A numerical study of cohomogeneity one manifolds

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Abstract: This dissertation explores numerical solutions for the cohomogeneity one Einstein and Ricci soliton equations when the principal orbits are $SU(3)/T^2$ and $Sp(3)/Sp(1)^3$. We present new numerical evidence for steady, expanding solitons as well as Einstein metrics with positive scalar curvature. In the case of steady solitons we produced a one-parameter family of solutions. In the expanding case, we generated a two-parameter family of solutions and in particular in the negative Einstein case we generated a one-parameter family of solutions. In the compact Einstein case we found numerical evidence for an infinite number of Einstein metrics.

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1 Overview

A Ricci soliton consists of a complete Riemannian metric g and a complete vector field X on a manifold which satisfy the equation

$$\mathrm{Ric}(g) + \frac{1}{2}L_X g + \frac{\epsilon}{2}g = 0$$

where L is the Lie derivative. If $\epsilon > 0$ the soliton is called **expanding**, if $\epsilon < 0$ **shrinking**, and **steady** if $\epsilon = 0$ [1]. In particular if the middle term on the left hand side is zero, then we have an Einstein metric. In the Einstein case, this reduces the soliton equation to $\mathrm{Ric}(g) = -\frac{\epsilon}{2}g$. This happens precisely when X is a Killing vector field for the metric g. Thus the scalar curvature is positive (resp negative) if $\epsilon < 0$ (resp $\epsilon > 0$). If $\epsilon = 0$, such Einstein metrics are called Ricci-flat.

Most known examples of Ricci solitons are of *gradient type*, that is, $X = \mathrm{grad}(u)$ for some smooth function u. In this case, the Ricci soliton equation becomes

$$\mathrm{Ric}(g) + \mathrm{Hess}(u) + \frac{\epsilon}{2}g = 0.$$

This dissertation presents numerical solutions in cases where the soliton is expanding, or steady and for the Einstein case with $\epsilon < 0$. We consider cohomogeneity one metrics in which case, the Einstein and soliton conditions reduce to a system of non-linear ordinary differential equations on the orbit space I with appropriate boundary conditions to ensure we have a smooth metric. As none of the ordinary differential equations have known analytic solutions we make use of computer algebra softwares and known numerical schemes to provide numerical solutions.

Various analytic solutions were found in the past. For the principal orbit $\frac{SO(p+1)\times SO(q+1)}{SO(p)\times SO(q)}$ there are results for compact Einstein metrics [6], Ricci flat and negative Einstein metrics [7], and steady soliton solutions [9][19]. For $U(n+1)/U(n)$, both [10] and [11] found results for the compact Einstein case as well the Ricci-flat metric with [4]. For $SO(n+1)/SO(n)$, there also exists steady soliton solutions due to Bryant. Finally for the case $Sp(2)/Sp(1)U(1)$, only the Ricci-flat solutions were generated [13].

In 1989, Bryant and Salamon showed that the cone $\mathbf{R}_+ \times \frac{SU(3)}{T^2}$ has an incomplete Ricci-flat metrics with G_2 holonomy [14]. In section 8 of Bohm's paper [6], he showed analytically that there are infinitely many Einstein metrics with positive scalar curvature on

S^k	$5 \leq k \leq 9$
$S^k \times S^\ell$	$5 \leq k + \ell \leq 9,\ 2 \leq k \leq \ell$
$S^{k+1} \times Q$	$3 \leq k+1 \leq 9 - \dim Q,\ 1 < \dim Q \leq 6$

Finally there are also numerical work done in [1] and [15]. The algorithm used in this dissertation was partly and independently developed by Jonathan Baker.

In section 2, 4, 3, and 5 we provide brief background material. In sections 6 and 7 we lay the groundwork for the numerics and finally section 8 presents the main result.

2 The Adjoint Representation and Isotropy Representation

Let G be a Lie group, then a manifold M is a **homogenous space** of G if the group G acts smoothly and transitively on M. Let K be a closed subgroup of G, it is possible to introduce a smooth structure on the set $G/K = \{gK : g \in G\}$ of all left cosets of K in G [3]. In particular we may select K to be an isotropy subgroup for the $G-$action on M.

A Riemannian manifold M whose isometry group $I(M)$ acts transitively is called a **Riemannian homogenous space**. The isometry group of a Riemannian manifold is a Lie group, this is a result of Myers-Steenrod [17].

Let G/K be a homogeneous space and $\pi : G \to G/K$ defined by

$$\pi(g) = gK$$

be the projection map. Let $\mathfrak{k}$ be the Lie algebra of $K \subset G$ and $\mathfrak{g}$ be the Lie algebra of G. Let $X \in \mathfrak{g}$ and $\exp(tX)$ be the corresponding one-parameter subgroup. The differential $d\pi_e : \mathfrak{g} \to T_o(G/K)$ where $o = \pi(e) = K$ can be computed in the following way,

$$d\pi_e(X) = \frac{d}{dt}(\pi \circ \exp(tX))\bigg|_{t=0} = \frac{d}{dt}(\exp(tX)K)\bigg|_{t=0}.$$

From this we see that $d\pi_e(\mathfrak{k}) = 0$, and in fact $\ker d\pi_e = \mathfrak{k}$ [20]. Furthermore, because $d\pi$ is onto [20], we obtain the canonical isomorphism,

$$\mathfrak{g}/\mathfrak{k} = \mathfrak{g}/\ker d\pi_e \approx T_o(G/K).$$

If K is compact, then there is an Ad_K-invariant inner product on $\mathfrak{g}$ by averaging. So since $\mathfrak{k} \subset \mathfrak{g}$, we can define an orthogonal complement $\mathfrak{p} = \mathfrak{k}^{\perp} \subset \mathfrak{g}$ through this inner product. Because of the quotient vector space structure, from the previous isomorphism, we also obtain the isomorphism,

$$\mathfrak{p} = \mathfrak{k}^{\perp} \approx T_o(G/K).$$

It is known that the isotropy representation of K in $T_o(G/K)$ is equivalent to the adjoint representation of K in $\mathfrak{p}$ through the commutative diagram.

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{Ad_G(k)} & \mathfrak{p} \\ \big\downarrow{\scriptstyle d\pi_e|_{\mathfrak{p}}} & & \big\downarrow{\scriptstyle d\pi_e|_{\mathfrak{p}}} \\ T_o(G/K) & \xrightarrow{(d\tau_k)_o} & T_o(G/K) \end{array}$$

Here $\tau_k\ (gK) = kgK$ is the left translation. The proof of this can be found e.g. in [3].

Hence we conclude that the isotropy representation of G/K can be computed by computing the adjoint representation of K in $\mathfrak{p}$ where $\mathfrak{p} = \mathfrak{k}^{\perp}$ denotes the complement of the Lie algebra of K in the orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^{\perp} = \mathfrak{k} \oplus \mathfrak{p}$.

3 Cohomogeneity One Manifold

Let G be a compact Lie group acting smoothly on the $n-$dimensional manifold M such that the orbit space M/G is connected.

Then by a theorem from [8], there exists a maximum orbit type G/K for G on M (i.e., K is conjugate to a subgroup of each isotropy). The union of orbits of type G/K is open and dense in M and its image in M/G is connected.

The maximum orbit type for orbits in M guaranteed by the above theorem is called the **principal orbit type** and orbits of this type are called **principal orbits**. Let $P \simeq G/K$ be a principal orbit and $Q \simeq G/H$ be any other orbit. Then K is conjugate to a subgroup of H, and so we may assume that $K \subset H$. Then there is an equivariant map $P \to Q$ is a fiber bundle projection $G/H \to G/K$ with fiber K/H. If $\dim P > \dim Q$ (i.e., $\dim K/H > 0$), then Q is called a **singular orbit** [8].

A **cohomogeneity one** manifold M is a Riemannian manifold acted on by a compact Lie group G with orbit space of dimension one. Here in this dissertation, we will restrict our discussion to the case where the orbit space is either a closed interval or a half open interval.

Now let (M, g) be a cohomogeneity one manifold of dimension $n+1$. Let $p \in M$ be a point in a principal orbit and $P = G/G_p$ be the corresponding principal orbit with isotropy subgroup G_p. Similarly we select singular point $q \in M$ and let $Q = G/G_q$ be the corresponding singular orbit with isotropy subgroup G_q. Define the projection map

$$\pi : M \to M/G = I.$$

Let $I = [0, a)$ and define $\hat{I} = \text{int}(I)$. The preimage $\pi^{-1}(\hat{I})$ consists of principal orbits and $\pi^{-1}(\{0\})$ is the singular orbit Q. Select a geodesic $\gamma : [0, \infty) \to M$ parametrized by arclength starting from the point q intersecting all principal orbits orthogonally[1]. All points on this geodesic in the principal orbits then have the same isotropy subgroup. Further, there is also a diffeomorphism

$$\begin{aligned} \phi : \hat{I} \times G/G_{\gamma(t)} &\to M_0 \subset M \\ \phi(t, gG_{\gamma(t)}) &= g\gamma(t). \end{aligned} \tag{1}$$

Here M_0 denotes the union of principal orbits in M, $\hat{I} = (0, \infty)$, and $G_{\gamma(t)}$ is the isotropy subgroup of the points $\gamma(t)$ for $t > 0$. Then $P_t = \phi(t, G/G_{\gamma(t)}) = G/G_{\gamma(t)}$ is the principal orbit passing through the point $\gamma(t)$.

Once again we take q from one of its singular orbit G/H where $H = G_q$ and consider its tangent space $T_q(G/H) \subset T_q(M)$ and its normal space $N_q(G/H) \subset T_q(M)$. In its isotropy subgroup H, select $h \in H$ and consider its derivative $dh : T_qM \to T_qM$.

Now since $dh|_{T_q(G/H)} : T_q(G/H) \to T_q(G/H)$ and dh acts by isometry, we have $dh|_{N_q(G/H)} : N_q(G/H) \to N_q(G/H)$ by invariance (as $N_q(G/H)$ is the orthogonal complement to $T_q(G/H)$). On each normal space $N_q(G/H)$, there exists $S_r^d \subset N_q(G/H)$ with a positive radius r where dh maps S_r^d into S_r^d by the isometry of dh. Now let us select and fix a non-zero element $v \in N_q(G/H)$. Set

$$L = \{dh(v) \; |h \in H\},$$

to be the image set of all derivatives of h applied to v. We claim $L = S_r^d$. First we have the inclusion $L \subset S_r^d$, but now suppose there is an element $w \in S_r^d - L$, then the projection $\pi : M \to M/G = I$ will map w into the orbit space. By a proposition from [8], which asserts that there is a one-to-one correspondence between the orbit space M/G and orbit space $N_q(G/H)/H$, the projection map will take w into a point different from $\pi(v)$.

But since M is a cohomogeneity one manifold, the orbit space is one-dimensional and all other vectors in the radial geodesic defined from q to v all mapped into a different point into the orbit space as each point belongs to a principal orbit. The element w

[1] A consequence of Gauss's Lemma, but we use the normal exponential map.

is at a fixed radius r from the point q, the same r of v by definition. This means that the point w cannot exist, hence $L = S^d_r \simeq H/K$ and so H acts transitively on S^d_r.

4 Cohomogeneity One Metrics

In this section we will describe cohomogeneity one Riemannian metrics on the space M. Using (1) and its constructions for this entire section, consider the metric g on $\hat{I} \times G/K$, that is the pullback of g through the map ϕ.

$$\phi^*(g) = dt^2 + g(t) \tag{2}$$

Here $g(t)$ as described in the introduction are the $G-$ invariant metrics on the hypersurfaces $G/K_{\gamma(t)}$.

Let $\mathfrak{p} \simeq T(G/G_{\gamma(t)})$ be the tangent space to $\gamma(t)$ of the hypersurface G/K, with $t \in \hat{I}$ and the adjoint representation of K in $\mathfrak{p}$. As discussed in section (2), if K is compact we may first select an $Ad_G(K)-$invariant inner product Q on $\mathfrak{p}$. Then we may decompose $\mathfrak{p}$ into $Q-$orthogonal real $Ad_G(K)-$irreducible subspaces,

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_r. \tag{3}$$

such that the first $k-$ direct summands $\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k \simeq T_K(H/K)$ and $\mathfrak{p}_{k+1} \oplus \cdots \oplus \mathfrak{p}_r \simeq T_H(G/H)$.

Lemma 4.1 *If each of the $\mathfrak{p}_i$s are pairwise inequivalent $Ad_G(K)$ representations, then Schur's lemma imply that $\langle , \rangle|_{\mathfrak{p}_i} = g_i^2 Q|_{\mathfrak{p}_i}$ for some functions g_i^2 to be determined.*

Proof:

The metric can be written as $g(X, Y) = \langle TX, Y \rangle |_{\mathfrak{p}}$ for some self-adjoint linear operator $T : \mathfrak{p} \to \mathfrak{p}$ and $\pi \circ T : \mathfrak{p}_{\mathfrak{i}} \to \mathfrak{p}_j$ where $\pi : \mathfrak{p} \to \mathfrak{p}_j$ is the projection map of $\mathfrak{p}$ onto the jth component of $\mathfrak{p}$.

The existence of T follows from the fact that if (e_i) is an orthonormal basis for background metric $\langle\rangle$ of $\mathfrak{p}$, then that means

$$\begin{aligned} g(X,Y) &= g\left(X, \sum_i \langle Y, e_i\rangle |_{\mathfrak{p}} e_i\right) \\ &= \sum_i g(X,e_i)\langle Y, e_i\rangle |_{\mathfrak{p}} \\ &= \sum_i \langle Y, e_i\rangle |_{\mathfrak{p}} g(X,e_i) \\ &= \left\langle Y, \sum_i g(X,e_i)e_i \right\rangle |_{\mathfrak{p}} \\ &= \langle Y, TX\rangle |_{\mathfrak{p}}. \end{aligned}$$

This map is linear and self adjoint because $\langle TX, Y\rangle |_{\mathfrak{p}} = g(X,Y) = g(Y,X) = \langle TY, X\rangle |_{\mathfrak{p}}$ and is $K-$invariant because

$$\begin{aligned} \langle T(kX), Y\rangle |_{\mathfrak{p}} &= g(kX,Y) \\ &= g(k^{-1}kX, k^{-1}Y) \\ &= g(X, k^{-1}Y) \\ &= \left\langle TX, k^{-1}Y\right\rangle |_{\mathfrak{p}} \\ &= \left\langle kTX, kk^{-1}Y\right\rangle |_{\mathfrak{p}} \\ &= \langle kTX, Y\rangle |_{\mathfrak{p}}. \end{aligned}$$

The linear map T is self-adjoint, therefore it has real eigenvalues λ. The map $(\pi \circ T - \lambda I)|_{\mathfrak{p}_i} : \mathfrak{p}_i \to \mathfrak{p}_i$ is also self-adjoint and $Ker(\pi \circ T - \lambda I) \neq 0$ is not trivial as λ is an eigenvalue and is an invariant subspace of $\mathfrak{p}_i$, therefore Schur's lemma implies that the non-injective map on $\mathfrak{p}_i$ is $Ker(\pi \circ T - \lambda I) = \mathfrak{p}_\mathfrak{i}$ meaning $\pi \circ T - \lambda I$ is the zero map. In other words, $\pi \circ T = \lambda I$. On the other hand if the kernel is trivial for a map $\pi \circ S : \mathfrak{p}_i \to \mathfrak{p}_j$, then this means $S \circ \pi$ is an linear isomorphism, but this cannot happens as $\mathfrak{p}_i$s are inequivalent.

■

In this dissertation, we will consider two cases where $r = 3$ and the action of $Ad_G(K)$ on the inequivalent $\mathfrak{p}_i$, so that such a decomposition is unique up to permutation and the summands $\mathfrak{p}_i$ are orthogonal with respect to the background metric/inner product Q, that is,

$$g(t)\big|_{\mathfrak{p}_i} \sim \langle,\rangle_t = g_i^2(t)h\big|_{\mathfrak{p}_1} \perp g_i^2(t)h\big|_{\mathfrak{p}_2} \perp \cdots \perp g_i^2(t)h\big|_{\mathfrak{p}_r}. \tag{4}$$

5 Ricci Soliton Equation Reduction

This part will borrow notations from [12]. Denote by $\hat{\nabla}$ and $\hat{R}$ the Levi-Civitia connection and curvature tensor of $\hat{g}$ respectively for our cohomogeneity Riemannian manifold $(\hat{M}, \hat{g})$.

Let $N = d\phi\left(\frac{\partial}{\partial t}\right)$. Then it is a unit normal field for the hypersurface family except at $t = 0$, and let $L(t)$ be the shape operator of the hypersurface $P_t = \phi(P \times \{t\})$ defined by

$$L(t)X = \hat{\nabla}_X N$$

for any $X \in TP_t$. We consider the shape operator as a one parameter family of endomorphisms on TP through the diffeomorphism ϕ described in the previous section. Then using the fact the bracket between $[N, X]$ is 0 for $X \in TP_t$ (because we are on a product manifold on M_0) we get,

$$\frac{\partial}{\partial t} g_t(X, Y) = 2g_t(L(t)X, Y) = 2(g \circ L)(X, Y).$$

Now using the Riccati equation for $L(t)$ and the Gauss-Codazzi equations for the hypersurfaces P_t, we obtain for $X, Y \in TP_t$ and an orthonormal basis $\{e_i\}$ of $\hat{g}$,

$$\begin{aligned}
\hat{\text{Ric}}(X, Y) &= \text{Ric}(X, Y) - \text{tr}(L)g(L(X), Y) - g\left(\frac{\partial L}{\partial t}(X), Y\right) \\
\hat{\text{Ric}}(X, N) &= \sum_i \langle (\nabla_{e_i} L)X, e_i \rangle - \langle (\nabla_{e_i} L)X)e_i, e_i \rangle \\
&= -\text{tr}(X \neg \delta^{\nabla} L) \\
\hat{\text{Ric}}(N, N) &= -\text{tr}\left(\frac{\partial L(t)}{\partial t}\right) - \text{tr}\left(L(t)^2\right)
\end{aligned}$$

Here $\delta^{\nabla} = \langle (\nabla_{e_i} L)X, e_i \rangle$ and $\neg$ is the interior product.

Let $r(t)$ denote the Ricci endomorphisms of TP of the metrics g_t, then $g(r_t(X), Y) = \text{Ric}(g_t)(X, Y)$. We do the same for $\hat{g}$ and denote $\hat{r}(t)$ for the Ricci endomorphisms for $\hat{g}$. The gradient Ricci soliton equation on M_0 now reduces to the system

Proposition 5.1

$$L' = u'L - tr(L)L + r - \epsilon I \tag{5}$$

$$tr(L') = u'' - tr(L^2) - \epsilon \tag{6}$$

$$-tr(X \neg \delta^{\nabla} L) = 0 \tag{7}$$

Remark 5.1 *Equations (5) refer to the components of the equation in the tangential direction to the hypersurfaces and (6) refer to the components in the $\frac{\partial}{\partial t}$ direction. Finally (7) represents the equation in mixed direction.*

In the case where $\hat{g}$ is an Einstein metric we have

Proposition 5.2

$$g' = 2gL \tag{8}$$

$$L' = -tr(L)L + r - \epsilon I \tag{9}$$

$$tr(L') = -tr(L^2) - \epsilon \tag{10}$$

$$-tr(X \neg \delta^{\nabla} L) = 0 \tag{11}$$

Remark 5.2 *If we take the trace of (9) and use (10), we obtain the conservation law equation*

$$s - tr(L)^2 + tr(L^2) = (n-1)\epsilon. \tag{12}$$

Here $s(t) = tr(r(t))$ denotes the scalar curvature of $g(t)$.

Remark 5.3 *Note that (8) implies $L = \frac{1}{2}g^{-1}g'$.*

Finally if there is a special orbit of dimension strictly smaller than n, and furthermore if (9) holds for a sufficiently smooth (C^3) metric and potential, then (6) also holds. If the well-known conservation law

$$\ddot{u} + (-\dot{u} + \mathrm{tr}(L))\dot{u} - \epsilon u = C \tag{13}$$

holds for some fixed constant C, then (10) also holds. Therefore we can use (13) in place of (10)

6 Isotropy Representation of $SU(3)/T^2$

This section concerns with the triplet $(G, H, K) = (SU(3), S(U(2)U(1)), T^2)$ where $S(U(2)U(1))/T^2 \simeq S^2$ and $SU(3)/S(U(2)U(1)) \simeq \mathbb{C}P^2$

Let us denote the Lie group of special matrices (unitary matrices with determinant 1) as

$$G = SU(n) = \{A \in U_n(\mathbb{C}) : A^*A = AA^* = I \text{ and } \det(A) = 1\}.$$

Here $A^* = A^{-1}$ denotes the the adjoint of A.

Let $A(t) \in SU(n)$ and suppose $A(0) = I$ and $B = A'(0)$. Differentiating $A^*A = AA^* = I$ and evaluating at $t = 0$ yields $B + B^* = 0$. In other words this is the set $\mathfrak{su}(n)$ of traceless skew-Hermitian matrices. The space $\mathfrak{su}(n)$ has dimension $n^2 - 1$.

Let $\mathfrak{g} = \mathfrak{su}(n)$ denote the Lie algebra of $SU(n)$,

$$\mathfrak{su}(n) = \{B \in \mathfrak{gl}(n, \mathbb{C}) : B = -B^*\}.$$

The objects B have the form

$$B = \begin{pmatrix} i\theta_1 & W_{12} & \dots & W_{1n} \\ -\overline{W}_{12} & i\theta_2 & \dots & W_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{W}_{1n} & -\overline{W}_{2n} & \dots & i\theta_n \end{pmatrix}$$

In the above W_{jk} are complex numbers.

Let us choose the background metric for $\mathfrak{su}(n)$ to be $Q(X,Y) = -\frac{1}{2}tr(XY)$ with $X, Y \in \mathfrak{k}$, then we see that $\mathfrak{su}(n)$ is spanned by the **orthonormal** basis

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \ddots & H_{ij} & 0 \\ 0 & -\overline{H}_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{1 \le i < j \le n}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i\theta_j & 0 & 0 \\ 0 & 0 & i\theta_k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{1 \le k < j \le n} : \sum \theta_i = 0, \right.$$
$$\left. H_{ij} \in \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\} \right\}$$

In particular for $n = 3$, we select

$$\left(\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&i&0\\i&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\begin{pmatrix}0&0&i\\0&0&0\\i&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&i\\0&i&0\end{pmatrix}\right)$$
$$= (X_1, X_2, X_3, X_4, X_5, X_6)$$

to be our orthonormal basis for $\mathfrak{p} = \mathfrak{k}^\perp$

Let $T^2 = \left\{\begin{pmatrix}e^{i\theta_1}&0&0\\0&e^{i\theta_2}&0\\0&0&e^{i\theta_3}\end{pmatrix} : \theta_1+\theta_2+\theta_3 = 0\right\}$ be the maximal torus of $SU(3)$. This is a torus because of the isomophism

$$\text{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) \to \text{diag}(e^{i(\theta_1)}, e^{i(\theta_2)}, \dots, e^{i(\theta_{n-1})})$$

that maps T onto the maximal torus $U(n-1)$.

The Lie algebra of T is

$$\mathfrak{t} = \left\{\begin{pmatrix}i\theta_1&0&0\\0&i\theta_2&0\\0&0&i\theta_3\end{pmatrix} : \theta_1+\theta_2+\theta_3 = 0\right\}.$$

Its isotropy representation is the map $Ad_G(K) : \mathfrak{k}^\perp \to \mathfrak{k}^\perp$

$$\begin{pmatrix}e^{i\theta_1}&0&0\\0&e^{i\theta_2}&0\\0&0&e^{i\theta_3}\end{pmatrix} X_i \begin{pmatrix}e^{-i\theta_1}&0&0\\0&e^{-i\theta_2}&0\\0&0&e^{-i\theta_3}\end{pmatrix}$$

Now we calculate for $i = 1 \dots 6$.

$$\begin{aligned}
tX_1t^{-1} &= \begin{pmatrix}e^{i\theta_1}&0&0\\0&e^{i\theta_2}&0\\0&0&e^{i\theta_3}\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}e^{-i\theta_1}&0&0\\0&e^{-i\theta_2}&0\\0&0&e^{-i\theta_3}\end{pmatrix}\\
&= \begin{pmatrix}0&e^{i\theta_1}e^{-i\theta_2}&0\\-e^{i\theta_2}e^{-i\theta_1}&0&0\\0&0&0\end{pmatrix}\\
&= \begin{pmatrix}0&e^{i(\theta_1-\theta_2)}&0\\-\overline{e^{i(\theta_1-\theta_2)}}&0&0\\0&0&0\end{pmatrix}\\
&= \cos(\theta_1-\theta_2)X_1 + \sin(\theta_1-\theta_2)X_2
\end{aligned}$$

$$\begin{aligned}
tX_2t^{-1} &= \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{i\theta_1}e^{-i\theta_2} & 0 \\ -e^{i\theta_2}e^{-i\theta_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & ie^{i(\theta_1-\theta_2)} & 0 \\ \overline{ie^{i(\theta_1-\theta_2)}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= -\sin(\theta_1-\theta_2)X_1 + \cos(\theta_1-\theta_2)X_2
\end{aligned}$$

$$\begin{aligned}
tX_3t^{-1} &= \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{i(\theta_1-\theta_3)} & 0 \\ -\overline{e^{i(\theta_1-\theta_3)}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \cos(\theta_1-\theta_3)X_3 + \sin(\theta_1-\theta_3)X_4
\end{aligned}$$

$$\begin{aligned}
tX_4t^{-1} &= \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & ie^{i(\theta_1-\theta_3)} & 0 \\ \overline{ie^{i(\theta_1-\theta_3)}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= -\sin(\theta_1-\theta_3)X_3 + \cos(\theta_1-\theta_3)X_4
\end{aligned}$$

$$\begin{aligned}
tX_5t^{-1} &= \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i(\theta_2-\theta_3)} \\ 0 & -\overline{e^{i(\theta_2-\theta_3)}} & 0 \end{pmatrix} \\
&= \cos(\theta_2-\theta_3)X_5 + \sin(\theta_2-\theta_3)X_6
\end{aligned}$$

$$tX_6t^{-1} = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & ie^{i(\theta_2-\theta_3)} \\ 0 & \overline{ie^{i(\theta_2-\theta_3)}} & 0 \end{pmatrix}$$
$$= -\sin(\theta_2-\theta_3)X_5 + \cos(\theta_2-\theta_3)X_6$$

$$\begin{aligned}
tX_1t^{-1} &= \cos(\theta_1-\theta_2)X_1 + \sin(\theta_1-\theta_2))X_2 \\
tX_2t^{-1} &= -\sin(\theta_1-\theta_2)X_1 + \cos(\theta_1-\theta_2)X_2 \\
tX_3t^{-1} &= \cos(\theta_1-\theta_3)X_3 + \sin(\theta_1-\theta_3)X_4 \\
tX_4t^{-1} &= -\sin(\theta_1-\theta_3)X_3 + \cos(\theta_1-\theta_3)X_4 \\
tX_5t^{-1} &= \cos(\theta_2-\theta_3)X_5 + \sin(\theta_2-\theta_3)X_6 \\
tX_6t^{-1} &= -\sin(\theta_2-\theta_3)X_5 + \cos(\theta_2-\theta_3)X_6
\end{aligned}$$

The complement $\mathfrak{k}^{\perp}$ is decomposed into three irreducible pieces $\mathfrak{k}_i^{\perp}$ for $i = 1\dots 3$ and each component $\mathfrak{p_i}^{\perp}$ is a two-dimensional vector subspace that can be denoted by $\mathfrak{p}_i = span\{X_i, X_{i+1}\} \subset \mathfrak{p}$ for $i = 2i-1\dots 2i$ and $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$

Therefore the isotropic representation is given by:

$$\begin{pmatrix}
\cos(\theta_1-\theta_2) & \sin(\theta_1-\theta_2) & 0 & 0 & 0 & 0 \\
-\sin(\theta_1-\theta_2) & (\cos(\theta_1-\theta_2) & 0 & 0 & 0 & 0 \\
0 & 0 & \cos(\theta_1-\theta_3) & \sin(\theta_1-\theta_3) & 0 & 0 \\
0 & 0 & -\sin(\theta_1-\theta_3) & \cos(\theta_1-\theta_3) & 0 & 0 \\
0 & 0 & 0 & 0 & \cos(\theta_2-\theta_3) & \sin(\theta_2-\theta_3) \\
0 & 0 & 0 & 0 & -\sin(\theta_2-\theta_3) & \cos(\theta_2-\theta_3)
\end{pmatrix},$$

and its smooth metric g is a 6×6 matrix with 2×2 block entries with numbers $a_i \in \mathbb{R}_{>0}$.

$$g(X,Y) = Y^t \begin{pmatrix} a_1I_2 & 0 & 0 \\ 0 & a_2I_2 & 0 \\ 0 & 0 & a_3I_2 \end{pmatrix} X.$$

In section 8, these values a_i shall be set to functions $a_i = g_i^2(t)$ to be evaluated.

6.1 Ricci Curvature of $\frac{SU(3)}{T^2}$

We use the following formulaes

Killing form

$$B(X,Y)_{SU(n)} = 2ntr(XY) \tag{14}$$

Ricci Curvature Tensor [5] (page 185)

$$r(X,X) = -\frac{1}{2}\sum_j \| [X, X_j] \|^2 - \frac{1}{2}B(X,X) + \frac{1}{4}\sum_{ij}([X_i,X_j]_p, X)^2 \tag{15}$$

Here [,] is the bracket of the Lie algebra and B is the Killing Form and (X_i) is an orthonormal basis with respect to the $G-$invariant metric g.

We calculate $r(X_1,X_1)$.

$$-\frac{1}{2}B\left(\frac{X_1}{\sqrt{a_1}}, \frac{X_1}{\sqrt{a_1}}\right) = -\frac{1}{2a_1}6tr(X_1^2) = -\frac{3}{a_1}(-2) = \frac{6}{a_1} \overset{a_i=4b_i}{=} \frac{3}{2b_1}.$$

Now

$$-\frac{1}{2}\sum_j \| [X_1, X_j] \|^2 = -\frac{1}{2}\left(\frac{2a_3}{a_1a_2} + \frac{2a_2}{a_1a_3}\right) \overset{a_i=4b_i}{=} -\frac{1}{4}\left(\frac{b_3}{b_1b_2} + \frac{b_2}{b_1b_3}\right)$$

.

Note $\left[\frac{X_3}{\sqrt{a_2}}, \frac{X_5}{\sqrt{a_3}}\right] = -\frac{1}{\sqrt{a_2a_3}}X_1$ and by anti-commutativity and the fact that $X_4 \in \mathfrak{p}_2$, we introduce a factor of 4 to account for $[X_3,X_5], [X_4,X_5], [X_3,X_6]$, and $[X_4,X_6]$

$$4\left(\left[\frac{X_3}{\sqrt{a_2}}, \frac{X_5}{\sqrt{a_3}}\right]_{\mathfrak{p}}, \frac{X_1}{\sqrt{a_1}}\right)^2 = 4\left(-\frac{1}{\sqrt{a_1a_2a_3}}g\left(X_1,X_1\right)\right)^2 = 4\frac{1}{a_1a_2a_3}a_1^2 = 4\frac{a_1}{a_2a_3} \overset{a_i=4b_i}{=} \frac{b_1}{b_2b_3}$$

So the sum

$$\frac{1}{4}\sum_{ij}([X_i,X_j]_{\mathfrak{p}}, X_1/\sqrt{a_1})^2 = \frac{1}{4}\frac{b_1}{b_2b_3}$$

Therefore in accordance with section 8, we get

$$r(X_1,X_1) = \frac{3}{2b_1} - \frac{1}{4}\left(\frac{b_3}{b_1b_2} + \frac{b_2}{b_1b_3}\right) + \frac{1}{4}\frac{b_1}{b_2b_3}$$

Likewise the calculations for $r(X_3,X_3)$ and $r(X_5,X_5)$ follow

$$r(X_3,X_3) = \frac{3}{2b_2} - \frac{1}{4}\left(\frac{b_1}{b_2b_3} + \frac{b_3}{b_1b_2}\right) + \frac{1}{4}\frac{b_2}{b_1b_3}$$

$$r(X_5, X_5) = \frac{3}{2b_3} - \frac{1}{4}\left(\frac{b_1}{b_2 b_3} + \frac{b_2}{b_1 b_3}\right) + \frac{1}{4}\frac{b_3}{b_1 b_2}$$

Remark 6.1 *Taking the trace of the above Ricci curvature tensor, we obtain the scalar curvature* $S_{SU(3)/T^2} = 3\left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}\right) - \frac{1}{2}\left(\frac{b_3}{b_1 b_2} + \frac{b_2}{b_1 b_3} + \frac{b_3}{b_2 b_3}\right)$ *with respect to the background metric* $V(X, Y) = -2tr(XY)$.

7 Isotropy Representation of $\frac{Sp(3)}{Sp(1)\times Sp(1)\times Sp(1)}$

This section concerns with the triplet $(G, H, K) = (Sp(3), Sp(2) \times Sp(1), Sp(1) \times Sp(1) \times Sp(1))$ where $\frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)\times Sp(1)} \simeq S^4$ and $\frac{Sp(3)}{Sp(2)\times Sp(1)} \simeq \mathbb{H}P^2$

Let us denote the Lie group of $n \times n$ unitary matrices by

$$U(n) = \{A \in GL_n(\mathbb{C}) : AA^* = I\}.$$

Here we identify the adjoint as the complex transpose $A^* = \overline{A}^t$ of A.

Let us denote the symplectic Lie group matrices with quaternion entries as

$$Sp(n) = \{A \in Gl_n(\mathbb{H}) : AA^* = I\}.$$

The conjugate of the quaternion $q = t+ix+yj+kz$ is $\overline{q} = t-ix-yj-iz$. Sometimes it is more convenient to use the equivalent definition

$$Sp(n) = \{A \in U(2n) : A^t J = JA^{-1}\},$$

where $J = diag(E, \dots, E)$ and $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The matrices in this space take the form

$$A = \begin{pmatrix} a_{11} & b_{11} & \dots & \dots & a_{1n} & -b_{1n} \\ -\overline{b_{11}} & \overline{a_{11}} & \dots & \dots & -\overline{b_{1n}} & \overline{a_{1n}} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{n1} & b_{n1} & \dots & \dots & a_{nn} & -b_{nn} \\ -\overline{b_{n1}} & \overline{a_{n1}} & \dots & \dots & -\overline{b_{nn}} & \overline{a_{nn}} \end{pmatrix}$$

Here each $a_{jk} = t_{jk} + ix_{jk}$ and $b_{jk} = y_{jk} + iz_{jk}$.

Let $q = t + ix + yj + kz \in \mathbb{H}$ be a quaternion, which is identified with the point $(t, x, y, z) \in \mathbb{R}^4$ or the point $(t + ix, y + iz) \in \mathbb{C}^2$. There is an isomorphism between the quaternions and certain 2×2 matrices given by

$$\phi : q \leftrightarrow \begin{pmatrix} t+ix & y+iz \\ \overline{-(y+iz)} & \overline{t+ix} \end{pmatrix} = \begin{pmatrix} t+ix & y+iz \\ -(y-iz) & t-ix \end{pmatrix} \in M_2(\mathbb{C})$$

In terms of real matrices we have,

$$\psi : q \leftrightarrow \begin{pmatrix} t & x & y & z \\ -x & t & -z & y \\ -y & z & t & -x \\ -z & -y & x & t \end{pmatrix} \in M_4(\mathbb{R}).$$

Denote $\mathfrak{sp}(3)$ by the Lie algebra of $Sp(3)$. Following the same procedures for the previous calculations in $SU(3)/T^2$ we obtain

$$\mathfrak{sp}(3) = \{B \in \mathfrak{g}(6;\mathbb{C}) : B + B^* = 0 \text{ and } B^tJ + JB = 0\}$$

The objects B have the form

$$B = \begin{pmatrix} R_{11} & W_{12} & \dots & W_{1n} \\ -W_{12}^* & R_{22} & \dots & W_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -W_{1n}^* & -W_{2n} & \dots & R_{nn} \end{pmatrix}$$

Here $R_{jj} = \begin{pmatrix} iu_j & -\overline{v_j} \\ v_j & -iu_j \end{pmatrix}$ and $W_{jk} = \begin{pmatrix} a_{jk} & b_{jk} \\ -\overline{b_{jk}} & \overline{a_{jk}} \end{pmatrix}$ with entries $u_j \in \mathbb{R}$ and $a_{jk}, b_{jk}, v_j \in \mathbb{C}$. The R_{jj} are obtained from the skew-Hermitian condition $B = -B^*$ and W_{jk} are obtained from $B^tJ + JB = 0$.

If one selects the background inner product $Q(X,Y) = -2tr(XY)$ where $X, Y \in \mathfrak{k}$, then we have the following orthonormal basis for $\mathfrak{sp}(n)$ from decomposing B:

$$\left\{ \frac{1}{2}\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \dots & E_{jj} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}_{1\le j\le n}, \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \ddots & H_{ij} & 0 \\ 0 & -H_{ij}^* & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{1\le i<j\le n} : \right.$$

$$E_{ii} \in \left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\},$$

$$\left. H_{ij} \in \left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\} \right\}$$

So in particular for the 3×3 case,

$$B = \begin{pmatrix} R_{11} & W_{12} & W_{13} \\ -W_{12}^* & R_{22} & W_{23} \\ -W_{13}^* & -W_{23}^* & R_{33} \end{pmatrix}$$

The Lie algebra of $K = Sp(1) \times Sp(1) \times Sp(1))$ is the subalgebra

$$\mathfrak{k} = \left\{ \begin{pmatrix} \mathfrak{sp}(1) & 0 & 0 \\ 0 & \mathfrak{sp}(1) & 0 \\ 0 & 0 & \mathfrak{sp}(1) \end{pmatrix} \right\}$$

So by direct sum decomposition, we obtain

$$\mathfrak{p} = \mathfrak{k}^{\perp} = \left\{ \begin{pmatrix} 0 & W_{12} & W_{13} \\ -W_{12}^* & 0 & W_{21} \\ -W_{13}^* & -W_{21}^* & 0 \end{pmatrix} \right\}$$

The adjoint representation of $Sp(1) \times Sp(1) \times Sp(1) = K$ in $\mathfrak{k}^{\perp}$ is given by the map

$$\phi_X : X \to kXk^{-1}$$

for $X \in \mathfrak{k}^{\perp}$.

We calculate the isotropy representation of the compact connected Lie group K by taking advantage of the fact that the representation is uniquely determined by its restriction to a maximal torus. This is a consequence of corollary 2.8 from [18]. We therefore consider,

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} X_i \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}.$$

Here $e^{i\theta_\alpha}$ are the unit complex numbers lying in $Sp(1)$. To make the calculation simpler, we shall identify each of the H_{ij} by its quaternion form and we will drop the constant $\frac{1}{2\sqrt{2}}$ for the remainder of the calculation and we will remember their presences in future calculations. For example, let

$$X_1 = \begin{pmatrix} 0 & H_{12} & 0 \\ -\overline{H_{21}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{with } H_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow q = 1$$

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & H_{12} & 0 \\ -\overline{H_{21}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{i\theta_1}e^{-i\theta_2} & 0 \\ -e^{i\theta_2}e^{-i\theta_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{i(\theta_1-\theta_2)} & 0 \\ -\overline{e^{i(\theta_1-\theta_2)}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \cos(\theta_1-\theta_2)X_1 + \sin(\theta_1-\theta_2)X_2$$

$$X_2 = \begin{pmatrix} 0 & H_{12} & 0 \\ -\overline{H_{21}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{with } H_{12} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \leftrightarrow q = i$$

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & H_{12} & 0 \\ -\overline{H_{21}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{i\theta_1}ie^{-i\theta_2} & 0 \\ e^{i\theta_2}ie^{-i\theta_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{i(\theta_1-\theta_2)}i & 0 \\ \overline{e^{i(\theta_1-\theta_2)}}i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= -\sin(\theta_1-\theta_2)X_1 + \cos(\theta_1-\theta_2)X_2$$

$$X_3 = \begin{pmatrix} 0 & H_{12} & 0 \\ -\overline{H_{21}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{with } H_{12} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \leftrightarrow q = k$$

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & H_{12} & 0 \\ -\overline{H_{21}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{i\theta_1}ke^{-i\theta_2} & 0 \\ e^{i\theta_2}ke^{-i\theta_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= e^{i(\theta_1+\theta_2)}X_3$$

$$= \cos(\theta_1+\theta_2)X_3 - \sin(\theta_1+\theta_2)X_4$$

$$X_4 = \begin{pmatrix} 0 & H_{12} & 0 \\ -\overline{H_{21}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ with } H_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow q = j$$

$$\begin{aligned}
&\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & H_{12} & 0 \\ -\overline{H_{21}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{i\theta_1} j e^{-i\theta_2} & 0 \\ e^{i\theta_2} j e^{-i\theta_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= e^{i(\theta_1+\theta_2)} X_3 \\
&= \cos(\theta_1+\theta_2) X_4 + \sin(\theta_1+\theta_2) X_3
\end{aligned}$$

$$X_5 = \begin{pmatrix} 0 & 0 & H_{13} \\ 0 & 0 & 0 \\ -\overline{H_{31}} & 0 & 0 \end{pmatrix} \text{ with } H_{13} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \leftrightarrow q = i$$

$$\begin{aligned}
&\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & H_{13} \\ 0 & 0 & 0 \\ -\overline{H_{31}} & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & e^{i(\theta_1-\theta_3)} i \\ 0 & 0 & 0 \\ \overline{e^{i(\theta_1-\theta_3)}} i & 0 & 0 \end{pmatrix} \\
&= \cos(\theta_1-\theta_3) \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -\bar{i} & 0 & 0 \end{pmatrix} + \sin(\theta_1-\theta_3) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&= \cos(\theta_1-\theta_3) X_5 - \sin(\theta_1-\theta_3) X_6
\end{aligned}$$

$$X_6 = \begin{pmatrix} 0 & 0 & H_{13} \\ 0 & 0 & 0 \\ -\overline{H_{31}} & 0 & 0 \end{pmatrix} \text{ with } H_{13} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow q = 1$$

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & H_{13} \\ 0 & 0 & 0 \\ -\overline{H_{31}} & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & e^{i(\theta_1-\theta_3)} \\ 0 & 0 & 0 \\ -\overline{e^{i(\theta_1-\theta_3)}} & 0 & 0 \end{pmatrix}$$
$$= \cos(\theta_1-\theta_3) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \sin(\theta_1-\theta_3) \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -\bar{i} & 0 & 0 \end{pmatrix}$$
$$= \cos(\theta_1-\theta_3)X_6 + \sin(\theta_1-\theta_3)X_5$$

$$X_7 = \begin{pmatrix} 0 & 0 & H_{13} \\ 0 & 0 & 0 \\ -\overline{H_{31}} & 0 & 0 \end{pmatrix} \text{ with } H_{13} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \leftrightarrow q = k$$

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & H_{13} \\ 0 & 0 & 0 \\ -\overline{H_{31}} & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & e^{i\theta_1}ke^{-i\theta_3} \\ 0 & 0 & 0 \\ e^{i\theta_3}ke^{-i\theta_1} & 0 & 0 \end{pmatrix}$$
$$= \cos(\theta_1+\theta_3) \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix} - \sin(\theta_1+\theta_3) \begin{pmatrix} 0 & 0 & j \\ 0 & 0 & 0 \\ j & 0 & 0 \end{pmatrix}$$
$$= \cos(\theta_1+\theta_3)X_7 - \sin(\theta_1+\theta_3)X_8$$

$$X_8 = \begin{pmatrix} 0 & 0 & H_{13} \\ 0 & 0 & 0 \\ -\overline{H_{31}} & 0 & 0 \end{pmatrix} \text{ with } H_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leftrightarrow q = j$$

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & H_{13} \\ 0 & 0 & 0 \\ -\overline{H_{31}} & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & e^{i\theta_1}je^{-i\theta_3} \\ 0 & 0 & 0 \\ e^{i\theta_3}je^{-i\theta_1} & 0 & 0 \end{pmatrix}$$
$$= \cos(\theta_1+\theta_3) \begin{pmatrix} 0 & 0 & j \\ 0 & 0 & 0 \\ j & 0 & 0 \end{pmatrix} + \sin(\theta_1+\theta_3) \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix}$$
$$= \sin(\theta_1+\theta_3)X_7 + \cos(\theta_1+\theta_3)X_8$$

$$X_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H_{23} \\ 0 & -\overline{H_{32}} & 0 \end{pmatrix} \text{ with } H_{23} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \leftrightarrow q = i$$

$$\begin{aligned}
&\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H_{23} \\ 0 & -\overline{H_{32}} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i(\theta_2-\theta_3)}i \\ 0 & \overline{e^{i(\theta_2-\theta_3)}i} & 0 \end{pmatrix} \\
&= \cos(\theta_2-\theta_3)X_9 - \sin(\theta_2-\theta_3)X_{10}
\end{aligned}$$

$$X_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H_{23} \\ 0 & -\overline{H_{32}} & 0 \end{pmatrix} \text{ with } H_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow q = 1$$

$$\begin{aligned}
&\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H_{23} \\ 0 & -\overline{H_{32}} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i(\theta_2-\theta_3)} \\ 0 & -\overline{e^{i(\theta_2-\theta_3)}} & 0 \end{pmatrix} \\
&= \sin(\theta_2-\theta_3)X_9 + \cos(\theta_2-\theta_3)X_{10}
\end{aligned}$$

$$X_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H_{23} \\ 0 & -\overline{H_{32}} & 0 \end{pmatrix} \text{ with } H_{23} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \leftrightarrow q = k$$

$$\begin{aligned}
&\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H_{23} \\ 0 & -\overline{H_{32}} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i(\theta_2+\theta_3)}k \\ 0 & e^{i(\theta_2+\theta_3)}k & 0 \end{pmatrix} \\
&= \cos(\theta_2+\theta_3)X_{11} - \sin(\theta_2+\theta_3)X_{12}
\end{aligned}$$

$$X_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H_{23} \\ 0 & -\overline{H_{32}} & 0 \end{pmatrix} \text{ with } H_{23} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow q = j$$

$$\begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H_{23} \\ 0 & -\overline{H_{32}} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i(\theta_2+\theta_3)}j \\ 0 & e^{i(\theta_2+\theta_3)}j & 0 \end{pmatrix}$$
$$= \cos(\theta_2+\theta_3)X_{11} + \sin(\theta_2+\theta_3)X_{12}$$

$$\begin{aligned}
kX_1 &= \cos(\theta_1-\theta_2)X_1 + \sin(\theta_1-\theta_2)X_2 \\
kX_2 &= -\sin(\theta_1-\theta_2)X_1 + \cos(\theta_1-\theta_2)X_2 \\
kX_3 &= \cos(\theta_1+\theta_2)X_3 - \sin(\theta_1+\theta_2)X_4 \\
kX_4 &= \sin(\theta_1+\theta_2)X_3 + \cos(\theta_1+\theta_2)X_4 \\
kX_5 &= \cos(\theta_1-\theta_3)X_5 - \sin(\theta_1-\theta_3)X_6 \\
kX_6 &= \sin(\theta_1-\theta_3)X_5 + \cos(\theta_1-\theta_3)X_6 \\
kX_7 &= \cos(\theta_1+\theta_3)X_7 - \sin(\theta_1+\theta_3)X_8 \\
kX_8 &= \sin(\theta_1+\theta_3)X_7 + \cos(\theta_1+\theta_3)X_8 \\
kX_9 &= \cos(\theta_2-\theta_3)X_9 - \sin(\theta_2-\theta_3)X_{10} \\
kX_{10} &= \sin(\theta_2-\theta_3)X_9 + \cos(\theta_2-\theta_3)X_{10} \\
kX_{11} &= \cos(\theta_2+\theta_3)X_{11} - \sin(\theta_2+\theta_3)X_{12} \\
kX_{12} &= \sin(\theta_2+\theta_3)X_{11} + \cos(\theta_2+\theta_3)X_{12}
\end{aligned}$$

Thus the isotropy representation is given by the 12×12 matrix

$$\begin{pmatrix} G_{11} & 0 & 0 \\ 0 & G_{22} & 0 \\ 0 & 0 & G_{33} \end{pmatrix}$$

with block sub-matrices G_{jj}

$$G_{11} = \begin{pmatrix} \cos(\theta_1-\theta_2) & \sin(\theta_1-\theta_2) & 0 & 0 \\ \sin(\theta_1-\theta_2) & \cos(\theta_1-\theta_2) & 0 & 0 \\ 0 & 0 & \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ 0 & 0 & \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}$$

$$G_{22} = \begin{pmatrix} \cos(\theta_1-\theta_3) & -\sin(\theta_1-\theta_3) & 0 & 0 \\ \sin(\theta_1-\theta_3) & \cos(\theta_1-\theta_3) & 0 & 0 \\ 0 & 0 & \cos(\theta_1+\theta_3) & -\sin(\theta_1+\theta_3) \\ 0 & 0 & \sin(\theta_1+\theta_3) & \cos(\theta_1+\theta_3) \end{pmatrix}$$

$$G_{33}=\begin{pmatrix}\cos(\theta_2-\theta_3) & -\sin(\theta_2-\theta_3) & 0 & 0\\ \sin(\theta_2-\theta_3) & \cos(\theta_2-\theta_3) & 0 & 0\\ 0 & 0 & \cos(\theta_2+\theta_3) & -\sin(\theta_2+\theta_3)\\ 0 & 0 & \sin(\theta_2+\theta_3) & \cos(\theta_2+\theta_3)\end{pmatrix}$$

7.1 Ricci Curvature of $Sp(3)/Sp(1)^3$

We use the following formula

Killing form

$$B(X,Y)_{Sp(n)}=2(n+1)tr(XY) \tag{16}$$

Ricci Curvature Tensor [5]

$$r(X,X)=-\frac{1}{2}\sum_j \parallel [X,X_j]_{\mathfrak{p}}\parallel^2-\frac{1}{2}B(X,X)+\frac{1}{4}\sum_{ij}([X_i,X_j]_p,X)^2 \tag{17}$$

Here [,] is the bracket of the Lie algebra and B is the Killing Form and (X_i) are an orthonormal basis with respect to the metric g.

Let

$$Y_j\in\left\{\frac{X_1}{\sqrt{a_1}},\frac{X_2}{\sqrt{a_1}},\frac{X_3}{\sqrt{a_1}},\frac{X_4}{\sqrt{a_1}},\frac{X_5}{\sqrt{a_2}},\frac{X_6}{\sqrt{a_2}},\frac{X_7}{\sqrt{a_2}},\frac{X_8}{\sqrt{a_2}},\frac{X_9}{\sqrt{a_3}},\frac{X_{10}}{\sqrt{a_3}},\frac{X_{11}}{\sqrt{a_3}},\frac{X_{12}}{\sqrt{a_3}}\right\}$$

Now

$$-\frac{1}{2}\sum_j\left\|\left[\frac{X_1}{\sqrt{a_1}},Y_j\right]_{\mathfrak{p}}\right\|^2=-\frac{1}{2}\left(4\frac{a_3}{2^2 2a_1a_2}+4\frac{a_2}{2^2 2a_1a_3}\right)\overset{a_i=(1/2)b_i}{=}-\frac{1}{2}\left(\frac{b_3}{b_1b_2}+\frac{b_2}{b_1b_3}\right).$$

$$-\frac{1}{2}\sum_j\left\|\left[\frac{X_5}{\sqrt{a_2}},Y_j\right]_{\mathfrak{p}}\right\|^2=-\frac{1}{2}\left(4\frac{a_3}{2^2 2a_1a_2}+4\frac{a_1}{2^2 2a_2a_3}\right)\overset{a_i=(1/2)b_i}{=}-\frac{1}{2}\left(\frac{b_3}{b_1b_2}+\frac{b_1}{b_2b_3}\right).$$

$$-\frac{1}{2}\sum_j\left\|\left[\frac{X_9}{\sqrt{a_3}},Y_j\right]_{\mathfrak{p}}\right\|^2=-\frac{1}{2}\left(4\frac{a_2}{2^2 2a_1a_3}+4\frac{a_1}{2^2 2a_2a_3}\right)\overset{a_i=(1/2)b_i}{=}-\frac{1}{2}\left(\frac{b_2}{b_1b_3}+\frac{b_1}{b_2b_3}\right).$$

$$-\frac{1}{2}B\left(\frac{X_1}{\sqrt{a_1}},\frac{X_1}{\sqrt{a_1}}\right)=-\frac{1}{2}\left(2(3+1)tr\left(\frac{X_1^2}{a_1}\right)\right)=-\frac{1}{2}\left(2\frac{4}{a_1}\left(-\frac{1}{2}\right)\right)=\frac{2}{a_1}\overset{a_i=(1/2)b_i}{=}\frac{4}{b_1}$$

$$-\frac{1}{2}B\left(\frac{X_5}{\sqrt{a_2}},\frac{X_5}{\sqrt{a_2}}\right)=-\frac{1}{2}\left(2(3+1)tr\left(\frac{X_1^2}{a_2}\right)\right)=-\frac{1}{2}\left(2\frac{4}{a_2}\left(-\frac{1}{2}\right)\right)=\frac{2}{a_2}\overset{a_i=(1/2)b_i}{=}\frac{4}{b_2}$$

$$-\frac{1}{2}B\left(\frac{X_9}{\sqrt{a_3}},\frac{X_9}{\sqrt{a_3}}\right)=-\frac{1}{2}\left(2(3+1)tr\left(\frac{X_9^2}{a_3}\right)\right)=-\frac{1}{2}\left(2\frac{4}{a_3}\left(-\frac{1}{2}\right)\right)=\frac{2}{a_3}\overset{a_i=(1/2)b_i}{=}\frac{4}{b_3}$$

Similar to subsection 6.1, we calculate

$$\begin{aligned}\left(\left[\frac{X_5}{\sqrt{a_2}},\frac{X_9}{\sqrt{a_3}}\right]_{\mathfrak{p}},\frac{X_1}{\sqrt{a_1}}\right)^2&=\left(-\frac{1}{\sqrt{a_1a_2a_3}}g\left(\frac{X_1}{2\sqrt{2}},X_1\right)\right)^2\\&=\left(\frac{1}{a_1a_2a_32^22}a_1^2\right)\\&=\left(\frac{a_1}{8a_2a_3}\right)\\&\overset{a_i=(1/2)b_i}{=}\frac{1}{4}\frac{b_1}{b_2b_3}\end{aligned}$$

So the sum

$$\frac{1}{4}\sum_{ij}([X_i,X_j]_{\mathfrak{p}},X_1/\sqrt{a_1})^2=\frac{1}{4}8\left(\frac{1}{4}\frac{b_1}{b_2b_3}\right)$$

$$r\left(\frac{X_1}{\sqrt{a_1}},\frac{X_1}{\sqrt{a_1}}\right)=-\frac{1}{2}\left(\frac{b_3}{b_1b_2}+\frac{b_2}{b_1b_3}\right)+\frac{4}{b_1}+\frac{1}{2}\frac{b_1}{b_2b_3}$$

$$r\left(\frac{X_5}{\sqrt{a_2}},\frac{X_5}{\sqrt{a_2}}\right)=-\frac{1}{2}\left(\frac{b_3}{b_1b_2}+\frac{b_1}{b_2b_3}\right)+\frac{4}{b_2}+\frac{1}{2}\frac{b_2}{b_1b_3}$$

$$r\left(\frac{X_9}{\sqrt{a_3}},\frac{X_9}{\sqrt{a_3}}\right)=-\frac{1}{2}\left(\frac{b_2}{b_1b_3}+\frac{b_1}{b_2b_3}\right)+\frac{4}{b_3}+\frac{1}{2}\frac{b_3}{b_1b_2}$$

Remark 7.1 *Taking the trace of the above Ricci curvature tensor, we obtain the scalar curvature* $S_{Sp(3)/Sp(1)^3}=16\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)-2\left(\frac{b_3}{b_1b_2}+\frac{b_2}{b_1b_3}+\frac{b_3}{b_2b_3}\right)$ *with respect to the background metric* $V(X,Y)=-tr(XY)$.

8 Numerics

We begin with the reduced Ricci Soliton equation. The Ricci Soliton equation for $SU(3)/T^2$ with the metric $\hat{g} = dt^2 + f_1(t)^2 h|_{\mathfrak{p}_1} + f_2(t)^2 Q|_{\mathfrak{p}_2} + f_3(t)^2 Q|_{\mathfrak{p}_3}$ reduces to

$$\frac{-f_1''}{f_1} + \frac{f_1'^2}{f_1^2} + u'\frac{f_1'}{f_1} - 2\left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3}\right)\frac{f_1'}{f_1} + \frac{3}{2f_1^2} + \frac{1}{4}\left(\frac{f_1^2}{f_2^2 f_3^2} - \frac{f_2^2}{f_1^2 f_3^2} - \frac{f_3^2}{f_1^2 f_2^2}\right) + \frac{\epsilon}{2} = 0.$$
$$\frac{-f_2''}{f_2} + \frac{f_2'^2}{f_2^2} + u'\frac{f_2'}{f_2} - 2\left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3}\right)\frac{f_2'}{f_2} + \frac{3}{2f_2^2} + \frac{1}{4}\left(\frac{f_2^2}{f_1^2 f_3^2} - \frac{f_1^2}{f_2^2 f_3^2} - \frac{f_3^2}{f_1^2 f_2^2}\right) + \frac{\epsilon}{2} = 0.$$
$$\frac{-f_3''}{f_3} + \frac{f_3'^2}{f_3^2} + u'\frac{f_3'}{f_3} - 2\left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3}\right)\frac{f_3'}{f_3} + \frac{3}{2f_3^2} + \frac{1}{4}\left(\frac{f_3^2}{f_1^2 f_2^2} - \frac{f_2^2}{f_1^2 f_3^2} - \frac{f_1^2}{f_2^2 f_3^2}\right) + \frac{\epsilon}{2} = 0.$$
$$u'' - u'^2 + 2u'\left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3}\right) - \epsilon u = C.$$

To run the Runge Kutta algorithm, we put the above in normal form by setting

$$\begin{array}{ll} z_1 = f_1 & z_2 = f_1' \\ z_3 = f_2 & z_4 = f_2' \\ z_5 = f_3 & z_5 = f_3' \\ z_7 = u & z_8 = u' \end{array}.$$

Then the equations become the first order system,

$$\begin{aligned}
z_1' &= z_2 \\
z_2' &= \frac{z_2^2}{z_1} + z_8 z_2 - 2z_2\left(\frac{z_2}{z_1} + \frac{z_4}{z_3} + \frac{z_6}{z_5}\right) + \frac{3}{2z_1} + \frac{z_1}{4}\left(\frac{z_1^2}{z_3^2 z_5^2} - \frac{z_3^2}{z_1^2 z_5^2} - \frac{z_5^2}{z_1^2 z_3^2}\right) + \frac{\epsilon}{2}z_1. \\
z_3' &= z_4 \\
z_4' &= \frac{z_4^2}{z_3} + z_8 z_4 - 2z_4\left(\frac{z_2}{z_1} + \frac{z_4}{z_3} + \frac{z_6}{z_5}\right) + \frac{3}{2z_3} + \frac{z_3}{4}\left(\frac{z_3^2}{z_1^2 z_5^2} - \frac{z_1^2}{z_3^2 z_5^2} - \frac{z_5^2}{z_1^2 z_3^2}\right) + \frac{\epsilon}{2}z_3. \\
z_5' &= z_6 \\
z_6' &= \frac{z_6^2}{z_5} + z_8 z_6 - 2z_6\left(\frac{z_2}{z_1} + \frac{z_4}{z_3} + \frac{z_6}{z_5}\right) + \frac{3}{2z_5} + \frac{z_5}{4}\left(\frac{z_5^2}{z_1^2 z_3^2} - \frac{z_3^2}{z_1^2 z_5^2} - \frac{z_1^2}{z_3^2 z_5^2}\right) + \frac{\epsilon}{2}z_5. \\
z_7' &= z_8 \\
z_8' &= C + z_8^2 - 2z_8\left(\frac{z_2}{z_1} + \frac{z_4}{z_3} + \frac{z_6}{z_5}\right) + \epsilon z_7
\end{aligned} \tag{18}$$

Remark 8.1 *Note that we have chosen the background metric $V(x,y) = -2tr(XY)$ for the above system 18*

Similarly the system for $Sp(3)/Sp(1)^3$ is,

$$
\begin{aligned}
z_1' &= z_2\\
z_2' &= \frac{z_2^2}{z_1} + z_8 z_2 - 4z_2\left(\frac{z_2}{z_1}+\frac{z_4}{z_3}+\frac{z_6}{z_5}\right) + \frac{4}{z_1} + \frac{z_1}{2}\left(\frac{z_1^2}{z_3^2 z_5^2} - \frac{z_3^2}{z_1^2 z_5^2} - \frac{z_5^2}{z_1^2 z_3^2}\right) + \frac{\epsilon}{2} z_1.\\
z_3' &= z_4\\
z_4' &= \frac{z_4^2}{z_3} + z_8 z_4 - 4z_4\left(\frac{z_2}{z_1}+\frac{z_4}{z_3}+\frac{z_6}{z_5}\right) + \frac{4}{z_3} + \frac{z_3}{2}\left(\frac{z_3^2}{z_1^2 z_5^2} - \frac{z_1^2}{z_3^2 z_5^2} - \frac{z_5^2}{z_1^2 z_3^2}\right) + \frac{\epsilon}{2} z_3.\\
z_5' &= z_6\\
z_6' &= \frac{z_6^2}{z_5} + z_8 z_6 - 4z_6\left(\frac{z_2}{z_1}+\frac{z_4}{z_3}+\frac{z_6}{z_5}\right) + \frac{4}{z_5} + \frac{z_5}{2}\left(\frac{z_5^2}{z_1^2 z_3^2} - \frac{z_3^2}{z_1^2 z_5^2} - \frac{z_1^2}{z_3^2 z_5^2}\right) + \frac{\epsilon}{2} z_5.\\
z_7' &= z_8\\
z_8' &= C + z_8^2 - 4z_8\left(\frac{z_2}{z_1}+\frac{z_4}{z_3}+\frac{z_6}{z_5}\right) + \epsilon z_7.
\end{aligned} \tag{19}
$$

Remark 8.2 *Note that we have chosen the background metric $V(x,y) = -tr(XY)$ for the above system 19*

8.1 Procedures and preliminaries

In both of our spaces $SU(3)/T^2$ and $\frac{Sp(3)}{Sp(1)Sp(1)Sp(1)}$, we will work with the initial conditions $(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) = (0, 1, a, 0, a, 0, b, 0)$ which will enforce the smoothness condition around the singular orbit. Each algorithm contains seven parameters which contain the initial conditions, the starting and ending point of the Runge Kutta algorithm, the constants ϵ, and C denoted by $astart, b, start, acc, limit, epsilon$, and C respectively.

Table 1: Table of Summary

	a	b	$u''(0)$	C	ϵ
Steady Soliton	> 0	$\neq 0$	$-1/3$	-1	0
Ricci Flat	> 0	0	n/a	0	0
Expanding Solitons	> 0	< 0	$b/3$	0	1
Negative Einstein	> 0	0	0	0	1
Compact Einstein	> 0	$b = 0$	$-b/3$	0	-1
Non-Compact Einstein	> 0	n/a	$-b/3$	0	-1

8.2 Procedure

We proceed as follows:

1. Generate finite polynomials (up to order 10, although only up to only order 5 were used at most) for the system 18 and 19 by Maple and the metric's components shown in the systems 18 and 19 are replaced by the polynomials.

2. We use the usual power series approximating method to eliminate the arbitrary coefficients using the prescribed initial conditions.

3. We use Maple to generate initial solutions for small values near 0 to reset the usual starting point of the 4th-Order Runge-Kutta. The identification is done through the human (author's) eyes.

The steps $1, 2$, and 3 are done to work around the irregular singular points in 18 and 19 at $t = 0$.

4. For the compact soliton case, we first plot the quantity of $SOL = z_1^2 + (z_2 + 1)^2 + z_4^2 + z_6^2 + z_8^2$ vs the initial values of a to locate values of SOL close to 0 which helps us find compact Einstein metrics.

The quantity hits 0 whenever $z_2 = -1$ and all other metrics hit 0 at the point of interest. By a permutation we also get a second SOL in the form of $z_2^2 + z_3^2 + (z_4 + 1)^2 + z_6^2 + z_8^2$ with similar construction. The derivatives of the metrics are plotted to ensure the numerical Einstein metric found satisfies the expected theoretical asymptotics.

8.3 The effects of homothety

This section begins with a discussion of the effects of homothety.

Let us consider the Ricci soliton equation and the conservation law once more

$$\mathrm{Ric}(\overline{g}) + \mathrm{Hess}(u) + \frac{\overline{\epsilon}}{2}\overline{g} = 0$$

$$\ddot{\overline{u}} + (-\dot{\overline{u}} + \mathrm{tr}(L))\dot{\overline{u}} - \epsilon\overline{u} = C.$$

By applying the transformation $\overline{g} = c^2 g$ for some $c > 0$, then above equations would change to

$$\mathrm{Ric}(c^2 g) + \mathrm{Hess}(u) + \frac{\epsilon}{2}c^2\frac{g}{c^2} = 0$$

$$\ddot{u} + (-\dot{u} + \mathrm{tr}(L))\dot{u} - \epsilon u = C.$$

Then the metric $\overline{g} = d\overline{t}^2 + \overline{g}_t$ becomes the metric $c^2 g = c^2 dt^2 + c^2 g_t$. Now this means that $\overline{t} \to ct$, then $\overline{f}_i(\overline{t})^2 = c^2 f_i^2(t) \implies \overline{f}_i(\overline{t}) = cf_i(t)$ and the smooth function u which has no association with the metric g will remain the same, that is $\bar{u}(\overline{t}) = u(t) = u(\overline{t}/c)$.

Their derivatives follow

$$\begin{cases} \bar{u}_{\overline{t}} = \frac{1}{c}u_t \\ \bar{u}_{\overline{t}\overline{t}} = \frac{1}{c^2}u_{tt} \end{cases} \qquad \begin{cases} \frac{d\overline{f_i}}{d\overline{t}^2} = \frac{1}{c}c\frac{df_i}{dt} = \frac{df_i}{dt} \\ \frac{d^2\overline{f_i}}{d\overline{t}^2} = \frac{1}{c}\frac{d^2 f_i}{dt^2} \end{cases}$$

So the above implies

$$\frac{(\overline{f}_i)_{\overline{t}}}{\overline{f}_i} = \frac{1}{c}\frac{(f_i)_t}{f_i},$$

and

$$\frac{(\overline{f}_i)_{\overline{t}\overline{t}}}{\overline{f}_i} = \frac{1}{c^2}\frac{(f_i)_{tt}}{f_i}.$$

There are two cases to consider,

Case 1: $\epsilon = 0$

By a homothetic change we may assume $C = -1$, then the conservation law implies

$$(d_1 + 1)\ddot{u}(o) = C + \epsilon u(0) \implies \ddot{u}(0) = \frac{-1}{d_1 + 1}$$

where d_1 stands for the dimension of $\mathfrak{p}_i$. This means that the initial value of $\ddot{u}(0)$ is fixed by the choice C. Therefore the quadruplet solution (f_1, f_2, f_3, f_4) can never include homothetic solutions.

Case 2: $\epsilon \neq 0$

If $\epsilon \neq 0$, then $\ddot{u}(0) = sgn(\epsilon)\frac{u(0)}{d_1+1}$ where we normalize ϵ to be $sgn(\epsilon)$ and instead of $u(0) = 0$, we choose $C = 0$ to rule out homothetic solutions.

Note that in the numerics, all $d_i = 2$ for $i = 1, 2, 3$.

9 $SU(3)/T^2$

The blue curves are the polynomial solutions and the red are the approximations from the Runge-Kutta.

9.1 Steady Solitons

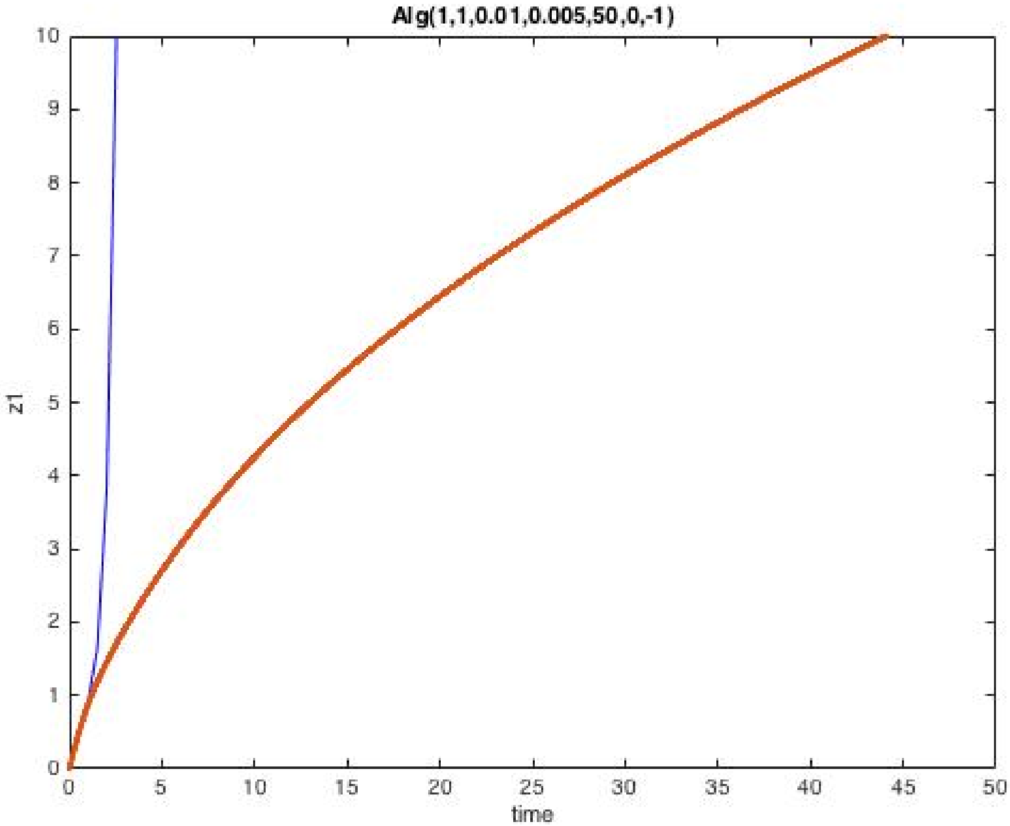

Figure 1: Steady Case of $SU(3)/T^2$ with $z_1, a = 1, b = 1$

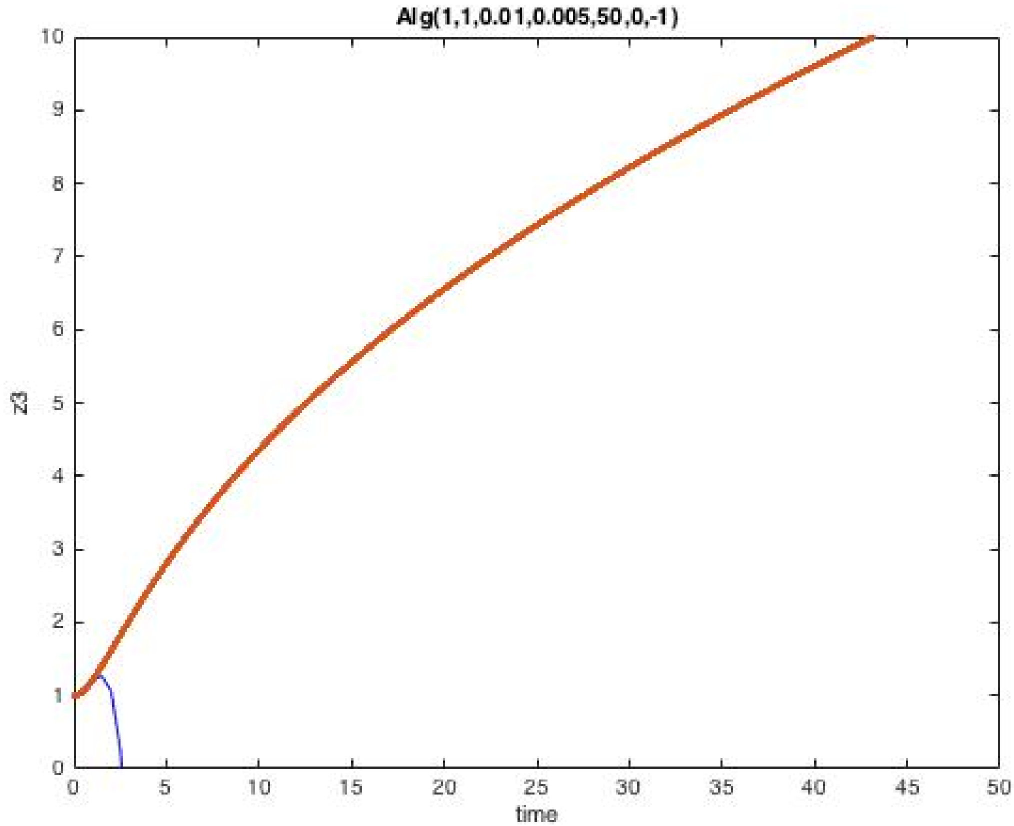

Figure 2: Steady Case of $SU(3)/T^2$ with $z_3, a = 1, b = 1$

Remark 9.1 *The curves in figure (1) and (2) are asymptotically* $\sim \sqrt{t}$*, indicating asymptotically paraboloidal behavior as in the case of the Bryant soliton and the steady soliton found in [9]*

Proposition 9.1 *[16] The soliton potential is strictly decreasing and strictly concave on* $(0, \infty)$

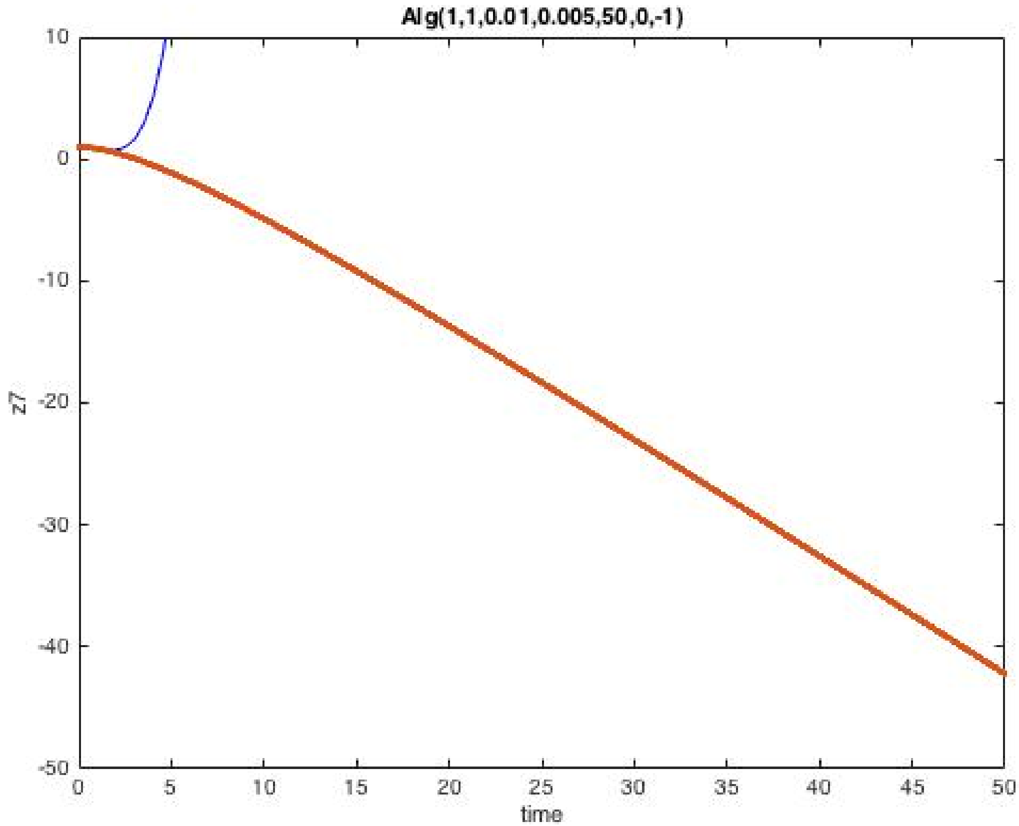

Figure 3: Steady Case of $SU(3)/T^2$ with $z_7, a = 1, b = 1$

Remark 9.2 *As in agreement with 9.1, the potential is strictly decreasing and concave down on* $(0, 50)$

Proposition 9.2 *[16] The mean curvature trL is strictly decreasing and satisfies $0 < trL < \frac{n}{t}$. The generalized mean curvature $\eta = -\dot{u} + trL$ is strictly decreasing and tends to $\sqrt{-C}$ as t tends to ∞.*

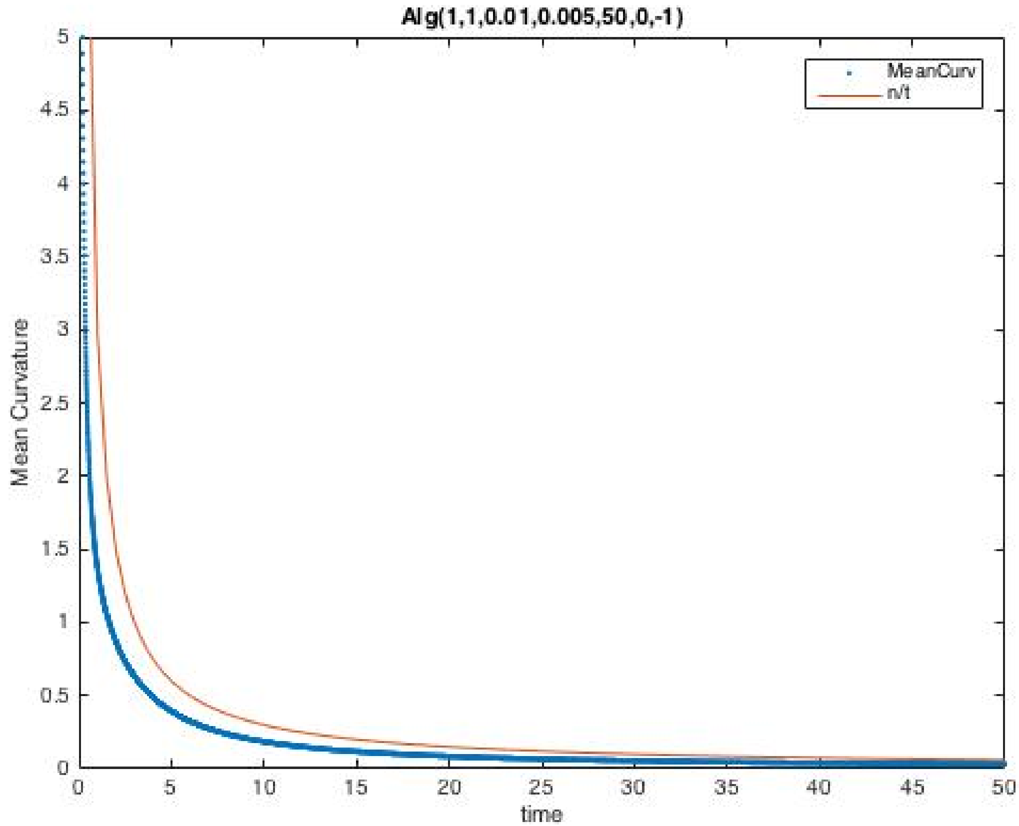

Figure 4: The Mean Curvature of $SU(3)/T^2$ in the Steady Case.

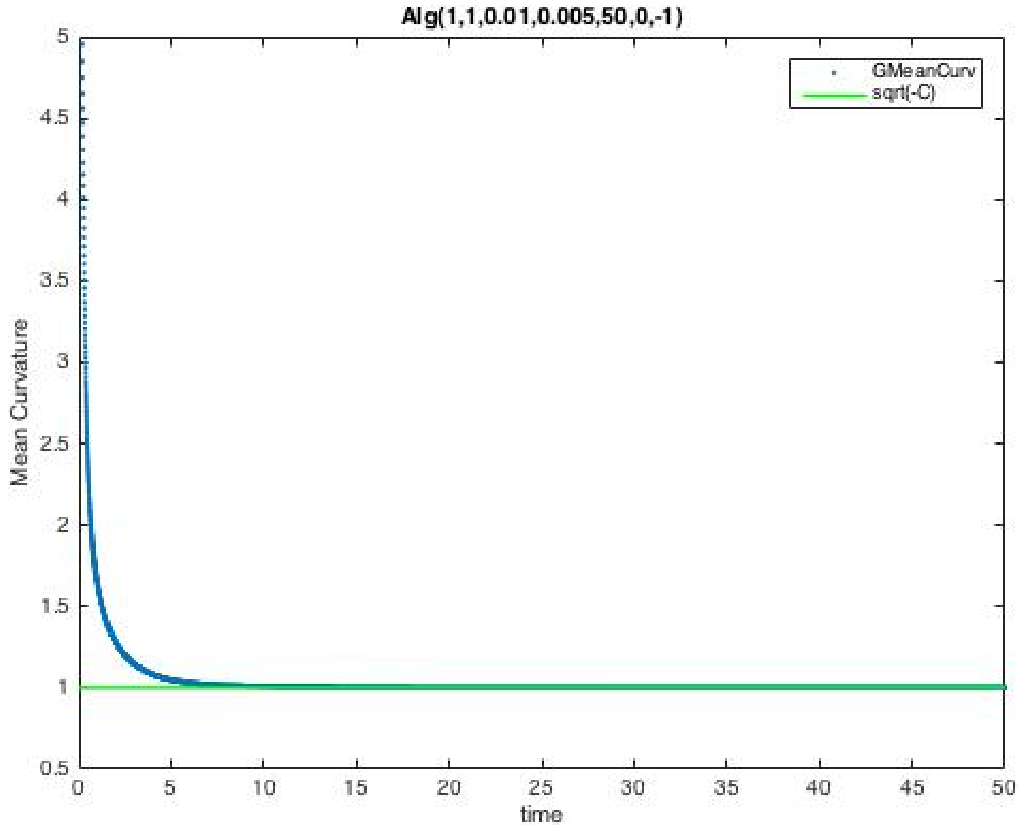

Figure 5: The Generalized Mean Curvature of $SU(3)/T^2$ in the Steady Case.

9.2 Ricci-flat

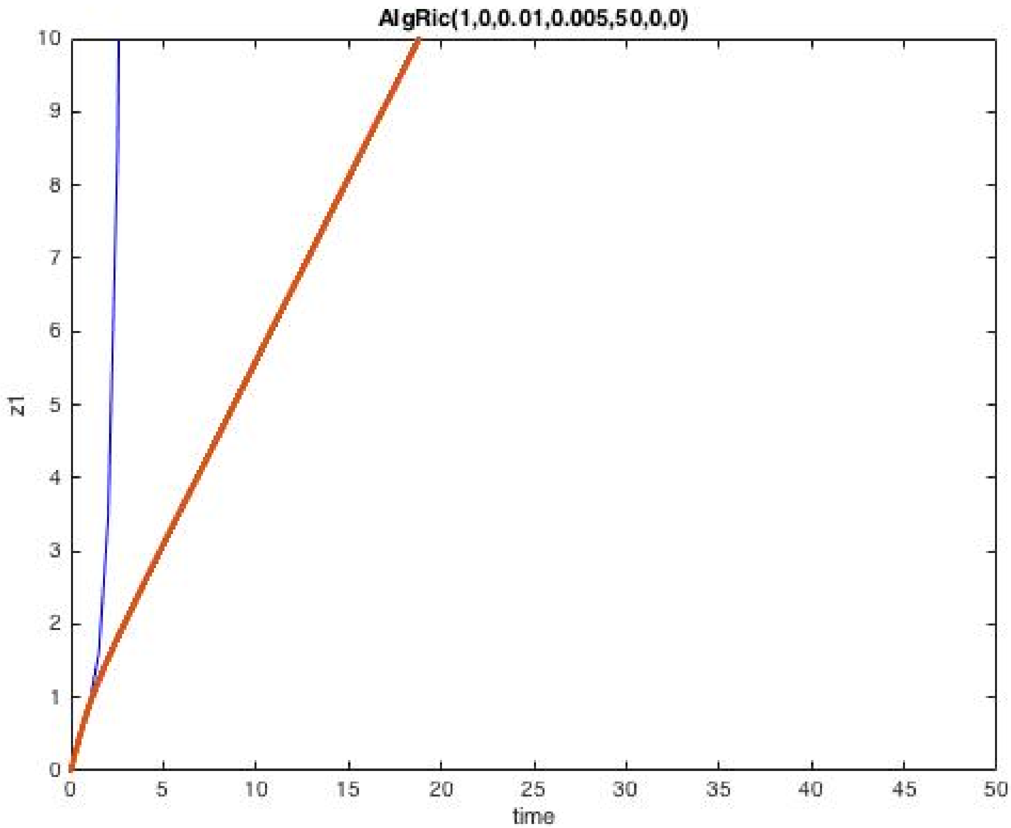

Figure 6: Ricci-flat Case of $SU(3)/T^2$ with $z_1, a = 1$

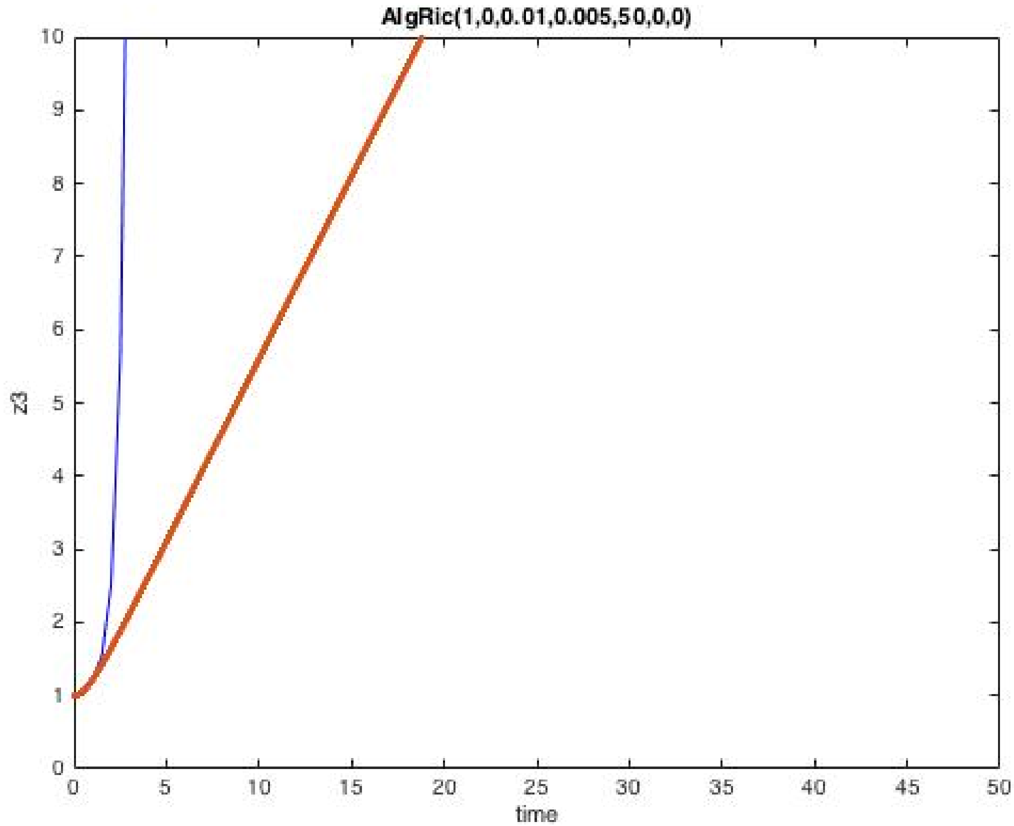

Figure 7: Ricci-flat Case of $SU(3)/T^2$ with $z_3, a = 1$

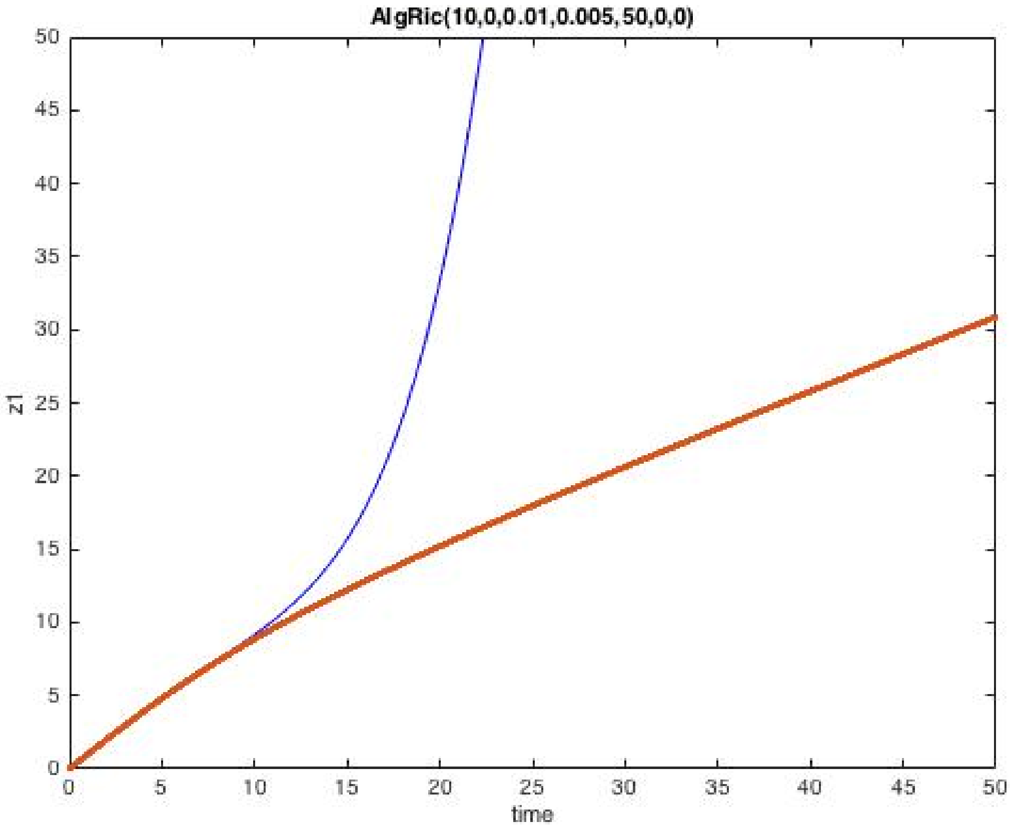

Figure 8: Ricci-flat Case of $SU(3)/T^2$ with $z_1, a = 10$

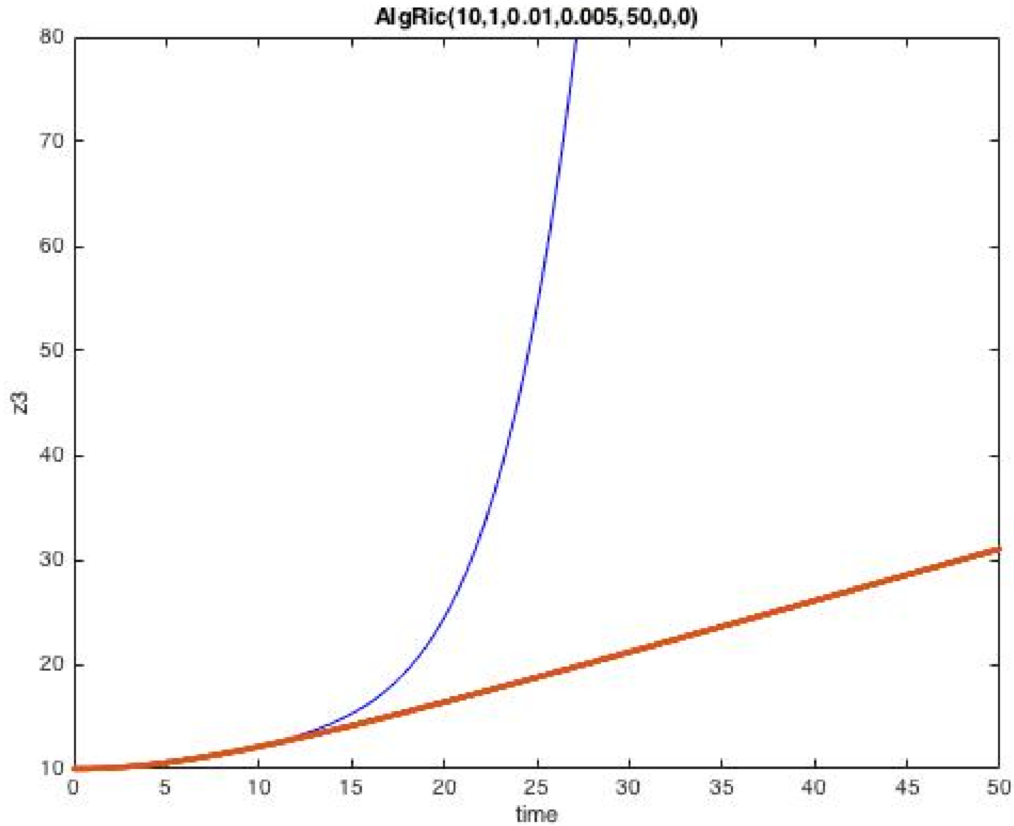

Figure 9: Ricci-flat Case of $SU(3)/T^2$ with $z_3, a = 10$

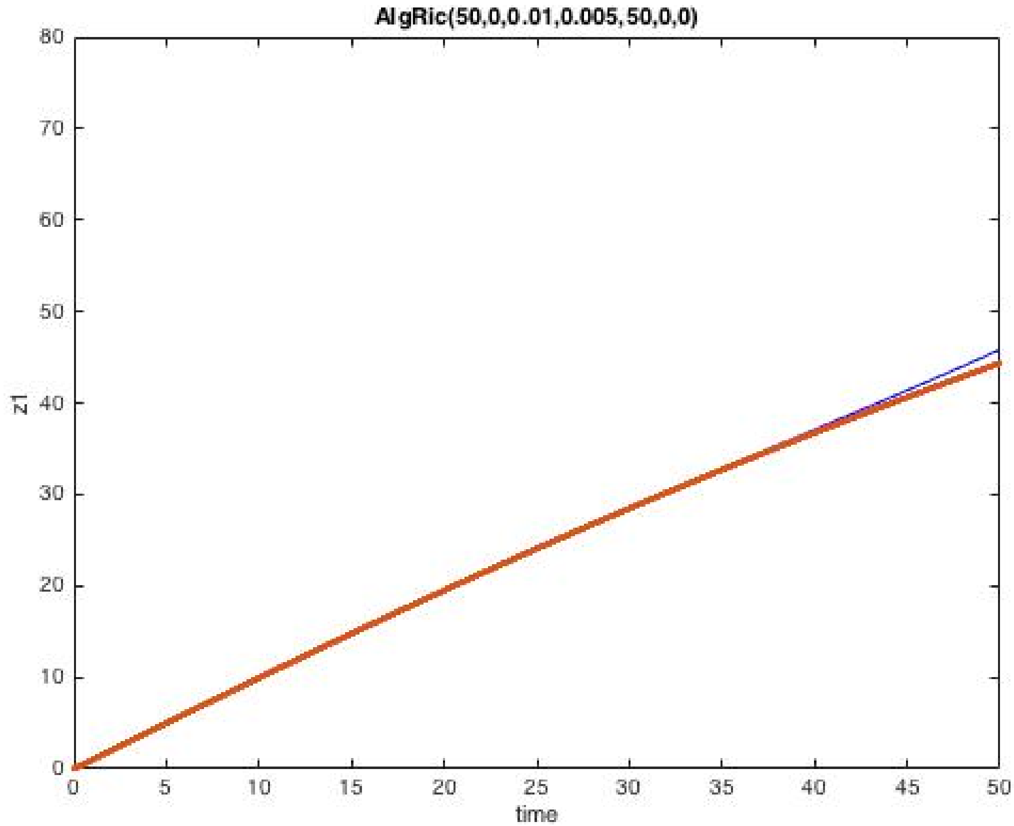

Figure 10: Ricci-flat Case of $SU(3)/T^2$ with $z_1, a = 50$

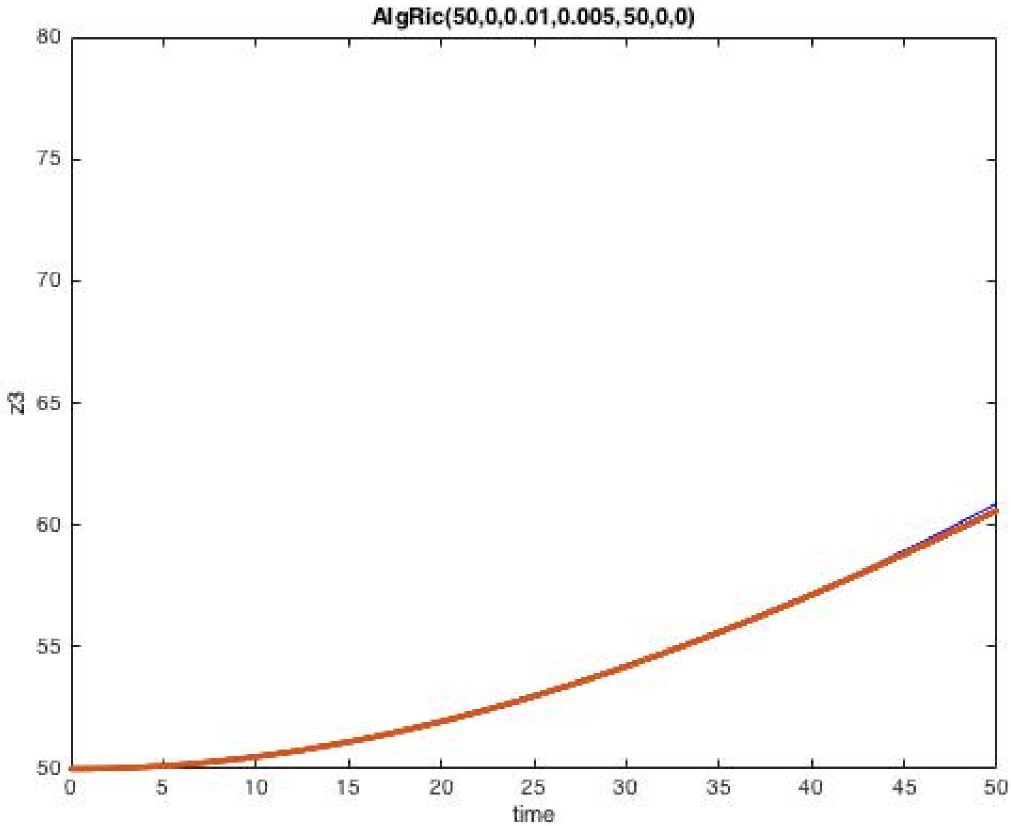

Figure 11: Ricci-flat case of $SU(3)/T^2$ with $z_3, a = 50$

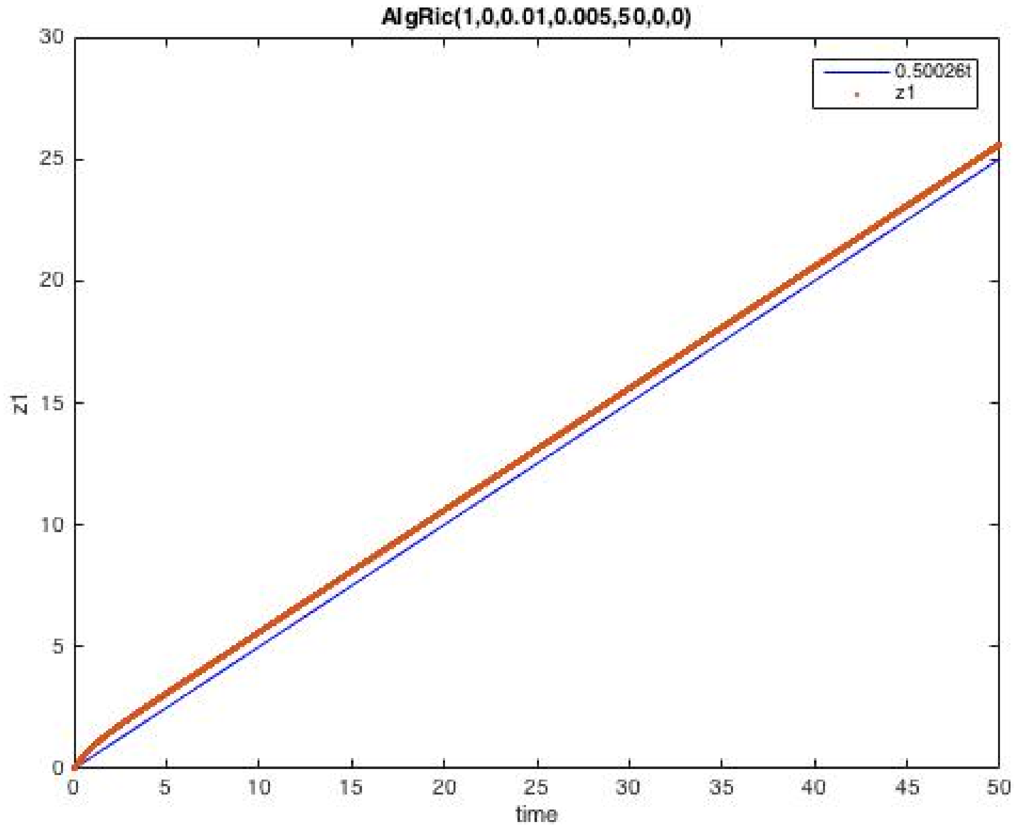

Figure 12: The graph of Ricci-flat of $SU(3)/T^2$ with $z_1, a = 1$ and its linear interpolation

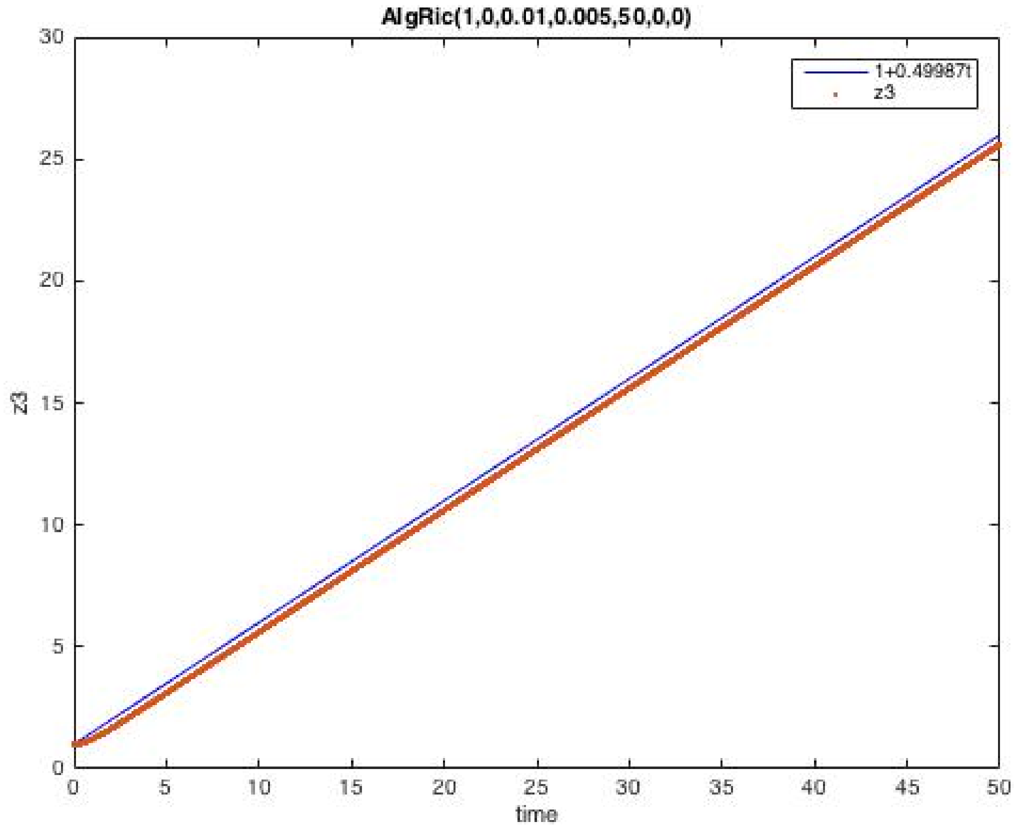

Figure 13: The graph of Ricci-flat of $SU(3)/T^2$ with $z_3, a = 1$ and its linear interpolation

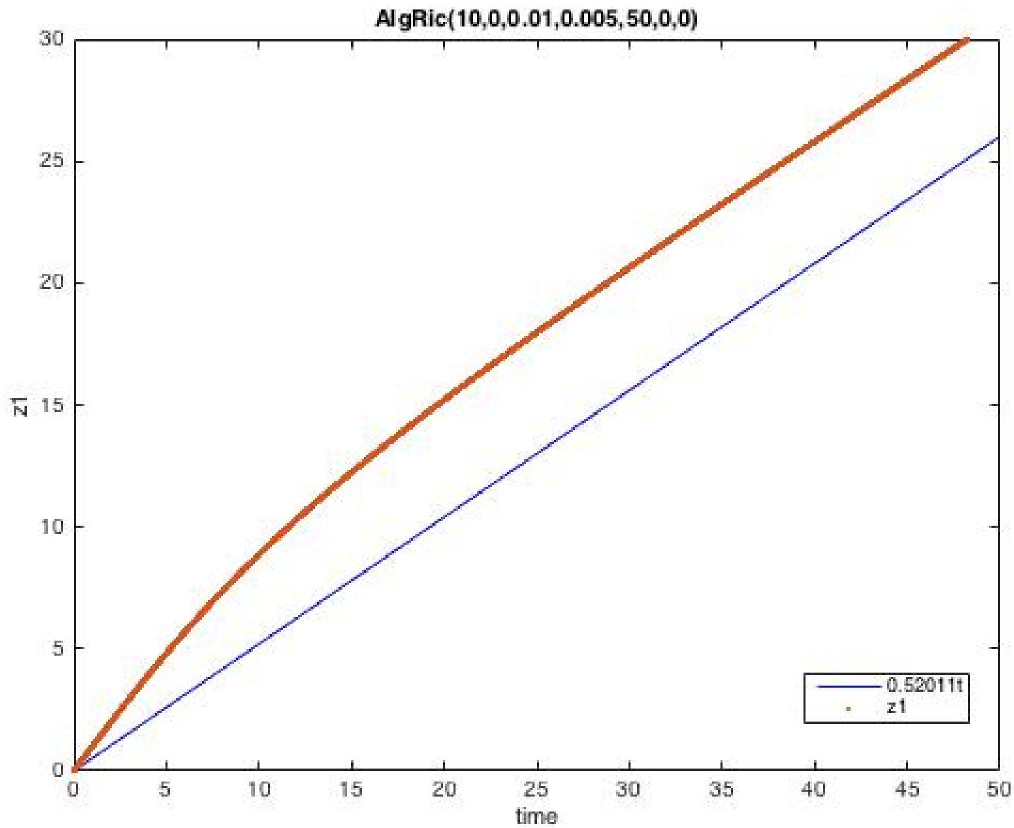

Figure 14: The graph of Ricci-flat of $SU(3)/T^2$ with $z_1, a = 10$ and its linear interpolation

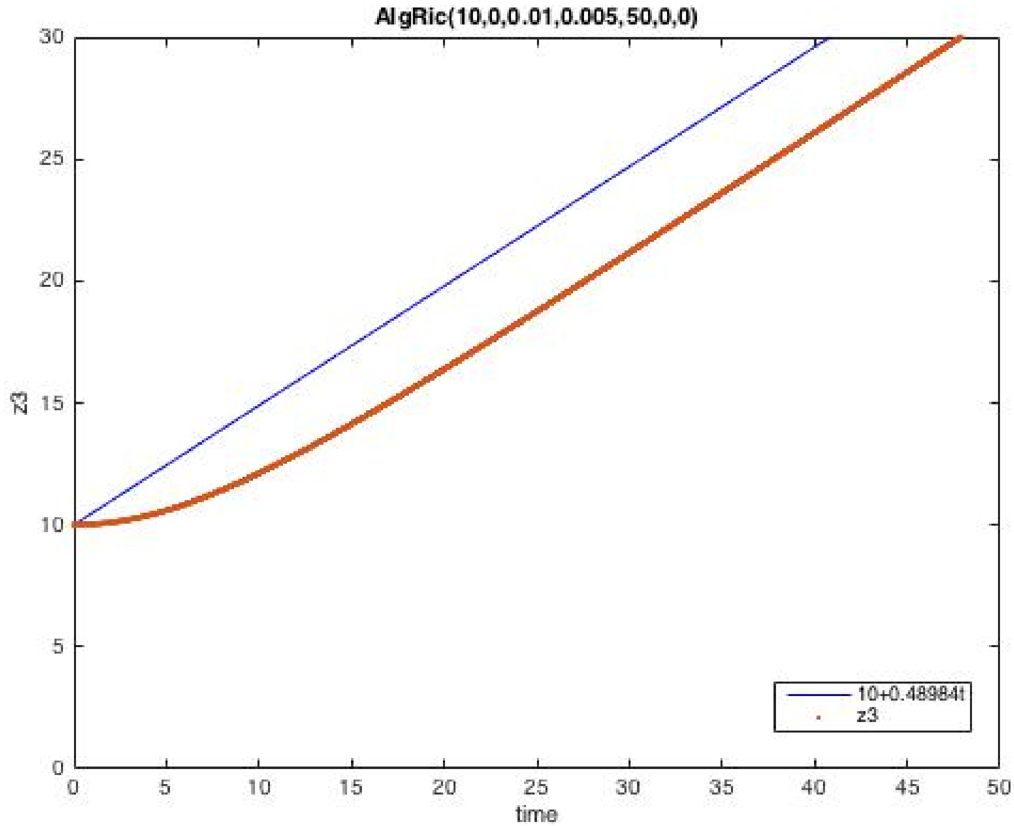

Figure 15: The graph of Ricci-flat of $SU(3)/T^2$ with $z_3, a = 10$ and its linear interpolation

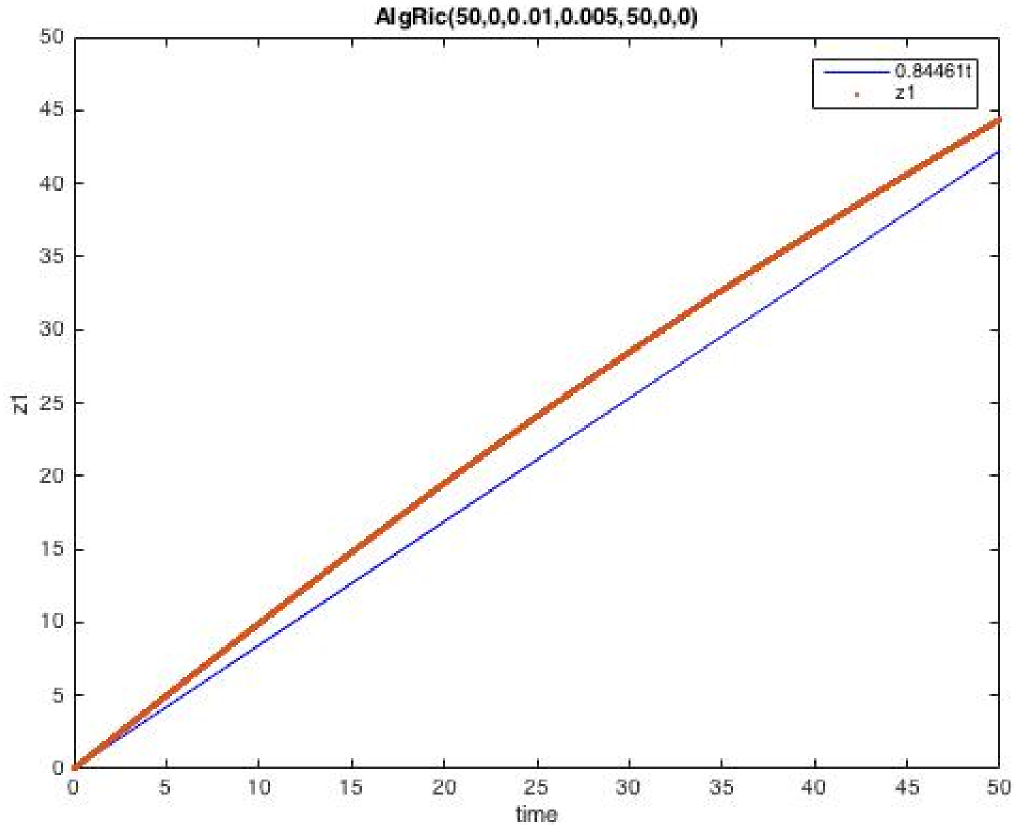

Figure 16: The graph of Ricci-flat of $SU(3)/T^2$ with $z_1, a = 50$ and its linear interpolation

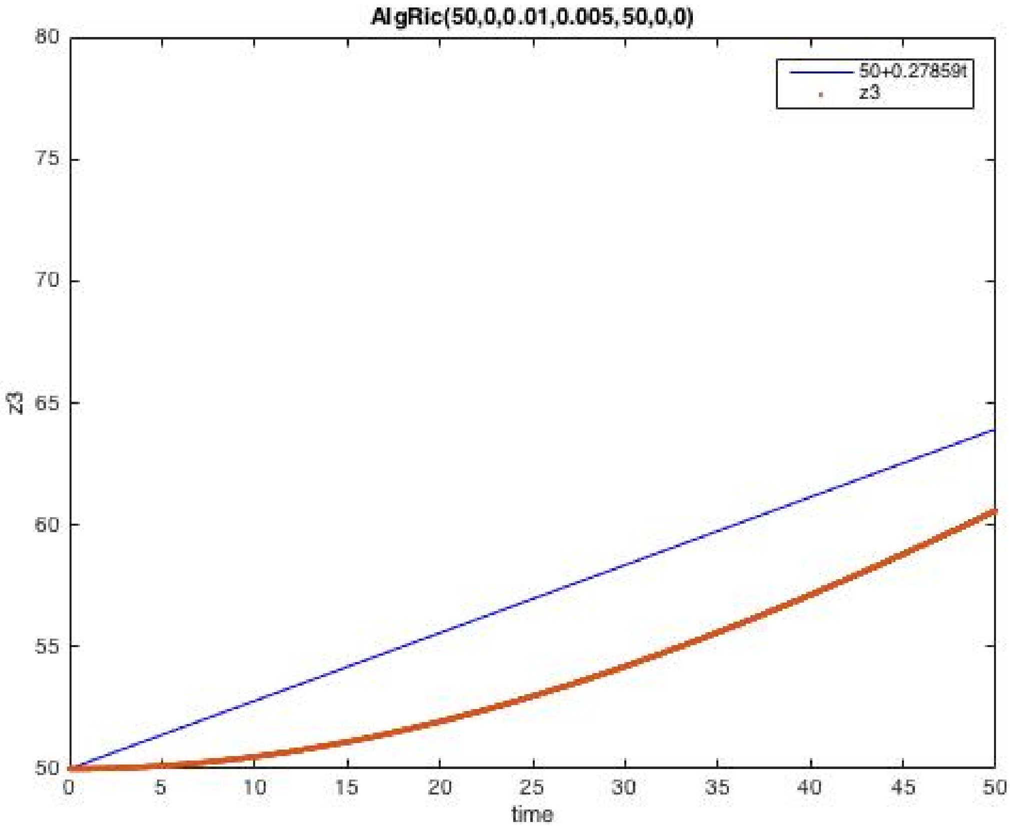

Figure 17: The graph of Ricci-flat of $SU(3)/T^2$ with $z_3, a = 50$ and its linear interpolation

Remark 9.3 *From these graphs, it is clear that figures 6, 7, 8, 9, 10, and 11 are asymptotically linear*

Remark 9.4 *We also see that we have a one-parameter family of solutions with the initial condition a being the parameter*

9.3 Expanding Solitons

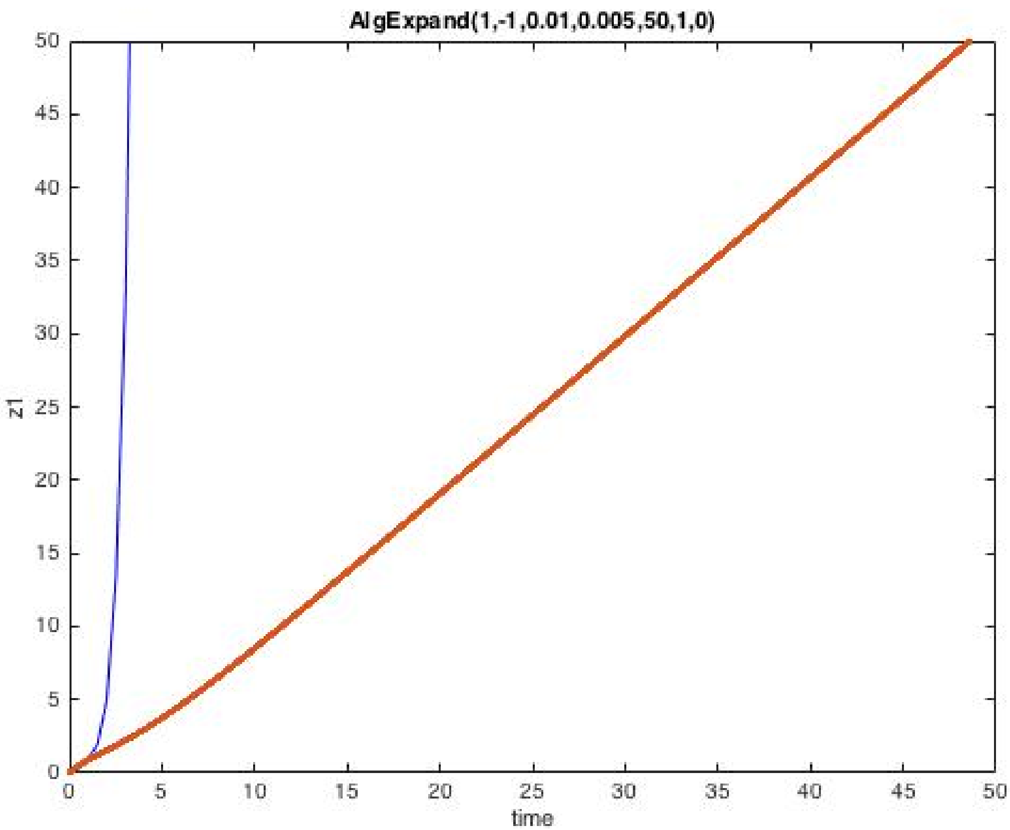

Figure 18: Expanding case of $SU(3)/T^2$ with $z_1, a = 1, b = -1$

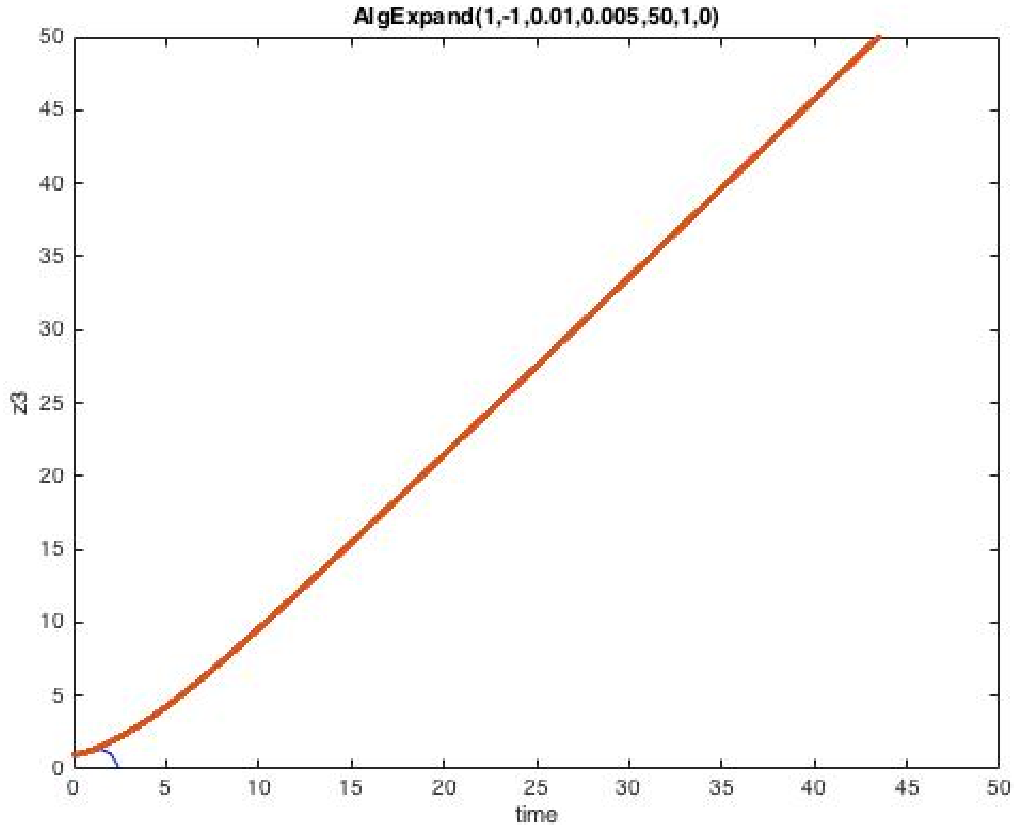

Figure 19: Expanding case of $SU(3)/T^2$ with $z_3, a = 1, b = -1$

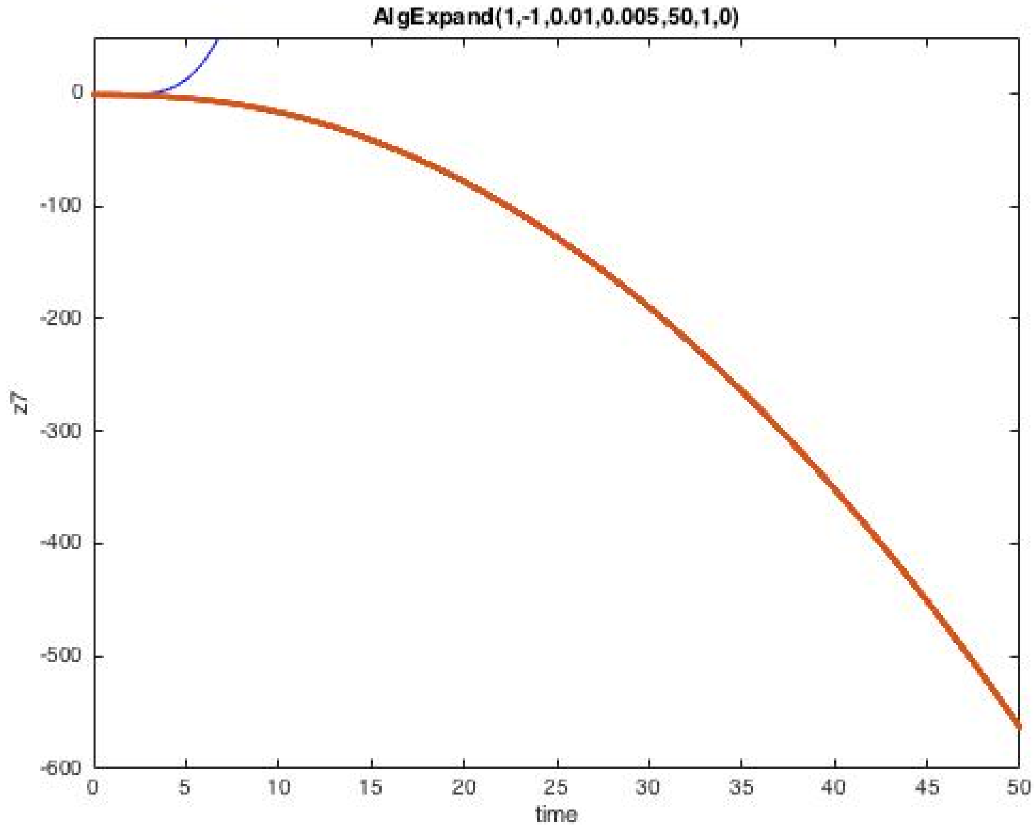

Figure 20: Expanding case of $SU(3)/T^2$ with $z_7, a = 1, b = -1$

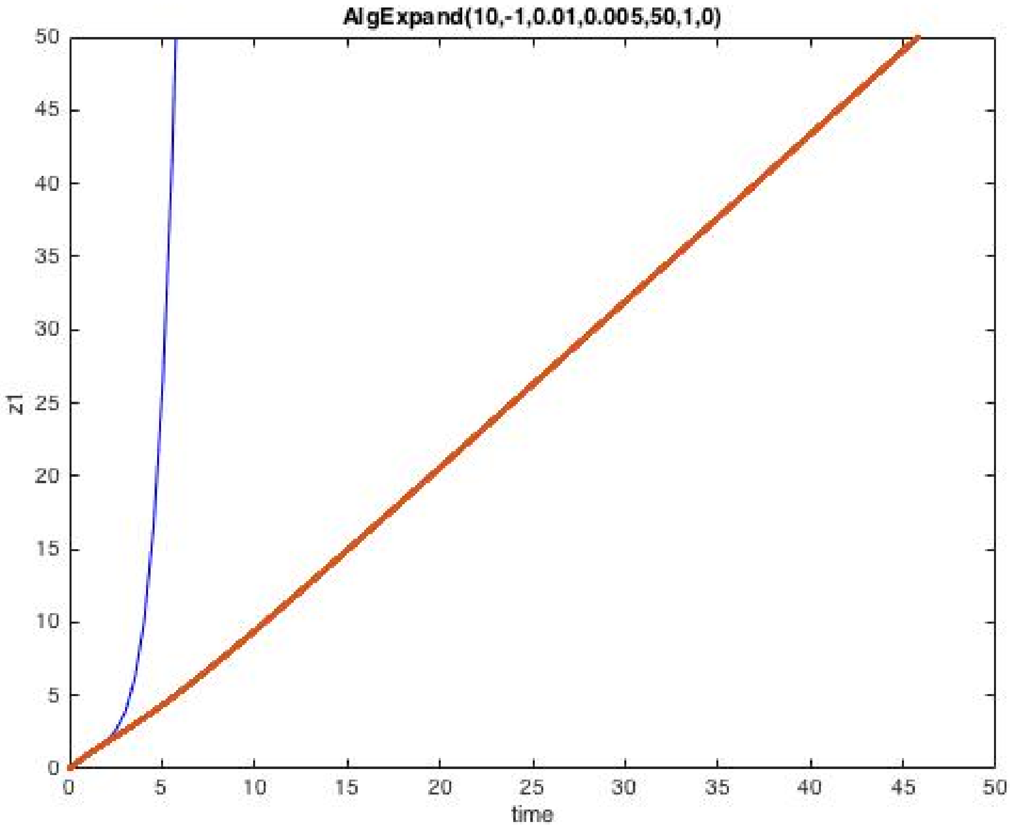

Figure 21: Expanding case of $SU(3)/T^2$ with $z_1, a = 10, b = -1$

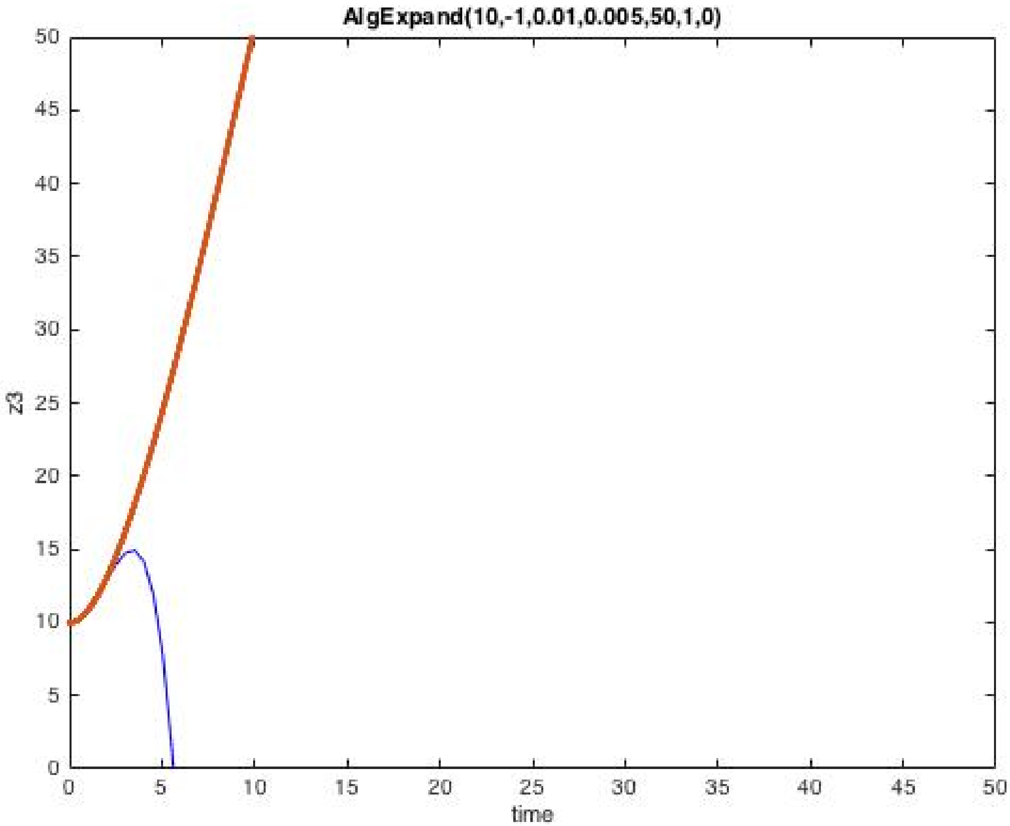

Figure 22: Expanding case of $SU(3)/T^2$ with $z_3, a = 10, b = -1$

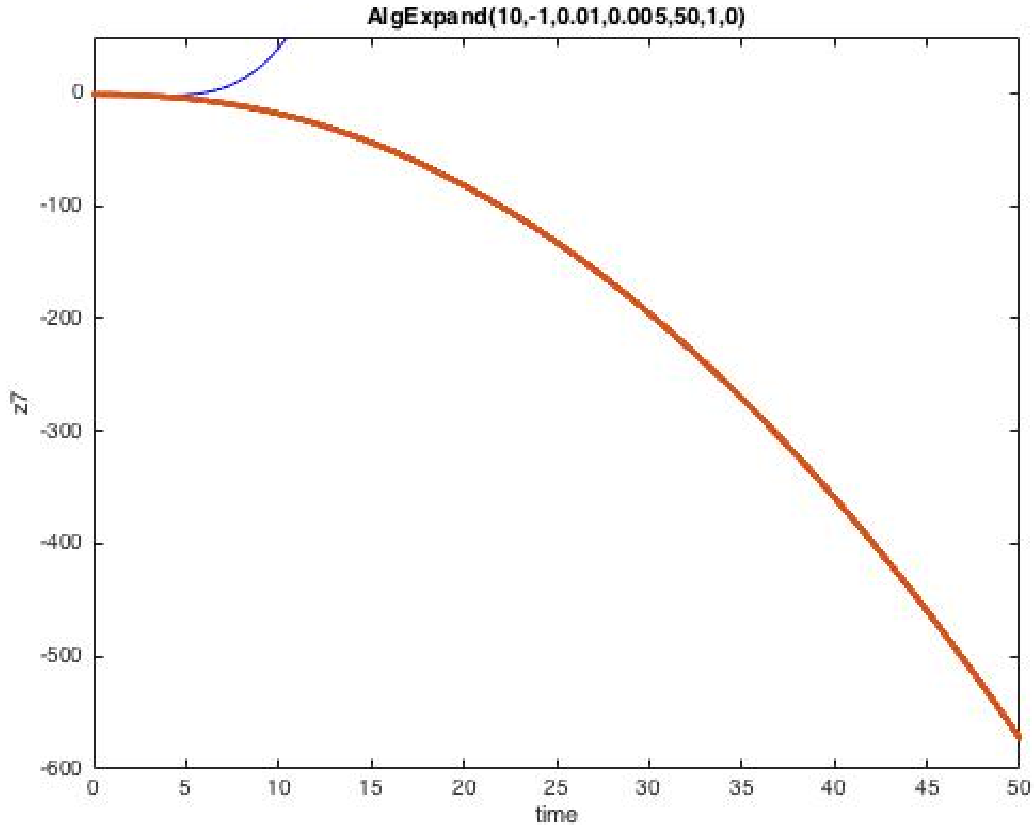

Figure 23: Expanding case of $SU(3)/T^2$ with $z_7, a = 10, b = -1$

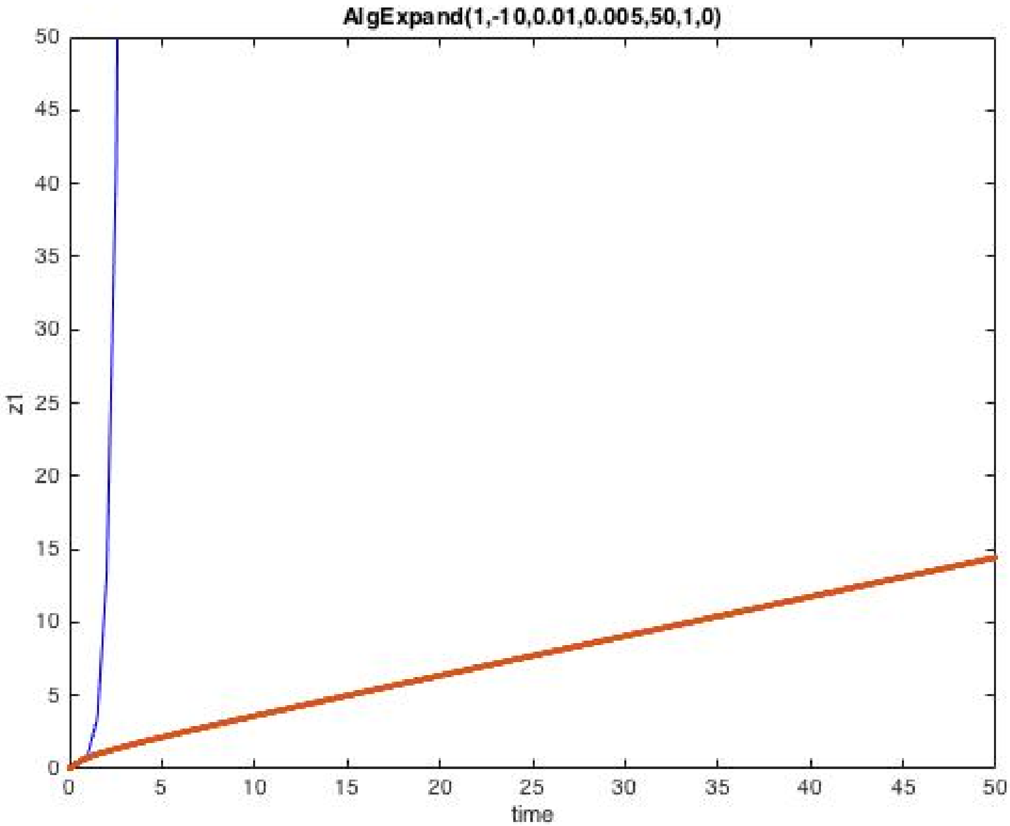

Figure 24: Expanding case of $SU(3)/T^2$ with $z_1, a = 1, b = -10$

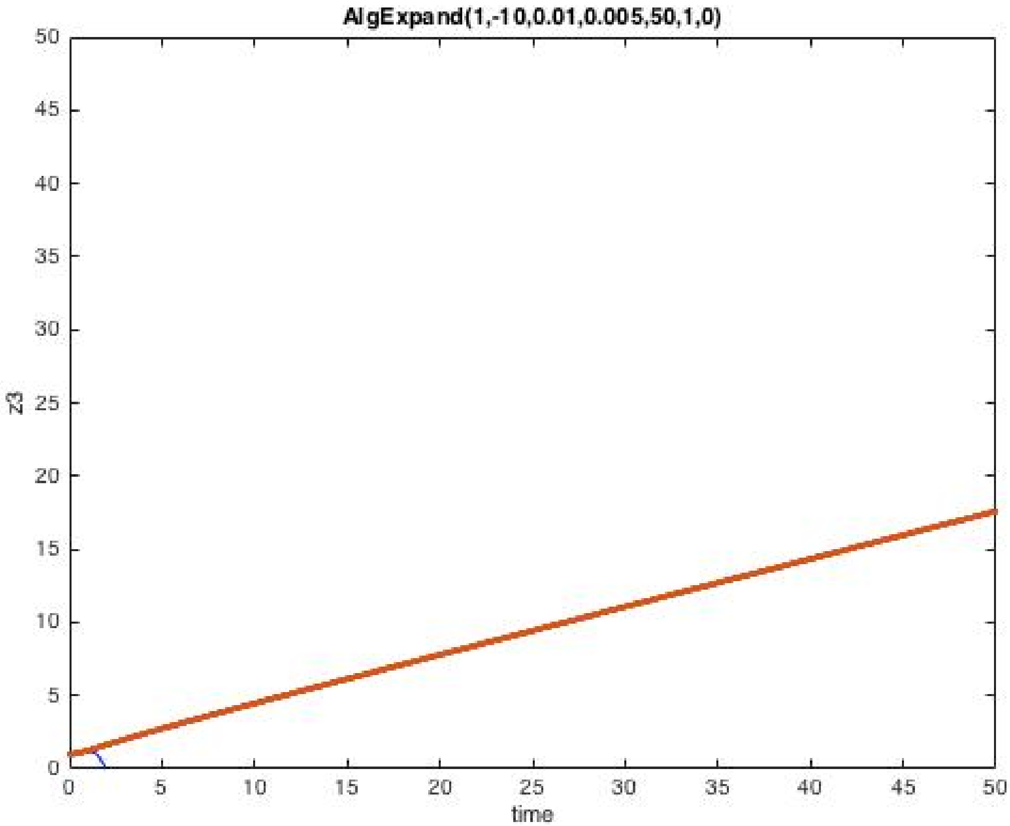

Figure 25: Expanding case of $SU(3)/T^2$ with $z_3, a = 1, b = -10$

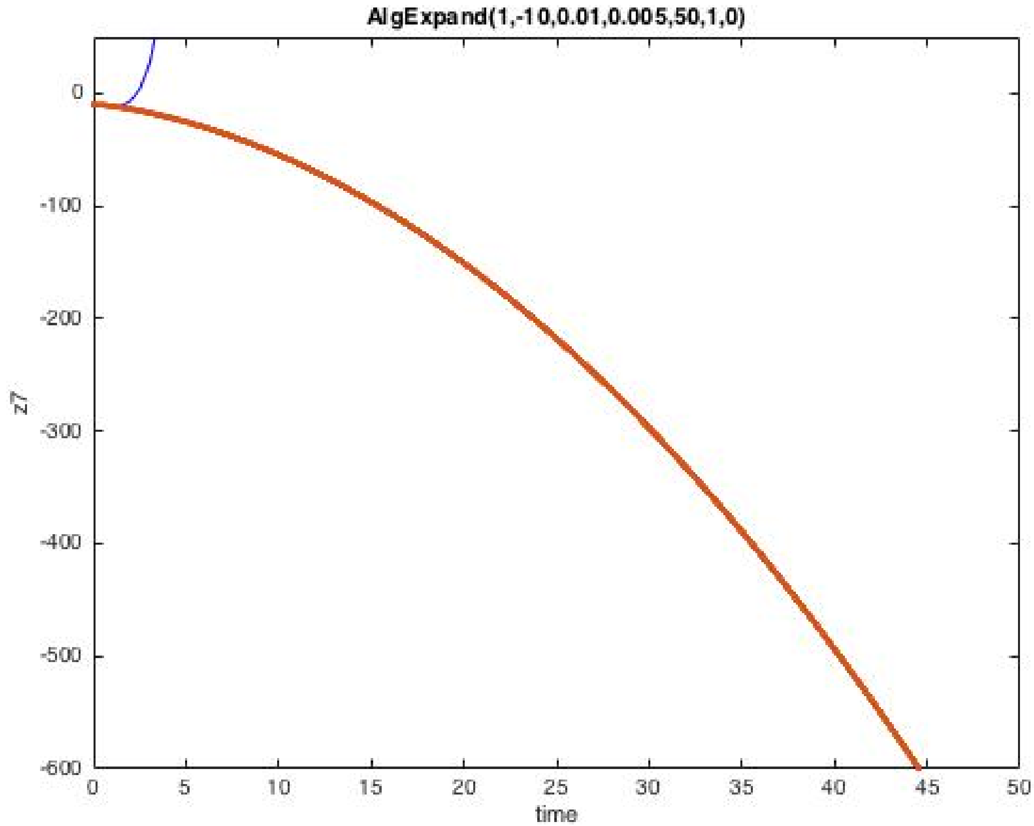

Figure 26: Expanding case of $SU(3)/T^2$ with $z_7, a = 1, b = -10$

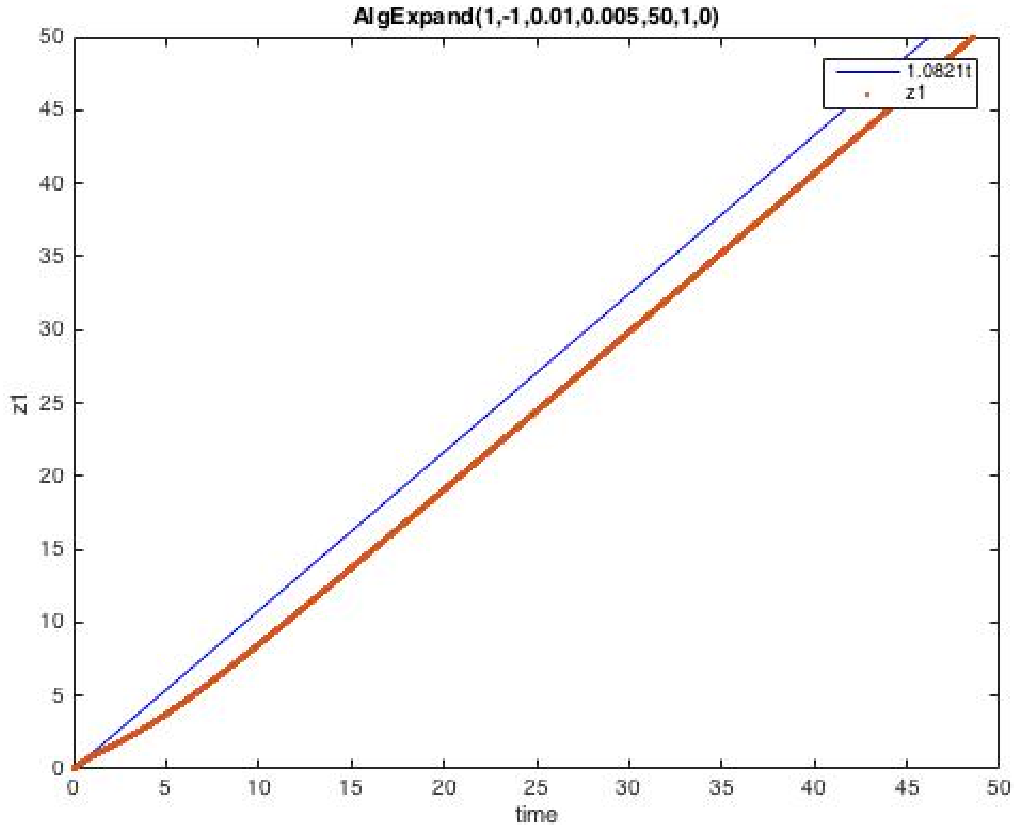

Figure 27: The graph of expanding case of $SU(3)/T^2$ with $z_1, a = 1, b = -1$ and its linear interpolation.

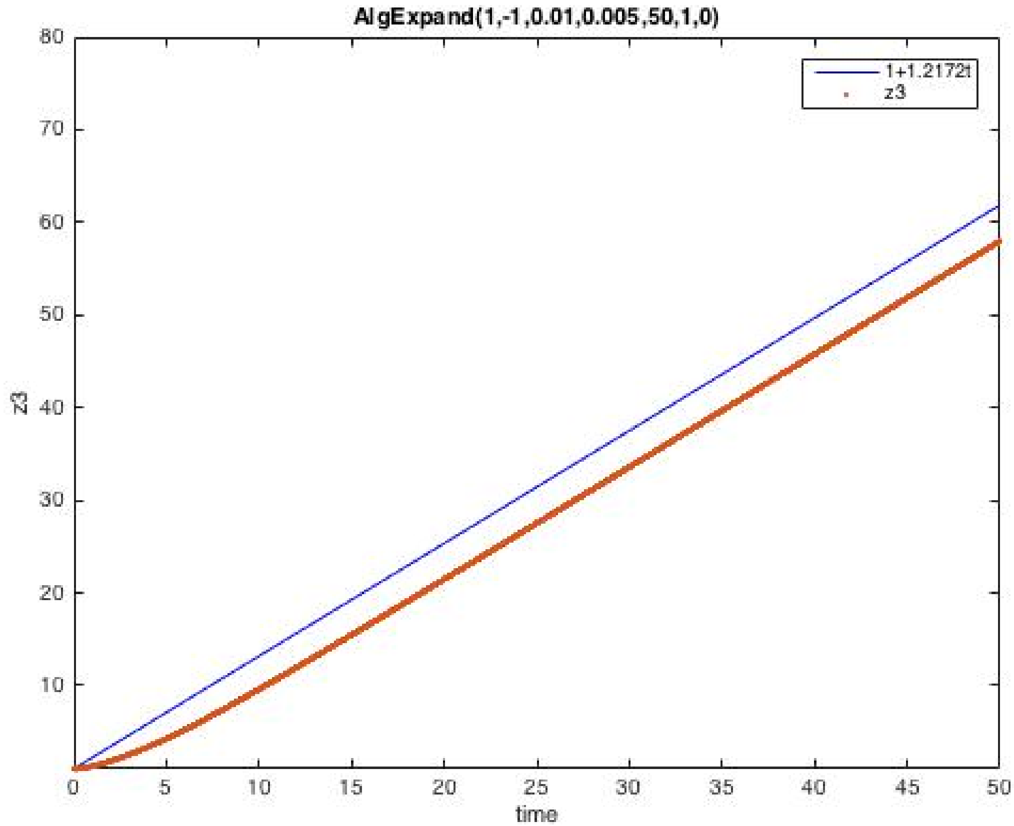

Figure 28: The graph of expanding case of $SU(3)/T^2$ with $z_3, a = 1, b = -1$ and its linear interpolation.

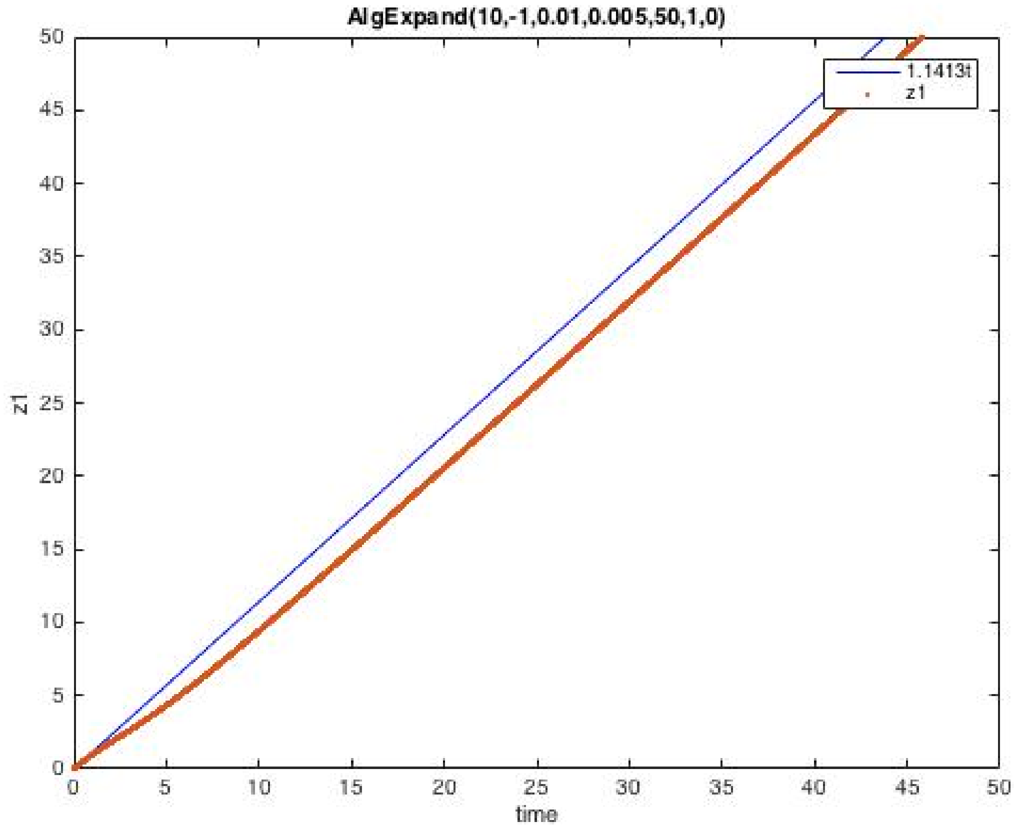

Figure 29: The graph of expanding case of $SU(3)/T^2$ with $z_1, a = 10, b = -1$ and its linear interpolation.

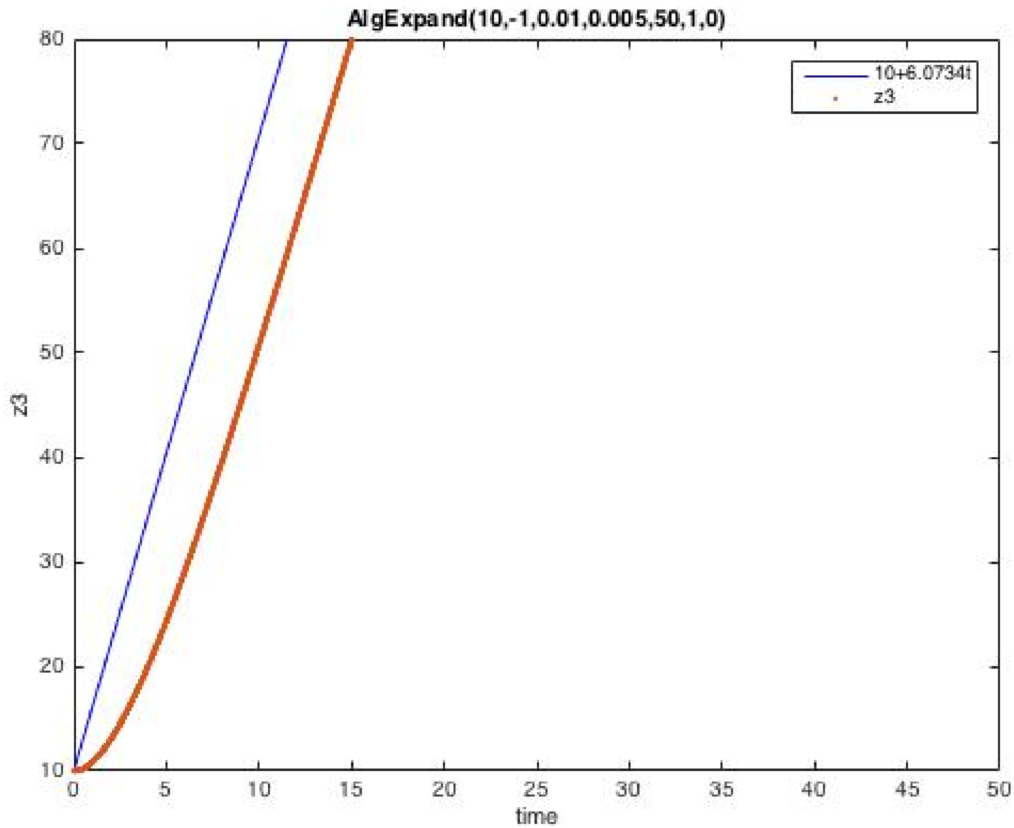

Figure 30: The graph of expanding case of $SU(3)/T^2$ with $z_3, a = 10, b = -1$ and its linear interpolation.

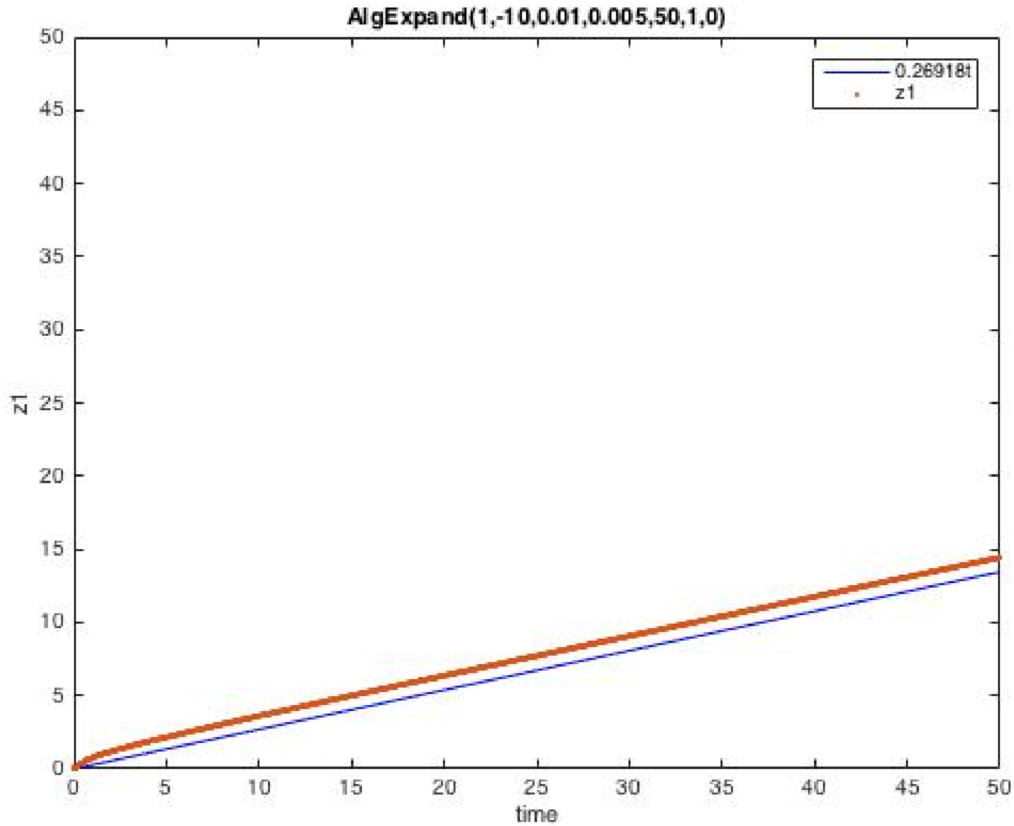

Figure 31: The graph of expanding case of $SU(3)/T^2$ with $z_1, a = 1, b = -10$ and its linear interpolation.

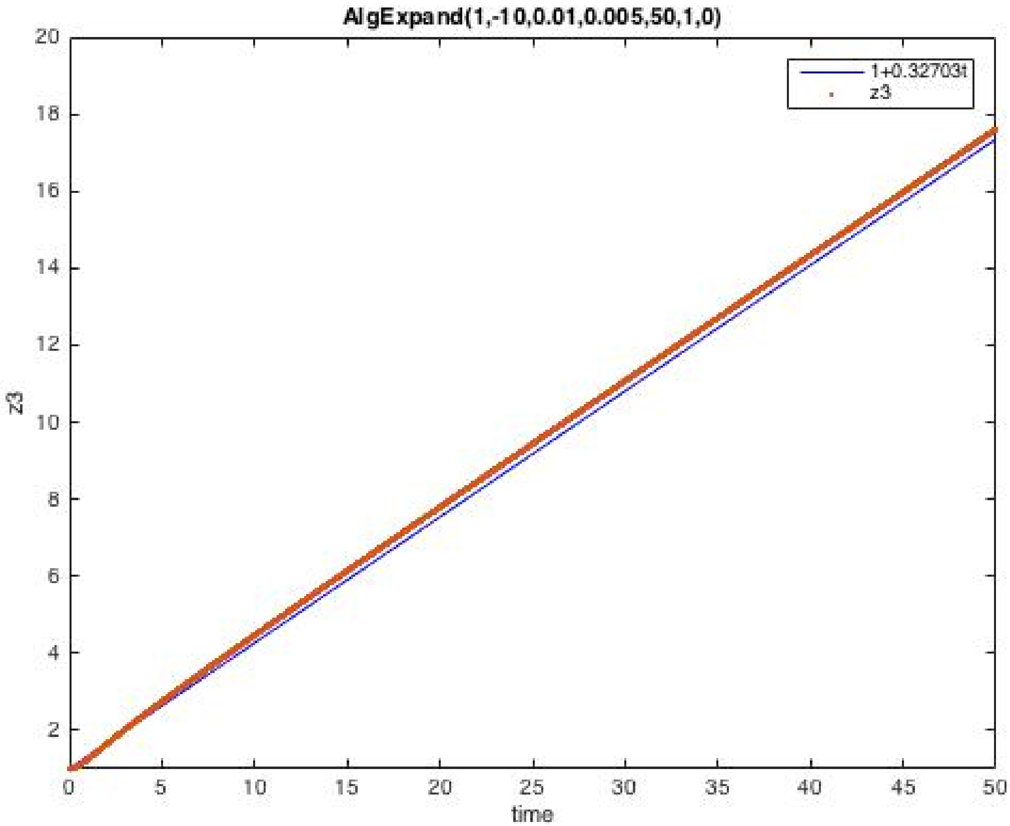

Figure 32: The graph of expanding case of $SU(3)/T^2$ with $z_3, a = 1, b = -10$ and its linear interpolation.

Remark 9.5 *The asymptotic linear behaviours of f_1, f_2, f_3 can also be observed in examples of [16], [9], [2], [19]*

9.4 Negative Einstein

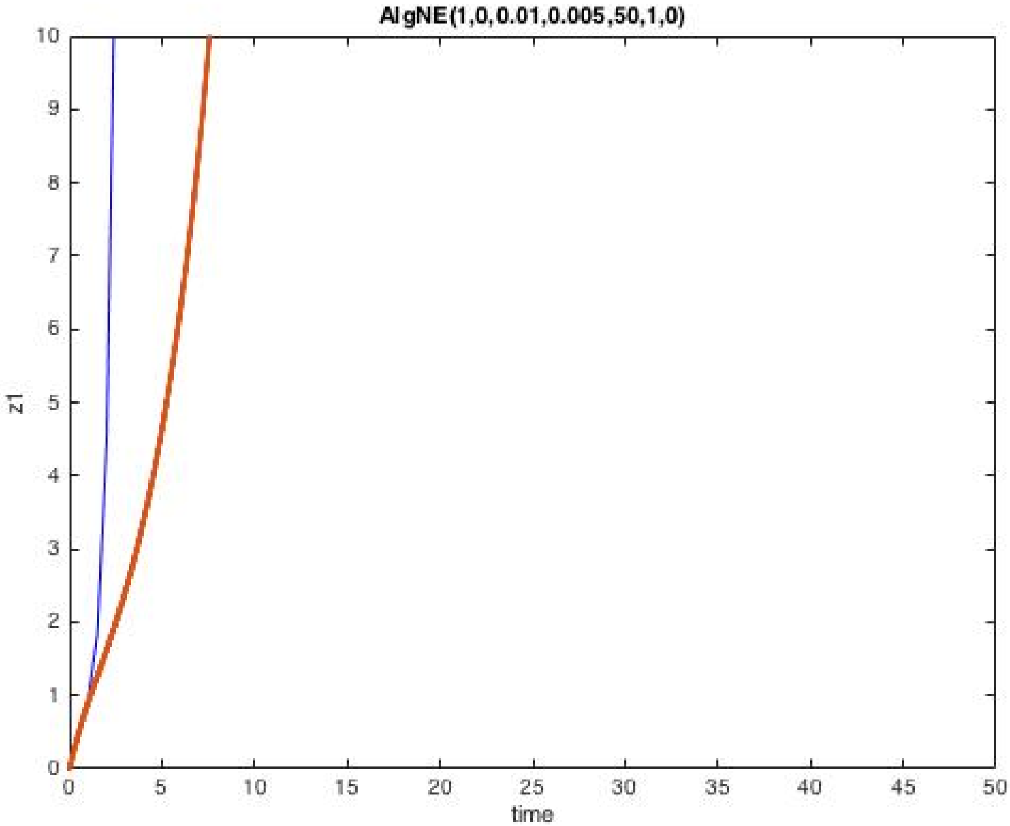

Figure 33: Negative Einstein case of $SU(3)/T^2$ with $z_1, a = 1, b = 0$

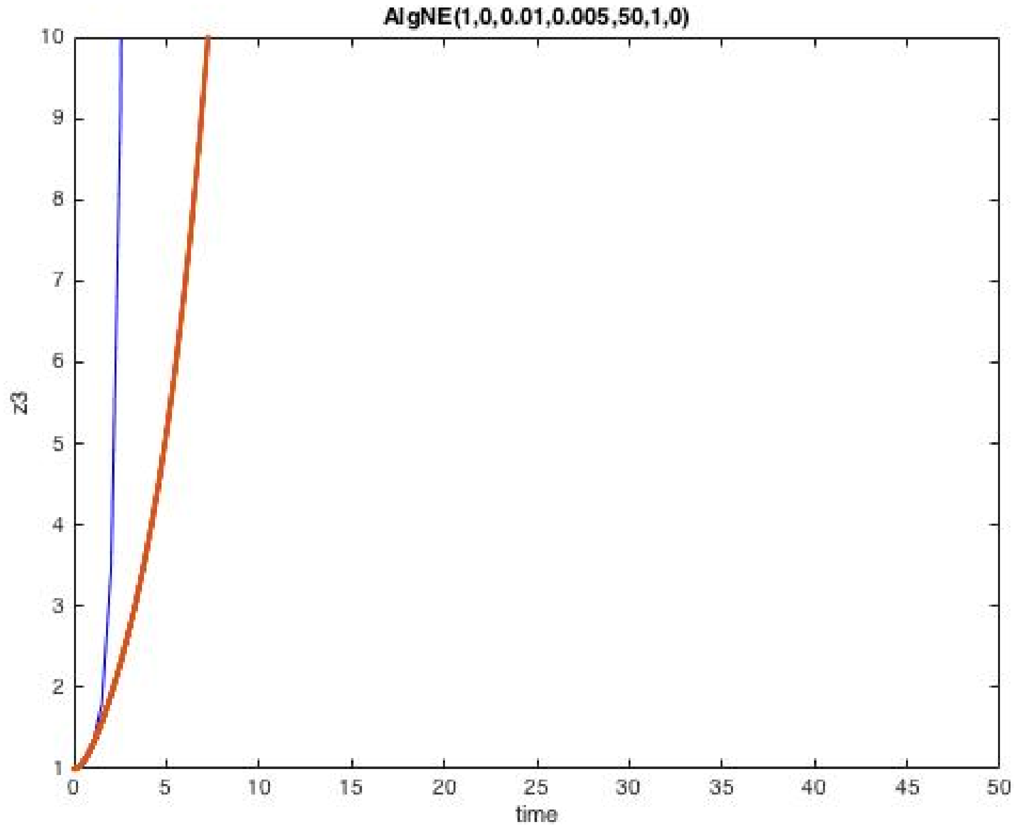

Figure 34: Negative Einstein case of $SU(3)/T^2$ $z_3, a = 1, b = 0$

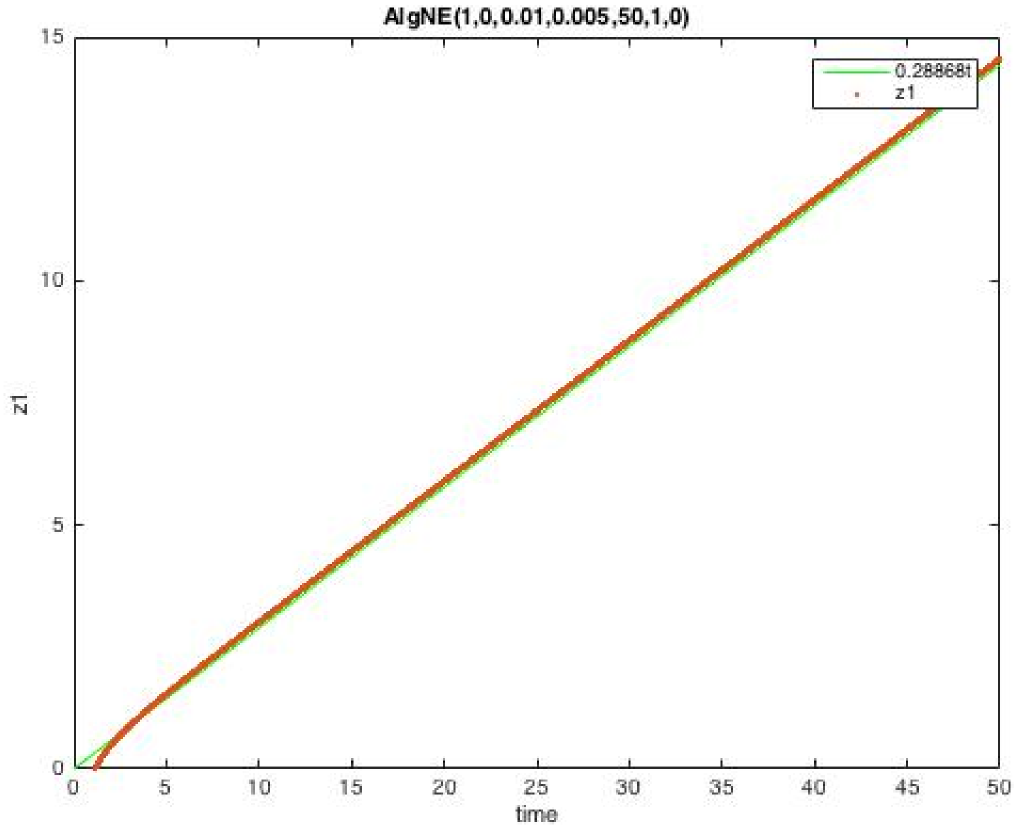

Figure 35: Log Graph of negative Einstein case of $SU(3)/T^2$ with $z_1, a = 1, b = 0$

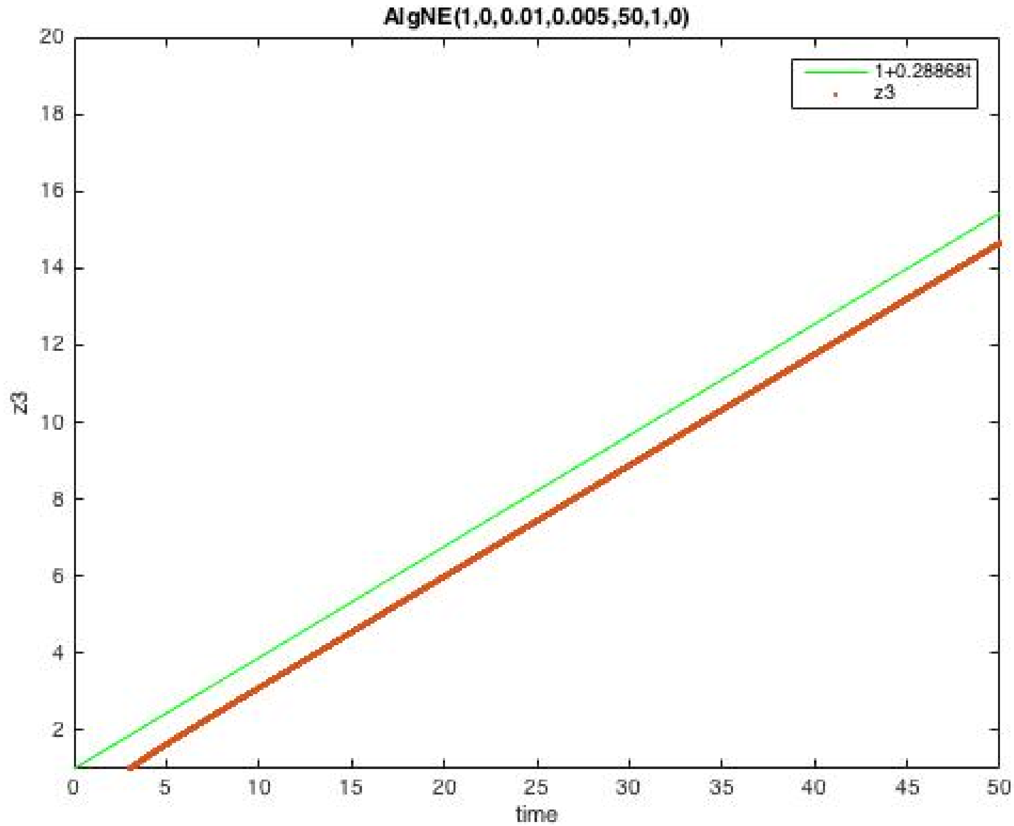

Figure 36: Log Graph of negative Einstein case of $SU(3)/T^2$ $z_3, a = 1, b = 0$

Remark 9.6 *Compare the figures 34 and 35 with their log graphs reveals that the solutions may be exponential.*

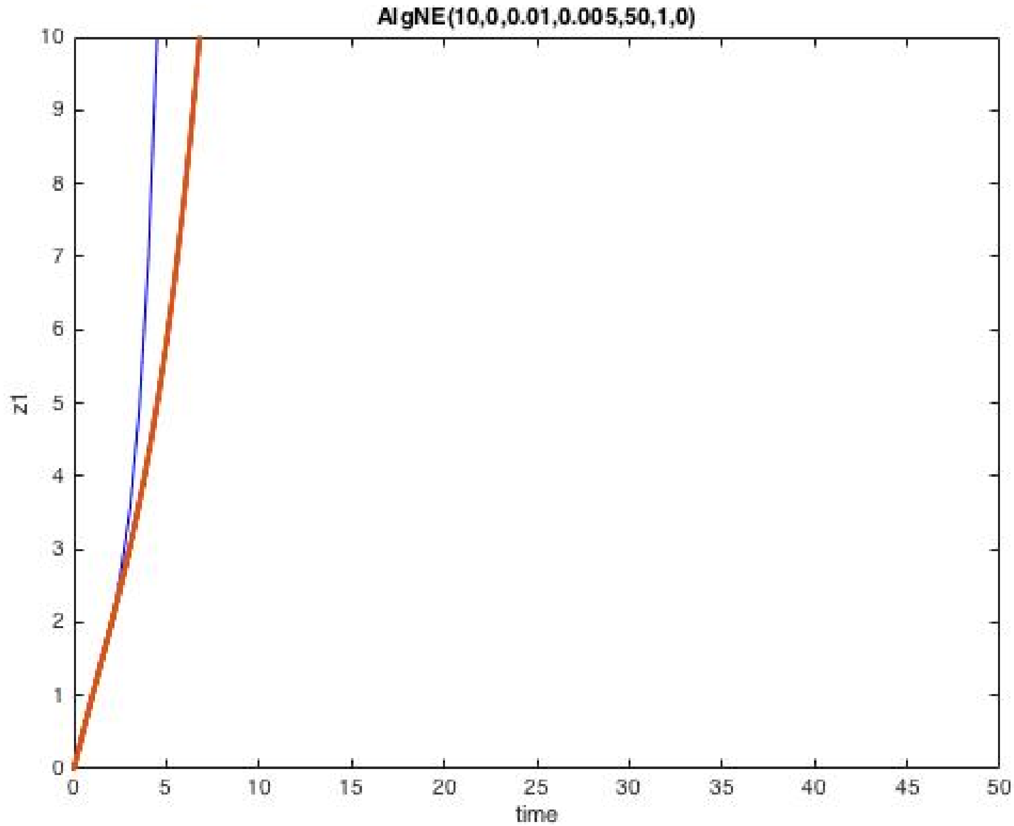

Figure 37: Negative Einstein case of $SU(3)/T^2$ $z_1, a = 10, b = 0$

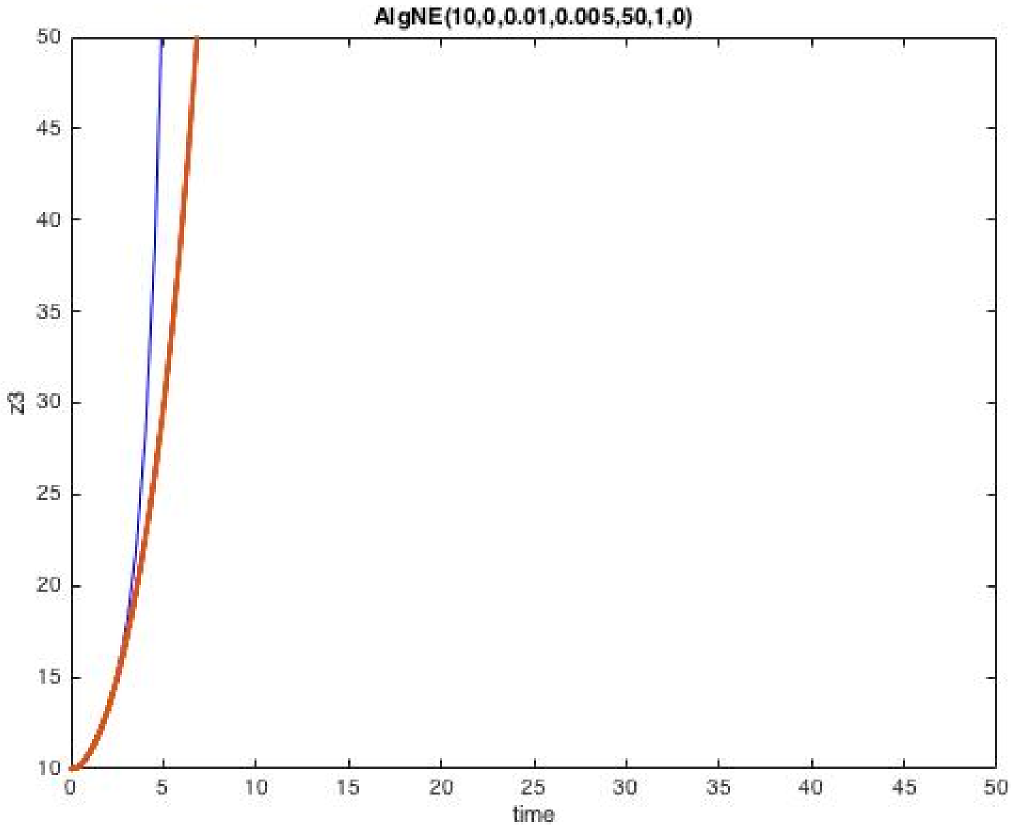

Figure 38: Negative Einstein case of $SU(3)/T^2$ $z_3, a = 10, b = 0$

9.5 Compact and non-compact Shrinking soliton

We did not find any numerical solutions for the case of non-compact shrinking solitons. However there are results for the compact Einstein solitons where there is an initial cluster of infinitely many Einstein metrics and a first independent Einstein metric.

The following is a graph of SOL vs initial value a which helps us detect Einstein metrics

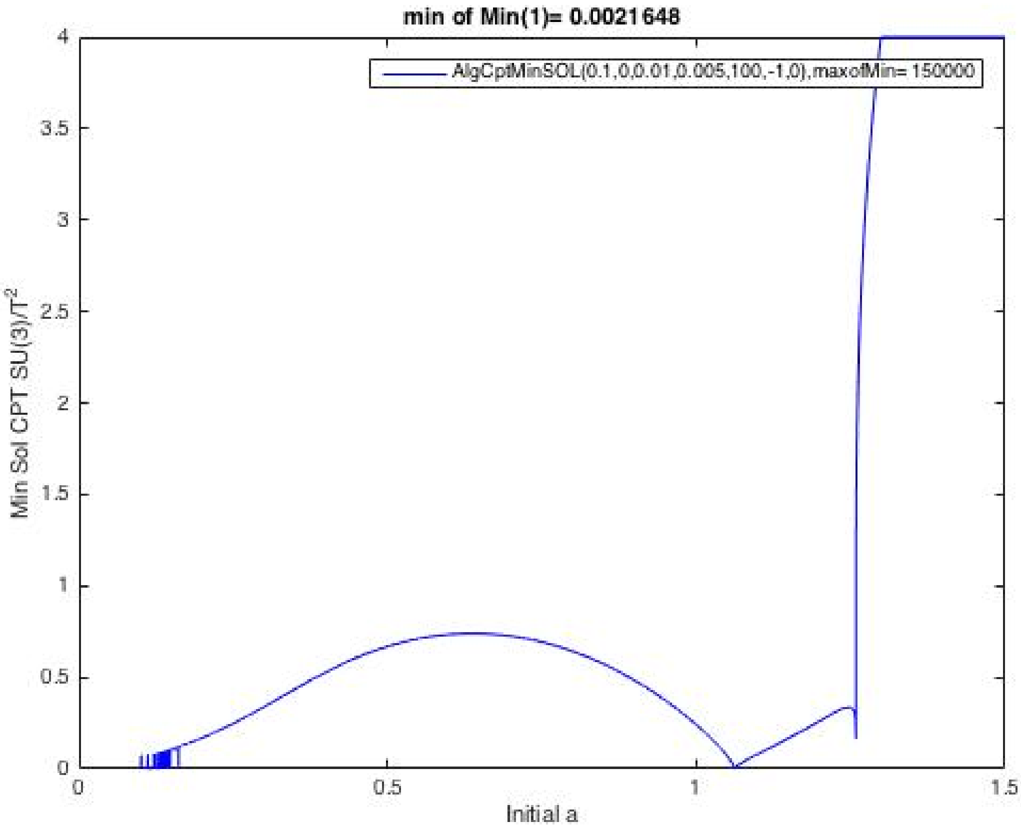

Figure 39: Compact Einstein case of $SU(3)/T^2$. Graph of SOL vs initial value a.

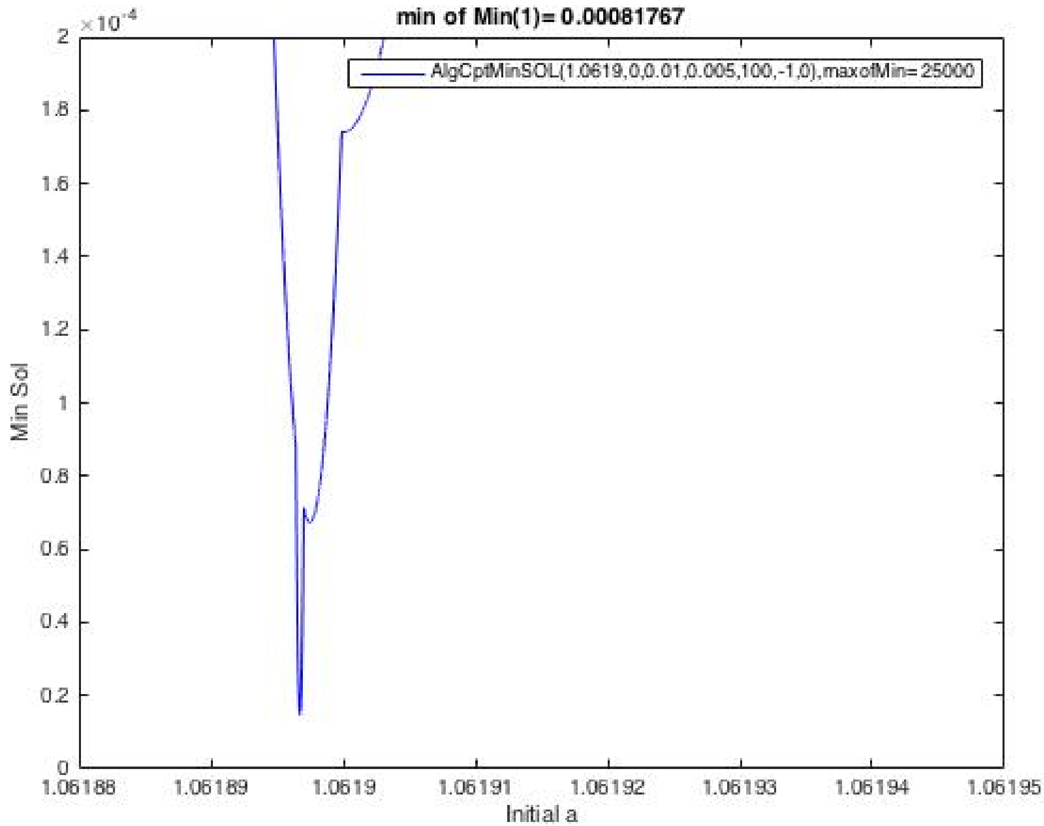

Figure 40: Compact Einstein case of $SU(3)/T^2$, Graph of SOL vs initial value a with a closer shot at the first Einstein metric

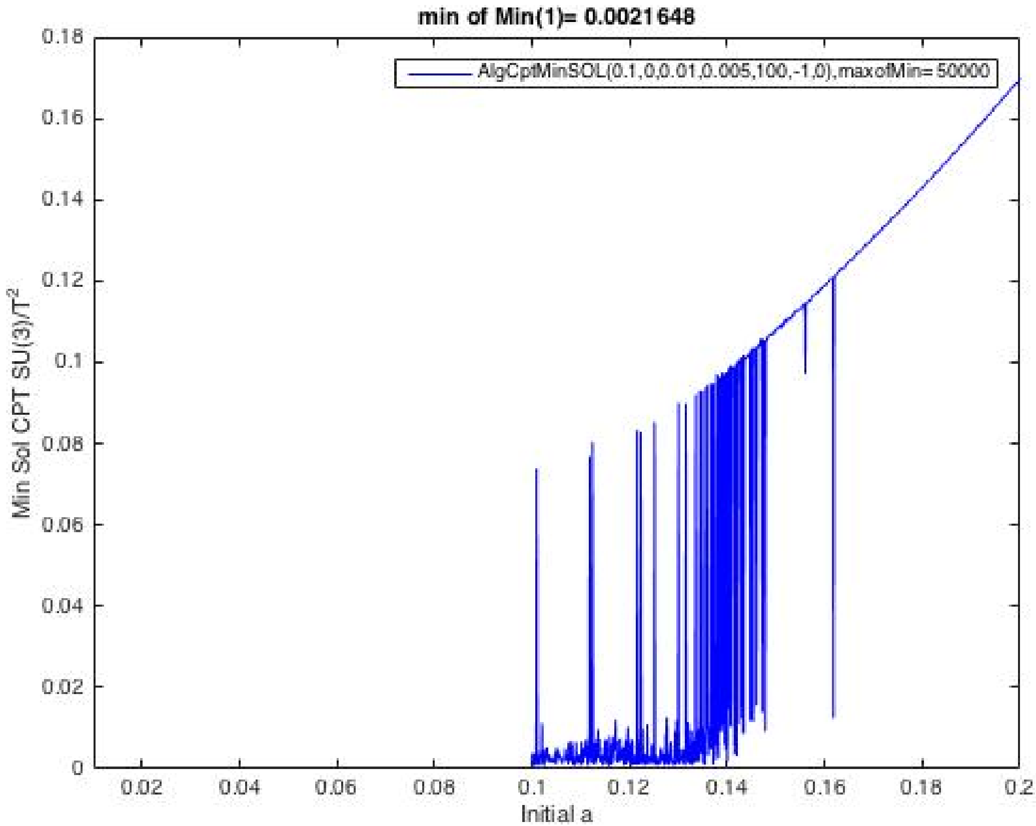

Figure 41: Cluster of figure 39

We estimate the point of interest at $a = 1.0619$. Note this is not a true value of the Einstein metric as we cannot numerically determine when $SOL = 0$. Hence we are only drawing values in a neighborhood of the Einstein metric which should have similar features.

It will shown that the results for this case is similar to $Sp(3)/Sp(1)^3$, this is due to a minor change in the soliton equation.

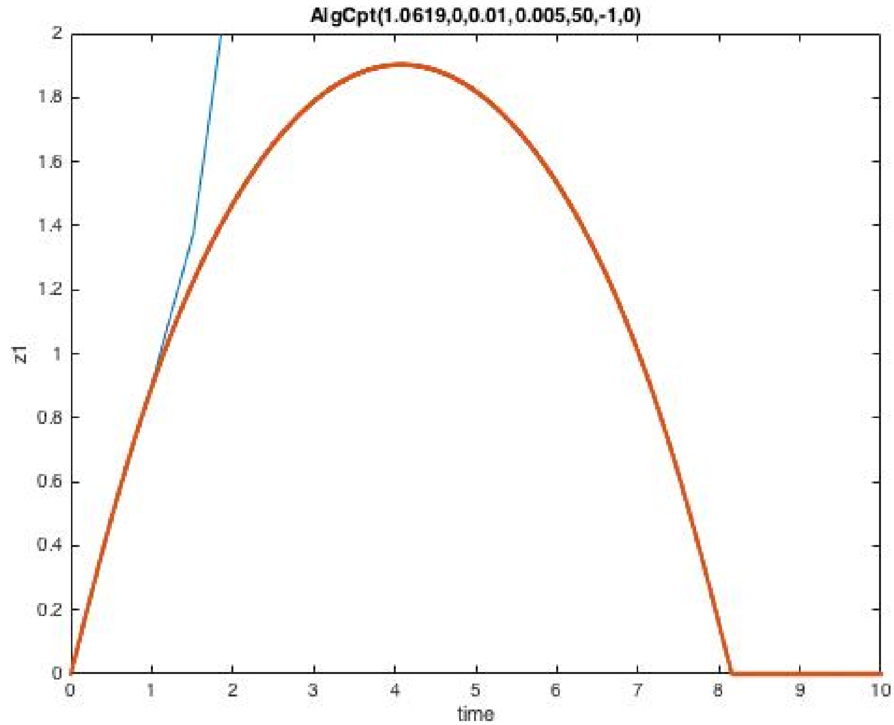

Figure 42: Compact Einstein case of $SU(3)/T^2$ with z_1, $a = 1.0619, b = 0$

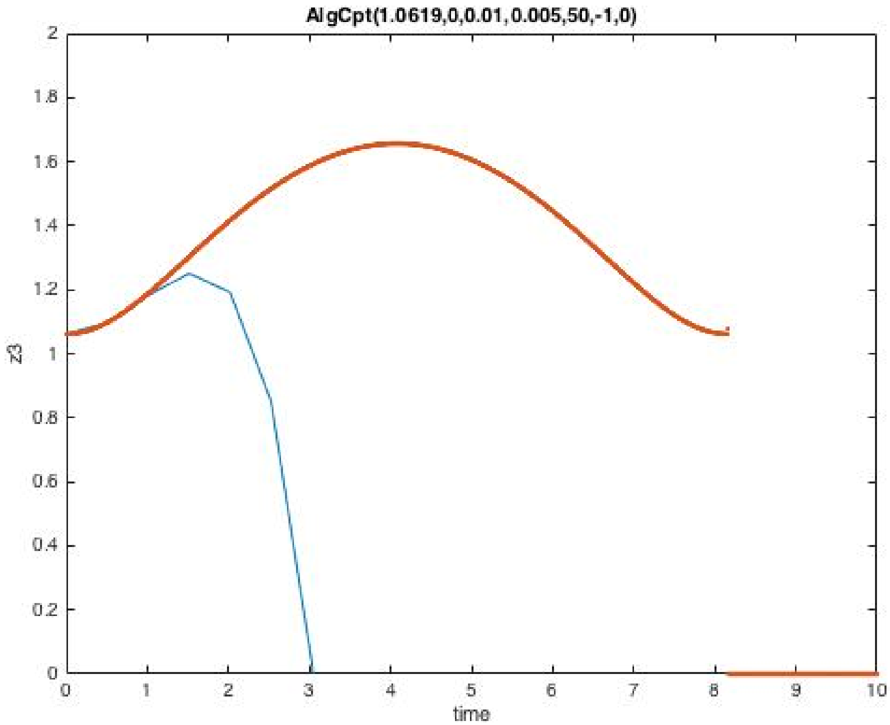

Figure 43: Compact Einstein case of $SU(3)/T^2$ with z_3, $a = 1.0619, b = 0$

As expected, the first graph of $z_1 = f_1$ closes up on the boundary point with negative slope and the graph of $z_3 = f_2$ ends with shows a negative slope.

10 $Sp(3)/Sp(1)Sp(1)Sp(1)$

This section features plots for the principal orbits of $Sp(3)/Sp(1)Sp(1)Sp(1)$. The remarks from $SU(3)/T^2$ carry over in this section.

10.1 Steady Solitons

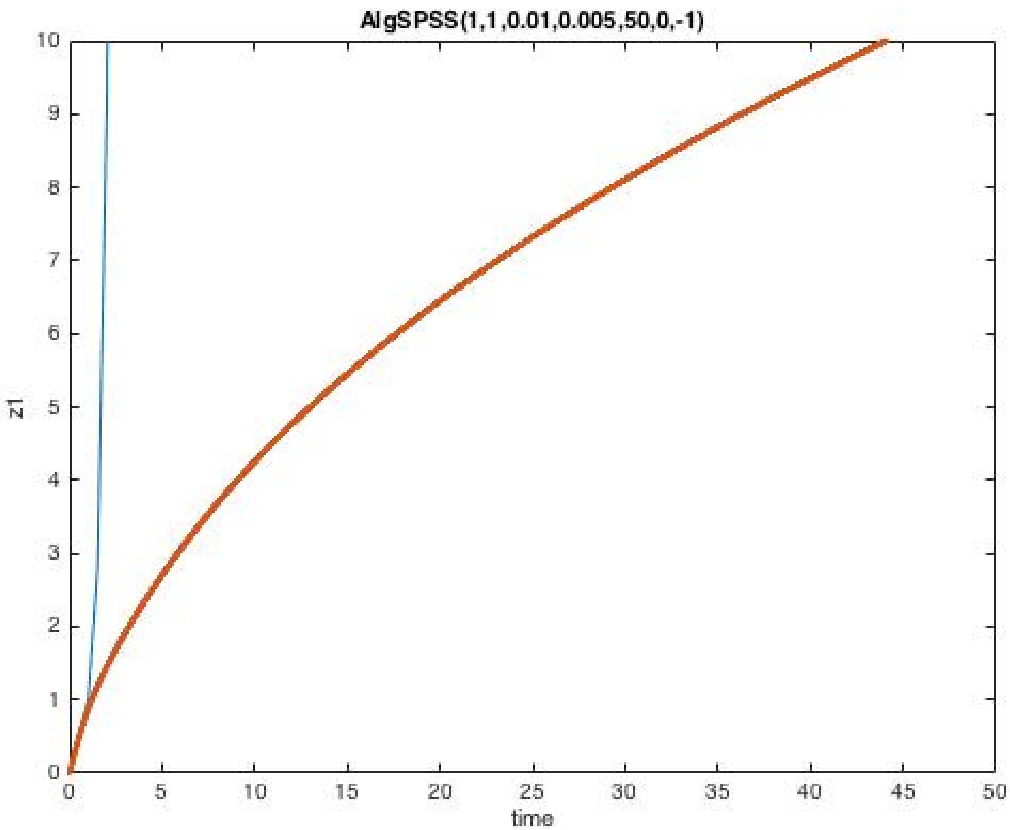

Figure 44: Steady Case of $Sp(3)/Sp(1)^3$ with $z_1, a = 1, b = 1$

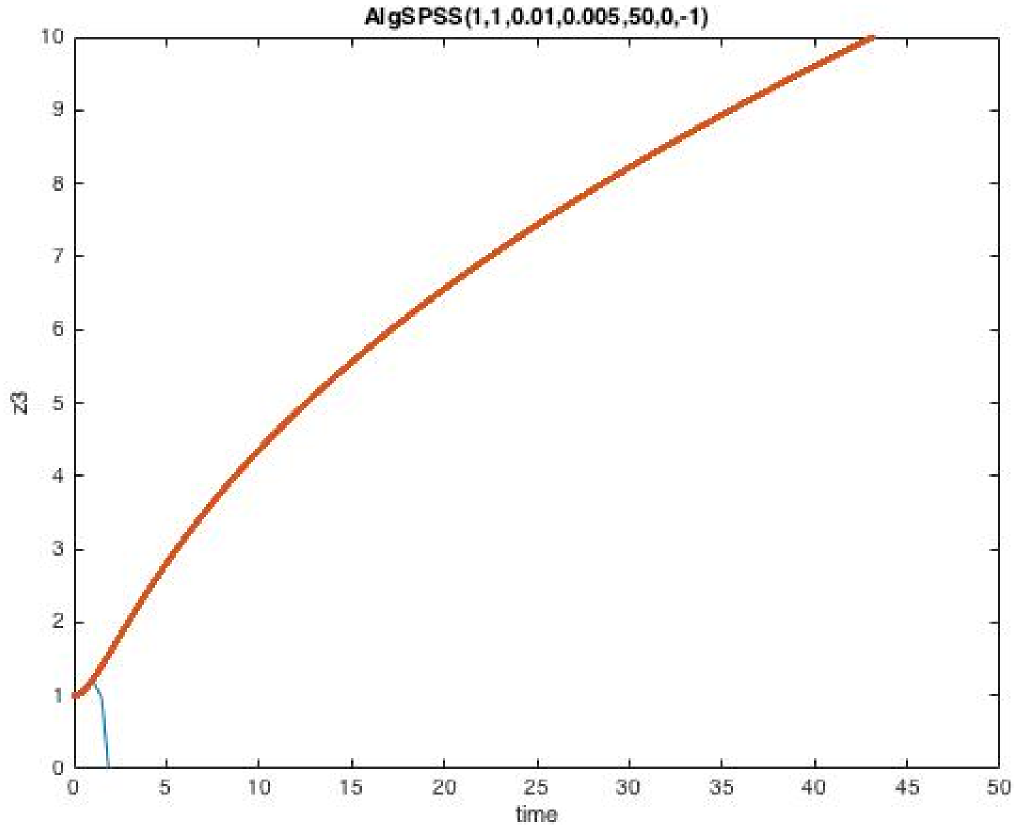

Figure 45: Steady Case of $Sp(3)/Sp(1)^3$ with $z_3, a = 1, b = 1$

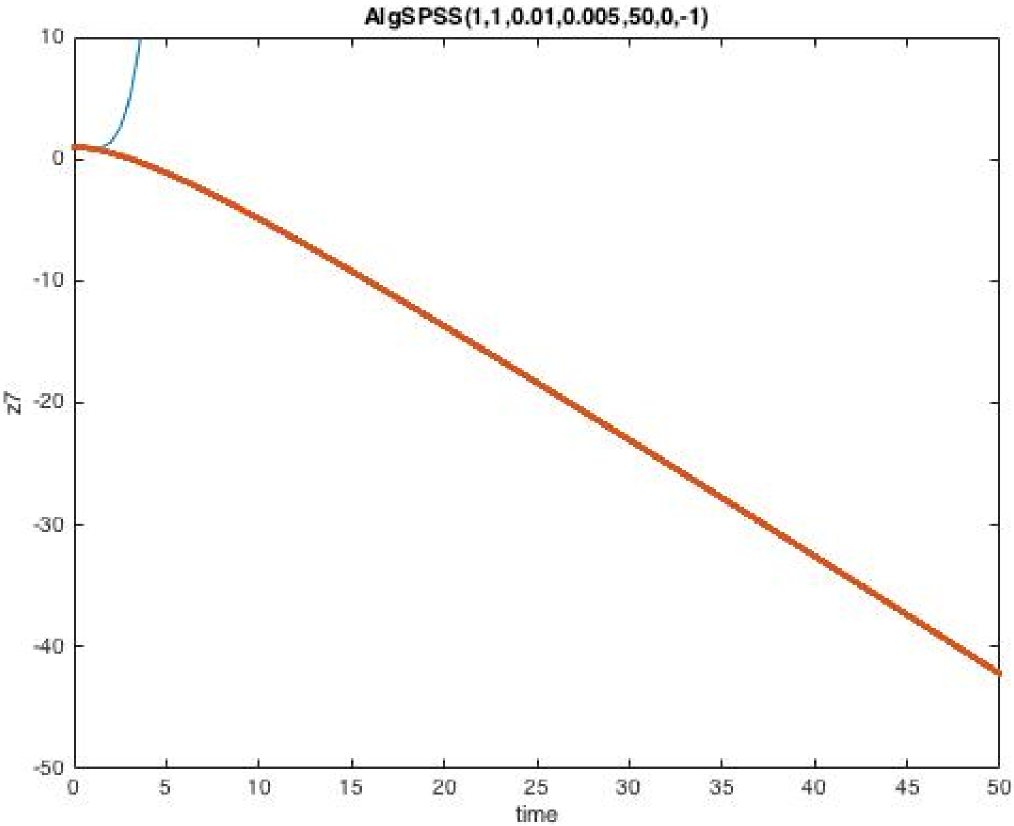

Figure 46: Steady Case of $Sp(3)/Sp(1)^3$ with $z_7, a = 1, b = 1$

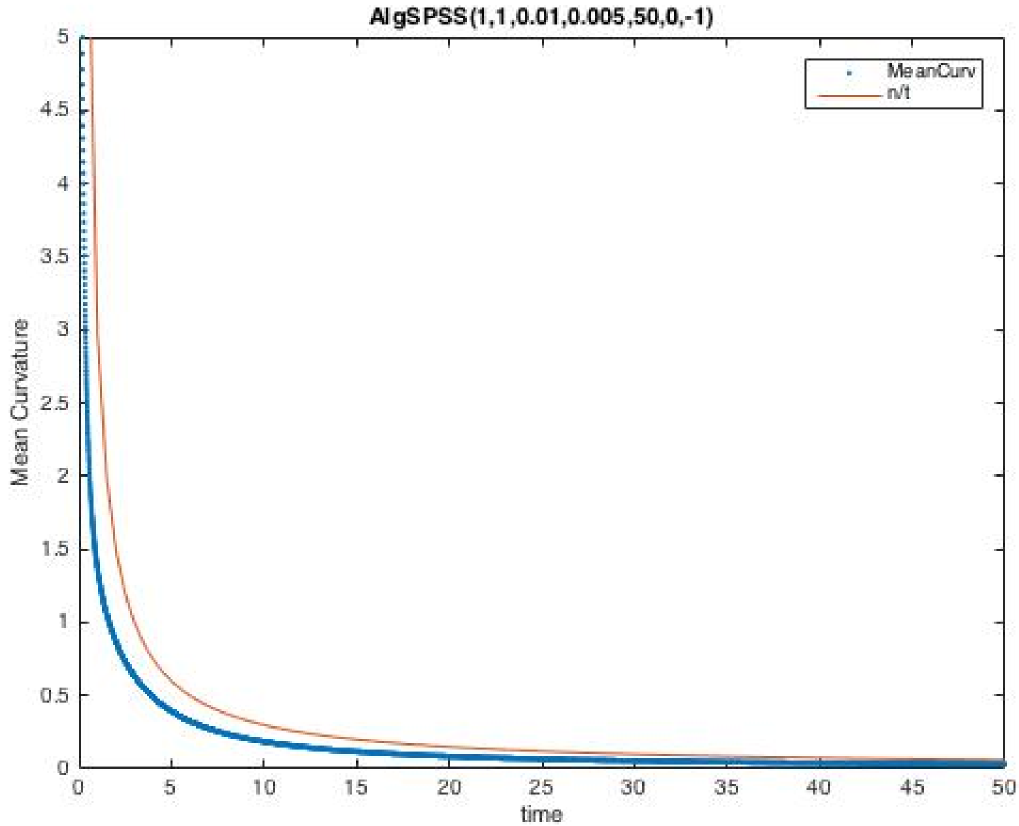

Figure 47: Mean Curvature of $Sp(3)/Sp(1)^3$

10.2 Ricci-flat

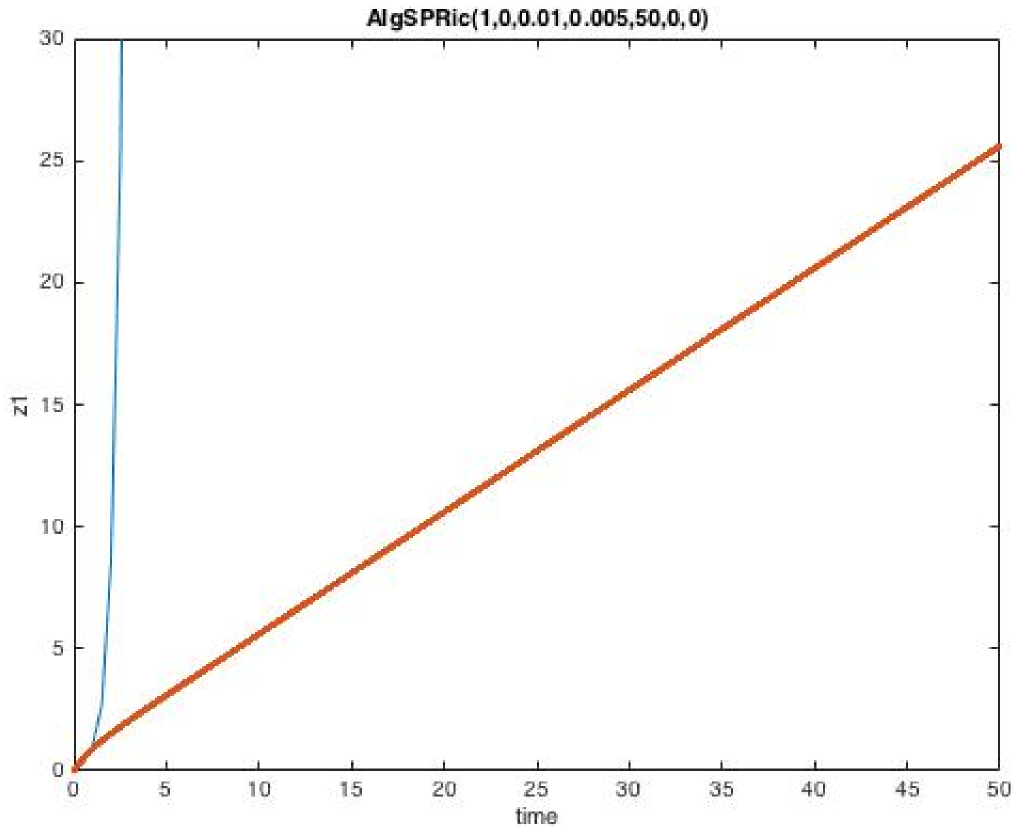

Figure 48: Ricci-flat Case of $Sp(3)/Sp(1)^3$ with $z_1, a = 1$

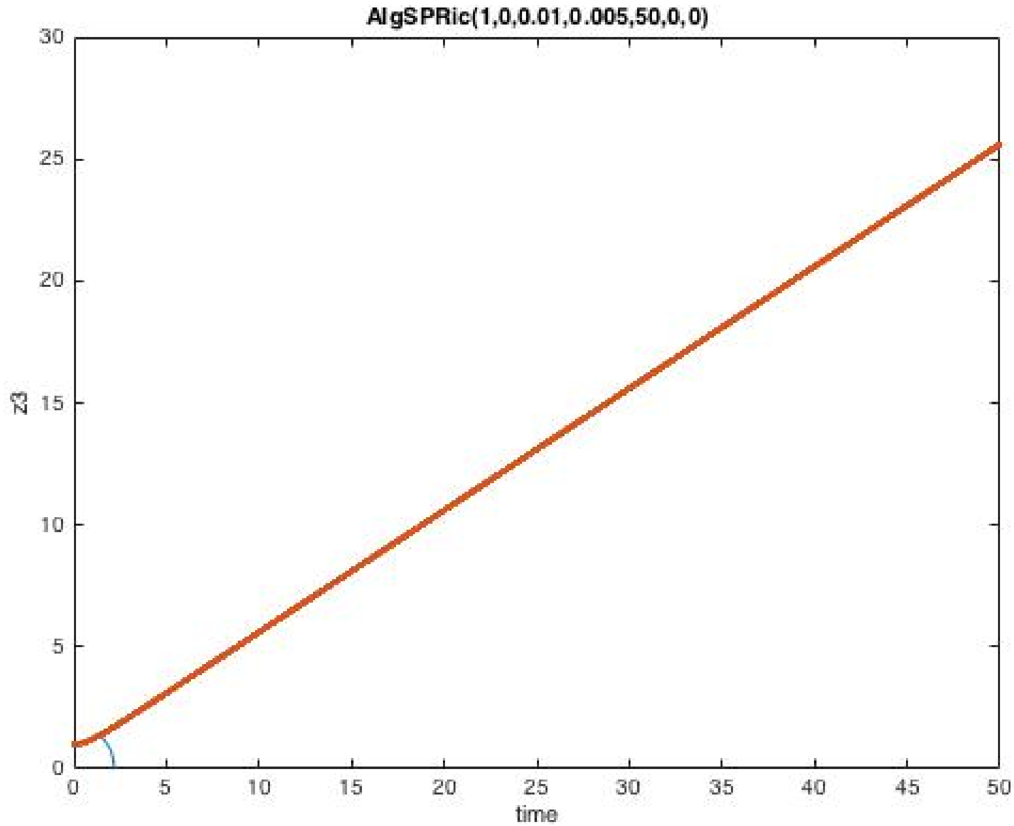

Figure 49: Ricci-flat Case of $Sp(3)/Sp(1)^3$ with $z_3, a = 1$

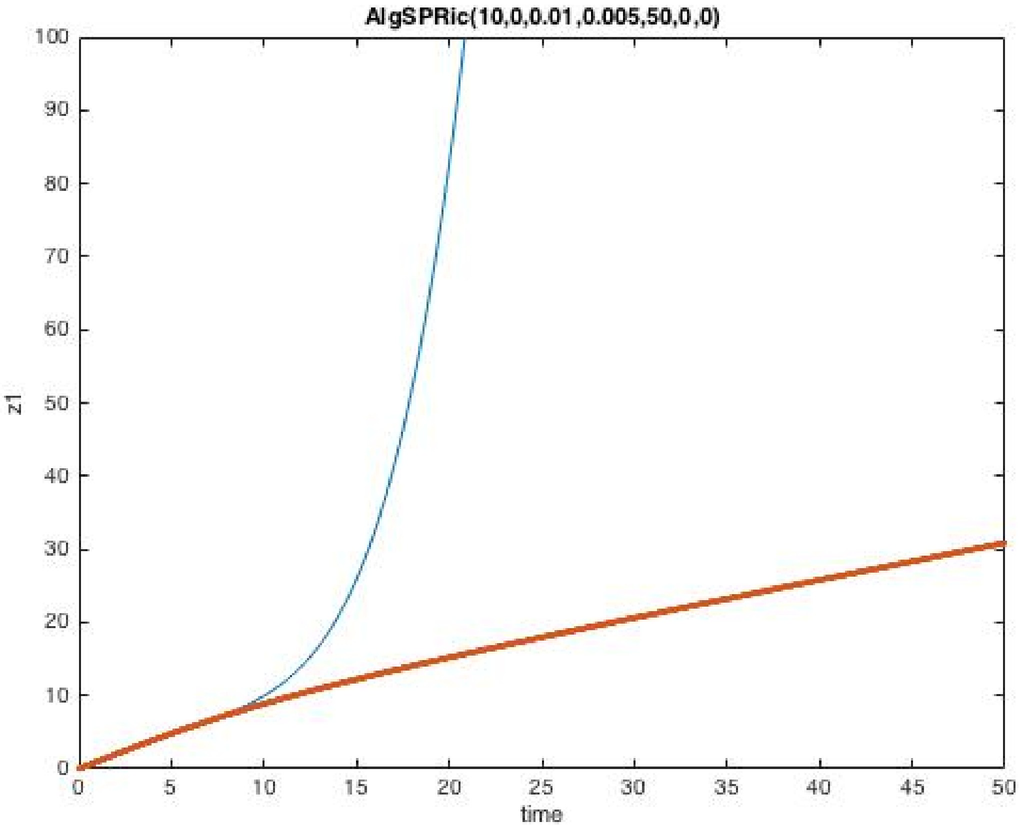

Figure 50: Ricci-flat Case of $Sp(3)/Sp(1)^3$ with $z_1, a = 10$

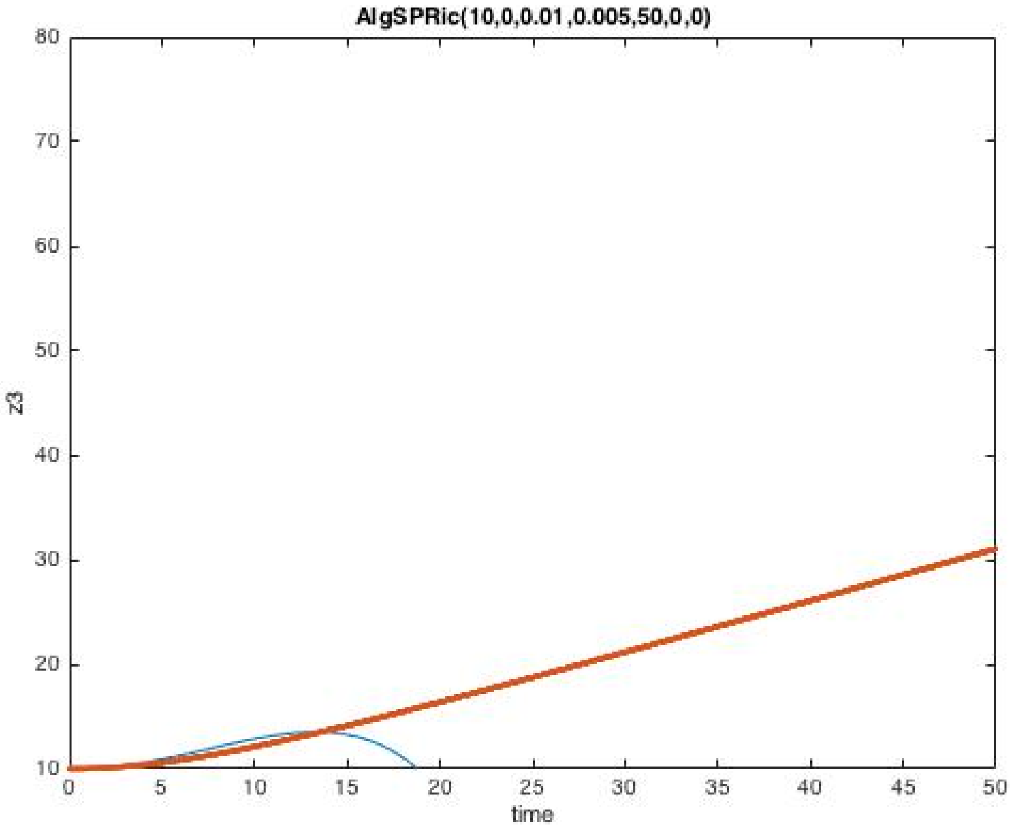

Figure 51: Ricci-flat Case of $Sp(3)/Sp(1)^3$ with $z_3, a = 10$

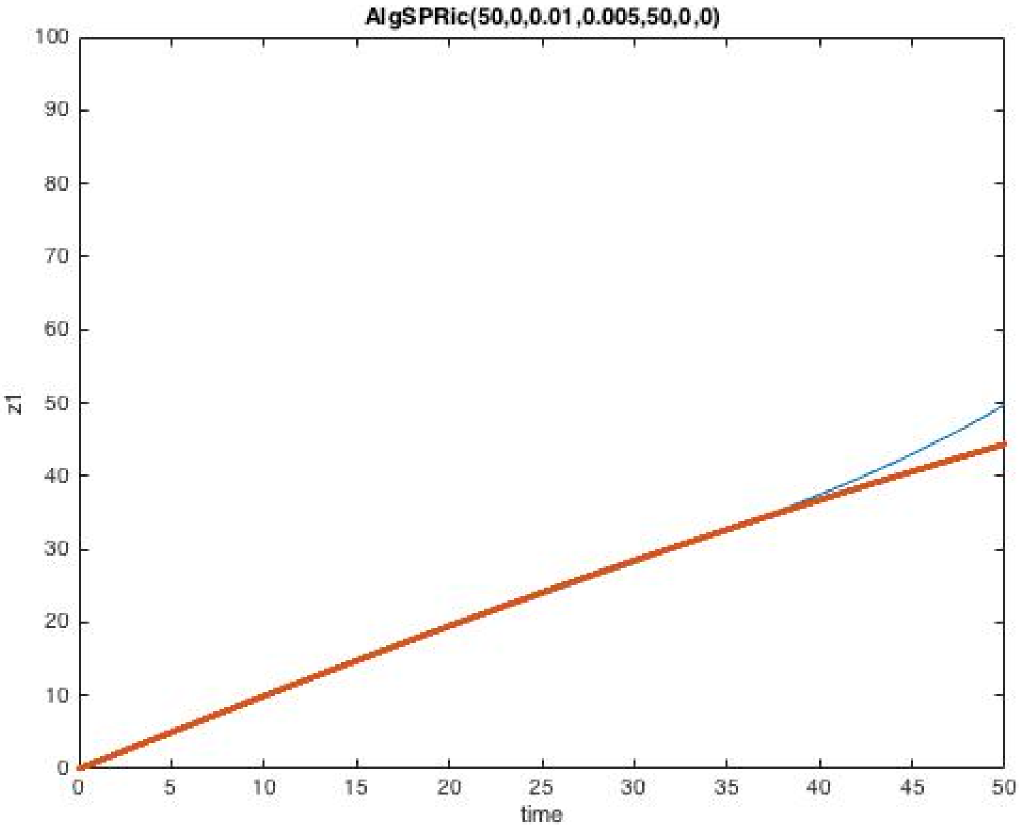

Figure 52: Ricci-flat Case of $Sp(3)/Sp(1)^3$ with $z_1, a = 50$

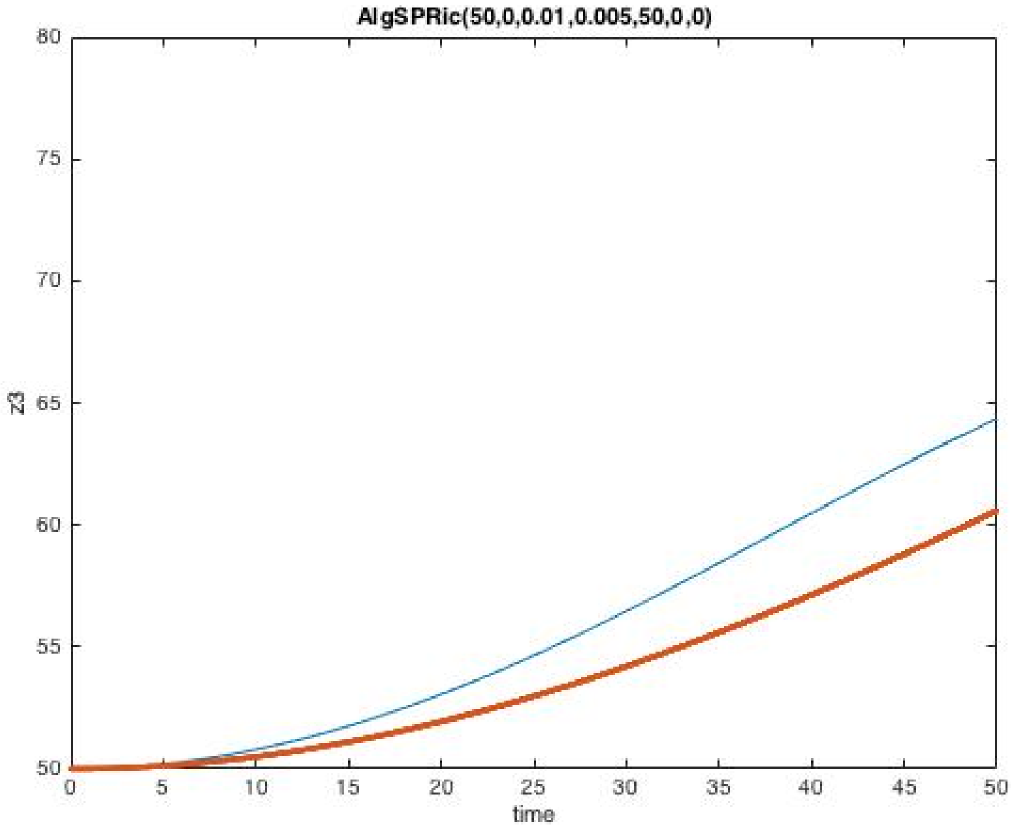

Figure 53: Ricci-flat Case of $Sp(3)/Sp(1)^3$ with $z_3, a = 50$

10.3 Expanding Solitons

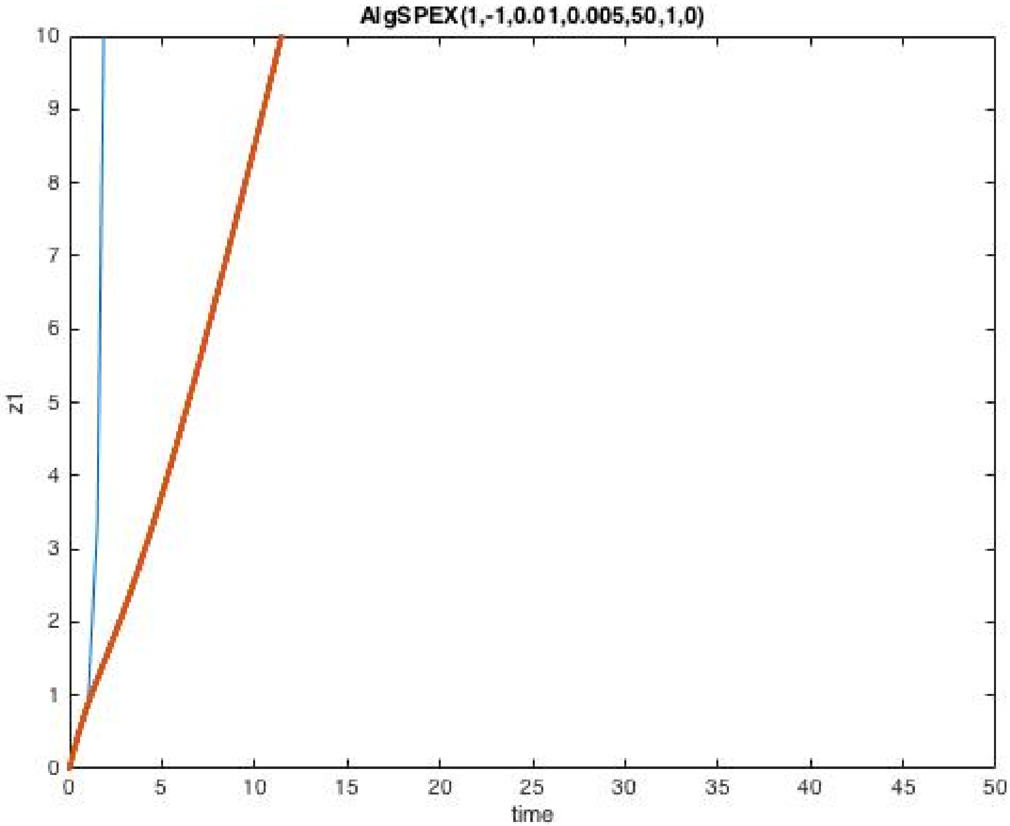

Figure 54: Expanding Case of $Sp(3)/Sp(1)^3$ with $z_1, a = 1, b = -1$

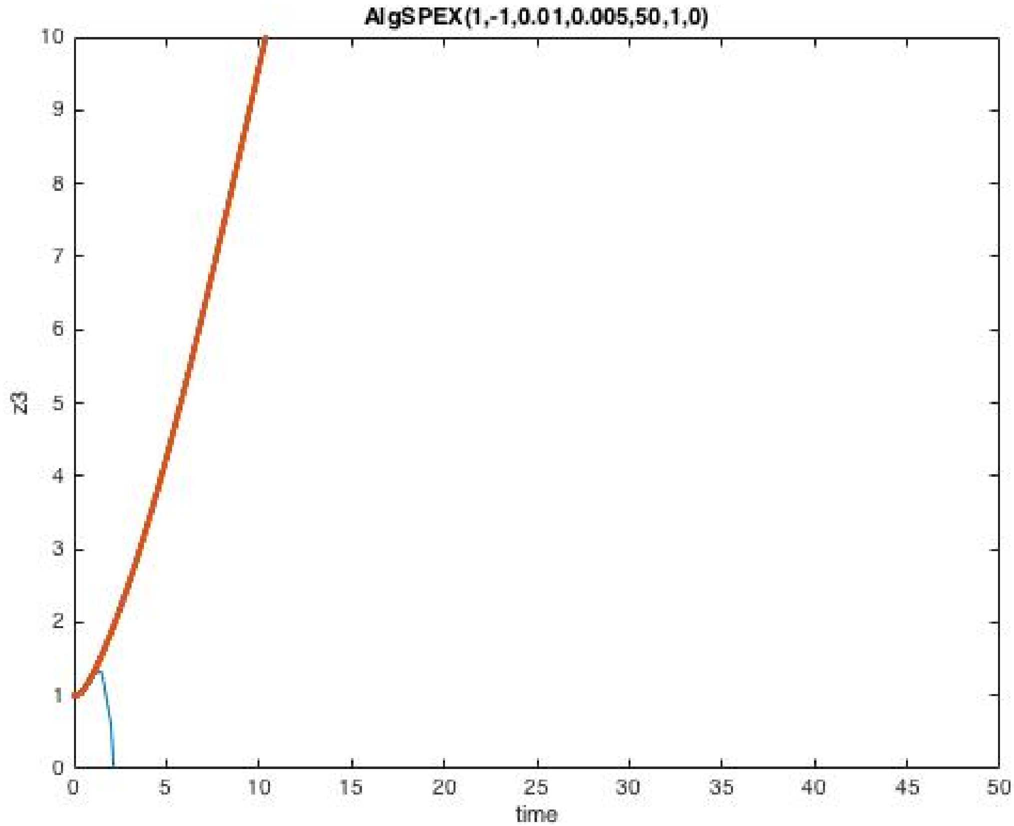

Figure 55: Expanding Case of $Sp(3)/Sp(1)^3$ with $z_3, a = 1, b = -1$

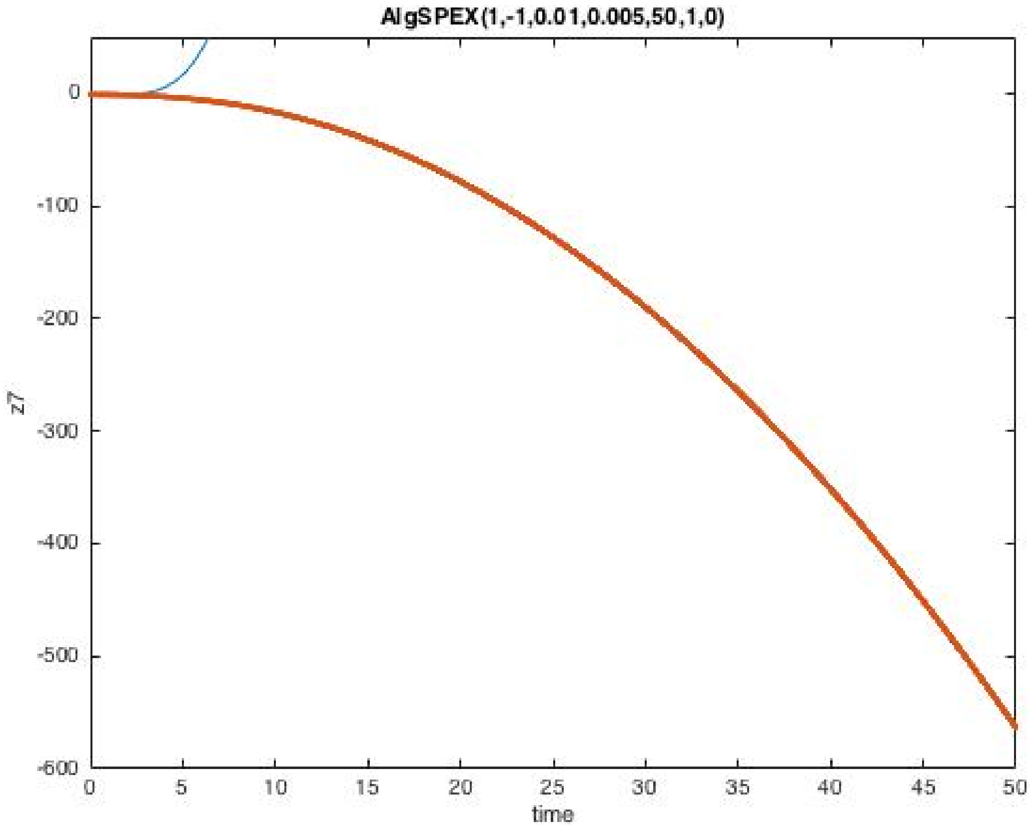

Figure 56: Expanding Case of $Sp(3)/Sp(1)^3$ with $z_7, a = 1, b = -1$

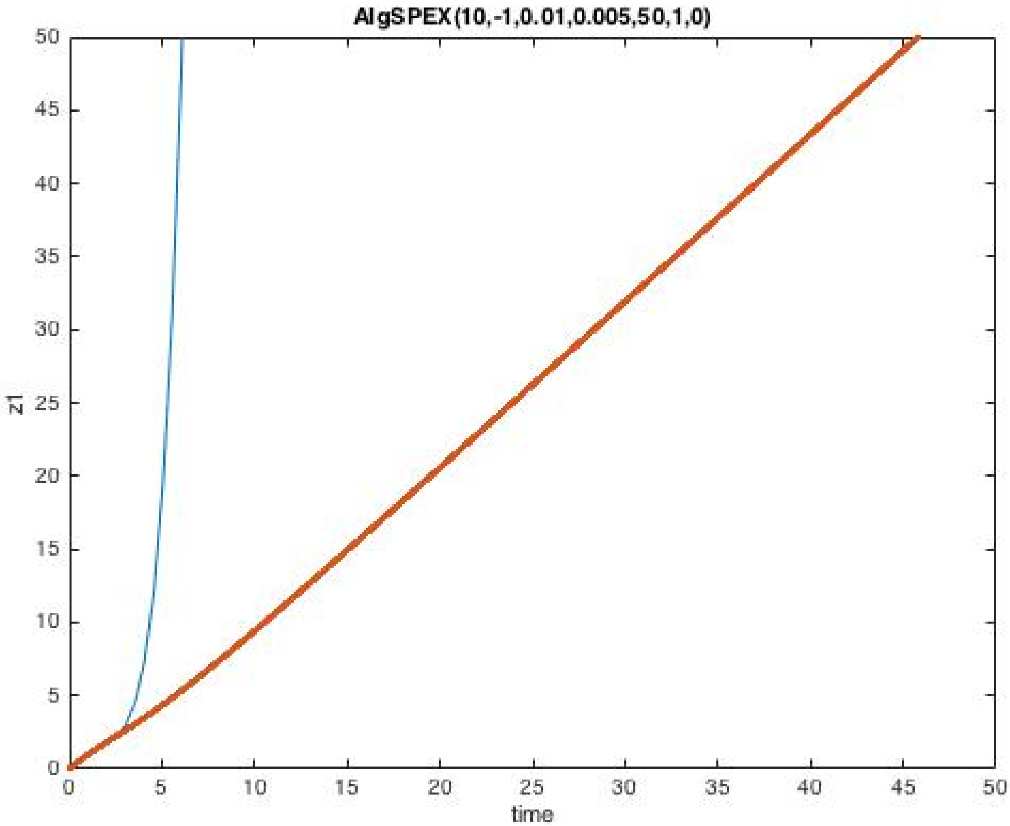

Figure 57: Expanding Case of $Sp(3)/Sp(1)^3$ with $z_1, a = 10, b = -1$

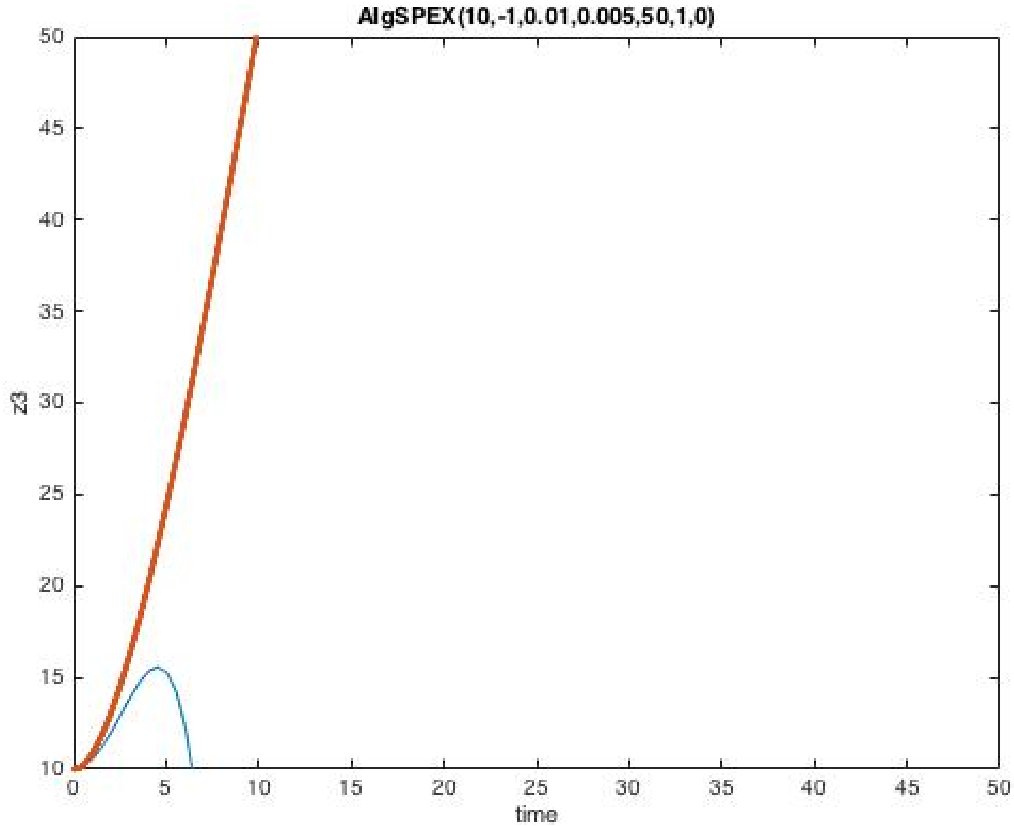

Figure 58: Expanding Case of $Sp(3)/Sp(1)^3$ with $z_3, a = 10, b = -1$

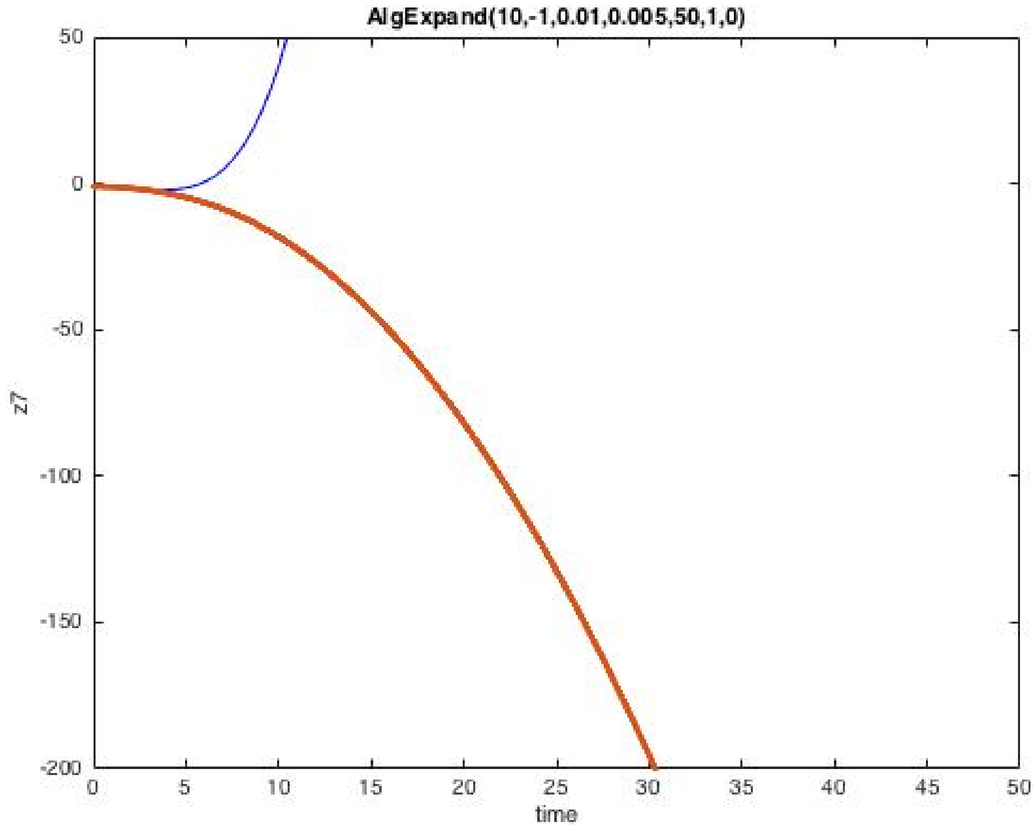

Figure 59: Expanding Case of $Sp(3)/Sp(1)^3$ with $z_7, a = 10, b = -1$

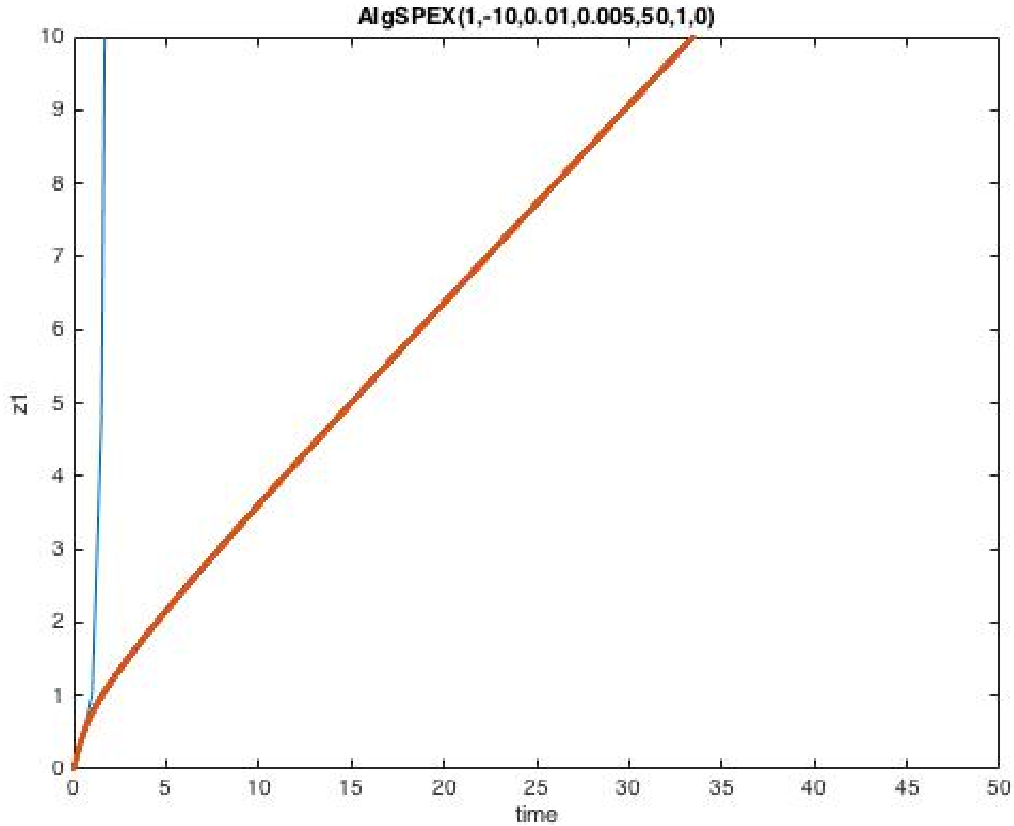

Figure 60: Expanding Case of $Sp(3)/Sp(1)^3$ with $z_1, a = 1, b = -10$

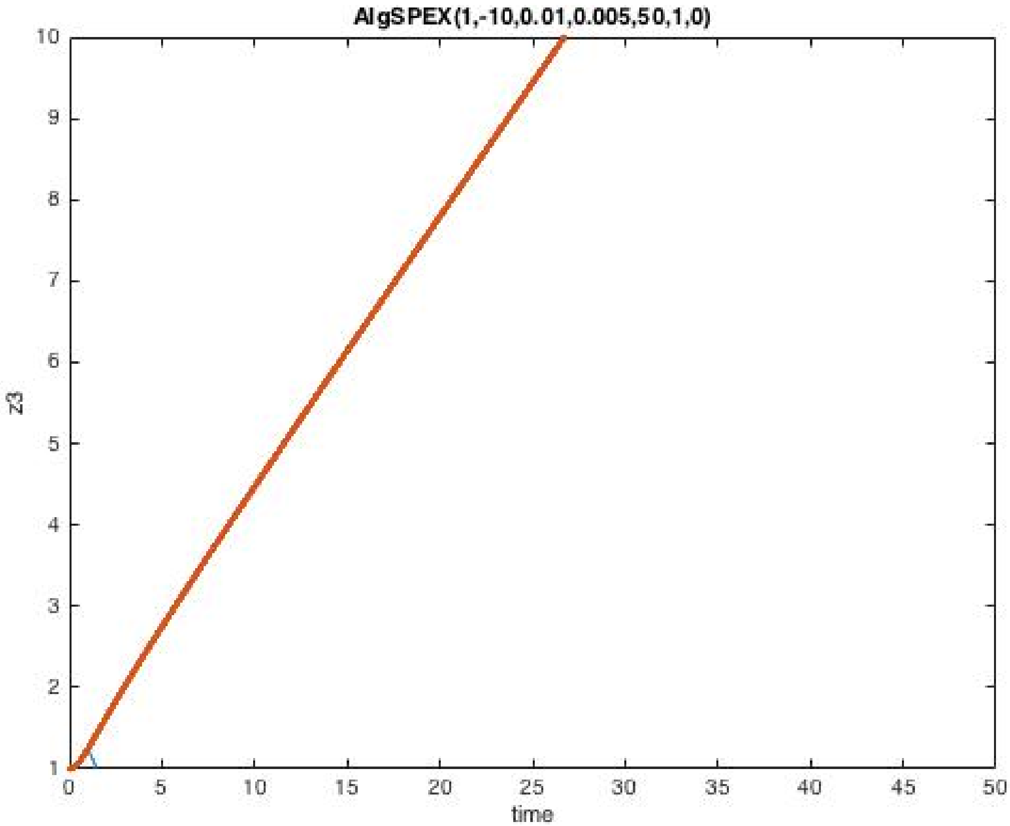

Figure 61: Expanding Case of $Sp(3)/Sp(1)^3$ with $z_3, a = 1, b = -10$

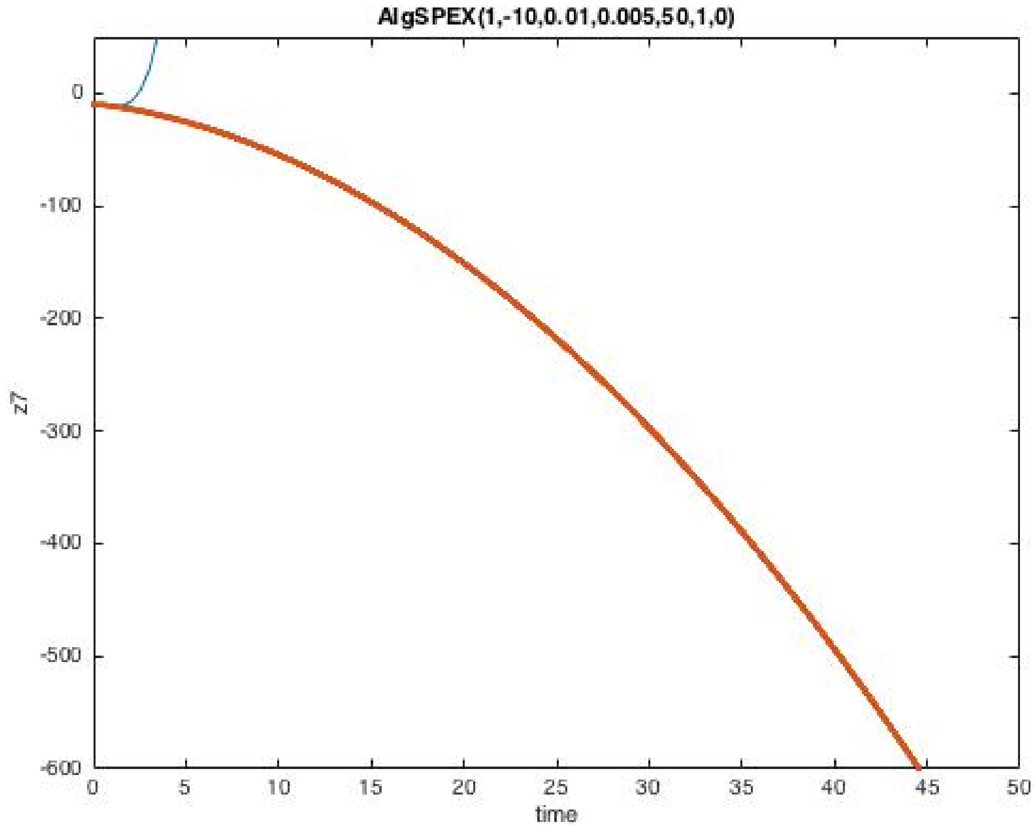

Figure 62: Expanding Case of $Sp(3)/Sp(1)^3$ with $z_7, a = 1, b = -10$

10.4 Negative Einstein

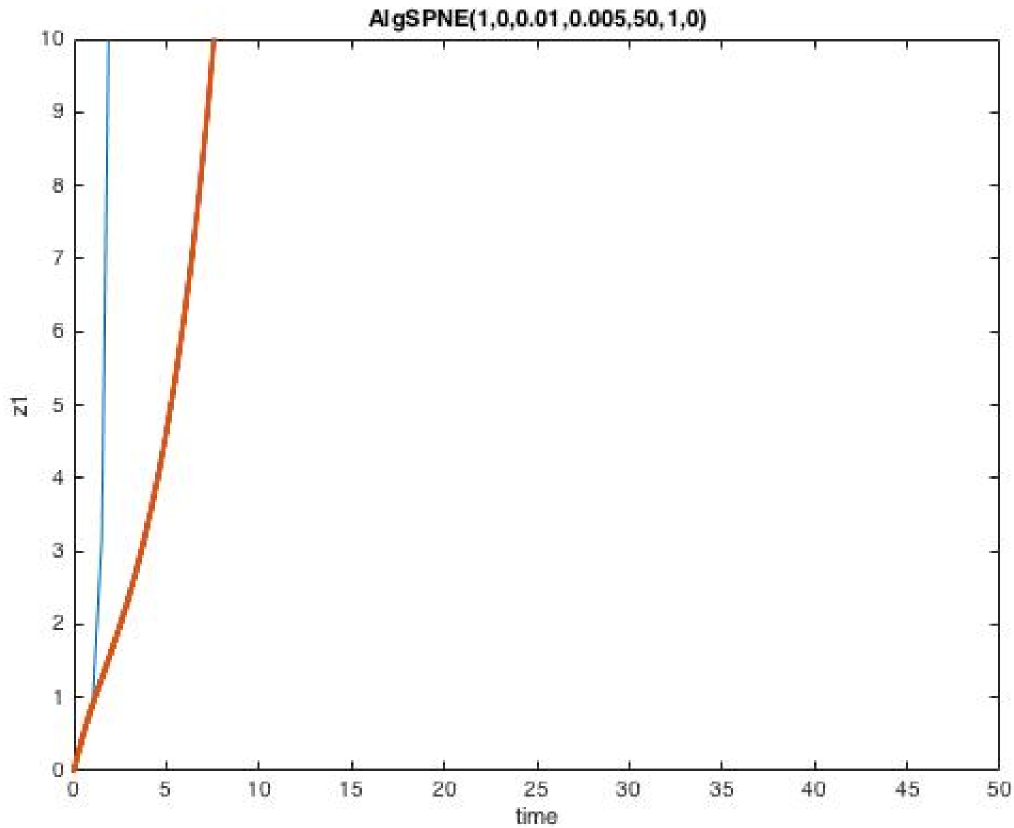

Figure 63: Negative Einstein of $Sp(3)/Sp(1)^3$ with $z_1, a = 1, b = 0$

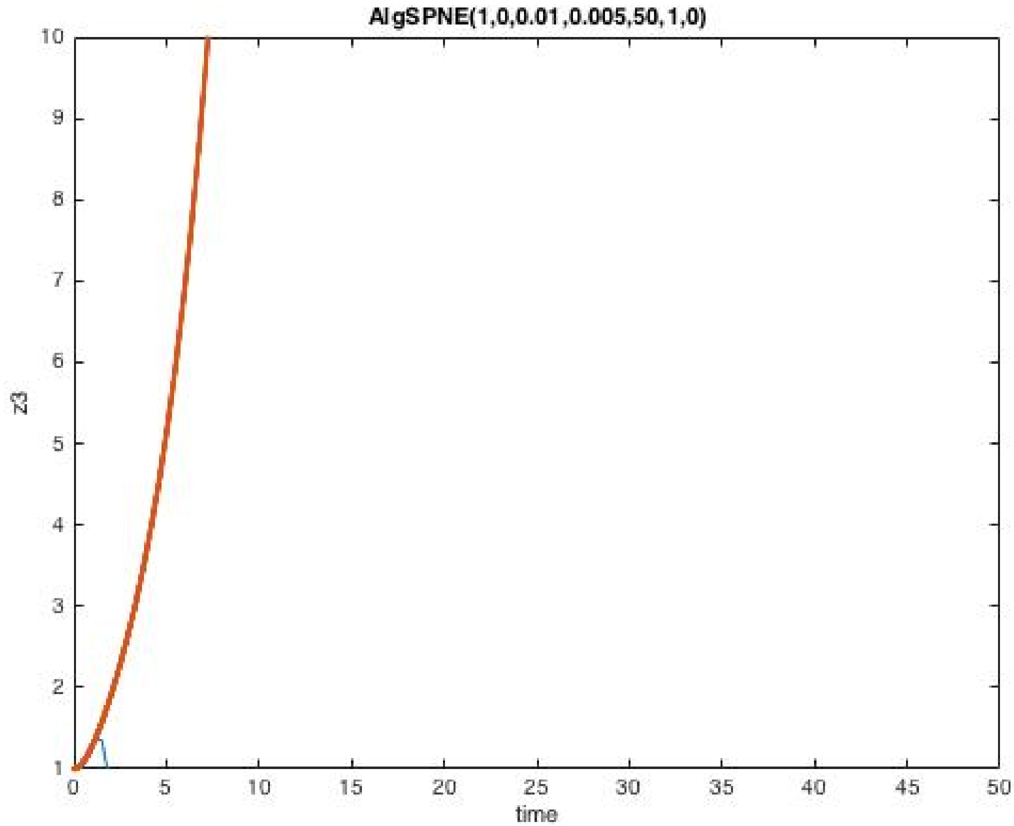

Figure 64: Negative Einstein of $Sp(3)/Sp(1)^3$ with $z_3, a = 1, b = 0$

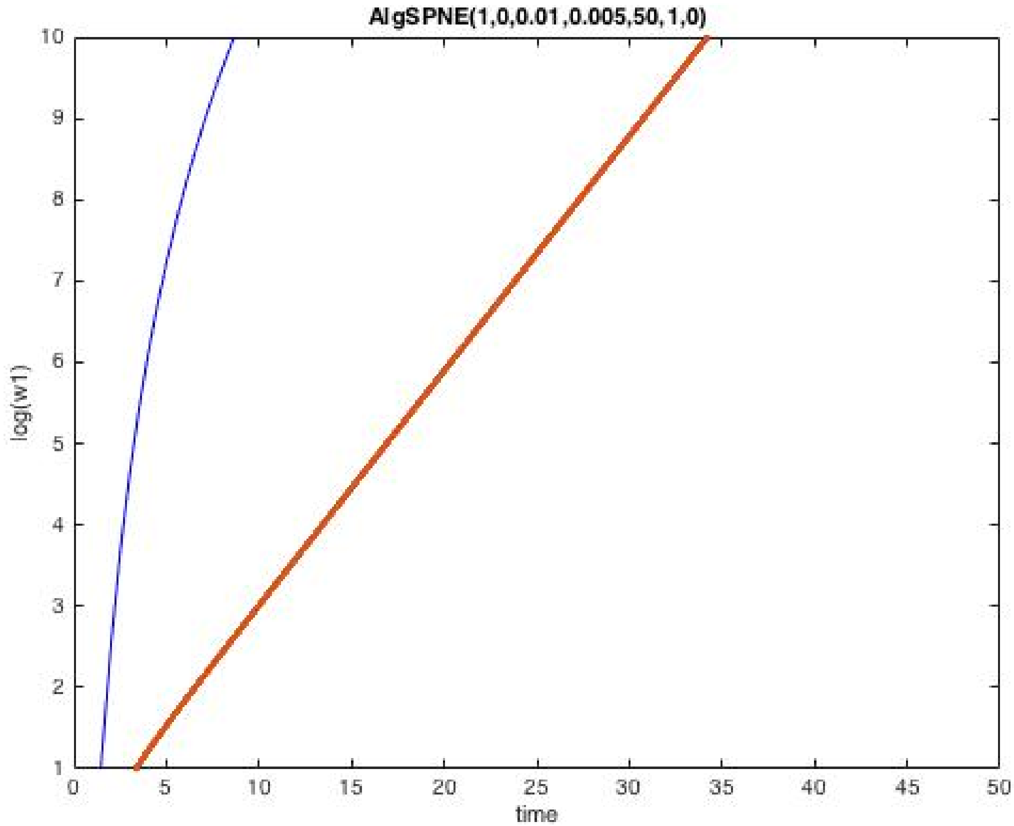

Figure 65: Log Graph of negative Einstein case of $Sp(3)/Sp(1)^3$ with $z_1, a = 1, b = 0$

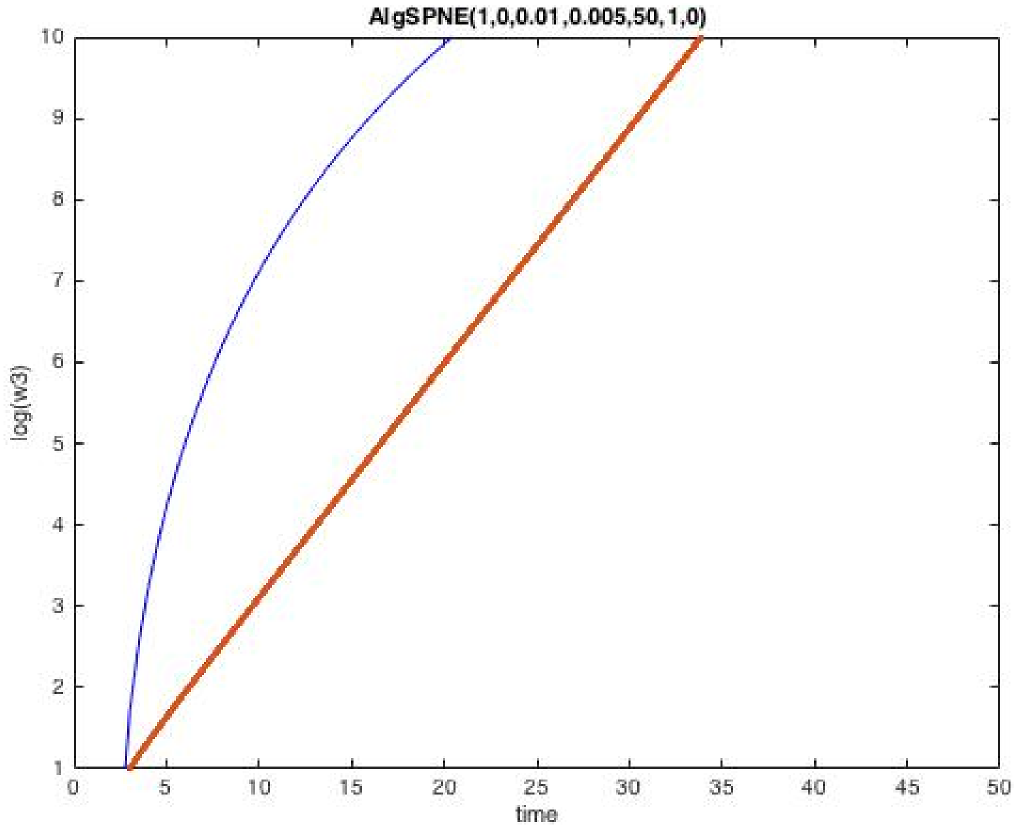

Figure 66: Log Graph of negative Einstein case of $Sp(3)/Sp(1)^3$ $z_3, a = 1, b = 0$

10.5 Compact and non-compact Shrinking soliton

There are (currently) no results for the non-compact shrinking case, but we have results for the Compact Einstein solitons

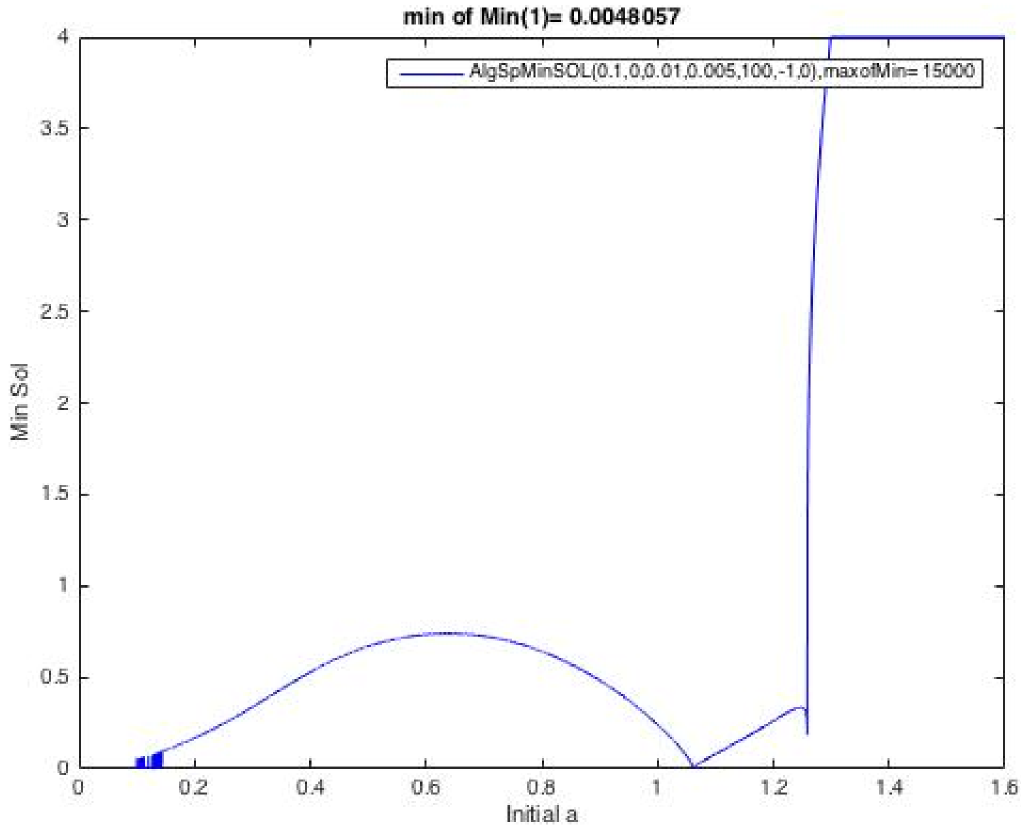

Figure 67: Compact Einstein case of $Sp(3)/Sp(1)^3$. Graph of SOL vs initial value a

This graph is used to locate the Einstein metrics described earlier in this chapter. As the human eye can see, there is a cluster of Einstein metrics near 0 and a more interesting point to the right of the cluster. Zooming into the region of interest on $[1, 1.2]$,

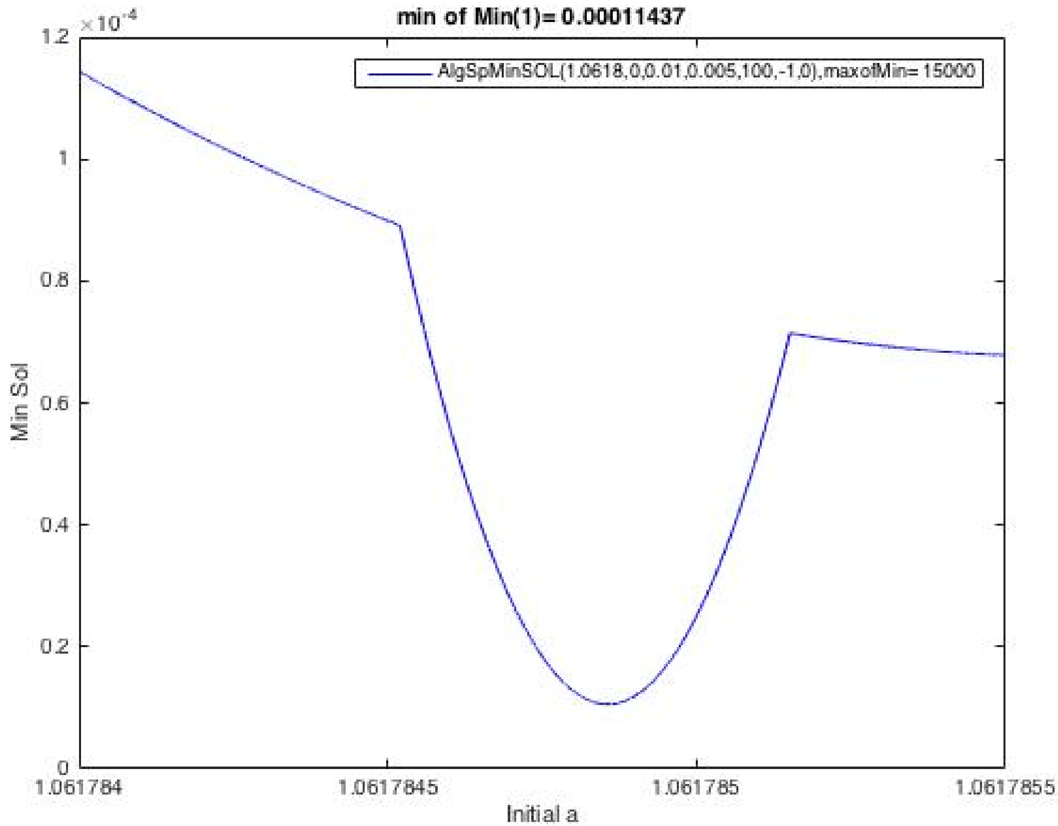

Figure 68: Compact Einstein case of $Sp(3)/Sp(1)^3$. Graph of SOL vs initial value a

Remark 10.1 *We estimate the point of interest at* $a = 1.0618$*. Note this is not a true value of the Einstein metric as we cannot numerically determine when* $SOL = 0$*. Hence we are only drawing values in a neighborhood of the Einstein metric which should have similar features. Interestingly this point is very close to the value for* $SU(3)/T^2$*. We suspect they may lie in the same neighborhood.*

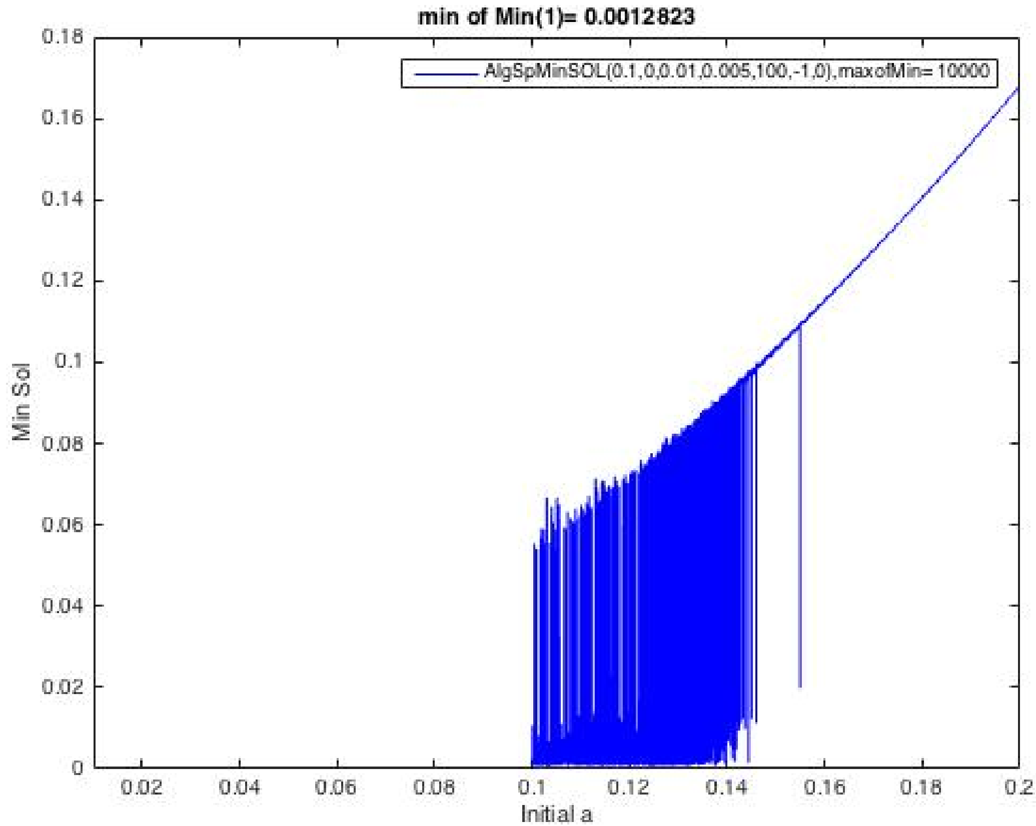

Figure 69: Cluster of figure 67

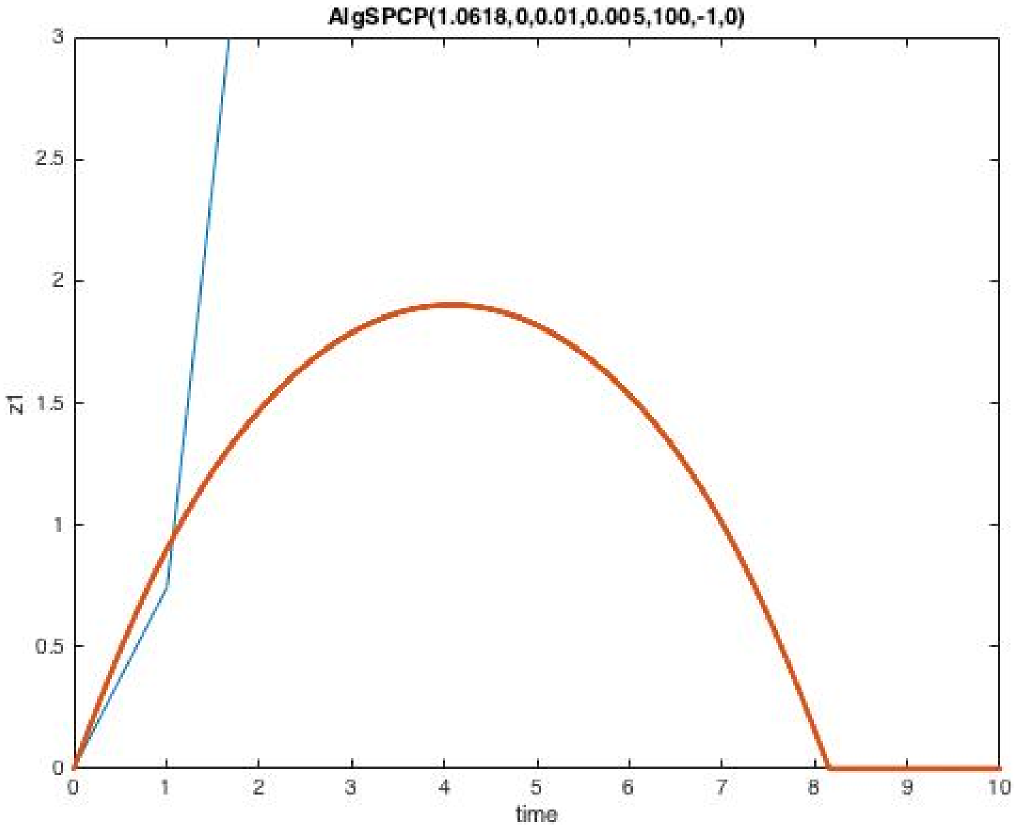

Figure 70: Compact Einstein case of $Sp(3)/Sp(1)^3$ with z_1, $a = 1.0619, b = 0$

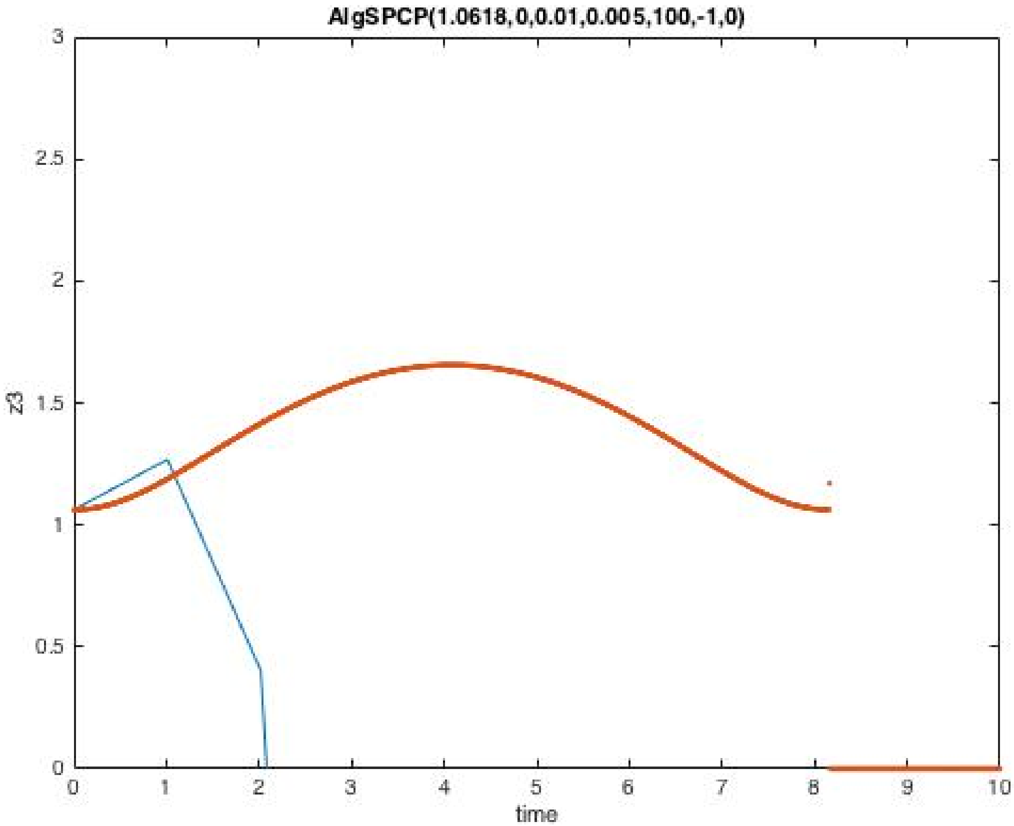

Figure 71: Compact Einstein case of $Sp(3)/Sp(1)^3$ with z_3, $a = 1.0619, b = 0$

As expected the first graph of $z_1 = f_1$ closes up on the boundary point and the graph of $z_3 = f_2$ shows a negative slope.

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