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ON

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By

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TITLE: Coulomb Effect on the Proton-Proton Low-Energy Scattering Parameters and Separable Potentials

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SCOPE AND CONTENTS:

We study the Coulomb effect on the proton-proton low-energy scattering parameters when all other effects are represented by a separable potential.

For this purpose, we present a formulation for the scattering of two particles via a separable potential. We treat the same problem when any potential, particularly a Coulomb potential or a separable potential, is added to the separable potential. The properties of scattering from a separable potential plus a (local or non-local) potential lead us to the possibility of obtaining a one term separable potential equivalent to a two term separable potential, and a model for the nuclear potential as a sum of a separable potential and a non-separable potential.

We determine, to the first order in $\frac{Me^2}{\beta}$ where β^{-1} is the range of the separable potential, the parameters for Yamaguchi's and Naqvi's separable potentials from proton-proton scattering data. We use these parameters to calculate the low-energy proton-proton scattering parameters when the Coulomb interaction is removed. Our results show that the shape dependence of these parameters are somewhat larger than obtained by Heller et al in their investigation on local potentials. Implications of our results concerning the charge symmetry and charge independence of the nuclear forces are discussed.

TABLE OF CONTENTS

I. INTRODUCTION

Since the hypothesis of Heisenberg and Condon (1), strong interaction is believed to be charge independent, the observed charge dependence should be accounted for by electromagnetic corrections. Charge symmetry is based on the equality of n - n and p - p forces; physically it states that, in the absence of electromagnetic forces, a system of nucleons behaves ecactly the same as its charge symmetric counterpart in the same quantum mechanical state. This is generalized by a stronger assumption, namely, charge independence, which equates the n - p forces to that of p - p and n - n in the same state (2).

A sensitive test of charge symmetry as well as charge independence of the nuclear interaction is provided by measurements of the low-energy nucleon-nucleon scattering parameters in the ^1S state, namely the scattering length and effective range. The advantage of using low-energy data is that only S-waves are important and accurate phase-shift determinations are possible (the tensor force is absent in the ^1S state and the theoretical treatment is independent of the shape of the potentials to first order (2)).

At low energies, the Coulomb interaction between the two nucleons is the major affect in breaking both charge symmetry and charge independence (3). At higher energies, the

Coulomb effects are less important, but the splitting between the p-n and p-p scattering lengths may be large. This is because the ^1S-scattering length is extremely sensitive to the change in the potential.

In order to compare the scattering parameters for p-p, p-n, n-n, one thus has to remove the effect of the Coulomb interaction for p-p. This is usually done by first assuming a p-p nuclear potential, V_s, which together with the Coulomb potential, V_c, reproduces the observed p-p scattering data, and then, solving the Schrödinger equation for V_s, without including the Coulomb potential, to obtain the scattering parameters. Thus, for the p-p system, there are two sets of scattering parameters, say, a_{sc} and r_{sc}, and a_s and r_s. a_{sc} and r_{sc} are directly determined from the observed phase shift and are supposed to be reproduced by solving the Schrödinger equation with $(V_s + V_c)$. a_s and r_s are the scattering parameters obtained only from V_s. The suffix sc implies that the relevant quantity comes from both strong and Coulomb interactions, whereas the quantity with the suffix s is due only to the strong nuclear interaction. a_s and r_s are the parameters for p-p that can be meaningfully compared with those of n-n and p-n.

As is obvious from the definition of a_s and r_s for p-p, they depend on the nuclear potential V_s. Although this dependence on V_s is believed to be very small, we now require a very precise knowledge of a_s and r_s, because all that concerns us about change symmetry or change independence are very small differences between quantities for p-p, p-n and n-n. Also the experimental data now available are very accurate so that even a very slight dependence of a_s and r_s on V_s can be meaningfully detected.

This problem, say the shape dependence of a_s and r_s has been examined for various local potentials that fit the p-p scattering data by Heller, Signel and Yodes (4). They found

$$a_s = -(16.6 \text{ to } 16.9)\text{F}$$

to be the spread of the probable value of a_s. Incidentally, our a_s corresponds to Heller et al's a_{nn} which they call the n-n scattering length. For clearity, we distinguish a_s and a_{nn}.

In addition to the effect of the Coulomb interaction, there are innumerable differences between p-p and n-n interactions (2, 5). Hence, removing the effect of the Coulomb interaction in the p-p scattering does not give the n-n scattering. It only gives quantities wh ich can be compared with corresponding quantities for n-n.

The prime purpose of this thesis is to estimate a_s and r_s when the strong nuclear interaction is represented by a non-local, separable potential. This problem was investigated by Harrington (6) who assumed a simple Yamaguchi-type separable potential.

We interupt our discussion of charge symmetry and charge independence to briefly discuss the significance of separable potentials. We define a separable potential a non-local potential which has the special form $V(\vec{r},\vec{r}') = g(\vec{r})\ g(\vec{r}')$ or a sum of several separable terms. The functions g(r)'s are called the form factors of the separable potentials.

When separable potentials are inserted in the Lippmann-Schwinger integral equations for the off-energy-shell two-nucleon partial wave scattering amplitude, the kernel of this equation becomes degenerate (or separable). Hence, the Lippman-Schwinger equation can be solved algebraically. The off-shell behaviour of the scattering amplitude is determined by the choice of the form factors of the separable potentials which can be made from a model for phase-shift (7-11). Separable potentials are extremely useful for three-body calculations since they reduce the Fadeev equations to an effective one-body problem; that is, they reduce the set of two dimensional integral euqations to a set of one dimensional integral equations (12).

When a Coulomb potential is added to a separable potential, it is shown by Harrington (6) that the most important property of the separable potential is preserved; namely, the solution of the Lippman-Schwinger equation can be given in closed form. The two particle scattering matrix $T = T_c + T_{sc}$. The main Coulomb effects are included in T_{sc} which can be obtained from the T_s matrix corresponding to the separable potential alone by replacing the form factors of the separable potential by their modified form factors.

If, instead of adding the Coulomb potential, we add a separable potential in order to have a two-term separable potential, we should arrive at the same results. We consider in this thesis the possibility of obtaining a one-term separable potential which is equivalent to a two-term separable potential. Of course, we can also add any non-separable potential to a separable potential, arriving at an 'equivalent separable potential'.

Despite of their successes in accounting for many properties of nucleon-nucleon scattering ~~for many~~ and the three-body problem, separable potentials seem to be purely phenomenological if not unrealistic. However, as shown by Lovelace (13), the T-matrix or, equivalently, the potential, near the resonances and bound states, is indeed separable. In view of the important property of scattering from a seperable potential plus a (separable or non-separable) potential as mentioned in the last paragraph, we can explain the separability of the nucleon-nucleon potential by saying that the latter is a sum of a separable potential and a non-separable potential. The separable part represents the resonances, bound states and other unknown effects of the nucleon-nucleon system. In this thesis we shall briefly consider this possible model for nucleon-nucleon potential.

We now return to the problem of determining a_s and r_s for p-p scattering. A simple one-attractive term separable potential used by Harrington (6) is not sufficient to describe the behaviour of the ^1S phase shift and does not provide a good fit to high energy p-p data. Hence, his results for $a_s = -17.9$ F, which is appreciably different from the value obtained by Heller et al, should be taken with a grain of salt. We have extended Harrington's work and consider separable potentials which fit the p-p scattering data up to about 300 MeV. We have tried two separable potentials; one is Naqvi's type and the other is a'modified Tabakin's type'.

The general format of this thesis is as follows. In part II, we will give briefly the general formulation for scattering of two particles via a separable potential and then apply it to the calculation of the scattering length and effective range for Yamaguchi's, Naqvi's, Tabakins' and modified Tabakin's potentials. The same problem is treated in more detail in part III where another potential is added to the separable potential. We consider scattering from a separable potential plus an repulsive Coulomb potential, and from two separable potentials, which leads us to a model for the nucleon-nucleon potential as a separable potential plus a non-separable potential. We use the approximation proposed by Harrington to relate the a_s and r_s to the corresponding proton-proton parameters. Numerical results and a discussion will be given in part IV.

II. SCATTERING FROM A SEPARABLE POTENTIAL

A. General Formalism

In this section, we shall study the properties of two particles of masses m_1 and m_2 interacting via a separable potential.

We suppose that the total hamiltonian of the system can be separated into:

$$H = H_o + V_s \tag{1}$$

where H_o is the free-hamiltonian and V_s is the interaction which we assume to be separable.

Let $|\phi_\alpha>$ the complete set of free-particles state, and $|\chi^\pm_\alpha>$ the outgoing and ingoing exact state. Then $|\chi^\pm_\alpha>$ is related to $|\phi_\alpha>$ by:

$$|\chi^\pm_\alpha> = |\phi_\alpha> + G_s^\pm(s)\ V_s|\phi_\alpha> \tag{2}$$

where

$$G_s(s) = (S - H \pm i\varepsilon)^{-1} \tag{3}$$

with s, the energy of the system.

The S-matrix elements are defined as:

$$S_{s,\beta\alpha} = <\chi_\beta^-|\chi_\alpha^+> \tag{4}$$

which can be shown to obey the equation (Ref. 14):

$$S_{s,\beta\alpha} = <\phi_\beta|\phi_\alpha> - 2\pi i\delta(S_\beta - S_\alpha)T_{s,\beta\alpha} \tag{5}$$

where

$$T_{s,\beta\alpha} = \langle\phi_\beta|V_s|\chi_\alpha^+\rangle \tag{6}$$

the operator T_s defined by:

$$T_{s,\beta\alpha} = \langle\Phi_\beta|T_s|\phi_\alpha\rangle \tag{7}$$

obeys the integral equation (Ref. 14):

$$T_s = V_s + V_s\ G_o^+(s)T_s \tag{8}$$

where $G_o(s)$ is the free particle resolvent

$$G_o^\pm(s) = (s - H_o \pm i\varepsilon)^{-1} \tag{9}$$

In this work, we shall be concerned only with uncoupled partial states; the vector spherical harmonic decompositions of the potential and the scattering operator are:

$$\begin{aligned} V_s &= \sum_{L,S,J,M} |Y_{LSJ}^M\rangle V_{s,L}^{SJ}\langle Y_{LSJ}^M| \\ T_s &= \sum_{L,S,J,M} |Y_{LSJ}^M\rangle T_{s,L}^{SJ}\langle Y_{LSJ}^M| \end{aligned} \tag{10}$$

Substituting (10) into (8) we obtain an equation for $T_{s,L}^{SJ}$

$$T_{s,L}^{SJ} = V_{s,L}^{SJ} + V_{s,L}^{SJ}\ G_o(s)T_{s,L}^{SJ} \tag{11}$$

Now we write $V_{s,L}^{SJ}$ as the sum of N separable terms:

$$V_{s,L}^{SJ} = \sum_{i=1}^{N} |g_{Li}^{SJ}\rangle\ \lambda_{Li}^{SJ}\ \langle g_{Li}^{SJ}| \tag{12}$$

where some of the function $g_{Li}^{SJ}(k) = \langle k|g_{Li}^{SJ}\rangle$ may be equal to zero.

From (2), we can deduce that the wave function is related to T_s matrix as:

$$|\chi_\alpha^\pm> = |\phi_\alpha> + G_o^\pm(s)\ T_s|\phi_\alpha> \tag{13}$$

Substituting (12) into (11) we can easily show that $T^{SJ}_{s,L}$ has a separable form similarly to (12). In momentum space, the equation (11) becomes:

$$T^{SJ}_{s,L}(s,k',k) = \Sigma_i\ \lambda^{SJ}_{Li}\ g^{SJ}_{Li}\ (k')\ g_{Li}(k)$$
$$+ \Sigma_i\ \frac{1}{2\pi^2} \int \lambda^{SJ}_{Li}\ g^{SJ}_{Li}\ (k')\ \frac{q^2\ g^{SJ}_{Li}(q)}{s(k) - s(9) + i\zeta}\ dq \tag{14}$$

Equation (14) is the Fredhom's integral equation with degenerate kernel, the solution of it has the form (Appendix B)

$$T^{SJ}_{s,L}\ (E,k',k) = \Sigma_{ij}\ g^{SJ}_{Li}\ (k'')\ K^{SJ}_{Lij}\ (s)\ g^{SJ}_{Lj}\ (k) \tag{15}$$

Dropping the indeces LSJ, we obtain for the matrix K

$$K(s) = \Lambda - \Lambda J(s)\ K(E) \tag{16}$$

where Λ is the diagonal matrix:

$$\Lambda_{ij} = \delta_{ij}\ \lambda_i \tag{17}$$

and the density of the matrix J(s) are given by:

$$J_{ij}(s) = \frac{1}{2\pi^2} \int_0^\infty \frac{k^2\ dk\ g_j(k)\ g_j(k)}{s(k) - s - i\varepsilon} \tag{18}$$

Solving (16) for K(s) we get:

$$K(s) = [1 + \Lambda J(s)]^{-1} \Lambda \tag{19}$$

We write the 'on-shell' partial waves amplitude as:

$$T^{SJ}_{s,L} = \frac{2\pi}{\mu k} \sin \delta^{SJ}_{s,L}(k)\, e^{i\delta^{SJ}_{s,L}(k)} \tag{20}$$

With (20) substituted into (15) we have:

$$-\frac{2}{\mu k} \sin \delta^{SJ}_{s,L}(k)\, e^{i\delta^{SJ}_{s,L}(k)} = \sum_{ij} g^{SJ}_{Li}(k)\, K^{SJ}_{Lij}\, g^{SJ}_{Lj}(k) \tag{21}$$

From (21) we see that:

$$\delta^{SJ}_{s,L}(k) = -\arg\{\det[1 + \Lambda^{SJ}_{L}\, J^{SJ}_{L}[s(k)]]\} \tag{22}$$

From the T_s matrix (15) we can easily write down the wave function, the desired cross-section. From the phase-shift $\delta^{SJ}_{s,L}(k)$ we can derive analytic expressions for the scattering length and effective range r_s etc.

B. Applications

To illustrate some of the features of the formulas developed so far, we shall apply them to the simple problem of calculating the S-wave phase shift for separable potenials proposed by Yamaguchi (Ref. 7), Naqvi (Ref. 10), Tabakin. We shall compute the scattering length and effective range for these potentials.

1. Yamaguchi's Potential

The most simple form of the singlet separable potential was first proposed by Yamaguchi in 1954:

$$<k'|V^{oo}_{s,o}|k'> = \lambda g(k') \; g(k) \tag{23}$$

where

$$g(k) = \frac{1}{k^2 + \beta^2} \tag{24}$$

Equation (15) gives:

$$T^{oo}_{s,o}(s,k',k) = \lambda g(k') \; g(k) K \tag{25}$$

with

$$K = \frac{\lambda}{1 + \lambda J(s)} \tag{26}$$

where

$$J[s(k)] = \frac{1}{2\pi^2} \int_o^\infty \frac{g^2 \; g^2(g) dg}{s(g) - s(k) - i\varepsilon} \tag{27}$$

Relation (22) gives:

$$\cot\delta_s = -\frac{Re(1 + \lambda J)}{\lambda ImJ} \tag{28}$$

Noting that $s(k) = \frac{k^2}{2\mu}$ where μ is the reduced mass, we can write (26) as:

$$J(k) = \frac{\mu}{\pi^2} \int_o^\infty \frac{g^2 \; dg}{(g^2 + \beta^2)(g^2 - k^2 - i\varepsilon)} \tag{29}$$

Using the representation:

$$\frac{P}{\chi - \chi_o} = \frac{1}{\chi - \chi_o - i\varepsilon} - i\pi \; \delta(\chi - \chi_o) \tag{30}$$

where P denotes the Cauchy principal value, we obtain:

$$Im \; J(k) = \frac{\mu}{2\pi} \frac{k}{(k^2 + \beta^2)^2} \tag{31}$$

$$\text{Re } J(k) = \frac{\mu}{\pi^2} P \int_0^\infty \frac{g^2 \, dg}{(g^2 + \beta^2)^2 (g^2 - k^2)} \tag{32}$$

The integral (31) can be evaluated by using a coutour integration. The result is:

$$\text{Re } J(k) = \frac{\mu}{4\pi} \frac{\beta^2 - k^2}{\beta(\beta^2 - k^2)^2} \tag{33}$$

From this:

$$k\cot\delta_s(k) = \frac{2\pi}{\mu\lambda} (k^2 + \beta^2)^2 - \frac{1}{2\beta} (\beta^2 - k^2) \tag{34}$$

To calculate the scattering length and effective range, we expand (33) in power series of k; the zero and second order in k give:

$$-\frac{1}{a_s} = -\frac{2\pi}{\mu\lambda} \beta^4 - \frac{\beta}{2} \tag{35}$$

$$\frac{1}{2} r_s = -\frac{4\pi}{\mu\lambda} \beta^2 - \frac{1}{2\beta} \tag{36}$$

2. Naqvi's Potential

We now apply our formulation to a more general separable potential with two terms. This potential has the general form (8).

$$\langle k' | V^{oo}_{s,o} | k \rangle = \lambda_1 g_1(k) \, g_1(k') + \lambda_2 g_2(k) \, g_2(k') \tag{37}$$

with

$$g_1(k) = \frac{1}{k^2 + \beta^2}$$

$$g_2(k) = \frac{k^2}{[(k - \alpha)^2 + \beta^2][(k + \alpha)^2 + \beta^2]} \tag{38}$$

Although these general form of $g_1(k)$ and $g_2(k)$ give excellent agreement with experiment (Ref. 8), when applied to scattering from separable potential plus coulomb potential, they give rise to very complicated transcendental functions which are difficult to deal with analytically. For simplification, we follow Naqŕi to take:

$$g_1(k) = \frac{1}{k^2 + \beta^2} \tag{39}$$

$$g_2(k) = \frac{k^2}{(k^2 + \beta^2)^2} \tag{40}$$

At low energy the potential should be attractive, λ_1 is taken to be negative and the form $g_1(k)$ and $g_2(k)$ were chosen so that in (37) the first term predominates the second at low energies. As experimental, S phase shift change sign at about 240 Mev, the value of λ_2 is adjusted to give repulsive phase-shift at 240 Mev, we have, according to (22):

$$\delta_s = -\arg \begin{vmatrix} 1 + \lambda_1 J_{11}(k) & \lambda_1 J_{12}(k) \\ \lambda_2 * J_{21}(k) & 1 + \lambda_2 J_{22}(k) \end{vmatrix} \equiv -\arg D \tag{41}$$

where

$$J_{11}(k) = \frac{\mu}{\pi^2} \int_0^\infty \frac{g^2 \, dg \, g_1^2(g)}{g^2 - k^2 - i\varepsilon} \tag{42}$$

$$J_{12}(k) = J_{21}(k) = \frac{\mu}{\pi^2} \int_0^\infty \frac{g^2 \, dg \, g_1(g) \, g_2(g)}{g^2 - k^2 - i\varepsilon} \tag{43}$$

$$J_{22}(k) = \frac{\mu}{\pi^2} \int_o^\infty \frac{g^2\ dg\ g_2^2(g)}{g^2 - k^2 - i\varepsilon} \tag{44}$$

Noting that:

$$g_2(k) = (1 + \beta^2 \frac{d}{d\beta^2})\ g_1(k) \tag{45}$$

It is easy to see that:

$$J_{12}(k) = J_1(k) + \frac{\beta^2}{2} J_1'(k) \tag{46}$$

with $J' = \frac{dJ}{d\beta^2}$; and

$$J_{22}(k) = J_1(k) + \beta^2 J_1'(k) + \frac{\beta^4}{6} J_1''(k) \tag{47}$$

we obtain, after some tedious calculations:

$$\cot \delta_s = -\frac{\text{Re } D(k)}{\text{Im } D(k)}$$

$$\text{Re } D(k) = B_o\{\lambda_1' + \lambda_2'(\frac{k^2}{K})^2 + \lambda_1' \lambda_2' \frac{\beta}{32K^2}(2 - \frac{K}{\beta^2})\}$$

$$\text{Im } D(k) = 1 + A_o\{\lambda_1' + \lambda_2'[1+\frac{2\beta^2}{K} + \frac{\beta^4}{K^2}]$$

$$+ \frac{\lambda_1' \lambda_2' \beta}{32\ K^2}(2 - \frac{K}{\beta^2})\} + \frac{\lambda_2' \beta}{K^2} [\frac{K}{8\beta^2}$$

$$- \frac{1}{32}(2 + \frac{K}{\beta^2})] - \frac{\lambda_1' \lambda_2'}{(16)^2\ \beta^2\ K^2}$$

$$\lambda_1' = \lambda_1\frac{\mu}{\pi}, \lambda_2' = \lambda_2\frac{\mu}{\pi}$$

$$K = k^2 + \beta^2, A_o = \frac{\beta^2 - k^2}{4\beta K^2}$$

$$B_o = \frac{k}{2K^2} \tag{48}$$

Expanding (48) in power series of k, we again obtain the expression for the scattering length and effective range. We find:

$$-\frac{1}{a_s} = -\frac{2\beta^4}{\lambda_1'}(1+\frac{\lambda_2'}{32\beta^3})^{-1}\{1+ \frac{\lambda_1'}{4\beta^3}(1+\frac{\lambda_2'}{8\lambda_1'}) + \frac{\lambda_1'\lambda_2'}{(16)^2\beta^6}\} \qquad (49)$$

$$\frac{1}{2}r_s = \frac{2\beta^2}{\lambda_1'}(1+\frac{\lambda_2'}{32\beta^3})^{-1}\{\frac{\lambda_1'}{4\beta^3}[3-\frac{\lambda_2'}{8\lambda_1'} + \frac{5\lambda_2'}{32\beta^3}] + \frac{\lambda_1'}{4\beta^4}(4+\frac{5\lambda_2'}{16\beta^3})(-\frac{1}{a_s})\} \qquad (50)$$

These expressions agree with the ones given by Naqvi for the scattering length and effective range. Note that if we let $\lambda_2' = 0$, we reproduce the expressions (35) and (36) corresponding Yamaguchi's potential.

3. Tabakin's potential

We now apply our formula to a one term separable potential containing both repulsive and attractive terms proposed recently by Tabakin (Ref. 15):

$$\langle k'|V_{s,o}^{oo}|k\rangle = g(k')\, g(k) \qquad (51)$$

with

$$g(k) = (k^2 - k_c^2)\frac{k_c^2 + d^2}{k_c^2 + b^2}\frac{1}{k^4 + a^4} \qquad (52)$$

We need the integral:

$$J(k) = \frac{\mu}{\pi^2} \int_0^\infty \frac{g^2\, dg\, g^2(g)}{g^2 - k^2 - i\varepsilon} \tag{53}$$

As before, we have:

$$\text{Im } J(k) = \frac{\mu k}{2\pi} g^2(k) \tag{54}$$

$$\text{Re } J(k) = \frac{\mu}{\pi^2} P \int_0^\infty \frac{g^2\, dg\, g^2(g)}{g^2 - k^2} \tag{55}$$

To calculate (55), we make a fractional decomposition for $g^2\ g^2(g)$ and use contour integration. The result is:

$$\text{Re } J(k) = \sum_{i=1}^{7} \delta_i\ J_i(k)$$

where

$$\delta_1 = 2\alpha\rho(\beta - b^2\gamma);\ \delta_2 = \alpha^2$$

$$\delta_3 = 2\alpha\tau(\beta - b^2\gamma);\ \delta_4 = 2\alpha(\sigma\beta + \gamma\tau)$$

$$\delta_5 = \beta^2;\ \delta_6 = 2\gamma\beta;\ \delta_7 = \gamma^2$$

$$\alpha = (d^2 - b^2)\frac{k_c^2 + b^2}{a^4 + b^4}$$

$$\beta = k_c^2 + (d^2 - b^2)\frac{k_c^2 b - a^4}{a^4 + b^4}$$

$$\gamma = -1 - (d^2 - b^2)\frac{k_c^2 + b^2}{a^4 + b^4}$$

$$\rho = -\frac{b^2}{a^4 + b^4} = -\sigma;\ \tau = \frac{a^4}{a^4 + b^4}$$

$$J_1(k) = \frac{\mu}{\pi^2} P \int_0^\infty \frac{dg}{(g^2+b^2)(g^2-k^2)} = -\frac{\mu}{2\pi b}\frac{1}{b^2+k^2}$$

$$J_2(k) = \frac{\mu}{\pi^2} P \int_0^\infty \frac{g^2 dg}{(g^2+b^2)^2(g^2-k^2)} = \frac{\mu}{4\pi b}\frac{b^2-k^2}{(b^2+k^2)^2}$$

$$J_3(k) = \frac{\mu}{\pi^2} P \int_0^\infty \frac{dg}{(g^4+a^4)(g^2-k^2)} = -\frac{\mu\sqrt{2}}{4\pi a^3}\frac{k^2+a^2}{k^4+a^4}$$

$$J_4(k) = \frac{\mu}{\pi^2} P \int \frac{g^2 dg}{(g^4+a^4)(g^2-k^2)} = \frac{\mu\sqrt{2}}{4\pi a}\cdot\frac{a^2-k^2}{a^4+k^4}$$

$$J_5(k) = \frac{\mu}{\pi^2} P \int_0^\infty \frac{g^2 dg}{(g^4+a^4)^2(g^2-k^2)} = \frac{\mu\sqrt{2}}{4a\pi}\frac{3a^6-5a^2k^4-k^6+a^2k^4}{a^4(a^4+k^4)^2}$$

$$J_6(k) = \frac{\mu}{\pi^2} P \int_0^\infty \frac{g^4 dg}{(g^4+a^4)^2(g^2-k^2)} = \frac{\mu\sqrt{2}}{16\pi a}\frac{a^6-k^6+3k^2a^2(a^2-k^2)}{a^2(a^4+k^4)^2}$$

$$J_7(k) = \frac{\mu}{\pi^2} P \int_0^\infty \frac{g^6 dg}{(g^4+a^4)^2(g^2-k^2)} = \frac{\mu\sqrt{2}}{16\pi a} - \frac{-(k^6+a^6) + 3a^2k^2(k^2+a^2)}{(k^4+a^4)^2} \tag{56}$$

The scattering length and effective range are:

$$-\frac{1}{a_s} = -\frac{2a^8b^4}{\lambda' k_c^2 d^4}\{1 + \frac{\lambda'}{b^3}[-\frac{\delta_1}{2} + \frac{\delta_2}{4} + \frac{\sqrt{2}}{4}\frac{b^3}{a^3}(-\delta_3 + \delta_4 + \frac{3}{4}\frac{\delta_5}{a^4} + \frac{\delta_6}{4a^2} - \frac{\delta_7}{4})]\} \tag{57}$$

$$\frac{1}{2}r_s = -\frac{2a^8b^2}{\lambda' k_c^2 d^4}\{2 + \frac{\lambda'}{b^3}[-\frac{\delta_1}{2} - \frac{\delta_2}{4} + \frac{\sqrt{2}}{2}\frac{b^3}{a^3}(-\frac{\delta_3}{a^2} + \delta_4 - \frac{\delta_5}{a^4} + \frac{\delta_6}{4a^2} - \frac{\delta_7}{4})$$

$$+ \frac{\sqrt{2}}{4} \frac{b^5}{a^5}(-\frac{\delta_3}{a^2} - \delta_4 - \frac{\delta_5}{a^4} + \frac{\delta_6}{4a^2} + \frac{3}{4}\delta_7)$$

$$- \frac{\lambda' d^2 k_c^2}{2a^8 b^2}(k_c^2 - d^2)(-\frac{1}{a_s})\}; \quad (\lambda' = \frac{\lambda\mu}{\pi}) \tag{58}$$

4. Modified Tabakin's Potential

Tabakin's potential, when the Coulomb potential is added, give rise to very highly transcendental functions which make our analysis difficult. For simplification, we shall modify Tabakin's potential as:

$$g(k) = \frac{k^2 - k_c^2}{(k^2 + a^2)(k^2 + b^2)}$$

$$= \frac{t_1}{k^2 + a^2} + \frac{t_2}{k^2 + b^2}$$

where

$$t_1 = \frac{k_c^2 + a^2}{a^2 - b^2} \; ; \; t_2 = - \frac{k_c^2 - b^2}{a^2 - b^2} \tag{59}$$

In this case:

$$\text{Re } J(k) = \frac{\mu t_1}{\pi} \frac{a^2 - k^2}{4a(a^2 + k^2)} + \frac{\mu t_2}{\pi} \frac{b^2 - k^2}{4b(b^2 + k^2)}$$
$$+ \frac{\mu t_1 t_2}{\pi(a + b)} \frac{ab - k^2}{(a^2 + k^2)(b^2 + k^2)} \tag{60}$$

The scattering length and effective range are given by:

$$- \frac{1}{a_s} = - \frac{2}{\lambda'} (\frac{ab}{k_c})^4 [1 + \lambda' (\frac{t_1}{4a^3} + \frac{t_2}{4b^3} + \frac{t_1 t_2}{ab(a+b)}]$$

$$\frac{1}{2} r_s = - \frac{2}{\lambda'} \frac{a^2 b^2 (a^2+b^2)}{k_c^4} \{2 + \lambda' \; [- \frac{t_1^2 b^2}{4a^3(a^2+b^2)}$$

$$- \frac{t_2^2 a^2}{4b^3(n^2+b^2)} + \frac{t_1^2 + t_2^2 a}{2ab(a^2+b^2)}$$

$$+ \frac{t_1 t_2 (a^2+b^2-ab)}{ab(a+b)(a^2+b^2)} - \frac{\lambda' k_c^2}{a^2 b^2 (a^2+b^2)} \quad (-\frac{1}{a_s}\} \qquad (62)$$

III. SCATTERING FROM A SEPERABLE POTENTIAL AND A POTENTIAL

A. General Formalism

In this section, we shall apply the conventional formulation of scattering from two potentials to the case where at least one of the potentials is separable. We shall study the properties of two particles of mass m_1 and m_2 interacting via a potential $V = V_s + V_a$ where V_s is a separable potential and V_a is any potential, local or non local.

Let $|\phi_\alpha>$; $|\Gamma_\alpha^{\pm}>$ eigenstate of H_o and V_a respectively and $|\Psi_\alpha^{\pm}>$ the outgoing and ingoing exact states, eigenstates of :

$$H = H_o + V_s + V_a \tag{1}$$

The formal relations between $|\Gamma_\alpha^{\pm}>$ and $|\phi_\alpha>$ is :

$$|\Gamma_\alpha^{\pm}> = |\phi_\alpha> + G_a^{\pm}(s_\alpha) \; V_a|\phi_\alpha> \tag{2}$$

and between $|\Psi_\alpha^{\pm}>$ and $|\phi_\alpha>$ is:

$$|\Psi_\alpha^{\pm}> = |\phi_\alpha> + G^{\pm}(s_\alpha)(V_s + V_a)|\phi_\alpha> \tag{3}$$

where

$$G_a^{\pm}(s) = (s_\alpha - H_o - V_a \pm i\varepsilon)^{-1} \tag{4}$$

$$G^{\pm}(s) = (s_\alpha - H \pm i\varepsilon)^{-1} \tag{5}$$

Relation (3) can be manipulated as:

$$|\Psi_\alpha^\pm\rangle = |\Gamma_\alpha^\pm\rangle + G^\pm(s)\ V_s|\Gamma_\alpha^\pm\rangle \qquad (6)$$

which may be an obvious result if we start with the complete set $|\Gamma_\alpha^\pm\rangle$.

The S-matrix elements are defined as:

$$S_{\beta\alpha} \equiv \langle\psi_\beta^-|\Psi_\alpha^+\rangle \qquad (7)$$

Inserting (6) into (7) we get:

$$\begin{aligned} S_{\beta\alpha} &= \langle\Gamma_\beta^-|\Gamma_\alpha^+\rangle + \langle\Gamma_\beta^-|G^+(s_\beta)\ V_s|\Gamma_\alpha^+\rangle \\ &+ \langle\Gamma_\beta^-|V_s\ G^+(s_\beta)\ V_s|\Gamma_\alpha^+\rangle \\ &+ \langle\Gamma_\beta^-|V_s\ G^-(s_\beta)\ G^+(s_\alpha)\ V_s|\Gamma_\alpha^+\rangle \qquad (8) \end{aligned}$$

Now we use the identity:

$$\begin{aligned} G^\pm(s) &= G_\alpha^\pm(s) + G^\pm(s)\ V_s\ G_a^\pm(s) \\ &= G_\alpha^\pm(s) + G_a^\pm(s)\ V_s\ G^\pm(s) \end{aligned}$$

for the second and the third terms in equation (8), we obtain:

$$\begin{aligned} S_{\beta\alpha} &= \langle\Gamma_\beta^-|\Gamma_\alpha^+\rangle \\ &+ \frac{1}{s_\alpha - s_\beta + i\varepsilon} \langle\Gamma_\beta^+|(1 + V_s\ G^+(s_\alpha))\ V_s|\Gamma_\alpha^+\rangle \\ &+ \frac{1}{s_\alpha - s_\beta + i\varepsilon} \langle\Gamma_\beta^-|V_s(1 + G^+(s_\beta)\ V_s)|\Gamma_\alpha^+\rangle \qquad (9) \\ &+ \langle\Gamma_\beta^-|V_s\ G^+(s_\beta)\ G^+(s_\alpha)\ V_s|\Gamma_\alpha^+\rangle \end{aligned}$$

The last term can be written as:

$$\frac{P}{S_\alpha - S_\beta} \langle \Gamma_\beta^- | V_s (G^+(S_\beta) - G^+(S_\alpha)) \; V_s | \Gamma_\alpha^+ \rangle$$

where we have chosen arbitrarily to write 'principal value' as the instruction for handling the apparent singularity at $S_\alpha = S_\beta$. If now in the second and the third term we write:

$$\frac{1}{S_\alpha - S_\beta + i\varepsilon} = \frac{P}{S_\alpha - S_\beta} + i\pi\delta(S_\alpha - S_\beta)$$

We see that the entire last term in the equation (9) is cancelled and we have:

$$S_{\beta\alpha} = \langle \phi_\alpha | \phi_\beta \rangle - 2\pi i \delta(s_\alpha - s_\beta) \times [T_{sa,\beta\alpha} + T_{a,\beta\alpha}] \tag{10}$$

with

$$T_{a,\beta\alpha} = \langle \phi_\beta | V_a | \Gamma_\alpha^+ \rangle \tag{11}$$

$$T_{sa,\beta\alpha} = \langle \Gamma_\beta^- | V_s | \Psi_\alpha^+ \rangle \tag{12}$$

The matrix elements $T_{a,\beta\alpha}$ is just the scattering amplitude corresponding to scattering from the potential V_a. We are interested mainly in $T_{sa,\beta\alpha}$ which can be written as:

$$T_{sa,\beta\alpha}(s) = \langle \Gamma_\beta^- | V_s + V_s \; G^+(s_\alpha) \; V_s) | \Gamma_\alpha^+ \rangle \tag{13}$$

The operator T_{sa} defined as:

$$T_{sa} = V_s + V_s \; G^+(s) \; V_s \tag{14}$$

Obeys the integral equation:

$$T_{sa} = V_s + V_s\, G_a^{+}(s)\, T_{sa} \tag{15}$$

Assuming the set $|\Gamma_\alpha^{\pm}>$ is complete, the equation (15) may be written as:

$$\begin{aligned} T_{sa,\beta\alpha} &= <\Gamma_\beta^{-}|V_s|\Gamma_\alpha^{+}> \\ &+ \sum_\gamma \frac{1}{s - s_\gamma^a + i\varepsilon} <\Gamma_\beta^{-}|V_s|\Gamma_\gamma^{-}>\, T_{sa,\gamma\alpha} \end{aligned} \tag{16}$$

where S_γ^a is the eigenvalue of the Hamiltonian $H_a = H_o + V_a$. In the absence of the potential V_a, V_s leads to an integral equation with degenerate kernel whose solutions are simple. In the presence of the potential V_a, equation (16) shows that we are led to an integral equation with V_s replaced by:

$$V_{sa,\beta\alpha}^{(\pm)} = \sum_{\mu\nu} <\Gamma_\beta|\xi_\nu>\, V_{s,\mu\nu}\, <\xi_\nu|\Gamma_\alpha^{(\pm)}> \tag{17}$$

where we have introduced a representation $\{\xi_\mu\}$. Because V_s is separable:

$$V_{s,\mu\nu} = \sum_{i=1}^{N} <\xi_\mu|g_i>\, \lambda_i <g_i|\xi_\nu> \tag{18}$$

Equation (17) shows that V_{sa} is also separable. Our problem is now reduced to the scattering problem from a separable potential which we have studied in Chapter II.

We should like to point out here that, instead of writing (3) as (6), we write it as:

$$|\Psi_\alpha^{\pm}> = |\chi_\alpha^{\pm}> + G^{\pm}(s)\, V_a|\chi_\alpha^{\pm}> \tag{19}$$

We obtain, instead of (10), the similar equation:

$$S_{\beta\alpha} = \langle\phi_\alpha|\phi_\beta\rangle - 2\pi i\delta[s_\alpha - s_\beta] [T_{as,\beta\alpha} + T_{s,\beta\alpha}] \tag{20}$$

with

$$T_{s,\beta\alpha} = \langle\phi_\beta|V_s|\chi_\alpha^+\rangle \tag{21}$$

$$T_{as,\beta\alpha} = \langle\chi_\beta|V_a|\Psi_\alpha^+\rangle \tag{22}$$

The operator T_s has been studied in Chapter II, T_{as} satisfies an equation similar to equation (15):

$$T_{as} = V_a + V_a\, G_s^+(s)\; T_{as} \tag{23}$$

Equation (23) is not necessarily an integral equation with degenerate kernel due to V_a being not necessarily separable. But if V_a is a separable potential with M terms, then $V_a + V_s$ is separable potential with N + M terms, then our formalism for scattering from two separable potentials is not necessary. We shall indicate later that, a two term separable potential or many terms separable potential may be equivalent to one term separable potential as far as the phase-shift is concerned.

B. Scattering from a Separable Potential and the Coulomb Potential

To illustrate the formulation developed so far, we shall study the problem of scattering from a separable from a separable potential and the Coulomb potential. This problem was first treated by Harrington.

To avoid the complications due to the extra bound states, we assume the Coulomb interaction is repulsive. We also cut off the Coulomb potential at a shielding radius R. We can then replace V_a in the above formulae by V_c, the Coulomb potential. T_c is just the usual Coulomb scattering operator. Our main object of interest is T_{sc}.

In momentum space, the partial matrix elements satisfies the integral equation (Appendix A).

$$T^{SJ}_{sc,L} = (s,k',k) = V^{SJ(+)}_{sc,L}(k',k) + \frac{1}{2\pi^2}\int_0^\infty \frac{q^2 dq\, V^{SJ(-)}_{sc,L}(k',q)\; T^{SJ}_{sc,L}(s,q,k)}{s(k) - s(q) + i\varepsilon} \tag{24}$$

With $V_{sc}(k',k)$ defined by:

$$V^{SJ}_{sc,L}(k',k) = \frac{1}{2\pi^2}\int_0^\infty q'^2 dq' \int_0^\infty q^2 dq \tag{25}$$

$$\langle \Gamma^{SJ-}_{CL}(k') | q' \rangle \; V^{SJ}_{s,L}(q',q) \langle q | \Gamma^{SJ\pm}_{CL}(k) \rangle$$

where the partial Coulomb states $|\Gamma^{SJ}_{C,L}\rangle$ is defined by:

$$|\Gamma_C\rangle = \sum_{L,S,J,M} |Y^M_{LSJ}\rangle |\Gamma^{SJ}_{c,L}\rangle \tag{26}$$

The unitary condition T_{sc} is satisfied if we write the on-shell Coulomb and total partial scattering matrix elements, respectively as:

$$T^{SJ}_{c,L}(k,k) = -\frac{2\pi}{\mu k} \sin \delta^{SJ}_{c,L}\; e^{i\delta^{SJ}_{c,L}(k)} \tag{27}$$

$$T^{SJ}_{c,L}(k,k) + T^{SJ}_{sc,L}[s(k),k,k)] + \frac{2\pi}{\mu k} \sin \delta^{SJ}_{c,L}(k)\ e^{i\delta^{SJ}_{c,L}(k)}$$

where $\delta^{SJ}_{c,L}(k)$ and $\delta^{SJ}_{L}(k)$ are real phase-shifts. Substituting the expression for $V^{SJ}_{s,L}(k',q)$ in (25) we obtain:

$$V^{SJ}_{sc,L}(k',k) = e^{i\delta^{SJ}_{c,L}(k')} \sum_i \lambda^{SJ}_{Li}\ g^{SJ}_{c,Li}(k')\ g^{SJ}_{c,Li}(k)\ e^{\pm i\delta^{SJ}_{c,L}(k)} \qquad (28)$$

Where we have defined:

$$g^{SJ}_{c,Li}\ e^{\pm i\delta^{SJ}_{c,L}(k)} = \frac{1}{2\pi^2} \int_0^\infty q^2 dq\ g^{SJ}_{Li}(g)\ \langle g | \Gamma^{SJ\pm}_{c,L} \rangle \qquad (29)$$

From (28), it is clear that $V_{sc,L}$ is also separable. The Coulomb wave function in configuration space has been studied by Yost, Wheeler and Breit (Ref. 16). We are concerned here with modified Coulomb wave functions, i.e. solution of the Schrödinger equation with a cut off rather than the exact Coulomb potential, which satisfies the asymptotic condition:

$$w_\ell(k,r)\ _{r\to\infty}\ \sin(kr - \frac{\ell\pi}{2} + \delta_{c,\ell}(k))$$

For $kR >> L(L+1) + \eta^2(k)$ and $r<R$:

$$W_L(k,r) = F_L(kr)$$

$$= (2i)^{-L-1}\ C_L(\eta)\ M i\eta,\ \tfrac{1}{2}(2ikr) \qquad (30)$$

where $M_{k,\mu}(z)$ is the Whittaker function.

$$C_L(\eta) = 2^L (2L+1)!^{-1}\ \Gamma(L+1-i\eta)\ e^{-\frac{1}{2}\pi\eta} \qquad (31)$$

is the barrier penetration factor, and:

$$\eta = \eta(k) = \frac{Me_1 e_2}{k} \tag{32}$$

The radial Coulomb wave function $F\ell(kr)$ has the asymptotic form:

$$F_o(kr) \simeq \sin\left(kr - \eta \ln 2kr - \frac{1}{2} l\pi + \sigma_L\right) \tag{33}$$

where σ_1, usually known as the Coulomb phase-shift, is given by:

$$\sigma_L = \arg \Gamma(L + 1 + i\eta) \tag{34}$$

The exact value of $\delta_{c,L}$ depends on the nature of the cut off. For k not to small, we expect (Ref. 14)

$$\begin{aligned} \sigma_{c,L} &= \sigma_L - \eta \ln 2kR \qquad L << kR \\ \delta_{c,L} &= 0 \qquad\qquad\qquad\quad L >> kR \end{aligned} \tag{35}$$

In order to make use of these known results for Coulomb wave function, it is convenient to express the $g^{SJ}_{c,Li}(k)$ defined by (29) in terms of the configuration space Coulomb wave functions.

Using the matrix elements for the transformations from $\{q\}$ representation to $\{r\}$ representation and the expansion of Coulomb waves, we can write (29) as:

$$g^{SJ}_{c,Li}(k) = \frac{1}{2\pi^2} \int q^2 dq\, g^{SJ}_{Li}(q) \times \int e^{iq.r}\, i^L\, w_L(kr)\, dr \tag{36}$$

Expanding $e^{iq.r}$ in series of harmonic functions, then simplifying with the aid of the closure relation for $y_L^M(\Omega)$ we obtain:

$$g^{SJ}_{c,Li}(k) = \frac{4\pi}{k} \int_0^\infty rdr\ G^{SJ}_{L,i}(r)\ w_L(k,r) \tag{37}$$

with

$$G^{SJ}_{L,i}(r) = \frac{(-)^L}{2\pi^2} \int_0^\infty q^2 dq\ g^{SJ}_{Li}(q)\ j_L(qr) \tag{38}$$

$j_L(kr)$ is spherical Bessel Functions.

Solution of the integral equation (24) can be written under the form (Appendix B)

$$T^{SJ}_{sc,L}(s,k',k) = e^{i\delta^{SJ}_{c,L}(k')} \Sigma\ g^{SJ}_{c,Li}(k')\ \tau^{SJ}_{L,ij}(s)\ g^{SJ}_{c,Lj}(k) e^{i\varepsilon^{SJ}_{c,L}(k)} \tag{39}$$

The matrix τ_{ij} is defined by:

$$\tau(s) = 1 + \Lambda I(s)\ ^{-1}\Lambda \tag{40}$$

where Λ is the diagonal matrix

$$\lambda_{ij} = \delta_{ij}\ \lambda_i \tag{41}$$

and the elements of matrix $I(s)$ are given by:

$$I_{ij}(s) = \frac{1}{2\pi^2} \int_0^\infty \frac{k^2 dk\ g_{c,i}(R)\ g_{c,j}(k)}{s(k) - s - i\varepsilon} \tag{42}$$

From (39) and (42) we see that, except for the Coulomb phase factors, $T^{SJ}_{sc,L}(s,k',k)$ can be obtained from the corresponding $T^{SJ}_{s,L}$-matrix elements in the absence of the Coulomb potential merely by replacing $g^{SJ}_{L,i}$ by $g^{SJ}_{c,Li}$.

Now, from (27) and (39) we deduce that:

$$\delta^{SJ}_{L} = \delta^{SJ}_{sc,L} + \delta^{SJ}_{c,L} \tag{43}$$

In examining the argument of (39), we can see early that:

$$\delta^{SJ}_{c,L} = -\arg\{\det[1 + \Lambda^{SJ}_{L}\, I^{SJ}_{L}(s)]\} \tag{44}$$

We have arrived at the results of Coulomb + short range potential scattering originally studied by Yost, Wheeler and Breit. In general the exact nature of the Coulomb screening is not known. As long as R is much longer than the range of V_s, however, under most experimental conditions, the differential cross-section is given accurately by (Ref. 14, page 263):

$$\frac{d\sigma}{d\Omega} = |f_c(\theta) + f_{sc}(\theta)|^2 \tag{45}$$

where

$$f_c = \frac{1}{k}\,\Sigma_L(2L + 1)\sin\sigma_L\, e^{i\sigma_L}\, P_L(\cos)$$

$$f_{sc} = \frac{1}{k}\,\Sigma_L(2L + 1)\sin\delta_{sc,L}\, e^{i\delta_{sc,L} + 2i\sigma_L} P_L(\cos\theta)$$

C. Scattering From Two Separable Potentials

In applications of separable potentials to Nucleon-Nucleon scattering, it is generally found that a many term separable potentials gives better agreements with experiments than a single term separable potential (Ref. 8). To obtain agreements with experimental phase-shifts up to 300 MeV, a two-term separable potential seems to account for known behaviours. As we have pointed out, as far as the phase-shift is concerned, the usual formulation for scattering from two potentials shows that a two terms separable potential may be equivalent to a one term separable potential. Let us now study the scattering problem via two separable potentials $V_s + U_s$ which, for simplification, we assume to be single term separable potentials.

Again we replace V_a in section A by U_s. Equations (10), (11) become :

$$S_{\beta\alpha} = \langle\phi_\alpha|\phi_\beta\rangle - 2\pi i\delta[s_\alpha - s_\beta] \times [T_{\beta\alpha} + \tau_{\beta\alpha}] \tag{47}$$

where

$$T_{\beta\alpha} = \langle\phi_\beta|U_s|\chi_\alpha^+\rangle \tag{48}$$

$$\tau_{\beta\alpha} = \langle\chi_\beta|V_s|\Psi_\alpha^+\rangle \tag{49}$$

$|\chi_\alpha^{(\pm)}\rangle$ are the outgoing and ingoing states in the absence of V_s, which we have studied in chapter II.

$$\tau_L^{SJ}(s,k',k) = W_L^{SJ+}(k',k) + \frac{1}{2\pi^2}\int_0^\infty \frac{q^2dq\; W_L^{SJ-}(k',q)\; \tau_L^{SJ}(s,q,k)}{s(k) - s(q) + i\varepsilon} \tag{50}$$

where

$$W_L^{SJ\pm}(k',k) = \frac{1}{2\pi^2} \int q^2 dq \int_0^\infty q^2 dq \tag{51}$$

$$<\chi_L^{SJ-}(k')|q'> U_{s,L}^{SJ}(q',q) <q|\chi_L^{SJ\pm}(k)>$$

where the partial wave $|\chi_L^{SJ}>$, according to equation (13) Chapter II, is given by :

$$|\chi_L^{SJ\pm}(k)> = |\phi_L^{SJ}(k)> + G_o^{\pm}(s(k)) \; T|\phi_L^{SJ}(k)>$$

or

$$<q|\chi_L^{SJ}(k)> = <q|\phi_L^{SJ}(k)>$$

$$+ \frac{1}{2\pi^2} \int_0^\infty q'^2 dq' \frac{<q'|\phi_L^{SJ}(k)>}{s(q') - s(k) + i\varepsilon} T_L^{SJ}[s(k),q,q'] \tag{52}$$

Now, if we write the on-shell T_L and τ_L as in equation (27), we find that we are led to an equivalent problem of scattering from a one-term separable potential whose form factor is given by (See equation 29) :

$$g_{U,L}^{SJ} \, e^{\pm i\delta_{U,L}^{SJ}(k)} = \frac{1}{2\pi^2} \int_0^\infty q^2 dq \; g_L^{SJ}(q) <q|\chi_L^{SJ\pm}(k) \tag{53}$$

where $g_L(q)$ is the form factor of the potential U_s and $\delta_{U,L}(k)$ is the partial phase-shift due to the potential U_s alone.

We have shown that a two term separable potential is equivalent to a one term separable potential whose form factor is obtained from one of two form-factors of the original two term separable potential by an integral transformation as (52). In doing this, we here assumed, as we did with the Coulomb states, the set $|\chi^{\pm}(k)\rangle$ is complete.

Our work here may have two applications. Firstly, a one term separable potential is always easier to handle than a two term separable potential. For some classes of problems which are too complicate to solve with a many term separable potential, we would then like to have a one term separable potential equivalent to it and work with this equivalent single term potential. Secondly, we can check with the conventional formulation for scattering from two potentials is exact. For two separable potentials, this problem is very easy because we can solve the problem, first exactly as scattering from two term separable potentials, and then, as scattering from two separable potentials; we then compare the results of our two approaches.

For a two term separable potential, we would choose our set $\{\phi\}$ the one corresponding to the term which yields a rather completeness for $\{\phi\}$. This term, in an intuitive way, would dominate over the other term in the range of interaction. If $\{\phi\}$ is complete, we would expect consistencies because mathematically we can use any complete set as our base for description of our system.

D. Nucleon-Nucleon Potentials

Despite their comparable success in Nucleon-Nucleon scattering analysis and three body calculations, separable potentials are purely phenomonological, if not unrealistic. The only theoretical justification for separable potentials is, as shown by Lovelace (Ref. 13), the approximation of the T matrix as separable near the resonances and bound states Now, in our analysis, scattering from a separable plus any potential is equivalent to scattering from a "modified separable potential". This suggests an explanation for the validity of separable potentials as Nucleon-Nucleon potentials. We imagine the Nucleon-Nucleon potentials contain a separable part which represents resonances, bound states and other unkown effects which we can approximate the potentials as separable, and a non separable part which is, for example, a local theoretical potential :

$$V = V_s + V_a \tag{54}$$

It is obvious that V_s depends on V_a. The most simple idea is that we can take for V_a a local theoretical potential (Yukawa, OPEP, TPEP, etc...). V_s is not known and again can be taken phenomonologically. In actual analysis this, of course, is not a clever approach because if (54) is equivalent to a pheonomological separable potential, we can simply take V as a separable potential for a much easier analysis. However, by studying the V_s in (54) we can have some understandings about the resonances and bound states if we believe in Lovelace's arguments. Anyway, we like to offer here only an explanation for Nucleon-Nucleon potentials as separable potentials.

F. Applications

We now apply some of the formulas developed in section B to the simple problem of calculating the phase-shift of scattering from a separable potential plus the Coulomb potential. We shall first treat a single one term separable potential of Yamaguchi's type, than a two term separable potential proposed by Naqvi and a single term separable potential containing repulsive and attractive parts (modified Tabakin's potential).

1. Yamaguchi's Potential

In Chapter II we have applied our formulas to a simple single separable potential of Yamaguchi's type. In this section we repeat the same procedure except we replace $g(k)$ by $g_c(k)$ and $J(k)$ by $I(k)$ with

$$I(k) = \frac{\mu}{\pi^2} \int_0^\infty \frac{q^2 dq \; g_c^2(q)}{q^2 - k^2 - i\varepsilon} \tag{55}$$

Using (38):

$$G(r) = \frac{1}{2\pi^2 r} \int_0^\infty \frac{q}{q^2 + \beta^2} \sin (qr) dr \tag{56}$$

The sine-Forrier transform (56) can be obtained easily:

$$G(r) = \frac{1}{4\pi r} e^{-\beta r}$$

Then (37) becomes:

$$\begin{aligned} g_c(k) &= \frac{1}{k}\int_0^\infty e^{-\beta r} w_o(kr)dr \\ &= \frac{1}{2ki}\int_0^\infty e^{-\beta r} C_o(\eta) M_{i\eta,\frac{1}{2}}(2ikr) \end{aligned} \tag{57}$$

The Laplace transform of Whittaker function is given in standard table (Ref. 22):

$$g_c(k) = g(k)\ C_o(\eta)\ e^{2\eta \tan^{-1}\frac{k}{\beta}} \tag{58}$$

with

$$C_o(\eta) = \left(\frac{2\pi\eta}{e^{2\pi\eta} - 1}\right)^{\frac{1}{2}} \tag{59}$$

We can write the integral (37) explicitly as:

$$I(k) = \frac{\mu}{\pi^2}\int_0^\infty \frac{q^2 dq\ C_o^2(\eta)\ e^{4\eta \tan^{-1}\frac{q}{\beta}}}{(q^2+\beta^2)^2(q^2-k^2-i\varepsilon)} \tag{60}$$

In the q-plane, $\tan^{-1}\frac{q}{\beta}$ has two branch points at $q = i\beta$ and $q = -i\beta$. Here we realize that the Coulomb interaction has the effect that it can change a pole of V_s into the branch point. To calculate I(k) we must cut the q plane in a convenient way and choose the value of $\tan^{-1}\frac{q}{\beta}$ in

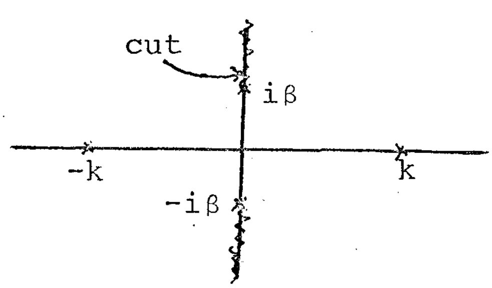

Fig. 1

a Rieman plane. It was shown (Ref. 17) that the Coulomb effect enters the dispersion relation in a way that if the left hand singularities in the neutron-neutron problem are represented by a single pole, then the proper statement of charge symmetry is that there will be a series of branch points in proton-proton problems.

Using (44) we obtain:

$$\cot_{sc}(k) = -\frac{\text{Re}(1 + \lambda I)}{\lambda \text{Im} I} \tag{61}$$

with

$$\text{Im}\, I(k) = \frac{\mu k}{2\pi} \frac{C_o^2(\eta)\, e^{4\eta \tan^{-1}\frac{k}{\beta}}}{(k^2 + \beta^2)^2} \tag{62}$$

$$\text{Re}\, I(k) \equiv R = \frac{\mu}{\pi^2} P \int_o^\infty \frac{q^2 dq\, C_o^2(\eta(q))\, e^{4\eta \tan^{-1}\frac{q}{\beta}}}{(q^2 + \beta^2)^2 (q^2 - k^2)} \tag{63}$$

The factor $4\eta \tan^{-1}\frac{q}{\beta} = \frac{4\mu e_1 e_2}{q} \tan^{-1}\frac{q}{\beta} << 1$ for all q in the case $\frac{\mu e_1 e_2}{\beta} << 1$. In this case the above formulas can be considerably simplified by expanding the factor $e^{4\eta \tan^{-1}\frac{q}{\beta}}$ which occurs in the integral (46) to obtain a perturbation for Re I(k).

The first term in the expansion is:

$$R_o(k) = \frac{\mu}{\pi^2} P \int_o^\infty \frac{q^2 dq\, C_o^2(\eta(q))}{(q^2 + \beta^2)^2 (q^2 - k^2)}$$

$$- \frac{k^2}{(k^2 + \beta^2)} \frac{\mu}{\pi^2} \int_o^\infty \frac{C_o^2(\eta(q))\, dq}{(q^2 + \beta^2)}$$

$$+ \frac{\beta^2}{\beta^2 + k^2} \frac{\mu}{\pi^2} \int_0^\infty \frac{C_o^2(\eta(q))dq}{(q^2+\beta^2)^2} + \frac{k^2}{(\beta^2+k^2)^2} \frac{P}{\pi^2} \int_0^\infty \frac{C_o^2(\eta)dq}{q^2 - k^2} \tag{64}$$

The first integral of (47) can be calculated with the aid of an integral representation of the Ψ-function. It can be verified that (Ref. 22)

$$\Psi[\eta(\beta)] = \ln\left(\frac{\mu e_1 e_2}{\beta}\right) - \frac{\beta}{2\mu e_1 e_2} - \frac{\beta^2}{\mu e_1 e_2} \frac{1}{\pi} \int_0^\infty \frac{dq\; C_o^2[\eta(q)]}{q^2 + \beta^2} \tag{65}$$

The second integral can be obtained by differentiating (48) with respect to β. The last integral is given by the formula:

$$h(\eta(k)) \equiv \mathrm{Re}\Psi(-i\eta) - \ln \eta = \frac{k^2}{\mu e_1 e_2} \frac{P}{\pi} \int_0^\infty \frac{dq\; C_o^2(\eta(a))}{q^2 - k^2} \tag{66}$$

The next term in the expansion is:

$$R_1(k) = 4\mu e_1 e_2 \frac{P}{\pi^2} \int_0^\infty \frac{qdq\; \tan^{-1}\frac{q}{\beta}\; C_o^2(\eta(q))}{(q^2+\beta^2)^2(q^2-k^2)} \tag{67}$$

We note that we shall keep terms only to order $\frac{\mu e_1 e_2}{\beta}$, ignoring terms of order $(\frac{\mu e_1 e_2}{\beta})^2 \ln (\frac{\mu e_1 e_2}{\beta})$, we can then replace $C_o^2(\eta(a))$ by 1, the integral becoming elementary. To avoid the difficulties due to the branch points of $\tan^{-1}$

$\frac{q}{\beta}$, we consider the integral

$$K = P \int_{-\infty}^{+\infty} \frac{z \ln (1 - i \frac{z}{\beta}) dz}{(z^2 + \beta^2)^2 (z^2 - k^2)}$$

which can be shown:

$$R_1(k) = \frac{8\mu e_1 e_2 \; iK}{\pi^2}$$

Using a contour integral in the z-plane (Appendix C) we obtain:

$$R_1(k) = \pi\mu e_1 e_2 [- \frac{1}{2\beta^2} + \frac{\ln 4 - \ln[\beta^2 + k^2) + 2 \ln \beta}{\beta^2 + k^2}] \quad (68)$$

Now, consistent with our approximation we write:

$$\Psi(\frac{\mu e_1 e_2}{\beta}) \simeq - \Gamma - \frac{\beta}{\mu e_1 e_2} \quad (69)$$

where Γ is the Euler constant; $\Gamma = 0.577215....$ With the aid of (65), (66), (68) and (69) we obtain:

$$R \equiv \text{Re } I(a) = \frac{\mu\beta}{\pi(\beta^2 + k^2)^2} \{\frac{\beta^2 - k^2}{4\beta^2} + \frac{\mu e_1 e_2}{\beta} \ln(\eta) + \frac{\mu e_1 e_2}{\beta} [\ln (4\mu e_1 e_2 \beta) - \ln(\beta^2 + k^2) + \Gamma]\} \quad (70)$$

Substituting (45) and (52) into (44), we obtain:

$$k \, C_o^2(\eta) \cot \delta_{sc} + 2\mu e_1 e_2 \, h(\eta) = k \cot \delta_s (1 - \frac{4\mu e_1 e_2}{k} \tan^{-1} \frac{k}{\beta}) - 2\mu e_1 e_2 [\ln(4\mu e_1 e_2 \beta) - \ln(\beta^2 + k^2)]$$

where $k \cot \delta_s$ is given by (II-32). The left hand side is the function used for the effective range expansion for the presence of Coulomb potential (Ref. 23). Expanding the right hand side of (53) in power series of k, we obtain the formulas for scattering length and effective range:

$$-\frac{1}{a_{sc}} = -\frac{1}{a_s}[1 - \frac{4\mu e_1 e_2}{\beta}) - 2\mu e_1 e_2 [\ln(\frac{4\mu e_1 e_2}{\beta}) + \Gamma] \tag{71}$$

$$\frac{1}{2}r_{sc} = \frac{1}{2}r_s[1 - \frac{4\mu e_1 e_2}{\beta}] + \frac{2\mu e_1 e_2}{\beta^2}[1 - \frac{2}{3\beta a_s}] \tag{72}$$

where a_s and r_s are given by (II-35) and (II-36).

2. Naqvi's Potential

The form factors of this potential are given by (II-39). From (58) we have:

$$gc_1(k) = g_1(k)\ C_o(\eta) e^{2\eta \tan^{-1}\frac{k}{\beta}} \tag{73}$$

From (II-45) we deduce that:

$$gc_2(k) = (1 + \beta^2 \frac{d}{d\beta^2})\ g_{c,1}(k) \tag{74}$$

Then:

$$I_{11}(k) = \frac{1}{2\pi^2}\int_0^\infty dq\ \frac{q^2 g_{c,1}^2(k)}{s(q) - s(k) - i\varepsilon} = I(k) \tag{75}$$

where I(k) is given by (62) and (70.

From:

$$g_{c1}(k)\; g_{c2}(k) = \left(1 + \frac{\beta^2}{2}\frac{d}{d\beta^2}\right) g_{c1}(k) \tag{76}$$

we obtain:

$$I_{12}(k) = I_{21}(k) = \frac{1}{2\pi^2}\int_0^\infty \frac{q^2 dq\; g_{c1}(q)\; g_{c2}(q)}{s(q) - s(k) - i\varepsilon}$$

$$= I(k) + \frac{\beta^2}{2} J'(k) \;\; ; \;\; J' = \frac{dJ}{d\beta^2} \tag{77}$$

From:

$$g_{c2}^2(k) = g_{c1}^2 + \beta^2 (g_{c1})' + \frac{\beta^4}{6}(g_{c1}^2)'' + \frac{\beta^4}{3}(2g_{c1}'^2 - g_{c1}\, g_{c1}'') \tag{78}$$

we obtain:

$$I_{22}(k) = J(k) + \beta^2 J'(k) + \frac{\beta^4}{6} J''(k) + R(k) \tag{79}$$

where, to the first order in α

$$R(k) = \frac{\alpha}{3}\frac{\beta^4}{2\pi^2}\int_0^\infty \frac{q^2 dq\; g_{c1}^2(q)}{(q^2+\beta^2)(s(q) - s(k) - i\varepsilon)} \tag{80}$$

In obtaining (79) we have used the relations.

$$g_{c1}'(k) = \frac{1+2}{k^2+\beta^2} g_{c1}(k)$$

$$g_{c1}''(k) = \frac{(1+\alpha)(2+\alpha)}{(k^2+\beta^2)^2} g_{c1}(k)$$

$$2g_{c1}'(k) - g_{c1}(k)\; g_{c1}''(k) \simeq \frac{\alpha}{k^2+\beta^2} g_{c1}^2(k), \alpha = \frac{\mu e_1 e_2}{\beta} \tag{81}$$

If we write I(k) as:

$$I(k) = A + i\,B$$

$$A = \frac{\beta^2 - k^2}{4\beta K^2} + \frac{\alpha\beta}{K^2}\{h(\eta) + \ln\frac{4\alpha\beta^2}{K} + \Gamma\}$$

$$B = C_o^2(\eta)\,\frac{k}{2K^2}\exp[4\eta\tan^{-1}\frac{k}{\beta}]$$

$$K = k^2 + \beta^2 \tag{82}$$

then:

$$\cot\delta_{sc}(k) = -\frac{\mathrm{Re}\ D(k)}{\mathrm{Im}\ D(k)} \tag{83}$$

where D(k) is the matrix defined as in (II-41)

$$\mathrm{Re}\ D(k) = 1 + \lambda_1' A + \lambda_2'\{A + \beta^2 A' + \frac{\beta^4}{6}(1 + \frac{\alpha}{3})A''$$

$$+ \lambda_1'\lambda_2'\ \frac{\beta^4}{6}\ (1 + \frac{\alpha}{3})(AA'' - BB'') - \frac{3}{2}(A'^2 - B'^2)\}$$

$$\mathrm{Im}\ D(k) = \lambda_1' B + \lambda_2'\{B + \beta^2 B' + \frac{\beta^4}{6}(1 + \frac{\alpha}{3})B''\}$$

$$+ \lambda_1'\lambda_2'\ \frac{\beta^4}{6}\ (1 + \frac{\alpha}{3})(AB'' + A''B) - 3A'B'\} \tag{84}$$

Let us write A as:

$$A = A_o + \alpha A_1 \tag{85}$$

where A_o, A_1 are defined by comparing (85) with (82).

Putting (85) into (84) and making use of (79), (80), (82), one can show, with some tedious manipulations, that:

$$\cot\delta_{sc} = -\frac{2(\mathrm{Re}\ D)_o}{(\mathrm{Im}\ D/B)} K^2 e^{4\eta\tan^{-1}\frac{k}{\beta}}$$

$$- 2\alpha\beta\{\ln\frac{4\alpha\beta^2}{K} + \Gamma\} \tag{86}$$

where $(\mathrm{Re}\ D)_o$ is obtained from (84) by discarding the terms αA_1 in the expressions of A, A' and A". Explicitly:

$$(\mathrm{Re}\ D)_o = 1 + A_o\{\lambda_1' + \lambda_2'[1 - \frac{2\beta^2}{K} + \frac{\beta^4}{K^2}(1 + \frac{\alpha}{3})]$$

$$+ \frac{\lambda_1'\lambda_2'\beta}{32K^2}(2 - \frac{K}{\beta^2}) - \frac{\alpha\beta^5\lambda_1'\lambda_2'}{6K^4}[-\frac{K}{2\beta^2} + \frac{K^2}{16\beta^4}(10 + \frac{K}{\beta^2})]\}$$

$$+ \frac{\lambda_2'\beta}{K^2}\frac{K}{8\beta^2} + \frac{\alpha\beta^2}{K}(-1 + \frac{K}{2\beta^2}) - \frac{1}{32}(1 + \frac{\alpha}{3})(2 + \frac{K}{\beta 2})$$

$$+ (1 + \frac{\alpha}{3})\frac{\alpha}{6}\frac{\beta 4}{K^2}(5 - \frac{2K}{\beta^2} - \frac{K^2}{2\beta^4})\}$$

$$+ \lambda_1'\lambda_2'\{\frac{-1}{16\beta^2K^2}(\frac{1}{16} + \frac{\beta^2\alpha}{2K} - \frac{\beta^4\alpha}{K^2}) - \frac{\alpha B^2}{K\beta^2}\frac{\beta^4}{6}\} \tag{87}$$

$$\frac{\mathrm{Im}\ D}{B} = \lambda_1' + \lambda_2'(\frac{k^2}{K})^2 + \lambda_1'\lambda_2'\frac{\beta}{32K^2}(2 - \frac{K}{\beta^2})$$

$$+ \alpha\frac{\beta^2}{K}[\lambda_2(-\frac{11}{6} + \frac{2\beta^2}{K})$$

$$- \lambda_1'\lambda_2'\frac{\beta^3}{K^3}(2 - \frac{2K}{\beta^2} + \frac{K^2}{8\beta^4} + \frac{K^3}{16\beta^6})] \tag{88}$$

We repeat that (86) is valid only to the first order in α. For completeness, we give here:

$$A' = -\frac{2}{K}A + \frac{\mu\beta}{K^3}[\frac{K^2}{8\beta^4} + \frac{k^2}{\beta^2}] \tag{89}$$

$$A'' = \frac{6}{K} A - \frac{\mu\beta}{\pi K^4} \{\frac{3}{16} \frac{K^2}{\beta^4}(3 + \frac{k^2}{\beta^2})$$

$$- \alpha[3 - 2 \frac{k^2}{\beta^2} - \frac{K^2}{2\beta^4}]\} \tag{90}$$

Expanding (86) in power series of k, we find:

$$kC_o^2(\eta)\cot \delta_{sc} + 2\alpha\beta h(\eta) = - \frac{1}{a_{sc}} + \frac{1}{2}r_{sc} k^2 + \ldots \tag{91}$$

with:

$$- \frac{1}{a_{sc}} = (1 - 4\alpha)(- \frac{1}{a_o}) - 2\alpha\beta(\ln 4\alpha + \Gamma) \tag{92}$$

$$\frac{1}{2}r_{sc} = (1 - 4\alpha)(\frac{1}{2}r_o) + \frac{4\alpha}{3\beta^2}(- \frac{1}{a_o}) + \frac{2\alpha}{\beta} \tag{93}$$

where

$$- \frac{1}{a_o} = - \frac{2\beta^4}{\lambda_1'}[1 + \frac{\alpha}{6} \frac{\lambda_2'}{\lambda_1'} + \frac{\lambda_2'}{16\beta^3}(\frac{1}{2} - 3\alpha)]^{-1}$$

$$\{1 + \frac{\lambda_1'}{4\beta^3}[1 + \frac{\lambda_2}{8\lambda_1}(1 - \alpha)] + \frac{\lambda_1'\lambda_2'}{\beta^6} \frac{1 + 6\alpha}{(16)^2}\} \tag{94}$$

$$\frac{1}{2}r_o = \frac{2\beta^2}{\lambda_1'}[1 + \frac{\alpha}{6} \frac{\lambda_2'}{\lambda_1'} + \frac{\lambda_2'}{16\beta^3}(\frac{1}{2} - 3\alpha)]^{-1}$$

$$\{\frac{\lambda_1'}{4\beta^3}[3 - \frac{\lambda_2'}{8\lambda_1'}(1 - \frac{77}{3} \alpha') + \frac{\lambda_2'}{32\beta^3}(1 + \frac{16\alpha}{3})]$$

$$+ \frac{\lambda_1'}{4\beta^4}[4 + \frac{\lambda_2}{\lambda_1}(3\alpha) + \frac{5\lambda_2'}{16\beta^3}(1 - 10\alpha)](- \frac{1}{a_o})\} \tag{95}$$

a_{sc} and r_{sc} are, by definition, the scattering length and effective range respectively. We note that if we let $\alpha = 0$, a_o and r_o becomes the a_s and r_s corresponding to Nagvi's potential studied in section II-B.

3. Modified Tabakin's Potential

This potential is given by (II-59). We have:

$$g_c(k) = t_1\, g_{c1}(k) + t_2\, g_{c2}(k)$$
$$g_{c1}(k) = \frac{C_o(\eta)}{k^2 + a^2} e^{2\eta \tan^{-1} \frac{k}{a}}$$
$$g_{c2}(k) = \frac{C_o(\eta)}{k^2 + b^2} e^{2\eta \tan^{-1} \frac{k}{b}} \qquad (97)$$

In this case:

$$\mathrm{Re}\, I(k) = I_1(k) + I_2(k) + I_3(k) \qquad (98)$$

where $I_1(k)$ and $I_2(k)$ are given by relation (70) and

$$I_3(k) = \frac{2\mu t_1 t_2}{\pi^2} P \int_o^\infty \frac{q^2 dq\, C_o^2(\eta) e^{2\eta[\tan^{-1} \frac{q}{a} + \tan^{-1} \frac{a}{b}]}}{(q^2 + a^2)(q^2 + b^2)(q^2 - k^2)} \qquad (99)$$

$$= XJ(k) + YK(k)$$

where

$$X = \frac{2t_1 t_2}{a^2 - b^2} = - Y$$
$$J(k) = \frac{\mu}{\pi^2} P \int_o^\infty \frac{q^2 dq\, C_o^2(\eta) e^{2\,[\tan^{-1} \frac{q}{a} + \tan^{-1} \frac{q}{b}]}}{(q^2 + b^2)(q^2 - k^2)} \qquad (100)$$
$$K(k) = \frac{\mu}{\pi^2} P \int_o^\infty \frac{q^2 dq\, C_o^2(\eta) e^{2\,[\tan^{-1} \frac{q}{a} + \tan^{-1} \frac{q}{b}]}}{(q^2 + b^2)(q^2 - k^2)}$$

We obtain immediately:

$$\mathrm{Im}\ I(k) = \frac{\mu k}{2\pi} g_c^2(k) \tag{101}$$

Assuming $\frac{\mu e_1 e_2}{a}$, $\frac{\mu e_1 e_2}{b} << 1$, we can expand the factor $\exp[2\ (\tan^{-1}\frac{q}{a} + \tan^{-1}\frac{q}{b})]$ to obtain a perturbation series for $I_3(k)$.

$$J_o(k) = \frac{\mu}{\pi^2} P \int_o^\infty \frac{q^2 dq\ C_o^2(\eta(q))}{(q^2 + a^2)(q^2 - k^2)} \tag{102}$$

Using (65), (66), we obtain:

$$J_o(k) = \frac{a\mu}{\pi(a^2 + k^2)} \{-\frac{1}{2} + \frac{\mu e_1 e_2}{a} h(\eta) + \frac{\mu e_1 e_2}{b} [\ln(\frac{\mu e_1 e_2}{b}) - \Psi(\eta)] \tag{103}$$

The next term in the expansion is:

$$J_1(k) = \frac{2\mu e_1 e_2}{\pi^2} P \int_o^\infty \frac{q dq\ C_o^2(\eta) \tan^{-1}\frac{q}{a}}{(q^2 + a^2)(q^2 - k^2)} + \frac{2\mu e_1 e_2}{\pi^2} P \int_o^\infty \frac{q dq\ C_o^2(\eta) \tan^{-1}\frac{q}{b}}{(q^2 + b^2)(q^2 - k^2)} \tag{104}$$

Using an contour integration, we obtain:

$$J_1(k) = \frac{\mu e_1 e_2}{2\pi(a^2 + k^2)} \{\ln[4(1 + \frac{b}{a})^2] + 2\ln(ab) - \ln[(k^2 + a^2)(k^2 + b^2)]\} \tag{105}$$

Then

$$J(k) = \frac{a\mu}{\pi(a^2 + k^2)} \{\frac{1}{2} + \frac{\mu e_1 e_2}{a} h(\eta)$$

$$+ \frac{\mu e_1 e_2}{a} [\ln(2\varepsilon(a + b)) - \frac{1}{2} \ln(k^2 + a^2)(k^2 + b^2) + \Gamma]\} \tag{106}$$

To obtain K(k), we interchange $a \rightarrow b$ in (106). Set:

$$K(a) = k^2 + a^2 \; ; \; K(b) = k^2 + b^2 \; ; \; \varepsilon = \mu e_1 e_2$$

From (99) and (100) we obtain:

$$I_3(k) = \frac{2\mu t_1 t_2}{(a^2 - b^2)} \{\frac{1}{2}[\frac{b}{K(b)} - \frac{a}{K(a)}]$$

$$+ \frac{\varepsilon(a^2 - b^2)}{K(a)\ K(b)} h(\eta) + \frac{\varepsilon(a^2 - b^2)}{K(a)\ K(b)}[\ln(2\varepsilon(a + b))$$

$$- \frac{1}{2} \ln[K(a)\ K(b)) + \Gamma]\} \tag{107}$$

From (99):

$$\text{Re } I(k) = \frac{t_1 \mu}{\pi K^2(a)} \{\frac{a^2 - k^2}{4a} + \varepsilon h(\eta) + \varepsilon[\ln(4\varepsilon a)$$

$$- \ln K(a) + \Gamma]\} + \frac{t_2}{\pi K^2(b)} \{\frac{a^2 - k^2}{4a} + \varepsilon h(\eta)$$

$$+ \varepsilon[\ln(4\varepsilon b) - \ln K(b) + \Gamma] + \frac{2\mu t_1 t_2}{kaK(b)}$$

$$\{\frac{ab - k^2}{2(a + b)} + \varepsilon h(\eta) + \varepsilon[\ln(2\varepsilon(a + b))$$

$$- \frac{1}{2} \ln K(a)\ K(b) + \Gamma]\} \tag{108}$$

As before:

$$\cot \delta_{sc}(k) = - \frac{1 + \lambda \mathrm{Re}\, I(k)}{\lambda \mathrm{Im}(k)} \tag{109}$$

The scattering length and effective range are given explicitly as:

$$- \frac{1}{a_{sc}} = - \frac{2a^4}{\lambda'} b^4 [t_1 b^2 (1 + \frac{2\varepsilon}{a}) + t_2 a^2 (1 + \frac{2\varepsilon}{b})]^{-2}$$

$$\{1 + \lambda' [\frac{t_1^2}{4a^3} + \frac{t^2}{4b^3} + \frac{t_1 t_2}{ab(a+b)}$$

$$+ \varepsilon\lambda' [\frac{t_1^2}{a^4}(\ln \frac{4}{a} + \Gamma) + \frac{t_2^2}{b^4}(\ln \frac{4\varepsilon}{b} + \Gamma)$$

$$+ \frac{2t_1 t_2}{a^2 b^2} (\ln(2\varepsilon(\frac{1}{a} + \frac{1}{b})) + \Gamma)]\} \tag{110}$$

$$\frac{1}{2} r_{sc} = - \frac{2a^2}{\lambda'} b^2 (a^2 + b^2) [t_1 b^2 (1 + \frac{2\varepsilon}{a}) + t_2 a^2 (1 + \frac{2\varepsilon}{b})]^{-2}$$

$$[2 + \lambda' \{- \frac{t_1^2 b^2}{4a^3(a^2 + b^2)} - \frac{t_2^2 a^2}{4b^3(a^2 + b^2)} + \frac{t_1 b + t_2 a}{2ab(a^2 + b^2)}$$

$$+ \frac{t_1 t_2}{ab(a+b)} \frac{a^2 + b^2 - ab}{a^2 + b^2}\} + \varepsilon\lambda' \{\frac{t_1^2}{a^2(a^2 + b^2)}$$

$$[2(\ln \frac{4\varepsilon}{a} + \Gamma) - \frac{b^2}{a^2}] + \frac{t^2}{b^2(a^2 + b^2)}$$

$$\times [2(\ln \frac{4\varepsilon}{b} + \Gamma) - \frac{a^2}{b^2}] + \frac{2t_1 t_2}{a^2 b^2} [\ln(2\varepsilon(\frac{1}{a} + \frac{1}{b}))$$

$$+ \Gamma - \frac{1}{2}]\} + \frac{\lambda'}{a^2 b^2 (a^2 + b^2)} \{t_1 b^2 (1 + \frac{2\varepsilon}{a})$$

$$+ t_2 a^2 (1 + \frac{2\varepsilon}{b})\}\{t_1 (1 + \frac{2\varepsilon}{a}) + t_2 (1 + \frac{2\varepsilon}{b})$$

$$- \frac{2t_1}{3} \frac{\varepsilon}{a} \frac{b^2}{a^2} - \frac{2t_2}{3} \frac{\varepsilon}{b} \frac{a^2}{b^2}\}\{- \frac{1}{a_{sc}}\}] \qquad (111)$$

IV. NUMERICAL RESULTS AND DISCUSSION

The proton-proton scattering length and effective range are very accurately determined from experiment (18), in our notation they read:

$$a_{sc} = -\ 7.786\ \text{F}, \quad r_{sc} = 2.840\ \text{F}.$$

These are consistent with the expressions for a_{sc} and r_{sc} corresponding to Yamaguchi's potential, equations (III-71) and (III-72), if:

$$\begin{aligned} \frac{\lambda\mu}{\pi} &= -\ 9.33 \times 4\pi^2 (0.2316)^3 \text{F}^{-3}, \\ \beta &= 1.081273\ \text{F}^{-1}. \end{aligned} \qquad (1)$$

For Naqvi's potential, we have adjusted the ratio $\frac{\lambda_2}{\lambda_1}$ in terms of β such that the S-wave phase shift has a change of sign at 240 MeV. Equations (III-92) and (III-93) then give the observed a_{sc} and r_{sc} if:

$$\begin{aligned} \frac{\lambda_1\mu}{\pi} &= -\ 21.98 \times 4\pi^2 (0.2316)^3 \text{F}^{-3}, \\ \beta &= 1.368914\ \text{F}^{-1} \\ \frac{\lambda_2}{\lambda_1} &= -\ 2.637038 \end{aligned} \qquad (2)$$

Using these parameters, we found, for Yamaguchi's potential:

$$a_s = -\ 17.596854\ \text{F}, \quad r_s = 2.969\ \text{F}$$

and for Naqvi's potential:

$$a_s = -\ 18.151\ \text{F}, \quad r_s = 2.930\ \text{F} \qquad (3)$$

which are only slightly different from the previous values for Yamaguchi's potential. This is not surprising if we recall that, in Naqvi's potential, the attractive term, identical with Yamaguchi's potential, dominates the repulsive term at low energies, and that the Coulomb effect is less important at high energies.

If we compare these values with Heller et al's results for local potential [a_s = - (16.6 16.9)F], we may conclude that the value of a_s for a non-local potential is smaller than that for a local potential.

For the difference $r_s - r_{sc}$, we obtain 0.09F which is much larger than the previous values, about 0.03 F, given by Breit (20) and by Noyes (21). Some difficulties in the theoretical interpretation of the difference between the effective range for p-p and p-n have been pointed out by Noyes (21) and by Leung and Nogami (5). Our r_s leads to a larger difference between p-p and p-n.

On the other hand, the neutron-neutron scattering length has been determined by an analysis of the final-state interaction of the nuclear reaction involving two neutrons:

$$H^3 + d \rightarrow He^3 + 2n \qquad (4)$$

with the result:

$$a_{nn} = -(16.1 \pm 1.5)F \qquad (5)$$

which agrees with our and with Heller et al's value for a_s. However, a combination of the experimental results for the energy distribution and the angular distribution of the reaction (19):

$$\pi^- + d \rightarrow n + n + \gamma \tag{6}$$

yields a somewhat different value,

$$a_{nn} = -(18.42 \pm 1.53)F \tag{7}$$

which is closer to our value for a_s than it is to Heller et al's.

We have also attempted a similar parameter determination for the 'modified Tabakin's potential'. Unfortunately, we have not obtained a result due to some difficulties in solving the system of simultaneous equations (III-110) and (III-111) numerically. It seems to possess highly unstable solutions and does not meet with the conditions for convergence required by existing elementary interation methods.

With the value for β we obtained for Yamaguchi's and Naqvi's potentials, $\frac{\mu e_1 e_2}{\beta} << 1$, the perturbation expansion used in our analysis should be valid. Of course, one can go to higher orders in $\frac{\mu e_1 e_2}{\beta}$, but this does not seem to improve our results. For a two or more term separable potential, the computation of the analytical expressions for the scattering length and effective range, especially in the presence of the Coulomb interaction, is extremely involved. To simplify the computation, a one term separable potential equivalent to a two term separable potential, as discussed in section (III-C), could be used.

We would like to point out here that an integral of type as found in equation (III-60) can be calculated exactly by making use of a summation formula due to plana (20). This would mean that our problem can be solved exactly. However, one may have difficulties in evaluating the residues at the singularities of encountered functions.

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APPENDIX A

INTEGRAL EQUATION FOR $T_{sc,L}$

We rewrite the equation for T_{sc}:

$$T_{sc}(E,k',k) = {}_c\langle k'(-)|V_s|k(+)\rangle_c + \sum_{k''} \frac{1}{E + i\varepsilon - E(k'')} \, {}_c\langle k'(-)|V_s|k''(-)\rangle_c \, T_{sc}(E,k'',k) \quad (1)$$

We make partial-wave decompositions of the matrix element.

$$T_{sc}(e,k',k) = \Sigma_L (2L+1) \, P_L(\hat{k}'.\hat{k}) \, T_{sc,L}(E,k',k)$$

$${}_c\langle k'(-)|V_s|k\pm\rangle_c = \sum_L (2L+1) \, P_L(\hat{k}'.\hat{k}) \, V^{(\pm)}_{sc,L}(k',k)$$

$$T_c(k',k) = \sum_L (2L+1) \, P_L(\hat{k}'.\hat{k}) \, T_{c,L}(k',k)$$

$$\langle k'|V_s|k\rangle = \Sigma_L (2L+1) \, P_L(\hat{k}'.\hat{k}) \, V_{sc,L}(k',k) \quad (2)$$

Substituting (2) into integral equation (1) we get:

$$\Sigma(2L+1) \, P_L(k',k) \, T_{sc,L} = \Sigma_L (2L+1) \, P_L(k',k) \, V^{(+)}_{sc,L}(k',k)$$

$$+ \sum_{L'L''} \sum_{k''} \frac{1}{E + i\varepsilon - E(k'')} (2L+1) \, P_L(\hat{k}'.\hat{k}'') \, V^{(-)}_{sc,L}(k',k'')$$

$$\chi \; (2L'+1) \, P_{L'}(\hat{k}''.\hat{k}) \, T_{sc,L''}(E,k'',k) \quad (3)$$

Replacing the $\Sigma_{k''}$ by the integral $\int \frac{1}{(2\pi)^3} k''^2 \, dk'' \, d\Omega(\hat{k}''.\hat{k})$, The second term of equation (3) becomes:

$$\sum_{LL'} \int (2L + 1)(2L' + 1)\, P_L(\hat{k}'.\hat{k}'')\, P_L(\hat{k}''.\hat{k})\, d\Omega(\hat{k}''.\hat{k})$$

$$\times \int V^{(+)}_{sc,L}(k',k'')\, T_{sc,L'}(E,k'',k)\, k''^2 dk'' \tag{4}$$

Using the addition theorem

$$\frac{2L + 1}{4\pi} P_L(\hat{k}',\hat{k}'') = \sum_{m=-L}^{m=L} Y^{m*}_L(\hat{k}'.\hat{k})\, Y^m_L(\hat{k}''.\hat{k})$$

and replacing $P'_L(\hat{k}'',\hat{k}) = \sqrt{\frac{4\pi}{2L'+1}}\, Y^o_{L'}(\hat{k}''.\hat{k})$, the first integral of (4) becomes:

$$\int (2L + 1)\, P_L(k'.k'')\, P_L(k''.k)\, d\Omega(k''.k) = 4\pi \sqrt{\frac{4\pi}{2L' + 1}}\, Y^o_L \delta_{LL'}$$

Then the integral (4) becomes:

$$\Sigma_L \frac{1}{2\pi^2}(2L + 1) \sqrt{\frac{4\pi}{2L + 1}}\, Y^o_L$$

$$\times \int_o^\infty \frac{V^{(-)}_{scmL}(k',k'')\, T_{sc,L'}(E,k'',k)\, k''^2 dk''}{E + i\varepsilon - E(k'')}$$

$$= \frac{1}{2\pi^2} \sum_L (2L + 1)\, P_L(\hat{k}',\hat{k})$$

$$\times \int_o^\infty \frac{V^{(-)}_{sc,L}(k',k'')}{E - E(k'') + i\varepsilon}\, T_{sc,L'}(E,k',k)\, k''^2 dR'' \tag{5}$$

Substituting (5) into equation (3) then equating the coefficient of P_L we obtain the integral equation for $T_{sc,L}$:

$$T_{sc,L}(E,k',k) = V^{+}_{sc,L}(k',k) + \frac{1}{2\pi^2}\int^{\infty} \frac{k''^2 dk'' \; V^{(-)}_{sc,L}(k',k'') \; T_{sc,L}(E,k',k)}{E - E(k'') + i\varepsilon} \tag{6}$$

APPENDIX B

SOLUTION OF THE INTEGRAL EQUATION FOR $T_{sc,L}$

Substituting $V_{sc,L}$ in the integral equation for $T_{sc,L}$ we obtain

$$T_{sc,L}(E,k',k) = e^{i\delta_{c,L}(k')} \Sigma_i \lambda_{L,i} g_{c,L,i}(k') g_{c,L,i}(k) e^{i\delta_{cL}(k)}$$

$$+ \frac{1}{2\pi^2} \int dk'' \sum_i \{g_{c,L,i}(k') e^{i\delta_{c,L}(k')}$$

$$\chi \frac{g_{c,L,i}(k'') e^{-i\delta_{c,L}(k'')} k''^2}{E + i - E(k'')}\} T_{sc,L}[E,k'',k] \qquad (1)$$

This is Fredham's integral equation with degenerate kernel whose solution has the form (22):

$$T_{sc,L}(E,k',k) = e^{i\delta_{c,L}(k')}$$

$$\chi \sum_i \{g_{c,L,i}(k') [1 + \frac{C_i e^{-i\delta_{c,L}(k)}}{2\pi^2 g_{c,L,i}(k)}] \lambda_{L,i}$$

$$\chi g_{c,L,i}(k)\} e^{i\delta_{c,L}(k)}$$

which can be put under the form:

$$T_{sc,L}(E,k',k) = e^{i\delta_{c,L}(k')}$$

$$\chi \sum_{i_j} \{g_{c,L,i}(k') [1 + \frac{C_i e^{-i\delta_{c,L}(k)}}{2\pi^2 g_{c,L,i}(k)}] \delta_{ij} \lambda_{L,i}$$

$$\chi g_{c,L,i}(k)\} e^{i\delta_{c,L}(k)}$$

if we set:

$$[1 - \frac{C_i \, e^{-i\delta_{c,L}(k)}}{2\pi^2 \, g_{c,L,i}(k)}] \, \delta_{ij} \, \lambda_{L,i} \equiv \tau_{ij}^{(L)}(E)$$

Then the solution of (1) can be set as:

$$T_{sc,L}(E,k',k) = e^{i\delta_{c,L}(k')}$$

$$\chi \, \Sigma_{ij} g_{c,L,i}(k') \, \tau_{ij}^{(L)}(E) \, g_{c,L,j}(k) \, e^{i\delta_{c,L}(k)} \tag{2}$$

To determine τ_{ij} we substitute (2) into (1), replace the simple summation in $V_{sL}^{(\pm)}$ by a double sum, then equate the corresponding coefficients. We obtain:

$$\tau_{ij}^{(L)}(L) = \delta_{ij} \, \lambda_{L,i} + \sum_{mn} \delta_{mn} \, \lambda_{L,n} \, \frac{1}{2\pi^2} \int_0^\infty \frac{q^2 dq \; g_{c,L,n}(q) \; g_{c,L,m}(q)}{E + i\varepsilon - E(q)} \, \chi \, \tau_{ij}^{(L)}(E) \tag{3}$$

In matrix notation, we obtain from (3) for the matrix τ (dropping the index L) the equation:

$$\tau(E) = \Lambda - \Lambda I(E) \, \tau(E) \tag{4}$$

where Λ is the diagonal matrix:

$$\Lambda_{ij} = \delta_{ij} \, \lambda_i \tag{5}$$

and the elements of the matrix I(E) are given by:

$$I_{ij} = \frac{1}{2\pi^2} \int_0^\infty \frac{k^2 dk\ g_{c,i}(k)\ g_{c,j}(k)}{e(k) - E - i\varepsilon} \qquad (6)$$

Solving (4) we get:

$$\tau(E) = [1 + \Lambda I(E)]^{-1} \Lambda \qquad (7)$$