HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

By

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree

Master of Science

McMaster University

September 1974

MASTER OF SCIENCE (1974) (Mathematics)

McMASTER UNIVERSITY Hamilton, Ontario

TITLE: Hardy-Littlewood Maximal Functions

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NUMBER OF PAGES: (vi), 76

SCOPE AND CONTENTS: The principal object of this study is to find weak and strong type estimates concerning functions in weighted L^p spaces and their maximal functions. We also apply these results to the study of convolution integrals.

ACKNOWLEDGEMENT

I would like to express my sincerest appreciation to my supervisor, Dr. Hans Heinig, for the invaluable guidance and encouragement he has given me during the preparation of this thesis.

Thanks are also due to McMaster University for their generous financial support that I have received as a graduate student.

Finally, I am indebted to Mrs. Linda Self who devoted her time and effort in the typing of this thesis.

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INTRODUCTION

The Hardy-Littlewood Maximal Functions are of considerable importance in the study of certain classes of functions convoluted with specific kernels, since they majorize many of these convolutions. This motivated the research in this area.

Chapter 1 collects and extends a number of inequalities involving the maximal functions in certain weighted L^p spaces.

In the second chapter, inequalities between the convolution of functions in weighted L^p spaces, $1 \leq p < \infty$, with certain kernels, and their maximal functions are studied. In the last half of Chapter 2, we study the case $0 < p < 1$, and collect the results of Chapter 1 and those of the first half of Chapter 2.

Finally, in the last chapter, the main result, Theorem 3.3, is proved, extending the original result of Fefferman and Stein.

CHAPTER I

Section A. Basic Definitions

The following is a list of standard notation and concepts that will be used throughout. Further notation and definitions will be introduced when needed.

$\mathbb{R} = (-\infty, \infty)$ denotes the set of real numbers, and $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^- = (-\infty, 0]$. $\mathbb{N}$ will be the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, while $\mathbb{T} = [0, 2\pi)$ denotes the "circle group". In the case of periodic functions on $\mathbb{R}$, with period 2π, $\mathbb{T}$ will be any interval of length 2π. If E is any given set, then E^n is the Cartesian product of n copies of E, $n \in \mathbb{N}$, thus $\mathbb{R}^{+n}$ is the product of n copies of $\mathbb{R}^+$ and $\mathbb{R}^n$ is the "n-dimensional Euclidean space". If $E \subseteq \mathbb{R}$, then $\lambda(E)$ denotes the Lebesgue measure of E, and $\int_E f(t)d\lambda(t) \equiv \int_E f(t)dt$ is the Lebesgue integral of a function f defined on E, whenever the integral is well defined. To denote any arbitrary positive measure on a measure space X, we use the symbols μ and σ, and if two functions, f and g, on X are equal except for a set of μ-measure zero, we say $f = g$ μ-a.e. On the n-product of the measure space X, ie X^n, we denote the product measure

$\mu x \mu x \ldots x \mu$ simply by μ if no confusion arises. Sets or functions that are Lebesgue measurable will be said to be measurable, and λ-a.e. will be abbreviated to a.e.

We define the following functions and operations: For $E \subseteq X$, we define the characteristic function, χ_E, by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \quad E \\ 0 & \text{if } x \notin E \end{cases}$$

Also, E^c denotes the complement of E in the (whole) space X. If E and E_1 are subsets at X, then $E_1 \setminus E$ will denote the set $\{x : x \in E_1 \text{ and } x \notin E\}$, while $\overset{\circ}{E}$ is the interior of E in the space X. For limit, we write lim, while $\overline{\lim}$ and $\underline{\lim}$ are the limit superior and limit inferior, respectively. Finally C(X) denotes the space of continuous functions on X. A, together with any subscripts, will denote a constant depending on indicated parameters, and is possibly different at each occurrence.

With these conventions, the spaces $L^p(X, d\lambda(x)) \equiv L^p(X)$ of Lebesgue-locally integrable functions on the set X are defined as follows:

<u>Definition 1.1</u>: If $0 < p \leq \infty$, then $f \in L^p(X)$ if and only if $\|f\|p < \infty$, where

$$\|f\|_p = \begin{cases} \left(\int_X |f(t)|^p dt\right)^{1/p}, & 0 < p < \infty \\ \operatorname*{ess\,sup}_{x \in X} |f(x)|, & p = \infty . \end{cases}$$

Here,

$$\operatorname*{ess\,sup}_{x\in X} |f(x)| = \inf\{h : \int_{\{x\ :\ |f(x)| > h\}} dx = 0,\ h > 0\} .$$

Although we speak of functions in $L^p(X)$, the fact is that we really mean equivalence classes of functions modulo a set of measure zero. This abuse of language is found widely in the literature, and will not be rectified here.

For $1 \leqslant p \leqslant \infty$, $\|\cdot\|_p$ defines a norm, under which $L^p(X)$ is a Banach space. For $0 < p < 1$, $\|\cdot\|_p^p$ is a metric, under which $L^p(X)$ is a Frechet space. These and other properties can be found, for example, in Royden [1], or Hewitt and Stromberg [8].

For any given real or complex valued function defined on X, let

$$E_h(f) = \{x : x \in X \text{ and } |f(x)| > h,\ h > 0\}$$

We now choose and fix a function $\emptyset$ such that it is measurable, non-negative and non-zero a.e. on X. This will be one of our weight functions in the sequel.

Definition 1.2: If $0 < p \leq \infty$, then define $L^p(X, \phi(t)dt) \equiv L^p(X,\phi)$ to be the class of (equivalence classes of) functions on X for which $\|f\|_{p,\phi} < \infty$, where

$$\|f\|_{p,\phi} = \begin{cases} \left(\int_X |f(t)|^p \phi(t)\,dt \right)^{1/p} & 0 < p < \infty \\ \operatorname*{ess\,sup}_{x \in X} |f(x)| & p = \infty \end{cases}$$

and

$$\operatorname*{ess\,sup}_{x \in X} f(x) = \inf\left\{ h : h > 0 \text{ and } \int_{E_h(f)} \phi(t)\,dt = 0 \right\}.$$

If $\phi(x) \equiv 1$ then these classes reduce of course to the Lebesgue spaces, $L^p(X)$.

Definition 1.3: If f is a locally integrable, non-negative function on $\mathbb{R}$, then define the following functions, wherever they exist:

$$f_1^*(x) = \sup_{\xi < x} \frac{1}{x-\xi} \int_\xi^x f(t)\,dt$$

$$f_2^*(x) = \sup_{x < \xi} \frac{1}{x-\xi} \int_\xi^x f(t)\,dt$$

$$f_0^*(x) = \max \{f_1^*(x),\ f_2^*(x)\}.$$

$f_0^*(x)$ is called the Hardy-Littlewood maximal function of f, and f_1^* and f_2^* are called the left and right maximal functions of f, respectively. Thus $\emptyset_i^*$, $i = 0,1,2$ denotes the respective maximal functions of the weight function $\emptyset$, wherever they exist.

In a manner analogous to Definition 1.2, we define for an arbitrary positive measure μ on a measure space X the classes $L^p(X, d\mu(x))$, $0 < p \leq \infty$. Examples of these classes are $L^p(X, \emptyset_i^*(x)dx) = L^p(X, \emptyset_i^*)$.

The proofs of the following three theorems show the close analogy that exists between weighted spaces and the Lebesgue spaces.

Theorem 1.4: (Holder's Inequality) If $p,q \geq 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, and if $f \in L^p(x,\emptyset)$, $g \in L^q(X,\emptyset)$ then the pointwise product $f \cdot g \in L^1(X,\emptyset)$ and

$$\|f \cdot g\|_{1,\emptyset} \leq \|f\|_{p,\emptyset}\|g\|_{q,\emptyset}.$$

Proof: If $p = \infty$ or $q = \infty$, then the theorem follows immediately.

If $1 < p < \infty$, apply Holder's Inequality for $L^p(X)$ as follows:

$$\int_X |f(t)g(t)|\,\phi(t)dt = \int_X |f(t)|\,\phi(t)^{1/p}\,|g(t)|\,\phi(t)^{1/q}dt$$
$$\leqslant \left(\int_X |f(t)|^p\phi(t)dt\right)^{1/p}\left(\int_X |g(t)|^q\phi(t)dt\right)^{1/q}.$$

Since the right side of this inequality is finite, by assumption, the use of Holder's Inequality is justified.

Theorem 1.5: (Minkowski's Inequality) If $f, g \in L^p(X,\phi)$ $1 \leqslant p \leqslant \infty$, then so is $f+g$, and

$$\|f+g\|_{p,\phi} \leqslant \|f\|_{p,\phi}+\|g\|_{p,\phi}.$$

Proof: The cases $p = 1$ and $p = \infty$ are obvious.

For $1 < p < \infty$, and any $x \in X$

$$|f(x)+g(x)|^p \leqslant 2^p(|f(x)|^p+|g(x)|^p)$$

so that $f+g \in L^p(x,\phi)$. Furthermore,

$$\int_X |f(t)+g(t)|^p\phi(t)dt \leqslant \int_X |f(t)+g(t)|^{p-1}|f(t)|\,\phi(t)dt$$
$$+\int_X |f(t)+g(t)|^{p-1}|g(t)|\,\phi(t)dt.$$

An application of Holder's Inequality shows that:

$$\int_X |f(t)+g(t)|^{p-1}|f(t)|\,\phi(t)dt \leqslant \|f\|_{p,\phi}\,\| |f+g|^{p-1}\|_{q,\phi}$$

and

$$\int_X |f(t)+g(t)|^{p-1} |g(t)| \phi(t)dt \leqslant \|g\|_{p,\phi} \| |f+g|^{p-1} \|_{q,\phi},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. But observe that

$$\| |f+g|^{p-1} \|_{q,\phi} = (\|f+g\|_{p,\phi})^{p/q}.$$

Hence, the above inequalities imply

$$\|f+g\|^p_{p,\phi} \leqslant (\|f\|_{p,\phi} + \|g\|_{p,\phi})(\|f+g\|_{p,\phi})^{p/q}$$

from which

$$\|f+g\|_{p,\phi} \leqslant \|f\|_{p,\phi} + \|g\|_{p,\phi}$$

follows.

<u>Theorem 1.6</u>: (Fatou's Lemma) Let $\{f_n\}_{n\geqslant 1}$ be a sequence of non-negative ϕ-measurable functions which converge a.e. on a set E to a function f. Then

$$\int_E f(t)\phi(t)dt \leqslant \underline{\lim}_{n\to\infty} \int_E f_n(t)\phi(t)dt.$$

<u>Proof</u>: By the definition of the integral of a non-negative function with respect to the measure $\phi(t)dt$, that is

$$\int_E f(t)\phi(t)dt = \sup_{\varphi}\int_E \varphi(t)\phi(t)dt$$

where φ ranges over all simple functions such that $0 \leq \varphi \leq f$, it is sufficient to show that for any such φ,

$$\int_E \varphi(t)\phi(t)dt \leq \underline{\lim}_{n\to\infty} \int_E f_n(t)\phi(t)dt.$$

If $\int_E \varphi(t)\phi(t)dt = \infty$, then there is a ϕ-measurable set $B \subseteq E$, and a constant $a > 0$ with $\int_B \phi(t)dt = \infty$ and $\varphi > a$ on B. If

$$B_n = \{x \in E : f_k(x) > a \text{ for all } k \geq n\},$$

then $\bigcup_{n=1}^{\infty} B_n \supseteq B$, since $\varphi \leq \underline{\lim}_{n\to\infty} f_n$, and $B_n \subseteq B_{n+1}$.

Therefore, $\lim_{n\to\infty}\int_{B_n} \phi(t)dt = \infty$, from which it follows that $\int_E f_n(t)\phi(t)dt = \infty$, since

$$\int_E f_n(t)\phi(t)dt \geq a\int_{B_n} \phi(t)dt.$$

If $\int_E \varphi(t)\phi(t)dt < \infty$, then there is a ϕ-measurable set $B \subseteq E$ such that $\int_B \phi(t)dt < \infty$ and $\varphi \equiv 0$ on B^c. Let $M = \max \varphi(x)$, choose $\varepsilon > 0$, and let

$$B_n = \{x \in E : f_k(x) > (1-\varepsilon)\varphi(x) \text{ for all } k \geqslant n\}$$

It is clear that $B_n \subseteq B_{n+1}$ and that $B \subseteq \bigcup_{n=1}^{\infty} B_n$ and hence $\{B \setminus B_n\}_{n=1}^{\infty}$ is decreasing with empty intersection. We recall that, if $\{E_i\}_{i=1}^{\infty}$ is a sequence of μ-measurable sets such that $E_i \supseteq E_{i+1}$, in some space X, with $\mu(E_1) < \infty$, then $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{i \to \infty} \mu(E_i)$, which is proved in, e.g., Royden [11, pp. 218 - 219]. It follows then that $\lim_{n \to \infty} \int_{B \setminus B_n} \phi(t)\,dt = 0$.

If n is chosen so large that for all $k \geqslant n$, $\int_{B \setminus B_n} \phi(t)\,dt < \varepsilon$, then

$$\begin{aligned}
\int_E f_k(t)\phi(t)\,dt &\geqslant \int_{B_k} f_k(t)\phi(t)\,dt \\
&\geqslant (1-\varepsilon)\int_{B_k} \varphi(t)\phi(t)\,dt \\
&\geqslant (1-\varepsilon)\int_E \varphi(t)\phi(t)\,dt - \int_{B \setminus B_k} \varphi(t)\phi(t)\,dt \\
&\geqslant \int_E \varphi(t)\phi(t)\,dt - \varepsilon\left[\int_E \varphi(t)\phi(t)\,dt + M\right]
\end{aligned}$$

Thus

$$\underline{\lim}_{n \to \infty} \int_E f_n(t)\phi(t)\,dt \geqslant \int_E \varphi(t)\phi(t)\,dt - \varepsilon\left[\int_E \varphi(t)\phi(t)\,dt + M\right]$$

which implies the theorem.

Remark 1: In Fatou's Lemma, the limit function, f, can be replaced by $\underline{\lim}_{n\to\infty} f_n$. Fatou's Lemma would then state that

$$\int_E \underline{\lim}\, f_n(t)\phi(t)dt \leqslant \underline{\lim}\int_E f_n(t)\phi(t)dt.$$

The alternate form of Fatou's Lemma, which follows readily from the above, is

$$\overline{\lim}\int_E f_n(t)\phi(t)dt \leqslant \int_E \overline{\lim}\, f_n(t)\phi(t)dt.$$

Remark 2: Minkowski's Inequality essentially shows that $\|\cdot\|_{p,\phi}$ and $\|\cdot\|_{p,\phi_i^*}$ are norms for $1 \leqslant p \leqslant \infty$ on their respective spaces. Moreover, one can show that these spaces are complete, and hence are Banach spaces.

Section B. Basic Theorems

For the sake of completeness, some useful theorems applied in the sequel are stated or proved in this section. These include the Calderon-Zygmund Lemma (or Riesz Rising Sun Lemma) and a theorem sometimes referred to as Chebyshev's Inequality.

Theorem 1.7: (Calderon-Zygmund Lemma [2, pp. 85 - 139])

If $y > 0$, and f is a non-negative function, integrable on $\mathbb{R}^n$, there exists a set of cubes, $I_k \subseteq \mathbb{R}^n$, $k \in \mathbb{N}$, such that, for $I = \bigcup_{k=1}^{\infty} I_k$, we have

1) $f(x) \leqslant y$ for $x \in I^c$;
2) $\mathring{I}_k \cap \mathring{I}_j$ is empty, $k \neq j$;
3) $y < \frac{1}{\lambda(I_k)} \int_{I_k} f(x)dx \leqslant 2^n y$

<u>Proof</u>: Decompose $\mathbb{R}^n$ into a mesh of cubes of common diameter, and disjoint interiors, such that, for each cube, I', in the mesh,

$$\frac{1}{\lambda(I')} \int_{I'} f(x)dx \leqslant y$$

Let I' be a fixed cube in the mesh, and divide it into 2^n cubes by bisecting the sides of I'. Then, for each new cube $I_i \subset I'$, either

$$\frac{1}{\lambda(I_i)} \int_{I_i} f(x)dx \leqslant y \quad \text{or} \quad \frac{1}{\lambda(I_i)} \int_{I_i} f(x)dx > y.$$

If the second alternative holds, the cube I_i is chosen to satisfy the theorem. If the first alternative holds, the process is repeated. Obviously, the I_i chosen satisfy condition 2). Also, since

$$y < \frac{1}{\lambda(I_i)} \int_{I_i} f(x)dx \leq \frac{2^n}{\lambda(I')} \int_{I'} f(x)dx \leq 2^n y,$$

property 3) is satisfied.

Let $I = \bigcup_{i=1}^{\infty} I_i$, where the union is over all cubes chosen by this process. Since the derivative of the indefinite integral of an integrable function equals the function a.e., then

$$f(x) = \lim_{\substack{x \in Q \\ \lambda(Q) \to 0}} \frac{1}{\lambda(Q)} \int_Q f(y)dy \quad \text{a.e.}$$

where the Q's are cubes centered at x. But each of the cubes that enter into our decomposition which contains an $x \in I^c$ is a cube for which the first alternative holds. Therefore, the theorem follows.

Remark 3: Properties 1) and 3) give the useful result that

$$\lambda(E_y(f)) < \frac{1}{y} \int_{E_y(f)} f(x)dx.$$

It's importance is in finding "weak type" estimates, which will be discussed later.

Definition 1.8: For an arbitrary measure μ on a measure space X, and for a function f on X let

$$D_f^{\mu}(y) \equiv \mu\{x: |f(x)| > y,\ y>0\} = \mu\left[E_y(f)\right].$$

We now deduce the following theorem, which can be found in, e.g., Hewitt and Stromberg [8, p. 421].

Theorem 1.9: If $E \subseteq X$, and if φ is a real valued, non-decreasing, differentiable function on $\mathbb{R}^+$ with $\varphi(0) = 0$, and f is a non-negative function defined on X such that $\int_E \varphi(f(x))\,du(x) < \infty$, then

$$\int_E \varphi(f(x))\,du(x) = \int_0^\infty \mu(E \cap E_t(f))\varphi'(t)\,dt,$$

where φ' denotes the derivative of φ.

Proof:
$$\begin{aligned}\int_E \varphi(f(x))\,d\mu(x) &= \int_X \chi_E(x)\varphi(f(x))\,d\mu(x)\\ &= \int_X \chi_E(x)\left(\int_0^{f(x)} \varphi'(t)\,dt\right)d\mu(x)\\ &= \int_X \chi_E(x)\left(\int_0^\infty \chi_{[0,f(x)]}(t)\varphi'(t)\,dt\right)d\mu(x)\\ &= \int_0^\infty \varphi'(t)\left(\int_X \chi_E(x)\chi_{[0,f(x)]}(t)\,d\mu(x)\right)dt\\ &= \int_0^\infty \varphi'(t)\ \mu(E \cap E_t(f))\,dt,\end{aligned}$$

where the interchange of order of integration is justified by Fubini's Theorem.

<u>Corollary:</u> (Chebyshev's Inequality) If $0 < p < \infty$, then

$$y^p D_f^\mu(y) \leq \int_X |f(x)|^p d\mu(x),$$

provided the right side is finite.

<u>Proof:</u> In Theorem 1.9, let $E = X$, and $\varphi(x) = x^p$, for $0 < p < \infty$. Clearly φ satisfies the conditions of Theorem 1.9, and therefore, for arbitrary function f on X, it follows that, for any $y > 0$,

$$\begin{aligned}\int_X |f(x)|^p d\mu(x) &= p\int_0^\infty \mu(X \cap E_f(t))\, t^{p-1} dt \\ &= p\int_0^\infty D_f^\mu(t)\, t^{p-1} dt \\ &\geq p\int_0^y D_f^\mu(t)\, t^{p-1} dt.\end{aligned}$$

Since $D_f^\mu(t)$ is non-increasing in t,

$$\begin{aligned}p\int_0^y D_f^\mu(t)\, t^{p-1} dt &\geq pD_f^\mu(y)\int_0^y t^{p-1} dt \\ &= pD_f^\mu(y)\,\frac{y^p}{p} \\ &= y^p D_f^\mu(y).\end{aligned}$$

This proves the corollary.

The following theorem is usually referred to as a converse to Holder's Inequality. The proof may be found, e.g., in Zaanen [13, p. 127].

<u>Theorem 1.10</u>: Let $f \in L^p(X,\mu)$, $1 \leq p \leq \infty$. Then

$$4) \quad \|f\|_{p,\mu} = \sup_g \left|\int_X f(x)g(x)\,d\mu(x)\right| = \sup_g \int_X |f(x)g(x)|\,d\mu(x)$$

where the supremum is taken over all functions g such that $\|g\|_{q,\mu} \leq 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

<u>Proof</u>: If $1 < p < \infty$, then by Holder's Inequality, it follows that

$$\left|\int_X f(x)g(x)\,d\mu(x)\right| \leq \|f\|_{p,\mu}\|g\|_{q,\mu} \leq \|f\|_{p,\mu}.$$

Moreover, this inequality is obvious for $p = 1$ or $p = \infty$. To complete the proof, it is sufficient to find one function, g, satisfying the hypothesis, for which

$$\|f\|_{p,\mu} = \left|\int_X f(x)g(x)\,d\mu(x)\right|$$

From the above inequality, it is obvious that if $\|f\|_{p,\mu} = 0$, the theorem is trivial. Hence, assume $\|f\|_{p,\mu} > 0$, $1 \leq p \leq \infty$. For $1 < p < \infty$, define the function $g(x)$ as follows:

$$g(x) = \frac{|f(x)|^{p-1}}{\|f\|_{p,\mu}^{p-1} \operatorname{sgn} f(x)},$$

where

$$\operatorname{sgn} f(x) = \begin{cases} \dfrac{f(x)}{|f(x)|}, & f(x) \neq 0 \\ 0, & f(x) = 0 . \end{cases}$$

Hence

$$\int_X |g(x)|^q d\mu(x) = \int_X \frac{|f(x)|^{q/p-1}}{\|f\|_{p,\mu}^{q(p-1)}} d\mu(x)$$

$$= \frac{1}{\|f\|_{p,\mu}^p} \int_X |f(x)|^p d\mu(x) = 1 ,$$

Therefore,

$$\left|\int_X f(x)g(x)d\mu(x)\right| = \left|\int_X \frac{f(x)\ |f(x)|^{p-1}}{\operatorname{sgn} f(x)\ \|f\|_{p,\mu}^{p-1}} d\mu(x)\right|$$

$$= \|f\|_{p,\mu}.$$

This completes the case $1 < p < \infty$.

For $p = 1$, let

$$g(x) = \begin{cases} \dfrac{1}{\operatorname{sgn} f(x)}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

then $\|g\|_{\infty,\mu} = 1$, from which it follows that

$$\left|\int_X f(x)g(x)\,d\mu(x)\right| = \int_X |f(x)|\,d\mu(x) = \|f\|_{1,\mu}.$$

For $p = \infty$, let $\varepsilon > 0$ be arbitrary. Then the set, $\{x: |f(x)| \geq \|f\|_{\infty,\mu} - \varepsilon\}$ contains a subset, E, of positive measure. Define $g(x)$ as

$$g(x) = \begin{cases} \dfrac{1}{\mu(E)\,\mathrm{sgn} f(x)}, & x \in E \\ 0, & x \in E^c \end{cases}$$

so that

$$\int_X |g(x)|\,d\mu(x) = \int_E \left|\frac{1}{\mu(E)\,\mathrm{sgn} f(x)}\right| d\mu(x) = 1.$$

Thus

$$\begin{aligned}\left|\int_X f(x)g(x)\,d\mu(x)\right| &= \left|\int_E f(x)\,\frac{1}{\mu(E)\,\mathrm{sgn} f(x)}\,d\mu(x)\right| \\ &= \frac{1}{\mu(E)}\int_E |f(x)|\,d\mu(x) \\ &\geq \frac{1}{\mu(E)}\int_E (\|f\|_{\infty,\mu} - \varepsilon)\,d\mu(x) \\ &= \|f\|_{\infty,\mu} - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows.

Remark 4: Since simple functions are dense in $L^q(X,\mu)$, $1 \leq q < \infty$, then for $1 < p \leq \infty$ the supremum can be taken over all simple functions, g, such that $\|g\|_{q,\mu} \leq 1$

Definition 1.11: Let T be a linear operator defined on simple functions on the measure space (X,μ). If, for $0 < p,\ q \leq \infty$,

$$\|Tf\|_{q,\mu} \leq A\|f\|_{p,\mu},$$

where A is independent of f, then T is said to be of strong type (p,q), or simply type (p,q).

If, for $0 < p,\ q < \infty$, and all $y > 0$,

$$y(D^{\mu}_{Tf}(y))^{1/q} \leq A\|f\|_{p,\mu},\ A \text{ independent of } f,$$

then T is said to be of weak type (p,q). If $q = \infty$, weak and strong types are defined to coincide.

Remark 5: Strong type (p,q) always implies weak type (p,q). To see this, note that, for $y > 0$

$$\begin{aligned}\left(\int_X |Tf(x)|^q d\mu(x)\right)^{1/q} &\geq \left(\int_{E_y(|Tf|)} |Tf(x)|^q d\mu(x)\right)^{1/q}\\ &\geq \left(\int_{E_y(|Tf|)} y^q d\mu(x)\right)^{1/q}\\ &= y(D^{\mu}_{Tf}(y))^{1/q}.\end{aligned}$$

The following theorem, which can be found in Hewitt and Stromberg [8, p. 423] is of considerable importance, and as an immediate consequence, implies that weak type (1,1) does not imply strong type (1,1).

Theorem 1.12: Let f be any non-negative, locally integrable function defined on $\mathbb{R}$. Then, with $E_h(f_i^*) = \{x : f_i^*(x) > h\}$, $i = 0,1,2$, $h > 0$,

$$5)\quad D_{f_i^*}(h) = \frac{1}{h}\int_{E_h(f)} f(x)dx, \quad i = 1,2$$

$$6)\quad D_{f_o^*}(h) \leq \frac{2}{h}\int_{E_h(f)} f(x)dx.$$

Proof: Inequality 6) follows immediately once 5) is established by noting that $E_h(f_o^*) = E_h(f_1^*) \cup E_h(f_2^*)$.

We consider first the case $i = 1$.

Note that the map $\varepsilon \to \frac{1}{x-\varepsilon}\int_{\varepsilon}^{x} f(t)dt$ is continuous on $(-\infty,x)$. Thus $E_h(f_1^*)$ is open and since any open set in $\mathbb{R}$ is the union of mutually disjoint open intervals (proved in, e.g., Bourbaki [1, p. 337]) we get $E_h(f_1^*) = \bigcup_{k=1}^{\infty}(a_k,b_k)$.

Consider (a_k,b_k), where k is fixed. For each $x \in (a_k,b_k)$, let

$$N_x = \{s : \int_s^x f(t)dt > h(x-s),\ s \in (a_k,x)\}.$$

<u>Claim</u>: N_x is non empty.

This claim is obvious if $a_k = -\infty$. Hence, suppose $a_k > -\infty$, and that N_x is empty for some $x \in (a_k, b_k)$. Then there must be an $w < a_k$, such that $\int_w^x f(t)dt > h(x-w)$, from which it follows that

$$\int_w^{a_k} f(t)dt = \int_w^x f(t)dt - \int_{a_k}^x f(t)dt$$
$$> h(x-w) - h(x-a_k)$$
$$= h(a_k - w).$$

Thus $a_k \in E_h(f_1^*)$ which is a contradiction since the interval (a_k, b_k) is open. Thus N_x is non-empty for each $x \in (a_k, b_k)$.

Let $S_x = \inf N_x$, and suppose that $S_x > a_k$. Continuity of the map $\varepsilon \to \frac{1}{x-\varepsilon}\int_\varepsilon^x f(t)dt$ implies that $\int_{S_x}^x f(t)dt = h(x-S_x)$. N_{S_x} is non empty and therefore, there must be some $y \in (a_k, S_x)$ such that

$$\int_y^{S_x} f(t)dt > h(S_x - y).$$

This implies that

$$\int_y^x f(t)dt > h(x-y)$$

contradicting $y < S_x$. Therefore, $S_x = a_k$, for all $x \in (a_k, b_k)$ and

$$\int_{a_k}^{x} f(t)\,dt \geqslant h(x-a_k).$$

Letting x tend to b_k shows that

$$\int_{a_k}^{b_k} f(t)\,dt \geqslant h(b_k-a_k).$$

If either a_k or b_k is infinite, 5) follows immediately. If (a_k, b_k) is bounded, then

$$\int_{a_k}^{b_k} f(t)\,dt \leqslant h(b_k-a_k),$$

since a_k, $b_k \notin E_h(f_1^*)$. Therefore, 5) follows, for $i = 1$.

By minor modifications of the above argument, the case $i = 2$ in 5) follows, which completes the proof.

Inequality 6) states that the map, T, defined by $Tf(x) = f_o^*(x)$ is of weak type (1,1), for any integrable function f. Let $f(x) = \chi_{[0,1]}(x)$, then $\int_{\mathbb{R}} f(x)\,dx = 1$, and

$$f_o^*(x) = 1 \text{ if } 0 \leqslant x \leqslant 1$$

$$= \frac{1}{x}, \text{ if } x > 1$$

$$= \frac{1}{1-x}, \quad x < 0 .$$

Hence

$$\int_{\mathbb{R}} f_o^*(x)\,dx \geq \int_1^\infty \frac{1}{x}dx = \infty.$$

Therefore, there does not exist any $A > 0$ such that

$$\int_{\mathbb{R}} f_o^*(x)\,dx \leq A\int_{\mathbb{R}} f(x)\,dx,$$

which proves that weak type, in general, does not imply strong type.

Next, we state, without proof, a general form of the continuous version of Minkowski's Inequality. See, for example, Dunford and Schwartz [3, p. 530].

Theorem 1.13: Let (X,μ) and (Y,σ) be positive measure spaces, and let f be a $\mu x\sigma$-integrable function on XxY. Then, for $p \geq 1$,

$$\int_X \left(\int_Y |f(x,y)|^p d\sigma(y)\right)^{1/p} d\mu(x) \geq \left(\int_Y \left(\int_X |f(x,y)|\,d\mu(x)\right)^p d\sigma(y)\right)^{1/p}$$

Finally in this section, the Marcinkiewicz Interpolation Theorem is stated. For a proof, see Stein

[12, pp. 272 - 274].

Theorem 1.14: (Marcinkiewicz Interpolation Theorem). Suppose T is a linear operator defined on simple functions, and $1 \leq p_i \leq q_i \leq \infty$, $i = 0,1$ and $p_o < p_1$, $q_o \neq q_1$. If T is of weak type (p_i, q_i), $i = 0,1$ then T is of strong type (p,q), where $\frac{1}{p} = \frac{1-\theta}{p_o} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_o} + \frac{\theta}{q_1}$, $0 < \theta < 1$.

Moreover

$$\|Tf\|_{q,\mu} \leq A\|f\|_{p,\mu},$$

for all $f \in L^p(X,\mu)$.

Remark 6: The case $p_1 = q_1 = 1$, $p_0 = q_0 = \infty$ is of special interest, for then the theorem reads:

If T is of weak types $(1,1)$ and (∞,∞), then T is of strong type (p,p), for all $1 < p < \infty$.

It also states that, if T is of weak type (p,p) for all $1 < p < \infty$, then it is of strong type (p,p).

Section C. Extensions of Inequalities of Maximal Functions

In this section, we extend the continuity property of the Hardy-Littlewood maximal functions from $L^p(\mathbb{R})$ to $L^p(\mathbb{R},\emptyset)$, $1 < p < \infty$, for certain, yet quite general, weight functions, $\emptyset$.

The inequalities are obtained via the Marcinkiewicz Interpolation Theorem, along the lines of the paper by C. Fefferman and E. M. Stein [5, pp. 107 - 114].

Lemma 1.15: If $f \in L^\infty(\mathbb{R},\emptyset_0^*)$, then $f_i^* \in L^\infty(\mathbb{R},\emptyset)$, $i = 0,1,2$, and

$$\|f_i^*\|_{\infty,\emptyset} \leq \|f\|_{\infty,\emptyset_0^*}.$$

Proof: We prove only the case $i = 1$, as the case $i = 2$ is similar, and the case $i = 0$ follows from the first two.

Without loss of generality, assume $f \in L^\infty(\mathbb{R},\emptyset_0^*)$ is non-negative. Approximate f by continuous functions g_α, such that $0 \leq g_\alpha \leq f$, and $\lim_{\alpha\to\infty} g_\alpha = f$ a.e. Thus, for fixed $\alpha > 0$, the set $E_h(g_\alpha)$ is open, from which it follows that

$$E_h(g_\alpha) = \bigcup_{k=1}^{\infty} (a_k,b_k).$$

these intervals being mutually disjoint. Letting $g^*_{\alpha 1}$ be the (right) maximal function of g_α, then for each $x \in (a_k,b_k)$, $g^*_{\alpha 1}(x) > h$, and hence

$$E_h(g_\alpha) \subseteq E_h(g^*_{\alpha 1}).$$

Assuming for the moment that for any $x \in (a_k,b_k)$

$$\phi^*_o(x) \geqslant \frac{1}{b_k-a_k}\int_{a_k}^{b_k}\phi(t)\,dt$$

(the proof will follow the lemma), then

$$\begin{aligned}\int_{E_h(g_\alpha)}\phi^*_o(x)\,dx &= \sum_k\int_{a_k}^{b_k}\phi^*_o(x)\,dx \\ &\geqslant \sum_k\int_{a_k}^{b_k}\Big(\frac{1}{b_k-a_k}\int_{a_k}^{b_k}\phi(t)\,dt\Big)dx \\ &= \sum_k\int_{a_k}^{b_k}\phi(t)\,dt = \int_{E_h(g_\alpha)}\phi(t)\,dt\end{aligned}$$

Hence

$$\begin{aligned}\Big\{h:\int_{E_h(g_\alpha)}\phi^*_o(x)\,dx = 0\Big\} &\subseteq \Big\{h:\int_{E_h(g_\alpha)}\phi(t)\,dt = 0\Big\} \\ &\subseteq \Big\{h:\int_{E_h(g^*_{\alpha 1})}\phi(t)\,dt = 0\Big\}.\end{aligned}$$

The last inclusion follows easily, noting that if $E_h(g_\alpha)$

has measure zero, then $E_h(g^*_{\alpha 1})$ is empty. Taking infimums yields

$$\|g^*_{\alpha 1}\|_{\infty,\emptyset} \leqslant \|g_\alpha\|_{\infty,\emptyset^*_o}.$$

As $\alpha\to\infty$, applying Fatou's Lemma completes the proof.

<u>Lemma 1.16</u>: Let f be a non-negative, measurable function on $\mathbb{R}$, and f^*_o its maximal function. Then for any finite interval $(a,b) \subseteq \mathbb{R}$,

$$f^*_o(x) \geqslant \frac{-1}{a-b}\int_a^b f(t)dt, \quad \text{for all } x \in (a,b).$$

<u>Proof</u>: We have

$$\begin{aligned} f^*_o(x) &= \max\left\{\sup_{\varepsilon<x} \frac{1}{x-\varepsilon}\int_\varepsilon^x f(t)dt,\ \sup_{x<\varepsilon} \frac{1}{x-\varepsilon}\int_\varepsilon^x f(t)dt\right\} \\ &\geqslant \max\left\{\frac{1}{x-a}\int_a^x f(t)dt,\ \frac{1}{b-x}\int_x^b f(t)dt\right\}, \end{aligned}$$

One of $\frac{1}{x-a}\int_a^x f(t)dt$, $\frac{1}{b-x}\int_x^b f(t)dt$ is larger. Assume, without loss of generality, that for fixed x

$$\frac{1}{b-x}\int_x^b f(t)dt \geqslant \frac{1}{x-a}\int_a^x f(t)dt$$

Therefore

$$(x-a)\int_x^b f(t)\,dt \geq (b-x)\int_a^x f(t)\,dt$$

$$\Rightarrow \quad (x-a)\int_x^b f(t)\,dt + b\int_x^b f(t)\,dt \geq (b-x)\int_a^x f(t)\,dt + b\int_x^b f(t)\,dt$$

$$\Rightarrow \quad (b-a)\int_x^b f(t)\,dt + x\int_x^b f(t)\,dt \geq b\int_a^b f(t)\,dt - x\int_a^x f(t)\,dt$$

$$\Rightarrow \quad (b-a)\int_x^b f(t)\,dt \geq (b-x)\int_a^b f(t)\,dt$$

$$\Rightarrow \quad \frac{1}{b-x}\int_x^b f(t)\,dt \geq \frac{1}{b-a}\int_a^b f(t)\,dt$$

Thus $f_0^*(x) \geq \frac{1}{b-a}\int_a^b f(t)\,dt$, for all $x \in (a,b)$.

This completes the proof of Lemma 1.15.

Lemma 1.17: If $f \in L^1(\mathbb{R}, \emptyset_o^*)$, then for $h > 0$,

$$D^{\emptyset}_{f_i^*}(h) \leq \frac{1}{h}\int_{E_y(f_i^*)} f(x)\,\emptyset_o^*(x)\,dx, \quad i = 0,1,2$$

Proof: The same approximation argument is used as in Lemma 1.15. Let $0 \leq g_\alpha \leq f$, where, without loss of generality, f is assumed non-negative and g_α continuous. We show only the case $i = 1$.

Let $E_h(g_{\alpha 1}^*) = \bigcup_{k=1}^{\infty} I_k$, where $I_k \cap I_j$ is empty for $k \neq j$, and $I_k = (a_k, b_k)$. The proof of Theorem 1.12 shows that

$$h \leq \frac{1}{b_k - a_k} \int_{a_k}^{b_k} g_\alpha(t)\,dt.$$

Also, from Lemma 1.16,

$$\phi_o^*(x) \geq \frac{1}{b_k - a_k} \int_{a_k}^{b_k} \phi(t)\,dt$$

Therefore

$$\int_{a_k}^{b_k} g_\alpha(x)\phi_o^*(x)\,dx \geq \int_{a_k}^{b_k} g_\alpha(x)\,dx \ \left(\frac{1}{b_k - a_k}\int_{a_k}^{b_k} \phi(t)\,dt\right)$$

$$= \int_{a_k}^{b_k} \phi(t) \ \frac{1}{b_k - a_k} \int_{a_k}^{b_k} g_\alpha(x)\,dx \ dt$$

$$\geq h\int_{a_k}^{b_k} \phi(t)\,dt.$$

Thus,

$$hD^{\phi}_{g^*_{\alpha 1}}(h) \leq \int_{E_h(g^*_{\alpha 1})} g_\alpha(x)\,\phi_o^*(x)\,dx$$

from which the Lemma follows.

<u>Theorem 1.18</u>: (Hardy's Inequality) If $0 \leq f \in L^p(\mathbb{R},\phi_o^*)$, $1 < p < \infty$, then $f_i^* \in L^p(\mathbb{R},\phi)$ for $i = 0,1,2$. Further,

$$\|f_i^*\|_{p,\emptyset} \leq A_p\|f\|_{p,\emptyset_o^*}, \quad i = 0,1,2.$$

<u>Proof</u>: Define the maps T_i on simple functions, f, by $T_i(f) = f_i^*$, $i = 0,1,2$. Then Lemma 1.15 states that T_i is of type (∞,∞), and Lemma 1.17 implies that T_i is of weak type (1,1). Therefore, by the Marcinkiewicz Interpolation Theorem, T_i is of strong type (p,p), $1 < p < \infty$, that is

$$\|f_i^*\|_{p,\emptyset} < A_p\|f\|_{p,\emptyset_o^*}, \quad i = 0,1,2,$$

and $f \in L^p(\mathbb{R},\emptyset_o^*)$. Since simple functions are dense in $L^p(\mathbb{R},\emptyset_o^*)$, the result follows.

<u>Corollary 1</u>: Let $X = (a,b) \subseteq \mathbb{R}$, and f and $\emptyset$ be zero in X^c. If $f \in L^p(X,\emptyset_o^*)$, $1 < p < \infty$, then $f_i^* \in L^p(X,\emptyset)$ and

$$\int_X (f_i^*(t))^p \emptyset(t)dt \leq A_p\int_X (|f(t)|)^p \emptyset_o^*(t)dt$$

<u>Proof</u>: Since

$$\int_X (f_i^*(t))^p \emptyset(t)dt = \int_{\mathbb{R}} (f_i^*(t))^p \emptyset(t)dt$$

and

$$\int_X (|f(t)|)^p \emptyset_o^*(t)dt = \int_{\mathbb{R}} (|f(t)|)^p \emptyset_o^*(t)dt,$$

the corollary follows immediately.

Corollary 2: If $\emptyset$ is an even function defined on $\mathbb{R}$, and non-increasing on $\mathbb{R}^+$, then for $1 < p < \infty$

$$\int_{\mathbb{R}} \left|\frac{1}{x}\int_o^x f(t)dt\right|^p \emptyset(x)dx \leq A_p \int_{\mathbb{R}} |f(x)|^p \left(\frac{1}{x}\int_o^x \emptyset(t)dt\right)dx$$

Proof: Without loss of generality, let f be non-negative. Then, if $x > 0$

$$\frac{1}{x}\int_o^x f(t)dt \leq f_1^*(x) \text{ and } \emptyset_o^*(x) \leq \frac{2}{x}\int_o^x \emptyset(t)dt.$$

Similarly, if $x < 0$,

$$\frac{1}{x}\int_o^x f(t)dt \leq f_2^*(x) \text{ and } \emptyset_o^*(x) \leq \frac{2}{x}\int_o^x \emptyset(t)dt.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}} \left(\frac{1}{x}\int_o^x f(t)dt\right)^p \emptyset(x)dx &= \int_{-\infty}^o \left(\frac{1}{x}\int_o^x f(t)dt\right)^p \emptyset(x)dx + \int_o^\infty \left(\frac{1}{x}\int_o^x f(t)dt\right)^p \emptyset(x)dx \\ &\leq \int_{-\infty}^o (f_2^*(x))^p \emptyset(x)dx + \int_o^\infty (f_1^*(x))^p \emptyset(x)dx \\ &\leq \int_{-\infty}^\infty (f_o^*(x))^p \emptyset(x)dx \ .\end{aligned}$$

Applying Theorem 1.18, then

$$\int_{\mathbb{R}} (f_o^*(x))^p \phi(x)\,dx \leqslant A_p \int_{\mathbb{R}} (f(x))^p \phi_o^*(x)\,dx$$

$$\leqslant A_p \int_{\mathbb{R}} (f(x))^p \left(\frac{1}{x}\int_o^x \phi(t)\,dt\right)dx .$$

Thus combining the last two inequalities yields the result.

Corollary 3: If, in addition to the hypotheses of Corollary 2

$$\frac{1}{x}\int_o^x \phi(t)\,dt \leqslant A\phi(x),$$

then for $1 < p < \infty$

$$\int_{\mathbb{R}} \left|\frac{1}{x}\int_o^x f(t)\,dt\right|^p \phi(x)\,dx \leqslant A_p \int_{\mathbb{R}} |f(x)|^p \phi(x)\,dx$$

Proof: Apply Corollary 2.

Remark 7: For any interval $(a,b) \subseteq \mathbb{R}$, Corollaries 2 and 3 can be modified in the same manner as Corollary 1 modifies Theorem 1.18.

Example (1): Let $\phi(x) = \frac{1}{\delta + x^\alpha}$, where $\delta = 0$ or 1 $x \in [0,\infty)$, and $0 \leqslant \alpha < 1$ is fixed, then

$$\frac{1}{x}\int_0^x \frac{1}{\delta+t}\,dt \leqslant \frac{A}{\delta+x^\alpha}.$$

To see this, consider the following three cases:

(a) If $\delta = 0$, then $\int_0^x \frac{1}{t^\alpha}dt = \frac{x^{1-\alpha}}{1-\alpha}$, thus

$$\frac{1}{x}\int_0^x \frac{1}{t^\alpha}dt \leqslant \frac{2}{1-\alpha}\,\frac{1}{x^\alpha}.$$

(b) If $\delta = 1$, and $0 < x \leqslant 1$,

$$\frac{1}{x}\int_0^x \frac{1+x^\alpha}{1+t^\alpha}dt \leqslant \frac{1}{x}\int_0^x 2dt \leqslant \frac{2}{1-\alpha}\ .$$

(c) If $\delta = 1$, and $1 \leqslant x < \infty$, then

$$\frac{1}{x}\int_0^x \frac{1+x^\alpha}{1+t^\alpha}dt \leqslant \frac{2}{x}\ x^\alpha\int_0^x \frac{1}{t^\alpha}dt$$

$$= \frac{2x^\alpha}{x}\,\frac{x^{1-\alpha}}{1-\alpha} = \frac{2}{1-\alpha}\ .$$

Therefore,

$$\frac{1}{x}\int_0^x \frac{1}{\delta+t^\alpha}dt \leqslant \frac{A}{\delta+x^\alpha}.$$

Hence, by the last theorem,

$$\int_{\mathbb{R}^+}(f_1^*(x))^p\ \frac{1}{\delta+x^\alpha}dx \leqslant A_p\int_{\mathbb{R}^+}\frac{|f(x)|^p}{\delta+x^\alpha}\,dx,$$

which was shown by Kaneko [9].

<u>Example (2)</u>: If $\phi(x) \equiv 1$, then $\phi_i^* \equiv 1$, $i = 0,1,2$ and Theorem 1.17 reduces to the Hardy-Littlewood maximal inequalities,

$$\int_{\mathbb{R}} (f_i^*(x))^p dx \leqslant A_p \int_{\mathbb{R}} |f(x)|^p dx.$$

For a direct proof, see Hewitt and Stromberg [8, p. 424].

For the rest of the chapter, we assume $\phi \in L^1(\mathbb{T})$ to be even and periodic and of period 2π. Also, for a function on $[0,2\pi]$, we denote it's maximal functions restricted to $[0,2\pi]$ by $f_i^*(x)$, $i = 0,1,2$. If f is extended periodically, with period 2π, to $[-\pi,3\pi]$, then we define

$$f_{1b}^*(x) = \sup_{-\pi \leqslant \xi < x \leqslant 3\pi} \frac{1}{x-\xi}\int_{\xi}^{x} f(t)dt,$$

with $f_{0b}^*(x)$ and $f_{2b}^*(x)$ also defined similarly.
We denote $f_{0b}^*(x)$ by $f_b^*(x)$.

With the above definitions, we have the following theorem:

<u>Theorem 1.19</u>: Let f be periodic, with period 2π, non-negative, and $f \in L^p(\mathbb{T},\phi_0^*)$, $1 < p < \infty$. Then

$$\int_{-\pi}^{3\pi} (f_{ib}^*(x))^p \phi(x)\,dx \leq A_p \int_0^{2\pi} (f(x))^p \phi_0^*(x)\,dx, \quad i = 0,1,2.$$

Proof: Since ϕ is an even function, one verifies that the following relations hold:

$$2\phi_0^*(x) \geq \phi_b^*(x), \quad x \in [0,2\pi],$$

$$2\phi_0^*(x+2\pi) \geq \phi_b^*(x), \quad x \in [-\pi,0],$$

$$2\phi_0^*(x-2\pi) \geq \phi_b^*(x), \quad x \in [2\pi,3\pi].$$

By Corollary 1 of Theorem 1.18, we have

$$\int_{-\pi}^{3\pi} (f_{1b}^*(x))^p \phi(x)\,dx \leq A_p \int_{-\pi}^{3\pi} (f(x))^p \phi_b^*(x)\,dx, \quad i = 0,1,2.$$

By the above observations on ϕ_0^* and ϕ_b^*, and by periodicity of f, we have

$$\int_{-\pi}^{3\pi} (f(x))^p \phi_b^*(x)\,dx \leq 4\int_0^{2\pi} (f(x))^p \phi_0^*(x)\,dx .$$

The theorem follows by combining the two inequalities.

Corollary 1: Under the conditions of Theorem 1.19,

$$\int_0^{2\pi} (f_i^*(x))^p \phi(x)\,dx \leq A_p \int_0^{2\pi} (f(x))^p \phi_0^*(x)\,dx, \quad i = 0,1,2$$

Proof: Use the obvious fact that $f_i^*(x) \leq f_{ib}^*(x)$ for $x \in [0,2\pi]$ and apply Theorem 1.19.

Remark 8: The condition that ϕ be even in the last theorem can be weakened to the condition that there exist an $A > 0$ such that

$$\phi_b^*(x) \leq A\ \phi_0^*(x) \qquad \text{if } x \in [0,2\pi]$$

$$\phi_b^*(x) \leq A\ \phi_0^*(x+2\pi) \qquad \text{if } x \in [-\pi,0]$$

$$\phi_b^*(x) \leq A\ \phi_0^*(x-2\pi) \qquad \text{if } x \in [2\pi,3\pi].$$

Remark 9: The interval $[-\pi,3\pi]$ does not appear to be a natural domain as the functions are defined on $\mathbb{T}$. However if we look at the Poisson function, $P(r,\theta-t)$, where

$$P(r,\theta-t) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}$$

$\theta \in [0,2\pi]$ and $t \in [-\pi,\pi]$, then $\theta-t \in [-\pi,3\pi]$. This widely used function will be discussed in the second chapter.

Using Theorem 1.19, we now deduce an extension of an inequality of Hardy and Littlewood, by modifying their argument. See e.g. Duren [4, p. 234].

<u>Theorem 1.20</u>: Let f be a periodic function, period 2π. If $f \in L^p(\mathbb{T},\phi_0^*)$, $1 < p < \infty$, and

$$f^*(x) = \sup_{0<|\varepsilon| \leq \pi} \left| \frac{1}{\varepsilon}\int_0^{\varepsilon} f(x+t)\,dt \right| ,$$

then $f^* \in L^p(\mathbb{T},\phi)$ and

$$\|f^*\|_{p,\phi} \leq A_p \|f\|_{p,\phi_0^*}.$$

<u>Proof</u>: Without loss of generality, f is assumed to be non-negative. Now define F_1 and F_2 by

$$F_1(x) = \sup_{0<\varepsilon\leq\pi} \frac{1}{\varepsilon}\int_{-\varepsilon}^{0} f(x+t)\,dt, \quad F_2(x) = \sup_{0<\varepsilon\leq\pi} \frac{1}{\varepsilon}\int_0^{\varepsilon} f(x+t)\,dt.$$

Consider F_1. A change of variable shows that

$$\begin{aligned} F_1(x) &= \sup_{0<\varepsilon\leq\pi} \frac{1}{\varepsilon}\int_{-\varepsilon}^{0} f(x+t)\,dt \\ &= \sup_{x-\pi\leq\varepsilon<x} \frac{1}{x-\varepsilon}\int_{\varepsilon}^{x} f(t)\,dt \end{aligned}$$

From this equality, it follows that $F_1(x) \leq f_{1b}^*(x)$, $x \in [0,2\pi]$, where f_{1b}^* is as in Theorem 1.19. Therefore,

$$\int_0^{2\pi} (F_1(x))^p \phi(x)\,dx \leq \int_0^{2\pi} (f_{1b}^*(x))^p \phi(x)\,dx.$$

Theorem 1.20 applied to the last inequality yields

$$\int_0^{2\pi}(F_1(x))^p\phi(x)dx \leqslant A_p\int_0^{2\pi}(f(x))^p\phi_0^*(x)dx.$$

A similar argument shows, first, that $F_2(x) \leqslant f_{2b}^*(x)$ for $x \in [0,2\pi]$, and then that

$$\int_0^{2\pi}(F_2(x))^p\phi(x)dx \leqslant A_p\int_0^{2\pi}(f(x))^p\phi_0^*(x)dx.$$

Since $f^*(x) \leqslant F_1(x) + F_2(x)$ for $x \in [0,2\pi]$, the theorem follows.

CHAPTER II

Section A. Inequalities Involving Convolutions and Maximal Functions for Weighted L^p-spaces, $1 \leq p < \infty$

In this chapter, we relate the Hardy-Littlewood maximal function to convolutions of functions with a suitable kernel. Specifically, we shall give conditions on the kernel, such that its convolution with a function f is majorized by the Hardy-Littlewood maximal function. The specific kernels of Fejer and Poisson will illustrate the results. We also investigate the boundary behaviour of these convolution integrals for a general class of functions.

ϕ is a weight function with restrictions specified when needed, and we consider the spaces $L^p(\mathbb{T},\phi)$, $L^p(\mathbb{T})$ and $L^p(\mathbb{T},\phi^*)$, $1 \leq p < \infty$.

Definition 2.1: If $0 \leq r < 1$, and $0 \leq \theta < 2\pi$, we define the Poisson kernel, $P(r,\theta)$ by

$$P(r,\theta) = \frac{1}{2\pi}\,\frac{1-r^2}{1-2r\cos\theta+r^2}\,.$$

The convolution of P with a measurable function f defined on $\mathbb{T}$ is called the Poisson integral of f, wherever it exists, and is denoted by u. That is,

$$u(r,\theta) = (P*f)(\theta) = \int_0^{2\pi} P(r,\theta-t)f(t)dt.$$

Remark 10: If f is extended to $\mathbb{R}$ periodically, then for any $\varepsilon \in \mathbb{R}$,

$$\int_{\varepsilon}^{\varepsilon+2\pi} P(r,\theta-t)f(t)dt = \int_0^{2\pi} P(r,\theta-t)f(t)dt.$$

Thus the choice of interval of integration can be changed whenever convenient, without changing the value of $u(r,\theta)$.

Lemma 2.2: If $0 \leq r < 1$, $\theta \in [0,\frac{\pi}{2}]$, $t \in [-\pi,\pi]$ then

$$\frac{1-r^2}{1-2r\cos(\theta-t)+r^2} \leq A \frac{1-r}{(1-r)^2+(\theta-t)^2},$$

where A is independent of θ, t, and r.

Proof: Let $\alpha = (\theta-t) \in [-\pi,\frac{3\pi}{2}]$. Assume first that $0 \leq r \leq \frac{1}{2}$. Then

$$\frac{1-r^2}{1-2r\cos\alpha+r^2} \leq \frac{1}{(1-r)^2} \leq 4.$$

Also,

$$\frac{\frac{1}{2}}{1+(\frac{3\pi}{2})^2} \leqslant \frac{1-r}{(1-r)^2+\alpha^2},$$

from which it follows that

$$\frac{1-r^2}{1-2r\cos\alpha+r^2} \leqslant 8\,(1+(\frac{3\pi}{2})^2)\ \frac{(1-r)}{(1-r)^2+\alpha^2}$$

For $\frac{1}{2} \leq r \leqslant 1$, and using the identity $\cos\alpha = 1-2\sin^2\frac{\alpha}{2}$, we have

$$\frac{1-r^2}{1-2r\cos\alpha+r^2} = \frac{1-r^2}{(1-r)^2+4r\sin^2\frac{\alpha}{2}} \leqslant \frac{1-r^2}{(1-r)^2+2\sin^2\frac{\alpha}{2}} \; .$$

It is easily verified that, for $\alpha \in \left[-\pi, \frac{3\pi}{2}\right]$,

$$\sin^2\frac{\alpha}{2} \geqslant (\frac{\sqrt{2}}{3\pi}\,\frac{\alpha}{2})^2$$

and that $1-r^2 \leqslant 2(1-r)$. Therefore,

$$\frac{1-r^2}{(1-r)^2+2\sin^2\frac{\alpha}{2}} \leqslant 2\frac{(1-r)}{(1-r)^2+(\frac{\alpha}{3\pi})^2} \leqslant 2\,(3\pi)^2\ \frac{1-r}{(1-r)^2+\alpha^2} \; ,$$

so that, with $A = 8(1+(\frac{3\pi}{2})^2)$, the lemma follows .

Theorem 2.3: Let f be a periodic function, with period 2π. If $u(r,\theta)$ is it s Poisson integral, and

$$f^*(t) = \sup_{0<|t-\varepsilon|\leq \pi} \left| \frac{1}{t-\varepsilon}\int_\varepsilon^t f(x)\,dx \right|,$$

then $|u(r,\theta)| \leq A f^*(\theta)$.

Proof: It suffices to consider non-negative f.

Fix $\theta \in [-\pi,\pi]$, and let

$$f_\theta(t) = \int_0^t f(x+\theta)\,dx.$$

Using the periodicity of f, rewrite $u(r,\theta)$ as

$$u(r,\theta) = \int_{-\pi}^{\pi} f(t)\; P(r,\theta-t)\,dt = \int_{-\pi}^{\pi} f(t+\theta)\; P(r,t)\,dt$$

Integrating the right hand side of the last equality yields

$$u(r,\theta) = \frac{1}{2\pi}\, \frac{f_\theta(t)\;(1-r^2)}{(1-2r\cos t+r^2)}\Bigg|_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\theta(t)\; \frac{(1-r^2)\,2r\sin t}{[1-2r\cos t+r^2]^2}\,dt$$

$$\equiv \qquad B_1 \qquad + \qquad B_2,$$

respectively. By the definitions of f^* and f_θ, it follows that $B_1 \leq f^*(\theta)$, since

$$\frac{1-r^2}{1+2r+r^2} = \frac{1-r}{1+r} \leq 1.$$

Now,

$$B_2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_\theta(t)(1-r^2)2r|\sin t|}{[1-2r\cos t+r^2]^2}\,dt$$

$$\leq f^*(\theta)\int_{-\pi}^{\pi} \frac{(1-r)2r|t\sin t|}{[1-2r\cos t+r^2]^2}\,dt$$

$$\leq 2Af^*(\theta)\int_{-\pi}^{\pi} \frac{(1-r)t^2}{[1-2r\cos t+r^2]^2}\,dt$$

$$\leq 2f^*(\theta)A\int_{-\pi}^{\pi} \frac{(1-r)}{(1-2r+r^2)+t^2}\,dt$$

$$\leq 2\pi f^*(\theta)A.$$

This completes the proof of the theorem.

Remark 11: If $u(r,\theta)$ is infinite at some point, θ, and for some r, it is easily verified that $f^*(\theta)$ is also infinite, and the inequality of Theorem 2.3 holds trivially. Thus the theorem applies to a very wide class of functions.

In the following, we consider kernels which satisfy the following:

Definition 2.4: A kernel $K(x,y)$, defined for $x \in \mathbb{R}^+$ and periodic in y, with period 2π, is said to satisfy condition $\mathcal{H}$ if

i) $\int_{-\pi}^{\pi} K(x,y)\,dy = 1$

ii) there exists a function, G, such that

$$|K(x,y)| \leq G(x,y) \text{ and } \int_{-\pi}^{\pi} G(x,y)\,dy \leq A_1$$

iii) for each $\delta > 0$,

$$\lim_{x \to \infty} \left(\int_{-\pi}^{-\delta} G(x,y)\,dy + \int_{\delta}^{\pi} G(x,y)\,dy \right) = 0$$

iv) $\int_{-\pi}^{\pi} \left| y \frac{\partial G(x,y)}{\partial y} \right| dy = A_2$

Theorem 2.5: If $K(x,y)$ satisfies condition $\mathcal{K}$, then for all $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$

$$\lim_{x \to \infty} \int_{-\pi}^{\pi} K(x,y-t) f(t)\,dt = f(y) \text{ a.e.}$$

Proof: To prove the result, we use [7, Theorem 44.8, pp. 636 - 637] of Hewitt and Ross. By this theorem, it suffices to show that

$$\lim_{x \to \infty} \int_{-\pi}^{\pi} K(x,y-t) f(x)\,dt = f(y)$$

for all f in a dense subset of $L^p(\mathbb{T})$ and that for all $f \in L^p(\mathbb{T})$,

$$\sup_{x} \int_{-\pi}^{\pi} K(x,t) f(y-t)\,dt < \infty \text{ a.e.}$$

Consider $g \in C(\mathbb{T})$, then conditions (i), (iii) and [7, Theorem 28.52, pp. 87 - 88] imply that

$$\lim_{x\to\infty} \int_{-\pi}^{\pi} K(x,y-t)g(x)dt = g(y)$$

in the $L^1(\mathbb{T})$-norm. To show that

$$\lim_{x\to\infty} \int_{-\pi}^{\pi} K(x,t)g(y-t)dt = g(y) \quad \text{pointwise,}$$

let $\varepsilon > 0$ be given. By (iii), there exists N, such that for $x \geq N$, and $\delta > 0$,

$$\left(\int_{-\pi}^{-\delta} G(x,t)dt + \int_{\delta}^{\pi} G(x,t)dt\right) < \varepsilon .$$

By continuity of g, we choose $\delta > 0$ such that for $t \in (-\delta,\delta)$,

$$|g(y-t) - g(y)| < \varepsilon .$$

Hence, by boundedness of g,

$$\left|\int_{-\pi}^{\pi} K(x,t) \ [g(y-t)-g(y)] \ dt\right|$$
$$\leq \left|\int_{-\pi}^{-\delta} K(x,t) [g(y-t)-g(y)] dt + \int_{\delta}^{\pi} K(x,t) [g(y-t)-g(y)] dt\right|$$
$$+ \int_{-\delta}^{\delta} |K(x,t)| \ |g(y-t)-g(y)| \ dt$$
$$\leq 2A\left(\int_{-\pi}^{-\delta} |K(x,t)| \ dt + \int_{\delta}^{\pi} |K(x,t)| dt\right) + \varepsilon \int_{-\delta}^{\delta} |K(x,t)| \ dt$$

$\leqslant 2A\varepsilon + \epsilon A_1$

where A is the bound of g.

It remains to show that, for all $f \in L^p(\mathbb{T})$,

$$\sup_x \left|\int_{-\pi}^{\pi} K(x,t)f(y-t)dt\right| < \infty.$$

Clearly, we may assume $f \geqslant 0$, then

$$\left|\int_{-\pi}^{\pi} K(x,t)f(y-t)dt\right| \leq \int_{-\pi}^{\pi} G(x,t)f(y-t)dt.$$

Let

$$f_y(h) = \int_0^h f(y-t)dt,$$

then integration by parts yields

$$\int_{-\pi}^{\pi} G(x,t)f(y-t)dt = f_y(t)G(x,t)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f_y(t)\frac{\partial G(x,t)}{\partial t}\,dt$$

$$\leq f^*(y)2\pi[G(x,\pi)-G(x,-\pi)]+f^*(y)\int_{-\pi}^{\pi}\left|t\frac{\partial G(x,t)}{\partial t}\right|dt.$$

Since the last integral is bounded, by (iv), it remains to show that $[G(x,\pi) - G(x,-\pi)]$ is bounded. To do this, we show that

$$\left|G(x,\pi) - G(x,-\pi) - \int_{-\pi}^{\pi} G(x,y)dy\right| \leqslant A_1 + A_2.$$

This, however, clearly follows from (iv) on integrating by parts.

Since $f^*(y)$ exists a.e. for $f \in L^p(\mathbb{T})$, $1 \leqslant p < \infty$, the result follows.

Remark 12: Suppose that $K(x,y)$ is defined on $[0,a) \times \mathbb{R}$, and that $t = \frac{1}{1-x}$, then substituting t for x makes K a function of t and y. Thus if $K(t,y)$ satisfies condition $\mathcal{K}$, the conclusion for $K(x,y)$ reads,

$$\lim_{x \to a^-} \int_{-\pi}^{\pi} K(x,y) f(h-y)\, dy = f(h) \text{ a.e.,}$$

for all $f \in L^p(\mathbb{T})$, $1 \leqslant p < \infty$.

Let $\{K_n(y)\}_{n=1}^{\infty}$ be a sequence of functions defined on $\mathbb{R}$. Define the function $K(x,y)$ by

$$K(x,y) = \begin{cases} K_1(y), & 0 \leqslant x < 1 \\ K_2(y), & 1 \leqslant x < 2 \\ \vdots & \\ K_n(y), & n-1 \leqslant x < n \\ \vdots & \end{cases}$$

Then, if $K(x,y)$ satisfies condition $\mathcal{K}$, the conclusion is stated as

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} K_n(t)f(y-t)dt = f(y) \text{ a.e.,}$$

for all $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$.

Example (1): It is well known that the Poisson kernel, $P(r,\theta)$, satisfies

$$\int_{-\pi}^{\pi} P(r,\theta)d\theta = 1$$

for all $0 \leq r < 1$. Also, by Lemma 2.2,

$$P(r,\theta) \leq A \frac{1-r}{(1-r)^2+\theta^2} \equiv G(r,\theta)$$

and obviously

$$\lim_{r\to 1^-} \left(\int_{-\pi}^{-\delta} G(r,\theta)d\theta + \int_{\delta}^{\pi} G(r,\theta)\,d\theta\right) = 0$$

for all $\delta > 0$. Since

$$G(r,\pi) = G(r,-\pi) = \frac{A(1-r)}{(1-r)^2+\pi^2} \leq \frac{A}{\pi^2}$$

then

$$\int_{-\pi}^{\pi} \theta \frac{\partial G(r,\theta)}{\partial \theta} d\theta = \int_{-\pi}^{\pi} \frac{A(1-r)2\theta^2}{((1-r)^2+\theta^2)^2} d\theta$$

$$\leqslant \int_{-\pi}^{\pi} \frac{2A(1-r)}{(1-r)^2+\theta^2}\, d\theta$$

$$\leqslant 2A(1-r) \int_{-\infty}^{\infty} \frac{1}{(1-r)^2+\theta^2}\, d\theta$$

$$\leqslant 2A\pi.$$

Hence, by Theorem 2.5 and Remark 12,

$$\lim_{r\to 1^-} \int_{-\pi}^{\pi} P(r,t) f(\theta-t)dt = f(\theta) \text{ a.e.,}$$

for all $f \in L^p(\mathbb{T})$, $1 \leqslant p < \infty$.

<u>Example (2)</u>: Let $K_n(t) = \frac{1}{n}\left[\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right]^2$, $n \in \mathbb{N}$, be the Fejer kernel. Again, it is known that $\int_{-\pi}^{\pi} K_n(t)dt = 1$, for all $n \in \mathbb{N}$. If we set

$$G_n(t) = \frac{4\pi^2 n}{(1+n|t|)^2},$$

then noting that $K_n(t) \leqslant \min\left\{n, \frac{\pi^2}{nt^2}\right\}$, $0 < t \leqslant \pi$, and that

$$G_n(t) > n, \quad 0 \leqslant t \leqslant \frac{1}{n},$$

$$G_n(t) \geqslant \frac{\pi^2}{nt^2}, \quad \frac{1}{n} \leqslant t \leqslant \pi,$$

implies $K_n(t) \leqslant G_n(t)$ for all $t \in [-\pi,\pi]$. Since, for any

$\pi \geq \delta > 0$ we have

$$\int_{-\pi}^{-\delta} G_n(t)dt + \int_{\delta}^{\pi} G_n(t)dt \leq \int_{-\pi}^{-\delta} \frac{4\pi^2 n}{(1+n\delta)^2}\, dt + \int_{\delta}^{\pi} \frac{4\pi^2 n}{(1+n\delta)^2}\, dt$$

$$= \frac{8\pi^2(\pi-\delta)n}{(1+n\delta)^2}$$

then it is clear that, for such a δ,

$$\lim_{n\to\infty} \left(\int_{-\pi}^{-\delta} G_n(t)dt + \int_{\delta}^{\pi} G_n(t)dt\right) = 0 .$$

Also

$$\int_{-\pi}^{\pi} G_n(t)dt = 4\pi^2 \int_{-\pi}^{\pi} \frac{ndt}{(1+n|t|)^2}$$

$$= 8\pi^2 \int_{0}^{\pi n} \frac{dt}{(1+t)^2}$$

$$= 8\pi^2 \left(1- \frac{1}{1+\pi n}\right) < 8\pi^2,$$

and, for $t \in (0,\pi]$,

$$|t\, G_n'(t)| = t\, \frac{2n^2 4\pi^2}{(1+n|t|)^3}$$

$$< \frac{8\pi^2 n}{(1+n|t|)^2} = 2\, G_n(t),$$

which implies that $\int_{-\pi}^{\pi} |t\, G_n'(t)|\, dt < 16\pi^2.$

Hence, by Theorem 2.5 and Remark 12,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} K_n(t)f(\theta-t)\,dt = f(\theta) \text{ a.e.,}$$

for all $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$.

Benjamin Muckenhoupt, in his paper "Weighted Norm Inequalities", [10], gives necessary and sufficient conditions on the weight function ϕ to ensure that the Poisson integral of a function $f \in L^1(\mathbb{T})$ converges in the $L^p(\mathbb{T},\phi)$ norm to the function f on the boundary. Suppose instead that $f \in L^1(\mathbb{T},\phi)$ and $K(x,y)$ is a kernel satisfying condition $\mathcal{H}$, then the following theorem gives sufficient conditions for norm convergence of the convolution integral.

Theorem 2.6: Let the weight function ϕ be integrable on $[0,2\pi]$. If $K(x,y)$ satisfies condition $\mathcal{H}$, and is also even in y, then for $f \in L^1(\mathbb{T},\phi)$

$$\lim_{x\to\infty}\int_0^{2\pi}\left|\int_0^{2\pi} f(y-t)K(x,t)\,dt\right|\phi(y)\,dy = \int_0^{2\pi}|f(y)|\,\phi(y)\,dy$$

Proof: Without loss of generality, f is assumed non-negative. By periodicity, it follows that

$$\int_0^{2\pi} f(y-t)K(x,t)\,dt = \int_0^{2\pi} f(t)K(x,y-t)\,dt.$$

Multiplying by $\phi(y)$ and integrating yields

$$\int_0^{2\pi}\left(\int_0^{2\pi} f(t)K(x,y-t)\,dt\right)\phi(y)\,dy = \int_0^{2\pi} f(t)\left(\int_0^{2\pi} K(x,y-t)\phi(y)\,dy\right)dt$$

where the formal interchange of order of integration will be justified by Fubini's Theorem.

We now apply first Fatou's Lemma, Theorem 1.6, and then Remark 1 to the right side of the last equality:

$$\text{(a)} \quad \lim_{x\to\infty} \int_0^{2\pi} f(t)\left(\int_0^{2\pi} K(x,y-t)\phi(y)\,dy\right)dt$$

$$\leq \int_0^{2\pi}\left(\overline{\lim_{x\to\infty}}\, f(t)\int_0^{2\pi} K(x,t-y)\phi(y)\,dy\right)dt$$

$$\leq \int_0^{2\pi} f(t)\phi(t)\,dt,$$

and

$$\underline{\lim_{x\to\infty}} \int_0^{2\pi} f(t) \int_0^{2\pi} K(x,y-t)\phi(y)\,dy\,dt$$

$$\geq \int_0^{2\pi}\left(f(t)\, \underline{\lim_{x\to\infty}} \int_0^{2\pi} K(x,t-y)\phi(y)\,dy\right)dt$$

$$= \int_0^{2\pi} f(t)\phi(t)\,dt\,.$$

Thus

$$\int_0^{2\pi} f(t)\phi(t)\,dt \leq \varliminf_{x\to\infty} \int_0^{2\pi} f(t)\left(\int_0^{2\pi} K(x,y-t)\phi(y)\,dy\right)dt$$

$$\leq \varlimsup_{x\to\infty} \int_0^{2\pi} f(t)\left(\int_0^{2\pi} K(x,y-t)\phi(y)\,dy\right)dt$$

$$\leq \int_0^{2\pi} f(t)\phi(t)\,dt$$

which proves the Theorem.

Note that, in the last theorem, Theorem 2.5 was applied to the convolution integral of K and ϕ, and that inequality (a) shows that $\int_0^{2\pi} f(t)K(x,y-t)\,dt$ exists a.e., at least for large x.

The following theorem shows that pointwise convergence of $\int_0^{2\pi} f(t)K(x,y-t)\,dt$ at y is a local property of f. It is an analogue of the Riemann Localization Principle.

Theorem 2.7: Let K(x,y) be even in y and satisfy condition $\mathcal{K}$. If $\phi \in L^1(\mathbb{T})$, and $f \in L^1(\mathbb{T},\phi)$ then convergence of $\int_0^{2\pi} f(t)K(x,y-t)\,dt$ at $y \in \mathbb{T}$ depends only on the behaviour of f in an arbitrarily small neighborhood of y.

Proof: Without loss of generality, we can assume $f \geq 0$. Since $f \in L^1(\mathbb{T},\phi)$, it is finite a.e., and measurable. Therefore, let $\{v_n(y-t)\}_{n\geq 1}$ be a sequence of simple functions on $\mathbb{T}$, such that $0 \leqslant v_n(y-t) \leqslant f(y-t)$ and $\lim\limits_{n\to\infty} v_n(y-t) = f(y-t)$ a.e.

Since for all $0 < \delta \leqslant \pi$, and G the majorant of K,

$$\lim_{x\to\infty} \left(\int_{-\pi}^{-\delta} G(x,y)\,dy + \int_{\delta}^{\pi} G(x,y)\,dy\right) = 0,$$

we consider $\int_{-\pi}^{-\delta} v_n(y-t)K(x,t)\,dt$. Since $v_n(y-t)$ is simple,

it is bounded, and thus has a maximum, say $A_n \geqslant 0$, on $\mathbb{T}$.
Therefore, for all $n \geqslant 1$,

$$\lim_{x\to\infty} \left|\int_{-\pi}^{-\delta} v_n(y-t)K(x,t)\,dt\right| \leqslant \lim_{x\to\infty} \int_{-\pi}^{-\delta} A_n G(x,t)\,dt$$

$$= 0$$

Similarly, $\lim\limits_{x\to\infty} \left|\int_{\delta}^{\pi} K(x,t)v_n(y-t)\,dt\right| = 0.$

Since $\lim\limits_{n\to\infty} v_n(y-t) = f(y-t)$, then for any $\rho > 0$,

there exists an $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$|v_n(y-t) - f(y-t)| < \rho,$$

except on a set E_ρ, with $\lambda(E_\rho) < \rho$. It follows that

$$0 = \lim_{x\to\infty} \int_{[\delta,\pi]\setminus E} v_n(y-t)\; G(x,t)\,dt$$

$$\geq \lim_{x\to\infty} \int_{[\delta,\pi]\setminus E} G(x,t)(f(y-t)-\rho)\,dt$$

$$= \lim_{x\to\infty} \int_{[\delta,\pi]\setminus E} f(y-t)\; G(x,t)\,dt = 0$$

Since ρ is arbitrary, it follows that

$$\lim_{x\to\infty} \left|\int_{\delta}^{\pi} f(y-t)K(x,t)\,dt\right| \leq \lim_{x\to\infty} \int_{\delta}^{\pi} f(y-t)G(x,t)\,dt = 0$$

Therefore,

$$\lim_{x\to\infty} \int_{0}^{2\pi} f(t)K(x,y-t)\,dt = \lim_{x\to\infty} \int_{0}^{2\pi} f(y-t)K(x,t)\,dt$$

$$= \lim_{x\to\infty} \left[\int_{\delta}^{\pi} f(y-t)K(x,t)\,dt + \int_{-\delta}^{\delta} f(y-t)K(x,t)\,dt + \int_{-\pi}^{-\delta} f(y-t)K(x,t)\,dt \right]$$

$$= \lim_{x\to\infty} \int_{-\delta}^{\delta} f(y-t)K(x,t)\,dt$$

which completes the theorem.

We now turn attention to Theorem 2.3, where it was shown that $|u(r,\theta)| \leq Af^{*}(\theta)$. The following theorem, which can be found in [14, p. 155], shows that this same property is true of the kernel $K(x,y)$ satisfying condition $\mathcal{H}$.

Theorem 2.8: If $K(x,y)$ satisfies condition $\mathcal{K}$ and $\int_{-\pi}^{\pi} K(x,t)f(y-t)dt$ exists a.e., then

$$\left|\int_{-\pi}^{\pi} K(x,t)f(y-t)dt\right| \leq Af^{*}(y).$$

Proof: Without loss of generality, let $f \geq 0$, then integrating by parts, we obtain, with

$$f_y(t) \equiv \int_0^t f(y-h)dh$$

that

$$\left|\int_{-\pi}^{\pi} f(y-t)K(x,t)dt\right| \leq \int_{-\pi}^{\pi} f(y-t)\ G(x,t)dt$$

$$\leq f_y(t)\ G(x,t)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f_y(t)\left|\frac{\partial G(x,y)}{\partial t}\right|dt$$

$$\leq f^{*}(y)A_1 + f^{*}(y)A_2 = Af^{*}(y).$$

Section B. Inequalities for the Case $0 < p < 1$

Up to this point, the case $0 < p < 1$ has been neglected, even though we have defined the corresponding L^p spaces. Since the $L^p(\mathcal{X},\emptyset)$, $0 < p < 1$, function spaces are of interest for specific weight functions, we conclude this chapter with some results concerning them.

We first consider the spaces $L^p(\mathbb{R},\emptyset)$ and $L^p(\mathbb{R},\emptyset_0^*)$. Then the following lemma will be used to establish some inequalities (see Hewitt and Ross [7, p. 425] and Heinig [6, pp. 8 - 11]).

Lemma 2.9: If f is a function defined on $\mathbb{R}$, then, for $0 < k < 1$,

$$D^{\emptyset}_{f_0^*}(y) \leq \frac{1}{(1-k)y}\int_{E_{yk}(f)} |f(x)|\,\emptyset_0^*(x)\,dx.$$

Proof: Without loss of generality, assume $f \geq 0$. Define $g(x) = f(x)$ if $f(x) > yk$ and $g(x) = 0$ otherwise. Then

$$\begin{aligned} f_1^*(x) &= \sup_{\varepsilon < x}\Big[\frac{1}{x-\varepsilon}\int_{\varepsilon}^{x} f(t)\,\chi_{E_{yk}(f)}(t)\,dt \\ &\qquad + \frac{1}{x-\varepsilon}\int_{\varepsilon}^{x} f(t)\,\chi_{E^c_{yk}(f)}(t)\,dt\Big] \\ &\leq g_1^*(x) + yk. \end{aligned}$$

Letting $N_s^1 = \{x : g_1^*(x) > s,\ s > 0\}$ then

$$E_y(f_1^*) = \{x : f_1^*(x) > y\} \subset \{x : g_1^*(x) + yk > y\} \equiv N^1_{y(1-k)}$$

Applying Lemma 1.17 with f replaced by g,

$$y\, D^{\phi}_{f_1^*}(y) \equiv y\int_{E_y(f_1^*)} \phi(x)\,dx \leq y\int_{N^1_{y(1-k)}} \phi(x)\,dx$$

$$\leq \frac{1}{1-k}\int_{N^1_{y(1-k)}} g(x)\ \ \phi_0^*(x)\,dx$$

$$= \frac{1}{1-k}\int_{N^1_{y(1-k)}\cap E_{yk}(f)} f(x)\ \ \phi_0^*(x)\,dx$$

$$\leq \frac{1}{1-k}\int_{E_{yk}(f)} f(x)\ \ \phi_0^*(x)\,dx .$$

Similarly

$$y\, D^{\phi}_{f_2^*}(y) \leq \frac{1}{1-k}\int_{E_{yk}(f)} f(x)\ \ \phi_0^*(x)\,dx .$$

Therefore

$$y\, D^{\phi}_{f_0^*}(y) = y\int_{E_y(f_0^*)} \phi(x)\,dx = y\int_{E_y(f_1^*)\cup E_y(f_2^*)} \phi(x)\,dx$$

$$\leq y\int_{E_y(f_1^*)} \phi(x)\,dx + y\int_{E_y(f_2^*)} \phi(x)\,dx$$

$$\leq \frac{2}{1-k}\int_{E_{yk}(f)} f(x)\ \ \phi_0^*(x)\,dx .$$

Lemmas 1.17 and 2.8 give weak type (1,1) estimates for the Hardy-Littlewood maximal function. The following lemma gives another estimate.

Lemma 2.10: If $0 < k < 1$, and $\emptyset$ is integrable on $\mathbb{R}$, then

$$\int_{\mathbb{R}} f_0^*(x)\emptyset(x)\,dx \leq \frac{1}{k}\int_{\mathbb{R}} \emptyset(x)\,dx + \frac{2}{1-k}\int_{\mathbb{R}} [f(x)\log^+ f\ \ (x)]\ \emptyset_0^*(x)\,dx$$

where $\log^+ f(x) = \log f(x)$ if $f(x) > 1$, and is zero otherwise.

Proof: Applying Theorem 1.9 and Lemma 2.9 yields

$$\int_{\mathbb{R}} f_0^*(x)\emptyset(x)\,dx = \int_0^\infty D^{\emptyset}_{f_0^*}(y)\,dy$$

$$= \int_0^{1/k} D^{\emptyset}_{f_0^*}(y)\,dy + \int_{1/k}^\infty D^{\emptyset}_{f_0^*}(y)\,dy$$

$$\leq \int_0^{1/k}\Big(\int_{E_y(f_0^*)} \emptyset(x)\,dx\Big)dy + \frac{2}{1-k}\int_{1/k}^\infty \frac{1}{y}\Big(\int_{E_{yk}(f)} f(x)\emptyset_0^*(x)\,dx\Big)dy$$

$$\leq \frac{1}{k}\int_{\mathbb{R}} \emptyset(x)\,dx + \frac{2}{1-k}\int_{\mathbb{R}} \emptyset_0^*(x) f(x)\Big(\int_{1/k}^\infty \chi_{E_{yk}(f)}(x)\ \frac{dy}{y}\Big)dx\,.$$

Since

$$\int_{1/k}^{\infty}\chi_{E_{yk}(f)}(x)\frac{dy}{y} = \int_{1/k}^{f(x)/k}\frac{dy}{y} = \log^{+}f(x),$$

the lemma follows.

<u>Theorem 2.11</u>: If f and ϕ are non-negative functions on $\mathbb{R}$ and ϕ is integrable, and if $f \in L^{1}(\mathbb{R},\phi_0^*)$, then $f_0^* \in L^{p}(\mathbb{R},\phi)$, $0 < p < 1$ and

$$\int_{\mathbb{R}}(f_0^*(x))^{p}\phi(x)\,dx \leq A_p\left(\int_{\mathbb{R}}\phi(x)\,dx\right)^{1-p}\left(\int_{\mathbb{R}}f(x)\phi_0^*(x)\,dx\right)^{p}$$

<u>Proof</u>: Let $\alpha > 0$, $0 < k < 1$, then by definition of $D^{\phi}_{f_0^*}(y)$, and Lemma 2.9,

$$\int_{\mathbb{R}}(f_0^*(x))^{p}\phi(x)\,dx = p\int_0^{\infty}y^{p-1}D^{\phi}_{f_0^*}(y)\,dy$$

$$= p\int_0^{\alpha/k}y^{p-1}D^{\phi}_{f_0^*}(y)\,dy + p\int_{\alpha/k}^{\infty}y^{p-1}D^{\phi}_{f_0^*}(y)\,dy$$

$$\leq p\int_0^{\alpha/k}y^{p-1}\left(\int_{E_y(f_0^*)}\phi(x)\,dx\right)dy + p\int_{\alpha/k}^{\infty}y^{p-1}D^{\phi}_{f_0^*}(y)\,dy$$

$$\leq \left(\frac{\alpha}{k}\right)^{p}\int_{\mathbb{R}}\phi(x)\,dx + \frac{2p}{1-k}\int_{\alpha/k}^{\infty}y^{p-2}\left(\int_{E_{yk}(f)}f(x)\phi_0^*(x)\,dx\right)dy$$

$$= \left(\frac{\alpha}{k}\right)^{p}\int_{\mathbb{R}}\phi(x)\,dx + \frac{2p}{1-k}\int_{\mathbb{R}}f(x)\phi_0^*(x)\left(\int_{\alpha/k}^{\infty}y^{p-2}\chi_{E_{yk}(f)}(x)\,dy\right)dx .$$

However,

$$\int_{\alpha/k}^{\infty} y^{p-2}\chi_{E_{yk}(f)}(x)\,dy = \int_{\alpha/k}^{f(x)/k} y^{p-2}\,dy$$

$$= \begin{cases} \dfrac{k^{1-p}}{1-p}\left[\alpha^{p-1}-(f(x))^{p-1}\right] & \text{if } f(x) > \alpha. \\ 0 \text{ if } 0 < f(x) < \alpha & \end{cases}$$

so that

$$\int_{\mathbb{R}} (f_0^*(x))^p \phi(x)\,dx \leq \left(\frac{\alpha}{k}\right)^p \int_E \phi(x)\,dx + \frac{2p}{(1-k)(1-p)}\left(\frac{\alpha}{k}\right)^{p-1} \int_{\mathbb{R}} f(x)\phi_0^*(x)\,dx .$$

A straight forward calculation shows that the right hand side of the last inequality is at a minimum for

$$\alpha = \left(\int_{\mathbb{R}} \phi(x)\,dx\right)^{-1} \frac{k}{1-k} \int_{\mathbb{R}} f(x)\phi_0^*(x)\,dx.$$

Upon substituting this α into the inequality, the result follows.

<u>Remark 13</u>: If ϕ and f are defined on any subinterval $X \subseteq \mathbb{R}$, and if E is any subinterval of X, with $\int_E \phi(x)\,dx < \infty$, then setting $\tilde{\phi} = \phi\chi_E$, and applying Theorem 2.11 to $\tilde{\phi}$ yields

$$\int_E \tilde{f}_0^*(x)\phi(x)\,dx \leq A_p\left(\int_E \tilde{\phi}(x)\,dx\right)^{1-p}\left(\int_X f(x)\tilde{\phi}_0^*(x)\,dx\right)^p .$$

Since $\tilde{\phi}_0^*(x) \leq \phi_0^*(x)$, it follows that

$$\int_E f_0^*(x)\phi(x)\,dx \leq A_p\left(\int_E \phi(x)\,dx\right)^{1-p}\left(\int_X f(x)\phi_0^*(x)\,dx\right)^p$$

We now unify the results of Chapters 1 and 2 by finding conditions, such that

$$\|K*f\|_{p,\phi} \leq A_p\|f\|_{p,\phi^*}.$$

Theorem 2.12: Let φ be a convex, increasing, non-negative function on $\mathbb{R}^+$. If $K(x,y) \geq 0$ satisfies condition $\mathcal{K}$ and is even in y, and

$$h(x,y) \equiv \int_{-\pi}^{\pi} K(x,t)f(y-t)\,dt$$

is finite a.e. for some f, then for $\phi \in L^1(\mathbb{T})$,

$$\int_{-\pi}^{\pi} \phi(y)\varphi(|h(x,y)|)\,dy \leq A\int_{-\pi}^{\pi} \phi^*(y)\varphi(|f(y)|)\,dy ,$$

provided the right side is finite.

Proof: Clearly

$$|h(x,y)| \leq \int_{-\pi}^{\pi} K(x,y-t)|f(t)|\,dt .$$

Therefore, by Jensen's Inequality,

$$\varphi(|h(x,y)|) \leq \varphi\left(\int_{-\pi}^{\pi} K(x,y-t)\,|f(t)|\,dt\right)$$

$$\leq \int_{-\pi}^{\pi} K(x,y-t)\varphi(|f(t)|)\,dt.$$

Multiplying by $\emptyset$ and integrating yields

$$\int_{-\pi}^{\pi} \emptyset(y)\varphi(|h(x,y)|)\,dy \leq \int_{-\pi}^{\pi} \emptyset(y)\left(\int_{-\pi}^{\pi} K(x,y-t)\varphi(|f(t)|)\,dt\right)dy$$

$$= \int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} \emptyset(y)K(x,y-t)\,dy\right)\varphi(|f(t)|)\,dt$$

$$\leq A\int_{-\pi}^{\pi} \emptyset^*(t)\varphi(|f(t)|)\,dt \;.$$

The second last step is justified by Fubini's Theorem, and the last step is just Theorem 2.8.

For the cases $0 < p < 1$ and $1 \leq p < \infty$, we have the following theorem.

<u>Theorem 2.13</u>: Under the hypotheses of Theorem 2.11,

(a) $\displaystyle\int_{-\pi}^{\pi} |h(x,y)|^p \emptyset(y)\,dy \leq A\int_{-\pi}^{\pi} |f(y)|^p \emptyset^*(y)\,dy$

for $1 \leq p < \infty$, provided the right side is finite.

For $0 < p < 1$, we have

$$\text{(b)} \quad \int_{-\pi}^{\pi} |h(x,y)|^p \phi(y)\,dy \leq A_p \left(\int_{-\pi}^{\pi} \phi(y)\,dy\right)^{1-p} \left(\int_{-\pi}^{\pi} f(x)\,\phi_0^*(x)\,dx\right)^p.$$

Proof: (a): In Theorem 2.12, let $\varphi(x) = x^p$, for $1 \leq p < \infty$ and (a) follows immediately.

(b): By Theorem 2.8, $|h(x,y)| \leq A_p f^*(y)$.

Applying Theorem 2.11 shows that

$$\int_{-\pi}^{\pi} |h(x,y)|^p \phi(y)\,dy \leq A \int_{-\pi}^{\pi} (f^*(y))^p \phi(y)\,dy$$

$$\leq A_p \left(\int_{-\pi}^{\pi} \phi(y)\,dy\right)^{1-p} \left(\int_{-\pi}^{\pi} f(y)\,\phi_0^*(y)\,dy\right)^p$$

for $0 < p < 1$.

The following two examples give another illustration of Theorem 2.12.

Example (1): If $\varphi(x) = e^x$, then

$$\int_{-\pi}^{\pi} e^{|h(x,y)|} \phi(y)\,dy \leq A \int_{-\pi}^{\pi} \phi_0^*(y)\, e^{|f(y)|}\,dy.$$

Example (2): If $\varphi(x) = \begin{cases} x \log x, & x > 1 \\ 0 & 0 \leq x \leq 1 \end{cases}$

then φ is convex, and, since

$$\log^+ x = \begin{cases} \log x, & x > 1 \\ 0 & , 0 \leq x \leq 1 \end{cases}$$

then

$$\int_{-\pi}^{\pi} |h(x,y)| \log^+ |h(x,y)| \phi(y)\,dy$$

$$\leq A \int_{-\pi}^{\pi} \phi_0^*(y)\, |f(y)| \log^+ |f(y)|\,dy \ .$$

CHAPTER III

In this chapter, some inequalities of Fefferman and Stein, [5, p.p. 107-114], are extended from the discrete case to the n-dimensional case.

We commence by introducing another function space.

Definition 3.1: For $0<p,r<\infty$, we define the spaces $L^{p,r}(\mathbb{R}^n x \mathbb{R}^{+m})$ to consist of all (equivalence classes of) functions $f(x,y)$ defined on $\mathbb{R}^n x \mathbb{R}^{+m}$ such that

$$\|f\|_{L^{p,r}} = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}} |f(x,y)|^r dy\right)^{p/r} dx\right)^{1/p}$$

is finite.

These spaces can be shown to be linear metric spaces by straightforward calculation.

Definition 3.2: Let $f(x,y)$ be defined on $\mathbb{R}^n x \mathbb{R}^m$. Define, for fixed $y \in \mathbb{R}^m$.

$$f^*(x,y) = \sup_Q \frac{1}{\lambda(Q)} \int_Q f(t,y)\,dt \text{ ,}$$

where the sup is taken over all cubes centered at x.

We now obtain the main theorem of this chapter, the proof of which is a modification of that of Fefferman and Stein. It s importance lies in the fact that it incorporates a Banach valued Hardy-Littlewood maximal theorem on $\mathbb{R}^n$, from which the result of Fefferman and Stein follows.

<u>Theorem 3.3</u>: If $f \in L^{p,r}(\mathbb{R}^n \times \mathbb{R}^{+m})$, $1 < p,r < \infty$, then

1) $\|f^*\|_{L^{p,r}} \leq A_{r,p} \|f\|_{L^{p,r}}$.

2) If $1 < r < \infty$, and $\alpha > 0$, then

$$\lambda\{x \in \mathbb{R}^n : (\int_{\mathbb{R}^{+m}} |f^*(x,y)|^r dy)^{1/r} > \alpha\} \leq \frac{A_r}{\alpha}\|f\|_{L^{1,r}} .$$

<u>Proof</u>: If the right hand side of either 1) or 2) is infinite, the inequality is trivial. Therefore, they are assumed finite. Without loss of generality, let $f \geq 0$.

The proof will now proceed as follows: (a): the case p=r in 1) ; (b): proof of 2) ; (c): proof of the case $p \leq r$ in 1) ; and finally, (d): the case $p \geq r$ in 1).

(a): p=r: By Fubini's Theorem, and the n-dimensional form of the Hardy- Littlewood maximal theorem, (see, for example, [5,p.p. 107-114]), it follows that

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}}(f^*(x,y))^r dy\right)^{r/r} dx\right)^{1/r}$$
$$= \left(\int_{\mathbb{R}^{+m}}\left(\int_{\mathbb{R}^n}(f^*(x,y))^r dx\right) dy\right)^{1/r}$$
$$\leq A_{r,r}\left(\int_{\mathbb{R}^{+m}}\left(\int_{\mathbb{R}^n}(f(x,y))^r dx\right) dy\right)^{1/r}$$
$$= A_{r,r}\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}}(f(x,y))^r dy\right) dx\right)^{1/r},$$

proving (a).

(b): Inequality 2): By the Calderon-Zygmund Lemma, there exists cubes, Q_k, with the properties: If

$$F(x) \equiv \left(\int_{\mathbb{R}^{+m}}(f(x,y))^r dy\right)^{1/r} \text{ and } Q \equiv \bigcup_k Q_k$$

and $\alpha > 0$, then

$$\sum_k \lambda(Q_k) < \frac{1}{\alpha}\int_{\mathbb{R}^n} F(x)\,dx ,$$

$F(x) \leq \alpha$ for all $x \in Q^c$, and

$$\frac{1}{\lambda(Q_k)}\int_{Q_k} F(x)\,dx \leq A\alpha ,$$

A depending on n. Now decompose f into two functions, h and g, such that $f(x,y) = g(x,y) + h(x,y)$, where

$$g(x,y) = f(x,y)\chi_{Q^c}(x) \ , \ h(x,y) = f(x,y)\chi_Q(x) .$$

Since

$$\left(\int_{\mathbb{R}^{+m}} (f^*(x,y))^r dy\right)^{1/r} \leqslant \left(\int_{\mathbb{R}^{+m}} (g^*(x,y))^r dy\right)^{1/r} + \left(\int_{\mathbb{R}^{+m}} (h^*(x,y))^r dy\right)^{1/r} ,$$

where we denote the first member of the right hand side by $G(x)$ and the second by $H(x)$, then 2) will be proved if we show that

$$3) \quad D_G^\lambda(\alpha) \leqslant \frac{A_1}{\alpha}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}} (f(x,y))^r dy\right)^{1/r} dx$$

and

$$4) \quad D_H^\lambda(x) \leqslant \frac{A_2}{\alpha}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}} (f(x,y))^r dy\right)^{1/r} dx .$$

From the obvious fact that

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}} (g(x,y))^r dy\right)^{1/r} dx) \leqslant \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}} (f(x,y))^r dy\right)^{1/r} dx$$

and since $F(x)\leqslant\alpha$ for all $x\in Q^c$, then

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}}(g^*(x,y))^r dy\right)dx \leqslant \alpha^{r-1}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}}(f(x,y))^r dy\right)^{1/r}dx.$$

Therefore, from the case $p=r$,

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}}(g^*(x,y))^r dy\right)dx \leqslant A^r_{r,r}\alpha^{r-1}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}}(f(x,y))^r dy\right)^{1/r}dx.$$

Since, by Chebyshev's Inequality,

$$\alpha^r D^{\lambda}_G(\alpha) \leqslant \int_{\mathbb{R}^n}(G(x))^r dx$$

then

$$D^{\lambda}_G(\alpha) \leqslant \frac{A^r_{r,r}}{\alpha}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}}(f(x,y))^r dy\right)^{1/r}dx$$

which is 3).

To prove 4), let

$$\tilde{f}(x,y) = \begin{cases} \frac{1}{(Q_k)}\int_{Q_k} f(t,y)\,dt, & x\in Q_k \\ 0 & , x\in Q^c. \end{cases}$$

Then, for $x \in Q_k$,

$$\left(\int_{\mathbb{R}^{+m}} (\tilde{f}(x,y))^r dy\right)^{1/r} \equiv \tilde{F}(x) = \left(\int_{\mathbb{R}^{+m}} \left[\frac{1}{\lambda(Q_k)}\int_{Q_k} f(x,y)\,dx\right]^r dy\right)^{1/r}$$
$$\leq \frac{1}{\lambda(Q_k)}\int_{Q_k} \left(\int_{\mathbb{R}^{+m}} (f(x,y))^r dy\right)^{1/r} dx,$$

by the continuous version of Minkowski's Inequality. From the fact that $\frac{1}{\lambda(Q_k)}\int_{Q_k} F(x)\,dx \leq A\alpha$, the right hand side of this inequality is $\leq A\alpha$, for $x \in Q$, and for $x \in Q^c$, $\tilde{f}(x,y) = 0$ and hence $\tilde{F}(x)=0$. Therefore, $\tilde{F}(Q)$ has support in Q, and is bounded by $A\alpha$. Since $\sum_k \lambda(Q_k) < \frac{1}{\alpha}\int_{\mathbb{R}^n} F(x)\,dx$, it follows that

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{+m}} (\tilde{f}(x,y))^r dy\right) dx \leq (A\alpha)^r \lambda(Q)$$
$$\leq A^r \alpha^{r-1} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{+m}} (f(x,y))^r dy\right)^{1/r} dx.$$

From the case $p=r$, and by Chebyshev's Inequality,

$$\alpha^r D^{\lambda}_{\tilde{F}^*}(\alpha) \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{+m}} (\tilde{f}^*(x,y))^r dy\right) dx,$$

it follows that

$$\lambda\left\{x \in \mathbb{R}^n : \left(\int_{\mathbb{R}^{+m}} (\tilde{f}^*(x,y))^r dy\right)^{1/r} > \alpha\right\} \leq \frac{A'}{\alpha}\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{+m}} (f(x,y))^r dy\right)^{1/r} dx.$$

This will imply 4) if there exists a constant A such that $h^*(x,y) \leq A\tilde{f}^*(x,y)$.

For any cube J, let $\tilde{J}$ denote a cube concentric with J, but having diameter 4n times as large. Let $\tilde{Q} = \bigcup_{k=1}^{\infty}\tilde{Q}_k$. Since $\sum_k \lambda(Q_k) < \frac{1}{\alpha}\int_{\mathbb{R}^n} F(x)dx$, it follows that

$$\lambda(\tilde{Q}) \leq \frac{A''}{\alpha}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{+m}}(f(x,y))^r dy\right)^{1/r}dx.$$

Since $h^*(x,y) = \sup_J \frac{1}{\lambda(J)}\int_J h(t,y)dt$, and for any fixed cube I with $x \in I$,

$$\frac{1}{\lambda(I)}\int_I h(x,y)\,dx = \frac{1}{\lambda(I)}\sum_{k\in\Lambda}\int_{Q_k\cap I} h(x,y)\,dx$$

where $\Lambda = \{k\in\mathbb{N}: \ Q_k\cap I \text{ is non-empty}\}$, and if $Q_k\cap I$ and $I\setminus\tilde{Q} \subseteq I\setminus\tilde{Q}_k$, are non-empty for some k, then $Q_k \subseteq \tilde{I}$, it follows

that

$$\begin{aligned}
\frac{1}{\lambda(I)}\sum_{k\in\Lambda}\int_{Q_k\cap I}(h(x,y))\,dx &\leq \frac{1}{\lambda(I)}\sum_{k\in\Lambda}\int_{Q_k}(h(x,y))\,dx \\
&\leq \frac{1}{\lambda(I)}\sum_{k\in\Lambda}\int_{Q_k}(\tilde{f}(x,y))\,dx \\
&\leq \frac{1}{\lambda(I)}\int_{\tilde{I}}(\tilde{f}(x,y))\,dx \\
&\leq \frac{A}{\lambda(\tilde{I})}\int_{\tilde{I}}\tilde{f}(x,y)\,dx \\
&\leq A\tilde{f}^*(x,y).
\end{aligned}$$

Therefore

$$\lambda\{x\in\tilde{Q}^c: \ (\int_{\mathbb{R}^{+m}}(h^*(x,y))^r dy)^{1/r} > A\alpha\}$$

$$\leq \lambda\{x\in\tilde{Q}^c: \ (\int_{\mathbb{R}^{+m}}(\tilde{f}^*(x,y))^r dy)^{1/r} > \alpha\}$$

$$\leq \frac{A}{\alpha}(\int_{\mathbb{R}^n}(\int_{\mathbb{R}^{+m}}(f(x,y))^r dy)^{1/r} dx.$$

Since

$$\lambda\{x\in\tilde{Q}: \ (\int_{\mathbb{R}^{+m}}(h^*(x,y))^r dy)^{1/r} > A\alpha\} \leq \lambda(\tilde{Q})$$

$$\leq \frac{A}{\alpha}\int_{\mathbb{R}^n}(\int_{\mathbb{R}^{+m}}(f(x,y))^r dy)^{1/r} dx,$$

then 4) follows. This completes the proof of 2).

(c): $1<p\leq r$: By (a), the operator T defined by

$$T(f(x,\cdot)) = (\int_{\mathbb{R}^{+m}}(f^*(x,y))^r dy)^{1/r}$$

has been shown to be of strong type (r,r). 2) can be restated to say T is of weak type (1,1). Hence, by the Marcinkiewicz Interpolation Theorem, T is of strong type (p,p) where $1<p<r$, that is

$$\|f^*\|_{L^{p,r}} \leq A_{r,p}\|f\|_{L^{p,r}}.$$

This is (c).

(d): To prove the case $p \geq r$, we observe [6, p.p 107-114] that for positive f and ϕ

$$\int_{\mathbb{R}^n} (f^*(x))^r \phi(x)\,dx \leq A_r \int_{\mathbb{R}^n} (f(x))^r \phi^*(x)\,dx\,.$$

Hence for fixed $y \in \mathbb{R}^{+m}$,

$$\int_{\mathbb{R}^n} (f^*(x,y))^r \phi(x)\,dx \leq A_r \int_{\mathbb{R}^n} (f(x,y))^r \phi^*(x)\,dx\,.$$

Therefore, by Fubini's Theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^{+m}} (f^*(x,y))^r dy \Big) \phi(x)\,dx \\
&\qquad = \int_{\mathbb{R}^{+m}} \Big(\int_{\mathbb{R}^n} (f^*(x,y))^r \phi(x)\,dx \Big) dy \\
&\qquad \leq A_r \int_{\mathbb{R}^{+m}} \Big(\int_{\mathbb{R}^n} (f(x,y))^r \phi^*(x)\,dx \Big) dy \\
&\qquad = A_r \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^{+m}} (f(x,y))^r dy \Big) \phi^*(x)\,dx.
\end{aligned}$$

Letting ϕ run over the unit ball of $L^q(\mathbb{R}^n)$, where $\frac{1}{q} + \frac{r}{p} = 1$, and applying the converse of Holder's Inequality for n-dimensional space, then

$$\|f^*\|^r_{L^{p,r}} \leq A'_{r,p/r} \|f\|^r_{L^{p,r}}$$

$$\leq A'_{r,p}\|f\|^r_{L^{p,r}} .$$

This completes (d), and thus the theorem is totally proved.

Corollary 1: (Fefferman and Stein) Let $f = (f_1, f_2, \ldots)$ be a sequence of functions on $\mathbb{R}^n$. Form the sequence f^*, whose k^{th} term is the maximal function of f_k. Then for $1<r,p<\infty$,

$$5)\quad \left(\int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty}[f_k^*(x)]^r\right)^{p/r}dx\right)^{1/p}$$

$$\leq A_{r,p}\left(\int_{\mathbb{R}^r}\left(\sum_{k=1}^{\infty}|f_k(x)|^r\right)^{p/r}dx\right)^{1/p}$$

provided the right hand side is finite.

For $1 < r < \infty$,

$$6)\quad \lambda\left\{x \in \mathbb{R}^n : \left(\sum_{k=1}^{\infty}\left[f_k^*(x)\right]^r\right)^{1/r} > \alpha\right\} \leq \frac{A_r}{\alpha}\int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty}|f_k(x)|^r\right)^{1/r}dx .$$

Proof: This follows by letting $m = 1$ and setting for $n \in \mathbb{N}$, $f(x,y) = f_n(x)$, $n-1 \leq y < n$, in Theorem 3.3.

Corollary 2: If $f \in L^p(\mathbb{R})$, $1 < p < \infty$, then $f_0^* \in L^p(\mathbb{R})$, and

$$\|f_0^*\|_p \leqslant A_p \|f\|_p.$$

<u>Proof</u>: In Corollary 1, let n = 1, and set f = (f,0,0,...), and the result follows.

We conclude by stating that Theorems 2.3 and 2.8 have their appropriate extensions if one considers convolutions of functions of two variables and the maximal functions defined in this chapter.

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