \leq k}^{-1} (a_k + b_k - c_k) p^k\}_p + d$$

for some $d \in \mathbb{Z}$. Since $\sum_{j \leq k}^{-1} (a_k + b_k - c_k) p^k$ is in $\mathbb{Z}_p$

$$\{\sum_{j \leq k}^{-1} (a_k + b_k - c_k) p^k\}_p = 0$$

and thus the proof is complete. □

Next, we define the additive characters:

Definition 5.3. *An additive character $\psi : K \to \mathbb{C}$ of the field K is a complex-valued continuous function ψ defined on K such that:*

- $|\psi(x)| = 1$ *for all* $x \in K$.
- $\psi(x + y) = \psi(x)\psi(y)$ *for all* $x, y \in K$.

If ψ is an additive character then it is easy to prove:

$$\psi(0) = 1, \quad \psi(-x) = \overline{\psi(x)} = (\psi(x))^{-1}, \quad \psi(nx) = \psi(x)^n$$

for all $x \in K$ and $n \in \mathbb{Z}$.

Next, we want to prove a lemma ([24], page 30) required for giving a nice form of additive characters.

Lemma 5.4. *Suppose $\psi : K \to \mathbb{C}$ is an additive character. Then there exists $m \in \mathbb{Z}$ such that $\psi(x) = 1$ for all $|x|_p \leq |\pi_0^m|_p$.*

Proof. By virtue of the conditions $\psi(0) = 1$, $|\psi(x)| = 1$ and ψ is a continuous function it is possible to choose a branch of the function $\ln(\psi(x)) = i\arg(\psi(x))$ such that it will be continuous at zero and $\arg(\psi(0)) = 0$. Thus there exists $m \in \mathbb{Z}$ such that $|\arg(\psi(x))| < 1$ if $|x|_p \leq |\pi_0^m|_p$.

Let $n \in \mathbb{Z}^{\geq 0}$. Taking into account that $|nx|_p \leq |\pi_0^m|_p$ if $|x|_p \leq |\pi_0^m|_p$ we conclude that

$$|\mathbf{arg}(\psi(x))| = \frac{1}{n}|\mathbf{arg}(\psi(nx))| < \frac{1}{n}$$

and thus $\mathbf{arg}(\psi(x)) = 0$ and $\psi(x) = 1$. $\square$

To give a nice description for additive characters, first we want to restrict our discussion to the additive characters on $\mathbb{Q}_p$. Suppose $\psi : \mathbb{Q}_p \to \mathbb{C}$ is such that

$$\psi(x) = e^{2\pi i\{\xi x\}_p}$$

for some $\xi \in \mathbb{Q}_p$. It follows from Lemma 5.2, ψ is an additive character. We want to prove that every additive character is of this form.

Fix a non-trivial additive character $\psi : \mathbb{Q}_p \to \mathbb{C}$. For $r \in \mathbb{Z}$ let $B_r = \{x \in \mathbb{Q}_p : |x|_p \leq p^r\}$. By Lemma 5.4, there exists $m \in \mathbb{Z}$ such that $\psi(x) = 1$ for all $x \in B_m$. We assume that the disc B_m is maximal so that as $\psi(x)$ is not trivial on B_n then $m < n$. Suppose $r \in \mathbb{Z}$ and $r > m$. We want to prove that the restriction of ψ on B_r, $\psi|_{B_r}$, is of the form

$$\psi(x) = e^{2\pi i\{\xi x\}_p}$$

for some $\xi \in \mathbb{Q}_p$ where $|\xi|_p \geq p^{-r+1}$. For this purpose, first we prove the following lemma which is proved partially in [24], page 31.

Lemma 5.5. *With the notation in the previous paragraph, if $s \in \mathbb{Z}$ and $m < s \leq r$ then there exists $n \in \{1, 2, ..., p^{r-m} - 1\}$ such that*

$$\psi(p^{-s}) = e^{2\pi i n p^{-s+m}}$$

where n does not depend on s. In particular, there exists $\xi \in \mathbb{Q}_p$ such that $p^{-r} < |\xi|_p \leq p^{-m}$ and

$\psi(p^{-s}) = e^{2\pi i \{\xi p^{-s}\}_p}$.

Proof. First, suppose $s = r$. Then

$$1 = \psi(p^{-m}) = \psi(p^{-r+r-m}) = (\psi(p^{-s}))^{p^{r-m}}.$$

The first equality is due to the fact that $\psi(x) = 1$ if $x \in B_m$. Thus $\psi(p^{-s})$ is one of the p^{r-m}th roots of unity. In other words, there exists $n \in \{0, 1, 2, ..., p^{r-m} - 1\}$ such that

$$\psi(p^{-s}) = e^{2\pi i n p^{-s+m}}$$

Claim: $\psi(p^{-s}) \neq 1$. In other words, $n \neq 0$ in the above equality. To prove the claim suppose $\psi(p^{-s}) = 1$. First, suppose $s = m + 1$. Let $x \in \mathbb{Q}_p$ such that $v(x) = -s = -m - 1$. Then

$$x = a_0 p^{-s} + \sum_{i=1}^{\infty} a_i p^{-s+i}$$

where a_0 and a_is are in $\{0, 1, 2, ...p - 1\}$ and $a_0 \neq 0$. Since $v(\sum_{i=1}^{\infty} a_i p^{-s+i}) \geq -m$ we have

$$\psi(\sum_{i=1}^{\infty} a_i p^{-s+i}) = 1$$

and hence

$$\psi(x) = \psi(a_0 p^{-s})\psi(\sum_{i=1}^{\infty} a_i p^{-s+i}) = \psi(a_0 p^{-s}) = \psi(p^{-s})^{a_0} = 1$$

and this contradicts the maximality of B_m.

Now let $s > m + 1$. Then

$$\psi(p^{-m-1}) = \psi(p^{-s} \cdot p^{s-m-1}) = \psi(p^{-s})^{p^{s-m-1}} = 1$$

and now we can use the previous case. This ends the proof of the claim.

Now we fix $n \in \{1, 2, ..., p^{r-m} - 1\}$ such that $\psi(p^{-r}) = e^{2\pi i n p^{-r+m}}$. Suppose $m < s < r$. Then

$$\psi(p^{-s}) = \psi(p^{-s+r-r}) = \psi(p^{-r})^{p^{-s+r}} = (e^{2\pi i n p^{-r+m}})^{p^{-s+r}}$$

and thus

$$\psi(p^{-s}) = e^{2\pi i n p^{-s+m}}.$$

Now let $\xi = p^m n$. Then $|\xi|_p = p^{-m}|n|_p > p^{-m}p^{-r+m} = p^{-r}$ and $|\xi|_p \leq p^{-m}$. Thus we have

$$\psi(p^{-s}) = e^{2\pi i \{\xi p^{-s}\}_p}$$

where $p^{-r} < |\xi|_p \leq p^{-m}$. ☐

Lemma 5.6. *Suppose $\psi : \mathbb{Q}_p \to \mathbb{C}$ is an additive character and $B_0 = \{x \in \mathbb{Q}_p : v(x) \geq 0\} = \mathbb{Z}_p$ is the maximal disc such that $\psi(x) = 1$ for all $x \in B_0$. Suppose $r > 1$ for some $r \in \mathbb{Z}$. Then there exists $\xi \in \mathbb{Q}_p$ such that*

$$\psi(x) = e^{2\pi i \{\xi x\}_p}$$

for all $x \in B_r$ where $p^{-r+1} \leq |\xi|_p \leq 1$.

Proof. By Lemma 5.5, there exists $\xi \in \mathbb{Q}_p$ such that $p^{-r} < |\xi|_p \leq 1$ and for all $0 < s \leq r$

$$\psi(p^{-s}) = e^{2\pi i \{\xi p^{-s}\}_p}.$$

Suppose $x \in B_r \setminus B_0$. Suppose $v(x) = -s$ for some $0 < s \leq r$. Then

$$x = a_0 p^{-s} + a_1 p^{-s+1} + ... + a_{s-1}p^{-1} + x'$$

where $x' \in B_0$, a_is are in $\{0, 1, ..., p-1\}$ and $a_0 \neq 0$. By additivity of ψ and the fact that $\psi(x) = 1$ for all $x \in B_0$

$$\begin{aligned}
\psi(x) &= \psi(a_0 p^{-s} + a_1 p^{-s+1} + ... + a_{s-1}p^{-1} + x') \\
&= \psi(a_0 p^{-s})\psi(a_1 p^{-s+1})...\psi(a_{s-1}p^{-1})\psi(x') \\
&= \psi(p^{-s})^{a_0}\psi(p^{-s+1})^{a_1}...\psi(p^{-1})^{a_{s-1}} \\
&= e^{2a_0\pi i \{\xi p^{-s}\}_p} e^{2a_1\pi i \{\xi p^{-s+1}\}_p} ... e^{2a_{s-1}\pi i \{\xi p^{-1}\}_p} \\
&= e^{2\pi i \{\xi x\}_p}.
\end{aligned}$$

□

Corollary 5.7. *Suppose $\psi : \mathbb{Q}_p \to \mathbb{C}$ is an additive character and B_m is the maximal disc such that $\psi(x) = 1$ for all $x \in B_m$. Suppose $r > m$ for some $r \in \mathbb{Z}$. Then there exists $\xi \in \mathbb{Q}_p$ such that*

$$\psi(x) = e^{2\pi i\{\xi x\}_p}$$

for all $x \in B_r$ where $p^{-r+1} \leq |\xi|_p \leq p^{-m}$.

Proof. Let $\chi(x) = \psi(p^{-m}x)$. It is easy to check that $\chi(x)$ is an additive character on $\mathbb{Q}_p$ and B_0 is the maximal disc such that $\chi(x) = 1$ for all $x \in B_0$. By the previous lemma, there exists $\xi' \in \mathbb{Q}_p$ such that

$$\chi(x) = e^{2\pi i\{\xi' x\}_p}$$

for all $x \in p^m B_r$ where $|\xi'|_p \geq p^{-r+m+1}$. Now let $x \in B_r$. Then $p^m x \in p^m B_r$ and hence

$$\psi(x) = \chi(p^m x) = e^{2\pi i\{\xi' p^m x\}_p}.$$

Let $\xi = \xi' p^m$. Then

$$|\xi|_p = |\xi'|_p p^{-m} \geq p^{-r+1}.$$

With the same method, we can conclude from $|\xi'|_p \leq 1$ that $|\xi|_p \leq p^{-m}$. □

Now we are ready to prove the main theorem of this chapter which is proved partially in [24], page 32.

Theorem 5.8. *Suppose $\psi : \mathbb{Q}_p \to \mathbb{C}$ is an additive character. Then there exists $\xi \in \mathbb{Q}_p$ such that*

$$\psi(x) = e^{2\pi i\{\xi x\}_p}$$

for all x .

Proof. Suppose B_m is the maximal disc such that $\psi(x) = 1$ for all $x \in B_m$. By Corollary 5.7, for $r = m + 1$, there exists $\xi \in \mathbb{Q}_p$ such that

$$\psi(x) = e^{2\pi i\{\xi x\}_p}$$

for all $x \in B_{m+1}$ where $v(\xi) = m$. Let $S_{m+2} = \{x \in \mathbb{Q}_p : v(x) = -m-2\}$. Then $S_{m+2} \cap B_{m+1} = \emptyset$ and $B_{m+2} = S_{m+2} \cup B_{m+1}$. First, suppose $x \in S_{m+2}$. There exists $a_0 \in \{1, 2, ..., p-1\}$ and $x' \in B_{m+1}$ such that

$$x = a_0 p^{-m-2} + x'.$$

Then

$$\psi(x) = \psi(a_0 p^{-m-2})\psi(x') = \psi(p^{-m-1})^{\frac{a_0}{p}}\psi(x') = (e^{2\pi i\{\xi p^{-m-1}\}_p})^{\frac{a_0}{p}} e^{2\pi i\{\xi x'\}_p}$$

$$= e^{2\pi i\{a_0 \xi p^{-m-2} + \xi x'\}_p} = e^{2\pi i\{\xi(a_0 p^{-m-2} + x')\}_p} = e^{2\pi i\{\xi x\}_p}$$

Thus for all $x \in B_{m+2}$

$$\psi(x) = e^{2\pi i\{\xi x\}_p}.$$

Continuing this process inductively, we can conclude

$$\psi(x) = e^{2\pi i\{\xi x\}_p}$$

for all $x \in \mathbb{Q}_p$ where $v(\xi) = m$. □

Now let's get back to the general case. Suppose $\psi : K \to \mathbb{C}$ is an additive character where K is a finite extension of $\mathbb{Q}_p$ of degree $n \in \mathbb{N}$. Then there are $z_1, ..., z_n \in K$ such that for all $x \in K$, $x = a_1 z_1 + ... + a_n z_n$ where $a_1, ..., a_n \in \mathbb{Q}_p$. Thus

$$\psi(x) = \psi(a_1 z_1 + ... + a_n z_n) = \psi(a_1 z_1) \cdot .... \cdot \psi(a_n z_n).$$

Fix $i \in [n]$. Let $\psi_i : \mathbb{Q}_p \to \mathbb{C}$ be such that $\psi_i(a) = \psi(a z_i)$. Then it is easy to check that ψ_i is an additive character on $\mathbb{Q}_p$. Now by Theorem 5.8, there is $\xi_i \in \mathbb{Q}_p$ such that

$$\psi_i(a) = e^{2\pi i\{\xi_i a\}_p}$$

for all $x \in \mathbb{Q}_p$. Thus if $x = a_1 z_1 + ... + a_n z_n \in K$ then

$$\psi(x) = e^{2\pi i\{\xi_1 a_1\}_p} \cdot ... \cdot e^{2\pi i\{\xi_n a_n\}_p}.$$

Chapter 6

p-adic Van Der Corput's Lemma

In this chapter, we state the p-adic Van Der Corput's Lemma proved by Cluckers in [7] and then we state and prove a modified version of this lemma which plays a key role in proving the main theorem. The Van Der Corput's Lemma is used by Stein in [23] to develope "the theory of oscillatory integrals of the first kind".

In [7], Cluckers proves a p-adic version of Van Der Corput's Lemma in one and multidimensional cases. In this chapter, first we give a generalized version of Cluckers' theorem in Theorem 6.12 and then we use the result to prove Theorem 6.14 which is the the main goal of this chapter.

Through out this chapter, we fix K as a finite extension of $\mathbb{Q}_p$ as we did in the previous chapters. R_v is the valuation ring of K, M_v is the maximal ideal of R_v and q is the cardinality of the residue field. Suppose x denotes one variable and let $K\{x\}$ be the ring of restricted power series over K in the variable x as we discussed in Chapter 2. Given $f(x) = \sum_i a_i x^i \in K\{x\}$, the $\sup_i\{|a_i|\}$ exists.

Definition 6.1. *Suppose $f(x) = \sum_i a_i x^i$ is a restricted power series. We define the Gauss norm of f, denoted by $||f||$, to be $\sup_i\{|a_i|\}$.*

Among restricted power series, there are some power series, Special Power series (SP), which can be approximated by their linear parts. The following definition is stated in [7] as Definition 2.2.

Definition 6.2. *A power series $\sum_i a_i x^i \in K\{x\}$ in one variable is called SP if $a_1 \neq 0$ and $a_j \in a_1 M_v$ for all $j > 1$. If $f \in K\{x\}$ is SP, we write $|f|_{SP}$ for $|a_1|$ which is equal to the Gauss norm of $f - f(0)$.*

To see why the SP power series can be approximated by their linear parts, Let $c \in R_v$ and $f(x) = \sum_i a_i x^i \in K\{x\}$ be SP. Since $f(c) - (a_0 + a_1 c)$ is convergent, there is $1 < m \in \mathbb{N}$ such that $a_m c^m = \sup_{i>1}\{a_i c^i\}$. Since $|c| \leq 1$, $|a_m c^m| \leq |a_m c|$. On the other hand, f is SP and thus $|a_m| < |a_1|$. Hence

$$|f(c) - (a_0 + a_1 c)| \leq |a_m c^m| \leq |a_m c| < |a_1 c| \leq |f(c)|.$$

Since $c \in R_v$ was arbitrary, for all $x \in R_v$ we have

$$|f(x) - (a_0 + a_1 x)| < |f(x)|.$$

Moreover, $|f'(x)| = |a_1|$.

The set of SP power series is a small subset of the restricted power series. We can convert some non-SP restricted power series to an SP series by using some affine transformations. The following definition is stated in [7] as Definition 2.3. For every integer $r > 0$, Let $M_v^r = \{d^r | d \in M_v\}$ and let $M_v' = \{d | |d| = 1\}$.

Definition 6.3. *Let $f(x) \in K\{x\}$. We define the SP-number of f to be the smallest integer $r \geq 0$ such that for all nonzero $c \in M_v^r$ and all $b \in R_v$, the power series*

$$f_{b,c}(x) := \frac{1}{c} f(b + cx)$$

is SP if such r exists, and define the SP-number of f as ∞ otherwise.

With regard to the definition, f is SP if and only if the SP number of f is zero. To see that, let $f(x) = \sum_i a_i x^i \in K\{x\}$ and let b_i be the ith coefficient of $\frac{1}{c}\sum a_i(b + cx)^i$. It is easy to check that

$$b_i = \sum_{m \geq i} n_m a_m c^{i-1} b^{m-i},$$

where $n_m \geq 0$ is an integer. Suppose f is SP and let $c \in M_v'$ and $b \in R_v$. By the previous discussion, $b_1 = \sum_{m \geq 1} n_m a_m b^{m-1}$. Since f is SP, $|a_m| < |a_1|$ for all $m > 1$ and hence $|b_1| = |a_1|$.

For any $i > 1$,

$$|b_i| \leq \max_{m \geq i}\{|n_m a_m c^{i-1} b^{m-i}|\} < |a_1| = |b_1|.$$

Thus $\frac{1}{c} f(b+cx)$ is SP. On the other hand if $\frac{1}{c} f(b+cx)$ is SP for all $c \in M_v'$ and $b \in R_v$, in particular it is SP when $b = 0$ and $c = 1$. In other words, $f(x)$ is SP.

The next lemma is proved by Cluckers in [7]. In this lemma, we find a useful relation between the Gauss norm, the SP-number and the cardinality of residue field.

Lemma 6.4. *[7, Lemma 2.4] Let $f(x) = \sum_i a_i x^i \in K\{x\}$ be such that $|f'(x)| \geq 1$ for all $x \in R_v$. Then the SP-number of f is an integer $r \geq 0$ satisfying*

$$q^{r-1} \leq ||f - f(0)||.$$

In addition, for all nonzero $c \in M_v^r$ and all $b \in R_v$, we have $|f_{b,c}|_{SP} \geq 1$.

Proof. We refer the reader to Cluckers, [7], Lemma 2.4. □

The following corollary extends the scope of Lemma 6.4.

Corollary 6.5. *Let $\epsilon > 0$ be an arbitrary real number and let $f(x) = \sum_i a_i x^i \in K\{x\}$. Suppose that $|f'(x)| \geq \epsilon$ for all $x \in R_v$. Then the SP-number of f is an integer $r \geq 0$ satisfying*

$$q^{r-1} \leq \frac{||f - f(0)||}{\epsilon}.$$

In addition, for all nonzero $c \in M_v^r$ and all $b \in R_v$, we have $|f_{b,c}|_{SP} \geq \epsilon$.

Proof. Let

$$\epsilon' = \min\{|x| : \ x \in K \ and \ |x| \geq \epsilon\}$$

and let $e \in K$ be such that $|e| = \epsilon'$. By definition, $|f'(x)| \geq |e|$ for all $x \in R_v$.

Let $g(x) = \frac{f(x)}{e}$. Then

$$|g'(x)| = |\frac{f'(x)}{e}| \geq 1$$

for all $x \in R_v$. By the previous lemma, the SP-number of g is an integer $r \geq 0$ satisfying

$$q^{r-1} \leq ||g - g(0)||.$$

Claim: The SP-numbers of g and f are the same.

Proof of claim: Suppose the SP-number of f is r and the SP-number of g is r'. Let $c \in M_v^r$ and $b \in R_v$. Since the SP-number of f is r, by Definition 6.3, $f_{b,c}(x)$ is SP. On the other hand

$$g_{b,c}(x) = \frac{1}{c} g(b + cx) = \frac{1}{ce} f(b + cx) = \frac{1}{e} f_{b,c}(x)$$

and thus $g_{b,c}$ is also SP. Thus $r' \leq r$. With the same argument we can prove $r \leq r'$ and hence the proof for the claim is complete.

By Definition 6.1,

$$||g - g(0)|| = \frac{||f - f(0)||}{|e|}.$$

Thus the SP-number of f is an integer $r \geq 0$ satisfying

$$q^{r-1} \leq \frac{||f - f(0)||}{|e|} = \frac{||f - f(0)||}{\epsilon'} \leq \frac{||f - f(0)||}{\epsilon}.$$

By the previous lemma, for all nonzero $c \in M_v^r$ and all $b \in R$

$$\frac{|f_{b,c}|_{SP}}{|e|} = |g_{b,c}|_{SP} \geq 1$$

and thus $|f_{b,c}|_{SP} \geq \epsilon' \geq \epsilon$. $\square$

In the next step, we are going to prove another lemma that we use to prove Van Der Corput's lemma. Before stating the lemma, we need to go through some preparations.

Definition 6.6. *Let $f(x) \in R_v\{x\}$ be a power series whose coefficients come from the valuation ring, R_v. We call f regular of degree $d \geq 0$ if $f(x)$ is congruent to a monic polynomial of degree d modulo the maximal ideal $M_v\{x\}$.*

Remark 6.7. *If $f = \sum_i a_i x^i \in K\{x\}$ is a restricted power series, Then, since $|a_i| \to 0$ as $i \to \infty$, there are a unique $d \geq 0$ and a unique $c \in K^\times$ such that cf becomes regular of degree d. If f is SP and $|a_0| \leq |a_1|$, since $a_1 \neq 0$, we see that c is a_1^{-1}. Since $|a_i| < |a_1|$ for all $i > 1$, f becomes $a_0 a_1^{-1} + x$ modulo the maximal ideal $M_v\{x\}$. Thus $d = 1$.*

The following is the statement of Weierstrass Preparation Theorem which can be found in [3] or [1].

Theorem 6.8. *Let $f \in R_v\{x\}$ be regular of degree d. Then there are a unique monic polynomial $w \in R[x]$ of degree d and a unique unit $u \in R_v\{x\}$ such that*

$$f = w.u$$

Now we are ready to prove the following lemma that Cluckers uses to prove Van Der Corput's lemma. The lemma and the sketch of the proof can be found in [7]. We give a more detailed version of the proof here.

Lemma 6.9. *[13, Lemma 2.8] Let $f(x) = \sum_i a_i x^i \in K\{x\}$. Suppose that f is SP. If there exists no $d \in R_v$ such that $f(d) = 0$, then*

$$|f(x)| = |a_0| > |f|_{SP}$$

for all $x \in R_v$. If there is $d \in R_v$ such that $f(d) = 0$, then

$$|f(x)| = |f|_{SP}.|x - d|.$$

In general, if $e \in R_v$ is such that $|f(e)|$ is minimal among the values $|f(x)|$ for $x \in R_v$, then one has for all $x \in R_v$

$$|f(x)| \geq |f|_{SP}|x - e|.$$

Proof. First we claim that $|a_0| \leq |a_1| = |f|_{SP}$ if and only if there exists $d \in R_v$ such that $f(d) = 0$. *Proof of claim:* Suppsoe $|a_0| \leq |a_1|$. Since f is a restricted power series and SP, by Remark 6.7, we can convert f to a regular power series of degree one by multiplying its coefficients by a_1^{-1}. For simplicity, let's assume that $|a_1| = 1$. By Weierstrass Preparation 6.8, there are a unique monic polynomial $w = b + x \in R_v\{x\}$ of degree one and a unique unit $u \in R_v\{x\}$ such that

$$f = w.u.$$

Now let $d = -b$, then $f(d) = 0$ and $d \in R_v$. For the other direction, suppose that there is $d \in R_v$ such that $f(d) = 0$. Since f is SP,

$$|f(d) - (a_0 + a_1 d)| \leq |f(d)|$$

and thus $a_0 + a_1 d = 0$. Hence $|a_0| \leq |a_1|$.
If there exists no $d \in R_v$ such that $f(d) = 0$, then $|a_0| > |a_1|$ and since $|a_j| < |a_1|$ for all $j > 1$, $|a_0| > |a_j|$ for all $j \geq 1$. Thus for all $x \in R_v$

$$|f(x)| = |a_0|$$

and this finishes the first case. Now suppose that there exists $d \in R_v$ such that $f(d) = 0$. Let $g(t) = f(t + d)$. Then g is SP and $g(0) = 0$. Let $g(t) = \sum_{i \geq 1} b_i t^i$. Then for $t \in R_v$

$$|g(t)| = |t|| \sum_{i \geq 1} b_i t^{i-1}|.$$

Since g is SP,

$$|g|_{SP} = |b_1| > |b_i t^{i-1}|$$

and thus

$$|g(t)| = |t|.|g|_{SP}$$

and this finishes the second case.

For the final statement, any $e \in R_v$ can serve in the first case and in the second case, we can take e to be d. $\square$

An oscillatory integral is an integral of the form

$$\int_{R_v} f(x) \psi(y.\phi(x)) |dx|$$

where ψ is an additive character on K as introduced in Chapter 5 and $|dx|$ is the normalized Haar measure on K as introduced in Chapter 3. The function ϕ is usually called the phase and f

the amplitude of the integral. For the many variables analogue, x or y can be tuples of variables and ϕ can be a tuple of K-valued functions, and then $y.\phi$ is the standard inner product. By using Theorem 5.8 and the paragraph after, we can assume that ψ is trivial on M_v and nontrivial otherwise.

The following lemma states that if the phase function of the oscillatory integral is SP, then the integral has arbitrarily quick decay at infinity.

Lemma 6.10. *[7, Lemma 3.2] Let $\phi(x) = \sum_i a_i x^i \in K\{x\}$ be SP. Then for all $y \in K$ with $|y| \geq |a_1|^{-1}$ one has*

$$\int_{R_v} \psi(y.\phi(x))|dx| = 0$$

and for y with $|y| < |a_1|^{-1}$ one has

$$\int_{R_v} \psi(y.\phi(x))|dx| = \psi(y.a_0).$$

Combining, one has

$$|\int_{R_v} \psi(y.\phi(x))|dx|| \leq q^{-1}|a_1|^{-1}|y|^{-1}$$

for all nonzero y.

Proof. We refer the reader to Cluckers, [7], Lemma 3.2. □

In [7], by using Lemmas 6.4, 6.9 and 6.10, Cluckers proves the following theorem which is the p-adic analogue of real version of Van Der Corput's lemma.

Theorem 6.11. *Let $\phi \in K\{x\}$ be a restricted power series such that for some $k \geq 1$ one has $|\phi^{(k)}(x)| \geq 1$ for all $x \in R_v$ ($\phi^{(k)}(x)$ is the kth derivative of ϕ). Then there is c such that for all $y \in K^\times$*

$$|\int_{R_v} \psi(y.\phi(x))|dx|| \leq c|y|^{-\frac{1}{k}},$$

where c only depends on k, the cardinality of the residue field and on the Gauss norm of $\phi - \phi(0)$.

Proof. We refer the reader to Cluckers, [7], Proposition 3.3. □

If instead of Lemma 6.4 we use Corollary 6.5, then by applying small modifications to Cluckers' proof we can prove the following version of Theorem 6.11. Here we have a weaker hypothesis on the derivative of the phase function and the constant c depends on M upper bound for the Gauss norm of $\phi - \phi(0)$.

Theorem 6.12. *Let $\phi \in K\{x\}$ be a restricted power series such that for some $k \geq 1$ and $\epsilon > 0$ one has $|\phi^{(k)}(x)| \geq \epsilon$ for all $x \in R_v$. Moreover, suppose that $||\phi - \phi(0)|| \leq M$ for some $M \in \mathbb{N}$. Then there is c such that for all $y \in K^\times$*

$$|\int_{R_v} \psi(y.\phi(x))|dx|| \leq c|y|^{-\frac{1}{k}},$$

where c only depends on k, the cardinality of the residue field, ϵ and on M.

Proof. Let q be the cardinality of the residue field. If $|y| < 1$, by using Lemma 3.8 and the fact that $|\psi| = 1$, we can choose any $c \geq q^{-1}$ as the desired upper bound. Hence, we can assume that $|y| \geq 1$. By applying Corollary 6.5 to $\phi^{(k-1)}(x)$, we can assume that the SP-number of $\phi^{(k-1)}$ is an integer $r \geq 0$ satisfying

$$q^{r-1} \leq \frac{||\phi^{(k-1)} - \phi^{(k-1)}(0)||}{\epsilon}.$$

Let c be a generator of M_v^r. Since $|c| = q^{-r}$, we have

$$|c|^{-1} \leq q\frac{||\phi^{(k-1)} - \phi^{(k-1)}(0)||}{\epsilon} \leq q\frac{||\phi - \phi(0)||}{\epsilon} \leq q\frac{M}{\epsilon} \qquad 6.12.1$$

by Corollary 6.5. Let b_i be a set of representatives of R_v/cR_v, for $i = 1, 2, ..., |c|^{-1}$. Let

$$\phi_{b_i,c,k}(x) = \frac{1}{c^k}\phi(b_i + cx).$$

Then each $\phi^{(k-1)}_{b_i,c,k}$ is SP and $|\phi^{(k-1)}_{b_i,c,k}|_{SP} \geq \epsilon$ by Corollary 6.5 and the chain rule. By Theorem 3.11,

$$\int_{R_v} \psi(y.\phi(x))|dx| = \sum_{i=1}^{|c|^{-1}} \int_{b_i+cR_v} \psi(y.\phi(x))|dx| = |c| \sum_{i=1}^{|c|^{-1}} \int_{R_v} \psi((c^k y).\phi_{b_i,c,k}(x))|dx|.$$

Let

$$I_i(y) := \int_{R_v} \psi((c^k y).\phi_{b_i,c,k}(x))|dx|.$$

Then

$$|\int_{R_v} \psi(y.\phi(x))|dx|| \leq |c| \sum_{i=1}^{|c|^{-1}} |I_i(y)|.$$

We proceed to prove the theorem by induction on k. For $k = 1$, by Lemma 6.10,

$$|I_i(y)| \leq q^{-1}|\phi_{b_i,c,k}|_{SP}^{-1}|cy|^{-1} \leq q^{-1}\epsilon^{-1}|c|^{-1}|y|^{-1}.$$

By Inequality 6.12.1,

$$|I_i(y)| \leq q^{-1}\epsilon^{-1}q\frac{M}{\epsilon}|y|^{-1} = \frac{M}{\epsilon^2}|y|^{-1}$$

and hence we are done in the case $k = 1$.

Suppose the theorem is true for all values up to $k-1$. By previous discussion, we know that $\phi_{b_i,c,k}^{(k-1)}$ is SP and $|\phi_{b_i,c,k}^{(k-1)}|_{SP} \geq \epsilon$ for each i. Fix i and suppose $|\phi_{b_i,c,k}^{(k-1)}(d)|$ is minimal for some $d \in R_v$ amonge the values $|\phi_{b_i,c,k}^{(k-1)}(x)|$ for $x \in R_v$. Up to translating by d, we can assume that $d = 0$. By lemma 6.9,

$$|\phi_{b_i,c,k}^{(k-1)}(x)| \geq |\phi_{b_i,c,k}^{(k-1)}|_{SP}|x - 0| \geq \epsilon|x| \qquad 6.12.2$$

for all $x \in R_v$. Let $\gamma \in R_v$ (γ will be specified later). We partition R_v into the ball

$$B_0 := \gamma R_v$$

and n balls of the form

$$B_j := d_j + n_j R_v$$

for d_j with $|d_j| > |\gamma|$ and n_j a generator of the ideal $d_j M_v$, $j = 1, 2, ..., n$, and where $n = (q-1)v(\gamma)$ ($v(x)$ is the valuation of x). By linearity of the integral we have

$$I_i(y) = \int_{R_v} \psi((c^k y).\phi_{b_i,c,k}(x))|dx| = \sum_{j=0}^{n} I_{ij}(y) \qquad 6.12.3$$

where

$$I_{ij}(y) := \int_{B_j} \psi((c^k y).\phi_{b_i,c,k}(x))|dx|.$$

Since $|\psi| = 1$ and the measure of B_0 is $|\gamma|$ (by Lemma 3.8),

$$|I_{i0}(y)| \leq \int_{B_0} |\psi((c^k y).\phi_{b_i,c,k}(x))||dx| = \int_{B_0} |dx| = |\gamma|.$$

For $j = 1, 2, ..., n$, by using change of variable theorem, Theorem 3.11, we can write

$$I_{ij}(y) = |n_j| \int_{R_v} \psi(c^k y.g_j(x))|dx|,$$

where

$$g_j(x) := \phi_{b_i,c,k}(d_j + n_j x).$$

By using the chain rule we have $g_j^{(k-1)}(x) = n_j^{k-1}\phi_{b_i,c,k}^{(k-1)}(d_j + n_j x)$ and hence, by using Inequality 6.12.2,

$$|g_j^{(k-1)}(x)| = |n_j^{k-1}\phi_{b_i,c,k}^{(k-1)}(d_j + n_j x)| \geq \epsilon |n_j^{k-1}||d_j + n_j x| = \epsilon |n_j^{k-1} d_j|.$$

Hence,

$$|\frac{g_j^{(k-1)}(x)}{n_j^{k-1} d_j}| \geq \epsilon$$

Moreover, by using the definitions of g_j and $\phi_{b_i,c,k}$ and the Inequality 6.12.1,

$$||g_j(x) - g_j(0)|| = ||\phi_{b_i,c,k}(d_j + n_j x) - \phi_{b_i,c,k}(d_j)|| = ||\frac{1}{c^k}\phi(b_i + c(d_j + n_j x)) - \frac{1}{c^k}\phi(b_i + c d_j)||$$

$$\leq |c^{-k}|||\phi(x) - \phi(0)|| \leq q^k \frac{M^{k+1}}{\epsilon^k}.$$

Hence, by applying the induction hypothesis in k to g_j,

$$|I_{ij}(y)| = |n_j|.|\int_{R_v} \psi(c^k y.g_j(x))|dx|| = |n_j|.|\int_{R_v} \psi((c^k n_j^{k-1} y d_j).\frac{g_j(x)}{n_j^{k-1} d_j})|dx||$$

$$\leq |n_j|.\tilde{c}_{k-1}.|c^k n_j^{k-1} y d_j|^{-\frac{1}{k-1}} = \tilde{c}_{k-1}.|c^k y d_j|^{-\frac{1}{k-1}}$$

where $\tilde{c}_{k-1}$ only depends on k, q, ϵ and M. By using Inequality 6.12.1 again,

$$|I_{ij}(y)| \leq \tilde{c}_{k-1}.|c|^{-\frac{k}{k-1}}.|y d_j|^{-\frac{1}{k-1}} \leq (q\frac{M}{\epsilon})^{\frac{k}{k-1}}.\tilde{c}_{k-1}.|y d_j|^{-\frac{1}{k-1}}.$$

Let

$$c'_{k-1} = (q\frac{M}{\epsilon})^{\frac{k}{k-1}}.\tilde{c}_{k-1}.$$

Therefore, we have

$$|I_{ij}(y)| \leq c'_{k-1}|yd_j|^{-\frac{1}{k-1}}$$

where c'_{k-1} only depends on k, q, ϵ and M. Now by using 6.12.3,

$$|I_i(y)| \leq |\gamma| + \sum_{j=1}^{n} |I_{ij}(y)| \leq |\gamma| + c'_{k-1}|y|^{-\frac{1}{k-1}} \sum_{i=1}^{n} |d_j|^{-\frac{1}{k-1}}.$$

By the definition of d_j, $0 \leq v(d_j) \leq v(\gamma)$. For each l with $0 \leq l \leq v(\gamma)$, there are $q-1$ different d_j with $v(d_j) = l$. Thus

$$\sum_{i=1}^{n} |d_j|^{-\frac{1}{k-1}} = (q-1) \sum_{l\geq 0}^{v(\gamma)-1} (q^{1/(k-1)})^l = (q-1)\frac{|\gamma|^{-1/(k-1)} - 1}{q^{1/(k-1)} - 1}$$

$$\leq |\gamma|^{-1/(k-1)} \frac{q-1}{q^{1/(k-1)} - 1}.$$

Therefore,

$$|I_i(y)| \leq |\gamma| + c''_{k-1}|\gamma y|^{-\frac{1}{k-1}}$$

where c''_{k-1} only depends on k, q, ϵ and M. Since $|y| \geq 1$, if we can choose $\gamma \in R_v$ such that

$$q^{-1}|y|^{-\frac{1}{k}} \leq |\gamma| < |y|^{-\frac{1}{k}},$$

then

$$|I_i(y)| \leq c'''_k|y|^{-\frac{1}{k}}$$

where c'''_k only depends on k, q, ϵ and M. By using the definition of I_i, we can find the desired bound for I and the proof is complete.

□

Remark 6.13. *Let $\{\phi_i\}_{i\in I} \subseteq K\{x\}$ be a an arbitrary set of restricted power series such that for some $k \geq 1$ one has $|\phi_i^{(k)}(x)| \geq \epsilon$ for all $x \in R_v$ and for all $i \in I$. Moreover, suppose that for* *all $i \in I$, $||\phi_i - \phi_i(0)|| \leq M$ for some $M \in \mathbb{N}$ .Then, since according to the previous theorem, c only depends on k, the cardinality of residue field, ϵ and on M, there is a single c such that for all $y \in K^\times$ and all $i \in I$*

$$|\int_{R_v} \psi(y.\phi_i(x))|dx|| \leq c|y|^{-\frac{1}{k}},$$

where c only depends on k, the cardinality of residue field, ϵ and on M.

The version of Van Der Corput's lemma stated in the previous two theorems applies when the domain of integration is the valuation ring R_v. To prove the main theorem, we need a more flexible version of these theorems. To be more precise, we need a modified version of Van Der Corput's lemma in which the domain of integration is any subanalytic set of the form $\{x \in K|\ |a| \leq |x| \leq |b|,\ x \in \lambda P_n\}$.

Theorem 6.14. *Let*

$$E = \{x \in K|\ |a| \leq |x| \leq |b|,\ x \in \lambda P_n\}$$

where $\lambda \in K^\times$, $a, b \in K$ and $n \in \mathbb{N}$. Let $\phi(x) = \sum_{i=0}^{\infty} a_i x^i$ be a power series which is convergent on E and suppose that for some $k \geq 1$, $|\phi^{(k)}(x)| \geq \epsilon$ for all $x \in E$. Then there is c such that

$$|\int_E \psi(y.\phi(x))|dx|| \leq c|y|^{-\frac{1}{k}}$$

for all $y \in K^\times$.

Proof. First we want to prove that without loss of generality we can assume that $\lambda = 1$. Let $\xi : K \to K$ be the analytic function defined by $\xi(x) = \lambda x$. Let

$$E' = \{x \in K|\ |a\lambda^{-1}| \leq |x| \leq |b\lambda^{-1}|, x \in P_n\}.$$

Then ξ is an analytic isomorphism from E' onto E. By Theorem 3.11, we have

$$\int_E \psi(y.\phi(x))|dx| = \lambda \int_{E'} \psi(y.\phi(\xi(x)))|dx|.$$

It is obvious that $\phi(\xi(x))$ is a convergent power series on E' and $|(\phi \circ \xi)^{(k)}(x))| \geq \epsilon|\lambda|^k$. Thus, the hypotheses of the theorem apply to $\phi \circ \xi$ and it suffices to prove the theorem where

$$E = \{x \in K|\ |a| \leq |x| \leq |b|,\ x \in P_n\}.$$

Let

$$Val_E = \{i \in \mathbb{Z}|\ |a| \leq (\frac{1}{p})^i \leq |b|,\ n|i\}.$$

Let $A^n = \{e_1, e_2, ..., e_m\}$ be a set of representatives in K for those elements of the residue field which have an nth root. For $i \in Val_E$ and $j \in [m]$, Let

$$E_{ij} = \{x \in K|\ x = e_j \pi_0^i + d \text{ for some } |d| \leq (\frac{1}{p})^{i+1}\},$$

where π_0 is the uniformizer of K ($|\pi_0| = \frac{1}{p}$). By using Theorem 2.2 and Lemma 2.9, It is easy to see that $E = \cup_{i,j} E_{ij}$ and thus

$$\int_E \psi(y.\phi(x))|dx| = \sum_{i,j} \int_{E_{ij}} \psi(y.\phi(x))|dx|.$$

For $i \in Val_E$ and $j \in [m]$ let $\xi_{ij} : K \to K$ be such that $\xi_{ij}(x) = \pi_0^{i+1} x + e_j \pi_0^i$. Then

$$\xi_{ij}(R_v) = E_{ij}$$

and ξ_{ij} defines an analytic isomorphism from R_v onto E_{ij}. By Theorem 3.11, we have

$$|\int_{E_{ij}} \psi(y.\phi(x))|dx|| = |\pi_0^{i+1}|.| \int_{R_v} \psi(y.\phi(\xi_{ij}(x))|dx||.$$

$\phi \circ \xi_{ij}(x)$ is a convergent power series on R_v and it is easy to see that

$$|(\phi \circ \xi_{ij})^{(k)}(x)| \geq \epsilon |\pi_0|^{k(i+1)}$$

and hence

$$\frac{|(\phi \circ \xi_{ij})^{(k)}(x)|}{|\pi_0|^{k(i+1)}} \geq \epsilon.$$

The coefficients of $\phi \circ \xi_{ij}$ are the summation of the terms of form $a_k \pi_0^{t(i+1)}(e_j\pi_0^i)^{k-t}$ for $t \in \{0, 1, 2, 3, ..., k\}$. Since $|e_j| = 1$ and by the definition of Val_E, we can easily check that $|\pi_0^{i+1}| \leq \frac{|b|}{p}$ and $|\pi_0^i| \leq |b|$. Thus for $k \neq 0$

$$|a_k \pi_0^{t(i+1)}(e_j\pi_0^i)^{k-t}| \leq |a_k|\frac{|b|^k}{p^t} \leq |a_k||b|^k \leq \sup_i\{|a_i||b|^i\}.$$

Since $\phi(b)$ is finite, the right side of above inequality is finite. Hence $||\phi \circ \xi_{ij} - \phi \circ \xi_{ij}(0)||$ is bounded above by $\sup_i\{|a_i||b|^i\}$ for all i and j.

Now by Remark 6.13, there is c' such that

$$\int_{R_v} \psi(y.\phi(\xi_{ij}(x))|dx| = \int_{R_v} \psi(\pi_0^{k(i+1)}y.\frac{\phi(\xi_{ij}(x))}{\pi_0^{k(i+1)}})|dx| \leq c'|\pi_0^{k(i+1)}y|^{-\frac{1}{k}} = c'\frac{|y|^{-\frac{1}{k}}}{|\pi_0^{(i+1)}|}$$

for all $y \in K^\times$ and all i and j. Thus

$$|\int_E \psi(y.\phi(x))|dx|| \leq \sum_{i,j} |\int_{E_{ij}} \psi(y.\phi(x))|dx|| = \sum_{i,j} |\pi_0|^{i+1}|\int_{R_v} \psi(y.\phi(\xi_{ij}(x))|dx||$$
$$\leq \sum_{i,j} c'|y|^{-\frac{1}{k}} = nc'|y|^{-\frac{1}{k}}.$$

in which n is the total number of indexes we have. Let $c = nc'$ and the proof is complete.

□

Chapter 7

Main Theorem

In this chapter, we prove the main theorem of this thesis by using Van Der Corput's lemma and its corollary which is proved in previous chapter as Theorem 6.14. First we restate the theorem.

Theorem 7.1. *Let $\phi : R_v^m \to K$ be an analytic map satisfying the hyperplane condition. Let $f \in C(R_v^m)$ be integrable and suppose ψ is an additive character. Let $\epsilon > 0$. Then there are real numbers $s < 0$ and $c > 0$ such that*

$$|\int_{R_v^m} f(x)\psi(y.\phi(x))|dx|| \leq c\min\{1, |y|^s\} + \epsilon$$

for all $y \in K^\times$. Moreover, s does not depend on ϵ while c does.

To prove the theorem we need to go through some preparation. The following lemma plays an important role in the proof of main theorem. The notation of this lemma comes from chapter four, Theorem 4.10.

Lemma 7.2. *Suppose $A \subseteq R_v^m$ is a cell over $\emptyset$ and let $g > 0$ be an integer. Let*

$$H(x) = t.(\prod_{j=1}^{m} |(x_j - c_j(x_1, ..., x_{j-1}))^{a_j} u_j^{a_j}|^{\frac{1}{n_j}})(\prod_{j=1}^{m} v(x_j - c_j(x_1, ..., x_{j-1}))^{s_j})$$

be a constructible function where $t = |w|$ or $t = v(w)$ for some $w \in K$ and $s_j \geq 0$ and a_j are *integers. Assume H is a positive valued map on A. Moreover, let $E \subseteq (K \setminus R_v) \cup \{0\} \times A$ be*

$$E = \{(\lambda, x) \mid |\lambda|^{-r} \leq |(x_j - c_j(x_1, ..., x_{j-1}))| \leq |\lambda|^r, for\ all\ j \in \{1, ..., m\}\}.$$

Then, if we take $r > 0$ to be small enough, there is a constant $c > 0$ such that

$$H(x) \leq c|\lambda|^g$$

for all $\lambda \in (K \setminus R_v) \cup \{0\}$ and $x \in E_\lambda$.

Proof. Let $I \subseteq [m]$ be such that $a_j \geq 0$ for all $j \in I$. Then

$$|(x_j - c_j(x_1, ..., x_{j-1}))| \leq |\lambda|^r \Longrightarrow |(x_j - c_j(x_1, ..., x_{j-1}))|^{\frac{a_j}{n_j}} \leq |\lambda|^{\frac{ra_j}{n_j}}$$

for all $j \in I$. On the other hand, if $j \in [m] \setminus I$,

$$|\lambda|^{-r} \leq |(x_j - c_j(x_1, ..., x_{j-1}))| \Longrightarrow |(x_j - c_j(x_1, ..., x_{j-1}))|^{\frac{a_j}{n_j}} \leq |\lambda|^{\frac{-ra_j}{n_j}}$$

For $j \in [m]$ let $b_j = \frac{ra_j}{n_j}$ if $j \in I$ and $b_j = \frac{-ra_j}{n_j}$ if $j \in [m] \setminus I$. Then

$$|(x_j - c_j(x_1, ..., x_{j-1}))|^{\frac{a_j}{n_j}} \leq |\lambda|^{b_j}$$

for all $j \in [m]$. To find an appropriate upper bound for $v(x_j - c_j(x_1, ..., x_{j-1}))^{s_j}$ first we notice that $v(x) = -\log_p |x|$. Let $J \subseteq [m]$ such that for $j \in J$, $v(x_j - c_j(x_1, ..., x_{j-1})) \geq 0$. Then

$$|\lambda|^{-r} \leq |(x_j - c_j(x_1, ..., x_{j-1}))| \Longrightarrow \log_p(|\lambda|^{-r}) \leq \log_p(|(x_j - c_j(x_1, ..., x_{j-1}))|)$$

$$\Longrightarrow -\log_p(|(x_j - c_j(x_1, ..., x_{j-1}))|) \leq -\log_p(|\lambda|^{-r}) \Longrightarrow v(x_j - c_j(x_1, ..., x_{j-1})) \leq r \log_p(|\lambda|)$$

$$\Longrightarrow v(x_j - c_j(x_1, ..., x_{j-1}))^{s_j} \leq (r \log_p(|\lambda|))^{s_j}$$

for all $j \in J$. On the other hand, if $j \in [m] \setminus J$

$$|(x_j - c_j(x_1, ..., x_{j-1}))| \leq |\lambda|^r \Longrightarrow \log_p(|(x_j - c_j(x_1, ..., x_{j-1}))|) \leq \log_p(|\lambda|^r)$$

$$\Longrightarrow -\log_p(|(x_j - c_j(x_1, ..., x_{j-1}))|) \geq -\log_p(|\lambda|^r) \Longrightarrow v(x_j - c_j(x_1, ..., x_{j-1})) \geq -r\log_p(|\lambda|)$$

$$\Longrightarrow -v(x_j - c_j(x_1, ..., x_{j-1})) \leq r\log_p(|\lambda|) \Longrightarrow (-1)^{s_j} v(x_j - c_j(x_1, ..., x_{j-1}))^{s_j} \leq (r\log_p(|\lambda|))^{s_j}.$$

Let $J_1 \subseteq [m] \setminus J$ such that s_j is an even positive integer for all $j \in J_1$ and let $J_2 \subseteq [m] \setminus J$ such that s_j is an odd positive integer for all $j \in J_2$. For $j \in J_1$

$$v(x_j - c_j(x_1, ..., x_{j-1}))^{s_j} \leq (r\log_p(|\lambda|))^{s_j}.$$

Without loss of generality we can assume $t \geq 0$. Since $H(x)$ is a positive valued function, the cardinality of J_2, must be an even number and thus $\sum_{j \in J_2} s_j$ is an even number. Hence

$$\prod_{j \in J_2} v(x_j - c_j(x_1, ..., x_{j-1}))^{s_j} = \prod_{j \in J_2} (-1)^{s_j} v(x_j - c_j(x_1, ..., x_{j-1}))^{s_j} \leq \prod_{j \in J_2} (r\log_p(|\lambda|))^{s_j}$$

By considering the above inequalities we can conclude

$$H(x) = (t \prod_{j \in [m]} |u_j|^{\frac{a_j}{n_j}})(\prod_{j \in [m]} |(x_j - c_j(x_1, ..., x_{j-1}))|)^{\frac{a_j}{n_j}}(\prod_{j \in [m]} v(x_j - c_j(x_1, ..., x_{j-1}))^{s_j})$$

$$\leq (t \prod_{j \in [m]} |u_j|^{\frac{a_j}{n_j}})(\prod_{j \in [m]} |\lambda|^{b_j})(\prod_{j \in [m]} (r\log_p(|\lambda|))^{s_j})$$

$$= (t \prod_{j \in [m]} r^{s_j} |u_j|^{\frac{a_j}{n_j}}).|\lambda|^{\sum_{j \in [m]} b_j}.(\log_p(|\lambda|))^{\sum_{j \in [m]} s_j}.$$

If we take r to be small enough such that $\sum_{j \in [m]} b_j \leq \frac{g}{2}$, then since $|\lambda| > 1$ we have

$$|\lambda|^{\sum_{j \in [m]} b_j} \leq |\lambda|^{\frac{g}{2}}.$$

On the other hand if we take r to be small enough and $c > 0$ to be large enough then

$$(t \prod_{j \in [m]} r^{s_j} |u_j|^{\frac{a_j}{n_j}})(\log_p(|\lambda|))^{\sum_{j \in [m]} s_j} \leq c|\lambda|^{\frac{g}{2}}.$$

By combining the last two equalities, we have

$$H(x) \leq c|\lambda|^g.$$

□

The following lemma gives us a powerful tool to find the decay rate of constructible functions. In [8], Cluckers and Miller prove a similar lemma in the real case. For stating and proving the lemma, we need the following definition.

Definition 7.3. *Suppose $\{E_\lambda\}_{\lambda \in K}$ is a family of subsets of a set $A \subseteq K^m$. Let $B \subseteq A$. We say $E_\lambda \to B$ as $|\lambda| \to \infty$ if $E_\lambda \subseteq E_{\lambda'}$ whenever $|\lambda| < |\lambda'|$ and*

$$B = \bigcup_{\lambda \in K} E_\lambda.$$

If $E_\lambda \to B'$ where $B' \setminus B$ has measure 0, we say $E_\lambda \to B$ almost everywhere and we write $E_\lambda \to B$ a.e.

Lemma 7.4. *Let $f \in C(U)$ for a subanalytic set $U \subset K^m$. Let f be integrable and $E \subset K \times U$ be a subanalytic set such that $E_\lambda \to \emptyset$ a.e. as $|\lambda| \to \infty$ where $E_\lambda = \{x \in U : (\lambda, x) \in E\}$ for $\lambda \in K$. Then there is $\alpha < 0$ such that*

$$\int_{E_\lambda} f(x)|dx| \ll min\{1, |\lambda|^\alpha\}$$

for all $\lambda \in K$.

Proof. By using Theorem 4.9, we can decompose U into cells A such that $f|_A$ is a finite sum $\sum_{i=1}^n H_i(x)$, where each H_i is a constructible function on A of the form given in Theorem 4.9. Moreover, by Theorem 4.9, we can choose A such that each H_i is integrable with a constant sign on A. It suffices to focus on $f|_A$. Moreover, since

$$|\int_{E_\lambda \cap A} f(x)|dx|| \leq \int_{E_\lambda \cap A} |f(x)||dx| \leq \sum_i \int_{E_\lambda \cap A} |H_i(x)||dx|$$

and $|H_i(x)|$ is a constructible function on A, it suffices to focus on the case where the integrand is

$|H_i|$ for some i. For suppose that for every i there are c_i and $\alpha_i < 0$ such that

$$\int_{E_\lambda \cap A} |H_i(x)||dx| \leq c_i min\{1, |\lambda|^{\alpha_i}\}.$$

Let $c = max\{c_i\}$ and $\alpha = max\{\alpha_i\}$. By using the properties of reciprocal functions, we know that for any $\lambda \in K$ either $|\lambda|^{\alpha_i} \geq 1$ for all i or $|\lambda|^{\alpha_i} < 1$ for all i. In the first case, since $min\{1, |\lambda|^{\alpha_i}\} = 1$ for all i, we have $min\{1, |\lambda|^{\alpha}\} = 1$ and

$$|\int_{E_\lambda \cap A} f(x)|dx|| \leq n.c. \min\{1, |\lambda|^{\alpha}\}$$

in which n is the number of indices. In the second case, since $|\lambda|^{\alpha_i} \leq |\lambda|^{\alpha}$ we have the desired property with the same upper bound as in the above inequality.
To prove the lemma for $|H_i(x)|$, first we notice that $|H_i(x)|$ is a constructible function since H_i does not change its sign. Let $\chi : K \times K^m \to \mathbb{R}$ be the characteristic function of $E_\lambda \cap A$. Obviously, $|\chi|$ is a constructible function and

$$\int_{E_\lambda \cap A} |H_i(x)||dx| \leq \int_A |H_i(x)|.|\chi(\lambda, x)||dx|.$$

Let $G(\lambda) := \int_A |H_i(x)|.|\chi(\lambda, x)||dx|$. Then by Theorem 4.5, $G(\lambda)$ is a constructible function. Moreover, when $|\lambda| \to \infty$, $E_\lambda \to \emptyset$ and so $G(\lambda) \to 0$. Now by Corollary 4.8, there is $\alpha_i < 0$ such that

$$G(\lambda) \ll \min\{1, |\lambda|^{\alpha_i}\}$$

and hence the proof is complete. □

In the main theorem, the phase function, $\phi(x)$, is an analytic map satisfying the hyperplane condition. We refer the reader to chapter one, Definition 1.2, for the definition of the hyperplane condition. To be able to use Van Der Corput's lemma to prove the theorem, we need to convert the hyperplane condition to a condition on partial derivatives of the phase function. That is the purpose of the following lemma. If $x = (x_1, x_2, ..., x_m)$ and $i = (i_1, i_2, ..., i_m) \in \mathbb{N}^m$, then we write x^i to mean $x_1^{i_1} x_2^{i_2} ... x_m^{i_m}$. We write $|i|$ which is defined to be $\sum_{j=1}^{m} i_j$.

Lemma 7.5. *Let $\phi(x) = \sum_{i\in\mathbb{N}^m} a_i x^i$ be a restricted power series satisfying the hyperplane condition where $x = (x_1, x_2, ..., x_m)$. Then there are a number $n \in \mathbb{N}$, $b_1, ..., b_n \in R_v^m$, $r_1, ..., r_n \in \{p^i | i \in \mathbb{Z}\}$, $\epsilon_1, ..., \epsilon_n \in \mathbb{R}$ and $k_1, ..., k_n \in \mathbb{N}^m$ with $|k_i| > 0$ such that for each $i \in [n]$*

$$|\partial_x^{k_i} \phi(x)| > \epsilon_i$$

if $|x - b_i| \leq r_i$, where $\partial_x^{k_i}\phi = (\prod_j \frac{\partial^{k_{i_j}}}{\partial x_j^{k_{i_j}}})\phi$. Moreover, we can choose the b_i and the r_i such that

$$R_v^m \subseteq \bigcup_{i\in[n]} \{x| \ |x - b_i| \leq r_i\}$$

Proof. Let $b \in R_v^m$. Consider the function $\phi(x) - \phi(b)$ mapping R_v^m into K. This function does not vanish identically on a neighborhood of b. For suppose V_b is an open set containing b such that

$$\phi(x) - \phi(b) = 0$$

for all $x \in V_b$. Then $V_b \subseteq \phi^{-1}(\{\phi(b)\})$. However $\{\phi(b)\} \subseteq K$ is a hyperplane and hence $\mu(\phi^{-1}(\{\phi(b)\})) = 0$ by the hyperplane condition. That is a contradiction since $\mu(V_b) > 0$.

We can write

$$\phi(x) = \sum_{i\in\mathbb{N}^m} a_i x^i = \sum_{i\in\mathbb{N}^m} a_i[(x - b) + b]^i = \sum_{i\in\mathbb{N}^m} c_i(x - b)^i$$

in which $c_i = \partial_x^i\phi(b)$. If $\partial_x^i\phi(b) = 0$ for all i with $|i| > 0$, $\phi(x) - \phi(b)$ would vanish identically on R_v^m which is not possible by previous argument. Thus there is $k \in \mathbb{N}^m$ with $|k| > 0$ such that $\partial_x^k\phi(b) \neq 0$. Let $\epsilon > 0$ be such that $|\partial_x^k\phi(b)| > \epsilon$. Since $|\partial_x^k\phi(x)|$ is continuous, there is $r \in \mathbb{R}$ such that

$$|\partial_x^k\phi(x)| > \epsilon$$

if $|x - b| \leq r$. Thus for every $b \in R_v^m$ there are r, ϵ and k satisfying the above inequality with the given condition. Since R_v^m is compact, there are $b_1, ..., b_n \in R_v^m$, $r_1, ..., r_n \in \{p^i | i \in \mathbb{Z}\}$, $\epsilon_1, ..., \epsilon_n \in \mathbb{R}$ and $k_1, ..., k_n \in \mathbb{N}^m$ with $|k_i| > 0$ such that

$$|\partial_x^{k_i}\phi(x)| > \epsilon_i$$

if $|x-b_i| \le r_i$ for all $i \in [n]$ and

$$R_v^m \subseteq \cup_{i\in[n]}\{x|\ |x-b_i| \le r_i\}.$$

□

Let $x = (x_1, x_2, ..., x_m)$ and let $V_{k,m}(K)$ be the K-vector space of homogeneous polynomials of degree k in x over K. In [7], Cluckers proves the following lemma that we are going to use to obtain a nicer description of partial derivatives of the phase function.

Lemma 7.6. *[7, Lemma 3.7] Let $\alpha > 0$ be an integer and $x = (x_1, ..., x_m)$ variables. Let K be an infinite field of characteristic $> \alpha$ or zero. Then the polynomials of the form*

$$(\xi.x)^\alpha$$

for $\xi \in K^n$, where $\xi.x = \sum_i \xi_i x_i$, span the K-vector space $V_{\alpha,m}(K)$ of homogeneous polynomials of degree α in x over K.

Proof. We refer the reader to Cluckers, [7]. Note that we can take ξ of length one. □

Now we are ready to prove the main theorem.

Proof of Theorem 7.1. Let $x = (x_1, ..., x_m)$. Since ϕ is analytic, it can be written as a convergent power series locally on R_v^m. By compactness of R_v^m, for some $n \in \mathbb{N}$, there are $b_1, ..., b_n \in R_v^m$ and $r_1, ..., r_n \in \{p^i | i \in \mathbb{Z}\}$ such that for each $j \in [n]$, there are $\{a_{ij}\}_{i\in\mathbb{N}^m}$ such that

$$\phi(x) = \sum_{i\in\mathbb{N}^m} a_{ji}(x-b_j)^i \qquad on\ |x-b_j| \le r_j$$

and $\{x||x-b_j| \le r_j\}_j$ forms an open cover for R_v^m.

For $j \in [n]$ let

$$g_j : R_v^m \to \{x||x-b_j| \le r_j\},$$

$$g_j(x) = r_j x + b_j.$$

Then g_j is an analytic isomorphism such that

$$\phi(g_j(x)) = \sum_{i\in\mathbb{N}^m} a_{ji}[(r_jx+b_j)-b_j]^i = \sum_{i\in\mathbb{N}^m} a_{ji}r_j^i x^i,$$

which is a convergent power series on R_v^m. Moreover, $\phi(g_j(x))$ satisfies the hyperplane condition since the image of a set of measure zero under an affine transformation has measure zero. By Theorem 3.11

$$|\int_{R_v^m} f(x)\psi(y.\phi(x))|dx|| \le |\sum_{j\in[n]}\int_{\{x||x-b_j|\le r_j\}} f(x)\psi(y.\phi(x))|dx|| \tag{7.1}$$

$$\le \sum_{j\in[n]} |r_j||\int_{R_v^m} f(g_j(x))\psi(y.\phi(g_j(x)))|dx||. \tag{7.2}$$

It is enough to prove the theorem for each integral in the above inequality. Thus, without loss of generality we can assume $\phi(x)=\sum_i a_ix^i$ is a restricted power series on R_v.

Since $\phi(x)$ is a restricted power series satisfying the hyperplane condition, Lemma 7.5 implies that there are $b_1,...,b_n \in R_v^m$, $r_1,...,r_n\in\{p^i|i\in\mathbb{Z}\}$, $\epsilon_1,...,\epsilon_n\in\mathbb{R}$ and $k_1,...,k_n\in\mathbb{N}^m$ with $|k_i|>0$ such that for each $i\in[n]$

$$|\partial_x^{k_i}\phi(x)|\ge\epsilon_i$$

if $|x-b_i|\le r_i$, where $|k_i|=\sum_j k_{i_j}$ and $\partial_x^{k_i}\phi=(\prod_j \frac{\partial^{k_{i_j}}}{\partial x_j^{k_{i_j}}})\phi$. Moreover, $\{x|\ |x-b_i|\le r_i\}$ forms an open cover for R_v^m. Thus, by using "change of variable theorem" and the same argument that we used before, we can assume that $\phi(x)$ is a restricted power series such that there is $k\in\mathbb{N}^m$ with $|k|>0$ and there is $\epsilon\in\mathbb{R}$ so that we have

$$|\partial_x^k\phi(x)|>\epsilon$$

for all $x\in R_v^m$.

Let $\alpha=|k|=\sum_j k_j$. By Lemma 7.6, there are $\xi_1,...,\xi_d\in K^m$ of length 1 such that the homogeneous polynomials

$$(\xi_i.x)^\alpha \quad i=1,2,...,d$$

form a basis for this vector space, where d is the dimension of $V_{\alpha,m}(K)$ as a K-vector scpace. Express the monomial x^k in this basis as

$$x^k = \sum_i e_i(\xi_i.x)^\alpha, \quad e_i \in K.$$

Let

$$\partial V_{\alpha,m}(K) = \{ \sum_{d_1+d_2+...+d_m=\alpha} e_{d_1,...,d_m}\partial_x^{(d_1,...,d_m)}\phi \mid \ e_{d_1,...,d_m} \in K,\ d_1,...,d_m \geq 0\},$$

where $\partial_x^{(d_1,...,d_m)}\phi = (\prod_i \frac{\partial^{d_i}}{\partial x_j^{d_i}})\phi$. Obviously, $\partial V_{\alpha,m}(K)$ is a vector space over K. Define

$$\sigma : V_{\alpha,m}(K) \longrightarrow \partial V_{\alpha,m}(K),$$

$$\sigma(\sum_{d_1+d_2+...+d_m=\alpha} e_{d_1,...,d_m}x_1^{d_1}...x_m^{d_m}) = \sum_{d_1+d_2+...+d_m=\alpha} e_{d_1,...,d_m}\partial_x^{d_1,...,d_m}\phi.$$

Then σ defines an isomorphism from $V_{\alpha,m}(K)$ onto $\partial V_{\alpha,m}(K)$ such that

$$\sigma((\xi_i.x)^\alpha) = (\xi_i.\nabla)^\alpha\phi,$$

where $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_m})$. Then, for $b \in R_v^m$,

$$\epsilon < |\partial_x^k\phi(b)| = |\sum_i e_i(\xi_i.\nabla)^\alpha\phi(b)| \leq \max_i\{|e_i(\xi_i.\nabla)^\alpha\phi(b)|\}.$$

Hence

$$|(\xi_i.\nabla)^\alpha\phi(b)| > \frac{\epsilon}{e_i}$$

for at least one i with $e_i \neq 0$. Since $|(\xi_i.\nabla)^\alpha\phi(x)|$ is a continuous function there is $r \in \{p^i|i \in \mathbb{Z}\}$ such that

$$|(\xi_i.\nabla)^\alpha\phi(x)| > \frac{\epsilon}{e_i}$$

if $|x-b| \leq r$. Hence by using a measure preserving affine transformation and change of variable theorem, as the first argument in the proof, we can assume

$$|(\partial^{\alpha}/\partial x_1^{\alpha})\phi(x)| > \epsilon$$

for all $x \in R_v^m$ and some $\epsilon \in \mathbb{R}^{\geq 0}$.

By Theorem 4.10, we can partition R_v^m into cells A over $\emptyset$ such that $f|_A$ is a finite summation of the functions of the form

$$H(x) = t.(\prod_{j=1}^{m} |(x_j - c_j(x_1, ..., x_{j-1}))^{a_j} u_j^{a_j}|^{\frac{1}{n_j}})(\prod_{j=1}^{m} v(x_j - c_j(x_1, ..., x_{j-1}))^{s_j})$$

where $t = |w|$ or $t = v(w)$ for some $w \in K$ and $s_j \geq 0$ and a_j are integers. We can choose the partitions such that $H(x)$ is integrable on A and does not change its sign. Without loss of generality, we can assume that $H(x)$ is positive on A. It is enough to prove the theorem for $H(x)$ over A. Moreover, if $\pi_1(A)$ is the projection of A on the first coordinate, then

$$\pi_1(A) = \{t \in K||a|\square_1|t - c_1| \leq |b|, t - c_1 \in \lambda P_n\}$$

with constants $n > 0$,$\lambda, c_1 \in K$, $a, b \in K^{\times}$, and $\square_1$ either $<$ or no condition. By using the affine transformation $g : K^m \to K^m$ with $g(x_1, ..., x_m) = (x_1 + c_1, ..., x_m)$ and by using the change of variable theorem we can assume c_1 is zero.

Let $E \subseteq \{(K \setminus R_v) \cup \{0\}\} \times A$ be:

$$E = \{(\lambda, x) \mid |\lambda|^{-r} \leq |(x_j - c_j(x_1, ..., x_{j-1}))| \leq \tau, \text{for all } j \in \{1, ..., m\}\}$$

where $\tau = b$ when $j = 1$ and $\tau = |\lambda|^r$ otherwise. Here r is a positive rational number, to be specified later. The idea of defining E in this way helps us to separate the finite and infinite parts of A. $A \setminus E_{\lambda} \to \emptyset$ a.e when $|\lambda| \to \infty$. By Lemma 7.4, there is $\gamma < 0$ and $c_1 > 0$ such that:

$$\int_{A \setminus E_{\lambda}} |H(x)||dx| \leq c_1 \min\{1, |\lambda|^{\gamma}\}$$

and since

$$|\int_{A\setminus E_\lambda} H(x)\psi(y.\phi(x))|dx|| \leq \int_{A\setminus E_\lambda} |H(x)||dx|$$

we can conclude

$$\int_{A\setminus E_\lambda} H(x)\psi(y.\phi(x))|dx| \ll \min\{1, |\lambda|^\gamma\}.$$

Thus it suffices to find a suitable bound for

$$|\int_{E_\lambda} H(x)\psi(y.\phi(x))|dx||.$$

Let $0 < g < \frac{1}{\alpha}$ and let $r > 0$ to be large enough such that Lemma 7.2 holds. Then by Lemma 7.2, there is $c_2 > 0$ such that $H(x) \leq c_2|\lambda|^g$ for all $\lambda \in (K \setminus R_v) \cup \{0\}$ and $x \in E_\lambda$. Thus

$$|\int_{E_\lambda} H(x)\psi(y.\phi(x))|dx|| \leq c_2|\lambda|^g.|\int_{E_\lambda} \psi(y.\phi(x))|dx||$$

and hence it is enough to find an appropriate upper bound for

$$|\int_{E_\lambda} \psi(y.\phi(x))|dx||.$$

Let $\pi_1(A)$ be the projection of A on the first coordinate and $\pi_{2:m}(A)$ be the projection of A on the last $m-1$ coordinate. By Fubini's theorem we have

$$|\int_{E_\lambda} \psi(y.\phi(x))|dx|| = |\int_{\pi_{2:m}(E_\lambda)} \int_{\pi_1(E_\lambda)} \psi(y.\phi(x))|dx_1||dx_2...dx_m||.$$

For each $a_2, ..., a_m \in R_v$, the Gauss norm of $\phi(x_1, a_2, ..., , a_m) - \phi(0, a_2, ..., a_m)$ is bounded by $||\phi(x) - \phi(0)||$. Hence by Theorem 6.14, there is c_3 such that

$$|\int_{\pi_1(E_\lambda)} \psi(y.\phi(x_1, a_2, ..., a_m))|dx_1|| \leq c_3|y|^{-\frac{1}{\alpha}}$$

for all $a_2, ..., a_m \in R_v$,$y \in K^\times$ and λ. Thus

$$|\int_{E_\lambda} \psi(y.\phi(x))|dx|| \leq |\int_{\pi_{2:m}(E_\lambda)} c_3|y|^{-\frac{1}{\alpha}}|dx_2...dx_m|| \leq c_3|y|^{-\frac{1}{\alpha}}$$

in which the last inequality is the result of $\mu(\pi_{2:m}(E_\lambda)) \leq 1$ (since $\pi_{2:m}(E_\lambda) \subseteq R^{m-1}$). Hence

$$|\int_{E_\lambda} H(x)\psi(y.\phi(x))|dx|| \leq c_3c_2|\lambda|^g|y|^{-\frac{1}{\alpha}}.$$

Let $|\lambda|$ be large enough such that

$$c_1|\lambda|^\gamma \leq \epsilon.$$

Now let $c = \max\{\int_A |H(x)||dx|, 1, c_2c_3|\lambda|^g\}$. Let $y \in K^\times$. If $|y| \leq 1$, then $1 \leq |y|^{-\frac{1}{\alpha}}$ and

$$|\int_A H(x)\psi(y.\phi(x))|dx|| \leq \int_A |H(x)||dx| \leq c = c\min\{1, |y|^{-\frac{1}{\alpha}}\}.$$

If $|y| > 1$, then

$$|\int_{A\setminus E_\lambda} H(x)\psi(y.\phi(x))|dx|| \leq c_1|\lambda|^\gamma$$

and

$$|\int_{E_\lambda} H(x)\psi(y.\phi(x))|dx|| \leq c_2c_3|\lambda|^g|y|^{-\frac{1}{\alpha}}$$

and thus

$$|\int_A H(x)\psi(y.\phi(x))|dx|| \leq c_2c_3|\lambda|^g|y|^{-\frac{1}{\alpha}} + c_1|\lambda|^\gamma \leq c_2c_3|\lambda|^g|y|^{-\frac{1}{\alpha}} + \epsilon \leq c|y|^{-\frac{1}{\alpha}} + \epsilon.$$

Hence

$$|\int_A H(x)\psi(y.\phi(x))|dx|| \ll \min\{1, |y|^{-\frac{1}{\alpha}}\} + \epsilon.$$

Let $s = -\frac{1}{\alpha}$ and the proof is complete.

The last statement in the theorem is the result of applying Theorem 6.14 in which the exponent only depends on the partial derivative of the phase function. □

As an interesting topic for future research in this area, one can think about finding a method to eliminate ϵ dependency in the main theorem. As another topic, one can think about the possibility of proving the main theorem in the case ϕ is defined on K instead of R_v.

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