PARASTATISTICS

FIRST AND SECOND QUANTIZATION THEORIES
OF
PARASTATISTICS

By

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"O Illustrious One, in one thing above all have I admired your teachings. Everything is completely clear and proved. You show the world as a complete, unbroken chain, an eternal chain, linked together by cause and effect. Never has it been presented so clearly, never has it been so irrefutably demonstrated. Surely every Brahmin's heart must beat more quickly, when through your teachings he looks at the world, completely coherent, without a loophole, clear as crystal, not dependent on chance, not dependent on the gods. Whether it is good or evil, whether life in itself is pain or pleasure, whether it is uncertain--that it may perhaps be this is not important--but the unity of the world, the coherence of all events, the embracing of the big and the small from the same stream, from the same law of cause, of becoming and dying: this shines clearly from your exalted teachings, O Perfect One. But according to your teachings, this unity and logical consequence of all things is broken in one place. Through a small gap there streams into the world of unity something strange, something new, something that was not there before and that cannot be demonstrated and proved: that is your doctrine of rising above the world, of salvation. With this small gap, through this small break, however, the eternal and single world law breaks down again. Forgive me if I raise this objection."

Herman Hesse: Siddharta
(Translated by Hilda Rosner)

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SCOPE AND CONTENTS: Although usually only two kinds of statistics, namely Bose-Einstein and Fermi-Dirac statistics, are considered in Quantum Mechanics and in Quantum Field Theory, other kinds of statistics, called collectively parastatistics, are conceivable. We critically review theoretical studies of parastatistics to date, point out and clarify several confusions.

We first study the "proofs" so far proposed for the symmetrization postulate which excludes parastatistics, emphasizing their ad hoc nature. Then, after exploring in detail the structure of the quantum mechanical theory of paraparticles, we clarify some confusions concerning the compatibility of parastatistics with the so-called cluster property, which has been an issue of controversy for several years. We show, following a suggestion of Greenberg, that the quantum mechanical theory of paraparticles can be formulated in terms of density matrix compatibly with the cluster property. We also discuss such topics as selection rules for systems with variable numbers of paraparticles, the connection between

statistics and permutation characters, and the classification of paraparticles.

For the quantum field theory of paraparticles, we study discrete representations of the para-commutation relations and illustrate in detail Greenberg and Messiah's theorem concerning Green's ansatzes . Also, fundamental topics such as the spin-statistic theorem, the TCP theorem and the observability of parafields are discussed on the basis of Green's ansatzes . Finally, we point out that the so-called particle permutation operators do not always define multi-dimensional representations of the permutation group both in first and second quantization theories. This questions the validity of the correspondence between the two theories which has recently been proposed.

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CHAPTER I

INTRODUCTION

Among the most interesting applications of Quantum Mechanics is the connection between Bose-Einstein (B.E.) and Fermi-Dirac (F.D.) statistics and the two choices (symmetric or anti-symmetric) of the permutation symmetry characters for the wavefunctions (wfn) of the many-body problem. However, the Schrödinger equation of the many-body problem also allows a host of solutions which are neither symmetric nor anti-symmetric in character, some of them would correspond to the statistics (called intermediate statistics) in which the maximal occupation numbers are neither 1 nor ∞. It becomes customary now to employ the term parastatistics to refer to all these solutions (in this sense, intermediate statistics are special parastatistics).

The possibility of parastatistics is perhaps the reason why the statement that the wfns of N-identical particles are either symmetric or anti-symmetric in particle variables is usually referred to as a postulate--the symmetrization postulate (S.P.). As a matter of fact, the S.P. has been experimentally verified only for photons (e.g. in black-body radiation), electrons (e.g. in atoms), neutrons and protons (e.g. in nuclei). For other elementary particles, about 200 in number, S.P. has been merely taken for granted and supplemented by Pauli's theorem. This theorem, proved in

Relativistic Quantum Field Theory, states that, if parastatistics are excluded a priori, integer-spin particles obey B.E. statistics whereas half-integer spin particles obey F.D. statistics.

Since it is certainly not prudent to exclude something that could exist in principle, the quantum theory of particles obeying parastatistics, called paraparticles, has frequently attracted interests of physicists in the realm of Elementary Particle Physics. Such a theory would be helpful in answering the question of whether any elementary particles obey parastatistics, and if none ever does, why Nature prefers only B.E. and F.D. statistics. Furthermore, even if some simple arguments may be found against parastatistics, the theory of paraparticles would still be useful in some physical contexts. For example, when electromagnetic interaction is absent or negligible, the neutron and the proton could possibly be treated as paraparticles.

The history of parastatistics goes back probably to 1940 when Gentile (1940, 1942) proposed the intermediate statistics in which the maximal number of particles in a non-degenerate state is neither 1 (F.D.) nor ∞ (B.E.), but instead a finite number p. The statistical properties of a classical "intermediate gas" have been discussed by Sommerfeld (1942), Schubert (1946), ter Haar (1952) and Guénault and McDonald (1962). Although of no direct applications to physical systems, the results obtained by these authors have helped in understanding the physical properties of B.E. and F.D. statistics. Unfortunately, these results, although interesting, would not help much in determining the statistics of elementary particles for, with only a few exceptions, experiments involving a large number of elementary particles are not feasible with the present technique. For this reason, we shall not attempt any discussion of the classical aspects of parastatistics in this thesis.

As in the case of B.E. and F.D. statistics, the question of parastatistics can be attacked in Quantum Mechanics or in Quantum Field Theory (see Dresden, 1963; Greenberg, 1966 for reviews). We shall discuss first the quantum mechanical theory, or the first quantization theory as it is often called.

The first quantization theory of paraparticles was first explored in detail by Messiah and Greenberg (1964) but some indications toward such a theory had already been outlined earlier by Dirac (1930). In this theory, it is required that dynamical states of a system of N-identical particles which are represented by wfns differing only by a permutation of the particles cannot be distinguished by any observation. Such a permutation is realized by an operator $U(\sigma)$ in the Hilbert space H^N of the many-body wfns, which. when acting on an N-body wfn, effectuates the permutation σ^{-1} of the particle variables, i.e.

$$U(\sigma) \langle x_1|k_1\rangle\langle x_2|k_2\rangle\ldots\langle x_N|k_N\rangle =$$

$$\langle x_{\sigma^{-1}1}|k_1\rangle\langle x_{\sigma^{-1}2}|k_2\rangle\ldots\langle x_{\sigma^{-1}N}|k_N\rangle$$

The starting point of the theory is then the assumption that physical observables must commute with all the $U(\sigma)$ (this is usually referred to as the indistinguishability postulate).

The permutations σ form what is called the group S_N (the composition law of the permutations satisfies certain properties called group properties) and the $U(\sigma)$ have the same composition law as the σ; it is said that the $U(\sigma)$ define a representation of S_N in H^N. This representation is highly reducible in the sense that one can decompose H^N into irreducible subspaces with respect to the $U(\sigma)$ and each subspace is said to support an irreducible representation (I.R.) of S_N. Two I.R.'s are said to be equivalent if the bases of their two supporting irreducible subspaces can be chosen so that the matrix representing each $U(\sigma)$ in one subspace is the same as in the other, otherwise they are said inequivalent (the irreducible subspaces themselves are correspondingly said equivalent or inequivalent). In this connection, there is an important lemma, called Schur's lemma, which can be stated in two parts: (i) an operator in an irreducible subspace which commutes with all the $U(\sigma)$ is a multiple of the identity operator, (ii) there is no non-zero transformation from an irreducible subspace to an inequivalent irreducible subspace which commutes with all the $U(\sigma)$.

Consider the subspace H[K] spanned by the $U(\sigma)K(x_1, x_2, \ldots x_N)$ where $K(x_1, x_2, \ldots x_N) = \langle x_1|k_1\rangle \langle x_2|k_2\rangle \ldots \langle x_N|k_N\rangle$. H[K] can be decomposed into irreducible subspaces by a standard method (which makes use of the Young diagrams). There are only two one-dimensional inequivalent irreducible subspaces--one consists of only the symmetric wfn and the other of only the anti-symmetric one--which correspond to B.E. and F.D. statistics respectively. There are, however, many linearly independent and equivalent multi-dimensional irreducible subspaces (the wfns in one of them have the same permutation symmetry character as the wfns in the others). Messiah and Greenberg proposed that the wfns representing the (pure) states of N-identical paraparticles belong to an irreducible subspace. However, there is no criterion for choosing a wfn among the others to represent the state of the system. They upheld that, since the observables commute with all the $U(\sigma)$ (indistinguishability postulate), according to the first part of Schur's lemma, measurable results do not depend on which wfn of the irreducible subspace is chosen to represent the state. They called an irreducible subspace a generalized ray and asserted that the state of N-identical paraparticles corresponds to a generalized ray. Using the second part of Schur's lemma, they also derived a super-selection rule which asserts that states represented by (coherent) mixtures of wfns belonging to different inequivalent irreducible subspaces are not physically realizable (observable).

We interrupt our review of historical developments in order to give some comments on Messiah and Greenberg's theory. It appears to us that Schur's lemma does not imply Messiah and Greenberg's result if the physical observables have domains of definition extended to many equivalent irreducible subspaces, i.e., if physical observables are operators that can transform vectors in one irreducible subspace to vectors in others (or linear combinations of them). Furthermore, in Messiah and Greenberg's theory, the fact that the state can be determined up to an irreducible subspace presupposes that the equivalent irreducible subspaces are physically distinguishable. This is quite questionable for, if only the permutation symmetry character is physically important, it may not be possible to distinguish the equivalent irreducible subspaces; the state may then be determined only up to a permutation invariant subspace, which is the direct sum of all the equivalent irreducible subspaces. We propose to explore the theory beyond the assumptions implicit in Messiah and Greenberg's theory.

The use of multi-dimensional representations of S_N allows the possibility of parastatistics but does not answer the question of whether paraparticles exist in Nature. Some efforts have been made to exclude parastatistics. Some proofs of the S.P. (Jauch, 1960; Jauch and Misra, 1961; Galindo, Morales and Nuñez-Lagos, 1962, Pandrès, 1962) have been rightly criticized by Messiah and Greenberg (1964); they involve either faulty formulation of the indistinguishability of identical particles in terms of properties of wfns, or they make assumptions of a rather technical nature about the algebra of observables, which are difficult to justify on physical grounds (into this category also fall a paper by Borcher , 1965, which treated the problem in the framework of the C^*-algebra approach to quantum theory and a paper by Girardeau, 1969, which involves some topological assumption related to the connectivity properties of the configuration space). The derivation of S.P. from the cluster property, as proposed by Casher, Frieder, Gluck and Peres (1965) and Steinmann (1966) has been, however, subjected to controversy.

Literally, the cluster property requires that the theory of an N-particle system yield the same results whether or not other far-away particles of the same species are taken into account, so long as their interactions with the system remains negligible. Steinmann (1966) argued that the cluster property allows a measurement to distinguish the wfns belonging to a generalized ray, in contradiction with Messiah and Greenberg's assertion that measurable results do not depend on which wfn of a generalized ray is chosen to represent the state. Hartle and Taylor (1968) and Arons (1969), on the contrary, claimed that the cluster property, in Steinmann's argument, actually allows the distinction between wfns belonging to different generalized rays.

Leaving aside the question of whether these authors are technically right, we wish to remark that they argued in the spirit of Messiah and Greenberg's theory which involves some implicit assumptions that we have pointed out. Furthermore, their conclusions depended on their assignments of the encountered wfns to the generalized rays; an incorrect conclusion could have been drawn from an incorrect assignment. Because the indistinguishability postulate is a condition imposed on the physical observables, we believe that the question of whether parastatistics is compatible with the cluster property could be answered, more unambiguously, by examining whether the physical observables allowed by the cluster assumption satisfy the indistinguishability postulate, for parastatistics. Even if an answer could be arrived at in this way, we should still make sure, before excluding parastatistics, that the indistinguishability postulate is the only expression of the indistinguishability of identical particles. We call to mind that the indistinguishability postulate has been derived from the assumption that the state of paraparticles can be represented by a wfn. But, due to the nature of paraparticles, one might have to represent the state by a density matrix. Would the indistinguishability postulate be still the only solution to the indistinguishability condition in the density matrix formalism?

Now, what is the connection between the I.R. of S_N and the statistics of paraparticles? The work of Okayama (1952) suggests that all I.R. associated with Young diagrams of at most p columns yield the intermediate statistics. Other I.R. associated with Young diagrams of unlimited numbers of columns, although providing no new statistics other than B.E.

and F.D. statistics, possess physical properties different from the completely symmetric wfns Stolt and Taylor (1970) have classified the paraparticles into I.R. of S_N: the particles are called parafermions (parabosons) of order p if their N-body states are associated with Young diagrams of at most p columns (p rows), paraparticles of infinite order if they are neither parabosons nor parafermions. This scheme of classification, we conjecture, is acceptable if it satisfies the self-consistent condition that many systems of paraparticles of one type form again a system of paraparticles of the same type.

So far, we supposed that the number of particles in the system is fixed in time. Messiah and Greenberg (1964) have also formulated the theory for the case in which the particles can be created or destroyed. It consists in taking the Hilbert space of dynamical states of the system as the Fock space $\mathfrak{F}(H)$, which is the direct sum of the H^N, $N = 0, 1, 2... \infty$. However, it is more difficult to formulate the indistinguishability of identical particles in $\mathfrak{F}(H)$ because the permutation operations are not defined in the whole $\mathfrak{F}(H)$ but in each component H^N of $\mathfrak{F}(H)$. It is required that the observables in each subspaces H^N satisfy the indistinguishability postulate. As in the case of a fixed number of particles, we should explore the theory beyond Messiah and Greenberg's scheme which led them to the notion of generalized rays. In addition, the derivation of selection rule involving operators of the whole $\mathcal{F}(H)$ should be subjected to critical considerations.

We stop now our development of the quantum mechanical theory in order to discuss the quantum field theory of paraparticles, or the second quantization theory as it is often called (it is also called parafield theory).

In the second quantization theory, the system of identical particles is described by the field operators $\psi(x)$ and $\psi^*(x)$, and no particle variables need to be introduced. One does not have to use the permutation operators to impose the indistinguishability of particles for the particles have no identity in the second quantization theory. The statistics of the particles reflects in the method of quantization one adopts. For example quantization with the commutation relation yields B.E. statistics whereas quantization with the anti-commutation relation yields F.D. statistics. One would then tend to start from the first quantization theory of paraparticles and follow the familiar Fock method to obtain, for para-statistics, the commutation relations from the permutation symmetry character of the wfns. However, this method of field quantization is quite formidable for it requires the knowledge of the matrix representations of S_N corresponding to the paraparticles under consideration. Furthermore, Galindo and Yndurain (1963) have shown that such a method would give no commutation relations of finite order which allow us to write, in an "almost normal" form, the state vectors, obtained by applying a certain number of creation and annihilation operators to the vacuum (i.e. to arrange them so that only a fixed finite number of successive creation operators stand to the right of one destruction operator) or to express the particle permutation operator as a polynomial in certain suitable operators. We recall, in this connection, that Okayama (1952) obtained from the representations of S_N some commutation relations, which have been shown by Kamefuchi and Takahashi (1962) to be unacceptable for field theory.

Green (1953) was the first to obtain some commutation relations from the Lagrange equation of motion for free fields by a simple modification of the free Hamiltonian. The Green method could be considered as an inspiration from Wigner's observation (1950) that the Heisenberg equation of motion of a harmonic oscillator does not determine the commutation relations for the operators p and q uniquely, (Wigner's analysis has been extended by O'raifeartaigh and Ryan (1963) and by Boulware and Deser (1963). These authors showed that the essential non-uniqueness of the commutation relations arises from the possibility of employing Fermi or para-Fermi statistics). Green's commutation relations, in terms of creation and annihilation operators, are as follows

$$[[a_k^*, a_\ell]_\pm, a_m] = -2\delta_{km}\, a_\ell$$

$$[[a_k, a_\ell]_\pm, a_m] = 0$$

These commutation relations (also called paracommutation relations) turn out to be general enough to accept as solutions the commutation relations obtained later by Volkov (1959) and by Kamefuchi and Takahashi (1962). The (+) sign referred to parabose and the (-) to parafermi commutation relations. There exists a generalized Pauli theorem which states that tensor fields must be quantized with the parabose commutation relation and spinor fields with the parafermi commutation relation(Dell'Antonio, Greenberg and Sudarshan, 1964).

The quantum field theory constructed on the para-commutation relations is now known as Parafield Theory (for reviews, see Dresden (1963) and Greenberg, 1966). In this theory one could construct the S-matrix by the usual procedure (Volkov, 1959; McCarthy, 1955). Some applications of Parafield Theory to elementary particles have been made by Feshback (1963) and by Kamefuchi and Strahdee (1963) with the desire to relate the strangeness quantum number to para-statistics or to find out whether any of the known elementary particles can be described by parafields. However, the results obtained by various authors did not agree with one another. The reason for this was perhaps the lack of an acceptable method of constructing an I.R. of the paracommutation relations in a Hilbert space. Although Green (1953) had early provided with some ansatzes for computing the expectation values of functions of fields, it was not until 1965 that Messiah and

Greenberg published a proof that Green's ansatzes actually exhaust all possible I.R. of the paracommutation relations of physical interest. With the aid of Green's ansatzes they also derived the selection rules which imply that no known elementary particles, except perhaps the hypothetical quarks (Greenberg, 1964; Mitra, 1966), are paraparticles associated with the parafields. This result has been strengthened by a theorem due to Araki, Greenberg and Toll (1966). In view of the importance of Green's ansatzes, their significance should be confirmed and clarified. Also, as in the case of the commutation relation and the anti-commutation relation, the non-Fock representations of the para-commutation relations, if existing, should be exhibited.

Green's ansatzes state that, within the framework of Lagrangian Field Theory, any parabose (parafermi) field can be expressed as a sum of a finite number p of ordinary bose (fermi) fields which obey the anomalous commutation relations (some different bose fields may anti-commute and some different fermi fields may commute). It has been proved (Kinoshita, 1958; Araki, 1961) that a set of ordinary fields obeying anomalous commutation relations can be transformed to a set of fields obeying normal commutation relations by a set of Klein transformations. Although these Klein transformations are essentially non-unitary, a question may arise as to whether a parafermi (parabose) field is physically distinct from a corresponding fermi field (a fermi field of the same mass and spin as the parafield) as far as measurable quantities are concerned. It might be of some help in understanding the properties of ordinary fields to see whether the results of Wightman's axiomatic field theory, transfered to parafields, are useful for the proofs of the T.C.P. and spin-statistic theorems for parafields.

We have seen that Parafield Theory has been developed independently of the quantum mechanical theory of paraparticles and owes no result to that theory. Is there by any chance some correspondence between the two theories? Previously, we formulated the first quantization theory in the configuration space but it would be more appropriate for our present discussion to formulate it in the abstract Hilbert space H^N spanned by tensors of the form $|k_1\rangle|k_2\rangle\ldots|k_N\rangle$. The effect of a $U(\sigma)$ in H^N is to shift the single particle state $|k_i\rangle$ at the j^{th} place of the tensor product to the σj^{th} place whatever that single particle state is. It has been proved by Yamada (1968) that the $U(\sigma)$ do not define multi-dimensional I.R. of S_N in the second quantized Hilbert space H^N spanned by the tensors $a^*_{k_1} a^*_{k_2} \ldots a^*_{k_N} |0\rangle$. This means that one cannot establish a correspondence between the first and the second quantization theories of paraparticles in the same way as for ordinary bosons and fermions. For this reason, several authors (Landshoff and Stapp, 1967; Ohnuki and Kamefuchi, 1969; Hartle and Taylor, 1968; Stolt and Taylor, 1970; Ohnuki and Kamefuchi, 1971) have resorted to the use of the so-called particle permutation operators $V(\sigma)$ defined as follows: acting on a tensor product of the first or the second quantization theory, $V(\sigma)$ replaces a k_i by $k_{\sigma i}$ whatever the places of k_i and $k_{\sigma i}$ are. Assuming that the $V(\sigma)$ are well-defined in both first and second quantization theories, a correspondence between the two theories has been proposed by giving the same physical significance to the $V(\sigma)$ in both theories. However, the physical interpretation of the $V(\sigma)$ has not always been agreed on by different authors. For example Landshoff and Stapp (1967) argued on physical grounds that the $V(\sigma)$ have observable effects and proposed a theory of identical particles, known as the unified theory, in which every physical observable is a function of the $V(\sigma)$. On the other hand, Ohnuki and Kamefuchi (1971) asserted that the $V(\sigma)$ are not physical observables for systems of interacting particles. The trouble with the particle permutations is, we think, besides their physical interpretation, they are queer objects when acting on an N-particle state, in which some of the k_i are equal: the statement "replace k_i by $k_{\sigma i}$" cannot be carried out uniquely when some of the k_i are equal. We would also wonder whether, in this case, the $V(\sigma)$ can define multi-dimensional representations of S_N. Since the works connected with the $V(\sigma)$ made use heavily of the group-theoretical properties of the $V(\sigma)$, it would be serious if the $V(\sigma)$ do not always define representations of S_N either in the first or in the second quantized Hilbert space.

The organization of this thesis is as follows:

In Chapter II, after stating what are known as the basic assumptions in quantum mechanics of identical particles, we discuss the nature of the symmetrization postulate following the works of Dirac (1930), of Jauch and Misra, of Galindo, Marales and Nuñez-Lagos and of Messiah and Greenberg.

In Chapter III, we give a precise mathematical formulation of the theory of N-identical particles in the language of group algebra of S_N. Special attention is given to the calculation of matrix elements of a physical observable and the preparation of paraparticle states. We try to exploit the theory with as few assumptions as we think they are reasonably allowed.

In Chapter IV, we discuss the connection between the I.R.'s of S_N and the statistics. In particular, we generalize the proof of a theorem due to Okayama and we check the self-consistency condition of Stolt and Taylor's classification of paraparticles.

In Chapter V, we study the cluster property as applied to the quantum mechanical theory of paraparticles. This chapter is self-contained and could be considered as an illustration of the theory developed in Chapter III with special consideration given to a 3-particle system.

In Chapter VI, we generalize the theory to a system with a variable number of particles. We give a critical comment on Messiah and Greenberg's selection rule and derive a selection rule which, we think, may explain the observed fact that electrons (photons), for example, are always fermions (bosons).

In Chapter VII, we review various schemes of second quantization leading to parastatistics. We first illustrate the problem by the example of an harmonic oscillator and then concentrate our attention on Green's and Kamefuchi and Takahashi's methods.

In Chapter VIII, we study the representations of the paracommutation relations. We exhibit the discrete (Fock or non-Fock) representations following the method of Wightman and Schweber (1955), we give a mathematical picture of Green's ansatzes and elaborate a proof of Messiah and Greenberg theorem concerning these ansatzes . Finally, as a verification of this theorem, we show the existance of parastatistics in any Fock representations.

In Chapter IX, we discuss some topics of parafields in the framework of Lagrangian theory and axiomatic theory. In particular, we discuss, with the aid of Klein's transformations, the T.C.P. theorem and spin-statistics theorem and the question of whether parafields are physically distinct from the ordinary fields.

In Chapter X, we study the correspondence between the first and the second quantization theories. We give a proof (which is similar to but somewhat simpler than Yamada's) that the U(σ) do not define multi-dimensional I.R. of

S_N in the second quantization theory and we study, in some detail, the mathematical nature of the particle permutation operators.

In Chapter XI, we summarize our results and their significance.

CHAPTER II

SYMMETRIZATION POSTULATE

1. Basic Assumptions

Consider a system of N identical particles of a certain species. Particles of other species may be present and can be viewed as some external interactions acting on this system. We shall suppose that the number N of particles is fixed in time.

The fact that the particles of our system are identical implies that no observation can distinguish them from one another, both in classical and in quantum theory. Yet there is a great difference in the classical and quantum description of such a system. In classical mechanics, one describes motion by specifying the orbits of the individual particles, and although they are indistinguishable from one another, if the initial conditions have been set, it makes perfect sense to say, at a given time, the particle 1 is moving along orbit A, particle 2 along orbit B, and so on. In quantum theory, however, instead of specifying orbits, one only can attribute wavefunctions to individual particles. The wavefunctions generally overlap in space so that one can no longer identify the particles by identifying the wavefunctions which they occupy. A quantum theory of identical particles must be such that the identity of the particles is completely irrelevant.

The starting point of the quantum theory of N identical particles is that of N distinguishable particles. Each particle in the system may be considered as a dynamical system by itself.

To treat all the particles on the same footing, one assumes that the Hilbert spaces of dynamical states of the particles, each considered as a dynamical system by itself, are all identical to a Hilbert space H spanned by a certain orthogonal basis $\{|k_i>\}$; i runs over discrete and continuous indices. This is of course a bold assumption since nothing guarantees that each particle is subjected to the same physical condition.

Consider now the Hilbert space H^N spanned by all the tensor products of the form

$$|K> = |k_{i_1}>|k_{i_2}>\ldots|k_{i_N}> \qquad (1)$$

(The letter K stands for the set $\{|k_{i_1}>, |k_{i_2}>, \ldots |k_{i_N}>\}$ of N single particle states.) For many statistical descriptions of an N-particle system, one assumes that H^N is an appropriate approximate Hilbert space of dynamical states of N-interacting particles. In other words, the basis given by (1) can be used as a complete set for dynamical description of N interacting particles.

Let x_1 be the set of dynamical variables characterizing the particle 1, x_2 characterizing the particle 2, etc. The configuration space for N particles is the tensor product space of tensors of the form

$$|x_1, x_2, \ldots x_N> = |x_1>|x_2>\cdots|x_N>$$

The configuration space wavefunction corresponding to the state $|K>$ given by (1) is

$$K(x_1, x_2, \ldots x_N) = <x_1, x_2, \ldots x_N|K>$$
$$= <x_1|k_{i_1}><x_2|k_{i_2}> \ldots <x_N|k_{i_N}> \qquad (2)$$

This interprets the fact that $|K\rangle$ is the state in which particle 1 occupies $|k_{i_1}\rangle$, particle 2 occupies $|k_{i_2}\rangle$ and so on. Thus, in the tensor product $|K\rangle$, the single particle state which appears at the j^{th} position can be understood as occupied by the particle j. With this convention, the position of a single particle state in the tensor product is important. Following Yamada (1968) we write $|K\rangle$ of eqn. (1) as

$$|K\rangle = \left| \begin{matrix} 1 & 2 & \ldots & N \\ k_{i_1} & k_{i_2} & \ldots & k_{i_N} \end{matrix} \right\rangle \tag{3}$$

where the figures 1,2,...N in the first line of the bracket simply indicate the places in which the $|k_{i_1}\rangle$, $|k_{i_2}\rangle \ldots |k_{i_N}\rangle$ are situated, and only the combinations of place 1 and the state $|k_{i_1}\rangle$, of 2 and $|k_{i_2}\rangle$, etc. are significant on the right hand side of (3), so that the absolute positions of the pairs $(1,k_{i_1})$, $(2,k_{i_2})$ ... and (N,k_{i_N}) do not matter at all. A state like $|K\rangle$ will be called an N-particle state specified by giving N pairs (j,k_i), j=1, 2,...N, and i takes N values.

2. <u>Place Permutation Operators</u>

To take into account of the indistinguishability of the particles, it is convenient to introduce the operators representing the permutations of the particles over the single particle states.

For each permutation σ of the numbers 1, 2,...N, we define a <u>place permutation operator</u> $U(\sigma)$ as

$$U(\sigma) \left| \begin{matrix} 1 & 2 \ldots & N \\ k_{i_1} & k_{i_2} \ldots & k_{i_N} \end{matrix} \right\rangle = \left| \begin{matrix} \sigma^{-1} 1 & \sigma^{-1} 2 & \ldots & \sigma^{-1} N \\ k_{i_1} & k_{i_2} & \ldots & k_{i_N} \end{matrix} \right\rangle \tag{4}$$

For example

$$U(23)|k_3>|k_2>|k_1> = U(23) \left| \begin{matrix} 1 & 2 & 3 \\ k_3 & k_2 & k_1 \end{matrix} \right\rangle$$

$$= \left| \begin{matrix} 1 & 3 & 2 \\ k_3 & k_2 & k_1 \end{matrix} \right\rangle$$

$$= \left| \begin{matrix} 1 & 2 & 3 \\ k_3 & k_1 & k_2 \end{matrix} \right\rangle = |k_3>|k_1>|k_2>$$

Thus, a place permutation operator shifts the single particle state at the j^{th} position to the $\sigma^{-1} j^{th}$ position whatever the single particle state is.

All the permutations σ form a group, called the symmetric group S_N. The operators $U(\sigma)$ obviously obey the same composition law as the permutations σ; they are said to define a representation of the group S_N. This representation is reducible in the sense that we can decompose H^N into a direct sum of subspaces which are irreducible with respect to the $U(\sigma)$. The rule for decomposing H^N into irreducible subspaces with respect to the $U(\sigma)$ is presented in Appendix A and is applied for classification of states of identical particles in Chapter III.

Other kinds of permutation operators can be introduced and will be discussed in Chapter X.

3. Indistinguishability Postulate

Suppose that $|\psi\rangle$ is a state of N-identical particles. The state $U(\sigma)|\psi\rangle$ differs from $|\psi\rangle$ by a permutation of the particles over the single particle states. Since the particles are indistinguishable from one another, none of the dynamical properties of the system is modified by such a permutation. This is the indistinguishability postulate which can be expressed as follows:

"Dynamical states represented by vectors which differ only by a permutation cannot be distinguished by any observation at any time."

If A represents a physical observable of the system of N identical particles, one requires

$$\langle\psi|A|\psi\rangle = \langle\psi|U^{-1}(\sigma)A\ U(\sigma)|\psi\rangle$$

for any $\sigma \in S_N$. From this it follows that all observables must commute with the $U(\sigma)$.

$$[A, U(\sigma)] = 0 \tag{5}$$

Eqn. (5) is sometimes referred to as the indistinguishability postulate.

4. Symmetrization Postulate

The indistinguishability postulate imposes invariance of all observables under the $U(\sigma)$. From this invariance property, important deductions can be made concerning the law of motions and the dynamical states of the system. Here we shall discuss the connection between the indistinguishability postulate and the symmetrization postulate (S.P.), which states that the wavefunctions describing identical particles must be either symmetric or anti-symmetric in all particle variables.

a) Finite dimensional case

As a first application of the indistinguishability postulate, the Hamiltonian H of the system must be permutation invariant

$$[U(\sigma), H] = 0, \quad \sigma \varepsilon S_N \tag{6}$$

From this equation and the equation of motion

$$i \frac{d}{dt} G(t,t_o) = HG(t,t_o)$$
$$G(t_o,t_o) = 1$$

where $G(t,t_o)$ is the evolution operator, one can show easily that $G(t,t_o)$ is also permutation invariant,

$$[U(\sigma), G(t,t_o)] = 0 \tag{7}$$

Now, all states $U(\sigma)|\psi\rangle$'s are degenerate with respect to all observables (exchange degeneracy), the state of the system is in general a linear combination of the $U(\sigma)|\psi\rangle$'s,

$$|\psi^\mu\rangle = \sum_\sigma \alpha_\sigma^\mu \, U(\sigma)|\psi\rangle \tag{5'}$$

where μ indicates the special linear combination. We are interested in linear combinations such that

$$\langle\psi^\mu|A|\psi^\mu\rangle = \langle\psi^\mu|U^+(\sigma)AU(\sigma)|\psi^\mu\rangle$$

for any observable and for any $\sigma\varepsilon S_N$. The apparatus of group theory enables us to find exhaustively these linear combinations. In section III.1, we shall write them out explicitly. Here, we are content to remark that the states of identical particles can be classified into irreducible representations of S_N.

To go further, one assumes there exists a maximal observation for the system. In a general way, a maximal observation would be such that other observation would either yield no new information or the new information would void some of the information previously obtained. If the Hilbert space of dynamical states is finite dimensional, the work of Dirac (1932) suggests that a maximal observation would be appropriately described by a complete set of commuting operators. There exists a non-degenerate ray which is the simultaneous eigenstate of all members of the complete set.

For identical particles, the condition of non-degeneracy is expressed by

$$U(\sigma)\,|\psi^{\mu}(K)\rangle = c_{\sigma}\,|\psi^{\mu}(K)\rangle$$

where

$$|c_{\sigma}|^{2} = 1$$

It is clear that states satisfying this must belong to one-dimensional representations of S_N. There are two such states: the completely symmetric state $|\psi^{S}(K)\rangle$ belonging to the identity representation and the completely anti-symmetric state $|\psi^{A}(K)\rangle$ belonging to the alternating representation. Explicitly,

$$|\psi^{S}\rangle = \frac{1}{\sqrt{N!}} \sum_{\sigma} U(\sigma)\,|\psi\rangle \tag{8}$$

$$|\psi^{A}\rangle = \frac{1}{\sqrt{N!}} \sum_{\sigma} \delta(\sigma)\; U(\sigma)\,|\psi\rangle \tag{8'}$$

where $\delta(\sigma)$ is the signature of σ which equals to 1 for an even permutation and -1 for an odd permutation.

Because the evolution operator is permutation invariant (equation (7)), the permutation symmetry property is conserved in time; no transition from a symmetric state to an anti-symmetric state is allowed and vice versa.

This selection rule applies for any observables; no observable has non-vanishing matrix elements between an anti-symmetric states and a symmetric state. This is the situation in which one expects that the selection rule becomes a super-selection rule; the states represented by coherent superposition of symmetric state vectors and anti-symmetric state vectors are not physically realizable. Thus, we state, "The states of a system containing identical particles are necessarily either all symmetric or anti-symmetric with respect to any permutation of the particles."

It is a consequence of (8) and (8') that particles whose N-body states are of the type (8) obey Bose-Einstein statistics and particles whose N-body states are of the type (8') obey Fermi-Dirac statistics. The symmetrization postulate is thus the postulate that only Bose-Einstein and Fermi-Dirac statistics are obeyed by identical particles.

b) General Case

In the previous section, we have seen how the symmetrization postulate is connected with the indistinguishability postulate and the assumption that there exists a maximal observation, in the finite dimensional case. In more important infinite dimensional case, Dirac's definition of states corresponding to maximal observation as simultaneous eigenfunctions of a complete set of commuting observables is not suitable because operators may possess continuous spectrum then eigenvectors do not exist in a Hilbert space. In this section, we shall discuss the generalization to infinite dimensional case using the results obtained by Jauch (1960)* and Jauch and Misra (1961).

(i) Superselection rules

In the general case, one considers the set $A = \{A_i\}$ of all bounded observables in a Hilbert space H. Observables are in general non-bounded, yet the restriction to bounded operators involves no loss of generality. This is so because observables are represented by self-adjoint operators for which there exist unique spectral resolutions and we can always use the spectral projectors, which are of course bounded, to replace operators.

It is usually assumed that A is irreduable in H. Yet as Wick, Wightman and Wigner (1952) point out, there are important physical systems in which this is not the case; H may be decomposed into certain orthogonal subspaces such that (i) the relative phase of the components of a state vector along these subspaces are intrinsically irrelevant and (ii) the matrix elements of all observables between these subspaces vanish. In such a case A is reducible in H and there exists

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* A more rigourous discussion has been given by Galindo, Morales and Nunez-Lagos (1962).

superselection rule; states represented by linear superposition of vectors belonging to different subspaces are not physically realizable. The subspaces are then called superselection sectors. For example, a state which is a superposition of states with different charge Q has never been observed. It also seems that every physically realizable state must be an eigenstate of B, the baryon number and $(-1)^F$, where F is an even integer for states of integer spin and an odd integer for states of half odd integer spin.

In a natural attempt to provide the simplest substitution for irreducibility, Wightman (1959) makes the hypothesis that the commutant[†] A' of A is commutative (that is sometimes called the hypothesis of commutative superselection rules). Then H breaks up into a direct sum (actually direct integral) of subspaces in which the relative phase of vectors in the different subspaces cannot be measured.

Jauch (1960), on the other hand, proposes to study the Von Neumann algebra O generated by A, defined as the double commutant of A,

$$O = A'' = (A')' \qquad (12)$$

It is the smallest Von Newman algebra containing A,

$$A \subset O \qquad (13)$$

An important property of a Von Neumann algebra is that it is identical with its double commutant.

$$O'' = O \qquad (14)$$

[†] By definition, the commutant A' is the set of all bounded operators which commute with all members of A.

Much of the structure of the physical system is already contained in the structure of the algebra O. For instance it is an essential feature of quantum theory that O is not a commutative algebra. In the finite dimensional case, as it is easily checked, the Von Newman algebra P generated by the observables $\{A_i\}$ coincides with the algebra of polynomials of a suitably chosen operator A. If $\{A_i\}$ is a complete set of commuting observables then it can be shown that (Jauch, 1960)

(i) P is maximal abelian, i.e.

$$P' = P$$

(ii) There exists a generating element g such that every element f in H can be represented by $f = P(A)g$ with some polynomial $P(A)$. In this case the operator A is said to have simple spectrum. Jauch (1960) transcripts this definition to the infinite dimensional case; a self-adjoint operator A is said to have simple spectrum if the linear manifold $\{Ng\}$ is dense in H, where N is the Von Neumann algebra generated by A.

Now the set of commuting observables generates a commutative agebra $M \subset O$. Jauch (1960) shows that M is maximal abelian, i.e. $M' = M$, if and only if the operators in M are generated by a single observable with simple spectrum. Then it is said that M is generated by a complete set of commuting observable in analogy with the finite dimensional case. Thus, the notion of a maximal observation finds its mathematical expression, in terms of maximal abelian algebra.

After these preliminaries about Von Neumann algebra out of the way, one can proceed to find what kind of structure of O would yield the superselection rules. The center Z of O will play an important role. It is defined as the set of elements in O which commute with all elements of O,

$$Z = O \cap O' \tag{14}$$

Consistent with the assumption that there exists a maximal observation, we assume there exists at least a maximal abelian algebra M contained in O,

$$M \subset O \tag{15}$$

From (15) it follows that

$$O' \subset M'$$

and since

$$M' = M$$

we have

$$O' \subset M \subset O \tag{16}$$

This says that O' is commutative and that the center Z coincides with O',

$$Z = O' \tag{17}$$

Since it can be shown that O' is a Von Newman algebra, Z is also a Von Neumann algebra. From the theory of the direct integral of Hilbert spaces*, Z determines a decomposition of

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* The theory of direct integral is due to Von Neumann (1949). It is a generalization of the concept of direct sum. The theorem which is used here is "every weakly closed commutative *-algebra

H into a direct integral and O is decomposed into invariant subalgebras, each acting irreducibly in the subspaces of H, if and only if the center Z is maximal abelian in O'. This is indeed the case because Z = O'. One concludes:

"The Hilbert space H can be decomposed into superselectors if there exists at least a complete set of commuting observables."

We remark that Wightman's hypothesis of commutative superselection rules can be proved if there exists a complete set of commuting observables. Loosely speaking, in this case all operators in A' are functions of operators of this complete set, hence they commute with one another.† Thus, Wightman's hypothesis has a direct physical interpretation.

(ii) Supersymmetries and Essential Observables

We now discuss, following Jauch and Misra (1961), the concept of supersymmetry. A unitary operator U is a supersymmetry if it is a non trivial operator which commutes with all members of the set A of all observables. That is

$$U \in A'$$

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* of bounded operators in H determine uniquely a decomposition of H into a direct integral."

† The equivalence between Wightman's hypothesis and Jauch's assumption that O has a maximal abelian subalgebra is rigourously proved by Galindo, Morales and Nunez-Lagos (1962).

Recalling that $\underline{A}'$ is a Von Neumann algebra, we have

$$O' = A''' = (A')'' = A'$$

therefore

$$U \in O' = Z \qquad (18)$$

One can show the converse of (18) ; if O' is non trivial, there must exist at least one unitary operator $U \in O'$. This follows from the property that every Von Neumann algebra can be generated by its unitary operators (Von Neumann, 1949).

Thus, a quantum mechanical system has supersymmetry if and only if the algebra O is reducible, that is, there exist superselection rules.

The supersymmetries are connected with essential observables. They are defined as the intersection of all complete sets of commuting observables. The essential observables generate a Von Neumann, algebra which is called the core C of O,

$$C = \bigcap_i M_i \qquad (19)$$

where $\{M_i\}$ is the set of all maximal abelian algebras contained in O. Jauch and Misra show that the core C is actually identical with the center Z of O,

$$C = Z = O' \qquad (20)$$

This says that observables in the center are essential observables.

One concludes: Superselection rules are connected with supersymmetries and essential observables.

(iii) Symmetrization Postulate

We are now in a position to relate the assumption that there exists a maximal observation with the symmetrization postulate.

For identical particle systems, the indistinguishability postulate requires that H carries a representation $U(\sigma)$ of the symmetric group S_N. The $U(\sigma)$'s commute with all observables. They are therefore supersymmetry of the system. According to (18), the $U(\sigma)$'s are contained in the center Z, hence they commute with one another,

$$U(\sigma)\ U(\sigma') = U(\sigma')\ U(\sigma) \tag{21}$$

for any $\sigma, \sigma' \in S_N$. Since only the identity and the alternating representations have this property, the vectors in H must belong to these representations.

Now, some self-adjoint operator in Z may not be observables because O is larger than the set of all observables. It may happen that the $U(\sigma)$'s are self adjoint operators in H (it turns out that this is the case), but they are not necessarily observables. However, some self-adjoint functions of the $U(\sigma)$'s may be observables. If we assume that the projectors onto symmetric states and anti-symmetric states,

$$S = \frac{1}{\sqrt{N!}} \sum_{\sigma} U(\sigma) \tag{22}$$

$$A = \frac{1}{\sqrt{N!}} \sum_{\sigma}{}' \delta(\sigma)\ U(\sigma) \tag{23}$$

are observables, then they are essential observables because they are elements of Z. They commute with all other observables so that one may call them superselection operators. States which are different eigenvectors of the superselection operators belong to different superselection sectors. Thus, states which are superpositions of symmetric and anti-symmetric states are not physically realizable.

In conclusion: The existance of superselection rules is consistent with the existance of at least one complete set of commuting observables for the system, the latter implies the summetrization postulate.

5. The Nature of the Symmetrization Postulate

a) Theoretical Foundation of the Symmetrization Postulate (S.P.)

Previously, we derived S.P. without much question about the assumptions involved. Here, we shall offer some critical comments.

(i) In the derivation of S.P., the notion of a complete set of commuting observables plays an important role. For one-particle system, such a set can be determined as consisting of position, spin and internal quantum numbers. For simple but more complicated systems, it is very difficult in general to specify what set of operators can serve as a complete set. For many particle systems, no set of operators is known to satisfy Dirac's requirement.

(ii) Strictly speaking, all experiments in elementary particle physics are collision experiments consisting of a set of one-body measurements. By a one-body measurement, it is meant a measurement performed on individual, widely separated, non-interacting particles. Quantities expressing the correlation effects between the particles cannot be measured directly. Informations about these quantities can be obtained only through the study of correlation between several sets of one-body measurements (e.g. angular correlation experiment), and the determination through such studies always requires some assumptions about the dynamical properties of the observed systems. Thus, in general, a set of one-body measurements, no matter how complete, does not yield a maximal observation. This has been discussed in details by Messiah and Greenberg (1964).

(iii) The physical meaning of the algebra generated by the observable has not been clarified. For example, one has to answer the question posed by Wigner as to what experiment corresponds to the operator $p+q$ (p: momentum, q: coordinate) Accepting that the mathematical technique employed previously is appropriate and that the assumption about maximal observation is reasonable, we still have to justify the following additional point. Previously, an operator A is said to have simple spectrum if the linear manifold {Ng} is dense in H, where N is the Von Neumann algebra, generated by A. Now, the linear manifold H' = {Of}, where f = $\{f_i\}$ is the set of all physically realizable state vectors, is a subspace of H. Physically, one would require {Ng} to be dense in H' instead of H. Then one cannot show that M is maximal abelian. In this case it does not follow that O' is commutative nor does the symmetrization postulate follow.

b) Experimental Foundation of the Symmetrization Postulate

It is clear that the theoretical foundation of the symmetrization postulate (S.P.) is of an ad hoc nature. The validity of S.P. should be tested by experiment. Historically, S.P. was introduced as a consistent way to account for the exclusion principle postulated by Pauli in 1925 as an explanation of the periodic table of chemical elements. Thus, electrons obey Fermi-Dirac statistics. From the study of black body radiations and the success of quantum electrodynamics, one concludes that photons are bosons. For nucleons, although the interaction between them have not been fully understood, their statistics reflect in the forbidden lines in the rotational spectra of homonuclear diatomic molecules since the lines do not depend on the details of nuclear forces. It can be shown* that, for such a molecule with spin I, the intensities of a line of odd angular momentum to that of the next line of even angular momentum is $\frac{I}{I+1}$ if the nuclei are fermions ($\frac{I+1}{I}$ if bosons). It turns out that the nucleons are fermions.

Since S.P. is well established for electrons, photon and nucleons, one tends to apply it to other particles without too much question. In fact, with present experimental techniques, one cannot determine the statistics of elementary particles unambiguously because, with a few exceptions, one cannot obtain large numbers of particles. In assigning the spin and parity

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* See for example Elton (1965), section 2.5.

to elementary particles, one usually takes for granted the usual spin statistics theorem. This theorem is again of an ad hoc nature because, to prove it[†], one excludes a priori other statistics different from Bose-Einstein and Fermi-Dirac statistics.

Tests of S.P. are proposed by Messiah and Greerberg. Since in practice, except for pions, it is very hard to produce systems containing more than two identical particles, they propose to test S.P. by looking for the S.P. violating terms in the state of two identical particles. The S.P. violating terms would contain incoherently both symmetric and anti-symmetric states. They found no direct evidence for the statistics of the particles K, Λ, Σ, $\underline{>}$ and μ.

In conclusion, we say that so far we find no compelling reason to suppose all particles obey S.P.

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† See for example Streater and Wightman (1964) section 4.4.

CHAPTER III

QUANTUM MECHANICS OF PARAPARTICLES

1. Classification of States of Identical Particles

We have stated at the beginning of Chapter II that one of our basic assumptions is that states of an N-identical particle system are defined in H^N. In this section we study a classification of these states.

It has already been remarked that an N-identical particle state is a linear combination of vectors of the form (eqn. II-5′):

$$|\psi[K]\rangle = \Sigma\, a(\sigma)\ U(\sigma)\ |K\rangle$$

where $|K\rangle$ is a tensor product

$$|K\rangle = |k_{i_1}\rangle|k_{i_2}\rangle \ldots\ |k_{i_N}\rangle$$

Suppose that the base $\{|k_i\rangle\}$ of the single particle Hilbert space H can be ordered in some way, then it is easy to see that H^N can be spanned by the base $\{U(\sigma)|K\rangle_s:\ |K\rangle_s \in H^N,\ \sigma \in S_N\}$ where $|K\rangle_s$ is given by the ordering

$$|k_{i_N}\rangle \geq |k_{i_{N-1}}\rangle \geq \ldots \geq |k_{i_2}\rangle \geq |k_{i_1}\rangle$$

It will be convenient to consider the group algebra $\mathfrak{U}$ which accept the $U(\sigma)$ as a base, i.e., the elements of $\mathfrak{U}$ are of the form

$$X = \sum_{\sigma} a(\sigma)\ U(\sigma)$$

The structure of the group algebra $\mathfrak{U}$ and its relation with the representations of S_N are summarized in Appendix A. The main features are:

$\mathfrak{U}$ is reducible and can be decomposed as

$$\mathfrak{U} = \bigoplus_{\mu} \mathfrak{U}^{\mu}$$

where the $\mathfrak{U}^{\mu}$ are the simple two-sided ideals. $\mathfrak{U}^{\mu}$ contains n_{μ} independent left ideals

$$\mathfrak{U}^{\mu} = \sum_{j=1}^{n_{\mu}} u_j^{\mu}$$

While the $\mathfrak{U}^{\mu}$ can be chosen uniquely, the choice of the u_j^{μ} is not unique: the decomposition of a two-sided ideal into n_{μ} independent left ideals is not unique. A possible choice of the left ideals is given by the idempotent ε_j^{μ} defined as

$$\varepsilon_j^{\mu} = \left(\frac{n_{\mu}}{N!}\right)^{\frac{1}{2}} P_j^{\mu} Q_j^{\mu}$$

where P_j^{μ} is a symmetrizer and Q_j^{μ} is the anti-symmetrizer of a standard Young tableau j of a Young daigram μ. The unit element U(e) of $\mathfrak{U}$ is decomposed into series of idempotents as

$$U(e) = \sum_{\mu} \varepsilon^{\mu}$$

$$\varepsilon^{\mu} = \sum_{j=1}^{n_{\mu}} \varepsilon_j^{\mu}$$

$\mathfrak{U}^{\mu}$ is obtained by left or right multiplications of every element of $\mathfrak{U}$ with ε^{μ}, u_j^{μ} is obtained by left multiplication with ε_j^{μ}. I.e. left multiplication with ε_j^{μ} is a projection into u_j^{μ}.

A state of the system can be written as

$$|\psi\rangle = \int dK\ f[K]\ X[K]\ |K\rangle_s$$

where $dK = dk_1\ dk_2\ \ldots\ dk_N$ is an integration (measure), $f[K]$ is a scalar function of the set $K = \{|k_{i_1}\rangle, |k_{i_2}\rangle, \ldots |k_{i_N}\rangle\}$ and $X[K]$ is an element of $\mathfrak{U}$, also a function of K. If every $X[K]$ of $|\psi\rangle$ belongs to a two-sided ideal $\mathfrak{I}^{\mu}$, the state $|\psi\rangle$ (then written as $|\psi^{\mu}\rangle$) will be said to be of μ-symmetry type. It can be shown that states of different symmetry types are orthogonal:

Let us define

$$U^{+}(\sigma) = U(\sigma^{-1})$$

so that $X^{+} = \Sigma\ \bar{a}\ (\sigma)\ U^{+}(\sigma)$, where $\bar{a}\ (\sigma)$ is the complex conjugate of $a(\sigma)$, then

$$X^{\mu^{+}} = (\varepsilon^{\mu}\ X)^{+} = X^{+}\ \varepsilon^{\mu^{+}}\ \epsilon\ \mathfrak{U}$$

Furthermore, it can be verified that $\varepsilon^{\mu^{+}} = \varepsilon^{\mu}$, hence $X^{\mu^{+}}\ \varepsilon\ \mathfrak{I}^{\mu}$ because $X^{\mu^{+}} = X^{+}\ \varepsilon^{\mu}$ and $\mathfrak{I}^{\mu}$ is a two-sided ideal. As a result, we have

$$X^{\mu^{+}}\ X^{\nu} = X^{\nu^{+}}\ \ X^{\mu} = 0,\ \mu \neq \nu \qquad (1)$$

Thus, we have

$$\langle\psi^{\mu}|\psi^{\nu}\rangle = \int dK\ dK'\ \bar{f}[K]\ f[K']\ {}_s\langle K|X^{\mu^{+}}[K]\ X^{\nu}[K']|K\rangle_s$$
$$= 0,\ \mu \neq \nu$$

upon using eqn. (1).

H^N is thus decomposable into direct sum of orthogonal subspaces of definite symmetry types as*

$$H^N = \bigoplus H^N_\mu \tag{2}$$

Each H^N_μ is invariant with respect to the $U(\sigma)$ but contains a finite number of irreducible subspaces. The projection operators onto the subspaces H^N_μ are the idempotents ε^μ generating the two-sided ideals of $\mathcal{U}$. Note, however, that the idempotents ε^μ_j generating the left ideals are not projection operators onto irreducible subspaces of H^N_μ.

An important point is that, with respect to the scalar product of the Hilbert space H^N, states belonging to different irreducible subspaces of a H^N_μ are not necessarily orthogonal. To see this, let us consider the scalar product of a state $X^\mu_j|\psi\rangle$ and $X^\mu_h|\phi\rangle$, where $X^\mu_i \in u^\mu_j$, $X^\mu_h \in u^\mu_h$ and $|\phi\rangle$, $|\psi\rangle \in H^N$. If $\langle\psi|X^\mu_i\, X^\mu_h|\phi\rangle$ vanishes for all $|\psi\rangle$ and $|\phi\rangle$ and for all X^μ_i, X^μ_h, we must have

$$X^{\mu^+}_i \; X_h = 0,\; i \neq h$$

for all X^μ_i, X^μ_h. In particular, we must have

$$X^{\mu^+}_i \varepsilon^\mu_h = 0, \text{ for all } X^\mu_i \in u^\mu_i \tag{3}$$

Since $X^{\mu^+}_i = \varepsilon^{\mu^+}_i X^+$, $X \in \mathcal{U}$ and since $\varepsilon^{\mu^+}_i$ is also an idempotent, the set r of all $X^{\mu^+}_i$ forms a right ideal. However, because ε^μ_h is the idempotent generating the left ideal u^μ_h, eqn. (3)

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* In this thesis, $\bigoplus$ denotes a direct sum of orthogonal subspaces, + denotes a direct sum of non-orthogonal subspaces.

implies that r is a left ideal* different from u_h^μ. Therefore, r is a two-sided ideal so that μ_i^μ is a two-sided ideal in contradiction with the hypothesis that it is a primitive left ideal.

In the sequel, we shall often concentrate our attention on a subspace H[K] spanned by the $U(\sigma)|K>_s$. H[K] can be decomposed as

$$H[K] = \bigoplus_\mu H^\mu[K] \tag{2^1}$$

$H^\mu[K]$ can be decomposed further into direct sum of irreducible subspaces with respect to the $U(\sigma)$,

$$H^\mu[K] = H_1^\mu[K] + \ldots + H_j^\mu[K] + \ldots + H_{n_\mu}^\mu[K]. \tag{2^2}$$

The above analysis implies that we cannot choose the left ideals such that the $H_j^\mu[K]$ are all mutually orthogonal for all K. We note that, when some of the single particle states in $|K>$ are equal, some of the subspaces $H_j^\mu[K]$ may vanish.

Although the $H_j^\mu[K]$ are not in general orthogonal, it is always possible to choose an orthogonal basis $\{\mu_j^i[K]\}$, for each $H_j^\mu[K]$, i.e., it is possible that

$$<\mu_j^i[K]\ \mu_j^{i'}[K]> = \delta_{ii'} \tag{4}$$

Since the representations of $U(\sigma)$ in different $H_j^\mu[K]$ are equivalent, we can choose the base $|\mu_j^i[K]>$ such that it satisfies (4) and such that the $H_j^\mu[K]$, for each μ, support identically

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* Remember that left multiplication with ε_h^μ is a projection onto u_h^μ.

the same irreducible representation of S_N, i.e., such that

$$U(\sigma)\,|\mu_j^i[K]> \;=\; \sum_{k=1}^{n_\mu} D_{ik}^{\mu}(\sigma)\,|\mu_j^k[K]> \qquad (5)$$

The scalar product of two vectors, $|\mu> = \sum_{i,j} \mu_j^i |\mu_j^i[K]>$ and $|\nu> = \sum \nu_j^i |\mu_j^i[K]>$, is given by

$$<\mu|\nu> = \sum_{i,j} \bar{\mu}_j^i \, \rho_{j\,\ell}^{i\,k} \, \nu_\ell^k$$

where

$$\rho_{j\,\ell}^{i\,k} = <\mu_j^i[K]|\mu_\ell^k[K]>$$

and the matrix $\rho = \{\rho_{j\,\ell}^{i\,k}\}$ is not completely diagonal. The result obtained previously implies that it is impossible to find a matrix T such that $T\rho T^{-1}$ is completely diagonal and $TU(\sigma)\,T^{-1}$ is represented by a direct sum of equivalent matrices, for all $\sigma \in S_N$.

In H[K], there exist a natural scalar product defined as

$$(\mu,\nu) = \sum_{i,j} \bar{\mu}_j^i \, \nu_j^i$$

This scalar product is related to the scalar product of the Hilbert space H^N by the relation

$$<\mu|\sigma> = (\mu,\rho\nu)$$

It is the scalar product $<\mu|\sigma>$ but not (μ,ν), which determines dynamical properties of the system. This point seems to escape from being emphasized in the literature.

From (2), it is convenient to classify identical particle states according to their symmetry types. Each symmetry type is associated with a representation of S_N which contains, in maximum, a number of equivalent irreducible representations equal to the dimension of the irreducible representations.

States of symmetry types associated with the one-dimensional representations describes the bosons or the fermions. We shall call para-particles those whose N-particle states are associated with multi-dimensional irreducible representations of S_N.

2. Matrix elements of Observables. A Superselection Rule

a) A Remark

Consdier the matrix element, between a state $|\psi^{\mu}[K]> \varepsilon H^{\mu}[K]$ and a state $|\psi^{\nu}[K]> \varepsilon H^{\nu}[K]$, of an observable A satisfying the indistinguishability of postulate

$$<\psi^{\mu}[K]|A|\psi^{\nu}[K]> = <\psi^{\mu}[K]|U^{+}(\sigma)\ A\ U(\sigma)|\psi^{\nu}[K]> \tag{6}$$

If, for some reason* we omit all but one equivalent irreducible subspaces of $H^{\mu}[K]$ and $H^{\nu}[K]$ in our dynamical description of the system, $|\psi^{\mu}[K]$ belongs definitely to an irreducible subspace and so does $|\psi^{\nu}[K]$, eqn. (6) can be transformed into a matrix equation.

$$D^{\mu}(\sigma)\ A = A\ D^{\nu}(\sigma)$$

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* One reason is the hypothesis that the equivalent irreducible subspaces are physically indistinguishable.

where $D^{\mu}(\sigma) = \{D^{\mu}_{ij}(\sigma)\}$ is the irreducible matrix representation of σ. Then Schur's lemma implies that $A = 0$ for $\mu \neq \nu$ and A is a multiple of the identity matrix for $\mu=\nu$; no transition between states of different symmetry types and every state of a irreducible subspace is degenerate with respect to all observables. For this reason, an irreducible subspace has been called a generalized ray.

However, there is no reason to omit the equivalent irreducible subspaces in our theory. $|\psi^{\mu}[K]$ and $|\psi^{\nu}[K]$ may have non-zero components in each irreducible subspaces, the operator A may have consequently domain of definition as the whole $H^{\mu}[K]$. Eqn. (6), in matrix form, would then be

$$\tilde{D}^{\mu}(\sigma)\ A = A\ \tilde{D}^{\nu}(\sigma) \tag{7}$$

where $\tilde{D}^{\mu}(\sigma) = \oplus n_{\mu} D^{\mu}(\sigma)$ $\tilde{D}^{\nu}(\sigma) = \oplus\ n_{\nu} D^{\nu}(\sigma)$ and n_{μ}, n_{ν} are the dimensions of the irreducible representations. Because $\tilde{D}^{\mu}(\sigma)$ and $\tilde{D}^{\nu}(\sigma)$ are not irreducible. Schur's lemma cannot be applied as done previously. This point escaped from attention in the literature of paraparticle theory.

b) A Superselection Rule

Consider now the matrix element $\langle\psi^{\mu}|A|\phi^{\nu}\rangle$ where $|\psi^{\mu}\rangle \ \varepsilon\ H^{N}_{\mu}$, $|\phi^{\nu}\rangle \ \varepsilon\ H^{N}_{\nu}$ and A is an observable. Making use of the definition of the ε^{μ} and remembering that $\varepsilon^{\mu +} = \varepsilon^{\mu}$, we have

$$\langle\psi^{\mu}|A|\phi^{\nu}\rangle = \langle\psi^{\mu}|\varepsilon^{\mu} A \varepsilon^{\nu}|\phi^{\nu}\rangle$$

because ε^{μ} is a function of the $U(\sigma)$, the indistinguishability postulate implies that ε^{μ} commutes with A, hence

$$<\psi^{\mu}|A|\phi^{\nu}> \;=\; <\psi^{\mu}|A\ \varepsilon^{\mu}\ \varepsilon^{\nu}|\phi^{\nu}>$$

$$= \;\; 0 \text{ if } \mu \neq \nu \qquad (8)$$

because $\varepsilon^{\mu}\ \varepsilon^{\nu} = 0$ if $\mu \neq \nu$. Eqn. (8) implies that the observables act irreducibly in each H^N_{μ}. This is the situation which leads to the following superselection rule:

"Each H^N_{μ} is a superselection Sector

This superselection rule is well-known but its derivation, as found in the literature, based on Schur's lemma, is quite unsatisfactory as pointed out in the above remark.

c) Matrix Elements

We study here in detail the matrix elements of a physical observable in a subspace $H[K]$. Since the observables are irreducible in $H^{\mu}[K]$, it is sufficient to consider operators defined in a subspace $H^{\mu}[K]$. With the choice of the base $\{|\mu^{i}_{j}[K]>\}$ satisfying (5), an observable A must satisfy

$$[A,\ \tilde{D}^{\mu}(\sigma)] = 0 \qquad (7^1)$$

where

$$\tilde{D}^{\mu}(\sigma) = \oplus\, n_{\mu}\ D^{\mu}(\sigma)$$

where $D^{\mu}(\sigma) = \{D^{\mu}_{ij}\}$. In $H^{\mu}[K]$, A can be represented by a matrix

$$A = \{\underset{\sim}{A}_{k\ell}\},\ k,\ell = 1,2,\ \dots\ n_{\mu}$$

where $\underset{\sim}{A}_{k\ell}$ is a $n_{\mu} \times n_{\mu}$ matrix,

$$\underset{\sim}{A}_{k\ell} = \{A^{i_k\ i_\ell}_{k\ell}\},\ i_k,\ i_\ell = 1,2,\ \dots\ n_{\mu}$$

Eqn. (7^1) requires

$$[\underset{\sim}{A}_{k\ell}, D^{\mu}(\sigma)] = 0$$

for all $\sigma \in S_N$. Since the $D^{\mu}(\sigma)$ are irreducible, Schur's lemma implies

$$A_{k\ell}^{i_k\ i_\ell} = A_{k\ell}\ \delta_{i_k\ i_\ell} \qquad (8)$$

where $A_{k\ell}$ are constants and $\delta_{i_k\ i_\ell}$ is the Kronecker -symbol. From this,

$$A|\mu_j^i[K]> \ = \ \sum_k A_{jk}|\mu_k^i[K] \qquad (9)$$

We see that the physical observable A is not a multiple of the identity operator on each irreducible subspace $H_j^{\mu}[K]$. On the contrary, A can mix vectors of a subspace $H_j^{\mu}[K]$ to vectors of other subspaces. This confirms our remark at the beginning of this section.

It is important to note that, due to the non-orthogonality of the $|\mu_j^i[K]$, the operator

$$\sum_{i,j} |\mu_j^i[K]><\mu_j^i[K]$$

is not the projection operator onto $H^{\mu}[K]$. Therefore, we have the inequality

$$A|\mu_j^i[K]> \ \neq \sum_{hk} |\mu_k^h[K]><\mu_k^h[K]|A|\mu_j^i[K]> \qquad (10)$$

I.e. (9) does not imply that $<\mu_k^h|A|\mu_j^i> \ = \ A_{jk}\ \delta_{hi}$ in contradiction with what is usually believed. It only implies that

$$(\mu_k^h[K],\ A\mu_j^i[K]) = A_{jk}\delta_{hi}$$

From eqn. (9) we obtain

$$<\mu_k^h[K]|A|\mu_j^i[K]> = \sum_k A_{jk} \rho_{kj}^{hi} \qquad (11)$$

which shows that the expectation value of an observable depends on the vectors of an irreducible subspace $H_j^\mu[K]$. It turns out that the states in an irreducible subspace are not degenerate with respect to all observables contrarily to what we expect. This result is due to the inclusion of the equivalent irreducible representations of the symmetry group*.

Among the operators (8), we consider those operators for which

$$A_{jk} = A_j \delta_{jk} \qquad (12)$$

These operators act as a miltiple of the identity operator on each irreducible subspace, every state in each irreducible subspace is degenerate with respect to these operators. It can be seen that operators satisfying (12) are linear combinations of the projection operators $P_j^\mu[K]$ onto the subspaces

* In the usual applications of group theory to physical problems, each irreducible representation occurs only once in the Hilbert space of dynamical states so that the states belonging to an irreducible representation are degenerate with respect to all observables which are invariant under the symmetry group.

$H_j^\mu[K]$ ($P_j^\mu[K]$ is the operator for which $A_j = 1$, $A_h = 0$ if $h \neq j$).

3. Preparation of the State

So far, we have studied the structure of the theory, based on the permutation invariance, which is a supersymmetry in the sense that it holds for every physical observables including the evolution operator G(t). We have seen that this supersymmetry implies that the physical observables are not irreducible in H^N. This is the situation which leads to the superselection rule stated earlier. For paraparticles, the quantum mechanics is more complicated than for bosons and fermions because the supersymmetry operators are not abelian.

A point to be discussed is the extent to which the state can be prepared by experiment. Usually, quantum mechanics deals with phenomena in which a maximum of information is available about the system under consideration. States of maximal information are often called pure states:* a pure state is characterized by the existance of a complete experiment that yields a result predictable with certainty when performed on the system, For example, linear polarization of light beam in a given plan is characterized by 100% transmission of each photon through a suitably oriented Nicol prism; no other state of polarization is fully transmitted by the same prism. Filtration through a Nicol prism defines a

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* See, in this connection, Fano (1957) for an excellent exposition.

state of polarization completely because beams thus filtered behave identically with respect to all other polarization analysers. An experiment that yields a unique pre-determined result for a system in a given state can be designed to act as a filter which leaves the system undisturbed. The experiment may then be repeated again and again on the same system, at least in principle, always with certainty as to its outcome. A pure state can, in fact, be prepared by subjecting systems to a filter-type experiment. Mathematically, a pure state is defined as an eigenstate of a variety of commuting hermitian operators. Given such a set of operators it proves possible in most cases to design, at least in principle, an experiment that constitutes a measurement of the corresponding observables.

For paraparticles, it may not be possible to prepare a pure state in the above sense. For example, no experiment could distinguish pure states differently only by a permutation of the particle variables. Nevertheless, the preparation of the paraparticle state would also consist in performing a set of compatible measurements, with the result that the state vector belongs to a common eigensubspace of the corresponding commuting observables. Since these observables are permutation invariant, the eigensubspace is also permutation invariant. With Messiah and Greenberg, we agree that the most complete preparation is achieved if the subspace is irreducible with respect to physical observables, that is, no other commuting observable could separate vectors of this subspace. However, we do not anticipate that the subspace is irreducible with respect to the permutation operators, the $U(\sigma)$. Furthermore, we assume that the projection operator onto the symmetry type belongs to a set of commuting observable otherwise the introduction of paraparticles would be completely fortuitous. Then, the least complete preparation would yield the result that the state belongs to a superselection sector $H^{\mu}[K]$ assuming, of course, that the set K is physically observable. We note that the subspace $H^{\mu}[K]$ is not irreducible with respect to the $U(\sigma)$. We cannot assert whether there exist other physical observables that can split the subspace $H^{\mu}[K]$, neither can we be sure that some of these observables, if existing, commute with the projection operator ε^{μ}.

Thus, we are faced with the larger indeterminacy of the state of the system. It is usually believed that this indeterminacy causes no difficulty if we assume that the preparation could yield an irreducible subspace, that is, that the projection operator onto a $H_j^{\mu}[K]$ is a commuting observable, because, measurable results do not depend on which state vector of $H_j^{\mu}[K]$ we choose to represent the system. This belief is based on mistaken application of Schur's lemma to the calculation of the expectation value of an observable. According to eqn. (11) measurable results do depend on the state vectors in $H_j^{\mu}[K]$.

In the following, we shall not assume that the projection operator onto an irreducible subspace of $H^{\mu}[K]$ is a physical observable. Such an assumption may not be correct because the choice of the irreducible subspace is not unique, therefore, the projection operators onto the irreducible subspaces can be chosen in many different ways; an operator which cannot be defined uniquely possesses unlikely any physical significance.

4. Density Matrix Description

So far, we have established only that the state belongs to a $H^{\mu}[K]$ but have not been able to specify to what extent it can be determined. If this situation is also the nature of the theory, that is, if no complete experiment can give a unique results predictable with certainty, we have to describe the system by a density matrix. Adopting the base defined previously, we can write the density matrix as

$$\rho^{\mu} = \sum_{ij} |\mu_j^i[K]\rangle \omega_j^i \langle \mu_j^i[K]| \tag{13}$$

where ω_j^i represent the probability in the incoherent superposition

$$|\psi^{\mu}[K]\rangle = \Sigma \quad \omega_j^i |\mu_j^i[K]\rangle \tag{14}$$

The indistinguishability postulate finds itself in the relation

$$\mathrm{Tr}(\rho^{\mu}A) = \mathrm{Tr}(U\rho^{\mu}U^{+}A) \tag{15}$$

where A is a physical observable and, for an operator O, the trace is defined as

$$\mathrm{Tr}O = \sum_{ij} \langle \mu_j^i[K]|O|\mu_j^i[K]\rangle \tag{16}$$

Note that, in our theory, the trace of an operator is not the usually defined trace of its representing matrix. Eqn. (15) is always satisfied if

$$A = UAU^+$$

for any $U(\sigma)$. We remark that Eqn. (15) is a condition imposed on A but not on ρ^μ. In fact, as pointed out by Greenberg (1972) if we choose ρ^μ such that $\rho^\mu = U\,\rho^\mu U^+$, for all $U(\sigma)$, then (15) is always satisfied.

The above remark permits us to build up a theory of identical particle in another direction.

The theory discussed so far is based on the condition (7), the indistinguishability postulate, imposed on the physical observables. However, it could be argued that, to formulate a theory, one should know <u>a priori</u> the physical observable quantities, then determine the state such that the symmetry property of the theory is satisfied. Thus, we may regard the indistinguishability of particles as a condition imposed on the states.

According to our discussion in the last section, it appears appropriate to describe the state of N-paraparticles by a density matrix which, in the chosen base, could be written as eqn. (13). Equation (15) is then considered as a condition imposed on a <u>"physical"</u> density matrix. This condition is satisfied if we choose the density matrix such that

$$[\rho^\mu, U(\sigma)] = 0, \quad \sigma \in S_N \tag{17}$$

This equation does not determine ρ^{μ} uniquely in $H^{\mu}[K]$; the matrix representing ρ^{μ} can be obtained from eqn. (9) by replacement of A by ρ^{μ}. Also, it is not the only solution of eqn. (15).

A trivial solution of eqn. (17) is

$$\rho_t^{\mu} = \sum_{ij} |\mu_j^i[K]\rangle\langle\mu_j^i[K]| \qquad (18)$$

Note that ρ_t is not the projection operator onto $H^{\mu}[K]$ although the latter is also a trivial solution of (17).

We note the similarity with the case of bosons and fermions: for bosons and fermions, the state of the system is described by a unique ray $|\psi\rangle$ so that $U(\sigma)|\psi\rangle$ should be the same ray as $|\psi\rangle$ because the physics is unaltered by the action of $U(\sigma)$; for paraparticles, the state is described by a density matrix ρ that would be required to satisfy $U(\sigma)\rho U^{+}(\sigma)=\rho$. This condition would determine ρ uniquely if we require ρ to be defined in an irreducible representation of S_N because, for such a representation, there exists only one invariant form due to Schur's lemma. However, this requirement does not seem compelling on physical grounds.

CHAPTER IV

CLUSTER ASSUMPTION

We cannot take the whole universe into account whenever we consider a particular system. That is why we considered so far a system of N identical particles of certain species although these are not the only particles of that species in the Universe. In treating this system as a distinct entity from the rest, we assume that the presence of other particles do not affect dynamical properties of the system because they are so far away from the system that their interaction with the system is negligible. A question arises whether such an assumption, called the cluster assumption, is consistent with the theory of identical particles discussed in Chapter III when certain correlation between N particles in the systems and others are established.

In practice, the particles of a system are all inside a certain spatial domain D. It is well known* that, if these particles are bosons or fermions, all other bosons or fermions outside D may simply be ignored so long as their interaction with the particles in the system remains negligible. We shall examine in this Chapter whether this result is also valid for paraparticles.

1. General Discussion

We consider first the example of a 3-particle system C

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* See for example Messiah (1965), Chapter XIV, 8.

consisting of a cluster C_1 of two particles and a particle C_2 sufficiently far away from C_1 so that the cluster assumption is applicable, i.e., it is possible to treat C_1 as a distinct system without taking C_2 into account at all. This means that it makes no difference whether we treat C_1 as an isolated system or a subsystem of C. We assume especially that all conservation laws, superselection rules, and the observable quantities found for C are applicable to C_1.

Let us consider first C_1 as a subsystem of C. We shall deal with the non-relativistic Schrödinger situation in which the state of the 3 particles, at time t, is represented by a wavefunction $\phi(x_1, x_2, x_3)$. Then the wavefunction of the 3 identical particle system is of the form

$$\psi^{\mu} (x_1, x_2, x_3) = \Sigma\ \alpha^{\mu}_{\sigma}\ U(\sigma)\ \phi\ (x_1, x_2, x_3)$$

where $U(\sigma)$, $\sigma \varepsilon S_3$, is the permutation operator defined as

$$U(\sigma)\ \phi\ (x_1, x_2, x_3) = \phi\ (x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3}) \qquad (1)$$

and α^{μ}_{σ} are scalar quantities. We suppose that C is in the parastatistic state belonging to the representation of S_3 associated with the triangular Young diagram. As is well known, this representation of S_3 is faithful so that the $U(\sigma)$ are not commutative. The wavefunction of C is an eigenfunction of the projection operator

$$P = \frac{1}{\sqrt{3}}\ \Sigma\ \chi\ (\sigma)\ U(\sigma) \qquad (2)$$

where $\chi(\sigma)$ is the character of σ in the representation associated with the triangular Young diagram.

Consider the operator Q defined as

$$Q = \frac{1}{2}\,[1+U(12)] \qquad (3)$$

which is the projection operator onto symmetry type of C_1. Since Q commutes with P, a measurement of Q, if possible, would not destroy the symmetry type of C. However, in the representation associated with the triangular Young diagram, Q does not commute with all U(σ) and consequently cannot be measurable (or physical observable) according to the indistinguishability postulate,(eqn. Chapter III-1),

$$[A,\, U(\sigma)] = 0 \qquad (4)$$

if A is an observable.

Let us consider now C_1 as a distinct system from C_2. In this case, C_2 has two symmetry types associated with the symmetric wavefunction and the anti-symmetric wavefunction. Q is just the projection operator onto the symmetric wavefunction, therefore, by assumption, it is a physical observable of C_1. This is in contradiction with the property of Q when considered as an operator of system C.

In order to escape from this contradiction, we have to restrict ourselves to states satisfying the symmetrization postulate for which Q satisfies (4) and thus can be accepted as a physical observable of C.

Thus, the indistinguishability postulate expressed as in eqn. 4 is incompatible with the cluster property. The question is whether this incompatibility constitutes a proof of the symmetrization postulate.

The answer depends on how seriously one relates eqn. 4 to the indistinguishability condition of identical particles.

If we regard eqn. 4 as an obvious relation for identical particles, then the cluster property provides a proof of the symmetrization postulate. However, if we consider eqn. 4 as a necessary condition, as derived by Messiah and Greenberg (1964) from the indistinguishability postulate, then we must know to what extent this derivation is valid.

Recalling that eqn. 4 was derived from the condition

$$\langle\psi|A|\psi\rangle = \langle\psi|U^{+}(\sigma)\ A\ U(\sigma)|\psi\rangle \qquad (5)$$

for every state $|\psi\rangle$ of N-identical particles. Clearly, $|\psi\rangle$ was taken to be a pure state (or coherent mixture of pure state) otherwise the expectation value of an observable cannot be written as in (5). However, as already discussed in Chapter III, due to the nature of paraparticles, it may not be possible to describe paraparticles by a pure state (or coherent mixture of pure state). We might have to describe paraparticles by an incoherent mixed state represented by a density matrix ρ. The indistinguishability of identical particles then finds itself in the expression

$$\mathrm{Tr}[\rho A] = \mathrm{Tr}[U^{+}(\sigma)\ \rho U(\sigma)\ A] \qquad (6)$$

Eqn. (4) is just a solution of this equation but not the only one. We may regard (6) as a condition imposed on ρ and choose an appropriate ρ to describe identical particles.

An obvious choice would be ρ such that

$$U^{+}(\sigma)\rho\, U(\sigma) = \rho \qquad (7)$$

We have shown in Chapter III that there exists at least one density matrix satisfying (7); the projection operator onto the invariant subspaces of paraparticle states. The theory of paraparticle formulated in terms of density matrix, therefore, possesses solutions.

Now, in the density matrix description of paraparticles, no condition is imposed on the physical observables; the incompatibility with the cluster property, as discussed previously, does not exist, neither does it constitute a proof of the symmetrization postulate.

This line of reasoning can be applied to an N-particle system C consisting of a cluster C_1 of M particles whose variable are x_1, x_2,...x_M and a cluster C_2 of N-M particles. Suppose that the system C is in the eigenstates of the projection operator ε^{μ} onto the symmetry type associated with the μ-representation of S_N,

$$P^{\mu} = \sqrt{\frac{n_{\mu}}{N!}} \sum_{\sigma\varepsilon S_N} \chi^{\mu}(\sigma)\; U(\sigma) \qquad (6)$$

where n_{μ} is the dimension of the representation and $\chi^{\mu}(\sigma)$ is the character. The state of the cluster C_1 can be determined when C_2 is far away from C_1 by a measurement of the projector operators

$$Q^{\nu} = \sqrt{\frac{n_{\nu}}{M!}} \sum_{\sigma\varepsilon S_M} \chi^{\nu}(\sigma)\; U(\sigma) \qquad (7)$$

where the ν are the representations of S_M obtained by the restriction of S_N to S_M.

2. An Illustration

We have shown that the cluster property is incompatible with the indistinguishability postulate, eqn. (4), for para-particles. It is so because the projection operator onto the symmetry type of a cluster, allowed to be physical observable by the cluster property, does not satisfy (4). In this section, we shall illustrate the three-particle situation in more details by constructing explicitly the three-particle states and, when conveniently, we shall point out some confusions in the literature concerning Steinman's argument.

Suppose that our 3-particle system C is found in the state belonging to the subspace spanned by $\phi(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3})$. We assume that the state of the system belong to representation of S_3 associated with the triangular Young diagram. A base of this representation can be constructed as

$$e^i_j = \frac{1}{\sqrt{3}} \sum_\sigma D_{ij}(\sigma)\; U(\sigma)\; \phi(x_1, x_2, x_3) \tag{9}$$

which obviously satisfy the relation

$$U(\sigma)\; e^i_j = \sum_k e^k_j\, D_{ki}(\sigma) \tag{10}$$

if $D(\sigma) = \{D_{ij}(\sigma)\}$ is the matrix representation of S_3 associated with the triangular Young diagram. The orthogonal $D(\sigma)$ are well known and is listed in Appendix B from which we obtain

$$e^1_1 = \frac{1}{2\sqrt{3}} [2\phi(x_1, x_2, x_3) + 2\phi(x_2, x_1, x_3) - \phi(x_1, x_3, x_2)$$

$$- \phi(x_3, x_2, x_1) - \phi(x_2, x_3, x_1) - \phi(x_3, x_1, x_2)]$$

$$e_2^2 = \frac{1}{2} [\phi(x_1, x_3, x_2) - \phi(x_3, x_2, x_1) + \phi(x_2, x_3, x_1) - \phi(x_3, x_1, x_2)]$$

$$e_2^1 = \frac{1}{2} [\phi(x_1, x_3, x_2) - \phi(x_3, x_2, x_1) - \phi(x_2, x_3, x_1) + \phi(x_3, x_1, x_2)] \quad (11)$$

$$e_2^2 = \frac{1}{2\sqrt{3}}[2\phi(x_1, x_2, x_3) - 2\phi(x_2, x_1, x_3) + \phi(x_1, x_3, x_2) + \phi(x_3, x_2, x_1) - \phi(x_2, x_3, x_1) - \phi(x_3, x_1, x_2)]$$

Assuming the usual normalization of the wavefunction $\phi(x_1, x_2, x_3)$ as

$$1 = \langle\phi(x_1, x_2, x_3), \phi(x_1, x_2, x_3)\rangle = \int dx_1\, dx_2\, dx_3\, \overline{\phi}(x_1, x_2, x_3)\, \phi(x_1, x_2, x_3)$$

We have, from (9):

$$\langle e^i_j, e^{i'}_{j'}\rangle = \frac{1}{3} \sum_{\sigma} D^{\mu}_{ji}(\sigma)\, D^{\mu}_{i'j'}(\sigma^{-1}) + \frac{1}{6} \sum_{\sigma \neq \sigma'} D^{\mu}_{ij}(\sigma^{-1})\, D^{\mu}_{j'i'}(\sigma'^{-1})\, \langle U(\sigma)\phi, U(\sigma')\phi\rangle$$

$$= \delta_{ii'}\, \delta_{jj'} + \frac{1}{6} \sum_{\sigma \neq \sigma} D^{\mu}_{ij}(\sigma'^{-1}) D^{\mu}_{j'i'}(\sigma'^{-1}) f(\sigma,\sigma')$$

where

$$f(\sigma,\sigma') = \int dx_1\, dx_2\, dx_3\, \overline{\phi}(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3})\, \phi(x_{\sigma' 1}, x_{\sigma' 2}, x_{\sigma' 3})$$

We see that, due to the second term in the above equation the e^i_j cannot be mutually orthogonal. In the sequel, we shall assume that

$$f(\sigma,\sigma') = 0, \quad \sigma \neq \sigma'$$

so that

$$\langle e_j^i, e_{j'}^{i'} \rangle = \delta_{jj'}\,\delta_{ii'} \tag{12}$$

We remark that the arguments in the following would not hold without this assumption.

A point to remark is that e_j^i, j fixed, i = 1, 2, span an irreducible representation space H_j which contains both the symmetric state and the anti-symmetric state with respect to U(12),

$$\begin{aligned} U(12)\, e_j^1 &= e_j^1 \\ U(12)\, e_j^2 &= -e_j^2 \end{aligned} \tag{13}$$

and, contrarily to what was claimed by Landshoff and Stapp (1967) by Hartle and Taylor (1968) and by Arons (1969), it is not possible to choose the base such that the symmetrical states belong to one irreducible representation space and the anti-symmetrical states belong to its equivalent irreducible space because (i) U(12) would be represented by the identity matrix in one representation (the one containing only symmetric states) in contradiction with the faithfulness of the representation associated to the triangular Young diagram, (ii) the base of H_j can be chosen such that H_1 and H_2 support identically the same irreducible representation; vectors in H_1 have the same transformation property with respect to U(12) as vectors in H_2 and (iii) the restriction of the representation of S_3 associated with the triangular representation is reducible and contains both the symmetric and anti-symmetric representations, as already stated by Steinmann (1966).

We calculate now the expectation value of a physical observable A satisfying the condition (4). The matrix representing A in $H = H_1 + H_2$ is

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

where A_{ij} is a 2 x 2 matrix. With the base given by (11), the $U(\sigma)$ is represented by

$$U(\sigma) = \left[\begin{array}{c|c} D(\sigma) & 0 \\ \hline 0 & D(\sigma) \end{array}\right]$$

Eqn. (4) requires

$$[A_{ij}, D(\sigma)] = 0, \quad \sigma \in S_3$$

Schur's lemma implies

$$A_{ij} = \lambda_{ij} I, \ \lambda_{ij} \text{ are scalars}$$

where I is 2 x 2 unit matrix,i.e.*

$$Ae^i_j = \sum_k \lambda_{jk} e^i_k$$

From this and from (12) we have

$$\langle e^i_j, A e^i_j \rangle = A_{jj} \tag{14}$$

for a physical observable A. This means that the expectation value of an observable is the same for all rays in a H_j. We call an H_j a generalized ray as defined by Messiah and Greenberg.

It was recognized by Steinmann that the projection operator onto the symmetric state of H_j is not physical observable by observing that (14) implies that no physical observable could distinguish the symmetric state of H_j from the

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* This result is independent of the assumption leading to eqn. (12).

anti-symmetric state. He then argued that the cluster property allows a measurement of the symmetry type of the cluster C_1, therefore, allows a distinction of the symmetric state from the anit-symmetric state in contradiction with the property of a generalized ray. Landshoff and Stapp, Hartle and Taylor and Arons, however, objected Steinmann's argument claiming that the symmetric states belong to representation different from the anti-symmetric states. We have remarked previously that this claim is in fact incorrect, therefore, it validates their objection to Steinmann's argument.

In Steinmann's argument, it must be assumed that the projection operators onto the H_j are not physical observables otherwise the cluster property, as Steinmann put it, would allow the distinction between the symmetric state of H_1, say, from the anti-symmetric state of H_2 (and there is no reason that it could not be so if the projection operators onto the H_j are physical observables). We shall confirm, in the following, Steinmann's argument relaxing this assumption.

Consider the space H_+ spanned by e_1^1 and e_2^1 and H_- spanned by e_1^2 and e_2^2. Eqn. (12) and eqn. (14) show that for each state

$$a(x_1, x_2, x_3) = \alpha\, e_1^1 + \beta\, e_2^1$$

where α, β are scalars , there exists a state

$$b(x_1, x_2, x_3) = \alpha\, e_1^2 + \beta\, e_2^2$$

such that the expectation value of an observable is the same for $a(x_1, x_2, x_3)$ as for $b(x_1, x_2, x_3)$. Thus, it is impossible to specify experimentally which one of the spaces H_+ and H_- the state of the system belongs to. We can express this by the statement that the projection operators onto H_+ and H_- are not physical observable quantities. This is precisely what we have shown in the last section because the projection operator onto H_+ is Q and that onto H_- is (1-Q).

We show now that the cluster property allows Q to be a physical observable. Suppose that our system C of 3-particles consists of a cluster C_1 and C_2 as before. Then the 3-particle state has the cluster form

$$\phi(x_1, x_2, x_3) = \psi(x_1, x_2)\ \chi(x_3)$$

where $\psi(x_1, x_2)$ describes C_1 and $\chi(x_3)$ describes C_2. By hypothesis, ψ does not overlap χ, i.e.

$$\psi(x_3, x_i) = \psi(x_i, x_3) = 0$$

$$\chi(x_i) = 0$$

for i = 1,2. Then (11) becomes

$$e_1^1 = \psi^+(x_1, x_2)\ \chi(x_3) \qquad (15)$$

$$e_2^2 = \psi^-(x_1, x_2)\ \chi(x_3) \qquad (16)$$

$$e_2^1 = e_1^2 = 0$$

where

$$\psi^{\pm}(x_1, x_2) = \frac{1}{2\sqrt{3}}\ [\psi(x_1, x_2) \pm \psi(x_2, x_1)]$$

The cluster property allows a measurement on C_1 without disturbing the symmetry type of C. This measurement is represented by an operator $A(x_1, x_2)$depending only on the variables of the particles in C_1. From (15) and (16) we have

$$\langle e_1^1, Ae_1^1\rangle = \int dx_1\, dx_2\, \bar{\psi}^+(x_1, x_2)\, A(x_1, x_2)\, \psi^+(x_1, x_2)$$

$$\langle e_2^2, Ae_2^2\rangle = \int dx_1\, dx_2\, \bar{\psi}^-(x_1, x_2)\, A(x_1, x_2)\, \psi^-(x_1, x_2)$$

Because of the superselection rule operating on C_1, C_1 must be either in $\psi^+(x_1, x_2)$ or in $\psi^-(x_1, x_2)$, the measurement of A should determine which of the $\psi^{\pm}(x_1, x_2)$ C_1 is in. Therefore, it should specify which of the H_+ and H_- the state of C belongs to. This is in contradiction with the physical indistinguishability of H_+ and H_- if C_1 is considered as a subsystem of C.

The derivation of S.P., based on the indistinguishability postulate, eqn. (4), and the cluster property, as presented in this section is valid only for the particular case for which eqn.(12) holds. Furthermore, it is base-dependent; an incorrect conclusion can be drawn from an incorrect construction of the base. This has been done actually in the literature. The derivation presented in section 1 holds in general case and base-independent.

So far, we have derived S.P. from the indistinguishability postulate expressed by eqn. (4) assuming that the state correspond to a ray in $H = H_1 + H_2$. However, it appears no less physically reasonable to describe the state by a density matrix ρ satisfying eqn. (6). We could choose ρ as

$$\rho = \sum_{ij} |e_j^i\rangle\langle e_j^i|$$

where $|e_j^i\rangle$ is the state corresponding to $e_j^i(x_1, x_2, x_3)$ of eqn. (11). To calculate the symmetry type of C_1, we calculate the expectation value of the operator U(12):

$$\begin{aligned}\langle U(12)\rangle &= Tr[U(12)\rho] \\ &= \langle e_1^1|U(12)|e_1^1\rangle + \langle e_1^2|U(12)|e_1^2\rangle \\ &+ \langle e_2^1|U(12)|e_2^1\rangle + \langle e_2^2|U(12)|e_2^2\rangle \\ &= 0\end{aligned}$$

upon using (12) and (13). Thus, ρ corresponds to the state of minimal information in similarity with the state of random spin. If we repeat Steinmann's experiment discussed earlier, we would find that the cluster C_1 has 50% of being in the symmetric state and 50% in the anti-symmetric state. No contradiction would arise from the measurement of the symmetry type of the cluster C_1.

3. Conclusion

We have seen that the cluster property is incompatible with condition (4) for an observable. Thus, the theory of paraparticles formulated from eqn. (4) is not acceptable. Instead, we should relax eqn. (4) and describe the state of the system by a density matrix satisfying eqn. (6) for any physical observable A.

CHAPTER V

CONNECTION BETWEEN SYMMETRY TYPES AND STATISTICS AND CLASSIFICATION OF PARAPARTICLES

1. Statistics of order p

States of identical particles can be classified according to irreducible representations of S_N. It is well known that the identity and alternating one-dimensional irreducible representations are connected with the Bose-Einstein and Fermi-Dirac statistics. Are higher dimensional representations connected with other statistics? In this section, we wish to establish the connection between the irreducible representations of S_N and a special statistics which we call statistics of order p. It is the statistics in which the maximal occupation number is an integer p. We will prove a theorem, first proved by Okayama (1952) for p=2, using the place permutation operators*.

Theorem:

"Particles whose N-body states belong to irreducible representations associated with Young diagrams of p columns obey statistics of order p".

Proof:

The proof is quite lengthly and follows the same line of reasoning of Okayama (1952).

*Okayama (1952) made use of the particle permutation operators which will be discussed in chapter X.

According to eqn. IV-9, a state of μ symmetry type is a linear combination of elements of the following matrix

$$|\psi^{\mu}(K)\rangle = \frac{n_{\mu}}{N!} \sum_{\sigma} D^{\mu}(\sigma^{-1})\ U(\sigma)|\psi(K)\rangle \qquad (1)$$

For our purpose, we record the one particle states and their positions in $|\psi(K)\rangle$. If the particles obey parastatistics of order p, $|\psi^{\mu}(K)\rangle$ should satisfy two conditions:

(i) $|\psi^{\mu}(K)\rangle$ vanishes identically when we equate more than p one-particle-states,

(ii) $|\psi^{\mu}(K)\rangle$ does not vanish identically when we equate p or less than p one particle states.

First, we show that it is sufficient to consider only the case in which we equate first q one-particle-states at position 1,2,...q in $|\psi(K)\rangle$ ($q>p$ or $q<p$). In fact, q states at arbitrary positions in $|\psi(K)\rangle$ can always be shifted to the first q positions by a permutation γ. It is possible to write every $\sigma \in S_N$ as a product $\tau\gamma$, $\tau \in S_N$, so that (1) becomes:

$$|\psi^{\mu}(K)\rangle = \frac{n_{\mu}}{N!} D^{\mu}(\gamma^{-1}) \sum_{\tau} D^{\mu}(\tau^{-1})\ U(\tau)|\psi'(K)\rangle \qquad (2)$$

where

$$|\psi'(K)\rangle = U(\gamma)|\psi(K)\rangle$$

Since $D(\gamma^{-1})$ is not a zero matrix, $|\psi(K)\rangle$ vanishes if and only if the sum on the right hand side of (2) vanishes when we equate the first q states in $|\psi'(K)\rangle$.

Now, we find among the representations of S_N those which satisfy condition (i) when we equate more than the first q one particle states. Consider the subgroup S_{p+1} of

S_N of all permutations of 1, 2, ..., p+1. Lagrange's theorem permits us to decompose the group S_N into a series of left cosets

$$S_N = S_{p+1} + \sigma_1 S_{p+1} + \ldots + \sigma_k S_{p+1}$$

where σ_i is an element of S_N not contained in $\sigma_{i-1} S_{p+1}$. Any element of S_N can be written as a product $\sigma_i \sigma_h$ with $\sigma_h \in S_{p+1}$ so that equation (1) becomes

$$|\psi^\mu(K)> = \frac{n_\mu}{N!} \sum_{i,h} D^\mu(\sigma_h^{-1} \sigma_i^{-1})\ U(\sigma_i)|U(\sigma_h)\rangle$$

where

$$|U(\sigma_h)> = U(\sigma_h)|\psi(K)>$$

Since the first more than p states are the same

$$|U(\sigma_h)> = |\psi(K)>$$

then

$$|\psi^\mu(K)> = \frac{n_\mu}{N!} (\sum_h D^\mu(\sigma_h^{-1}))(\sum_i D^\mu(\sigma_i^{-1})|U(\sigma_i)>)$$

The requirement that $|\psi^\mu(K)>$ vanishes identically is equivalent to

$$\sum_h D^\mu(\sigma_h^{-1}) = \sum_h D^\mu(\sigma_h) = 0 \qquad (3)$$

We seek for representation of S_{p+1} satisfying equation (3). Firstly, equation (3) requires

$$\sum_h \chi^\mu(\sigma_h) = 0 \qquad (4)$$

where $\chi^\mu(\sigma_h)$ is a character. Since the sum of all characters divided by the total number of elements of a group is the number of times the identity representation is found in a representation, equation (4) requires that we must exclude the identity representation. Secondly, since the matrix $\sum_h D^\mu(\sigma_h)$ commutes with all $D^\mu(\sigma_h)$'s, in view of Schur's lemma it is a multiple of the identity if D is irreducible,

$$\sum_h D^\mu(\sigma_h) = \lambda I^\mu$$

The exclusion of the identity representation implies $\lambda \equiv 0$ in order that (3) is satisfied. Hence, condition (i) is satisfied for any irreducible representation associated with Young diagrams of at most p columns.

It remains to satisfy condition (ii) when we equate the first less than p states. Again, we write $|\psi^\mu(K)>$ as

$$|\psi^\mu(K)> = \frac{n_\mu}{N!} \left(\sum_h D^\mu(\sigma_k^{-1})\right)\left(\sum_j D^\mu(\sigma_j)\,|\psi(\sigma_j)>\right)$$

where $\sigma_k \in S_q$, $q \leq p$. Consider first the case $q=p$, then $\sigma_k \in S_p$. Condition (ii) is equivalent to

$$\sum_k D^\mu(\sigma_k) \neq 0 \qquad (5)$$

Since $|U(\sigma_j)>$'s are independent and $D^\mu(\sigma_j)$'s are not identically equal to zero, the only representation which satisfied (5), as a result of our previous reasoning concerning equation (3), is the identity representation which is associated with the horizontal Young diagram of p blocks. Since we can add to the identity representation any other representation,

$$D^\mu = I \oplus D^\nu\,, \qquad (6)$$

condition (5) is still satisfied, we require that the representation of S_p must contain the identity representation at least once. Obviously, this condition satisfied condition (ii).

also for the case $q<p$.

The representation of S_p are known, the representation of S_{p+1} for elements which are contained in S_p can be obtained from them by applying the branching law*: the representation of S_{p+1} associated with a Young diagram T^{μ}, for elements contained in S_p, is a direct sum of all representations associated with the Young diagrams obtained from T^{μ} by regular removal of one block. Because we require that the representations of S_p contain the identity representation, the representation S_{p+1} must contain a representation associated with a Young diagram which yields a single row of p columns after removal of one block. Thus, the representation of S_{p+1} must contain the irreducible representation associated with the Young diagram of p blocks in the first row, and only one block in the second row.

The representation of S_N for elements contained in S_{p+1} can be built up by successive application of the branching law. We require that the representations of S_N are those associated with Young diagrams which, after successive removal of (N-p-1) blocks, yield at least one diagram of the type mentioned above. The necessary and sufficient condition for such a diagram is that it has p columns.

We make the following remark:

(i) The connection between statistics and the symmetry

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* See for example Hamermesh (1962), page 215.

types is not unique (all symmetry types associated with Young diagrams of p columns are connected with the statistics of order p). The symmetry type plays a role more important than that played by the statistics in the sense that the former, but not the latter, determine the physical property of the system.

(ii) Condition (ii) on page 69 seems to be too restrictive because it implies that particles obeying statistics of order p do not have N-body state with $N<p$ because in this case we cannot obtain Young diagrams of p columns. To overcome this difficulty we can

(a) drop the condition (ii) and accept all Young diagrams of at most p columns.

(b) impose no restriction on the symmetry types of the systems of less than p particles. i.e., for $N>p$, we take states satisfying the theorem and for $N<p$, we take states of any symmetry type.

Whether (a) or (b) has to be adopted depends on the nature of the system.

2. Classification of Paraparticles

Can we assign to each species of identical particles symmetry type of N-body states unambiguously? The superselection rule discussed in section III-1 implies that for each system of identical particles of a species, there exists a unique symmetry type of the N particle states, but does not assert that the symmetry type is the same for all systems of

N particles of the same species. A different question arises whether there is any relation between the symmetry type of the N particle states and that of the M particle states. Suppose for example a 3-particle system has the symmetry type associated with the triangular Young diagram  . What is the symmetry type of the 2-particle systems: symmetric type associated with Young diagram  or anti-symmetric type associated with  ? The superselection rule forbids coherent linear combination of states of two different symmetry types. There is no known reason for prefering one particular symmetry type to the other. The question should be answered by experiment.

Thus, we cannot answer on the theoretical ground alone the posed questions. We may expect, however, certain class of symmetry types is to describe certain kind of paraparticles so that we can classify the paraparticles according to the associated class of symmetry type. In general, in order to define these classes of symmetry types, one has to impose certain physical conditions. Stolt and Taylor (1970) classify the paraparticles by requiring them to satisfy the cluster assumption. We shall take Stolt and Taylor's classification as definition of special types of paraparticles:

(i) The paraparticles are called <u>parabosons (or para-fermions) of order p</u> if their N-particle states have symmetry types associated with Young diagrams of at most p rows (or columns).

(ii) paraparticles of infinite order if they are not parabosons or parafermions.

Parafermions satisfy condition (i) on page 69 but not condition (ii). Bosons (or fermions) are parabosons (or parafermions) of order 1.

3. Consistency of the Classification

For the sake of consistency, we demand that many systems of parabosons for parafermions of the same species in interaction form a system of parafermions (or bosons) (we assume they conserve the numbers of identical particles). Let us see whether this condition can be met.

Consider, without loss of generality, an N-parafermion system in interaction with an M-parafermion system. Suppose when isolated one system has $(N-\mu)$ symmetry type and the other has $(M-\nu)$ symmetry type. We demand that the system of N+M parafermions is in the states of symmetry type associated with Young diagrams of at most p columns where p is the order of parafermions under consideration.

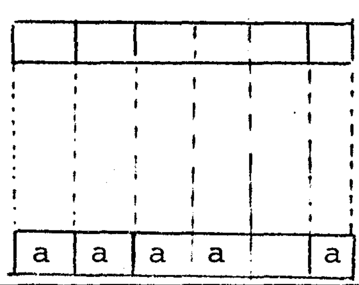
The states of N+M particles must have the symmetry types associated with the representations found in the decomposition of the outer product $(N-\mu) \otimes (M-\nu)$ of representations (N,μ) of S_N with representation $(M-\nu)$ into irreducible representations $(N+M,\gamma)$'s of S_{N+M},

$$(N-\mu) \otimes (M-\nu) = \sum_{\gamma} (N+M-\gamma) \tag{1}$$

We must show that, among the $(N+M,\gamma)$'s, there is at least one associated with Young diagram which has no more than p columns.

The rule for finding all $(N+M,\gamma)$'s are well known:*
(i) draw the Young diagram for $(N-\mu)$ representation, (ii) in the $(M-\nu)$ Young diagram assign the same symbol a to all boxes in the first row, the same symbol b to all boxes in the second row, etc. (iii) apply symbols a to the $(N-\mu)$ Young diagram and enlarge it in all possible ways subjected to the rule that no two a's appear in the same column and that the resultant graph be regular (iv) repeat with the b's, etc. (v) after all symbols have been added to $(N-\mu)$ Young diagram, select those tableaux in which the symbols, read from right to left in the first row, then the second row, etc. form a lattice permutation of a's, b's, etc.

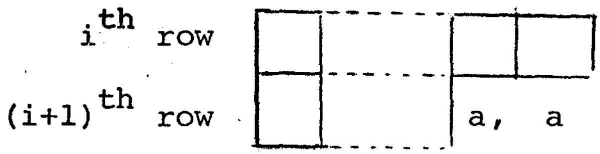
Now, we wish to show that, among the Young tableaux obtained by applying rule (iii), there exists at least one in which the symbol a does not appear in the first row. In fact, let us suppose that the $(N-\mu)$ Young diagram is square. Then the symbol a can be assigned by enlarge the $(N-\mu)$ Young diagram as



(2)

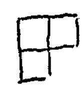
* See Hamermesh (1962)

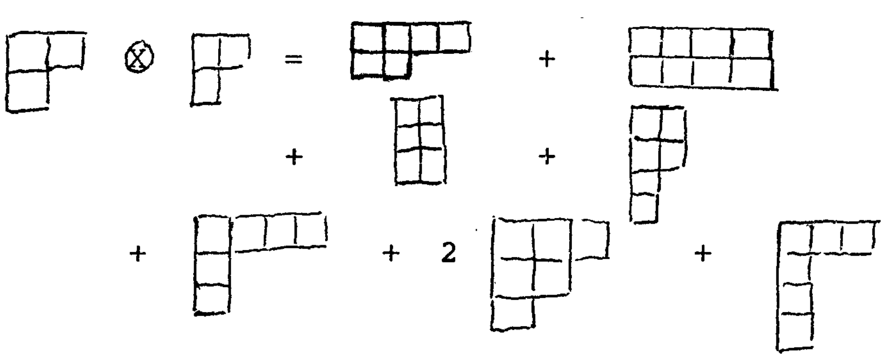
In general, the i^{th} row of (N-μ) Young diagram is shorter than the $(i+i)^{th}$ row so that we can assign a as

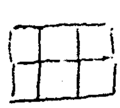
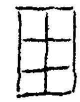
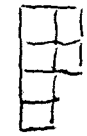


It is easy to convince oneself that the a's can be put into the boxes in this way without putting any a on the first row. Applying rule (iv) and (v) to this Young tableau whose number of columns is equal to that of (N-μ) Young diagram. The representation given by the obtained Young tableau is thus acceptable for the state of (N+M) parafermions.

For parabosons, the number of columns is not limited, it is very easy to check the consistency of the classification.

Consider the example of two -particle systems whose symmetry type is associated with the Young diagram  . We can consider these paraparticles as parabosons or parafermions of order 2. The symmetry types of the 6-particle system are given by the decomposition



If the particles are parabosons, then the symmetry type  or  must be chosen, if they are parafermions then the symmetry types  or  must be chosen.

Thus, in general, the choice of the symmetry types for a system of many systems of paraparticles is not unique. However, it can be easily shown that the choice is unique for systems of bosons or fermions by applying the rules on page 76.

CHAPTER VI

SYSTEMS WITH VARIABLE NUMBER OF PARAPARTICLES

1. Indistinguishability Postulate

So far, we discussed only quantum mechanics of systems with a fixed number of identical particles. When the number of particles is not conserved, the state vectors of the system are assumed to be in the Fock space $\mathcal{F}(H)$ which is the direct sum of all space H^N constructed from H with the convention that H° denotes the space of constant functions,

$$\mathcal{F}(H) = \bigoplus_{N=0}^{\infty} H^N \tag{1}$$

Any vector in $\mathcal{F}(H)$ is of the form

$$|\psi> = |\psi^\circ> + |\psi^1> + \cdots + |\psi^N> + \ldots$$

where $|\psi^\circ> \varepsilon H^\circ$, $|\psi^1> \varepsilon H^1$, etc. We can define in $\mathcal{F}(H)$ projection operators Σ^N onto H^N as follows

$$\Sigma^N|\psi> = |\psi^N>$$
$$\Sigma^N|\psi^M> = 0 \qquad \text{if} \qquad N \neq M \tag{2}$$

We have

$$\Sigma^N = \bigoplus_{\mu} \Sigma^N_\mu \tag{3}$$

where Σ^N_μ is the projection operator onto the subspace of H^N of μ-symmetry type. A state is specified by $(N-\mu)$ and will be called the state of $(N-\mu)$ symmetry type. Furthermore.

$$\Sigma^N_\mu = \Sigma^N_{\mu,1} + \cdots + \Sigma^N_{\mu,j} + \cdots + \Sigma^N_{\mu,n_\mu} \tag{4}$$

where $\Sigma^N_{\mu,j}$ is the projection operator onto an irreducible subspace of the decomposition (III-2^2).

The indistinguishability postulate can be formulated in $\mathcal{F}(H)$. Some care must be exercised because the permutation operators $U(\sigma)$ are not defined in the whole $\tilde{\mathcal{F}}(H)$ but instead in each subspace H^N. The equations of operators, however, must be applied to vectors in $\mathcal{F}(H)$. Following Messiah and Greenberg (1964) we make use of the projection operators Σ^N to impose the indistinguishability postulate on an observable A^N in each subspace H^N. Note that

$$A^N = \Sigma^N A^N \Sigma^N \qquad (5)$$

The indistinguishability postulate would find itself in the expression in each subspace H^N

$$[A^N, U^N(\sigma)] = 0 \qquad (6)$$

where $U^N(\sigma)$ are the permutation operators previously defined in H^N.

Consider an observable A^M in H^M and let $G(t)$ be the evolution operator. Since Σ^N is certainly a physical observable, the operator $\Sigma^N G^+(t) A^M G(t) \Sigma^N$, being product of physical observables, is also a physical observable. Furthermore, $\Sigma^N G^+(t) A^M G(t) \Sigma^N$ is a physical observable in H^N, eqn. (6) requires

$$[\Sigma^N G^+(t) \Sigma^M A^M \Sigma^M G(t) \Sigma^N, U^N(\sigma)] = 0 \qquad (7)$$

for all $\sigma \in S_M$, where we have made use of the fact that $A^M = \Sigma^M A^M \Sigma^M$.

2. Messiah Greenberg's Selection Rule

It is usually believed that eqn. (7) implies the following selection rule derived by Messiah and Greenberg (1964):

"Transitions from a symmetry type to a symmetry type of smaller dimension* are absolutely forbidden". We wish to give a critical comment on the derivation of this selection rule.

Messiah Greenberg's selection rule has been derived in the following fashion:

"Consider a transition from a state in an irreducible subspace $H^N_{\mu,j}[K]$, say, to a state in an irreducible subspace $H^M_{\nu,k}[K']$ (the definition of $H^N_{\mu,j}[K]$ and $H^M_{\nu,k}[K']$ were given in Chapter III); K is an N-particle state and K' is an M-particle state. The transition under consideration is obviously described by the operator.

$$Q = \Sigma^M_{\nu,k}[K']\ G(t)\ \Sigma^N_{\mu,j}[K] \qquad (8)$$

where $\Sigma^N_{\mu,j}[K]$ and $\Sigma^M_{\nu,k}[K']$ are the projectors onto $H^N_{\mu,j}[K]$ and $H^\mu_{\mu,k}[K']$.

From eqn. (7) taken with $\Sigma^M_{\nu,k}[K']$ for the observable A^M, it follows that Q^+Q is invariant under all $U^N(\sigma)$. Since in $H^N_{\mu,j}[K]$, due to Schur's lemma, there exists only one invariance, the operator $\Sigma^N_{\mu,j}[K]$, we must have

$$Q^+Q = c\ \Sigma^N_{\mu,j}[K] \qquad (9)$$

where c is a constant.

Let n_μ be the dimension of $H^N_{\mu,j}[K]$ and m_ν be that

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* By definition, the dimension of a symmetry types is the dimension of the irreducible representation of the symmetric group to which it is associated.

of $H^M_{\nu,k}[K']$. Then Q can be represented by an n x m matrix and Q^+ can be represented by an m x n matrix. Eqn. (9) writes as

$$\sum_m Q^*_{mn'} Q_{nm} = c\,\delta_{n'n} \tag{10}$$

where $Q = \{Q_{nm}\}, 1<n<n\mu,\ 1<m<m_\nu$, and $\delta_{n',n}$ is the Kroneckor symbol.

The ranks of the matrix Q and Q^+ are at most equal to m_ν. This implies that the rank of Q^+Q is also at most equal to m_ν. However, the rank of $\Sigma^N_{\mu,j}[K]$ is equal to n_ν. Hence, if $m_\nu<n_\mu$, eqn. (9) or (10) implies that c = 0 or equivalently Q = 0. From this follows Messiah Greenberg selection rule.

A weak point of this derivation is that the operators $\Sigma^M_{\nu,k}[K']$ and $\Sigma^N_{\mu,j}[K]$ must be assumed to be physical observables, an assumption which is not necessarily valid as shown in section III.3. However, this weak point can be remedied easily. In fact, we should consider a transition from a state in an invariant subspace $H^N_\mu[K]$ to a state in an invariant subspace $H^M_\nu[K']$, which is determined by the operator

$$T = \Sigma^M_\nu[K']\ G(t)\ \Sigma^N_\mu[K] \tag{11}$$

where $\Sigma^N_\mu[K]$ and $\Sigma^M_\nu[K']$ are projectors onto $H^N_\mu[K]$ and $H^M_\nu[K']$ respectively. Then T^+T must be permutation invariant so that it could be represented by the matrix

$$T^+T = \begin{bmatrix} \lambda_{11}I & \lambda_{12}I & & \lambda_{1n_\mu}I \\ \lambda_{21}I & \lambda_{22}I & & \lambda_{2n_\mu}I \\ & & & \\ \lambda_{n_\mu 1}I & \lambda_{n_\mu 2}I & & \lambda_{n_\mu n_\mu}I \end{bmatrix} \tag{12}$$

where λ_{ij} are constants and I is the $n_\mu \times n_\mu$ identity matrix. The above argument about the rank of T^+T could be applied here to obtain Messiah and Greenberg's selection rule. Now we can show that Messiah and Greenberg's selection rule implies the conservation of the dimension of the symmetry type. In fact, if $T \neq 0$, then $T^+ \neq 0$. But, an argument similar to the above leads us to the result that $T^+ \neq 0$ only if $n_\mu > m_\nu$. This, coupled with the condition $n_\mu < m_\nu$, yields $n_\mu = m_\nu$. From this follows the conservation law which we shall call Messiah and Greenberg's conservation law : The dimension of the symmetry type is conserved in any transition.

3. Carpenter's Selection Rule

In this section, we present a derivation of a selection rule similar to that obtained by Carpenter (1970) for S-matrix within the framework of Landshoff and Stapp's (1967) unified theory of identical particles. Our selection rule will be applicable to the evolution operator G(t) and derivable from eqn. (6).

Consider two transitions described by T defined by eqn. (11) and R defined by

$$R = \Sigma_{\nu}^{M}[K''] \; G(t) \; \Sigma_{\gamma}^{N}[K] \tag{13}$$

where $\Sigma_{\gamma}^{N}[K]$ is the projector onto the symmetry type (N,γ).

The operator $R^{+}T$ is a physical observable in $H^{N}[K]$ so it must satisfy eqn. (6). i.e.

$$[R^{+}T, \; U^{N}(\sigma)] = 0 \tag{14}$$

for all $U^{N}(\sigma)$. But $R^{+}T$ is a mapping from $H_{\mu}^{N}[K]$ to $H_{\gamma}^{N}[K]$, eqn. (14) implies that it must be a zero mapping as a consequence of the superselection rule operating between the symmetry types in $H^{N}[K]$. That is

$$R^{+}T = 0 \tag{15}$$

Now Messiah and Greenberg's conservation law implies that T is a mapping of $H^{N}[K]$ onto $H^{M}[K']$. Putting $K' = K''$, we see that if

$$T|\psi^{N}[K]\rangle \neq 0 \tag{16}$$

for any $|\psi^{N}[K]\rangle \in H^{N}[K]$, then eqn. (15) requires that

$$R^{+}|\psi^{M}[K']\rangle = 0$$

for any $|\psi^{M}[K']\rangle \in H^{M}[K']$, or equivalently

$$R = 0 \tag{17}$$

Eqns. (16) and (17) imply the following selection rule: If an N-particle system of symmetry type μ undergoes a transition to an M-particle system of symmetry type ν, then it cannot make transitions to other M-particle systems of a different symmetry type. We shall call this Carpenter's selection rule.

4. Discussion

Carpenter's selection rule has been derived for systems of identical particles of one species. However, as already stated in the beginning of this thesis, the presence of particles of other species can be viewed as external interactions acting on our considered system. Thus, Carpenter's selection rule may be applied to each species of particle in a complicated system.

Consider an N-boson (fermion) system. Carpenter's selection rule states that if the system can make a transition to an M-boson (fermion) system, then it cannot make a transition to an M-fermion (boson) system. We wish to show that if the particles can be created or destroyed freely then it is always possible for an N-boson (fermion) system to make a transition to an M-boson (fermion) system; M is arbitrary. This would mean that a system of bosons always remains to be a system of bosons; a fact that has been observed.

To prove the stated proposition, we perform the following experiment:

Suppose that we have an N-boson system in our laboratory. Since the bosons can be created freely, we create M'bosons of the same species in a region very far from the laboratory. This creation, according to the cluster assumption, would not disturb the symmetry type of the N-bosons in the laboratory. The system N + M' bosons is again a system of bosons. Here, an objection can be raised: The system of N + M' particles may not be a boson system. We take the condition that two system of bosons form a system of bosons as a definition of bosons. By an appropriate preparation of the wave packet of the initially created M'bosons, we could make these M'bosons enter, at some time, into the laboratory through a window. Since the symmetry type of the system of N + M'bosons is conserved, we would have at the time the M'bosons enter the laboratory, a system of N + M' bosons in the laboratory. To an observer inside the laboratory, the N-boson system have made a transition to a system of N + M' = M bosons. Since M' can be arbitrary, M is arbitrary and greater than N. To show that the N-boson system can make a transition to M-boson system with M<N, we suppose that the wave packet of the N-bosons system has been prepared such that M' bosons could escape from

the laboratory through the window and travel to a region very far away from the laboratory, where they can be destroyed. This destruction, again according to the cluster assumption, cannot disturb the symmetry type of the system of $N - M'' = M$ remaining particle in the laboratory. To the observer inside the laboratory, the whole process appears as if the N-boson system has made a transition to an M-boson system, $M<N$.

CHAPTER VII

SECOND QUANTIZATION THEORY OF PARAPARTICLES

In this chapter, we shall discuss the second quantization theory of paraparticles. This theory, however, has been developed independently of the first quantization theory discussed so far and owes no result to that theory.

In second quantization theory, the system of identical particles of a species is described by the field operators $\psi(x)$ and $\psi^*(x)$ and no particle variable need to be introduced. One does not have to use the permutation operators to impose the indistinguishability of the particles since the particles have no identity in the second quantization theory. The statistics of the particles are reflected in the method of quantization one adopts: quantization with the commutation relation yields Bose-Einstein statistics, quantization with the anti-commutation relation yields Fermi-Dirac statistics. The theory of second quantization of paraparticles consists in adopting algebraic rules, different from the usual commutation or anti-commutation relations, for the field $\psi(x)$, $\psi^*(x)$, while still reserving the equation of motion. This may imply that the equation of motion in fact does not determine uniquely the algebraic rules for the operators representing observables.

The possibility that the equation of motion does not determine uniquely the commutation rules for the operators $\underset{\sim}{p}$ and $\underset{\sim}{q}$ was first discussed by Wigner (1950) in the example of a harmonic oscillator. Wigner's analysis has been extended by O'raifeartaigh and Ryan (1963) and by Boulware and Deser (1963). Before going into the second quantization theory of paraparticles, we shall illustrate the content of this theory in the method of quantization of a harmonic oscillator.

1. Quantization of a Harmonic Oscillator

Consider a linear harmonic oscillator of unit mass and frequency $1/2\pi$. The classical Hamiltonian of this system is

$$H = \frac{1}{2}(p^2 + q^2) \tag{1}$$

and the Lagrange equation of motion is

$$\dot{q} = p, \ \dot{p} = -q \tag{2}$$

Let a and a* be defined as

$$a = \frac{q + ip}{\sqrt{2}} \tag{3}$$

$$a^* = \frac{q - ip}{\sqrt{2}} \tag{4}$$

Then the Hamiltonian can be written as $H = a^*a$ (5)

equation of motion (2) become

$$\dot{a} = - ia \tag{6}$$

$$\dot{a}^* = ia^* \tag{7}$$

The passage to quantum theory consists of

(i) replacing the c-number variables a, a* by operators defined in a Hilbert space

$$a \rightarrow a_{op}$$

$$a \rightarrow a^*_{op} \qquad (8)$$

In making this change in H, however, some care must be exercised. In the classical theory, the ordering of a and a* in H is unimportant so that we could write the classical H as

$$H = \mu aa^* + (1-\mu)a^*a$$

$$= a^*a + \mu[a,a^*] \qquad (9)$$

with

$$0 \leq \mu \leq 1 \qquad (10)$$

to guarantee the positive definiteness of H.

The ordering of a and a*, however, is important in quantum theory. Hence, to obtain the most general quantum mechanical Hamiltonian we should make the replacement (8) in the H of (9) rather than the H of (5),

$$H_{op} = a^*_{op}\, a_{op} + \mu\, [a_{op},\, a^*_{op}] \qquad (11)$$

(ii) usually postulating the commutation relation

$$[a_{op},\, a^*_{op}] = 1 \qquad (12)$$

The results of this method of quantization is well known. We mention here two points. Firstly, there exists a unique infinite dimensional Dirac representation of a_{op} and a^*_{op}; secondly, if we use the Heisenberg rule

$$\dot{O}_{op} = i[H_{op},\, O_{op}] \qquad (13)$$

then

$$\dot{a}_{op} = -i\, a_{op} \tag{14}$$

$$\dot{a}^*_{op} = i\, a^*_{op} \tag{15}$$

which coincides with the classical Lagrange equations of motions (6) and (7).

The point of importance is that one can alternatively carry out the step (ii) by a different method. One does not adopt any commutation rule for a_{op} and a^*_{op} but rather demand that the Heisenberg rule (13) together with the Hamiltonian H_{op} of (11) yield the equation of motion for a_{op} and a^*_{op} identical to the classical Lagrange equations, that is, one demands that

$$[H_{op}, a_{op}] = - a_{op} \tag{16}$$

$$[H_{op}, a^*_{op}] = a^*_{op} \tag{17}$$

with H_{op} given by (11),

$$H_{op} = a^*_{op}\, a_{op} + \mu[a_{op}, a^*_{op}], \quad (11^1)$$

$$o \leq \mu \leq 1$$

From now on we drop the subscript "op" on the a_{op} and a^*_{op}. A recurring problem in quantum theory is that of finding the representations of the a and a* obeying (16) and (17) in a Hilbert space. In general, the commutators [a*, a] are different in each representation. The results due to O'Raifeartaigh and Ryan (1963) can be classified into three cases depending on μ and E_o, the lowest eigenvalue of H_o (zero point energy):

Case 1: $\mu = 0$, $E_o = 0$

The representations of a are of finite dimension $D = 1,2 \ldots$ with

$$|a_{r+1,r}|^2 = r; \quad (1 \le r \le D)$$

and all other matrix elements are zero so that

$$(a^+)^{D-1} \phi_o \neq 0$$

$$(a^+)^D \phi_o = 0$$

where ϕ_o is eigenvector of H_o corresponding to eigenvalue E_o.

The representation $D = 2$ is just the Fermi oscillator and the representation $D \to \infty$ is the normal Bose oscillator. All other cases give para-Fermi statistics in which the occupation number is $(D-1)$

Case 2: $\frac{1}{2} > \mu > 0$, $E_o = \mu \pm \frac{N}{\left(\frac{1-\mu}{\mu}\right)^N \mp 1}$

The representations are of even dimension N corresponding to (+) sign or odd dimension N corresponding to (-) sign. Boulware and Deser (1963) show that, in this case, there exist a transformation U such that $A = Ua$ admits the representation of Case 1, and that the Hamiltonian becomes

$$H = A^+A + E_o(\mu, N)$$

so that its zero point energy is shifted by $E_o(\mu, N)$.

Case 3: $\mu \neq 0$, $E_o \neq 0$

This includes the case $\mu = \frac{1}{2}$ treated by Wigner (1950). The representations of a are infinite dimensional and, as showed by Boulware and Deser, can be transformed to the normal Bose oscillator by a transformation whose effect on the Hamiltonian is to shift its zero point energy.

The commutation relation $[a,a^*]$ are different in all three cases. We have seen that the essential non-uniqueness of the commutation relation arises from the possibility of employing Fermi or para-Fermi statistics. All other apparently possible commutation relations merely shift the zero-point energy.

2. Second Quantization of Paraparticles

In this section we shall discuss various known methods of field quantization allowing parastatistics. We shall consider in some detail the two well-known methods due to Green (1953) and to Kamefuchi and Takahashi (1962)and shall briefly touch upon the others. To avoid complications arising from anti-particles, we shall restrict ourself to quantization of a non-relativistic field.

a) Green's Method

Green studies paraparticles by requiring that the free Hamiltonian be properly symmetrized as

$$H_o = \frac{1}{4} \int d^3x \, [\underline{\nabla}\psi^+(x,t), \ \underline{\nabla}\psi(x,t)]_{\pm} \qquad (18)$$

For a reason which will become clear later, H_o is symmetrized (+ sign) for parabosons and anti-symmetrized (- sign) for parafermions.

Green's method consists in requiring that the field $\psi(x,t)$ satisfied the Heisenberg equation:

$$\frac{\partial\psi(x,t)}{\partial t} = i[H_o, \psi(x,t)] \tag{19}$$

Let $\{|k>\}$ be a complete set of one particle states and consider the Fourier expansions

$$\psi(x,t) = \sum_k <x,t|k> a_k \tag{20}$$

$$\psi^+(x,t) = \sum_k <x,t|k> a_k^+ \tag{21}$$

In terms of the creation and annihilation operators, a_k^+ and a_k, the free Hamiltonian take the form

$$H_o = \frac{1}{2} \Sigma\omega_k [a_k^+, a_k]_\pm \tag{22}$$

where

$$\omega_k = \frac{1}{2} \int d^3x \underline{\nabla}(<x,t|k>) \tag{23}$$

and the Heisenberg equation (19) becomes

$$[H_o, a_k] = \omega_k a_k^+ \tag{24}$$

Green finds that the necessary and sufficient condition for which (24) is satisfied is

$$[\ [a_k^+, a_\ell]_\pm, a_m] = -2\delta_{km} a_\ell \tag{25}$$

and that this relation supplemented by the relation

$$[\ [a_k, a_\ell]\ , a_m] = 0 \tag{26}$$

may be used as the condition of quantization for paraparticles.

From (25), (26) and the relations obtained from them by taking the hermitean conjugate, it can be verified that there exists an operator defined to a constant by

$$N_k = \frac{1}{2} [a_k^+, a_k]_\pm + \text{constant} \tag{27}$$

which has the property of a number operator,

$$[N_k, a_\ell^+] = \delta_{k\ell}\, a_k^+ \tag{28}$$

$$[N_k, a_\ell] = -\, \delta_{k\ell}\, a_k \tag{29}$$

$$[N_k, N_\ell] = 0 \tag{30}$$

A different way to obtain Green's commutation relations, (25) and (26), is proposed by Bialynicki-Birula (1963) who demands that equations (28), (29) and (30) be invariant under any unitary transformation of the one particle states,

$$|'k\rangle = \sum_\ell (\delta_{k\ell} - \alpha_{k\ell})|\ell\rangle$$

where the unitary condition in terms of the infinitesimal parameters $\alpha_{k\ell}$ has the form

$$\alpha_{k\ell} + \alpha_{k\ell} = 0 \quad .$$

Under this transformation, a_k and a_k^+ transform as

$$'a_k = \sum_\ell (\delta_{k\ell} + \alpha_{k\ell}) a_\ell$$

$$'a_k^+ = \sum_\ell (\delta_{k\ell} - \alpha_{k\ell}) a_\ell^+$$

An explicit construction of a_k and a_k^+ satisfying Green's commutation relations is provided by Green's ansatzes (1953):

For parabosons, consider a set of operator $a_k^{(\alpha)}$, $\alpha = 1,2 \ldots p$, obeying the relations

$$[a_k^{(\alpha)}, a_\ell^{(\alpha)+}]_- = \delta_{k\ell}$$

$$[a_k^{(\alpha)}, a_\ell^{(\alpha)}]_- = [a_k^+, a_\ell^{(\alpha)+}]_- = 0 \qquad (31)$$

$$[a_k^{(\alpha)}, a_\ell^{(\beta)}]_+ = [a_k^{(\alpha)+}, a_\ell^{(\beta)}]_+ = 0, \ \alpha \neq \beta$$

Then a_k, a_k^+ defined as

$$a_k = \sum_{\alpha=1}^{p} a_k^{(\alpha)} \qquad (32)$$

$$a_k^+ = \sum_{\alpha=1}^{p} a_k^{(\alpha)+}$$

satisfy Green's commutation relations. For para-fermions, the (+) and (-) signs in (31) are interchanged. (32) will be called Green's ansatz of order p.

With the usual definition of the vacuum state $|0\rangle$ for the Green components $a_k^{(\alpha)}$, $a_k^{(\alpha)+}$, i.e.

$$a_k^{(\alpha)} |0\rangle = 0$$

Green's ansatzes give

$$a_k a_\ell^+ |0\rangle = p\delta_{k\ell}|0\rangle \qquad (33)$$

This equation together with (25), (26) and their hermitian conjugate are sufficient for calculating the expectation value of any function of a_k and a_k^+.

Given the irreducible representation of $a_k^{(\alpha)}$, $a_\ell^{(\alpha)+}$ satisfying (31), the representation of Green's para-commutation relation rules via Green's ansatzes , are not in general irreducible. However, in the next chapter we shall show that Green's ansatzes yield all irreducible representations with a unique vacuum state. Therefore, (33) is the general rule for calculating the expectation values. In the following, we shall frequently use Green's ansatzes and shall discuss the meaning of these ansatzes for field theory in Chapter VIII.

b. Kamefuchi and Takahashi's Method

Kamefuchi and Takahashi (1962) determine the commutation relation by considering the unitary infinitesimal, linear transformation which leaves the following quantities invariant

$$\Lambda_{\pm} = \sum_{k=1} [a_k^{+}, a_k] \tag{34}$$

namely, the transformation

$$a_k' = G^{+} a_k G \tag{35}$$

where

$$G = 1 - i \sum_{\ell m} N_{\ell m} \S_{\ell m} - \frac{1}{2} \sum_{\ell m} L_{\ell m} \eta_{\ell m} - \frac{1}{2} i \sum_{\ell m} M_{\ell m} \S_{\ell m}$$

with the conditions

$$\begin{aligned} &\S^{*}_{km} = \S_{mk}, \ \eta^{*}_{km} = \S_{mk} \\ &\eta_{km} \pm \eta_{mk} = 0 \\ &L^{+}_{\ell m} = M_{m\ell} \\ &M_{\ell m} \pm M_{m\ell} = 0 \end{aligned} \tag{36}$$

and

$$[a_k, N_{\ell m}] = \delta_{k\ell}\, a_m$$

$$[a_k, L_{\ell m}] = \delta_{k\ell}\, a^+_m \pm \delta_{km}\, a^+_\ell \qquad (37)$$

$$[a_k, M_{\ell m}] = 0$$

The upper and lower signs are referred to S-type and R-types transformations respectively.

Kamefuchi and Takahashi demand that these transformations are representations of a group so that G satisfies the closure relations which establish a Lie algebra of the generators $N_{\ell m}$ and L_{mn}. It turns out that the operators a_k and a_k^+ are subjected to unitary transformation corresponding to the group O(2f) or $S_p(2f)$, the generators of these transformations $N_{\ell m}$, $L_{\ell m}$, M_{mm} satisfying (36), (37) and the following relations

$$[N_{k\ell}, N_{mn}] = \delta_{\ell m}\, N_{kn} - \delta_{kn}\, N_{m\ell}$$

$$[L_{k\ell}, L_{mn}] = 0$$

$$[M_{k\ell}, M_{mn}] = 0$$

$$[L_{k\ell}, N_{mn}] = -\, \delta_{kn}\, L'_{m\ell} \pm \delta_{\ell n}\, L_{mk} \qquad (38)$$

$$[M_{k\ell}, N_{mn}] = \delta_{km}\, M_{n\ell} \mp \delta_{\ell m}\, M_{nk}$$

$$[L_{k\ell}, M_{mn}] = -\, \delta_{km}\, N_{\ell m} \pm \delta_{kn}\, N_{\ell m} - \delta_{\ell n}\, N_{km}$$
$$\pm\, \delta_{\ell n}\, N_{km}$$

Here, the upper and lower signs correspond to the cases O(2f) and Sp(2f) respectively.

It can be shown that the simplest form of the generators as functions of a_i's and a_i^+'s are given by

$$N_{k\ell} = K[a_k^+, a_\ell]_{\mp}$$

$$L_{k\ell} = K[a_k^+, a_\ell^+]_{\mp} \tag{39}$$

$$M_k = K[a_k, a_\ell]_{\mp} \tag{40}$$

where K is a real constant. The operator

$$N_k = N_{kk} + \text{const.}$$

satisfies (28), (29), (30) and thus can be identified as the number operator. Furthermore, with a conventional normalization of a_k and a_k^+ one can take $K = \pm\frac{1}{2}$. We consider various cases:

<u>Case 1:</u> $K = \frac{1}{2}$, R type

Ohnuki, Yamada and Kamefuchi (1971) point out that the set of operators $N_{k\ell}$, $L_{k\ell}$, $M_{k\ell}$, when supplemented by a_k and a_k^+, form the Lie algebra for O(2f+1). This coincides with a result obtained by Ryan and Sudarshan (1963), namely, that the algebra of a set of operators a_k and a_k^+, k = 1, 2 ... f, is isomorphic to the Lie algebra of O(2f+1). This suggests that case 1 corresponds to Green's para-Fermi quantization. In fact, Bialynicki-Birula (1963) points out that Green's ansatzes satisfy all the commutation relations found by K.T. Since we shall show in

Chapter VII that Green's ansatzes actually exhaust all possible irreducible representation with unique vacuum state of Green's commutation relations, Green's quantization is included in Kamefuchi and Takahashi's method.

Case 2: $K = \frac{1}{2}$, S type

This corresponds to Green's parabose quantization due to the fact that the last remark for case 1, about Green's ansatzes , is applicable to this case.

In both cases 1 and 2, one can prove that, for the representation characterized by the parameter s ,the number operator defined as

$$N_k = \frac{1}{2}[a_k^+, a_k]_{\mp} \pm \frac{1}{2}s(s-1)$$

has zero as its lowest eigenvalue. Defining the vacuum state $|0\rangle$ as

$$a_k|0\rangle = 0 \text{ and } N_k|0\rangle = 0$$

one obtains

$$a_k \, a_\ell^+ \, |0\rangle = (s-1) \, \delta_{k\ell}|0\rangle$$

This relation together with (37) enables one to evaluate the vacuum expectation values of product of field operators. Comparing this result with the results given by Green's ansatzes , equations (33), one arrives at the equivalence of Case 1 and Case 2 to Green's quantization method.

Case 3: $K = -\frac{1}{2}$, R type or S type

This case corresponds to the method proposed by

Kamemova and Kraev.(1971) Ohnuki, Yamada and Kamefuchi show that this case is not applicable to field theory because it yields negative norms for the completely anti-symmetric N-particle states or negative eigenvalues for the number operators.

Further generalization of the method of quantization may be obtained by adopting, instead of (39) and (40), a different ansatz for the generators $N_{k\ell}$, $L_{k\ell}$, $M_{k\ell}$. K.T. (1966) also considering the following ansatz for the generators

$$N_{k\ell} = \sum_{\substack{\alpha=1,2...\\ m,m',m''}} (A_m^{(\alpha)} a_k^+ ... a_\ell C_{m'...}^{(\alpha)} C_{m''...}^{(\alpha)} - A_{m...}^{(\alpha)} a_\ell B_{m'}^{(\alpha)} a_k^+ C_{m''}^{(\alpha)})$$

$$L_{k\ell} = \sum_{\substack{\alpha=1,2...\\ m,m',m''}} (A_m^{(\alpha)} a_k B_{m'...}^{(\alpha)} a_\ell C_{m''}^{(\alpha)} - A_{m...}^{(\alpha)} a_\ell^+ B_{m'...}^{(\alpha)} a_k^+ C_{m''...}^{(\alpha)}) \qquad (41)$$

$$M_{k\ell} = \sum_{\substack{\alpha=1,2\\ m,m',m''}} (A_{m...}^{(\alpha)} a_k B_{m'...}^{(\alpha)} a_\ell C_{m''}^{(\alpha)} - A_{m...}^{(\alpha)} a_\ell B_{m'...}^{(\alpha)} a_k C_{m''...}^{(\alpha)}$$

Here $A_{m...}^{(\alpha)}$, $B_{m'...}^{(\alpha)}$ and $C_{m''...}^{(\alpha)}$ are functions of a_m's and a_m^+'s such that (i) the expression $\sum_{m,m',m''} A_{m...}^{(\alpha)} B_{m'...}^{(\alpha)} C_{m''}^{(\alpha)}$ as a whole remains invariant under the orthogonal transformations of a_k's and a_k^+'s, (ii) the resulting $N_{k\ell}$, $L_{k\ell}$, and $M_{k\ell}$ satisfy the conditions (36) and (iii) the operators $N_k \equiv N_{kk}$ become traceless.

K.T. find that this ansaltz yields the commutations relations for a_k and a_k^+ which take the forms different from those of Green. The same result is also obtained by Ohnuki who finds that the bound states of particles associated to Green's method do not obey Green's conditions of quantization. It is not known whether these bound states obey the quantization conditions by the ansatz (41) or others within the K.T. framework of quantization.

Note that Green's and T.K.'s para-Fermi quantization, in the one dimensional limit, does not correspond to the quantization of the harmonic oscillator considered by O'raifeartaigh and Ryan (Case i discussed in section 1) in contrast to the claim of Boulware and Deser that the non-uniqueness of the commutation relations for a harmonic oscillator arises from the possibility of para-Fermi statistics in Green's sense. In fact, Green's para-Fermi ansaltz does not satisfy equation (17) (with $\mu = 0$ for case 1 on page 92). To see this, let us write the first hand side of equation (17) using Green ansatz

$$[a^* a, a^*] = a - 2 \sum_{\alpha} a^{(\alpha)*} a^{(\alpha)} a^{(\alpha)*} + \sum_{\substack{\alpha\neq\beta \\ \gamma=\beta}} a^{(\alpha)*} a^{(\beta)} a^{(\gamma)*} - \sum_{\substack{\alpha\neq\beta \\ \alpha'=\beta}} a^{(\gamma)*} a^{(\alpha)*} a^{(\beta)}$$

This equation does not agree with (17) because of the presence of the last 3 terms which do not cancel one another when acting on any state vector, a fact easily verified by letting them act on the vacuum.

c) Other Methods of Quantization

Different sets of commutation rules are given by Volkov (1959), by Okayama (1952). Volkov studies the properties of a fields associated to the commutation rules:

$$a_k\, a_\ell^+\, a_m \mp a_m\, a_\ell^+\, a_k = 2\delta_{k\ell}\, a_m \mp 2\delta_{m\ell}\, a_k$$

$$a_k^+\, a_\ell\, a_m \mp a_m\, a_\ell\, a_k^+ = \mp 2\delta_{k\ell}\, a_m$$

$$a_k\, a_\ell\, a_m \mp a_m\, a_\ell\, a_k = 0 \qquad (42)$$

These commutation relations are satisfied by Green's ansaltzes of order 2 so that they are just a representation of Green's commutation relations.

Okayama (1952) proposes a set of commutation relations which permit statistics of occupation number $n_{max} = 2$ as follows

$$a_k\, a_\ell^+\, a_m - a_\ell^+\, a_m\, a_k = \delta_{k\ell}\, a_m$$

$$\sum_{(perm)} a_k\, a_\ell\, a_m = 0$$

$$a_k\, a_k\, a_k^+ + 2\, a_k^+\, a_k\, a_k = 2a_k$$

$$a_k\, a_k^+\, a_k^+\, a_k^+ + a_k\, a_k^+\, a_k^+ + a_k^+\, a_k^+\, a_k\, a_k = 2$$

T.K. (1962) proves that this algebra allows only a trivial solution so that it cannot be applied to field theory.

We note that there exists a different method of quantization due to Roman and Aghassi. Kamefuchi (1966)

point out that this method can be obtained by modifying Okayama's method in an appropriate way. Roman and Aghasshi's method has not been applied to relativistic field theory.
In the following chapters,we shall study field theory resulted from Green's quantization and the terminology para-commutation relations will be referred to Green's commutation relations.

CHAPTER VIII

REPRESENTATIONS OF GREEN'S COMMUTATION RELATIONS

We propose to study in this chapter all representations of Green's commutation relations. We shall exhibit for the para-Fermi commutation relations, a maze of infinite number of inequivalent irreducible representations among them figures the Fock representation used in second quantization theory. We shall obtain the representations given by Green's ansatzes from the well-known representations of the commutation and anti-commutation relations. We shall prove that Green's ansatzes exhaust all possible irreducible representations of Green's commutation relations with a unique vacuum state. Finally, as a verification of this result, we shall prove the existance of para-Fermi statistics in any representations with a unique vacuum state.

1. Discrete Representations of the Para-Fermi Commutation Relations

We propose to study, in this section, the representations of the para-Fermi commutation relations,

$$\left[\frac{1}{2}\,(a_j^* \;,\; a_k) \;,\; a_\ell\right] = -\,\delta_{j\ell}a_k \;, \tag{1}$$

$$\left[\frac{1}{2}\,(a_j \;,\; a_k) \;,\; a_\ell\right] = 0 \;, \tag{2}$$

in a Hilbert space. Unlike the anti-commutation relations, (1) and (2) do not imply that the a_j and a_j^* are necessarily bounded. Unbounded operators, as is well known, cannot be defined everywhere. We tacitly assume that a_j and a_j^* are defined at least in certain domain satisfying the condition which permit unique closed linear extension of a_j and a_j^* in the whole space H.

Let

$$n_j = \frac{1}{2} [a_j^* , a_j] \tag{3}$$

then (1) and (2) imply:

$$[n_j , a_j^*] = a_j^* \tag{4}$$

$$[n_j , a_j] = -a_j \tag{5}$$

$$[n_j , n_k] = 0 \tag{6}$$

Note that (3)(4) and (5) form the Lie algebra of O(3) (Jordan, Mukunda and Pepper 1963). In fact, if one puts

$$a_j^* = J_1 + iJ_2, \; a_j = J_1 - iJ_2 , \; n_j = J_3$$

then one has

$$[J_k , J_\ell] = i\varepsilon_{k\ell m} J_m , \; k, \ell, m = 1,2,3$$

which is the algebra of the angular momentum. It is well-known from the theory of angular momentum that the operator n_j is self-adjoint admitting the spectral resolution:

$$n_j = \sum_J p_j^{(J)} , \; J=0 , \frac{1}{2} , 1, \frac{3}{2} \ldots \tag{7}$$

$$p_j^{(J)} = \sum_{m=-J}^{J} m \, p_j^{(J,m)} \tag{8}$$

$$\sum_{Jm} p_j(J,m) = 1 \tag{9}$$

Furthermore

$$P_j^{(J,m)} = \left[J(J+1) - m(m+1)\right]^{-\frac{1}{2}} a_j^* P_j^{(J,m-1)} a_j \qquad (10)$$

or

$$P_j^{(J,m)} = \left[J(J+1) - m(m+1)\right]^{\frac{1}{2}} a_j P^{(J,m+1)} a_j^* \qquad (10^1)$$

Let us consider the self-adjoint operator

$$N_j = n_j + J\underset{\sim}{1} \qquad (11)$$

and its spectral resolution

$$N_j = \sum_J P_j^{(J)}, \; J = 0, \frac{1}{2} \ldots \qquad (12)$$

$$P_j^{(J)} = \sum_{n=0}^{2J} nP_j^{(J,n)} \qquad (13)$$

with

$$P_j^{(J,n)} = \left[J(J+1)-(n-J)(n-J+1)\right]^{-\frac{1}{2}} a_j^* P^{(J,n-1)} a_j$$

$$= \prod_{r=1}^{n} \left\{ J(J+1)-(r-J)(r-J+1)\right\}^{-\frac{1}{2}} (a_j^*)^n P^{(j,0)} (a_j)^n \qquad (14)$$

or

$$P^{(J,n)} = \left[J(J+1)-(n-J)(r-J+1\right]^{\frac{1}{2}} a_j P_j^{(J,n+1)} a_j^*$$

$$= \left\{\prod_{r=n}^{2J} J(J+1)-(r-J)(r-J+1)\right\}^{\frac{1}{2}} (a_j)^{2J-n} P_j^{(J,2J)} (a_j^*)^{2J-n} \qquad (14^1)$$

We note that $P_j^{(J,n)} P_k^{(J,m)} = P_k^{(J,m)} P_j^{(J,n)}$ as a consequence of (6) and that none of $P_j^{(J,n)}$ ever vanishes.

Following Wightman and Schweber (1955), we define a set of projection operators, $E_\alpha^{(J)}$, where α stands for the infinite sequence of integer α_1, α_2. With $\alpha_j = 0,1,2, \ldots 2J$. α can be regarded as the real number whose binary fraction expansion is $\alpha_1, \alpha_2 \ldots$, i.e. $\sum_{n=1}^{\alpha} 2^{-n} \alpha_n$. $E_\alpha^{(J)}$ is defined as

$$E_\alpha^{(J)} = \lim_{M\to\infty} E^{(J,M)} = \lim_{M\to\infty} \prod_{j=1}^{M} P_j^{(J,\alpha_j)} \qquad (15)$$

The physical significance of $E_{(\alpha)}^{(J)}$ is that it is the operator whose occupation number distribution is $\alpha_1, \alpha_2 \ldots$ As in the case of the anti-commutation relations, some of the $E_\alpha^{(J)}$ may vanish for a representation of a_j and a_j^* in a separable Hilbert space H. A representation in which all E_α vanish is called <u>continuous</u> and a representation in which no trivial subspace on which all $E_\alpha^{(J)}$ vanish is called <u>discrete</u>. In the following we shall concentrate only on discrete representations. It is quite easy to show that all properties found* by WS for the anti-commutation relations hold also for the para-Fermi commutation relations. They are:

<u>Lemma 1</u>

$E_{\alpha'}^{(J)} E_\alpha^{(J)} = 0 = E_\alpha^{(J)} E_{\alpha'}^{(J)}$ if $\alpha \neq \alpha'$.

$\sum_\alpha E_\alpha^{(J)}$ is a projection operator. $(\sum_\alpha E_\alpha^{(J)})$ H and $(1- \sum_\alpha E_\alpha^{(J)})$H are manifolds of H invariant under the a_j and a_j^*. In $(\sum_\alpha E_\alpha^{(J)})$H the representation is discrete, in $(1- \sum_\alpha^\alpha E_\alpha^{(J)})$H it is continuous. A necessary condition that two representations be unitary equivalent is that $E_\alpha^{(J)}$ be equivalent for all ∞. Representations corresponding to different J are inequivalent and $E_\alpha^{(J)} E_{\alpha'}^{(J')} = E_{\alpha'}^{(J')} E_\alpha^{J} = 0$ if $J \neq J'$.

Proof:

The proof of every statement except the last one of this lemma may be proceeded in the same way as given by WS*. For later use, we sketch here the main argument of the proof.

The manifold belonging to $E_\alpha^{(J)}$ is characterized by the fact that if

$$E_\alpha^{(J)}\phi = 0 \tag{16}$$

then

$$P_j^{(J,\alpha_j)}\phi = \phi \tag{17}$$

for all j = 1,2, ... Consequently, if $\alpha \neq \alpha'$, $E_\alpha^{(J)} E_{\alpha'}^{(J)} = E_{\alpha'}^{(J)} E_\alpha^{(J)} = 0$ The sum $\sum_\alpha E_\alpha$ is a projection operator since it is the sum of orthogonal projection operators. From the relations

$$[P_j^{(J,\alpha_j)}, a_j^*] = a_j^* \tag{18}$$

$$[P^{(J,\alpha_j)}, a_j] = - a_j \tag{19}$$

it follows that if ϕ satisfy (16) and hence (17) then $a_j \phi$ and $a_j^* \phi$ satisfy (17) with α_j changed by one and therefore (16) for some $E_{\alpha'}^{(J)}$. The net result is that ϕ satisfying (16) for some α are carried by application of a_j or a_j^* into ϕ satisfying (16) for some other α. Consequently, $(\sum_\alpha E_\alpha^J)H$ is invariant under a_j and a_j^*.

The representation of the a_j and a_j^* is continuous in $(1- \sum_\alpha E_\alpha^{(J)})$ follows from the definition of $(1- \sum_\alpha E_\alpha^{(J)})$ and is discrete in $(\sum_\alpha E_\alpha^{(J)})H$ follows from the fact that the vanishing

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* Hereafter we shall use the abbreviation W.S. for Wightman and Schweber (1955).

of E_α ϕ for all α implies the vanishing of ϕ.

If two representations are unitarily equivalent there exists an operator U such that

$$E_\alpha^{(J)} = U\, E_\alpha^{1(J)}\, U^{-1} \tag{21}$$

From the theory of angular momentum, it is obvious that representations corresponding to different J are inequivalent and that $E^{(J)}\, E^{(J')} = E^{(J')} E^{J} = 0$ if $J \neq J'$. The representation of a_j and a_j^* is thus the direct sum of representations each of which is characterized by a positive integer p=2J and is in term direct sum of a discrete and a continuous representations.

Although many of $E_\alpha^{(J)}$ may vanish, if one is non-vanishing so are an infinity of others. In order to deal with this situation, WS define equivalence class, $[\alpha]$, of the sequence α: α and α' lie in the same equivalent class if they differ in at most a finite number of digits.

<u>Lemma 2</u>

All $E_\alpha^{(J)}$ whose α's belong to the same equivalence class have the same dimension. Furthermore

$$(a_j)^n\, E^{(J)}_{\alpha_1,\ \alpha_2 \ldots \alpha_j \ldots}\, (a^*_j)^n = E_{\alpha_1\ \alpha_2 \ldots \alpha_{j-n}} \tag{22}$$

$$(a_j^*)^n\, E^{(J)}_{\alpha_1\ \alpha_2 \ldots \alpha_j \ldots}\ (a_j)^n = E_{\alpha_1\ \alpha_2 \ldots \alpha_{j+n}}$$

With the definition that $E_{\alpha_1\ \alpha_2 \ldots} = 0$ for α_j negative.

<u>Proof</u>

Let ϕ be a non-null proper function of E_α. If $\alpha_j = 0$

consider $a_j\phi$, if $\alpha_j \neq 0$ consider $a_j^*\phi$ and $a_j\phi$. As we have seen in Lemma 1, these are non null proper function of $E_{\alpha'}$, where α' is different from α by only in the j^{th} place, i.e., $\alpha_j' = \alpha_j + 1$. Furthermore, if $\alpha = 0$, $a_j^*\phi \neq 0$ since if $a_j^*\phi = 0$ (14^1) implies $P^{(J,0)}\phi = 0$, a contradiction. If $\alpha_j \neq 0$, $a_j^*\phi \neq 0$, and $a_j\phi \neq 0$ since if $a_j^*\phi = 0$, $a_j\phi = 0$, (14) and (14^1) imply $P_j^{(J,\alpha_j)}\phi = 0$, a contradiction. Consequently, a_j^* is a non singular mapping of the manifold of E_α into the manifold of $E_{\alpha'}$, and a_j is a non singular mapping of the manifold of $E_{\alpha'}$ into the manifold of E_α. Thus, the manifolds of E_α and $E_{\alpha'}$ have the same dimension. In fact, the mappings are one to one for every solution of $P_j^{(J)}\psi = \psi$ is of the form $\psi = a_j^*\phi$ where $P_j^{(J)}\phi = 0$ (set $\phi = a_j\psi$) while every solution of $P_j^{(J)}\psi = 0$ is of the form $a_j\phi$ where $P_j^{(J)}\phi = \phi$ (set $\phi = a_j^*\psi$).

Lemma 3:

Let $[\alpha]$ and $[\alpha']$ be distinct equivalence class of dual fractions. Then $\sum_{\beta \,\varepsilon [\alpha]} E_\beta$ and $\sum_{\beta \,\varepsilon [\alpha']} E_\beta$ are orthogonal projectors whose manifolds, $(\sum_{\beta \,\varepsilon [\alpha]} E_\beta)H \equiv m(\alpha)$ and $(\sum_{\beta \,\varepsilon [\alpha']} E_\beta)H$, are invariant under the a's and a^*'s. The representations induced in $m(\alpha)$ and $m(\alpha')$ are equivalent if one of them is non trivial.

The representation induced in $m(\alpha)$ is a direct sum of irreducible equivalent representation.

Proof:

The proof of the first two statements can be given in the same way as WS. The last statement can be checked as follows.

Consider the representation of the a's and a^*'s in $m(\alpha)$. If the manifolds belonging to the E_β, $\beta \in [\alpha]$ are one dimensional, then the representation is irreducible because it is cyclic and contains a projection operator onto the cyclic vector (Haag and Schroer; 1962). In fact, every vector of $m(\alpha)$ is a cyclic vector of the representation. If the manifolds belonging to E_β are greater in dimension than one, we pick up an orthogonal basis in one of them, ϕ_{α_i}, $i = 1, 2, \ldots$ Then the argument of WS shows that $m(\alpha)$ is a direct sum of orthogonal closed linear manifolds, R_i's spanned by vectors of the form $\Pi\alpha\phi_{\alpha_i}$ (i fixed), and that the representation of a's and a^*'s in the R_i's are all unitarily equivalent.

Lemma 1, 2, and 3 can be summarized in

Theorem 1

Every discrete representation of the para-Fermi commutation relations is a direct sum of irreducible representations.

The number of inequivalent irreducible representation is infinite, each corresponds to one class $[\alpha]$ of binary fraction. Among these representations only for the one whose equivalence class contains zero does a no particle state (vacuum state) and the number operator, $\sum_{j=1}^{\infty} N_j$, exist. This is the Fock representation used in the theory of second quantization.

2. Representations given by Green's ansaltzes

Consider a set of operators b_j, b_j^*, $j = 1,2,...$ in a Hilbert space H satisfying the anti-commutation relations

$$b_j^* b_k + b_k b_j^* = \delta_{jk} \tag{22}$$

$$b_j b_k + b_k b_j = 0 \tag{23}$$

Let a set of operator $b_j^{(\alpha)}$, $b_j^{(\alpha)*}$, $= 1, 2, ..., p$ in the subspace $H^p = H \otimes H \otimes ... \otimes H$ (p times) of the tensor power space $\overset{p}{\otimes} H$ defined as follows:

$$b_j^{(\alpha)} (\phi_1 \otimes \phi_2 \otimes ... \otimes \phi_\alpha \otimes ... \otimes \phi_p) = \phi_1 \otimes \phi_2 \otimes ... \otimes b_j \phi_\alpha \otimes \phi_p \tag{24}$$

$$b_j^{(\alpha)*} (\phi_1 \otimes \phi_2 \otimes ... \otimes \phi_\alpha \otimes ... \otimes \phi_p) = \phi_1 \otimes \phi_2 \otimes ... \otimes b_j^* \phi_\alpha \otimes ... \otimes \phi$$

i.e. $b_j^{(\alpha)}$ and $b_j^{(\alpha)*}$ effect only the α^{th} factor in the tensor products. Since the space H^p is dense in $\overset{p}{\otimes} H$, the $b_j^{(\alpha)}$ and $b_j^{(\alpha)*}$ have unique closed linear extension in the whole space $\overset{p}{\otimes} H$. Clearly,

$$b_j^{(\alpha)*} b_k^{(\alpha)} + b_k^{(\alpha)} b_j^{(\alpha)*} = \delta_{jk} \tag{25}$$

$$b_j^{(\alpha)} b_k^{(\alpha)} + b_k^{(\alpha)} b_j^{(\alpha)} = 0 \tag{26}$$

$$b_j^{(\alpha)}, b_k^{(\beta)*} = b_j^{(\alpha)}, b_k^{(\beta)} = 0, \alpha \neq \beta \tag{27}$$

Let a_j and a_j^*, $j=1, 2, ...,$ a set of operators in $\overset{p}{\otimes} H$ defined as

$$a_j = \sum_{\alpha=1}^{p} b_j^{(\alpha)} \tag{28}$$

$$a_j^* = \sum_{\alpha=1}^{p} b_j^{(\alpha)*} \tag{29}$$

Then relations (25), (26), (27) imply a_j, a_j^* satisfy the para-Fermi commutation relations. In this way, we obtain a representation, induced by the representations of the anti-commutation relations, which coincides with the representations given by the Green's ansatz (Green 1953).

It is natural to ask the questions as to whether (i) the representations as given by (28), (29) are irreducible and (ii) exhaust all possible representations. As to (i), one can be contented to subrepresentations obtained by the restriction of a_j, a_j^* to closed invariant subspace of $\overset{p}{\otimes}H$. Instead of trying to answer (ii) for every representations induced by every representations of b_j, and b_j^*, we shall consider the representations which admit a unique vacuum state ϕ_o.

By letting equation (1) act on the vacuum state ϕ_o, Greenberg and Messiah (1965) finds that any irreducible representation of the para-Fermi (Bose) commutation relations with a unique vacuum satisfies the relations

$$a_j \phi_o = 0 \tag{30}$$

$$a_j a_k^* \phi_o = p\, \delta_{jk}\, \phi_o \tag{31}$$

where p is a positive integer. They observe that (30) and (31) imply the representations given by Green's ansatz actually exhaust all possible representations with a unique vacuum state. We shall give a proof for this statement with the aid of 2 lemma:

<u>Lemma 4</u>:

Equations (1), (30) and (31) are sufficient to eliminate

a_j from any function of both a_j and a_j^*.

Proof:

It is sufficient to justify the lemma for functions of the form

$$f(n) = a_k a_\ell^* a_n^* \ldots a_n^* \quad (32)$$

Taking the self-adjoint of equation (1) we have

$$f(2) = - a_k a_\ell^* a_m^*$$

$$= 2\delta_{km} a_\ell^* + a_m^* a_\ell^* a_k - a_m^* a_k a_\ell^* - a_\ell^* a_k a_m^* \quad (33)$$

Let (33) act on the vacuum and using (31) we have

$$f(2)\phi_o = (2\delta_{km} a_\ell^* - p_{k\ell} a_m^* - p\delta_{km} a_\ell^*)\phi_o \; (1) \quad (34)$$

The annihiliation operator a_k has been eliminated on the left hand side of (34). For f(3), we right-multiply (33) by a_k^*

$$f(3) = 2\delta_{km} a_\ell^* a_h + a_m^* a_h^* a_k a_h^*$$
$$- a_m^* a_h a_\ell^* a_h^*$$

using (31) we have

$$f(3)\phi_o = (- 2\delta_{km} a_\ell^* a_h^* + p\delta_{kh} - f(2))\ \phi_o$$

where $f(2)\phi_o$ contains only creation operator acting on the vacuum state (equation 34). To obtain $f(n)\phi_o$, we right-multiply f(2) by a product of (n-2) creation operators. Using the same method we have applied to eliminate a_j's in $f(2)\phi_o$ and $f(3)\phi_o$ we can eliminate all a_j's in $f(n)\phi_o$.

Lemma 5:

Two representations with vacuum state are unitarily equivalent if the vacuum expectation values of any polynomial in the creation and annihilation operators are equal in two representations.

Proof:

This lemma is a consequence of a theorem in the theory of symmetric ring (Naimark, 1960) which states that a cyclic representation $x \to A_x$ with cyclic vector ϕ_o is uniquely determined to within unitarily equivalence by the positive functional $f(x) = (A_x\phi_o, \phi_o)$. It is useful to illustrate this theorem for representation of polynomial algebra of creation and annihilation operators.

Let the vacuum state vector in the two representation H and H' be ϕ_o and ϕ'_o respectively. The vector in H and H' are the closures of vectors of the form $p(a^*)\phi_o$ and $p(a^*)\phi'_o$. We define a mapping from H to H' by assigning to each vector $\psi_i = p_i(a^*)\phi_o$ of H the vector $\psi'_i = p_i(a^*)\phi'_o$ of H'. We have

$$(\psi_i, \psi_j) = (\phi_o, p_i(a)\ p_j\ (a^*)\phi_o)$$

$$= (\phi_o, p_h(aa^*)\phi_o) \qquad (35)$$

and

$$(\psi'_i, \psi'_j) = (\phi'_o, p_h(aa^*)\phi'_o) \qquad (36)$$

where p_h is a polynomial. If the vacuum expectation value of any polynomial of aa^* in H is equal to that in H', then (35) and (36) imply $(\psi_i,\psi_j) = (\psi'_i,\psi'_j)$. The mapping is thus an unitary mapping. Furthermore, this mapping maps a dense set $p(a^*)\phi_o$ of H into a dense set $p(a^*)\phi'_o$ of H', it can be linearly extended to unitary mapping of H into H'. We conclude that the two representations are equivalent.

Theorem 2:

Every irreducible representation with a unique vacuum is unitarily equivalent to a representation given by the Green's ansaltz.

Proof:

Let ϕ'_o be the unique vacuum which is the p time tensor product of the vacuum of H. It is easy to verify that the Green's ansatzes give

$$a_j\, a_k^*\, \phi'_o = p\, \delta_{jk}\, \phi'_o \tag{37}$$

which coincides with (31). By lemma 1, all the anihilation operators in $p(aa^*)\phi_o$ and $p(aa^*)\phi'_o$ can be eliminated in a unique way. Now, for free fields, all states $p(a^*)\phi_o(p(a^*)\phi'_o)$ are orthogonal to the vacuum $\phi_o(\phi'_o)$ so that lemma 1 leads to the result that the expectation values of all $p(aa^*)$ in any representation (with unique vacuum) are equal to those in $\overset{p}{\otimes} H$. By lemma 2, all irreducible representations with a unique vacuum are unitarily equivalent to the representation (with vacuum) in $\otimes^p H$.

So far, we only deal with the para-Fermi commutation relations in this section. For para-Bose commutation relation, we take H for the representation space of the commutation relations

$$b_j^{*'} b_k' - b_k' b_j^{*'} = \delta_{jk} \tag{38}$$

$$b_j' b_k' - b_k' b_j' = 0 \tag{39}$$

and define $b_j^{(\alpha)'}$, $b_j^{(\alpha)'*}$ as (3), (4). Clearly $b_j^{(\alpha)'}$ and $b_j^{(\alpha)'*}$ satisfy

$$b_j^{*(\alpha)'*} b_k^{(\alpha)'} - b_k^{(\alpha)'} b_k^{(\alpha)'*} = \delta_{jk} \tag{40}$$

$$b_j^{(\alpha)'} b_k^{(\alpha)'} - b_k^{(\alpha)'} b_j^{(\alpha)'} = 0 \tag{41}$$

$$\left[b_j^{(\alpha)'}, b_k^{(\beta)*'}\right]_- = \left[b_j^{(\alpha)'}, b_k^{(\beta)'}\right]_- = 0 \tag{42}$$

Using a set of klein transformations (H. Araki, 1961), we can change $b_j^{(\alpha)'}$ to $b_j^{(\alpha)}$ and $b_j^{(\alpha)'*}$ to $b_j^{(\alpha)*}$ satisfying (25), (26) and (27) with all (-) sign changed to (+) sign. The a_j and a_j^* defined as (28) and (29) satisfy the para Bose commutation relations. Lemma 1, Lemma 2, and theorem 2 hold also for para Bose case.

3. Existance of para-Fermi Statistics

In the representation given by Green's ansatzes , it is easy to verify, for para-Fermi case, that

$$(a_j^*)^n \phi_o = 0 \text{ if } n \geq p+1$$

which exhibits the existance of para-Fermi statistics in which the maximal occupation number is p. As a verification of theorem 2, let us prove the existance of para-Fermi statistics in any representation with unique vacuum ϕ_o.

First, we verify the relation

$$a(a^*)^n \phi_o = n\ (p-n+1)(a^*)^{n-1}\ \phi_o \tag{43}$$

for para-Fermi case. In fact this relation is verified for n=1, n=2. Suppose it is verified for n=3, 4... m-1. We prove that it is verified for n=m.

Let us right-multiply the equation

$$a(a^*)^2 = -2a^* + 2a^* a a^* - (a^*)^2 a$$

by $(a^*)^{m-2}$ to get

$$a(a^*)^m = -2(a^*)^{m-1} + 2a^* a (a^*)^{m-1}$$
$$- (a^*)^2 a (a^*)^{n-2}$$

Let this equation act on the vacuum state, using (43) for n=m-1, m-2, we have

$$a(a^*)^m \phi_o = [-2 + 2(m-1)(p-m+2)$$
$$- (m-2)(p-m+3)](a^*)^{m-1} \phi_o$$

which is nothing but equation (43).

Equation (43) shows that

$$a(a^*)^n \phi_o = 0$$

for $n \geq p+1$. Since ϕ_o is unique this implies

$$(a^*)^n \phi_o = C_n \phi_o, \; n \geq p+1 \qquad (44)$$

where C_n is a constant,

$$|C_n|^2 = || a^n \phi_o ||^2$$
$$= (\phi_o, (a^n)(a^*)^n \phi_o)$$

Using again (43), we obtain

$$|C_n|^2 = \prod_{j=1} j\ (p-j+1) \tag{45}$$

which shows that $C_n = 0$ for $n \geq p+1$. Consequently, (44) implies

$$(a^*)^n\ \phi_o = 0,\ n \geq p+1.$$

Thus, for para-Fermi case, p is the maximum occupation number of the corresponding para-Fermi statistics.

CHAPTER IX

PARAFIELD THEORY

This chapter is devoted to parafield theory based on Green's quantization and on the results of representation theory of Green's commutation relations in chapter VIII. We shall study parafields within the framework of Lagrangian theory and of axiomatic theory.

1. Lagrangian Parafield Theory

a) Relativistic Parafields

The method of quantization presented in chapter VII can easily be applied to a relativistic field theory. The Fourrier decompositions (VII-20) and (VII-21) should be extended to take account of the creation and annihilation operators of anti-particles b_k and b_k^*. Then for the commutation relation between a_k and a_k^* and for those between b_k and b_k^* one adopts the same commutation relations as in the non-relativistic case. To obtain the commutation relations involving a_k, a_k^*, b_k and b_k^*, one can consider b_k(or b_k^*) as one of the a_k (or a_ℓ^*) with a suffix different from any of those of the a_k and then apply to commutation relations for a_k and a_k^*.

Contrary to the non-relativistic case where both para-Fermi and parabose commutation relations can be applied to one and the same Schrödinger field, there exists a generalized Pauli theorem concerning spin and statistics. (Kamefuchi and Takahashi, 1962; Kamefuchi and Strahdee, 1963; Dell' Antonio, Greenberg and Sudarshan, 1964). This theorem*, which we shall prove in section 2, states that tensor fields must be quantized according to the parabose scheme and spinor fields according to the parafermi scheme. Hence, for a scalar field $\phi(x)$, the following commutation relation hold

$$[[\phi^*(x),\ \phi(y)]_+,\ \phi(z)] = -2\Delta(x-z)\phi(y)$$

$$[[\phi(x),\ \phi(y)]_+,\ \phi(z)] = 0 \qquad (1)$$

and for a Dirac field $\psi(x) = \{\psi_\mu(x)\}$,

$$[[\overline{\psi}_\alpha(x),\ \psi_\beta(y)]_-,\ \psi_\gamma(z)] = -2S_{\alpha\gamma}(x-z)\psi_\beta$$

$$[[\psi_\alpha(x),\ \psi_\beta(y)]]_-,\ \psi_\gamma(z)] = 0 \qquad (2)$$

where $\Delta(x-z)$ and $S_{\alpha\gamma}(x-z)$ are the familiar Green's functions (distribution) encountered in the ordinary Bose and Fermi field theory. Further commutation relations can be obtained from (1) and (2) by taking the hermitian conjugate or using the Jacobi identity.

Scharfstein (1963) realized that the commutation relations (1) and (2), at equal time, can be derived from

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* The ordinary Pauli theorem (see, for example, Streat and Wightman, 1964, page 148) states that, if the parastatistics are excluded, tensor fields must be quantized by the commutation relation and spinor fields by the anti-commutation relation.

Schwinger action principle. Recalling that, in the context of Schwinger action principle, the generators $G(\psi)$ $(\tilde{G}(\phi))$ and $G(\overline{\psi})$ $(\tilde{G}(\phi^*))$ which generate the infinitesimal transformation of the spinor field $\psi(x)$ (tensor field $\phi(x)$) satisfy the following equations.

$$[\psi(\underset{\sim}{x}),\ G(\underset{\sim}{\psi})] = i\delta\psi(\underset{\sim}{x}), \tag{3}$$

$$[\overline{\psi}(\underset{\sim}{x}),\ G(\psi)] = -i\delta\psi(\underset{\sim}{x}) \tag{4}$$

$$[\psi(\underset{\sim}{x}),\ G(\overline{\psi})] = [\overline{\psi}(\underset{\sim}{x}),\ G(\psi)] = 0 \tag{5}$$

and similar equations for $\phi(x)$, $\phi^*(x)$, $G(\phi)$ and $\tilde{G}(\phi^*)$ Scharfstein showed that, if one defines

$$G(\psi) = i\int[\overline{\psi}(\underset{\sim}{x}),\ \gamma_5\delta\psi(\underset{\sim}{x})]_-d^3x \tag{6}$$

$$G(\overline{\psi}) = i\int[\gamma_5\ \psi(\underset{\sim}{x}),\ \delta\overline{\psi}(\underset{\sim}{x})]_-d^3x \tag{7}$$

$$\tilde{G}(\phi) = \int[\phi^*(\underset{\sim}{x}),\ \delta\phi(\underset{\sim}{x})]_+d^3x \tag{8}$$

$$\tilde{G}(\phi^*) = [\phi(x),\ \delta\phi^*(\underset{\sim}{x})]_+d^3x \tag{9}$$

then eqns. (1) and (2), taking at equal time, are the solutions of eqns. (3), (4) and (5). He also proved the generalized Pauli theorem by showing that some inconsistency arises if in eqns. (6), (7), (8) and (9) the commutations (anti-commutators) are replaced by the anti-commutators (commutators).

The results of the representation theory of the para-commutation relations suggest that Green's ansatzes can be applied to represent a relativistic parafield. Let $\phi^{(\alpha)}(x)$, $\alpha=1,2, \ldots, p$ be a set of tensor fields obeying the commutation rules

$$[\phi^{(\alpha)}(x),\ \overset{*}{\phi}{}^{(\alpha)}(y)]_- = \Delta(x-y)$$

$$[\phi^{(\alpha)}(x),\ \phi^{(\alpha)}(y)] = 0 \tag{10}$$

$$[\overset{*}{\phi}{}^{(\alpha)}(x),\ \phi^{(\beta)}(y)]_+ = [\phi^{(\alpha)}(x),\ \phi^{(\beta)}(y)]_+ = 0, \alpha \neq \beta$$

Then a parabose field can be represented as

$$\phi(x) = \sum_{\alpha=1}^{p} \phi^{(\alpha)}(x) \tag{11}$$

The $\phi^{(\alpha)}(x)$ are called Green components of the parabose field of order p.

The same presentation (11) holds also for a parafermi field of order p with all (-) signs changed to (+) signs and $\Delta(x-y)$ changed to $S(x-y)$.

b) Case of Several Different Interacting Fields

So far, we have considered only the case of a single free parafield. It is necessary to generalize the formalism to the more realistic case of several different interacting fields.

In the conventional field theory, it is usually assumed that different spinor fields anti-commute while tensor fields commute with different tensor fields and spinor fields. Such an arrangement is usually referred to as <u>nomal case</u>. Even within the framework of ordinary field theory, there are also other possibilities. Some of the spinor fields may be allowed to commute with other spinor fields and some tensor fields may be allowed to anti-commute with other tensor fields or spinor fields. Such an arrangement is referred to as an anomalous case. A main difference of the anomalous case from the normal case is that, in general, additional restrictions have to be set on the interaction Hamiltonian in order to preserve locality, the practical consequence of which is the occurence of some additional conservation rules, modulo two. (Araki, 1964)

For parafields, there also exists the reciprocity relations between the form of the Hamiltonian and the commutation relations between different fields. For example, as pointed out by Kamefuchi and Strathdee, the choice of the normal case as mentioned above for parafields would be inconsistent with the following equation deduced from relativistic invariance,

$$\partial_\mu \psi^a(x) = i[\psi^a(x), P_\mu]$$

where P_μ is the 4-momentum operator, if one considers a Fermi-type interaction between the spinor fields ψ^a, ψ^b... It is thus necessary to generalize the commutation relation between parafields. The simplest generalization is probably the one made by Greenberg and Messiah who demand the following

i) The left-hand side must have the form

$$[[A,B]_\varepsilon ,C]_\eta$$

with ε, $\eta = \pm 1$, and the right hand side must be linear (to preserve covariance).

(ii) When the internal pair $[A,B]_\varepsilon$ refers to the same field, its ε must have the form related to the number operator ($\varepsilon = +1$ for para-Bose, $\varepsilon = -1$ for para-Fermi) and it must commute with $C(\eta=-1)$ if C refers to another field.

(iii) These relations must be satisfied by ordinary Bose or Fermi fields.

Applying these conditions, Greenberg and Messiah find the following set of commutation relations

$$[[a_k^*, a_\ell]_{\varepsilon^a}, b_m] = 0 \tag{12a}$$

$$[[a_k, a_\ell]_{\varepsilon^a}, b_m] = 0 \tag{12b}$$

$$[[a_k^*, a_\ell^*]_{\varepsilon^a}, b_m] = 0 \tag{12c}$$

$$[[b_m, a_k^*]_\eta, a_\ell]_{-\eta\varepsilon^a} = 0 \tag{12d}$$

$$[[a_\ell, b_m]_\eta, a_k^*]_{\eta\varepsilon^a} = 2\eta\delta_{k\ell}\, b_m \tag{12e}$$

where ε^a, $\varepsilon^b = \pm 1$ depending whether the field $\phi^a(\phi^b)$ is of para-Bose type or para-Fermi type. The important point is that, owing to condition (iii), the same value η must be taken everywhere. The set of commutation relations corresponding to $\eta = +1$ is called <u>relative para-Bose</u>, that corresponding to $\eta = -1$ is called <u>relative para-Fermi</u>. All other commutation relations can be obtained from the set (12) by hermitian conjugate and by the generalized Jacobi identity

$$[[A,B]_{\varepsilon},C]_{-} + [[C,A]_{\eta},B]_{-\eta\varepsilon} + \eta\varepsilon[[B,C]_{\eta},A]_{-\eta\varepsilon} = 0$$

The problem of finding Fock representations satisfying the commutation relations (1), (2) and (12) is solved by the following theorem due to Greenberg and Messiah (1964).

Theorem 1:

All Fock representations are given by Green's ansatzes

$$a_k = \sum_{\alpha=1}^{p} a_k^{(\alpha)}, \quad b_k = \sum_{\alpha=1}^{p} b_k^{(\alpha)} \tag{13}$$

where for each pair of components belonging to the same field, one assumes the commutation rules of Green's ansaltz (Section VIII-2), i.e., the para-Bose rule if $\varepsilon = +1$ and the para-Fermi rule if $\varepsilon = -1$ and for each pair $a_k^{(\alpha)}$, $b_m^{(\beta)}$. We assume the para-Bose rule if $\eta = +1$ and the para-Fermi rule if $\eta = -1$, that is

$$\begin{aligned} &[a_k^{(\alpha)}, b_m^{(\alpha)}]_{-\eta} = 0, \quad [a_k^{(\alpha)}, b_m^{(\alpha)*}] = 0 \\ &[a_k^{(\alpha)}, b_m^{(\beta)}]_{\eta} = 0, \quad [a_k^{(\alpha)}, b_m^{(\beta)*}]_{\eta} = 0, \quad \alpha \neq \beta \end{aligned} \tag{14}$$

Proof:

The proof follows the same line of argument as given in Section VIII-2. It consisting in proving that any representation with unique vacuum state satisfied the conditions

$$a_k\, a_\ell^*\, \phi_o = p\delta_{k\ell}\, \phi_o \tag{15a}$$

$$b_m\, b_n^*\, \phi_o = p\delta_{mn}\, \phi_o \tag{15b}$$

$$a_k\, b_m^*\, \phi_o = 0 \tag{15c}$$

$$b_m\, a_k^*\, \phi_o = 0 \tag{15d}$$

which are satisfied by Green's ansaltzes and in applying lemma 4 and lemma 5 of section VIII-2.

Conditions (15) can be proved by letting (12d), with a and b interchanged, act on the vacuum state,

$$b_\ell(a_k\, a_m^*\, \phi_o) = 0 \text{ for all } k,\ \ell,\ m$$

From the uniqueness of the vacuum state, one has

$$a_k\, b_m^*\, \phi_o = c_{km}\, \phi_o$$

Using (1) and (2) in terms of a_k^* and b_k, one gets

$$[[a_k,\ a_\ell^*]_{\varepsilon^a},\ a_\ell\, b_m^*]_- = -2\varepsilon^a\, a_k\, b_m^* \tag{16}$$

and using

$$a_k\, a_\ell^*\, \phi_o = p^{(a)}\, \delta_{k\ell}\, \phi_o$$

$$b_m\, b_n^*\, \phi_o = p^{(h)}\, \delta_{mn}\, \phi_o$$

obtained by letting (1) and (2) act on the vacuum, the two members of (16) applied to ϕ_o give

$$o = -2\varepsilon^a\, a_k\, b_m^*\, \phi_o$$

which proves (15c). (15d) is proved in the same way.

Equation

$$[b_m, a_k^*]_\eta \, a_k \, b_m^* = 2\varepsilon^a [a_k \, a_k^* - b_m \, b_m^*]$$

$$+ \varepsilon^a \, \varepsilon^b \, a_k \, b_m^* \, [b_m, a_k^*]_\eta$$

applied to ϕ_o, taking account of (15c), (15d) gives

$$0 = (a_k \, a_k^* - b_m \, b_m^*) \phi_o$$

$$= (p^{(a)} - p^{(b)}) \phi_o$$

hence

$$p^{(a)} = p^{(b)} = p$$

and (15a) and (15b) are proved. (QED)

An important point of theorem 1 is that all the parafields must have the same order p. Therefore, the commutation relations (12) should be applied only to parafields of the same order.

Different commutation relations must be adopted for parafields of different orders. It is assumed they commute or anti-commute.

The fields are divided into families, each containing all parafields of the same order p. The <u>normal commutation relations</u> are determined uniquely by the conditions:

(i) Inside a family the relative commutation rules are all para (trilinear type).

(ii) The relative rules are of Fermi-type if and only if both fields are Fermi or para-Fermi.

Each fields can be expanded into Green's components as

$$\phi_i(x) = \sum_{\alpha=1}^{p_i} \phi_i^{(\alpha)}(x) \tag{17}$$

The normal commutation rules can then be expressed in a general form as

$$\phi_i^{(\alpha)}(x) \quad \phi_j^{(\beta)}(y) = (-)^{\delta_{p_i p_j}(1 + \delta_{\alpha\beta}) + \tau_{ij}} \phi_j^{(\beta)}(y) \ \phi_i^{(\alpha)}(x)$$

for $(x-y)^2 < 0$, where, $\delta_{p_i p_j}$ and $\delta_{\alpha\beta}$ are Kronecker symbols and τ_{ij} is a function which takes 0 when both $\psi_i^{(\alpha)}$ and $\psi_j^{(\beta)}$ are Bose fields or when one of them is a Bose field and takes 1 when both of them are Fermi fields.

The Hilbert space G in which Green's component fields act is larger than the physical Hilbert space H in which the parafields act. It is believed that G has no physical interpretation but is only a convenient mathematical space for physical description.

c) Observability of Parafields

We have seen that, within the framework of Lagrangian theory, a parafield can be expanded in terms of a set of Bose of Fermi fields obeying anomalous commutation relations. On the other hand, it has been shown (Kinoshita, 1958; Araki, 1961) that a set of Bose and Fermi fields obeying anomalous commutation relations can be transformed into a set of Bose and Fermi fields obeying the normal commutation relations by a set of Klein transformations (Klein, 1938). These Klein transformations $\psi_i^{(\alpha)}(x) \rightarrow \psi_i^{(\alpha)'}(x)$, contrary to the symmetries, cannot be represented by unitary or anti-unitary operators so that the fields $\psi_i^{(\alpha)'}(x)$ are in general physically distinct from the fields $\psi_i^{(\alpha)}$. A question arises whether the field

$$\psi_i(x) = \sum_{\alpha=1}^{p_i} \psi_i^{(\alpha)}(x)$$

and the field

$$\psi_i'(x) = \sum_{\alpha=1}^{p_i} \psi_i^{(\alpha)'}(x) \qquad (18)$$

are physically distinct. In other words, we want to ask the following question. If, in the parafield theory, a certain function, say $F(\psi)$ is observable and corresponds to certain well-defined measurement in the laboratory, then the Klein transformations can be regarded as leading to a new theory in which $F[\psi']$ corresponds to the very set of measurement. Do the two theories predict the same result for all experiments? To answer this question, a detailed knowledge of the observables in the theory is required. In general, there is not much change that the two theories are physically equivalent (Streater and Wightman, 1964).

We shall study the case of a single interacting para-Fermi field.

Since all relevant informations can be derived from the S-matrix, we study the transformation property of the S-matrix under the Klein transformations. In the interaction picture, the S-matrix is written formally as

$$S = P\exp[-i\int d^4x\, H_I(x) \tag{19}$$

where $H_I(x)$ is the interaction Hamiltonian written in terms of the free field. We require that $H_I(x)$ satisfies the locality (or integrability condition),

$$[H_I(x), H_I(y)]_- = 0 \text{ for } (x-y)^2<0 \tag{20}$$

in the space H. In order that the theory be a local Lagrangian field theory, we must require further that locality should hold also between $\phi(x)$ and $H_I(x)$ in the space H, i.e.

$$[\phi(x), H_I(y)]_- = 0 \text{ for } (x-y)^2<o \tag{21}$$

(It is known (Takahashi and Umezawa, 1953) a theory which satisfied (20) but not (21) leads to essentially non-local results).

Because Araki, Greenberg and Toll (1966) have already shown that locality in H is equivalent to locality in G, we can express equations (19) and (20) in terms of Green's components via Green's ansaltz. Hence, eqn. (19) and (20) yield

$$\sum_{m,m'} [H_I^{(m)}(\psi^{(\beta)}(x)], H_I^{(m')}(\psi^{(\beta)}(y)]_- = 0 \tag{20^1}$$

$$\sum_{\alpha=1} \sum_{m} [\psi^{(\alpha)}(x), H_I^{(m)}(\psi^{(\beta)}(y))]_- = 0 \tag{21^1}$$

where (m) labels those terms of $H_I(x)$ which are generated by Green's ansaltz. Ohnuki and Kamefuchi (1968) show that (20^1) and (21^1) imply

Theorem 2:

$H_I(x)$ consists only of terms of the type $[\psi(x),\psi(x)]_- \times [\psi(x),\psi(x)]_- \ldots [\psi(x),\psi(x)]_-$ where $\psi(x)$ stands for either $\psi(x)$ or $\overline{\psi}(x)$.

Now, the Klein transformation defines, for some set of $\psi^{(\alpha)}$, $\psi^{(\alpha)'}(x) = -\psi^{(\alpha)}(x)$ if the domain of definition of that very set belong to some part of G. Under such a transformation, $[\psi(x),\psi(x)]$ in general is changed, so H_I is changed, hence the S-matrix is changed. I.e., the theory in which $\psi(x)$ is replaced by $\psi'(x)$ predicts different result for S-matrix. But $\psi'(x)$ is just an ordinary Fermi field, the parafield $\psi(x)$ is physically distinct from a Fermi field. In other words, the paraphermia is physically observable; experiment can in principle tell us whether a particle is just boson or fermion or a paraparticles.

2. Axiomatic Parafield Theory

Consider a field $\phi(x)$ satisfying the Wightman Axioms about the domain D and continuity, the Lorentz transformation law and the local commutativity

$$[\psi(f),\ \psi(g)]_{\pm},\ \psi(h)] = 0$$

if the support of f and the support of g are space-likely separated from the support of h (i.e. f(x)h(y) = 0 and

$g(x)h(y) = 0$ if $(x-y)^2<0)$ when the left hand side is applied to any vector in D, where ψ may stand for its self-adjoint ψ^*. In terms of unsmeared field, the local commutativity axioms is simply

$$[[\psi(x), \psi(y)]_{\pm}, \psi(z)] = 0$$

$$(x-z)^2<0, \ (y-z)^2<0 \qquad (22)$$

We have seen that a free parafield admits the Green expansion (17) as a result of Chapter VIII. For interacting parafields, it is not known whether the local commutativity postulate does imply an expansion as (17). The expansion, however, satisfies (22) so that it can be used to define a special class of interacting parafields which we shall call special parafields.

We remark that the special parafield theory involves a set of Bose and Fermi fields obeying anomalous commutation rules. As already stated, such a theory can be transformed to a theory with normal commutation rules by means of a set of Klein transformations without changing the physical content (all vacuum expectation values are changed by a factor ±1) so that the connection between spin statistics and T.C.P. theorem also hold in the anomalous case. Because the special parafield theory is a sub-theory of ordinary Bose and Fermi fields obeying anomalous commutativity, we obtain

Theorem 3:

"The usual connection between spin and statistics and the T.C.P. theorem holds for special para-Bose and para-Fermi fields."

This theorem has also been proved by Dell'Antonio, Greenberg and Sudarshan (1964) without relying on the Klein transformations.

For non-special parafields, the theorem is not obvious although all properties of the Wightman functions which do not depend on the local commutivity still hold. Let us consider the case of Volkov's parafield satisfying the local commutativity as follows

$$\phi(x)\phi(y)\phi(z) \pm \phi(z)\phi(y)\phi(x) = 0 \tag{23}$$

$$(x-z)^2<0, \quad (y-z)^2<0$$

which is the generalization of a Lagrangian parafield of order 2 (see equation VII-42).

Consider, in particular, the relation

$$\phi(x)\phi^*(y)\phi(z) \pm \phi(z)\phi^*(y)\phi(x) = 0$$

$$(x-y)^2<0, \quad (y-z)^2<0 \tag{24}$$

Using the cluster decomposition property*, we find that (24) implies

$$[(\psi_o, \phi(x)\ \phi^*(y)\ \psi_o) \pm (\psi_o, \phi^*(y)\ \phi(x)\ \psi_o]\ (\psi_o, \phi(z)\ \psi_o) = 0 \tag{25}$$

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* See Streater and Wightman (1964), page 111

where ψ_o is the vacuum state vector. It is known that if ψ_o is invariant under the proper Lorentz group then $(\psi_o, \phi(x)\psi_o)$ vanishes for all field other than a proper scalar field*. Thus, for a proper scalar field, (25) requires

$$(\psi_o, \phi(x)\,\phi^*(y)\psi_o) \pm (\psi_o, \phi^*(y)\,\phi(x)\psi_o) = 0 \tag{26}$$

The argument used in the proof of the spin statistics theorem for ordinary fields can be repeated here to show that (26) implies

Theorem 4:

"The (-) sign in (23) must be chosen for a proper scalar field, i.e., the spin statistic theorem holds for proper scalar fields".

Now, let us see whether (24) implies weak locality condition, i.e.

$$(\psi_o, \phi(x_1)\,\phi(x_2)\,\ldots\phi(x_n)\psi_o) = \pm(\psi_o, \phi(x_n)\,\ldots\phi(x_2)\,\phi(x_1)\psi_o) \tag{27}$$

for all $(x_i - x_j)^2 < 0$.

Because equation (18) implies that the vacuum expectation value of any product of fields is changed by a factor ±1 under any interchange of 2 fields at odd or even positions, (27) is satisfied if n is odd (one check this by reversing the orders of all fields at odd positions then at even positions). However, this is not true for even n as seen

* See Roman (1970), page 284, problem 5.1.

from the simple example

$$(\psi_o, \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\psi_o) =$$

$$\pm\ (\psi_o, \phi(x_1)\,\phi(x_4)\,\phi(x_3)\,\phi(x_2)\psi_o)$$

where it is not possible to pull $\psi(x_1)$ to the extreme right of the product of fields. Thus, because we do not know the rule for interchanging a field at even position with a field at odd position, we cannot obtain (27) for even n. Thus, we cannot assert whether the weak local condition is satisfied by parafields of order 2.

CHAPTER X

CORRESPONDENCE BETWEEN THE FIRST AND SECOND QUANTIZATION THEORIES

1. States of N Paraparticles in Parafield Theory

In the ordinary field theory, the states $a^*_{k_1} a^*_{k_2} \ldots a^*_{k_N}|0\rangle$ form a complete set in the space $\mathcal{H}^N$, the Hilbert space of dynamical states of N quanta of the field. That is,

$$1 = \frac{1}{M} \int dK\, a^*_{k_1} a^*_{k_2} \ldots a^*_{k_N}|0\rangle\langle 0|a_{k_N} \ldots a_{k_1} \qquad (1)$$

so that a normalized N particle state is given by

$$|\psi\rangle = \frac{1}{\sqrt{M}} \int dK \psi(k_1, k_2, \ldots k_N)\left(\prod_{i=1}^{N} a^*_{k_i}|0\rangle\right) \qquad (2)$$

where M is the normalization constant and dK stands for $dk_1\, dk_2 \ldots dk_N$, and

$$\psi(k_1, k_2, \ldots k_N) = \frac{1}{\sqrt{M}} \langle 0|a_{k_N} \ldots a_{k_1}|\psi\rangle$$

Furthermore, in ordinary field theory, the normalized function $\psi(k_1, k_2, \ldots k_N)$ has the same permutation property as the wavefunction of N identical particles so that the former can be identified with the latter. The ordinary field theory is completely equivalent to the quantum mechanical theory of identical bosons or fermions.

In parafield theory, we expect that (1) also holds, and hence (2), because it is a result of the irreducibility of the operators a_k, a_k^* and the self adjointness of the number operator. For N=3 and for para-Fermi fields of order 1 and 2, eqn. (1) can be verified as follows. Using the relation

$$a_k \, |o\rangle = 0$$

$$a_k \, a_\ell^* \, |0\rangle = p\delta_{k\ell} \, |0\rangle$$

and the commutation relations, we obtain

$$\langle 0|a_i a_j a_k a_\ell^* a_m^* a_n^*|0\rangle =$$

$$(p-2)[p(p-2)\delta_{km}\delta_{jn}\delta_{il} + p\delta_{km}\delta_{j\ell}\delta_{in}]$$

$$+(p-2)\delta_{kn} \; [p(p-2)\delta_{jm}\delta_{i\ell} + p\delta_{j\ell}\delta_{im}]$$

$$+p\delta_{k\ell} \; [p(p-2)\delta_{jn}\delta_{im} + p\delta_{jm}\delta_{in}]$$

$$-2\delta_{kn} \; [p(p-2)\delta_{j\ell}\delta_{im} + p\delta_{jm}\delta_{i\ell}]$$

from which it follows that

$$\frac{1}{M} \int didjdk a_k^* a_j^* a_i^* \; |0\rangle\langle 0|a_i a_j a_k a_\ell^* a_m^* a_n^*|0\rangle$$

$$= \frac{1}{M} \{(p-2)[p(p-2) \; a_m^* a_n^* a_\ell^* + p \; a_m^* a_\ell^* a_m^*]$$

$$+ \; (p-2)[p(p-2) \; a_n^* a_m^* a_\ell^* + p \; a_n^* a_\ell^* a_m^*]$$

$$+ \; p[p(p-2) \; a_\ell^* a_n^* a_m^* + p \; a_\ell^* a_m^* a_n^*]$$

$$- \; 2[p(p-2) \; a_n^* a_\ell^* a_m^* + p \; a_n^* a_m^* a_\ell^*] \qquad (3)$$

For a para-Fermi field of order 1,

$$[a_m, a_\ell] = 0$$

and

$$p = 1$$

the right hand side of eqn. 3 is $a_\ell^* a_m^* a_n^* |0\rangle$ if we take $M = 2$. For a para-Fermi field of order 2,

$$a_\ell a_m a_n + a_n a_m a_\ell = 0$$

$$p = 2$$

The right hand side of (3) is again $a_\ell^* a_m^* a_n^* |0\rangle$ if we take $M = 8$.

Thus, the state $|\psi\rangle$ given by (2) can be taken as the definition of an N-paraparticle state in parafield theory.

2. Permutation Symmetry in Parafield Theory

As in the first quantization theory, we shall write an N-particle state as

$$|K\rangle = a_{k_{i_1}}^* \; a_{k_{i_2}}^* \ldots a_{k_{i_N}}^* \; |0\rangle$$

$$= |k_{i_1} \; k_{i_2} \; \ldots \; k_{i_N}\rangle$$

$$= \left| \begin{matrix} 1 & 2 \;.... & N \\ k_{i_1} & k_{i_2} & k_{i_N} \end{matrix} \right\rangle \quad (4)$$

where the numbers 1,2, ... N of the first line in the bracket indicate the position of $a_{k_{i_1}}^*$, $a_{k_{i_2}}^*$, ..., $a_{k_{i_N}}^*$ in the state $|K\rangle$.

then we can define a place permutation operator as follows

$$U(\sigma)\,|K> \;=\; \left|\begin{matrix} \sigma^{-1}1 & \sigma^{-1}2 & \ldots & \sigma^{-1}N \\ k_{i_1} & k_{i_2} & \ldots & k_{i_N} \end{matrix}\right\rangle \qquad (5)$$

Contrary to the ordinary field theory, the application of the U(σ) in general yields independent states, i.e., the U (σ)|K> are, in general, linearly independent. We shall show in this section that, unlike in the first quantization theory, <u>the U(σ) do not define multi-dimensional representations of S_N in the second quantization theory</u>.

We consider first a para-Fermi field of order 2 satisfying the commutation relation:

$$a_k^* a_\ell^* a_m^* + a_m^* a_\ell^* a_k^* = 0 \qquad (5)$$

Suppose that a representation of S_N can be built up from vectors of the form $|\phi>=\Sigma\, a(\sigma)\, U(\sigma)\, |K>$. Relation (5) implies that a permutation of two creation operators at even positions or at odd positions in the states U(σ) |K> changes them by a factor -1. i.e.

$$U(i,\, i+2)\ |\phi> = -|\phi> \ ,\ i=1,2\ldots N-2$$

Clearly, all the permutations (i, i+2) must be represented by the operator -1, where 1 is the unit operator. In other words, the representation must be unfaithful. As is well-known, a representation of a group is a homomorphism of the group to a set of operators. The kernel of the homomorphism must always be an invariant subgroup. If the kernel consists of the unit element only, this is a faithful representation, and if otherwise the representation is not faithful. On the other hand, the invariant subgroups are completely known as listed in Appendix C. The irreducible representations of S_N whose kernel is S_N or A_N is, of course, the identity or the alternating representation. For $N \neq 4$, it can be concluded on account of Appendix C that an unfaithful representation must be either the identity representation or the alternating representation, i.e., a one-dimensional representation. This result is well-known.* Thus, the representations of S_N defined by the $U(\sigma)$, if existing in the Hilbert space of para-Fermi field of order 2, cannot be multi-dimensional.

The proof can be proceeded in the same way for a paraBose field of order 2 by observing that the commutation relation, in this case, implies that a permutation of two creation operators at even or odd positions do not change the $U(\sigma)\ |K\rangle$.

The proof can be easily generalized for parafields of any order following an observation made by Green (1953) that, for parafields of order p, the creation operators in the $U(\sigma)\ |K\rangle$ are divided into p groups such that the exchange

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* See for example Boerner (1963)

of two creation operators in the same group changes the $U(\sigma)\ |K>$ only by a factor of ± 1. (+1 for paraBose fields and -1 for para-Fermi fields (two creation operators in the same group are always separated by p creation operators).

Green's observation also allows us to conclude that the Hilbert space of paraBose fields supports only the identity representation of S_N and that of para-Fermi fields only the alternating representation.

Thus, the second quantization theory of para-particles is not equivalent to the first quantization theory. The quantum mechanical theory Hilbert space always supports a multi-dimensional representation of S_N defined by the $U(\sigma)$ whereas the quantum field Hilbert space does not.

The fact that the $U(\sigma)$ may not be unitarily represented in the second quantized Hilbert space was first observed by Galindo and Yndurain (1963). Yamada (1968) provided a proof of the non-equivalence between the first and the second quantization theory of para-particles. Our proof is similar to but simpler than Yamada's.

3. Particle Permutation Operators in First and Second Quantization Theories of Paraparticles

It is possible to define, both in first and second quantization theories, a different kind of permutation operators, called the particle permutation operators (Landshoff and Stapp, 1967; Yamada, 1968), as follows

$$V(\sigma) \left| \begin{matrix} 1 & 2 & \ldots & N \\ k_{i_1} & k_{i_2} & & k_{i_N} \end{matrix} \right\rangle = \left| \begin{matrix} 1 & 2 & \ldots & N \\ k_{i_{\sigma 1}} & k_{i_{\sigma 2}} & & k_{i_{\sigma N}} \end{matrix} \right\rangle \quad (6)$$

The following discussion is applied for both first and second quantization theories, except when otherwise stated.

A particle permutation operator $V(\sigma)$ replaces k_{i_m} by $k_{i_{\sigma m}}$ whatever the position of k_{i_m} is. Clearly, the particle permutation operators depend on the way in which we label the k_i by the symbols $i_1, i_2, \ldots i_N$. Consider for example the three particle state $|k_1\ k_2\ k_3\rangle$. If we label k_1 by k_{i_1}, k_2 by k_{i_2} and k_3 by k_{i_3}, then

$$V(23)\ |k_1\ k_2\ k_3\rangle = |k_1\ k_3\ k_2\rangle \quad (7)$$

However, if we label k_1 by k_{i_2}, k_2 by k_{i_3} and k_3 by k_{i_1} then

$$\begin{aligned} V(23)\ |k_1\ k_2\ k_3\rangle &= V(23)\ |k_{i_2}\ k_{i_3}\ k_{i_1}\rangle \\ &= |\ k_{i_3}\ k_{i_2}\ k_{i_2}\ \rangle \\ &= |\ k_2\ k_1\ k_3\ \rangle \end{aligned}$$

which is different from eqn. (7). It has been suggested by Stolt and Taylor (1970) that one can define the particle permutation operators with the aid of the labelling satisfying the condition

$$k_{i_1} \leq k_{i_2} \leq \ldots \leq k_{i_N}$$

assuming that a certain ordering of the k_i has been adopted. However, as noted by Steinmann (1971), this still does not determine the particle permutation operators uniquely because the equal k_i can be labelled by the i_j in many different ways. For example, consider the state $|k_1\ k_2\ k_2>$ and suppose $k_2 > k_1$. One can write this state as $|k_{i_1}\ k_{i_2}\ k_{i_3}>$ or equally well as $|k_{i_1}\ k_{i_3}\ k_{i_2}>$ (with $k_{i_1} = k_1$, $k_{i_2} = k_{i_3} = k_2$) and obtain different results for the $V(\sigma)$. For this reason, Steinmann (1971) rejected any physical interpretation of the $V(\sigma)$ although they may be useful sometimes in some calculations.*

Despite this difficulty with the $V(\sigma)$, one finds a good deal of works on the connection between the first quantization theories, which made use of the $V(\sigma)$ (Landshoff and Stapp, 1967; Ohnuki and Kamefuchi, 1969; Hartle and Taylor, 1969; Carpenter, 1970; Ohnuki and Kamefuchi, 1971). The idea was to classify the second quantized N-paraparticle states into irreducible representations of S_N defined by the $V(\sigma)$, assuming that these representations exist, exactly in the same fashion as the classification of the first quantized N-paraparticle states into irreducible representations of S_N defined by the $U(\sigma)$ (See Chapter III). The correspondence between the first quantized states and the second quantized states was proposed

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* For example, Dirac (1930) found that the $V(\sigma)$ are convenient variables in the determination of the energy level of N electrons to first order in perturbation method.

by adopting the same classification with the aid of particle permutation operators for both first and second quantization theories. A second quantized state belonging to an irreducible representation (defined by the $V(\sigma)$) corresponds to a first quantized state belonging to the same irreducible representation, also defined by the $V(\sigma)$.

Leaving aside the question of physical interpretation of the $V(\sigma)$, we wish to raise an objection to the correspondence between the first and the second quantization theories, as proposed in the literature, by showing that the particle permutation operators do not, in general, define multi-dimensional representations of S_N, both in first and second quantization theories.

It sufficies to consider an N-particle state $|K\rangle$ in which some of the k_{i_j} are equal. We suppose, for definite, that $k_{i_1} = k_{i_2} = \ldots = k_{i_M}$. Let S_M be the group of permutations of 1,2, ... M. If the subspace $\mathcal{H}^N[K]$ spanned by the $V(\sigma)|K\rangle$ supports a representation of S_N defined by the $V(\sigma)$, then all the $\sigma \in S_M$ must be represented by the unit operator because the permutations over the equal k_{i_j} do not change the vectors at all. Therefore, the representation is unfaithful. Such a representation, as proved in section 1, is not multi-dimensional.

Thus, the space $\mathcal{H}^N$ of second quantized states (or first states) contains subspaces which do not support multi-dimensional representations of S_N defined by the $V(\sigma)$. This observation, of course, questions the valadity of numerous published results concerning the correspondence between the first and second quantization theories of paraparticles.

4. Right Regular Representations

The particle permutation operators are frequently identified with the permutation operators $U'(\lambda)$ defined as follows

$$U'(\lambda) \left| \begin{matrix} \sigma^{-1} 1 & \sigma^{-1} 2 & \sigma^{-1} N \\ k_{i_{\pi 1}} & k_{i_{\pi 2}} & k_{i_{\pi N}} \end{matrix} \right\rangle =$$

$$\left| \begin{matrix} \sigma^{-1} \lambda 1 & \sigma^{-1} \lambda 2 \ \ldots & \sigma^{-1} \lambda N \\ k_{i_{\pi 1}} & k_{i_{\pi 2}} & k_{i_{\pi N}} \end{matrix} \right\rangle \qquad (8)$$

This identification, however, should be understood properly:

Consider the action of a particle permutation operator on a vector of the form

$$|X[K]\rangle = \sum_{\sigma} a(\sigma)\ U(\sigma)\ |K\rangle,$$

i.e., consider

$$V(\pi)\ |X[K]\rangle = \sum_{\sigma} a(\sigma)\ V(\pi)\ U(\sigma)\ |K\rangle \qquad (9)$$

Previously, we have pointed out that there is an "embarras du choix" for the $V(\pi)$ if some of the k_{i_j} in $|K\rangle$ are equal.

To avoid this difficulty, we consider only the case for which all the k_{i_j} in $|K>$ are distinct

It is usually believed that

$$[U(\sigma), V(\pi)] = 0 \tag{9}$$

for all σ and π, both in first and second quantization theories of paraparticles, so that one can write (9) as

$$V(\pi) \; |X[K]> \; = \sum_{\sigma} a(\sigma) \; U(\sigma) \; V(\pi) \; |K> \tag{10}$$

It can be seen easily that, for each $V(\pi)|K>$, there exists a σ_{π} ε such that

$$V(\pi)|K> \; = U(\sigma_{\pi})|K> \tag{11}$$

Hence, (10) writes as

$$V(\pi)|X[K]> \; = \sum_{\sigma} \; a(\sigma) \; U(\sigma) \; U(\sigma_{\pi})|K>$$

$$= U'(\sigma_{\pi})|K> \tag{12}$$

This shows that a particle permutation operator is just an operator $U'(\lambda)$.

The operators $U'(\lambda)$ coincide with the operators $V(\pi)$ for those $|K>$ in which all the k_{i_j} are distinct but the $U'(\lambda)$ are determined unambigously also for those $|K>$ with some equal k_{i_j}. This suggests that we should consider the operators $U'(\lambda)$ instead of the $V(\pi)$. However, in doing this, we face two difficulties:

(i) Looking at eqn. (12), we see that the commutation relations do impose some relations between the $U'(\lambda)$, as they do between the $U(\sigma)$ (in eqn. (12), all the $U(\sigma_\pi)|K\rangle$ are not linearly independent because of the commutation relations). Like the $U(\sigma)$, these relations imply that multi-dimensional representations* defined by the $U'(\lambda)$ do not exist in the Hilbert space of second quantized states of paraparticles.

(ii) Following the same line of reasoning as in the last two paragraphs of section 3, we can show that, when some of the k_{i_j} in $|K\rangle$ are equal, there exists no multi-dimensional representation defined by the $U'(\lambda)$ (eqn. (12) implies that $U(\sigma_\pi)|K\rangle = |K\rangle$ if σ_π is a permutation of the places occupied by equal k_{i_j}).

Result (i) questions the validity of eqn. (9) in the second quantization theory. To see this, we shall show that if eqn. (9) holds any representation defined by the $V(\pi)$ must coincide with a representation defined by the $U'(\lambda)$. In fact, consider a representation defined by the $V(\pi)$ and given by the base $\{|e^i[K]\rangle\}$, i=1,2, ... h,

$$|e^i[K]\rangle = \sum_\pi e^i(\pi)\ V(\pi)|K\rangle \tag{13}$$

where $e^i(\pi)$ are scalars(We suppose again that all the k_{i_j} in K are distinct). According to eqn. (10), eqn. (13)

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* These representations, if existing, coincide with the right regular representation of S_N, a mathematical terminology in the algebraic representation theory of S_N.

can be written as

$$|e^i[K]> \;=\; \sum_{\pi} \; e^i(\pi)\; U(\sigma_\pi)\,|K> \tag{14}$$

Now if eqn. (9) holds, we have from (9)

$$V(\lambda)\,|e^i[K]> \;=\; \sum_{\pi} \; e^i(\pi)\; U(\sigma_\pi)\; V(\lambda)\,|K>$$

Again, eqn. (11) yields

$$V(\lambda)\,|e^i[K]> \;=\; \sum_{\pi} \; e^i(\pi)\; U(\sigma_\pi)\; U(\sigma_\lambda)\,|K>$$

$$= U'(\sigma_\lambda)\,|e^i[K]>$$

This means that the base $\{|e^i[K]>\}$ constitutes also a representation defined by the $U'(\sigma)$. I.e. if no multi-dimensional representation defined by the $U'(\sigma)$ exists then neither does a multi-dimensional representation defined by the $V(\pi)$. However, it is quite easy to construct a multi-dimensional representation defined by the $V(\pi)$. For example, consider a three particle state of a para-Fermi field of order 2

$$|k_{i_1}\; k_{i_2}\; k_{i_3}> \;=\; a^*_{k_{i_1}}\; a^*_{k_{i_2}}\; a^*_{k_{i_3}}\;|0>,\; k_{i_1} \neq k_{i_2} \neq k_{i_3}$$

and consider

$$|e^1> \;=\; a^*_{k_{i_1}}\; a^*_{k_{i_2}}\; a^*_{k_{i_3}}\,|0> \;=\; -\; a^*_{k_{i_3}}\; a^*_{k_{i_2}}\; a^*_{k_{i_1}}\;|0>$$

$$|e^2> \;=\; a^*_{k_{i_2}}\; a^*_{k_{i_3}}\; a^*_{k_{i_1}}\,|0> \;=\; -\; a^*_{k_{i_1}}\; a^*_{k_{i_3}}\; a^*_{k_{i_2}}\;|0>$$

$$|e^3> \;=\; a^*_{k_{i_3}}\; a^*_{k_{i_1}}\; a^*_{k_{i_2}}\,|\;> \;=\; -\; a^*_{k_{i_2}}\; a^*_{k_{i_1}}\; a^*_{k_{i_3}}\;|0>$$

Let

$$|e_1'> = \frac{1}{\sqrt{3}} [|e_1>+|e_2>+|e_3>]$$

$$|e_2'> = \frac{1}{\sqrt{2}} [|e_2>-|e_3>]$$

$$|e_3'> = \frac{2}{3} [|e_1>- \frac{1}{2} (|e_1>+|e_2>]$$

It is easy to check that $|e_1'>$ constitutes a one dimensional representation defined by the $V(\pi)$ $|e_2'>$ and $|e_3'>$ constitute a two-dimensional representation defined by the $V(\pi)$. This example suggests that eqn. (9) is violated in the second quantization theory. In fact, for a para-Fermi field of order e, we have

$$V(13)|k_{i_1}\ k_{i_2}\ k_{i_3}> = -\ |k_{i_1}\ k_{i_2}\ k_{i_3}>$$

so that

$$U(12)\ V(13)|k_{i_1}\ k_{i_2}\ k_{i_3}> = -|k_{i_2}\ k_{i_1}\ k_{i_3}> \tag{15}$$

On the other hand

$$V(13)\ U(12)|k_{i_1}\ k_{i_2}\ k_{i_3}> = V(13)|k_{i_2}\ k_{i_1}\ k_{i_3}>$$

$$= |k_{i_2}\ k_{i_3}\ k_{i_1}>$$

Comparing this with (15) we have

$$U(12)\ V(13) \neq V(13)\ U(12)$$

which violates eqn. (9).

Thus, if the commutation relations are applied at each stage of the calculation, the particle permutation operators do not commute with the place permutation operators in the second quantization theory.

CHAPTER XI

CONCLUSIONS

We have seen in chapter II that the Symmetrization Postulate (S.P.) is related to the assumption that there exists a complete set of commuting observables characterizing a maximal observation. More precisely, the maximal observation is compatible with the existance of the superselection rule connected with the commuting supersymmetries which, when applied to the $U(\sigma)$, lead to the S.P. Besides the question of whether the mathematical techniques employed to arrive at this result were physically appropriate, we agreed with Messiah and Greenberg that the assumption about maximal observation was too strong to be verified in the present status of Quantum Mechanics and experimental technique.

We then formulated, in chapter III, a theory of identical particles not obeying S.P. in the language of the algebraic representation theory of finite groups. We begin by classifying the states of identical particles into the two-sided ideals of the group algebra which accepts the $U(\sigma)$ as a base (a two-sided ideal contains a whole class of I.R.'s of S_N): each two-sided ideal corresponds to a symmetry type of identical particle states. We distinguished the "physical" scalar product of the many-body problem Hilbert space H^N from the cartesian scalar product and emphasized that it is the former, but not the latter, which determines the physical properties of the particles. We found that, with respect to the physical scalar product, states of different symmetry types, and only these, are always orthogonal and that there exists a superselection rule operating between the symmetry types. This superselection rule is, however, connected with the non-commuting supersymmetries, which shows that the assumption about maximal observation, though compatible with the superselection rules, is by no means responsible for the existance of all superselection rules. It is also well-known but its derivation as found in the literature based on Schur's lemma ,we criticized, is quite unsatisfactory due to the inclusion of the equivalent I.R's into our physical problem. We remarked that this inclusion renders invalid the statement, also usually found in the literature, that the expectation value of a physical observable is the same for all normalized states belonging to an irreducible representation of S_N (defined by the place permutation operators). This remark had been subsequently confirmed by our calculation of the matrix representing a physical observable. We pointed out that, with respect to the base chosen for the theory of paraparticles, the elements of this matrix are not the "physical" matrix elements of the observables (for example, the diagonal elements are not the expectation values of the observables). The reason for this is that the basis vectors are not always mutually orthogonal with respect to the physical scalar products. It seems to us that this simple but important point has not been recognized. Consequently, the structure of the theory of paraparticles is quite different from what is usually believed. In examining the extent to which the state of a paraparticle system can be prepared by experiment,we only assumed that the symmetry type is physically observable. We agreed with Messiah and Greenberg that the most complete preparation is acheived if it yields the result that the state belongs to a common eigensubspace of a set of commuting observables. However, we did not find any reason to anticipate that the eigensubspace is irreducible with respect to the U(σ). Moreover, even if we did,the theory does not possess the property of the generalized rays, namely, measurable results do not depend on which rays of a generalized ray is chosen to represent the state. Our statement, which concerns the least complete preparation rather than the most complete one, has been that the eigensubspace is the non-zero eigenvalued eigensubspace of the projection operator onto a symmetry type. We think that, for paraparticles, it might be necessary to represent the state by a density matrix as shown in Chapter III rather than by a ray. In the density matrix description, however, the indistinguishability postulate is only one solution of the indistinguishability of identical particles, another solution could be the permutation invariance of the density matrices. A theory of identical particles could be built up from the permutation invariance of the density matrices.

In chapter IV, we found that the idistinguishability postulate is incompatible with the cluster property for parastatistics. Our result agrees with Steinmann's but not with Arons' and Hartle and Taylor's. Our approach made more direct use of the indistinguishability postulate than these authors', therefore, our result should be more unambigous. Furthermore, these authors relied on the property of the generalized rays which, as we have repeatedly stated is not valid for every measurement. Upon reconsidering Steinmann's argument, we found that it is valid and leads to our result only under special assumption about the orthogonality property of the 3-particle wavefunction. Our approach, furthermore, offers a very easy generalization to N-particle systems. While quite sure that the indistinguishability postulate is incompatible in the cluster property, we did not claim that this constitutes a proof of the symmetrization postulate. On the contrary, we think that it is necessary, or perhaps more correct, to use the theory in which the state is represented by a permutation invariant density matrix. No incompatibility with the cluster assumption arises in such a theory.

In chapter V, in establishing the connection between the permutation symmetry types and the intermediate statistics (which we called statistics of order p), we found that many symmetry types correspond to the same intermediate statistics and that the conditions which define these statistics are too restrictive for physical applications. Thus, it is more physically reasonable to classify paraparticles according to the permutation symmetry types than the statistics. We argued that Stolt and Taylor's classification of paraparticles into I.R's of S_N is self-consistent by showing that the outer product of two I.R.'s corresponding to one kind of paraparticles always contains at least one I.R. corresponding to the same kind of paraparticles.

In chapter VI, we studied the implication of the indistinguishability postulate in systems with variable numbers of particles. We found that the well-known selection rule first derived by Messiah and Greenberg, remains applicable in our theory and then derived a selection rule which we called Carpenter's selection due to its similarity with that obtained by Carpenter for S-matrix in Landshoff and Stapp's theory. This selection rule, which states that an N-particle system of a certain symmetry type can make transition to an M-particle state of one and only one symmetry type, is capable of explaining the observed fact that electrons for example are always fermions. We do not take this selection rule for a superselection rule for, due to the cluster property, we do not wish to impose the indistinguishability postulate on all physical observables, although it can be imposed on the evolution operator (of course, with the aid of the projection operators onto subspaces of fixed number of particles).

We began our discussion of Parafield Theory in chapter VII in which we reviewed various schemes of second quantization with the emphasis on Green's and Kamefuchi and Takahashi's methods. Kamefuchi and Takahashi's method, while capable of accounting for the paracommutation relations, also yields different commutation relations (C.R.). In this connection, we recall that it has been shown (Ohmuki, 1966) that bound states of particles associated to the paracommutation relations do not obey the para C.R.'s. It is interesting to find out whether they obey other C.R.'s obtained by Kamefuchi and Takahashi's method.

In chapter VIII, we studied the discrete representations of the parafermi C.R.'s following the method of Wightman and Schweber and focussed our attention on the representation given by Green's ansatzes. We have exhibited an infinite number of inequivalent I.R.'s among them only the Fock I.R. used in the second quantization theory possesses a unique vacuum state. We showed that the representation given by Green's ansatzes are those induced by representations of the anti C.R. or the C.R. in the tensor power space $\otimes^p H$ of the representation space H of the (anti) C.R. In particular, we illustrated in details the proof of a theorem, due to Greenberg and Messiah, which states that Green's ansatzes exhaust all I.R.'s of the para C.R.'s with a unique vacuum. Finally, as a verification of this theorem, we proved the existance of parafermi statistics in any Fock representations.

In chapter IX, we studied the class of parafields generated by Green's ansatzes with the aid of the Klein transformations. We made it clear that, although a parafield theory can be transformed to a theory of ordinary Bose and Fermi fields, a parafield theory predicts different results from the same theory in which every parafields are replaced by ordinary fields. This means that it is possible to determine experimentally whether a particle is associated with a parafield or an ordinary field. However, a physical significance of Green's ansatzes would be that parafields may not describe the true paraparticles but instead particles with some hidden variables reflected in the Green component fields. One of our results asserts that these hidden variables yield observable effects.

In chapter X., perhaps our most significant "result" was the awareness that many published results concerning the correspondence betwen the first and the second quantization theories cannot be trusted. The reason for this is that these results have been obtained by heavy use of the group-theoretical properties of the particle permutation operators, but we have showed that these operators do not define multidimensional representations of S_N. Our analysis admittedly is scanty but have clarified many aspects of the particle permutation operators which we think, have been a source of confusions. The particle permutation operators usually identified with the right regular representations of the place permutation operators, but strangely enough, it has not been recognized that, in the second quantization theories, the para C.R.'s imply the same thing for both left and right regular representations:

no multidimensional representations of S_N exist. We have adopted Yamada' definition of particle permutation operators which we believe expresses precisely what the people mean by particle permutation operators. It turns out that, in the first quantization theory where the particle permutation operators coincide with the operators defining the right regular representation, whenever the particle permutation operators can be defined unambigously (in many cases, we are faced with an "embarras du choix" for the particle permutation operators). The fact that multidimensional representations of S_N do not exist in the right regular representation in the second quantization theory has suggested to us, and we have verified, that the particle permutation operators do not commute with the place permutation operators in this theory. This is also a difference between the first quantization theory and the second quantization theory.

In summary, we have gained an insight into the structures of the first and the second quantization theories of paraparticles. Several confusions in the first quantization theories, the significence and applications of Green's ansatzes, and the nature of the permutation operators in the two theories have been clarified. While having not attempted an answer to the question of whether any existing elementary particles are para, we have made it quite clear that the attempts to rule out parastatistics have not been successful.

REFERENCES

Araki, H., J. Math. Phys. 2, 267 (1964)

Araki, H., Greenberg, O.W. and Toll, J.S., Phys. Rev., 142, 1017 (1966)

Arons, M.E., Lettere Nuovo Cimento, 1, 911 (1969)

Boerner, H., Representation of Groups (North-Holland, Amsterdam), chapter III and IV (1963

Boulware, D.G. and Deser, S., Nuovo Cimento, 30, 230 (1963)

Bialynicki-Birula, I., Nucl. Phys., 49, 605 (1963)

Carpenter, K.M., Ann. Phys., 60, 1 (1970)

Casher, A. Frieder O., Gluck, M. and Peres A., Nucl. Phys., Nucl. Phys., 66, 632 (1965)

Dresden, M., in Brandeis Summer Lectures in Theoretical Physics (Brandeis University), vol. 2 (1963)

Dell' Antonio, O.F., Greenberg, O.W. and Sudarshan, E. C. G., in Group Theoretical Concepts and Method in Elementary Particle Physics (ed. F. Gursey, Gordon and Breach Publishers, Inc., New York, 403 (1964)

Dirac, P.A.M., The Principle of Quantum Mechanics (Oxford, Fourth Edition) (1958)

Elton, L.R.B., Introductory Nuclear Physics (W.B. Saunders Co., Philadelphia) (1966)

Fano, U., Rev. Mod. Phys., 29, 74 (1957)

Galindo, A., Morales A. and Nunez-Lagos, R., J. Math. Phys., 3, 324 (1962)

Galindo, A. and Yndurain, F.J., Nuovo Cimento, 30, 1040 (1963)

Green, H.S., Phys. Rev. 90, 270 (1953)

Greenberg, O.W., Phys. Rev. Letters, 13, 598 (1964)

Greenberg, O.W.,in Proceedings of Conference on Mathematical Theory of Elementary Particles, ed. R. Goodmand and E. Segal (London-New York: M.I.T. - Press) (1966)

Greenberg, O.W., private communication (1972)

Greenberg, O.W., and Messiah, A.M.L., Phys. Rev., 138,B1155 (1965)

Gentile, G., Nuovo Cimento, 17, 493 (1940)

Gentile, G., Nuovo Cimento, 19, 106 (1942)

Girardeau, M.D., Phys. Rev. 139, B500 (1965)

Girardeau, M.D., J. Math. Phys., 10, 1302 (1969)

Guenault, A.M. and MacDonald, D.K.C., Molecular Physics, 5, 524 (1962)

Haag, R. and Schroer, B., J. Math. Phys. 3, 248 (1962)

Hamermesh, M., Group Theory (Addison-Wesley) (1962)

Hartle, J.B., and Taylor, J.R., Phys. Rev., 178, 2043 (1968)

Jauch, J.M., Helv. Phys. Acta, 33, 711 (1960)

Jauch, J.M., and Misra, B., Helv. Phys. Acta, 34, 699 (1961)

Jordan, T.F., Mukunda, N. and Pepper, S.V., J. Math. Phys. 4, 1089 (1963)

Kamefuchi, S.. Nuovo Cimento, 45B, 6912 (1966)

Kamefuchi, S., and Takahashi, Y., Nucl. Phys. 36, 177 (1962)

Kamefuchi, S., and Takahashi, Y., Progr. Theoret. Phys. (Kyoto), Suppl. 37, and 38, 244 (1966)

Kamefuchi, S., and Strathdee, J., Nucl. Phys., 42, 1966 (1963)

Kademova, K.V., and Kraev, M.M., Nucl. Phys., B26, 326 (1971)

Kademova, K.V., and Kraev, M.M., Phys. Letters 34B, (1971)

Kinoshita, T., Phys. Rev., 110, 978 (1958)

Klein, O., J. Phys. Rad., 9, 1 (1938)

Landshoff, P.V., and Stapp, H.P., Ann. Phys. 43, 72 (1967)

Messiah, A.M.L., Quantum Mechanics (North-Holland, Amsterdam, (1965), vol II, 601

Messiah, A.M.L., and Greenberg O.W., Phys. Rev. 136, B248 (1964)

Mitra, A.N., Phys. Rev. 151, 1168 (1966)

Naimark, M.A., Normed Rings, translated by L.F. Boron (P. Noordhoff, Ltd., 1960), chapter IV.

Nishijima, K., Progr. Theor. Phys. (Kyoto), 5, 187 (1950)

Okayama, T., Prog. Theo. Phys. (Kyoto), 1, B19 (1952)

Oneda, S. and Umezawa, H., Prog. Theor. Phys. (Kyoto), 9, 685, (1953)

Ohnuki, Y., Progr. Theor. Phys. (Kyoto) suppl. 37 and 38, 285 (1966)

Ohnuki, Y., and Kamefuchi, S., Phys. Rev. 170, 1279 (1968)

Ohnuki, Y. and Kamefuchi, S., Ann. Phys. 51, 337 (1969)

Ohnuki , Y. and Kamefuchi, S., Ann. Phys. 65, 19, (1971)

Ohnuki, Y., Yamada, M. and Kamefuchi, S., Phys. Letters, 36B, 51, (1971)

O'raifeartaigh, L. and Ryan C., Prov. Royal Irish Acad. 47, 207 (1963)

Roman, P., Introduction to Quantum Field Theory (North-Holland, Amsterdam, 1970). page 284, problem 5.1.

Ryan, C. and Sudarshan, E.C.G., Nucl. Phys. 47, 207 (1963)

Scharfstein, H., Nuovo Cimento, 30, 740 (1963)

Scharfstein, H., J. Math. Phys. 7, 1707 (1966)

Schubert, G., Z. Naturf., 1, 113 (1946)

Sommerfeld, A., Ber. Dtsch. Chem. Ges., 75, 1988 (1942)

Stolt, R.H. and Taylor, J.R., Nucl. Phys., B19, 1 (1970)

Stolt, R.H. and Taylor, J.R., Phys. Rev., D1, 2226 (1970)

Stolt, R.H. and Taylor, J.R., Nuovo Cimento, 5A, 185 (1971)

Streater, R.F. and Wightman, A.S., PCT, Spin ∝ Statistics and All That (W.A. Benjamin, New York - Amsterdam 1964)

Steinmann, O., Nuovo Cimento 44A, 755 (1966)

Steinmann, O., private communication (1971)

Ter Haar, D., Physica, 18, 199 (1952)

Volkov, D.V., Soviet Phys. (JETP), 9, 1107 (1959)

Volkov, D. V., Soviet Phys. (JETP), 11, 375 (1960)

Von Neumann,J., Ann. Math. 50, 401 (1949)

Weyl, H., Classical Groups (Princeton University Press, 1966) chapter III

Wick, G.C., Wightman, A.S. and Wigner, E.P., Phys. Rev. 88, 101 (1952)

Wightman, A.S., Nuovo Cimento Suppl., 14, 8 (1959)

Wightman, A.S., and Schweber S.A., Phys. REv. 98, 812 (1955)

Wigner, E.P., Phys. Rev., 77, 711 (1950)

APPENDIX A

SOME RESULTS OF THE ALGEBRAIC REPRESENTATION THEORY OF A FINITE GROUP

In this appendix, we summarize some results of the theory of group algebra of a finite group which are relevant for our discussion in Chapter III. We shall not give proofs to all our statements since they can all be found in the literature[†]

The vector space $\mathfrak{U}$ constructed on the basis formed by the elements of a finite group G is called the group algebra of G. An element of $\mathfrak{U}$ is of the form

$$X = \sum_{\sigma} a(\sigma)\sigma, \quad \sigma \in G \qquad \text{(A.1)}$$

a(σ)'s are constants.

There can exist, within the algebra $\mathfrak{U}$, a linear set $\mathfrak{J}$ which, with two elements, contains their product, $\mathfrak{J}$ will be called a subalgebra of $\mathfrak{U}$. The most interesting subalgebras are the left (or right) ideals: if x is an element of a left (right) ideal, and $y \in \mathfrak{U}$, yx (or xy) will also be an element of the left (or right) ideal. A subalgebra which is simultaneously a left and a right ideal is called a two-sided ideal or an invariant subalgebra. The invariance property can be expressed conveniently as $\mathfrak{U}\mathfrak{J} \subseteq \mathfrak{J}$, $\mathfrak{J}\mathfrak{U} \subseteq \mathfrak{J}$

An algebra is said to be reducible if it is the sum of two subalgebras $\mathfrak{J}$ and $\mathfrak{T}$ satisfying $\mathfrak{J}\mathfrak{T} = \mathfrak{T}\mathfrak{J} = \phi$, $\mathfrak{J} \cap \mathfrak{T} = \phi$ (an empty intersection). $\mathfrak{U}$ is called the direct sum of $\mathfrak{J}$ and $\mathfrak{T}$, with the notation

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[†]See for example Boerner (1963), Chapter III & IV; Weyl (1966), Chapter III.

$$\mathfrak{U} = \mathcal{S} \oplus \mathcal{T}$$

One can show that the condition $\mathcal{S}\mathcal{T} = \mathcal{T}\mathcal{S} = \phi$ is equivalent to requiring the invariance of $\mathcal{S}$ and $\mathcal{T}$ (or $\mathcal{S}$ and $\mathcal{T}$ are two sided ideals).

An element $\varepsilon \in \mathfrak{U}$ such that $\varepsilon^2 = \varepsilon$ is called an idempotent. An idempotent ε is said to be primitive if there exists no idempotent δ such that $\varepsilon\delta = \delta\varepsilon = \varepsilon$.

A left ideal is called primitive if it contains no left ideal other than itself. A two-sided ideal is called simple if it contains no other two-sided ideal. A simple two-sided ideal may contain a finite number of left ideal s_1, $s_2 \ldots s_r$ in which case one writes

$$\mathcal{S} = s_1 + s_2 + \ldots + s_r$$

It can be shown that any left ideal is generated by a primitive idempotent ε, i.e. any element of the left ideal can be written as $X\varepsilon$, $X \in \mathfrak{U}$. ε is called the generating unit of the left ideal. A two-sided ideal can also be generated by an idempotent (which is not primitive). The following important theorem is well known in the theory of group algebra.

Theorem A-1

"The group algebra is reducible and decomposable into a series of simple two sided ideals,

$$\mathfrak{U} = \mathfrak{U}^1 \oplus \mathfrak{U}^2 \oplus \ldots \oplus \mathfrak{U}^r, \qquad (A.2)$$

each of them contains a finite number of equivalent primitive left ideals (which can be mapped into one another).

$$\mathfrak{U}^{\mu} = u_1^{\mu} + u_2^{\mu} + \ldots u_{n_\mu}^{\mu} \qquad (A.3)$$

Let ε^{μ} be the generating unit of $\tilde{\mathcal{U}}^{\mu}$ and ε^{μ}_{j} the generating unit of u^{μ}_{j}. The following decompositions hold

$$e = \varepsilon^{1} + \varepsilon^{2} + \ldots + \varepsilon^{\mu} + \ldots + \varepsilon^{r} \qquad \text{(A.4)}$$

$$\varepsilon^{\mu} = \varepsilon^{\mu}_{1} + \varepsilon^{\mu}_{2} + \ldots \varepsilon^{\mu}_{j} + \ldots \varepsilon^{\mu}_{n\mu} \qquad \text{(A.5)}$$

with

$$\left.\begin{aligned} &\varepsilon^{\mu}\varepsilon^{\nu} = 0,\ \mu \neq \nu \\ &\varepsilon^{\mu}_{i}\ \varepsilon^{\mu}_{j} = 0,\ i \neq j \end{aligned}\right. \qquad \text{(A.6)}$$

where e is the unit element of $\tilde{\mathcal{U}}$.

The vector space $\mathcal{U}$ supports a representation of the group G, called the <u>regular representation</u>. Each element $\pi\varepsilon G$ corresponds to a linear transformation $\tilde{\pi}$: $x \rightarrow y$ given by

$$y = X\pi = \sum_{\sigma} a(\sigma)\sigma\pi$$

$$= \sum_{\sigma} a(\sigma\pi^{-1})\sigma$$

and effectuated by the matrix whose element are the $a(\sigma\pi^{-1})$ which is the representative of π in $\tilde{\mathcal{U}}$.

The matrix representing π, with the basis chosen, has no non-zero element on its principal diagonal except when π=e for $\sigma\pi^{-1} = \sigma$ implies $\sigma = e$. Therefore the characters of a regular representation are zero except the one corresponding to the unit element which is equal to g, the number of elements of G. Given a representation μ of G, the number of time it is found in the regular representation is given by (with X denoting the character):

$$c_{\mu} = \frac{1}{g} \sum_{\sigma} \overline{\chi}_{\mu}\ (\sigma)\ X(\sigma)$$

$$= \frac{1}{g}\ \overline{\chi}_{\mu}\ (e)\ X(e) = n_{\mu}$$

n_μ being the dimension of the μ-representation. Thus, any irreducible representation of G is contained in the regular representation, each representation occuring a number of times equal to its dimension.

From this, there follows a method of finding all irreducible representation of G. Since the irreducibility covers the notion of invariant subspace, we can obtain all irreducible representations by determining all primitive left ideals. Equivalent left ideals yield equivalent irreducible representations. With a special choice of the basis of $\mathfrak{A}$, the matrices representing an element of $\mathfrak{A}$, in particular an element of G, are the same in every left ideals. In this connection, it can be seen that each two-sided ideal yields an unequivalent irreducible representation.

The set of elements of $\mathfrak{A}$ which commute with all elements of $\mathfrak{A}$ is called the <u>center</u> $\mathfrak{C}$ of $\mathfrak{A}$. Since any element $X \in \mathfrak{A}$ can be written as

$$X = X^1 + X^2 + \ldots + X^\mu + \ldots + X^r$$

where $X^\mu \in \mathfrak{A}^\mu$ and

$$X^\mu X^\nu = X^\nu X^\mu = 0, \quad \mu \neq \nu$$

Schur's lemma implies that, if $X \in \mathfrak{C}$, then $X^\mu = \lambda^\mu \varepsilon^\mu$, λ^μ is constant. The center $\mathfrak{C}$ consists of elements of the form

$$X = \lambda^1 e^1 + \lambda^2 \varepsilon^2 + \ldots + \lambda^r e^r \qquad \text{(A.7)}$$

A different basis for the center $\mathfrak{C}$ can be obtained by noting that an $X \in \mathfrak{C}$ is characterized by

$$\sigma^{-1} X \sigma = X$$

or

$$\frac{1}{g} \sum_{\sigma} \sigma^{-1} X \sigma = X$$

if $X = \Sigma\, a(\sigma)\sigma$, we have

$$\Sigma\, \sigma^{-1} X \sigma = \sum_{\sigma,\lambda} a(\lambda)\sigma^{-1} \lambda \sigma$$

$$= \Sigma\, a(\lambda) (\sum_{\sigma} \sigma^{-1} \lambda \sigma)$$

Let us denote k_p the sum of elements of the p^{th} class of G. Obviously,

$$\Sigma\, \sigma^{-1} \lambda \sigma = \frac{g}{g_p} k_p,$$

g_p being the number of elements of the p^{th} class, so that X appears as a linear combination of the elements of the k_p,

$$X = \lambda_1 k_1 + \lambda_2 k_2 + \ldots + \lambda_r k_r \qquad (A.8)$$

We summarize the results concerning the center in the

Theorem A.2

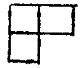
"The center of the group algebra is a r dimensional vector space, r being the number of classes of the group, with a basis constructed on the idempotents generating the simple two-sided ideals or on the sums of elements of the classes."

For the symmetric group S_N, the problem of finding all left ideals has been solved completely long time ago. The primitive idempotent is given by

$$\varepsilon^{\mu} = \frac{n_{\mu}}{N!} PQ$$

where P is the Young symmetrizer of all the rows in a standard Young tableau and Q is the Young anti-symmetrizer of all columns.

APPENDIX B

IRREDUCIBLE REPRESENTATIONS OF S_3 ASSOCIATED WITH THE TRIANGULAR YOUNG DIAGRAM 

(e)	(12)	(23)	(13)	(123)	(132)
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

See Hammermesh (1962), page 224.

APPENDIX C

INVARIANT SUBGROUPS OF THE SYMMETRIC GROUP S_N

N	Invariant Subgroups			
2	S_2,	{e}		
3	S_3,	A_3,	{e}	
4	S_4,	A_4	V_4	{e}
≥ 5	S_N	A_N	{e}	

{e} is the subgroup of S_N consisting only of the unit element. A_N is the alternating group of degree N. i.e. subgroup of S_N consisting only of even permutations. V_4 is the set {e,(12)(34), (13)(24), (14)(23)} usually called Klein's group.