ATOMIC COMPACTNESS IN QUASI-PRIMITIVE CLASSES OF STRUCTURES

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Master of Science

McMaster University
April 1974

MASTER OF SCIENCE (1974)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Atomic Compactness in Quasi-Primitive Classes of Structures

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NUMBER OF PAGES: iv, 68

ACKNOWLEDGEMENTS

For patient hours of thought-provoking discussion, the author thanks her supervisor, Dr. Bernhard Banaschewski. For a little confidence, many helpful words and the inspiring comment that the exercise herein set down is "a great service to those of us who frequently say '... and, of course, this also holds for structures'", she also thanks his colleague, Dr. Evelyn Nelson.

Finally gratitude is expressed to Ms. Jane Fabricius for her efficient typing of this manuscript.

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INTRODUCTION

The origin of this work lies in a statement made by B. Banaschewski and E. Nelson in [4] that all proofs presented in that paper on equational compactness of algebras could be readily modified to give analogous results for atomic compactness of structures; that is, sets endowed with a family of relations as well as a family of operations.

Their work represents a highly successful attempt at proving by purely algebraic methods the major results proved model-theoretically by Taylor in [7]. It is these methods which are employed here in their natural extensions to the theory of structures. The most useful tool in moving from algebras to structures is a generalization, due to Evelyn Nelson, of the kernel of a homomorphism called the relation kernel: For a structure homomorphism $f: A \to B$ the relation kernel of f is the disjoint union of the set of pairs of elements identified by f (ie. kerf) with the disjoint union of the sets of sequences in A which are mapped pointwise by f into the corresponding relation on B. A striking example of the naturality of this generalization is the fact that, simply by replacing "kernel" by "relation kernel", we obtain a Homomorphism Decomposition Theorem for structures.

A slight deviation from the practice of straight-forward generalization is the setting with respect to which our results are obtained. The classes of structures which might be called varieties, ie. those which are productive, hereditary and closed under formation of quotients, are noted to be unnecessarily restrictive. Instead we work, quite successfully, within quasi-primitive classes of structures ie. classes which are productive, hereditary and closed under formation of up-directed colimits.

Whereas Taylor's aim is stated as characterization of those varieties which have "only" a set of subdirectly irreducible (respectively, pure-irreducible) members, that of Banaschewski and Nelson can be viewed as characterization of those varieties with enough (respectively, purely-enough) equationally compact algebras. It is the latter emphasis that is adopted here.

The actual presentation of results herein follows the ideology and style of [2] and [3]. The characterizations of quasi-primitive classes having, in certain senses, "enough" atomic compact structures are given without the many cardinality results of similar theorems for algebras in [7] and [4]. The significant point here is that good characterizations of such classes may still be obtained without determining explicit cardinality bounds, as in [7] and [4], for the small atomic compact structures. A key component of this alternate method of proof is the uniqueness of atomic compact hulls.

This presentation is entirely self-contained. In Chapters 1 and 2 are given basic definitions and remarks about structures and their homomorphisms including the significant observation that, for structures, monomorphisms are one-one maps and not necessarily embeddings. The relation kernel of a homomorphism is defined and both a Homomorphism Decomposition Theorem and a First Isomorphism Theorem are stated and proved for structures. Chapter 3 provides some basic facts about the category of structures of a given type and their homomorphisms discussing, in particular, updirected colimits and freeness in this category and some special full subcategories.

In Chapter 4 the concept of quasi-primitive classes is introduced and studied as a suitable setting for our discussion. Here a version of Birkhoff's Subdirect Representation Theorem [5, Thm. 20.3] relativized to a fixed quasi-primitive class is proved.

The focus of Chapter 5 is on the parallel developments for monomorphisms, embeddings and pure embeddings of concepts intimately related to atomic compactness.

Finally, in Chapter 6, we define the notion of an atomic compact structure providing a large class of examples of such, in the class of all structures whose underlying set can be endowed with a compact Hausdorff topology compatible with its operations and relations, and also a helpful characterization theorem. Then we introduce the notion of atomic compact hull and a characterization theorem for the existence of such for a given structure in a given quasi-primitive class. This characterization leads to an analogue of Taylor's result about the representation of equationally compact algebras [6 , Cor. 5.8] which is a major tool for reaching our ultimate goal.

Chapters 7 and 8 contain the main results of this work, the characterization of quasi-primitive classes of structures with, in three senses, "enough" atomic compact structures.

CHAPTER I

STRUCTURES AND THEIR HOMOMOPHISMS

This preliminary chapter contains some basic definitions and facts in the theory of structures.

Definition 1.1: A <u>type of structures</u> is a pair (τ,σ) where $\tau = (n_\lambda)_{\lambda\varepsilon\Lambda}$ and $\sigma = (m_\rho)_{\rho\varepsilon P}$ are families of cardinals. A <u>structure of type (τ,σ)</u> is a triple $A = (X,(f_\lambda)_{\lambda\varepsilon\Lambda}, (g_\rho)_{\rho\varepsilon P})$ where X is a set, each $f_\lambda\colon X^{n_\lambda} \to X$, and each g_ρ is a subset of X^{m_ρ}. X is the <u>underlying set of A</u>, f_λ is the <u>λ-th operation of A</u>, g_ρ is the <u>ρ-th relation of A</u>. If P is empty, A is called a <u>(universal) algebra of type τ</u>; if Λ is empty, A is called a <u>relational system of type σ</u>.

In order to clearly indicate the underlying set, operations, and relations of a specific structure A, the notation will be as follows: $X = |A|$, $f_\lambda = \lambda_A$, $g_\rho = \rho_A$.

Within this discussion all structures will be <u>finitary</u>; that is all n_λ, m_ρ will be finite cardinals.

Definition 1.2: For structures A,B of type (τ,σ) a <u>homomorphism from A to B</u> is a set map $f\colon |A| \to |B|$ for which $\lambda_B f^{n_\lambda} = f\lambda_A$ for each $\lambda\varepsilon\Lambda$ and $f^{m_\rho}(\rho_A) \subseteq \rho_B$ for each $\rho\varepsilon P$. Notation: $f\colon A \to B$.

Remark 1.3: Because the identity map is a homomorphism and the composition of two homomorphisms is a homomorphism, we can speak of the category of all structures (of type (τ,σ)) and their homomorphisms. Notation: $\mathcal{S}(\tau,\sigma)$.

Definition 1.4: A <u>substructure</u> of a structure A of type (τ,σ) is a structure B of same type such that $|B|$ is a subset of $|A|$, $\lambda_B = \lambda_A / |B|^{n_\lambda}$ for each $\lambda\varepsilon\Lambda$ and $\rho_B = \rho_A \cap |B|^{m_\rho}$ for each $\rho\varepsilon P$. Notation: $B \subseteq A$.

Remark 1.5:

1. If a subset Y of $|A|$ is such that $\lambda_A(x_1,\ldots,x_{n_\lambda})$ is in Y whenever $x_1,\ldots,x_{n_\lambda}$ are in Y, for each $\lambda\varepsilon\Lambda$, then Y is the underlying set of a substructure of A.

2. The set-theoretic intersection of a family of substructures is again a substructure.

3. The set-theoretic union of an updirected family of substructures is again a substructure.

Proof of 2,3: Consider a family $(A_\alpha)_{\alpha\varepsilon\Phi}$ of substructures of a structure A. By Remark 1.5.1., it **suffices** to show that, for each $\lambda, \lambda_A(x_1,\ldots,x_{n_\lambda}) \varepsilon \bigcap |A_\alpha|$ whenever $x_1,\ldots,x_{n_\lambda}$ are in $\bigcap |A_\alpha|$, but this is clearly guaranteed by the fact that each A_α is a substructure of A.

Consider an updirected family $(A_\alpha)_{\alpha\varepsilon\Phi}$ of substructures of A. Again, it **suffices** to show that $\lambda_A(x_1,\ldots,x_{n_\lambda}) \ \varepsilon \bigcup |A_\alpha|$ whenever $x_1,\ldots,x_{n_\lambda}$ are in $\bigcup |A_\alpha|$. Because n_λ is finite, there exists $\alpha\varepsilon\Phi$ with $x_1,\ldots,x_{n_\lambda}$ in $|A_\alpha|$. Then $\lambda_A(x_1,\ldots,x_{n_\lambda}) = \lambda_{A_\alpha}(x_1,\ldots,x_{n_\lambda})$ is in $|A_\alpha|$, hence, in $\bigcup |A_\alpha|$.

Definition 1.6: If $f\colon A \to B$ in $\mathcal{J}(\tau,\sigma)$, the <u>image of f</u> is the substructure of B with underlying set $f(|A|)$.
Notation: Im(f). A homomorphism $f\colon A \to B$ is an <u>embedding</u> iff it is an isomorphism of A with Im(f).

These special maps have certain nice properties.

Remark 1.7:

1. The composite of two embeddings is an embedding.

2. If fg is an embedding, then g is an embedding.

and, hence,

3. If f has a left inverse, then f is an embedding.

Definition 1.8: Let $(A_\alpha)_{\alpha\varepsilon\Phi}$ be a family of structures of type (τ,σ). The <u>product structure of the A_α</u> is a triple $A = (\Pi|A_\alpha|,\ (\lambda_A),\ (\rho_A))$ where each λ_A is defined by $p_\alpha\lambda_A = \lambda_{A_\alpha} p_\alpha^{\,n_\lambda}$ for each α (where the p_α are the projection maps) and and each ρ_A is defined by $x\varepsilon\rho_A$ iff $p_\alpha^{\,m_\rho}(x)\varepsilon\rho_{A_\alpha}$ for each α.

Remark 1.9: A is the categorical product in $\mathcal{J}(\tau,\sigma)$ of the family $(A_\alpha)_{\alpha\varepsilon\Phi}$. Notation: $A = \Pi A_\alpha$.

Definition 1.10: For $f\colon A \to B$, the <u>kernel of f</u> is $\{(a,b)\varepsilon|A|^2 \mid f(a) = f(b)\}$. Notation: kerf.

Remark 1.11: Kerf, with operations and relations the restriction of those on $A\times A$ $(=A^2)$, is a substructure of A^2.

With the use of this concept, we can prove the first part of the following statement.

Remark 1.12:

1. Monomorphisms in $\mathcal{S}(\tau,\sigma)$ are exactly one-one homomorphisms.

2. Monomorphisms are not necessarily embeddings.

Proof: 1. Assume that $f: A \to B$ is one-one. Consider $g,h: C \to A$ such that $fg = fh$. Then, since $fg(x) = fh(x)$, $g(x) = h(x)$ for all $x \varepsilon C$; hence, $g = h$.

Assume that $f: A \to B$ is a monomorphism. Consider x,y elements of A with $f(x) = f(y)$; that is, $(x,y) \varepsilon \ker f$. For p,q the restrictions to $\ker f$ of the projections of A^2 to A, $fp = fq$; hence, $p = q$. In particular $x = p(x,y) = q(x,y) = y$.

2. Consider the set N of natural numbers. Define N_1 to be the structure $(n,=)$, N_2 the structure $(N,\leq)$. Then, the identity map is obviously one-one but there are elements which are not related by $=$, yet are related by $\leq$.

CHAPTER II

CONGRUENCES AND RELATION KERNELS

The concepts introduced in this chapter are analogues of set-theoretic definitions and results.

Definition 2.1: A congruence on a structure A is a substructure of A^2 which is an equivalence relation.

Remark 2.2:

1. The set-theoretic intersection of a family of congruences is again a congruence.

2. The set-theoretic union of an up-directed family of congruences is again a congruence.

Proof: 1. For a family $(\theta_\alpha)_{\alpha\varepsilon\Phi}$ of congruences on a structure A, by Remark 1.5.2., $\bigcap\theta_\alpha$ is again a substructure of A^2. But, clearly, $\bigcap\theta_\alpha$ is also still an equivalence relation; so, we are finished.

2. Consider an updirected family $(\theta_\alpha)_{\alpha\varepsilon\Phi}$ of congruences on A. Then $\theta = \bigcup\theta_\alpha$ is, by Remark 1.5.3, a substructure of A^2. Clearly, θ is reflexive and symmetric. If $(x,y)\varepsilon\theta$, $(y,z)\varepsilon\theta$, there exist α,β in Φ with (x,y) in θ_α, (y,z) in θ_β. The updirectedness of the family provides $\gamma\varepsilon\Phi$ with (x,y), (y,z) and, hence, (x,z) in θ_γ and so in θ.

Definition 2.3: For a congruence θ on a structure A, the <u>quotient structure of A modulo θ</u> is the triple $B = (|A|/\theta, (\lambda_B), (\rho_B))$ where each λ_B is defined by $\lambda_B v^{n_\lambda} = v\lambda_A$ for $v\colon |A| \to |A|/\theta$, the natural quotient map, and each ρ_B is defined by $\rho_B = v^{m_\rho}(\rho_A)$.

Remark 2.4:

1. B is a structure of type (τ,σ). Notation: $B = A/\theta$

2. As for algebras, congruences are exactly the kernels of homomorphisms.

Because of their behaviour, the following special maps are worthy of study.

Definition 2.5: An onto homomorphism $f\colon A \to B$ is a <u>quotient map</u> iff $\rho_B = f^{m_\rho}(\rho_A)$ for each $\rho \varepsilon P$.

We now introduce a generalization of the kernel of a homomorphism which is useful in numerous instances.

Definition 2.6:

1. For a structure $A, A^{\#} = |A|^2 \sqcup \coprod_{\rho\varepsilon P} |A|^{m_\rho}$ (the coproduct in <u>Set</u>, of $|A|^2$ with the coproduct, in <u>Set</u>, taken over the indexing set of the relations of A, of the corresponding powers of $|A|$).

2. For $f\colon A \to B$, the <u>relation kernel of f</u> is $\ker f \sqcup \coprod_{\rho\varepsilon P} (f^{m_\rho})^{-1}(\rho_B)$ (which is a subset of $A^{\#}$). Notation: Rkerf.

Utilizing this concept, we obtain a generalization of a well-known theorem about algebras which is itself a direct analogue of a set-theoretic result.

Proposition 2.7: If $f\colon A \to B$ and $g\colon A \to C$ onto, then there exists a homomorphism $h\colon C \to B$ with $f = hg$ iff $\mathrm{Rker}g \subseteq \mathrm{Rker}f$. Moreover, $\mathrm{Im}(h) = \mathrm{Im}(f)$ and $\mathrm{Rker}h = g^{\#}(\mathrm{Rker}f)$ (where $g^{\#} = g^2 \sqcup \coprod_{\rho\varepsilon P} g^{m_\rho}$).

Proof: If there exists such an h, then $\mathrm{Rker}f = \mathrm{Rker}hg$ which contains Rkerg.

Assuming $\mathrm{Rker}g \subseteq \mathrm{Rker}f$, define $h(x) = f(\bar{x})$, where, because g is onto, there does exist $\bar{x}\varepsilon A$ with $g(\bar{x}) = x$. If $g(\bar{x}) = x = g(\bar{\bar{x}})$, then $f(\bar{x}) = f(\bar{\bar{x}})$; so, h is well-defined. Clearly, $f = hg$. Then, for each $\lambda\varepsilon\Lambda, \lambda_B h^{n_\lambda} g^{n_\lambda} = \lambda_B f^{n_\lambda} = f\lambda_A = hg\lambda_A = h\lambda_C g^{n_\lambda}$; so, because g^{n_λ} is onto, $\lambda_B h^{n_\lambda} = h\lambda_C$. For each $\rho\varepsilon P$, $(g^{m_\rho})^{-1}((h^{m_\rho})^{-1}(\rho_B)) = (f^{m_\rho})^{-1}(\rho_B)$ which contains $(g^{m_\rho})^{-1}(\rho_C)$; hence $(h^{m_\rho})^{-1}(\rho_B) \subseteq \rho_C$; so, h is a structure homomorphism.

Clearly, $\mathrm{Im}(h) = \mathrm{Im}(f)$ and $\mathrm{Rker}h = g^{\#}(\mathrm{Rker}f)$.

Corollary 2.8: If $f\colon A \to B$ and $g\colon A \to C$ quotient map, there exists $h\colon C \to B$ with $f = hg$ iff $\ker g \subseteq \ker f$. In this case, too, $\mathrm{Im}(h) = \mathrm{Im}(f)$, $\mathrm{Rker}h = g^{\#}(\mathrm{Rker}f)$.

Proof: It suffices to show that, when g is a quotient map, $\ker g \subseteq \ker f$ implies $\mathrm{Rker}g \subseteq \mathrm{Rker}f$.

For $\rho\varepsilon P$, take $x_1,\ldots,x_{m_\rho}$ with $(g(x_1),\ldots,g(x_{m_\rho}))\varepsilon\rho_C$. Because g is a quotient map, there exists $(\bar{x}_1,\ldots,\bar{x}_{m_\rho})\varepsilon\rho_A$ with $g(\bar{x}_i) = g(x_i)$, $i \leq m_\rho$.

Then $(f(x_1),\ldots,f(x_{m_\rho})) = (f(\bar{x}_1),\ldots,f(\bar{x}_{m_\rho}))\varepsilon\rho_B$.

Corollary 2.9: If $f\colon A \to B$ quotient map, then $A/\mathrm{ker}f \cong \mathrm{Im}(f)$.

Proof: Because the natural quotient map $v\colon A \to A/\mathrm{ker}f$ is a quotient map, there exists $g\colon A/\mathrm{ker}f \to B$ with $gv = f$, $\mathrm{Im}(g) = \mathrm{Im}(f)$. It suffices to show that g is an embedding. Because $R\mathrm{ker}g = v^{\#}(R\mathrm{ker}f)$, in particular, kerg is trivial. For any $(gv(x_1),\ldots,gv(x_{m_\rho}))\varepsilon\rho_B$, $(gv(x_1),\ldots,gv(x_{m_\rho}))$ $= (f(x_1),\ldots,f(x_{m_\rho}))$; hence, because f is a quotient map, there exists $(\bar{x}_1,\ldots,\bar{x}_{m_\rho})\varepsilon\rho_A$ with $(x_i,\bar{x}_i)\varepsilon\mathrm{ker}f$, $i \leq m_\rho$, and, thus, $(v(x_1),\ldots,v(x_{m_\rho}))\varepsilon\rho_{A/\mathrm{ker}f}$.

Proposition 2.7 will be referred to as the Homomorphism Decomposition Theorem and its second corollary as the First Isomorphism Theorem.

CHAPTER III

BASIC PROPERTIES OF $\mathcal{J}(\tau,\sigma)$

Some basic facts about limits and colimits in $\mathcal{J}(\tau,\sigma)$ are essential to this discussion.

Proposition 3.1: $\mathcal{J}(\tau,\sigma)$ is complete and cocomplete.

Proof: It is sufficient to show that $\mathcal{J}(\tau,\sigma)$ has products, equalizers, coequalizers and coproducts.

As has been previously noted, the product structure of a family of structures in $\mathcal{J}(\tau,\sigma)$ is the categorical product.

For $f,g\colon A \to B$, define $E = \{a \in A \mid f(a) = g(a)\}$. Then E is a substructure of A and, coupled with the natural injection into A, is the equalizer of the pair (f,g).

For $f,g\colon A \to B$, define θ to be the congruence generated by $\{(f(a),g(a)) \mid a \in A\}$. Then B/θ, coupled with the natural quotient map, is the coequalizer of the pair (f,g).

Take $(A_\alpha)_{\alpha\in\Phi}$ a family in $\mathcal{J}(\tau,\sigma)$. Take R a representative set of those structures of type (τ,σ) which are generated by at most Σ card A_α elements (where card A_α is the cardinality of $|A_\alpha|$). Take $H = \{u = (u_\alpha)_{\alpha\in\Phi} \mid u_\alpha\colon A_\alpha \to D_u,\ D_u \in R,\ \bigcup u_\alpha(A_\alpha)$ generates $D_u\}$. Then, for each $\alpha\in\Phi$, consider $\prod_{u\in H} u_\alpha\colon A_\alpha \to \prod_{u\in H} D_u$ defined by $p_u \pi u_\alpha = u_\alpha$ where p_u is the u-th projection. Take E, the substructure of ΠD_u generated by $\bigcup_\alpha \pi u_\alpha(|A_\alpha|)$. We have a family $(j_\alpha)_{\alpha\varepsilon\Phi}$ of morphisms from A_α to E defined by $j_\alpha(a) = \pi u_\alpha(a)$ for all $a\varepsilon A_\alpha$. Then $((j_\alpha),E)$ is the coproduct of $(A_\alpha)_{\alpha\varepsilon\Phi}$: take a family $(g_\alpha: A_\alpha \to C)$ in $\mathcal{f}(\tau,\sigma)$. Then there exists a $u\varepsilon H$ and a $D_u\varepsilon R$ with $u = (u_\alpha)$, $\bigcup u_\alpha(A_\alpha)$ generating D_u and an isomorphism $k: D_u \to C$ with $ku_\alpha = g_\alpha$ for each $\alpha\varepsilon\Phi$. Define $h: E \to C$ by $h = kp_u/E$. Then $hj_\alpha = g_\alpha$ and h, is, in fact, uniquely determined by this equation.

In our discussion, special kinds of colimits, namely updirected colimits, will be used extensively. We define these as follows.

Definition 3.2: For a category $\mathcal{C}$ and a functor $D: I \to \mathcal{C}$ (called a diagram in $\mathcal{C}$), a colimit of D is a pair $((h_i)_{i\varepsilon I},A)$ with $h_i: D(i) \to A$ in $\mathcal{C}$ for all $i\varepsilon I$ and $h_i = h_jD(\alpha)$ for all $\alpha: i \to j$ in I such that, for any family $(g_i: D(i) \to B)_{i\varepsilon I}$ in $\mathcal{C}$ with $g_i = g_jD(\alpha)$ for all $\alpha: i \to j$ in I, there exists a unique $f: A \to B$ in $\mathcal{C}$ with $g_i = fh_i$ for all $i\varepsilon I$. If I is an updirected partially ordered set, $((h_i),A)$ is an <u>updirected</u> <u>colimit</u>.

Remark 3.3: If $((f_i),A)$ and $((g_i),B)$ are colimits in a category $\mathcal{C}$ of the diagram $D: I \to \mathcal{C}$, there exists a unique isomorphism $f: A \to B$ such that $ff_i = g_i$ for all $i\varepsilon I$. Because of this we speak of "the" colimit of a diagram.

Proposition 3.4: For an updirected family $((h_{\alpha\beta}: A_\alpha \to A_\beta)_{\alpha\leq\beta}, (A_\alpha)_{\alpha\varepsilon I})$ in $\mathcal{S}(\tau,\sigma)$, $((h_\alpha A_\alpha \to A), A)$ its colimit in $\mathcal{S}(\tau,\sigma)$, if each $h_{\alpha\beta}$ is an embedding, then each h_α is an embedding.

Proof: It suffices, by the above remark, to find one construction of the colimit of this family for which the colimit homomorphisms are embeddings.

For each $\alpha\varepsilon I$, define B_α to be the intersection of the family of equalizers of the maps $p_\beta, h_{\alpha\beta}p_\alpha$ for all $\beta\geq\alpha$ where the p_α are the projection maps from ΠA_a to A_α. Then define, for each α, a new structure $C_\alpha = (|B_\alpha|, (\lambda_{B_\alpha}), (\rho_{C_\alpha}))$ where an element x of $|B_\alpha|^{m_\rho}$ is in ρ_{C_α} iff $p_\beta^{m_\rho}(x)\varepsilon\rho_{A_\beta}$ for all $\beta\geq\alpha$. Finally, for each $\alpha\varepsilon I$, define $q_\alpha: C_\alpha \to A_\alpha$ by $q_\alpha = p_\alpha/B_\alpha$.

For $x\varepsilon|B_\alpha|, \gamma\geq\beta\geq\alpha$, $h_{\beta\gamma}p_\beta(x) = h_{\beta\gamma}h_{\alpha\beta}p_\alpha(x) = h_{\alpha\gamma}p_\alpha(x) = p_\gamma(x)$ so $B_\alpha \subseteq B_\beta$ whenever $\alpha\leq\beta$. Furthermore, for $\beta\geq\alpha$, and $x_1,\ldots,x_{m_\rho}$ in C_α with $(p_\gamma(x_1),\ldots,p_\gamma(x_{m_\rho}))\varepsilon\rho_{A_\gamma}$ whenever $\gamma\geq\beta$, because $p_\gamma(x_i) = h_{\alpha\gamma}p_\alpha(x_i)$, $i \leq m_\rho$, and $h_{\alpha\gamma}$ is an embedding, $(p_\alpha(x_1),\ldots,p_\alpha(x_{m_\rho}))\varepsilon\rho_{A_\alpha}$ and, hence, $(p_\beta(x_1),\ldots,p_\beta(x_{m_\rho})) = (h_{\alpha\beta}p_\alpha(x_1),\ldots,h_{\alpha\beta}p_\alpha(x_{m_\rho}))\varepsilon\rho_{A_\beta}$. Thus, $C_\alpha \subseteq C_\beta$ whenever $\alpha\leq\beta$. Thus $C = \bigcup_{\alpha\varepsilon I} C_\alpha$ is a structure.

If we define $\theta_\alpha = \ker q_\alpha$ for each $\alpha\varepsilon I$, because $\ker q_\alpha = \ker h_{\alpha\beta}q_\alpha = \ker(q_\beta/C_\alpha) = (\ker q_\beta)/C_\alpha = \theta_\beta/C_\alpha$ if $\alpha\leq\beta$, the θ_α are an updirected family of congruences whose union θ is a congruence on C.

The homomorphisms q_α are quotient maps: Any a in A_α is $q_\alpha((x_\beta)_{\beta\varepsilon I})$ where $x_\alpha = a$, $x_\beta = h_{\alpha\beta}(a)$, $\beta\geq\alpha$. If $(a_1,\ldots,a_{m_\rho})\varepsilon\rho_{A_\alpha}$, then $(h_{\alpha\beta}(a_1),\ldots,h_{\alpha\beta}(a_{m_\rho}))\varepsilon\rho_{A_\beta}$ whenever $\beta\geq\alpha$; so, the preimage of a under $q_\alpha^{m_\rho}$ is in ρ_{C_α}. Thus, by the First Isomorphism Theorem, for each α there exists an isomorphism s_α with $q_\alpha = s_\alpha v_\alpha$ where v_α is the natural quotient map $C_\alpha \to C_\alpha/\theta_\alpha$. Also, for each α, because the restriction v/C_α of the natural quotient map $C \to C/\theta$ is a quotient map, there exists an embedding u_α: $C_\alpha/\theta_\alpha \to C/\theta$ with $v/C_\alpha = u_\alpha v_\alpha$.

We claim that $((h_\alpha), C/\theta)$, where each $h_\alpha = u_\alpha s_\alpha^{-1}$, is colimit of $((h_{\alpha\beta}),(A_\alpha))$. For $\alpha\leq\beta$, $h_\beta h_{\alpha\beta}q_\alpha = h_\beta q_\beta/C_\alpha$

$= u_\beta s_\beta^{-1}q_\beta/C_\alpha = u_\beta v_\beta/C_\alpha = v/C_\alpha = u_\alpha v_\alpha = h_\alpha q_\alpha$, and so,

$h_\beta h_{\alpha\beta} = h_\alpha$. For a family $(g_\alpha\colon A_\alpha \to D)$ in $\mathcal{J}(\tau,\sigma)$ with $g_\alpha = g_\beta h_{\alpha\beta}$ whenever $\alpha\leq\beta$, the homomorphisms $g_\alpha q_\alpha$ are such that $g_\beta q_\beta/C_\alpha = g_\alpha q_\alpha$; so, we can define f: $C \to D$ by $f(x) = g_\alpha q_\alpha(x)$

if $x\varepsilon C_\alpha$. Then each θ_α is contained in kerf; hence, so is θ, and we have g: $C/\theta \to D$ with $f = gv$. And $g_\alpha = gh_\alpha$ for all $\alpha\varepsilon I$.

With the help of this fact, we can establish a convenient way of viewing updirected colimits in $\mathcal{S}(\tau,\sigma)$.

Remark 3.5: Consider an updirected family $((h_{\alpha\beta}: A_\alpha \to A_\beta)_{\alpha\leq\beta}, (A_\alpha)_{\alpha\in I})$ in $\mathcal{S}(\tau,\sigma)$. Define $\theta_\alpha = \bigcup\{\ker h_{\alpha\beta} | \alpha\leq\beta\}$. Then define a structure $B_\alpha = (|A_\alpha/\theta_\alpha|, (\lambda_{A_\alpha/\theta_\alpha}), (\rho_{B_\alpha}))$ where $\rho_{B_\alpha} = \bigcup_{\beta\geq\alpha} u_\alpha{}^{m_\rho}((h_{\alpha\beta}{}^{m_\rho})^{-1}(\rho_{A_\beta}))$. Then, for each $\alpha\in I$, define $u_\alpha: A_\alpha \to B_\alpha$ by $u_\alpha(x) = v_\alpha(x)$ where v_α is the natural quotient map $A_\alpha \to A_\alpha/\theta_\alpha$.

Whenever $\alpha\leq\beta$, $R\ker u_\alpha \subseteq R\ker u_\beta h_{\alpha\beta}$: $\ker u_\alpha = \theta_\alpha \subseteq \ker u_\beta h_{\alpha\beta}$, and, if $(u_\alpha(x_1),\ldots,u_\alpha(x_{m_\rho}))\in\rho_{B_\alpha}$, there exist $\bar{x}_1,\ldots,\bar{x}_{m_\rho}$ in A_α with $(x_i,\bar{x}_i)\in\ker h_{\alpha\gamma_i}$ for some $\gamma_i\geq\alpha$ and $(h_{\alpha\gamma_0}(\bar{x}_1),\ldots,h_{\alpha\gamma_0}(\bar{x}_{m_\rho}))\in\rho_{A_{\gamma_0}}$ for some $\gamma_0\geq\alpha$. Take $\gamma\geq\beta,\gamma_j, 0\leq j\leq m_\rho$. Then $h_{\alpha\gamma}(\bar{x}_i) = h_{\alpha\gamma}(x_i)$ and so $(h_{\alpha\gamma}(x_1),\ldots,h_{\alpha\gamma}(x_{m_\rho}))\in\rho_A$; so, we have $h_{\alpha\beta}(x_i)$, $i\leq m_\rho$, in A_β with $(h_{\beta\gamma}(h_{\alpha\beta}(x_1)),\ldots,h_{\beta\gamma}(h_{\alpha\beta}(x_{m_\rho})))\in\rho_{A_\gamma}$ forcing $(u_\beta h_{\alpha\beta}(x_1),\ldots,u_\beta h_{\alpha\beta}(x_{m_\rho}))\in\rho_{B_\beta}$. Therefore, by Homomorphism Decomposition Theorem, there exists $\bar{h}_{\alpha\beta}: B_\alpha \to B_\beta$ with $\bar{h}_{\alpha\beta}u_\alpha = u_\beta h_{\alpha\beta}$ whenever $\alpha\leq\beta$. Moreover, each $\bar{h}_{\alpha\beta}$ is an embedding: $\ker\bar{h}_{\alpha\beta} = u_\alpha{}^2(\ker u_\beta h_{\alpha\beta})$ which is trivial; so, $\bar{h}_{\alpha\beta}$ is one-one. Take $u_\alpha(x_1),\ldots,u_\alpha(x_{m_\rho})$ in B_α with $(\bar{h}_{\alpha\beta}u_\alpha(x_1),\ldots,\bar{h}_{\alpha\beta}u_\alpha(x_{m_\rho}))$ $= (u_\beta h_{\alpha\beta}(x_1),\ldots,u_\beta h_{\alpha\beta}(x_{m_\rho}))\in\rho_{B_\beta}$. Then there exist

$\bar{x}_1,\ldots,\bar{x}_{m_\rho}$ in A_β with $h_{\alpha\gamma_i}(x_i) = h_{\beta\gamma_i}(\bar{x}_i)$ for some $\gamma_i \geq \beta$, $i \leq m_\rho$ and $(h_{\beta\gamma_0}(\bar{x}_1),\ldots,h_{\beta\gamma_0}(\bar{x}_{m_\rho}))\varepsilon\rho_{A\gamma_0}$ for some $\gamma_0 \geq \beta$.
Take $\gamma \geq \gamma_j$, $0 \leq j \leq m_\rho$. Then $h_{\alpha_\gamma}(x_i) = h_{\beta_\gamma}(\bar{x}_i)$, $i \leq m_\rho$, and so, $(h_{\alpha_\gamma}(x_1),\ldots,h_{\alpha_\gamma}(x_{m_\rho}))\varepsilon\rho_{A_\gamma}$ forcing $(u_\alpha(x_1),\ldots,u_\alpha(x_{m_\rho}))\varepsilon\rho_{B_\alpha}$.

Consider the colimit $((\bar{h}_\alpha),A)$ of the family $((\bar{h}_{\alpha\beta}),(B_\alpha))$ in $\mathcal{f}(\tau,\sigma)$. We will first show that A, together with the homomorphisms $h_\alpha = \bar{h}_\alpha u_\alpha$, is (up to isomorphism) the colimit in $\mathcal{f}(\tau,\sigma)$ of $((h_{\alpha\beta}),(A_\alpha))$ and then examine some properties of this colimit $((h_\alpha),A)$.

Firstly, for any pair α,β in I with $\alpha \leq \beta$, $h_\alpha = \bar{h}_\alpha u_\alpha = \bar{h}_\beta \bar{h}_{\alpha\beta} u_\alpha = \bar{h}_\beta u_\beta h_{\alpha\beta} = h_\beta h_{\alpha\beta}$. Next, take a family $(g_\alpha\colon A_\alpha \to A_\beta)$ with $g_\alpha = g_\beta h_{\alpha\beta}$ whenever $\alpha \leq \beta$. Then $\mathrm{Rker}u_\alpha \subseteq \mathrm{Rker}h_{\alpha\beta} \subseteq \mathrm{Rker}g_\beta h_{\alpha\beta} = \mathrm{Rker}g_\alpha$ for each α; so there exists $\bar{g}_\alpha\colon B_\alpha \to B$ with $\bar{g}_\alpha u_\alpha = g_\alpha$, for each α. Then $\bar{g}_\alpha u_\alpha = g_\alpha = g_\beta h_{\alpha\beta} = \bar{g}_\beta u_\beta h_{\alpha\beta} = \bar{g}_\beta h_{\alpha\beta} u_\alpha$ and, hence, because u_α is onto, $\bar{g}_\alpha = \bar{g}_\beta h_{\alpha\beta}$ whenever $\alpha \leq \beta$. Then, by the colimit properties of $((\bar{h}_\alpha),A)$, there exists a unique homomorphism $g\colon A \to B$ with $g\bar{h}_\alpha = \bar{g}_\alpha$ and, hence, $gh_\alpha = g_\alpha$.

Note that, because each $\bar{h}_{\alpha\beta}$ is an embedding, so is each $\bar{h}_\alpha$; thus, A_α is isomorphic to $\mathrm{Im}(\bar{h}_\alpha)$.

Finally, we discuss what the colimit looks like. $\bigcup \mathrm{Im}(\bar{h}_\alpha)$ (which is the union of an up-directed set of substructures of A and, hence, a substructure of A), together with homomorphisms

f_α which are the corestrictions of the $\bar{h}_\alpha$ to $\mathrm{Im}(\bar{h}_\alpha)$, is the colimit of the family $((\bar{h}_{\alpha\beta}),(B_\alpha))$ (and, hence, isomorphic to A): Whenever $\alpha\leq\beta$, for $x\varepsilon B_\alpha$, $f_\alpha(x) = \bar{h}_\alpha(x) = \bar{h}_\beta\bar{h}_{\alpha\beta}(x)$ $= f_\beta\bar{h}_{\alpha\beta}(x)$. Take a family $(g_\alpha: A_\alpha/\theta_\alpha \to B)$ in $\mathcal{J}(\tau,\sigma)$ with $g_\alpha = g_\beta\bar{h}_{\alpha\beta}$ whenever $\alpha\leq\beta$. Any $x\varepsilon \bigcup \mathrm{Im}(\bar{h}_\alpha)$ is of the form $x = \bar{h}_\alpha(y)$ for some $\alpha\varepsilon I$ and some $y\varepsilon B_\alpha$. Define $g(x) = g_\alpha(y)$. If $\bar{h}_\alpha(y) = x = \bar{h}_\beta(z)$, because each $\bar{h}_\gamma$ is one-one, $\bar{h}_{\alpha\gamma}(y) = \bar{h}_{\beta\gamma}(z)$ for some $\gamma\geq\alpha,\beta$, and so, $g_\alpha(y) = g_\gamma h_{\alpha\gamma}(y)$ $= g_\gamma h_{\beta\gamma}(z) = g_\beta(z)$; thus, g is well-defined. The fact that g is a homomorphism arises, also, from the updirectedness of I. The uniqueness of g stems from the fact that the $\mathrm{Im}(\bar{h}_\alpha)$ generate $\bigcup \mathrm{Im}(\bar{h}_\alpha)$. But $\mathrm{Im}(\bar{h}_\alpha) = \mathrm{Im}(h_\alpha)$ because the u_α are onto. So A is isomorphic to $\bigcup \mathrm{Im}(h_\alpha)$.

Also the colimit homomorphisms h_α are such that $\mathrm{Rker}h_\alpha = \bigcup_{\beta\geq\alpha}\mathrm{Rker}h_{\alpha\beta}$: $\ker h_\alpha = \ker\bar{h}_\alpha u_\alpha = \ker u_\alpha = \bigcup_{\beta\geq\alpha}\ker h_{\alpha\beta}$. For $\rho\varepsilon P$, $(h_\alpha{}^{m_\rho})^{-1}(\rho_A) = (u_\alpha{}^{m_\rho})^{-1}(\rho_{B_\alpha})$ because each $\bar{h}_\alpha$ is an embedding. By definition of ρ_{B_α}, this gives $(h_\alpha{}^{m_\rho})^{-1}(\rho_A)$ $= (u_\alpha{}^{m_\rho})^{-1}\bigcup_{\beta\geq\alpha}u_\alpha{}^{m_\rho}((h_{\alpha\beta}{}^{m_\rho})^{-1}(\rho_{A_\beta}))$ which certainly contains $\bigcup_{\beta\geq\alpha}(h_{\alpha\beta}{}^{m_\rho})^{-1}(\rho_{A_\beta})$. For $(x_1,\ldots,x_{m_\rho})$ in $(h_\alpha{}^{m_\rho})^{-1}(\rho_A)$, because $\bar{h}_\alpha$ is an embedding, $(u_\alpha(x_1),\ldots,u_\alpha(x_{m_\rho}))\varepsilon\rho_{B_\alpha}$. So, there exist $\bar{x}_1,\ldots,\bar{x}_{m_\rho}$ in A_α with $(h_{\alpha\beta}(\bar{x}_1),\ldots,h_{\alpha\beta}(\bar{x}_{m_\rho}))\varepsilon\rho_{A_\beta}$ and

$h_{\alpha\gamma}(x_i) = h_{\alpha\gamma}(\bar{x}_i)$ for some $\beta,\gamma \geq \alpha$. Then, for $\delta \geq \beta,\gamma$, $(h_{\alpha\delta}(x_1),\ldots h_{\alpha\delta}(x_{m_\rho}))\varepsilon\rho_{A_\delta}$ which says $(x_1,\ldots,x_{m_\rho})$ is in $\bigcup_{\beta\geq\alpha}(h_{\alpha\beta}{}^{m_\rho})^{-1}(\rho_{A_\beta})$.

In short, the updirected colimit of an updirected family $((h_{\alpha\beta}), (A_\alpha))$ in $\mathcal{J}(\tau,\sigma)$ can be viewed as $((h_\alpha), \bigcup \mathrm{Im}(h_\alpha))$ where $\mathrm{Rker}h_\alpha = \bigcup_{\beta\geq\alpha}\mathrm{Rker}h_{\alpha\beta}$.

This observation is useful in proving the following fact about the updirected colimit of a family of embeddings. First we must explain this term. For an updirected family $((h_{\alpha\beta}),(A_\alpha))$ in $\mathcal{J}(\tau,\sigma)$ with colimit $((h_\alpha),A)$ and a family $(u_\alpha: B \to A_\alpha)$ in $\mathcal{J}(\tau,\sigma)$ for which $u_\beta = h_{\alpha\beta}u_\alpha$ whenever $\alpha\leq\beta$, $h_\alpha u_\alpha = h_\beta u_\beta$ for all α,β in I. The homomorphism $u = h_\alpha u_\alpha$ for all $\alpha\varepsilon I$ is called the <u>updirected colimit of the u_α</u>.

Proposition 3.6: The updirected colimit of a family of embeddings is an embedding.

Proof of proposition: Using notation as above, $\mathrm{Rker}u = \mathrm{Rker}(h_\alpha u_\alpha) = (u_\alpha{}^{\#})^{-1}(\mathrm{Rker}h_\alpha) = (u_\alpha{}^{\#})^{-1}\bigcup_{\beta\geq\alpha}\mathrm{Rker}h_{\alpha\beta}$ $= \bigcup_{\beta\geq\alpha}\mathrm{Rker}h_{\alpha\beta}u_\alpha = \bigcup_{\beta\geq\alpha}\mathrm{Rker}u_\beta$. Thus, if each u_β is an embedding, keru is trivial and, for each ρ, $(u^{m_\rho})^{-1}(\rho_A) = \bigcup_{\beta\geq\alpha}(u_\beta{}^{m_\rho})^{-1}(\rho_{A_\beta})$ $= \rho_B$; so, u is an embedding.

The following definition will not be utilized immediately but is a basic concept in the theory of structures which should be mentioned here.

Definition 3.7: For a structure A of type (τ,σ), a <u>subdirect representation of A</u> is a pair $((A_\alpha)_{\alpha\varepsilon I},\Phi)$ where A_α are structures of type (τ,σ), $\Phi: A \to \Pi A_\alpha$ is an embedding and $p_\alpha\Phi$ is onto for all $\alpha\varepsilon I$ where p_α is the α-th projection map.

Certain kinds of subclasses of the class of all structures of a given type will prove important in this discussion. Among these are the classes defined as follows.

Definition 3.8: 1. A subclass Σ of $\mathcal{J}(\tau,\sigma)$ will be called <u>productive</u> iff any structure isomorphic to a product of structures in Σ is itself in Σ.

2. A subclass Σ of $\mathcal{J}(\tau,\sigma)$ will be called <u>hereditary</u> iff any structure A for which there exists an embedding $A \to B$ where $B\varepsilon\Sigma$ is itself in Σ.

The significance of such classes of structures is evident in the study of special structures which we now define.

Definition 3.9: For a class Σ of structures of type (τ,σ) and a set X, a structure A in $\mathcal{J}(\tau,\sigma)$ is <u>free over X relative to Σ</u> iff X generates A and, for every $B\varepsilon\Sigma$, every (set) map $f: X \to |B|$, there exists a $g: A \to B$ with $g/X = f$.

Remark 3.10: If Σ is the empty class, any structure is free over any of its generating sets relative to Σ. If Σ contains only the empty structure (assuming the type permits), any structure is trivially free relative to Σ over any of its non-void generating sets and the structure with empty generating set is itself empty, hence, free over $\emptyset$ relative to Σ. If Σ contains only trivial (ie. one-element) structures, because there is only one map from any set to a one-element set, again, any structure is free over any of its generating sets relative to Σ. And, finally, this is still true if Σ contains only the empty structure and trivial structures.

So, the more interesting situation is when Σ contains nontrivial structures in which case we obtain the following result.

Proposition 3.11: If Σ is a productive, hereditary subclass of $\mathcal{S}(\tau,\sigma)$ which contains nontrivial structures, for any set X, there exists a structure A in Σ free over X in Σ.

Proof: Consider a representative set R (up to isomorphism) of structures in Σ which are generated by at most card X elements. Put $H = \{u: X \to |D_u| \mid D_u \varepsilon R, u(X) \text{ generates } D_u\}$. We have a map $\underset{u\varepsilon H}{\Pi} u: X \to \underset{u\varepsilon H}{\Pi} |D_u|$ defined by $\Pi u(x) = (u(x))_{u\varepsilon H}$ and can take the substructure A of ΠD_u generated by $\Pi u(X)$ and the map j from X to A which is the corestriction of Πu to A.

For $B\varepsilon\Sigma$, $f: X \to |B|$, there exists $u\varepsilon H$, $g: D_u \to B$ with

$gu = f$ where D_u is the structure in R isomorphic to the substructure of B generated by $f(X)$. Then $gp_u/A: A \to B$ (where p_u is the u-th projection of ΠD_u) with $gp_u/A_j = f$.

Now j is one-one: Because Σ contains non-trivial structures the maps from X to any one of these distinguish the points of X and, hence, the $u \varepsilon H$ distinguish the points of X. For this reason there exists a structure C, containing X, isomorphic to A via a map which replaces each $j(x)$ by x. This structure C, by the remarks in the preceding paragraph, is free over X relative to Σ.

Note that nowhere in this construction have we surpassed the bounds of our productive, hereditary class Σ.

Remark 3.12: If A,B are free structures relative to Σ over a set X, there exists a unique isomorphism from A to B mapping X identically.

A further notion, patterned after the topic of free structures, will be especially useful in the study of atomic compactness and purity.

Definition 3.13: 1. If an extension B of a structure A is generated by A and some subset X of B, we say X generates B over A.

2. An extension B of a structure A is called a free extension of A by set X relative to a subclass Σ

of $\mathcal{f}(\tau,\sigma)$ iff X generates B over A and for any map, $f: X \to |C|$ and homomorphism $g: A \to C$ with $C\varepsilon\Sigma$, there exists $h: B \to C$ with $h/X = f, h/A = g$.

By a proof exceedingly similar to that used for free structures, we prove the existence of such structures in non-trivial productive, hereditary subclasses of $\mathcal{f}(\tau,\sigma)$.

Proposition 3.14: If Σ is a productive, hereditary subclass of $\mathcal{f}(\tau,\sigma)$ which contains nontrivial structures then, for any $A\varepsilon\Sigma$, any set X disjoint from $|A|$, there exists a free extension of A by X relative to Σ.

Proof: Consider a representative set R (up to isomorphism) of the structures in Σ which are generated by at most card A + card X elements. Put $H = \{u=(u_0,u_1) \mid u_0: A \to D_u, u_1: X \to |D_u|, D_u\varepsilon R$ generated by $u_0(A) \cup u_1(X)\}$. We have maps $\underset{u\varepsilon H}{\sqcap} u_0: A \to \underset{u\varepsilon H}{\Pi} D_u$ and $\underset{u\varepsilon H}{\sqcap} u_1: X \to \underset{u\varepsilon H}{\Pi} |D_u|$ and can take B the substructure of ΠD_u generated by $\sqcap u_0(A) \cup \sqcap u_1(X)$ and $i: A \to B$ and $j: X \to |B|$ the corestrictions of $\sqcap u_0, \sqcap u_1$ respectively.

For $C\varepsilon\Sigma$, $f: X \to |C|$, $g: A \to C$, there exists $u\varepsilon H$ and a monomorphism $h: D_u \to C$ with $f = hu_1$, $g = hu_0$ (ie. D_u is isomorphic to the substructure of C generated by $f(X) \cup g(A)$). Then $p_u/_B i = u_0$ and $p_u/_B j = u,$; so, $hp_{u/B} i = g$ and $hp_u/_B j = f$.

Now, i,j are one-one: Since Σ contains non-trivial structures, the maps from X and A to any of these distinguish points; hence, so do the u_0, u_1. Hence, there exists a structure D containing A and X which is isomorphic to B by an isomorphism mapping i(a) to a and j(x) to x. By the remarks in the preceding paragraph, it is clear that this D is a free extension over A by X relative to Σ.

Note that nowhere in this construction have we strayed outside our productive, hereditary class Σ.

Remark 3.15: If B,C are free extensions of A by X relative to Σ there exists a unique isomorphism from B to C mapping A and X identically.

CHAPTER IV

QUASI-PRIMITIVE CLASSES

The primary setting for our discussion will be certain classes of structures.

Definition 4.1: A subclass Σ of $\mathcal{S}(\tau,\sigma)$ is <u>quasi-primitive</u> iff it is productive, hereditary and, also, closed under the formation of up-directed colimits; that is, if $((h_{\alpha\beta}),(A_\alpha))$ is an updirected family in Σ with colimit $((h_\alpha),A)$, then A is in Σ, too.

Examples: 1. The class of partially ordered abelian groups is a quasi-primitive subclass of the class of structures of type ((0,1,2); (2)).

2. The class of partially ordered rings is a quasi-primitive subclass of the class of structures of type ((0,1,2,2); (2)).

3. The class of graphs is a quasi-primitive subclass of the class of relational systems of type (2).

Remark 4.2: If this discussion were to be a direct analogue of the results in [4] the setting would be classes of structures closed under formation of quotient structures, substructures and products (probably called <u>primitive classes</u>).

But such classes are too restrictive for our purposes because they do not include in their number even such a familiar class of structures as that of partially ordered abelian groups: Although Z, the set of integers, is a partially ordered subgroup of the additive abelian group R of reals with the usual ordering, and Z determines a congruence on R for which we denote the quotient structure by R/Z, this R/Z is not a partially ordered abelian group.

But, fortunately, quasi-primitive classes retain some significant properties of primitive classes.

Firstly, because we are interested in remaining within a quasi-primitive class Σ under certain constructions, we must define a <u>Σ-congruence</u> on a structure A as kernel θ of a homomorphism from A to a structure in Σ. Noting that a quasi-primitive class Σ in $\mathcal{S}(\tau,\sigma)$ is a full, reflective subcategory of $\mathcal{S}(\tau,\sigma)$, we define the following: The <u>Σ-quotient</u> of A by θ is the reflection of A/θ in Σ. Notation: $A/\!\!/\theta$. There is then a natural map, called the <u>natural quotient map</u>, from A to $A/\!\!/\theta$.

Clearly the intersection of a family of Σ-congruences is again such as is the union of an up-directed family of Σ-congruences.

Proposition 4.3: Quasi-primitive classes are complete and cocomplete.

Proof: Consider a quasi-primitive class Σ determining a full subcategory of $\mathcal{S}(\tau,\sigma)$. Because Σ is productive and hereditary, it has products and equalizers as given in Proposition 3.1.

As for coproducts, if we take for R a representative set of structures in Σ which are so generated and then proceed with the construction as in Proposition 3.1, we obtain coproducts in Σ.

It remains to be shown that Σ has coequalizers. For $f,g\colon A \to B$ in Σ, take θ the Σ-congruence generated by $\{(f(a),g(a)) \mid a \in A\}$. Then $B/\!\!/\theta$, coupled with the natural quotient map $B \to B/\!\!/\theta$, is the coequalizer of f,g in Σ.

The concept, introduced earlier, of subdirect irreducibles seems to have little future. What does seem to be of use is the following.

Definition 4.4: 1. A <u>weak subdirect representation</u> of a structure A of type (τ,σ) is a pair $(\Phi,(A_\alpha))$ where (A_α) is a family in $\mathcal{S}(\tau,\sigma)$, $\Phi\colon A \to \Pi A_\alpha$ a monomorphism and each $p_\alpha\Phi$ onto (for p_α projections).

2. A structure A is <u>weak subdirectly irreducible in</u> productive, hereditary <u>Σ</u> iff, for each subdirect representation $(\Phi,(A_\alpha))$ of A, where each $A_\alpha \in \Sigma$ some $p_\beta\Phi$ is one-one as well as onto.

Remark 4.5: For Σ a quasi-primitive class, a structure A is weak subdirectly irreducible in Σ iff the identity congruence Δ of A is completely meet-irreducible in the Σ-congruence lattice of A (that is, Δ cannot be expressed as the intersection of a family of Σ-congruences not containing Δ). Assume Δ completely meet-irreducible, then, for a weak subdirect representation $(\Phi,(A_\alpha))$ in Σ of A, $\Delta = \ker\Phi = \bigcap \ker p_\alpha\Phi$; hence, some $\ker p_\alpha\Phi$ is trivial ie. some $p_\alpha\Phi$ is a monomorphism. Now, assume A is weak subdirectly irreducible in Σ. Consider a family (θ_α) of Σ-congruences on A with $\bigcap\theta_\alpha = \Delta$. Define $\Phi\colon A \to \Pi A/\!\!/\theta_\alpha$ by $p_\alpha\Phi = v_\alpha$ for each α (where v_α is the α-th natural quotient map). Then $(\Phi,(A/\!\!/\theta_\alpha))$ is a weak subdirect representation in Σ of A; so some $\ker p_\alpha\Phi = \theta_\alpha$ is trivial.

On this result and a well-known result of lattice theory hinges the following generalization of the Birkhoff Representation Theorem for algebras.

Proposition 4.6: For a quasi-primitive class Σ (of finitary structures) any structure in Σ has a weak subdirect representation by weak subdirect irreducibles in Σ.

Proof: The lattice-theoretic contribution to this result is the fact that every element of an algebraic lattice is the meet of completely-meet-irreducible elements (this is sometimes known as McCoy-Fuchs theorem). The finitariness of the structure, call it A, and the fact that Σ is closed under updirected colimits, guarantees that its Σ-congruence lattice is algebraic. Consider a family (θ_α) of Σ-congruences on A whose intersection is Δ, the identity congruence on A.

Then each $A /\!\!/ \theta_\alpha$ is weak subdirectly irreducible because its identity congruence is completely-meet-irreducible. And, for $\Phi: A \to \Pi A /\!\!/ \theta_\alpha$ defined by $p_\alpha \Phi = v_\alpha$ for each α, $(\Phi, (A /\!\!/ \theta_\alpha))$ is a weak subdirect representation of A.

CHAPTER V

SPECIAL MAPS

In this section we shall study certain special homomorphisms in $\mathcal{J}(\tau,\sigma)$ which play a role in the discussion of atomic compactness, the notion central to this discourse.

Definition 5.1: 1. $f: A \to B$ in Σ is an <u>essential monomorphism in Σ</u> iff it is a monomorphism and, whenever gf is a monomorphism for some $g: B \to C$ in Σ, g itself must be a monomorphism.

2. $f: A \to B$ in Σ is an <u>essential embedding in Σ</u> iff it is an embedding and, whenever gf is an embedding for some $g: B \to C$ in Σ, g itself must be an embedding. If the natural embedding of a substructure B of A into A is an essential embedding, A is an <u>essential extension</u> of B.

Remark 5.2: 1. For Σ a quasi-primitive class, essential monomorphisms in Σ may be characterized as follows.

A monomorphism $f: A \to B$ is an essential monomorphism in Σ iff, for any Σ-congruence θ on B, θ is trivial whenever $(f^2)^{-1}(\theta)$ trivial: If f is an essential monomorphism and $(f^2)^{-1}(\theta)$ is trivial, then $\ker vf = (f^2)^{-1}(\theta)$ is trivial for v the natural quotient map $B \to B/\!/\theta$. This says that vf is a monomorphism; so, v is a monomorphism; hence, θ is trivial. If f is a monomorphism with the property that $(f^2)^{-1}(\theta)$ trivial implies θ trivial for all Σ-congruences on B, for $g: B \to C$ in Σ with gf one-one, $\ker gf = (f^2)^{-1}(\ker g)$ is trivial; so, $\ker g$ is trivial forcing g to be one-one.

2. Unfortunately, there does not seem to be a suitable analogue of this characterization for essential embeddings.

Remark 5.3: A composite of essential maps (monomorphisms or embeddings) is again such.

Both of these classes of maps possess a property which is sometimes called (E3) [1].

Proposition 5.4: For a quasi-primitive class Σ, if $f: A \to B$ is a monomorphism in Σ there exists $g: B \to C$ onto in Σ for which gf is an essential monomorphism.

Proof: Put $I = \{\theta \mid \theta\ \Sigma\text{-congruence on } B \text{ with } (f^2)^{-1}(\theta) \text{ trivial}\}$. Then I is inductive. Take θ a maximal element of I. We claim that $vf: A \to B/\!\!/\theta$, where v is the natural quotient map $B \to B/\!\!/\theta$, is an essential monomorphism: Firstly, θ was picked in such a way as to make vf one-one. For Ψ a Σ-congruence on B/θ with $(vf)^{-2}(\Psi)$ trivial, $(v^2)^{-1}(\Psi)$ is a Σ-congruence on B containing θ whose inverse image with respect to f^2 is trivial. The maximality of θ guarantees that $(v^2)^{-1}(\Psi) = \theta$ so Ψ is trivial.

Proposition 5.5: For a quasi-primitive class Σ, if $f\colon A \to B$ is an embedding in Σ, there exists $g\colon B \to C$ onto in Σ for which gf is an essential embedding in Σ.

Proof: We begin in the manner of the proof of Proposition 5.4, but carry the procedure one step further.

Put $I = \{\theta \mid \theta\Sigma\text{-congruence on } B,\ A \to B/\!\!/\theta \text{ embedding}\}$. Then I is inductive. Take θ a maximal element of I. Now, put $J = \{C \mid \text{identity } i\colon B/\!\!/\theta \to C \text{ is a homomorphism}, C\varepsilon\Sigma, ivf\colon A \to C \text{ embedding}\}$. Then J also is inductive. Take C a maximal element of J. We claim that $uf\colon A \to C$, where u acts like the natural quotient map v and, hence, is onto, is an essential embedding: Firstly, C was picked in such a way as to make uf an embedding. Next, take $g\colon C \to D$ in Σ with guf embedding. Of course, kergu contains θ. So, by Corollary 1 of the Homomorphism Decomposition Theorem, there exists $h\colon B/\!\!/\text{kergu} \to D$ with $h\mu = gu$. But $uf\colon A \to B/\!\!/\text{kergu}$ is an embedding in Σ for μ the natural quotient map $B \to B/\!\!/\text{kergu}$; so the maximality of θ forces $\theta = \text{kergu}$.

For the same reason, there exists $k\colon B/\!\!/\theta \to D$ with $kv = h\mu$. But $giv = gu = h\mu = kv$; hence, $gi = k$. $\text{Kerk} = v^2(\text{kerh}\mu)$ which is trivial; so, gi and, hence, g is one-one.
For $\rho\varepsilon P$, $\rho_C \subseteq (g^{m_\rho})^{-1}(\rho_D)$. But the structure $\bar{C} = (|C|, (\lambda_C), ((g^{m_\rho})^{-1}(\rho_D)))$ is in Σ, the identity $i\colon B/\!\!/\theta \to \bar{C}$ is a homomorphism, and $ivf\colon A \to \bar{C}$ is an embedding; so, by

maximality of C, $(g^{m}\rho)^{-1}(\rho_D) = \rho_C$. Hence, g is an embedding.

The notion of essential maps is intricately related to the following special structures.

Definition 5.6: 1. For a substructure A of B, A is a <u>retract of B</u> iff there exists f: B → A with f/A the identity on A.

2. A structure A in a quasi-primitive class Σ is an <u>absolute retract in Σ</u> iff any monomorphism A → B in Σ has a left inverse.

3. A structure A in a quasi-primitive class Σ is an <u>$\mathcal{E}$-absolute retract in Σ</u> iff any embedding A → B in Σ has a left inverse.

(The letter $\mathcal{E}$ will subsequently appear often when we are dealing with the class of embeddings in a category of structures.)

Note that A is an $\mathcal{E}$-absolute retract in Σ iff it is a retract of each of its extensions in Σ.

As hinted, these classes of objects may be characterized by reference to essential maps.

Proposition 5.7: For any quasi-primitive class Σ, the following are equivalent:

1. A is an absolute retract in Σ.

2. There exist no proper essential monomorphisms A → B in Σ, that is, any essential monomorphism from A is an isomorphism.

Proof: 1. => 2. Assume A is an absolute retract in Σ. Take $f: A \to B$ an essential monomorphism in Σ. Then there exists $g: B \to A$ with gf the identity on A. Because gf is a monomorphism so is g. Thus $gfg = g$ implies fg is the identity on B; so, f is an isomorphism.

2. => 1. Assume A is domain of no proper essential monomorphism in Σ. Take $f: A \to B$ monomorphism in Σ. Then, by Proposition 5.4, there exists $g: B \to C$ in Σ with gf an essential monomorphism from A and, hence, an isomorphism. Clearly $(gf)^{-1}g$ provides a left inverse for f.

Proposition 5.8: For any quasi-primitive class Σ, the following are equivalent:

1. A is an $\mathcal{E}$-absolute retract in Σ
2. There exist no proper essential embeddings $A \to B$ in Σ.

Proof: 1. => 2. Assume A is an $\mathcal{E}$-absolute retract in Σ. Take $f: A \to B$ an essential embedding from A in Σ. Then f has a left inverse g which is necessarily an embedding and $gfg = g$ implies fg is the identity on B; so, f is an isomorphism.

2. => 1. Assume A is domain of no proper essential embedding in Σ. Take an embedding $f: A \to B$ in Σ. Then, by Proposition 5.5, there exists $g: B \to C$ onto in Σ

with gf an essential embedding, hence, an isomorphism. $(gf)^{-1}g$ serves as left inverse for f.

The abundance of essential maps (monomorphisms or embeddings) from a structure is a significant property of that structure; so we give names to those structures with "not too many" essential maps.

Definition 5.9: Σ quasi-primitive class.

1. $A \varepsilon \Sigma$ is <u>essentially bounded</u> in Σ iff there is only a set, up to isomorphism, of essential monomorphisms from A in Σ.

2. $A \varepsilon \Sigma$ is <u>$\mathcal{E}$-essentially bounded</u> in Σ iff there is only a set, up to isomorphism, of essential embeddings from A in Σ.

3. Σ is <u>essentially bounded</u> iff every structure in Σ is essentially bounded in Σ.

4. Σ is <u>$\mathcal{E}$-essentially bounded</u> iff every structure in Σ is $\mathcal{E}$-essentially bounded in Σ.

One of the fortunate properties of such structures is given in the next two propositions.

Proposition 5.10: For a quasi-primitive class Σ, any structure A which is essentially bounded in Σ is domain of some essential monomorphism $A \to C$ where C is an absolute retract in Σ.

Proof: Because A is essentially bounded in Σ there is, in particular, only a set I of non-isomorphic structures B in Σ for which the identity map from A to B is an essential monomorphism. This is equivalent to the existence of a set J of non-isomorphic structures C in Σ for which the identity map from A to C is an essential monomorphism and all $|C|$ are subsets of some fixed set X. (We need only pick an infinite set X with cardinality greater than that of each member of I). Order the set J by inclusion, as subalgebras, of underlying algebras and set inclusion of relations. Then J is inductive. Take C a maximal element of $J, i: A \to C$ the identity.

We claim that C is an absolute retract in Σ. Take $f: C \to D$ a monomorphism in Σ. Then, by Proposition 5.4, there exists $g: D \to E$ onto in Σ with gfi an essential monomorphism. The essentialness of i gives that gf is one-one; so we have $\bar{E}$, isomorphic to E and actually containing C, obtained by replacing $gf(x)$ by x for each $x \in C$. Call this isomorphism h. Then $hgfi$, which is the identity on C and, hence, on A, is still essential. Then there exists F containing C in J with an isomorphism $k: \bar{E} \to F$ over C: card $\bar{E}$ < card X and card C < card X by the choice of X) and, because X is infinite, there are enough elements of X outside C to be put in one-one correspondence with the members of $\bar{E}$ outside C and this is how we obtain F. But the maximality of C gives us that $C = F$. Then $khg: D \to C$ with $khgf$ the identity on C.

Proposition 5.11: For a quasi-primitive class Σ, any structure A which is $\mathcal{E}$-essentially bounded in Σ is domain of some essential embedding $A \to C$ where C is an $\mathcal{E}$-absolute retract in Σ.

Proof: As in Proposition 5.10, there exists a set J of non-isomorphic structures C in Σ for which the identity map from A to C is an essential embedding and all $|C|$ are subsets of some fixed suitably large set X. We order J by inclusion as substructures. Then J is inductive.

In view of Proposition 5.5, exactly the same argument works here to show that a maximal member of J is an $\mathcal{E}$-absolute retract in Σ.

We now introduce another class of maps in $\mathcal{S}(\tau,\sigma)$ which is critical in the study of atomic compactness.

Definition 5.12: $f: A \to B$ in $\mathcal{S}(\tau,\sigma)$ is a <u>pure embedding</u> iff, for any finite subset K of $A[X]^{\#}$, $A[X]$ the absolutely free extension of A by a set X, if there exists $g: A[X] \to B$ over f (ie. with $g/A = f$) with K contained in Rkerg, then there exists $h: A[X] \to A$ over A (ie. mapping A identically) with K contained in Rkerh. If the natural embedding of a substructure A of B into B is a pure embedding, B is a <u>pure extension</u> of A.

Now there is a series of appropriate comments to be made about this concept.

Remark 5.13: The idea of a pure extension B of A is an algebraic formulation of the model-theoretic statement: Any finite set of atomic formulae with constants in A which is satisfiable in B is already satisfiable in A.

Remark 5.14: It should be noted that the $A[X]$ appearing in the definition of a pure embedding may be considered, without changing the content of the condition, to be free extensions of A by X in a fixed hereditary, productive class containing B.

Remark 5.15: Any pure embedding is an embedding: For $f: A \to B$ pure, take x,y in A with $f(x) = f(y)$. This says $\{(x,y)\} \subseteq \mathrm{Rker} f$; hence, by purity, there exists $g: A[X] \to A$ over A with $\{(x,y)\} \subseteq \mathrm{Rker} g$. So $x = g(x) = g(y) = y$. Take x in A_{m_ρ} with $f^{m_\rho}(x) \varepsilon \rho_B$. Then $\{x\} \subseteq \mathrm{Rker} f$; hence, by purity, there exists $g: A[X] \to A$ over A with $\{x\} \subseteq \mathrm{Rker} g$. So, $g^{m_\rho}(x) = x \varepsilon \rho_A$.

Remark 5.16: An embedding $f: A \to B$ is pure iff B is a pure extension of $\mathrm{Im}(f)$. Assume that B is a pure extension of $\mathrm{Im}(f)$. Consider finite $K \subseteq A[X]^{\#}$ with $g: A[X] \to B$ over f for which K is in $\mathrm{Rker} f$. If we take $\bar{f}: A[X] \to \mathrm{Im}(f)[X]$ over X mapping A by f, then $\bar{f}^{\#}(K)$ is a finite subset of $\mathrm{Im}(f)[X]^{\#}$. For $\bar{g}: \mathrm{Im}(f)[X] \to B$ which maps $\mathrm{Im}(f)$ identically, X by g, $\bar{f}^{\#}(K)$ is contained in $\mathrm{Rker}\bar{g}$. Because B is a pure extension of $\mathrm{Im}(f)$, there exists $\bar{h}: \mathrm{Im}(f)[X] \to \mathrm{Im}(f)$ over $\mathrm{Im}(f)$ with $f^{\#}(K)$ in $\mathrm{Rker}\bar{h}$. Now, because f is an embedding it possesses a left inverse k from $\mathrm{Im}(f)$. For $h = k\bar{h}f: A[X] \to A$

over A, K is contained in Rkerh.

Now assume that f is a pure embedding. Consider finite K in $\mathrm{Im}(f)[X]^{\#}$ and $g: \mathrm{Im}(f)[X] \to B$ over Im(f) with K in Rkerg. For k the left inverse of f from Im(f), put $\bar{k}: \mathrm{Im}(f)[X] \to A[X]$ over X mapping Im(f) by k. Then $\bar{k}^{\#}(K)$ is a finite subset of $A[X]^{\#}$. For $\bar{g}: A[X] \to B$ over f mapping X by g, $\bar{k}^{\#}(K)$ is in Rker$\bar{g}$. So, there exists $\bar{h}: A[X] \to A$ over A with $(\bar{k}^{\#})(K)$ in Rker$\bar{g}$. For $\tilde{f}$ the corestriction of f to Im(f), $h = \tilde{f}\bar{h}\bar{k}: \mathrm{Im}(f)[X] \to \mathrm{Im}(f)$ over Im(f) is such that K is in Rkerh.

Pure embeddings retain some admirable properties of embeddings.

Proposition 5.17: 1. Composite of two pure embeddings is a pure embedding.

2. If fg is a pure embedding, then g is a pure embedding.

3. The updirected colimit of a family of pure embeddings is a pure embedding.

Proof: 1. If $g: A \to B$, $f: B \to C$ are pure embeddings, K a finite subset of $A[X]^{\#}$ for some set X, $h: A[X] \to C$ over fg with K in Rkerh, then for $\bar{g}: A[X] \to B[X]$ over X mapping A by g and $\bar{f}: B[X] \to C$ over f mapping X by h, $(\bar{g}^{\#})(K)$ is in Rker$\bar{f}$ which says, by purity of f, that there exists $k: B[X] \to B$ over B with $(\bar{g}^{\#})(K)$ in Rkerk. For $\bar{k}: A[X] \to B$ over g mapping X by k, K is in Rker$\bar{k}$; hence, there exists $A[X] \to A$ over A whose relation kernel contains K.

2. For $f: B \to C$, $g: A \to B$, with fg a pure embedding, consider finite subset K of $A[X]^{\#}$ with $h: A[X] \to B$ over g whose relation kernel contains K. Then $fh: A[X] \to C$ over fg with K in Rkerfh; so, by purity of fg, there exists $A[X] \to A$ over A whose relation kernel contains K.

3. For an updirected family $((h_{\alpha\beta}),(A_{\alpha}))$ in $\mathcal{S}(\tau,\sigma)$ with colimit $((h_{\alpha}),A)$, consider a family $(u_{\alpha}: B \to A_{\alpha})$ in $\mathcal{S}(\tau,\sigma)$ with $h_{\alpha\beta}u_{\alpha} = u_{\beta}$ whenever $\alpha\leq\beta$ for which each u_{α} is a pure embedding. We claim that $u = h_{\alpha}u_{\alpha}: B \to A$ for all α is a pure embedding: Consider, for some set X, a finite subset K of $B[X]^{\#}$ and $f: B[X] \to A$ over u with K in Rkerf. Because (τ,σ) is finitary, there exists a finite subset Y of X with K contained in $B[Y]^{\#}$. But we know that $A = \bigcup \mathrm{Imh}_{\alpha}$ and $\mathrm{Rkerh}_{\alpha} = \bigcup_{\beta\geq\alpha} \mathrm{Rkerh}_{\alpha\beta}$ for each α. Then, because Y is finite and the indexing set of the A_{α} updirected, there exists an index β with $f(Y)$ inside $\mathrm{Im}(h_{\beta})$. Define $g: B[X] \to A_{\beta}$ over u_{β} such that $h_{\beta}g(x) = f(x)$ for all x in Y. Then K is in $\mathrm{Rkerh}_{\beta}g$; so, $g^{\#}(K)$ is in Rkerh_{β}. However, because $g^{\#}(K)$ is finite, there exists $\gamma\geq\beta$ with $g^{\#}(K)$ in $\mathrm{Rkerh}_{\beta\gamma}$ and, thus, K in $\mathrm{Rkerh}_{\beta\gamma}g$. But $h_{\beta\gamma}g: B[X] \to A_{\alpha}$ over $h_{\beta\gamma}u_{\beta} = u_{\gamma}$; so, the purity of u_{γ} provides $h: B[X] \to B$ over B with K in Rkerh.

Corollary 5.18: In $\mathcal{S}(\tau,\sigma)$ any map with a left inverse is a pure embedding.

Proof: This follows immediately from 2.

To round out this chapter we define a few more concepts which will prove to be intimately related to atomic compactness.

Definition 5.19: Let Σ be a quasi-primitive class in $\mathcal{S}(\tau,\sigma)$.

1. A structure A in Σ is <u>pure-injective in Σ</u> iff for any pair of maps $f\colon B \to A$ in Σ and $g\colon B \to C$ a pure embedding in Σ, there exists $h\colon C \to A$ in Σ with $hg = f$.

2. A structure A in Σ is a <u>pure-absolute retract in Σ</u> iff it is a retract of each of its pure extensions in Σ.

3. A pure embedding $f\colon A \to B$ in Σ is a <u>pure-essential embedding in Σ</u> iff whenever gf is a pure embedding for $g\colon B \to C$ in Σ g is necessarily an embedding. For A substructure of B, B in Σ, if the natural embedding $A \to B$ is pure-essential, B is a <u>pure-essential extension</u> of A in Σ.

Some comments about these notions are in order.

Remark 5.20: 1. Any injective (in category-theoretic sense) structure in Σ is pure-injective in Σ.

2. Products and retracts of pure-injectives in Σ are pure-injective in Σ: This argument follows immediately from the properties of products, retracts and the definition of pure-injectives.

3. An absolute retract in Σ is an $\mathcal{E}$-absolute retract in Σ which is a pure-absolute retract in Σ.

4. A map which is pure and an essential embedding in Σ is pure-essential in Σ.

5. The composite of pure-essential embeddings is not necessarily pure-essential: Taylor, in [8], gives an example to show that pure-essential extensions need not be transitive.

The class of pure-essential maps also possesses the property (E3).

Proposition 5.21: For a quasi-primitive class Σ, if $f\colon A \to B$ is a pure embedding in Σ, there exists $g\colon B \to C$ onto in Σ for which gf is a pure-essential embedding.

Proof: Put $I = \{\theta \mid \theta\ \Sigma\text{-congruence on } B, vf\colon A \to B/\!\!/\theta$ pure embedding$\}$. Then, by 5.17.3, I is inductive. Take θ a maximal element of I. Now, put $J = \{C \mid$ identity $i\colon B/\!\!/\theta \to C$ is a homomorphism, $C \in \Sigma$, $ivf\colon A \to C$ pure embedding$\}$. Then J also is inductive. Take C a maximal element of J. We claim that $uf\colon A \to C$, where $u = iv$ and, hence, onto, is a pure-essential embedding in Σ. Firstly, C was picked in such a way as to make uf a pure embedding. Then, take $g\colon C \to D$ in Σ with guf pure. By an argument exactly parallel to that for essential embeddings, g is shown to be an embedding.

CHAPTER VI

ATOMIC COMPACT STRUCTURES

In this chapter we define the concept central to this discussion.

Definition 6.1: A structure A in $\mathcal{J}(\tau,\sigma)$ is *atomic compact* iff, for any subset K of $A[X]^{\#}$, A[X] the absolutely free extension of A by a set X, there exists f: $A[X] \to A$ over A with K contained in Rkerf whenever this is true for all finite subsets of K. If A is an algebra, it is said to be *equationally compact*.

Remark 6.2: This notion, as is that of purity, is an algebraic formulation of a model-theoretic statement: A system of atomic formulae with constants in A is satisfiable in A iff it is finitely satisfiable in A.

It will be helpful to realize that, again as for purity, the condition for atomic compactness of a structure in a productive, hereditary class Σ is equivalent to the condition obtained by replacing "absolutely free extension" by "free extension in Σ".

Lemma 6.3: For Σ a productive, hereditary class of structures, a structure A in Σ is atomic compact iff, for any free extension $A[X]$ of A by X in Σ, for any subset K of $A[X]^{\#}$, there is a map $A[X] \to A$ whose relation kernel contains K exactly when this is true for all finite subsets of K.

Proof: For a set X, let $A[X]_o$ be the absolutely free extension of A by X, $A[X]$ the free extension of A by X in Σ. Note the existence of an onto map $u: A[X]_o \to A[X]$ over A,X with the property that any $f: A[X]_o \to B$ for $B\varepsilon\Sigma$ factors uniquely through u via the map $A[X] \to B$ extending f/A, f/X.

Now assume that A is atomic compact and consider a subset K of $A[X]^{\#}$ for which to every finite subset F of K there corresponds f_F: $A[X] \to A$ over A with F inside $Rkerf_F$. The ontoness of u (and, hence, $u^{\#}$) allows us to choose a subset K_o of $A[X]_o^{\#}$ with $u^{\#}(K_o) = K$. Then the hypothesis on K guarantees that each finite subset F_o of K_o is in the relation kernel of some $A[X]_o \to A$ which, in turn, assures this for K_o (because A is atomic compact). A map f: $A[X]_o \to A$ with K_o in Rkerf factors (uniquely) through u via g as mentioned above. And K is in Rkerg.

Now assume the "relative atomic compactness" condition for A and consider K, a subset of $A[X]_o$, whose finite subsets have the appropriate property. Then, for any finite subset G of $u^{\#}(K)$, we may choose a finite subset F of K with $u^{\#}(F) = G$. By assumption there exists f_F: $A[X]_o \to A$ over A with F in $Rkerf_F$

and, thus, G is in Rkerg$_F$ for $f_F = g_F u$. It follows that there exists $f: A[X] \to A$ over A with $u^{\#}(K)$ in Rkerf and, hence, K in Rkerfu.

Some rationalization for the study of such structures, in the form of assurances that some ("non-trivial") such do exist, is valuable at this stage. In aid of this, we introduce more definitions.

Definition 6.4: 1. A <u>topological structure of type (τ,σ)</u> is an ordered four-tuple $(X,(f_\lambda)_{\lambda\varepsilon\Lambda}(g_\rho)_{\rho\varepsilon P},\mathfrak{T})$ where $A = (X,(f_\lambda),(g_\rho))$ is a structure of type (τ,σ), $\mathfrak{T}$ is a topology on X with respect to which the operations $f_\lambda: X^{n_\lambda} \to X$ are continuous and the relations g_ρ, as subsets of X^{m_ρ}, are closed in the product topology.

2. A structure A of type (τ,σ) is <u>compactable</u> iff, either $\Lambda = \phi$ and there exists a compact topology $\mathfrak{T}$ on $|A|$ for which $(A;\mathfrak{T})$ is a topological structure or $\Lambda \neq \phi$ and there exists a compact Hausdorff topology $\mathfrak{T}$ on $|A|$ for which $(A;\mathfrak{T})$ is a topological structure.

Examples 6.5: 1. Any finite structure is compactable by the discrete topology on its underlying set.

2. [0,1] with the usual ordering is compactable as a partially ordered set by the usual topology on [0,1].

3. $[0,1]$ with the usual ordering and multiplication is compactable as a partially ordered monoid by the usual topology on $[0,1]$.

Proposition 6.6: Any compactable structure is atomic compact.

Proof: For A, a compactable structure in $\mathcal{S}(\tau,\sigma)$, $A[X]$ the absolutely free extension of A by a set X, consider a subset K of $A[X]^{\#}$ for which to any finite subset F of K there corresponds a map $A[X] \to A$ over A whose relation kernel contains F. (We prove the result for the case $\Lambda \neq \phi$, noting the fact that, if $\Lambda = \phi$, the Hausdorff property is not needed).

To each element u of the compact Hausdorff space $|A|^X$ (with the product topology) there corresponds $\bar{u}: A[X] \to A$ over A extending u. For each p in $A[X]$, we can define $\hat{p}: |A|^X \to A$ by $\hat{p}(u) = \bar{u}(p)$ for each u in $|A|^X$. Then each $\hat{p}$ is continuous: If $p = a \varepsilon A$, $\hat{p}(u) = a$ for all u ie. $\hat{p}$ is constant, hence continuous. If $p = x \varepsilon X$, $\hat{p}(u) = u(x)$ ie. p is evaluation at x which is continuous. Then, if $p = \lambda_A(q_1,\ldots,q_{n_\lambda})$ where each $\hat{q}_i$ is continuous, then

$\hat{p} = \lambda_A(q_1 \sqcap \ldots \sqcap q_{n_\lambda})$ which is continuous. So the set of all

$p \varepsilon A[X]$ with $\hat{p}$ continuous is a substructure of $A[X]$ containing A,X, hence is $A[X]$ itself.

For each (p,q) in $A[X]^2$, define $E_{pq} = \{u \varepsilon |A|^X | \hat{p}(u) = \hat{q}(u)\}$; for each $\rho \varepsilon P$, each $(p_i)_{i \leq m_\rho} \varepsilon A[X]^{m_\rho}$, define

$E_{(p_i)} = \{u\varepsilon|A|^X|(\hat{p}_i(u))\varepsilon\rho_A\}$. Then each E_{pq} is closed, for E_{pq} is the inverse image under the continuous map $u \mapsto (\hat{p}(u),\hat{q}(u))$ of the diagonal of $|A|^2$ which is closed because $|A|$ is Hausdorff. And each $E_{(p_i)}$ is closed, for $E_{(p_i)}$ is the inverse image under the continuous map $u \mapsto (\hat{p}_i(u))$ of ρ_A which is closed because A is a topological structure. So $\{E| E = E_{pq}$ for $(p,q)\varepsilon K$ or $E = (p_i)_{i\leq m_\rho}$ for $(p_i)\varepsilon K\}$ is a system of closed sets of $|A|^X$ with finite intersection property. Thus, there does exist some $u\varepsilon|A|^X$ with K a subset of $\mathrm{Rker}\bar{u}$.

If $\Lambda = \phi$, $|A[X]|$ is simply $A \cup X$. For this reason for any subset K of $A[X]^\#$, there exists $f: A[X] \to A$ over A with $K \subseteq \mathrm{Rker} f$ iff this condition is satisfied for a subset K' of $A[X]^\#$ which is obtained from K as follows: K' contains exactly the elements of the form $(p_i)_{i\leq m_\rho}$ from K modified according to the pairs (p,q) in K; that is, for each pair (p,q) in K, each occurence of q in one of these m_ρ-sequences in K is replaced by p. Thus, in this case, we need only consider sets $E_{(p_i)}$ and so we need not employ the Hausdorff property.

Corollary 6.7: Any finite structure is atomic compact.

If we call a subset E of $|A|^X$ algebraic iff it is the intersection of a family of E_{pq} and E_x as defined above, then the family of algebraic subsets of $|A|^X$ is a closure system.

If this closure system is such that any set of its members for which every finite subset has non-void intersection itself has non-void intersection, we will call it, in analogy with the topological situation, compact. Then, we have that a structure A is atomic compact iff the closure system of its algebraic subsets for any set X is compact. In this manner, the notion of atomic compactness is a structural analogue of that of topological compactness.

Now that we believe these objects to be worthy of scrutiny we will extend the methods of this scrutiny by means of a characterization result.

Proposition 6.9: For a structure A in a quasi-primitive class Σ, the following conditions are equivalent:

1. A is atomic compact.
2. A is pure-injective in Σ.
3. A is a pure-absolute retract in Σ.
4. A has no proper, pure-essential extensions in Σ.

Proof: 1. => 2. Assume that A is atomic compact. Consider $f: B \to A$ in Σ and $g: B \to C$ pure in Σ. Take a set X, disjoint from $|A|, |B|$ and large enough that there exists an onto map $h: B[X] \to C$ over g. Our aim is to factor some map $B[X] \to A$ through h in a suitable way.

Define $k: B[X] \to A[X]$ over X extending f. Take a finite subset F of $k^{\#}(\mathrm{Rkerh})$. Then there exists a finite subset $\bar{F}$ of Rkerh with $F = k^{\#}(\bar{F})$. Because g is pure, we have a map $u: B[X] \to B$ over B with $\bar{F}$ in Rkeru. For $v: A[X] \to A$ over A extending fu/X, $vk = fu$. Therefore, $F = k^{\#}(\bar{F}) \subseteq k^{\#}(\mathrm{Rkeru}) \subseteq k^{\#}(\mathrm{Rkerfu}) = k^{\#}(\mathrm{Rkervk}) = \mathrm{Rkerv}$. But A is atomic compact, so there exists, in fact, $w: A[X] \to A$ over A with $k^{\#}(\mathrm{Rkerh})$ contained in Rkerw and, hence, Rkerh contained in Rkerwk. By the Homomorphism Decomposition Theorem, we have $\ell: C \to A$ with $\ell h = wk$. But then $\ell g = \ell h_{/B} = wk_{/B} = f$.

2. => 3. Assume that A is pure-injective in Σ. Then, in particular, for an arbitrary pure extension B of A in Σ, there exists $g: B \to A$ with $g_{/A}$ the identity on A.

3. => 4. Assume that A is a pure-absolute retract in Σ. Consider B a pure-essential extension of A in Σ. Then, by assumption, there exists $g: B \to A$ mapping A identically. The pure-essentialness of the extension, coupled with the fact that the identity map is a pure embedding, then gives us that g is an embedding; hence, A = B.

4. => 1. Assume that A has no proper pure-essential extensions in Σ. Take $A[X]$ a free extension of A in Σ and a subset K of $A[X]^{\#}$ for which to each finite subset F of K there corresponds a map $f_F: A[X] \to A$ over A whose relation kernel contains F.

For each finite subset F of K, define a structure $A[X]_F$ with same underlying set and operations as $A[X]$ and relations enlarged by the addition of all members of $F \cap A[X]^{m_\rho}$ for each $\rho \varepsilon P$. Consider θ_F the Σ-congruence on $A[X]$, and, hence, on $A[X]_F$, generated by $F \cap A[X]^2$. Then, for i_F: $A[X] \to A[X]_F$ the identity, v_F the natural quotient map from $A[X]_F$ to $A[X]_F /\!\!/ \theta_F$, $u_F = v_F i_F$: $A[X] \to A[X]_F /\!\!/ \theta_F$, $\mathrm{Rker} u_F$ is certainly contained in $\mathrm{Rker} f_F$; so, we have g_F: $A[X]_F /\!\!/ \theta_F \to A$ with $g_F u_F = f_F$. Because f_F maps A identically, each u_F/A is left-invertible and, hence, pure. Now $\mathrm{Rker} u_F$ is contained in $\mathrm{Rker} u_G$ whenever F is contained in G, giving h_{FG}: $A[X]_F /\!\!/ \theta_F \to A[X]_G /\!\!/ \theta_G$ with $h_{FG} u_F = u_G$. For u: $A[X] \to A[X]_K /\!\!/ \theta_K$, $\bigcup \mathrm{Rker} u_F = \mathrm{Rker} u$, so there exists a family $(h_F: A[X]_F /\!\!/ \theta_F \to A[X]_K /\!\!/ \theta_K)$ with each $h_F u_F = u$. $((h_F), A[X]_K /\!\!/ \theta_K)$ is the updirected colimit of the updirected family $((h_{FG}), (A[X]_F /\!\!/ \theta_F))$ in Σ; so u/A is the updirected colimit of pure embeddings and, thus, is itself pure.

Now we invoke Proposition 5.20 to get $w\colon A[X]_K/\theta_K \to B$ onto in Σ with wu/A pure-essential. By our assumption, wu/A must then be an isomorphism. Now we have a map, specifically $(wu/A)^{-1}wu$, from $A[X]$ to A over A. But K is in Rkeru and, hence, in the relation kernel of this map.

Before we give corollaries to this theorem we will explain a choice of definition which perhaps until now seemed to be arbitrary. If we were to directly generalize previous notions, it would have been natural to define two notions of "pure-essentialness" -- the one we have defined and another as follows: A map $f\colon A \to B$ is a <u>weak pure-essential embedding in Σ</u> iff it is a pure embedding and any map $g\colon B \to C$ in Σ for which gf is a pure embedding is necessarily a monomorphism. If the natural embedding $i\colon A \to B$ of A into an extension B in Σ is weak pure-essential in Σ, B is a <u>weak pure-essential extension</u> of A <u>in Σ</u>. But, in the situations under consideration, these notions give identical results. For instance, with regard to the above result, we realize that a structure has no proper pure-essential extensions in Σ iff it has no proper weak pure-essential extensions in Σ.

Corollary 6.10: Any absolute or ξ-absolute retract in Σ is atomic compact.

Corollary 6.11: Products and retracts of atomic compact structures are atomic compact.

Proof: This stems from the fact that such is easily proven for pure-injectives in Σ by employing the properties of products and retracts, respectively.

Corollary 6.12: Any maximal pure-essential extension in Σ is atomic compact.

Proof: We show that a maximal pure-essential extension B of A in Σ is a pure-absolute retract in Σ.

Consider a pure extension C of B in Σ, $i: B \to C$ the natural embedding. Then $i_{/A}$ is a pure embedding and so, by Proposition 5.20, there exists $g: C \to D$ onto in Σ with $gi_{/A}$ pure-essential. Then, D gives rise to a pure-essential extension of A, hence gi is an isomorphism. So $(gi)^{-1}\ g: C \to B$ mapping B identically.

To certain structures there correspond in a natural way, atomic compact structures.

Definition 6.13: In a quasi-primitive class Σ, an <u>atomic compact hull</u> of a structure A <u>in Σ</u> is an atomic compact pure-essential extension of A in Σ.

The existence of an atomic compact hull in Σ for a structure can be characterized in familiar terms.

Proposition 6.14: For Σ a quasi-primitve class, $A \varepsilon \Sigma$, the following are equivalent:

Proof: 1. A has an atomic compact hull in Σ.

2. A has a pure embedding into an atomic compact structure in Σ.

3. A has, up to isomorphism, only a set of pure-essential extensions in Σ.

Proof: 1. => 2. is obvious.

2. => 3. Let $f: A \to B$ be pure in Σ where B is atomic compact, C any pure-essential extension of A in Σ with $i: A \to C$ the natural embedding. Because B is pure-injective in Σ, there exists $g: C \to B$ with $gi = f$. The pure-essentialness of i guarantees that g is an embedding. Thus, C is isomorphic to a substructure of B. But there is, up to isomorphism only a set of such.

3. => 1. In view of Proposition 5.20, the argument of Propositions 5.10 and 5.11 provide us with a pure-essential embedding of A into a pure-absolute retract in Σ which is, of course, atomic compact.

Note that again here the notions of pure-essential and weak pure-essential are indistinguishable: A structure has only a set of non-isomorphic pure-essential extensions in Σ iff it has only a set of non-isomorphic weak pure-essential extensions in Σ.

Corollary 6.15: Atomic compact hulls, as far as they exist, are unique up to isomorphism.

Proof: Let B be an atomic compact hull of A in Σ. Then A also has an atomic compact hull C which is a maximal pure-essential extension of it in Σ. By pure-injectivity of B, there exists f: $C \to B$ over A which, by pure-essentialness is an embedding and then, by maximality of C, is onto. Thus, the atomic compact hulls of A in Σ are, up to isomorphism, the maximal pure-essential extensions of A in Σ and, hence, any two are isomorphic. In the following, n will be the cardinal number $\aleph_0$ + cardΛ + cardP.

Note that any structure generated by fewer than n elements has at most n elements.

Lemma 6.16: For any extension B of structure A there exists a structure C such that $A \subseteq C \subseteq B$, card$C \leq n$ + cardA, B is a pure extension of C.

Proof: This result is an immediate consequence of the (downward) Löwenheim-Skolem theorem (see Grätzer, [5 , p.236]) because elementary extensions are pure extensions.

Out of the class of all atomic compact structures of a given type certain are fundamental in some sense.

Definition 6.17: An atomic compact structure is <u>small</u> iff it is the atomic compact hull of a structure with at most n elements.

Remark 6.18: There exist, up to isomorphism, at most 2^n small atomic compact structures: This fact results from the uniqueness of atomic compact hulls and the fact that there are at most 2^n non-isomorphic structures with at most n elements.

The fundamental nature of these is explained in the next result.

Proposition 6.19: Any atomic compact structure A has a subdirect representation $h: A \to \Pi A_\alpha$ where

1. each A_α is a small atomic compact substructure of A
2. each A_α is a retract of A via $p_\alpha h$, and
3. h is a pure embedding.

Proof: For any finite subset F of A, put B_F the substructure of A generated by F. By Lemma 6.16, there exists a structure C_F with $B_F \subseteq C_F \subseteq A$, $\text{card} C_F \leq n + \text{card } B_F = n$. Because C_F has a pure atomic compact extension in Σ, by Proposition 6.14, it has an atomic compact hull, call it A_F, in Σ. The pure-injectivity of A in Σ gives a map from A_F to A over C_F which, by the pure-essentialness of A_F, is necessarily an embedding; so we may assume without loss of generality that A_F is a substructure of A.

The pure-injectivity of A_F provides a map $f_F: A \to A_F$ over C_F for which the pure-essentialness of A_F says that f_F/A_F is an embedding. The maximality of A_F as pure-essential extension of C_F guarantees that f_F/A_F is also onto, hence an isomorphism. Then $h_F = (f_F/A_F)^{-1} f_F \colon A \to A_F$ is a a retraction of A to A_F.

We claim that $h = \sqcap h_F \colon A \to \Pi A_F$ is a pure embedding: For a free extension $A[X]$ of A and a finite subset K of $A[X]^{\#}$ with $f\colon A[X] \to \Pi A_F$ over h whose relation kernel contains K, there exists a finite subset F of A with K in $B^{\#}$ where B is the substructure of $A[X]$ generated by B_F and X.
Define $g\colon A[X] \to A$ over A as extension of $p_F f_{/X}$. Then $p_F f_{/B} = g_{/B}$; so K, being in $B^{\#}$ and $\mathrm{Rker} p_F f$, is in $\mathrm{Rker} u_{/B}$ and, hence, in $\mathrm{Rker} u$.

CHAPTER VII

QUASI-PRIMITIVE CLASSES AND ATOMIC COMPACT STRUCTURES

This section presents characterizations, analogous to those given for equational classes of algebras in [2] of quasi-primitive classes which, in a certain sense, possess "enough" atomic compact structures.

Within these characterizations we will need a couple of notions not yet presented.

Definition 7.1: For a class Σ of structures, a <u>class of cogenerators of Σ</u> is a subclass Φ of Σ such that for any pair of distinct maps $f,g\colon A \to B$ in Σ there exists a $C \varepsilon \Phi$ and a map $h\colon B \to C$ in Σ such that $hf \neq hg$.

Remark 7.2: Φ is a class of cogenerators of a productive, hereditary class Σ iff every member of Σ has a monomorphism into a product of members of Φ: If the latter is true, for $f,g\colon A \to B$, distinct, take $u\colon B \to \Pi C_\alpha$ a monomorphism with $C_\alpha \varepsilon \Phi$ and $a \varepsilon A$ with $f(a) \neq g(a)$, then $uf(a) \neq ug(a)$ so $p_\alpha uf(a) \neq p_\alpha ug(a)$ for some α. If Φ is a class of cogenerators of Σ, take the structure in Σ free on one element x. For any distinct elements a,b of A in Σ, there exist $f,g\colon F \to A$ with $f(x) = a, g(x) = b$. The cogenerator property provides $u_B\colon A \to B \varepsilon \Phi$

with $u_B(fa) \neq u_B g(b)$. For $u = \sqcap u_B: A \to \Pi\{B \varepsilon \Phi |$ there exists $u_B: A \to B\}$, $\ker u = \bigcap \ker u_B$ which is trivial; so, u is the desired monomorphism.

In analogy with this concept, we introduce a related notion.

Definition 7.3: A subclass Φ of a productive, hereditary class Σ of structure is a <u>class of $\mathcal{E}$-cogenerators of Σ</u> iff every member of Σ has an embedding into a product of members of Φ.

We also specify the sense of "enough": A class of structures has <u>enough</u> objects of a certain kind iff each member of the class is domain of a monomorphism into one of the special objects which is itself in the class. A class has <u>$\mathcal{E}$-enough</u> objects of a certain kind iff each member of the class is domain of an embedding into such an object.

Proposition 7.4: For a quasi-primitive class Σ, the following are equivalent:

1. Σ has enough atomic compact structures.
2. There is, up to isomorphism, only a set of weak subdirect irreducibles in Σ.
3. Σ has a set of cogenerators.
4. Σ is essentially bounded.
5. Σ has enough absolute retracts.

Proof: 2. => 3. follows from the comment that, in view of Proposition 4.6, the weak form of the Birkhoff Representation Theorem, the weak subdirect irreducibles in Σ form a class of cogenerators of Σ. 4. => 5. has been proved as Proposition 5.10, and 5. => 1. is obvious. So we are left to show 1. => 2. and 3. => 4.

1. => 2. Take A, a weak subdirect irreducible in Σ, $f: A \to B$ a monomorphism in Σ with B atomic compact. B, by Proposition 6.19, has a subdirect representation $g: B \to \Pi B_\alpha$ in Σ which provides a weak subdirect representation $gf: A \to \Pi B_\alpha$ of A in Σ. The weak subdirect irreducibility of A in Σ then implies that some $p_\alpha gf$ is a monomorphism. So, from each weak subdirect irreducible in Σ, there is a monomorphism to a small atomic compact structure of which there are at most 2^n such, but there is only a set of non-isomorphic structures with this property.

3. => 4. Take $A \varepsilon \Sigma$, $f: A \to B$ an essential monomorphism in Σ. By assumption there exists a monomorphism $g: B \to \Pi G_\alpha$ where the G_α are members of the set $\Phi = \{G_\alpha | \alpha \varepsilon I\}$ of cogenerators of Σ. We want to provide a cardinality bound for B independent of B and the particular G_α used. For distinct elements a,b of A, there exists an index $\gamma = \gamma_{ab}$ such that $p_\gamma gf(a) \neq p_\gamma gf(b)$. For $J = \{\gamma = \gamma_{ab} | (a,b) \varepsilon A^2 \setminus \text{diagonal of } A\}$, we have a map from $\prod_{\alpha\varepsilon I} G_\alpha$ to the partial product $\prod_{\gamma\varepsilon J} G_\gamma$

defined by $q_\gamma h = p_\gamma$ for q_γ the γ-th projection $\Pi G_\gamma \to G_\gamma$. Then hgf is a monomorphism by the construction of J; so hg is a monomorphism.card$J \leq \aleph_o$ + cardA; so, for k the supremum of cardG_α for $\alpha\epsilon I$, card B $\leq k^{\aleph_o}$ + card A.

From this result can be extracted, with little effort, some facts about cardinalities.

Corollary 7.5: For any essentially bounded quasi-primitive class Σ, 1. Σ has a set of cogenerators Φ with at most 2^n elements,

if we put k the supremum of the cardinalities of small atomic compact structures

2. any codomain of an essential monomorphism from A in Σ has cardinality $\leq k^{\aleph_o + \text{card A}}$, and

3. every weak subdirect irreducible in Σ has cardinality $\leq$ k.

Proof: The set of cogenerators mentioned in 1. can be taken as the set of small atomic compact structures of which there are at most 2^n. Results 2.3, are completely developed within the proof of the proposition.

About quasi-primitive classes with ξ-enough atomic compact structures, extremely similar things can be said.

Proposition 7.6: For a quasi-primitive class Σ, the following are equivalent:

1. Σ has $\mathcal{E}$-enough compact structures.
2. Σ has a set of $\mathcal{E}$-cogenerators.
3. Σ is $\mathcal{E}$-essentially bounded.
4. Σ has $\mathcal{E}$-enough $\mathcal{E}$-absolute retracts.

Proof: 1. => 2. Each A in Σ can be embedded in an atomic compact structure B in Σ which can, in turn, be embedded into a product of small atomic compact structures of which there are, up to isomorphism, at most 2^n.

2. => 3. Take $A \varepsilon \Sigma$, $f: A \to B$ an essential embedding in Σ. By assumption there exists an embedding $g: B \to \Pi G_\alpha$ where the G_α are members of the set $\Phi = \{G_\alpha | \alpha \varepsilon I\}$ of $\mathcal{E}$-cogenerators of Σ. We want to provide a cardinality bound for B independent of B and the particular G_α used. For distinct elements a,b of A there exists an index $\gamma = \gamma_{ab}$ such that $p_\gamma gf(a) \neq p_\gamma gf(b)$. Also for each x, in A^{m_ρ}, not in ρ_A there exists an index $\gamma = \gamma_x$ such that $(p_\gamma gf)^{m_\rho}(x)$ is not in ρ_{G_γ}.

For J the collection of these indices γ_{ab} and γ_x, we have a map $h: \prod_{\alpha \varepsilon I} G_\alpha \to \prod_{\gamma \varepsilon J} G_\gamma$ defined by $q_\gamma h = p_\gamma$ for q_γ the γ-th projection $\Pi G_\gamma \to G_\gamma$. Then hgf is an embedding; so, by the essentialness of f, hg is an embedding. card J $\leq m_A$ = card(A×A∖diagonal of A) + Σcard$(A^{m_\rho} \setminus \rho_A)$; so, for k the supremum of card G_α for $\alpha \varepsilon I$, card B $\leq k^{m_A}$.

3. => 4. has been proved as Proposition 5.11.

4. => 1. is obvious.

Again, we can immediately list some cardinality results.

Corollary 7.7: For any $\mathcal{E}$-essentially bounded quasi-primitive class Σ, 1. Σ has a set of $\mathcal{E}$-cogenerators with at most 2^n elements,

2. If k is the supremum of the cardinalities of small atomic compact structures, any domain of an essential embedding from A in Σ has cardinality $\leq k^{m_A}$.

CHAPTER VIII

QUASI-PRIMITIVE CLASSES AND ATOMIC COMPACT HULLS

What has been said about quasi-primitive classes where every member is domain of a monomorphism or embedding into an atomic compact structure has a "pure" analogue.

We must introduce "pure" forms of familiar notions.

Definition 8.1: 1. A pure representation of a structure A in $\mathcal{S}(\tau,\sigma)$ is a pair $(h,(A_\alpha))$ where $h\colon A \to \Pi A_\alpha$ is a pure-embedding in $\mathcal{S}(\tau,\sigma)$, $p_\alpha h$ is onto for each α.

2. A structure A in a quasi-primitive class Σ is pure-irreducible in Σ iff for any pure representation $h\colon A \to \Pi A_\alpha$ where all A_α are in Σ, some $p_\alpha h$ is a monomorphism.

In [7], W. Taylor proves, for structures, a pure analogue of Birkhoff's Representation Theorem which will be referred to as the Pure Representation Theorem. The same argument gives us that any structure in a quasi-primitive class Σ of (finitary) structures has a pure representation by pure-irreducibles in Σ.

We will say that a class of structures has purely-enough objects of a certain kind iff any member of the class can be purely embedded into such an object. For additional convenience, a subclass Φ of a class Σ of structures will be called a <u>class of pure cogenerators of Σ</u> iff any member of Σ can be purely embedded in a product of members of Φ and a class Σ of structures <u>pure-essentially bounded in Σ</u> iff each member of it has "only" a set of pure-essential extensions in Σ.

Armed with these terms to tidy up its statement, we state the main proposition of this section.

Proposition 8.2: For a quasi-primitive class Σ, the following are equivalent:

1. Σ has purely-enough atomic compact structures.

2. There is, up to isomorphism, only a set of pure-irreducibles in Σ.

3. Σ has a set of pure cogenerators.

4. Σ is pure-essentially bounded.

Proof: 1. => 2. Take A a pure-irreducible in Σ, $f: A \to B$ a pure embedding in Σ with B atomic compact. B, by Proposition 6.19, has a pure representation $g: B \to \Pi B_\alpha$ in Σ where the B_α are small atomic compact structures. The pure-irreducibility of A then implies that some $p_\alpha gf$ is a monomorphism. So each pure-irreducible in Σ has a one-one, onto map to a small atomic compact structure of which there at most 2^n and, hence, there are only a set of pure-irreducibles in Σ.

2. => 3. By the modified version of the Pure Representation Theorem, the pure-irreducibles in Σ form a

set of pure-cogenerators of Σ.

3. => 4. Take a pure-essential extension B in Σ of A in Σ, $i: A \to B$ the natural embedding. By assumption there exists a pure embedding $g: B \to \Pi G_\alpha$ where the G_α are members of the set $\Phi = \{G_\alpha | \alpha \varepsilon I\}$ of pure-cogenerators of Σ. We want to provide a cardinality bound for B independent of B and the particular G_α used. On I, consider the equivalence relation $R = \{(\alpha,\beta) | p_\alpha g = p_\beta gi\}$. Take J a representative set of the equivalence classes of this relation and put $s: I \to J$ the map from $\alpha \varepsilon I$ to the element β of J for which $(\alpha,\beta)\varepsilon R$. Then we have maps $h: \prod_{\alpha\varepsilon I} G_\alpha \to \prod_{\beta\varepsilon J} G_\beta$ defined by $q_\beta h = p_\beta$ for each $\beta\varepsilon J$, where p_α is the α-th projection $\prod_{\alpha\varepsilon I} G_\alpha \to G_\alpha$ and q_β the β-th projection $\prod_{\beta\varepsilon J} G_\beta \to G_\beta$, and $k: \prod_{\beta\varepsilon J} G_\beta \to \prod_{\alpha\varepsilon I} G_\alpha$ defined by $p_\alpha k = q_{s(\alpha)}$ for each $\alpha\varepsilon I$. Note that $khgi = gi$ because, for each α, $p_\alpha khgi = q_{s(\alpha)} hgi = p_{s(\alpha)} gi = p_\alpha gi$.

We claim that hfi is a pure embedding: Consider a free extension $A[X]$ of A and a finite subset K of $A[X]^{\#}$ with $u: A[X] \to \Pi G_\beta$ over hf_i with K in Rkeru. Then $ku: A[X] \to \Pi G_\alpha$ over $khfi = fi$ with K in Rkerku. But fi is pure, so we have $v: A[X] \to A$ over A with K in Rkerv.

Now, the pure-essentialness of B guarantees that hf is an embedding. In addition, note that the map from J to $\{p_\beta fi | \beta\varepsilon J\}$ which takes β to $p_\beta fi$ is one-one; so, card J $\leq$ card$\{A \to G | G\varepsilon\Phi\}$. If we denote the latter by m_A, we now have card B $\leq k^{m_A}$ for k the supremum of cardinalities of members of Φ.

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