POSITIVE DEFINITE FUNCTIONS FOR THE CLASS $L_p(G)$

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SCOPE AND CONTENTS: This thesis contains four main theorems on Positive definite functions for the class $L_p(G)$ over compact and locally compact groups. Information as to how the class P(F) varies with F is provided by those theorems. Some topological properties of the set $P(L_p) \bigcap L_q$ have been considered. Two results analogous to those of E. Hewitt and M. A. Naimark have also been proved.

PREFACE

There are several notions of positive definiteness for functions on topological groups e.g. the notion of Bochner type positive definite functions and integrally positive definite functions. It is of some interest to observe that a change in the class of functions in the definition of integrally positive definite function produces a different class of positive definite functions.

The purpose of this thesis is to study the functions which are positive definite for the classes L_p $(1 \leq p < \infty)$ over compact and locally compact groups. It is worth mentioning here that the papers of James Stewart [7] and Edwin Hewitt [4] inspired new observations in this direction.

Chapter 1 contains basic definitions and facts from Harmonic Analysis which we shall need in the development of some ideas in the subsequent chapters.

Chapter 2 and Chapter 3 present some new material regarding positive definite functions for the class $L_p(G)$, $1 \leq p < \infty$, over compact and locally compact groups.

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CHAPTER 1

Preliminaries

The preliminary material is presented here in a condensed form for future reference. Many of the ideas developed in the ensuing chapters will depend upon these facts. See [1], [2], [3], [5].

1. Measure and Integration

Definition 1.1: If X is a topological space and Θ is the family of all open subsets of X, then the family of Borel sets in X is defined as the smallest σ-algebra of sets containing Θ. If $\mathfrak{M}$ is a σ-algebra of subsets of X and μ is a countably additive measure on $\mathfrak{M}$ (or simply a measure), then the triple $(X, \mathfrak{M}, \mu)$ is called a measure space. If $\mu(X) < \infty$, then $(X, \mathfrak{M}, \mu)$ is called a finite measure space and μ a finite measure. If X is the union of a countable family of sets in $\mathfrak{M}$ each having finite measure, then $(X, \mathfrak{M}, \mu)$ and μ are called σ-finite. An extended real valued function f on X is $\mathfrak{M}$-measurable if the set $\left\{x \in X: f(x) > \alpha\right\}$ belongs to $\mathfrak{M}$ for every real α. A complex valued function f on X is $\mathfrak{M}$-measurable if its real and imaginary parts are $\mathfrak{M}$-measurable. If X is a topological space and $\mathfrak{M}$ is the σ-algebra of Borel sets of X, an $\mathfrak{M}$-measurable function is called Borel measurable.

Definition 1.2: Let μ be a measure on $\mathcal{M}$. Let f be any function on X with values in $[0, \infty]$. Then the Lebesgue integral $\int_X f(x)\, d\mu(x)$ [or simply $\int_X f\, d\mu$] is defined as

$$\sup \left\{ \sum_1^n [\inf (f(x): x \varepsilon A_k]\, \mu(A_k): \left\{A_1, A_2, \dots A_n\right\} \text{ is a partition of } X \text{ with each } A_k \varepsilon \mathcal{M}. \right\}$$

Definition 1.3: Let X be any topological space, and $C(X)$ denote the set of all bounded complex valued continuous functions on X. For $f \varepsilon C(X)$ let

$$\| f \|_u = \sup \left\{ |f(x)| : x \varepsilon X \right\}.$$

Let $C_o(X)$ denote the set of all complex valued continuous functions f on X such that for every $\varepsilon > 0$, there exists a compact subset F of X [depending upon f and ε] such that $|f(x)| < \varepsilon$ for all $x \varepsilon F'$.

Let $C_{oo}(X)$ denote the set of all complex valued continuous functions f on X such that there exists a compact subset F of X [depending upon f] such that $f(x) = 0$ for all $x \varepsilon F'$.

Remark 1.4: $C_o(X)$ is a linear subspace of $C(X)$ and $C_{oo}(X)$ is a linear subspace of $C_o(X)$. $C(X)$ and $C_o(X)$ are complete in the metric induced by the norm $\| \cdot \|_u$. $C_{oo}(X)$ is a dense linear subspace of $C_o(X)$ [see Theorem 1.5].

If X is a compact topological space, then $C_{oo}(X)$, $C_o(X)$ and $C(X)$ coincide. If X is locally compact, non compact, Hausdorff space, then $C_{oo}(X) \subset C_o(X) \subsetneqq C(X)$; the equality $C_{oo}(X) = C_o(X)$, may obtain even for locally compact normal spaces. However, if G is locally compact, non-compact group, then we have $C_{oo}(G) \subsetneqq C_o(G)$.

Theorem 1.5: Let X be a locally compact Hausdorff space. Every function in $C_o(X)$ can be arbitrarily uniformly approximated by functions in $C_{oo}(X)$.

Definition 1.6: If $A \subset X$ and $\mu(A) = 0$, we call A a null set. If $A \cap F$ is μ-null for every compact subset F of X, then A is said to be a locally μ-null set. A property that holds on X except for a locally μ-null set is said to hold locally μ-almost everywhere. A complex valued function f on X that vanishes μ-almost everywhere is called a μ-null function.

Definition 1.7: Let $\mathfrak{M}$ be a σ-algebra of subsets of X containing all open sets. A non-negative measure μ on $\mathfrak{M}$ is called regular if and only if for every open set V, we have

$$\mu(V) = \sup\left\{\mu(F): \ F \text{ is compact and } F \subset V\right\}$$

and for all $A \in \mathfrak{M}$, we have

$$\mu(A) = \inf\left\{\mu(V): \ V \text{ is open and } V \supset A\right\}.$$

Theorem 1.8: (Riesz Representation Theorem)

Let X be a locally compact Hausdorff space, I, a non-negative linear functional on $C_{oo}(X)$. Then there is a unique non-negative regular Borel measure μ defined on the Borel subsets of X such that $\mu(F) < \infty$ for each compact subset F and

$$I(f) = \int_X f \, d\mu \quad \text{for all} \quad f \in C_{oo}(X).$$

<u>Definition 1.9</u>: (Integration on Product spaces)

Let X and Y be locally compact Hausdorff spaces. Then $X \times Y$ with the product topology is also locally compact Hausdorff. Let I, J be non-zero non-negative linear functionals on $C_{oo}(X)$ and $C_{oo}(Y)$ respectively. Let f be a complex valued function on $X \times Y$ such that for some $x_o \in X$, the function $\emptyset : y \longrightarrow f(x_o, y)$ is in $C_{oo}(Y)$. We shall write $J_y(f(x_o, y))$ for $J(\emptyset)$. We can define similarly $I_x(f(x,y_o))$.

<u>Theorem 1.10</u>: For every $f \in C_{oo}(X \times Y)$, we have $J_y(f(x, y)) \in C_{oo}(X)$ and $I_x(f(x, y)) \in C_{oo}(Y)$. Also (i) $I_x[J_y(f(x, y))] = J_y[I_x(f(x, y))] = K(f)$. The functional K defined by (i) is a non-negative, non-zero, linear functional on $C_{oo}(X \times Y)$. K is called the product of the functionals I and J and is written $K = I \times J$.

By Riesz representation theorem, there exists a measure $i \times \eta$ such that

$$K(f) = \int f \, d\,(i \times \eta).$$

$i \times \eta$ is called the product of the measures i and η induced respectively by i and η.

<u>Theorem 1.11</u>: [Fubini's theorem] Let f be in $L_1(X \times Y, i \times \eta)$. Then $f(x, y)$ as a function of x is in

$L_1(X, i)$ for η-almost all $y \in Y$, and $f(x, y)$ as function of y is in $L_1(Y, \eta)$ for i-almost all $x \in X$. Furthermore the function of x defined by $x \longrightarrow \int_Y f(x, y)\, d\eta(y)$ where the integral exists and 0 elsewhere is in $L_1(X, i)$; similarly for $\int_X f(x,y)\, di(x)$.

Finally we have

$$\text{(i)} \quad \int_{X \times Y} f(x, y)\, di \times \eta\,(x, y) = \int_X \int_Y f(x, y)\, d\eta(y)\, di(x)$$

$$= \int_Y \int_X f(x, y)\, di(x)\, d\eta(y)$$

2. <u>The spaces L_p $(1 \leq p < \infty$</u>

<u>Definition 1.12</u>: Let p be a positive real number and let $(X, \mathfrak{M}, \mu)$ be a measure space. Let f be a complex valued $\mathfrak{M}$-measurable function defined μ-a.e. on X such that $\int_X |f|^p\, d\mu < \infty$. Then we say that $f \in L_p(X, \mathfrak{M}, \mu)$. We define the symbol $\|f\|_p$ by

$$\|f\|_p = \left[\int_X |f|^p\, d\mu\right]^{1/p}$$

For $1 \leq p < \infty$, the function $f \longrightarrow \|f\|_p$ on L_p satisfies the axioms of a norm except the positivity requirements: $\|f\|_p > 0$ if $f \neq 0$. If we agree that $f = g$ means $f(x) = g(x)$ for μ-almost all $x \in X$, i.e. L_p actually consists of equivalence classes of functions (Two functions such that $\|f - g\|_p = 0$ are taken to be identical in L_p), $L_p(1 \leq p < \infty)$ is a normed linear space.

<u>Theorem 1.13</u>: (Hölder's inequality). Let $f \varepsilon L_p$ and $g \varepsilon L_q$ where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \varepsilon L_1$ and we have

(i) $\left| \int fg \; d\mu \right| \leq \int |fg| \; d\mu$

(ii) $\int |fg| \; d\mu \leq \|f\|_p \; \|g\|_q$.

<u>Theorem 1.14</u>: (Minkowski's inequality)

For $1 \leq p < \infty$ and $f, g \varepsilon L_p$, we have

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

<u>Theorem 1.15</u>: For $1 \leq p < \infty$, L_p is a complex Banach space i.e. in metric $\rho(f, g) = \|f - g\|_p$, L_p is a complete metric space.

<u>Theorem 1.16</u>: For $1 \leq p < \infty$, the conjugate space of L_p is L_q, in the sense that for every bounded linear functional T on L_p there is a function $g \varepsilon L_q$ such that

(i) $T(f) = \int_X f \, g \, d\mu$ for all $f \varepsilon L_p$ and $\|T\| = \|g\|_q$

<u>Theorem 1.17</u>: If $\mu(X) < \infty$ and $0 < p < q$, then $L_q \subset L_p$ and the inequality

$$\|f\|_p \leq \|f\|_q \; (\mu(X))^{\frac{1}{p} - \frac{1}{q}}$$

holds for all $f \varepsilon L_q$.

<u>Theorem 1.18</u>: If $f \varepsilon L_p \cap L_q$ where $0 < p < q < \infty$ and if $p < r < q$, then $f \varepsilon L_r$.

Theorem 1.19: Let X be a locally compact Hausdorff space. Then $C_{oo}(X)$ is a dense subspace of $L_p(X, \mathcal{M}, \mu)$.

Theorem 1.20: Let $(X, \mathcal{M}, \mu)$ be a measure space, let p be a real number such that $p > 1$ and let f be a $\mathcal{M}$-measurable function on X such that (i) $\left\{ x \in X: \ f(x) \neq 0 \right\}$ is the union of a countable number of sets in $\mathcal{M}$ having finite measure and (ii) $fg \in L_1(X, \mathcal{M}, \mu)$ for all $g \in L_p(X, \mathcal{M}, \mu)$. Then $f \in L_q(X, \mathcal{M}, \mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

3. Haar measure and convolution

Definition 1.21: Let G be a set that is a group and also a topological space. G is called a topological group if

(i) the mapping $(x, y) \longrightarrow xy$ of $G \times G$ onto G is a continuous mapping of the Cartesian product $G \times G$ onto G.

(ii) the mapping $x \longrightarrow x^{-1}$ of G onto G is continuous .

In terms of open sets, condition (i) asserts that for every neighbourhood U of xy, there are neighbourhoods V and W of x and y respectively such that $VW \subset U$. Condition (ii) asserts that for every neighbourhood U of x^{-1}, there is a neighbourhood V of x such that $V^{-1} \subset U$.

On a locally compact Hausdorff topological group, Haar measure is the analogue of Lebesgue measure on the real line. On the real line, as additive group, Haar measure coincides with Lebesgue measure. We shall define first what Haar integral is.

Definition 1.22: (a) Let E be a linear space. A real valued function f on E is said to be a functional.

(b) A functional is said to be additive (or subadditive) if for each pair $x, y \varepsilon E$, $f(x + y) = f(x) + f(y)$ (or $f(x + y) \leq f(x) + f(y)$).

(c) A functional is said to be positive homogeneous if for each $x \varepsilon E$ and real $\lambda \geq 0$, $f(\lambda x) = \lambda(f(x)$. If for each real λ and $x \varepsilon E$, $f(\lambda x) = \lambda f(x)$, then f is said to be homogeneous.

(d) A homogeneous additive functional is said to be linear. A linear functional I on a linear space E is non-negative if $I(f) \geq 0$ for all $f \geq 0$.

<u>Definition 1.23</u>: Let G be any group and let $\mathfrak{F}$ be a set of functions on G. For a fixed element $a \varepsilon G$, let ${}_af[f_a]$ be the function on G such that ${}_af(x) = f(ax)$ $[f_a(x) = f(xa)]$ for all $x \varepsilon G$. Then ${}_af[f_a]$ is called the left translate [right translate] of f by a. Suppose that $f \varepsilon \mathfrak{F}$ and $a \varepsilon G$ imply ${}_af \varepsilon \mathfrak{F}$ $[f_a \varepsilon \mathfrak{F}]$. Let I be any function on $\mathfrak{F}$ such that $I(f) = I(af)[I(f) = I(f_a)]$ for all $f \varepsilon \mathfrak{F}$ and $a \varepsilon G$. Then I is said to be left invariant or invariant under left translations [right invariant or invariant under right translations].

<u>Theorem 1.24</u>: Let G be a locally compact Hausdorff topological group. Then there exists a non-trivial (i.e. not identically zero), non-negative, left invariant, positive homogeneous, additive, functional I on $C_{oo}^{+}(G)$ and I is unique upto a multiplicative constant.

<u>Definition 1.25</u>: I can be extended uniquely to a linear functional on $C_{oo}(G)$ also denoted by I and the extension is

necessarily left invariant. I is called <u>left Haar integral</u> on $C_{oo}(G)$. The measure λ induced by the left Haar integral I (Riesz Representation Theorem) is called the <u>Left Haar measure</u>, which is unique upto a multiplicative constant. λ satisfies the following conditions:

(1) $\lambda(V) > 0$ for all non-void open set $V \subset G$ $(\lambda \neq 0)$

(2) $\lambda(aB) = \lambda(B)$ for all Borel set $B \subset G$, $a \varepsilon G$.

<u>Theorem 1.26</u>: Let G be a locally compact group with a left Haar measure λ. Then $\lambda(G)$ is finite if and only if G is compact. [If G is compact, we will always normalize left Haar measure by the requirement that $\lambda(G) = 1$].

<u>Theorem 1.27</u>: Let G be a locally compact group and let I be a left Haar integral on C_{oo}. For $f \varepsilon C_{oo}^{+}$, $f \neq 0$ and for $x \varepsilon G$, let $\Delta(x) = \frac{I(f_{x}-1)}{I(f)}$: Then Δ depends only upon x and not upon I or f. The function Δ is continuous, positive through out G, and satisfies the functional equation

$$\Delta(xy) = \Delta(x)\,\Delta(y) \quad \text{for all } x, y \,\varepsilon\, G$$

<u>Definition 1.28</u>: The function Δ is called the <u>modular function</u> of the locally compact group G. G is called <u>unimodular</u> if and only if $\Delta = 1$ on G.

<u>Theorem 1.29</u>: Every compact Hausdorff topological group is unimodular, and a locally compact Hausdorff topological abelian group is unimodular.

Theorem 1.30: Let f be a non-negative λ-measurable function on G. Then f^* is also λ-measurable $(f^*(x) = f(x^{-1}))$ and

(i) $$\int_G f^* d\lambda = \int_G f \frac{1}{\Delta} d\lambda$$

and (ii) $$\int_G f\, d\lambda = \int_G f^* \frac{1}{\Delta} d\lambda.$$

If $f \varepsilon L_1(G)$, then $f^* \frac{1}{\Delta} \varepsilon L_1(G)$, (ii) holds and $\|f\|_1 = \|f^* \frac{1}{\Delta}\|_1$.

Theorem 1.31: For $f, g \varepsilon L_1(G)$, we have

(i) $$f{*}g(x) = \int_G f(y)\, g(y^{-1}x)\, dy$$

(ii) $$f{*}g(x) = \int_G f(xy)\, g(y^{-1})\, dy$$

(iii) $$f{*}g(x) = \int_G \Delta(y^{-1})\, f(xy^{-1})\, g(y)\, dy$$

(iv) $$f{*}g(x) = \int_G \Delta(y^{-1})\, f(y^{-1})\, g(yx)\, dy.$$

Equality holds for λ-almost all $x \varepsilon G$.

Theorem 1.32: Let $f \varepsilon L_1(G)$ and $g \varepsilon L_p(G)(1 \le p \le \infty)$. Then

(i) $$f{*}g(x) = \int_G f(xy)g(y^{-1})\, dy = \int_G f(y)\, g(y^{-1}x)\, dy$$

exists and is finite for all $x \varepsilon G \cap N'$, where N is λ-null if $p < \infty$ and is locally λ-null if $p = \infty$. The function $f{*}g$ is in $L_p(G)$ and

(ii) $\|f*g\|_p \le \|f\|_1 \|g\|_p$

If $f \in L_p(G)$ and $\Delta^{-1/q} g \in L_1(G)$, the integral

(iii) $f*g(x) = \int_G \Delta(y^{-1})\, f(xy^{-1})\, g(y)\, dy = \int_G f(xy)\, g(y^{-1})\, dy$

$$= \int_G f(y)\, g(y^{-1}x)\, dy.$$

exists and is finite for all $x \in G \cap N'$. The function $f*g$ is in $L_p(G)$ and

(iv) $\|f*g\|_p \le \|f\|_p \, \|\Delta^{-1/q} g\|_1$ where $\frac{1}{p} + \frac{1}{q} = 1$.

<u>Theorem 1.33</u>: Let p and p' be real numbers such that $1 < p < \infty$, $1 < p' < \infty$, $\frac{1}{p} + \frac{1}{p'} > 1$; and let $r = \frac{pp'}{p+p'-pp'}$ so that $\frac{1}{p} + \frac{1}{p'} - \frac{1}{r} = 1$.

Let f and g be Borel measurable functions on G such that $f \in L_p(G)$, $g \in L_{p'}(G)$, $g^* \in L_{p'}(G)$, and $\|g\|_{p'} = \|g^*\|_{p'}$.

Then for λ-almost all $x \in G$, the integral

(i) $f*g(x) = \int_G f(xy)\, g(y^{-1})\, dy$

exists and is finite. The function $f*g$ is in $L_r(G)$ and the inequality

(ii) $\|f*g\|_r \le \|f\|_p \, \|g\|_{p'}$

obtains.

<u>Definition 1.34</u>: An algebra A over a field F is a vector space over F which is also a ring and in which the mixed

associative Law relates scalar multiplication to ring multiplication:

$$(\lambda x)y = x(\lambda y) = \lambda(xy).$$

If the multiplication is commutative, then A is called a commutative algebra.

Definition 1.35: A Banach algebra is an algebra over the complex numbers, together with a norm under which it is a Banach space and which is related to multiplication by the inequality:

$$\|xy\| \leq \|x\| \, \|y\|.$$

Theorem 1.36: Let G be a compact group. For $1 \leq p \leq \infty$, the function space $L_p(G)$ [with normalized Haar measure] is a Banach algebra under convolution. That is, if $f, g \in L_p(G)$, then $f*g$ is in $L_p(G)$ and

(i) $\|f*g\|_p \leq \|f\|_p \, \|g\|_p$.

The following results due to J. Stewart [5] are recorded for use in the next chapter.

Theorem 1.37: $P(C_{oo}) = P(L_c^p)$ for every $p \geq 2$ where L_c^p is the set of functions in L_p^p with compact support. [See Def. 2.1]

Theorem 1.38: Let $1 \leq p \leq 2$ and $q = p/2(p-1)$. If $f \in P(L_c^2)$ and f is locally in L^q, then $f \in P(L_c^p)$.

Definition 1.39: Let E be a real normed space and let K be a cone which is a subset of E with the following properties:

(1) $K + K \subset K$

(2) $\alpha K \subset K$ for all $\alpha \geq 0$ and

(3) $K \cap (-K) = \{0\}$.

The natural partial ordering $\geq$ is associated with the cone K i.e. $a \geq b$ in case $a - b \in K$. The dual E^* of E is also partially ordered by the <u>dual cone</u> $K^* = \left\{ f \in E^*: f(x) \geq 0 \text{ for all } x \in K \right\}$.

The cone K is said to <u>generate</u> E or to be a generating cone in case every element in E can be written as the difference of two elements in K i.e. $E = K - K$.

E is said to be <u>(O) - complete</u> (order complete) in case any upward directed subset with an upperbound (with respect to $\geq$) has the supremum.

E is said to be <u>quasi - (O) - complete</u> in case any sequence $\left\{a_i\right\}$, such that $0 \leq a_1 \leq a_2 \ldots \leq a$ and $a_{i+j} - a_i \leq \varepsilon_i a$ with $\varepsilon_i \downarrow 0$, has the supremum.

A real linear space E is said to be a <u>vector lattice</u> if E is a lattice by a partial order relation $x \leq y$ satisfying the conditions:

$x \leq y$ implies $x + z \leq y + z$

$x \leq y$ implies $\alpha x \leq \alpha y$ (or $\alpha x \geq \alpha y$) for every $\alpha \geq 0$ (or $\alpha \leq 0$).

A complete normed vector lattice is called a <u>Banach Lattice</u> in case its norm satisfies the following condition:

$$|a| \leq |b| \text{ implies } \|a\| \leq \|b\| \text{ where } |a| = a \vee (-a).$$

We quote here the following theorem of T. Ando [8] for future reference:

<u>Theorem 1.40</u>: (a) K generates E if and only if E^* is quasi-order complete.

(b) K^* (dual cone) generates E^* if and only if E is quasi - (O) - complete.

<u>Theorem 1.41</u>: (ALAOGLU) Let E be a real or complex normed linear space. Then the unit ball $S^* = \left\{ L \in E^* : \|L\| \leq 1 \right\}$ in E^* is compact in the weak*-topology of E^*.

CHAPTER 2

Positive definite functions for the class L_p over compact groups

Definition 2.1: Let F be a set of complex valued measurable functions on a Hausdorff locally compact group G. Let λ be the left Haar measure on G(normalized by $\lambda(G) = 1$ if G is compact). For brevity we shall write dx in place of $d\lambda(x)$ and $d(x,y)$ in place of $d(\lambda x \lambda)(x,y)$.

A complex valued function (Borel measurable) ϕ on G is called positive definite for F if

$$\int_{G x G} \left| \phi(y^{-1}x)\, \overline{f(y)}\, f(x) \right| \, d(x,y) < \infty,$$

and

$$\int_{G x G} \phi(y^{-1}x)\, \overline{f(y)}\, f(x)\, d(x,y) \geq 0 \quad \text{for all} \quad f \in F.$$

The class of functions which are positive definite for F will be denoted by $P(F)$. Clearly $F_1 \subset F_2$ implies that $P(F_1) \supset P(F_2)$.

We wish to show that any $L_p (1 \leq p \leq \infty)$ function which is positive definite for the class Coo, is also positive definite for the class L_q for some q. A precise statement of our result follows;

Theorem 2.2: If G is a compact Hausdorff topological group, then for $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$

$$P(Coo) \cap L_p = P(L_q) \cap L_p.$$

The proof of this result is based on the following two Lemmas.

<u>Lemma 1</u>: Let G be compact and $1 \leq p < \infty$. If $f_n \longrightarrow f$ in L_p and $g_n \longrightarrow g$ in L_p, then

$f_n * g_n \longrightarrow f * g$ in L_p, where $*$ denotes the convolution.

<u>Proof</u>: Under the conditions in the Lemma, $L_p(G)$ by <u>1.36</u> is a Banach algebra with respect to the convolution $*$. Consequently $f_n * g_n$ and $f * g$ are in $L_p(G)$. It follows therefore

$$\| f_n * g_n - f * g \|_p = \| f_n * (g_n - g) + (f_n - f) * g \|_p$$

$$\leq \quad \| f_n \|_p \, \| g_n - g \|_p + \| f_n - f \|_p \, \| g \|_p$$ by <u>1.14</u> and <u>1.36</u> (i).

The sequence $\left\{ \| f_n \|_p \right\}$ is bounded so that as $n \to \infty$, we obtain

$f_n * g_n \longrightarrow f * g$ in $L_p(G)$.

<u>Lemma 2</u>: If the function $(x,y) \longrightarrow \phi(y^{-1}x)\, \overline{f(y)}\, f(x)$ is in $L_1(G \times G, \lambda \times \lambda)$, then

$$\int_{G \times G} \phi(y^{-1}x)\, \overline{f(y)}\, f(x)\; d(x,y) = \int_G \overset{*}{f} * f(x)\, \phi(x)\; dx$$

where $f^*(x) = \overline{f(x^{-1})}$, and $f \in L_1(G)$.

<u>Proof</u>: By Fubini's theorem <u>(1.11)</u> the above integral on the left in the Lemma, can be re-written as

$$\iint_{GG} \phi(y^{-1}x)\ f(x)\ dx\ \overline{f(y)}\ dy = \iint_{GG} \phi(x)\ f(yx)\ dx\ \overline{f(y)}\ dy$$

$$= \iint_{GG} f(yx)\ \overline{f(y)}\ dy\ \phi(x)\ dx$$

Since by Theorem 1.29 every compact Hausdorff topological group is unimodular, we have by Theorem 1.31 (iv)

$$\overset{*}{f} * f(x) = \int_G \overline{f(y)}\ f(yx)\ dy, \quad (f^* \varepsilon\ L_1(G) \text{ by Theorem 1.30}).$$

Hence by another citation of (1.11), $(\overset{*}{f} * f)\ \phi$ is in $L_1(G)$ and

$$\int_{GxG} \phi(y^{-1}x)\ \overline{f(y)}\ f(x)\ d(x,y) = \int_G \overset{*}{f} * f(x)\ \phi(x) dx$$

Proof of the theorem: Since by Theorem 1.19 $Coo \subset L_q(G)$, $1 < q < \infty$, we get $P(Coo) \supset P(L_q)$ and then

$$P(Coo) \cap L_p \supset P(L_q) \cap L_p.$$

(It may be remarked here that classes $P(F) \cap L_p$, $F = Coo$ or L_q are non-null because they contain at least the non-negative constants.)

To prove the opposite inclusion, suppose $\phi\ \varepsilon\ P(Coo) \cap L_p$ and let $f\ \varepsilon\ L_q \subset L_1$, (because G is compact and $\lambda(G) < \infty$, λ being the Haar measure, by Theorem 1.26).

The denseness of Coo in L_q ensures the existence of a sequence $\{f_n\}$ in Coo such that

(*) $$\|f_n - f\|_q \longrightarrow 0 \quad \text{and hence} \quad \|f_n^* - f^*\|_q \longrightarrow 0.$$

Since $\emptyset \varepsilon P(Coo)$, we have

$$\int_{GxG} \left| \emptyset(y^{-1}x)\, \overline{f_n(y)}\, f_n(x) \right| \, d(x,y) < \infty$$

and

$$\int_{GxG} \emptyset(y^{-1}x)\, \overline{f_n(y)}\, f_n(x) \, d(x,y) \geq 0$$

for each $f_n \varepsilon Coo$.

By Lemma 2

$$\iint_{GG} \emptyset(y^{-1}x)\, \overline{f_n(y)}\, f_n(x) \, dx\, dy = \int_G f_n^* * f_n(x)\, \emptyset(x)\, dx \geq 0$$

for each n.

Next we claim that if $f \varepsilon L_q \subset L_1$ and $\emptyset \varepsilon L_p$, then the integral

$$\int_{GxG} \left| \emptyset(y^{-1}x)\, \overline{f(y)}\, f(x) \right| d(x,y) \quad < \infty.$$

By Theorem <u>1.32</u>

$$\overline{f} * \emptyset(x) = \int_G \overline{f(y)}\, \emptyset(y^{-1}x) dy$$

exists and is finite for λ-almost all $x \varepsilon G$ and is a function in $L_p(G)$. Since $f \varepsilon L_q$, it follows by Hölder's inequality <u>(1.13)</u> that the integral

$$\iint_{GG} \left| \overline{f(y)} \right| \, \left| \emptyset(y^{-1}x) \right| \, dy \, \left| f(x) \right| \, dx < \infty.$$

Fubini's theorem implies that the integral

$$\int_{G\times G} \left| \emptyset(y^{-1}x)\ \overline{f(y)}\ f(x) \right| \, d(x,y)$$

exists and is finite for $\emptyset \in L_p$ and $f \in L_q$.

Hence by Lemma 2,

$$\iint_{GG} \emptyset(y^{-1}x)\ \overline{f(y)}\ f(x)\ dx\ dy = \int_G f^{*}*f(x)\ \emptyset(x)\ dx.$$

To complete the proof of the theorem, we must show that the above integral is non-negative.

To see this we appeal to Lemma 1 and the Hölder's inequality, for by (*)

$$\left| \int_G (f_n^{*} * f_n)\ \emptyset\ d\lambda - \int_G (f^{*} * f)\ \emptyset\ d\lambda \right|$$

$$\leq \int_G \left| (f_n^{*} * f_n - f^{*} * f)\ \emptyset \right| d\lambda$$

$$\leq \left\| f_n^{*} * f_n - f^{*} * f \right\|_q \left\| \emptyset \right\|_p \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Hence $\int_G f^{*} * f(x)\ \emptyset(x)\ dx$, being the limit of a non-negative sequence $\int_G f_n^{*} * f_n(x)\ \emptyset\ (x)\ dx$ is non-negative.

Consequently $\emptyset \in P(L_q) \cap L_p$, so that we have established

$$P(Coo) \cap L_p = P(L_q) \cap L_p.$$

Remark 2.3: It may be noted that the above Theorem remains valid if Coo, the space of continuous functions with compact support, is replaced by any dense subspace of L_q. Moreover by 1.4, since G is compact,

$$P(Coo) = P(Co) = P(C) \text{ and hence } P(C) \cap L_p = P(L_q) \cap L_p.$$

Corollary 2.4: If $1 \leq p \leq 2$, and $q = p/p-1$, then

$$P(Coo) \cap L_q = P(L_2) \cap L_q = P(L_p) \cap L_q.$$

Proof: G being compact with Haar measure λ, we have by Theorem 1.17

$$Coo \subset L_2 \subset L_p \quad \text{which implies}$$

$$P(Coo) \cap L_q \supset P(L_2) \cap L_q \supset P(L_p) \cap L_q.$$

Following the method of proof in Theorem 2.2 it can be shown that

$$P(Coo) \cap L_q \subset P(L_p) \cap L_q$$

and hence the equality follows.

Remark 2.5: For $1 \leq p < \infty$, and compact G, Theorem 2.2 gives another way of looking at the theorem of Weil [P. 1311, 7], replacing $P(Coo) \cap L_p$ by the class $P(L_q) \cap L_p$, where $p^{-1} + q^{-1} = 1$

The following theorem is proved in case p and q are not necessarily conjugate real numbers.

Theorem 2.6: Let G be compact with Haar measure λ.

Let $1 \leqslant p \leqslant 2$ and $q = p/2(p-1)$. Then

$$P(Coo) \cap L_q = P(L_2) \cap L_q = P(L_p) \cap L_q .$$

Proof: Consider the case $1 < p < 2$. By Theorem 1.17 and 1.19, since $2 > p$, we have $Coo \subset L_2 \subset L_p$.

Hence

$$P(Coo) \cap L_q \supset P(L_2) \cap L_q \supset P(L_p) \cap L_q \quad .$$

Let $\emptyset \in P(Coo) \cap L_q$ and let $f \in L_p$. If we let $p' = p$ in Theorem 1.33, then

$$\frac{1}{r} = \frac{2}{p} - 1 = 1 - (2 - \frac{2}{p}) = 1 - \frac{2(p-1)}{p} .$$

The hypothesis yields

$$\frac{1}{r} = 1 - \frac{1}{q}$$

i.e. r and q are conjugate real numbers greater than 1. We may also note $\frac{1}{p} + \frac{1}{p} > 1$ by hypothesis.

Theorem 1.30 implies that if G is compact and $g \in L_p(G)$, then g^* also belongs to $L_p(G)$, because $\Delta = 1$ by Theorems 1.29, 1.30

Letting $g = f^*$, we conclude by Theorem 1.33, that the function $f*f^*$ exists and is finite and belongs to $L_r(G)$.

Since $\emptyset \in L_q(G)$, it follows by Holder's inequality (1.13), that the integral

$$\int_G f*f^*(x) \quad \emptyset(x) \, dx$$

exists and is finite for λ-almost all $x \in G$ and for all $f \in L_p(G)$.

As in Theorem 2.2

$$\iint_{GG} \phi\,(y^{-1}x)\ \overline{f(y)}\ f(x)\ dx\ dy = \int_G f*f^*(x)\ \phi(x)\ dx$$

for all $f \in L_p(G)$.

It remains to show that the above integral is non-negative. To see this let $\{f_n\}$ be a sequence in Coo such that

$$\lim_{n\to\infty} \| f_n - f \|_p = 0, \quad \text{Also} \quad \lim_{n\to\infty} \| f_n^* - f^* \|_p = 0.$$

$\phi \in P(Coo) \cap L_q$ implies as before

$$\iint_{GG} \phi(y^{-1}x)\ \overline{f_n(y)}\ f_n(x)\ dx\ dy \qquad (a)$$

$$= \int_G f_n*f_n^*\ (x)\ \phi(x)\ dx \geq 0 \quad \text{for each } f_n \in Coo.$$

Theorem 1.33 says $f_n*f_n^*$ is in $L_r(G)$ and by Minkowski inequality (1.14) ($L_r(G)$ is a Banach algebra)

$$\| f_n*f_n^* - f*f^* \|_r$$

$$= \|(f_n - f) * f_n^* + f*(f_n^* - f^*)\|_r$$

$$\leq \|(f_n - f) * f_n^*\|_r + \|f * (f_n^* - f)\|_r$$

Appealing once more to Theorem (1.33)

$$\| f_n*f_n^* - f*f^* \|_r$$

$$\leq \|f_n - f\|_p\ \ \|f_n^*\|_p + \|f\|_p\ \ \|f_n^* - f^*\|_p \qquad (b)$$

Since the sequence $\left\{\|f_n^*\|_p\right\}$ is bounded, the limit of the last line in (b) is zero. By Hölder's inequality therefore

$$\left|\int_G (f_n * f_n^*)\, \emptyset \, d\lambda - \int_G (f * f^*)\, \emptyset \, d\lambda\right|$$

$$\leq \int_G \left|(f_n * f_n^* - f * f^*)\right| \, |\emptyset| \, d\lambda$$

$$\leq \|f_n * f_n^* - f * f^*\|_r \; \|\emptyset\|_q$$

By (a) and (b) it follows

$$\int_G f * f^*(x)\, \emptyset(x)\, dx$$

is non-negative for all $f \varepsilon L_p(G)$ which implies that

$$\emptyset \; \varepsilon \; P(L_p) \cap L_q .$$

We have therefore shown

$$P(Coo) \cap L_q = P(L_2) \cap L_q = P(L_p) \cap L_q .$$

<u>Remark</u>: For $p = 1$, $q = \infty$ and $p = 2$, $q = 1$ the result follows by Theorem 2.2 and Theorem 1.32 (iii) respectively.

For any locally compact group Edwin Hewitt [3] has proved that if any Borel measurable function f on G is positive definite for the class $L_1(G)$, then it is an L_∞ function. That is

$$f \; \varepsilon \; P(L_1) \text{ implies } f \; \varepsilon \; L_\infty .$$

His proof is based on the fact that the algebra $L_1(G)$ is factorable: every element in $L_1(G)$ is the convolution product of two elements in $L_1(G)$. The generalization of this result, i.e.

every function which is positive definite for the class $L_p(G)$ is an L_q-function where $\frac{1}{p} + \frac{1}{q} = 1$, seems to be difficult because the function space $L_p(G)$, $1 < p \leq \infty$, for compact infinite group, is a convolution algebra that is not factorable.

Nevertheless under special circumstances we have the following anologous result.

Theorem 2.7: If G is a compact discrete group, then for $1 < p \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\emptyset \in P(L_p(G)) \text{ implies } \emptyset \in L_q(G).$$

The proof is based on the following observations.

Definition: If the Haar measure of G is normalized by giving each point the measure 1 and f and g are any two complex valued functions on G, then the convolution of f with g, is given by

$$f * g(x) = \int_G f(xy)\, g(y^{-1})dy = \sum_{y \in G} f(xy)\, g(y^{-1})$$

$$= \int_G f(y)\, g(y^{-1}x)dy = \sum_{y \in G} f(y)\, g(y^{-1}x)$$

It is well-knonw that $L_1(G)$ has an identity if and only if G is discrete. We have the following Lemma:

Lemma 2.8: If G is a compact discrete (i.e. finite) group, then $L_p(G)$ has a unit, the unit of $L_1(G)$.

Proof: We know that the convolution algebra $L_p(G)$ is a subalgebra of $L_1(G)$ that is a Banach algebra with respect to a norm of its own. The function $e(x)$ which is 1 at $x = e$, the identity of G and zero elsewhere, is an identity of $L_1(G)$ because

$$f*e(x) = \int_G f(y)\, e(y^{-1}x)\, dy = \sum_{y\varepsilon G} f(y)\, e(y^{-1}x)$$

$$= f(x).$$

Thus the function $e(x)$ serves as an identity of $L_p(G)$. Consequently we observe that every $f \varepsilon L_p(G)$ is the convolution of any two elements in $L_p(G)$. Since, in general

$$L_p(G) * L_p(G) \subset L_p(G),$$

we have proved the following:

Lemma 2.9: If G is a compact discrete group, then $L_p(G)$ $(1 < p \leq \infty)$ is factorable i.e.

$$L_p(G) * L_p(G) = L_p(G).$$

Proof of Theorem 2.7: Let $\emptyset \varepsilon P(L_p)$. Then

$$\int_{GxG} \left| \emptyset(y^{-1}x)\, \overline{f(y)}\, f(x) \right| \, d(x,y) < \infty$$

and

$$\int_{GxG} \emptyset(y^{-1}x)\, \overline{f(y)}\, f(x)\, d(x,y) \geq 0 \quad \text{for all} \quad f \varepsilon L_p(G).$$

As in the proof of Lemma 2 (Theorem 2.2) $\overset{*}{f} * f(x) = \int_G \overline{f(y)}\, f(yx)dy$ for all $f \varepsilon L_p(G)$.

(i) $(f^{*} * f)\,\phi \;\varepsilon\; L_1(G)$ and (ii) $\int_G f^{*} * f\,(x)\,\phi\,(x)\,dx \geq 0$

for all $f \;\varepsilon\; L_p(G)$.

Since $f * g = \frac{1}{4} \sum_{K=0}^{3} i^{K} (f^{*} + i^{-K}g)^{*} * (f^{*} + i^{-K}g)$,

we see by (i) that $(f * g)\,\phi \;\varepsilon\; L_1(G)$ for all $f, g \;\varepsilon\; L_p(G)$.

By Lemma 2.9 every $h \;\varepsilon\; L_p(G)$ is equal to $f * g$, for some $f, g \;\varepsilon\; L_p(G)$. Hence $h\,\phi \;\varepsilon\; L_1(G)$ for all $h \;\varepsilon\; L_p(G)$. We may assume $h \neq 0 \;\; \forall\, x \;\varepsilon\; G$.

By Theorem 1.20, it follows that

$$\phi \;\varepsilon\; L_q(G).$$

It is well known that for $1 \leq p < \infty$, the conjugate space of L_p is L_q (in the sense that for every bounded linear functional T on L_p, there is a function $g \;\varepsilon\; L_q$ such that $T(f) = \int f\,g\,d\lambda$ and $\|T\| = \|g\|_q$). We have the following definition:

<u>Definition</u>: Let G be locally compact. For functions $f_1, f_2, \ldots f_n \;\varepsilon\; L_p(G)$ and $\phi_0 \;\varepsilon\; L_q(G)$, let

$$U\,(f_1, f_2, \ldots f_n; \varepsilon; \phi_0)$$
$$= \left\{ \phi \;\varepsilon\; L_q(G) : \left| \int_G f_j\,\phi\,d\lambda - \int_G f_j\,\phi_0\,d\lambda \right| < \varepsilon,\; j \;\varepsilon \left\{1, 2, \ldots n\right\} \right\}$$

Taking all sets (neighbourhoods) $U\,(f_1, f_2, \ldots f_n; \varepsilon, \phi_0)$ as a basis for open sets, we get <u>weak*-topology</u> for $L_q(G)$. (The integrals in U exist by Hölder's inequality.)

We prove the following result:

<u>Theorem 2.10</u>: For $1 \leq p < \infty$ and $p^{-1} + q^{-1} = 1$, $P(L_p) \cap L_q$ is a weakly closed set in $L_q(G)$, where G is a compact group.

<u>Proof</u>: Let $\phi \in P(L_p) \cap L_q$. Then $\iint_{GG} \phi(y^{-1}x)\, \overline{f(y)}\, f(x) dx\, dy$ exists and is a non-negative for all $f \in L_p(G)$. As in Theorem 2.2 we can write

$$\text{(i)} \qquad \iint_{GG} \phi(y^{-1}x)\, \overline{f(y)}\, f(x)\, dx\, dy = \int_G f^* * f(x)\, \phi(x)\, dx \geq 0$$

Define

$$T: L_q(G) \longrightarrow \mathbb{R} \quad \text{as follows}$$

$$T(\phi) = \int_G f^* * f(x)\, \phi(x)\, dx, \quad \text{where} \quad f^* * f \in L_p(G).$$

Then obviously T is a linear functional on $L_q(G)$. We shall show T is continuous in the weak*-topology as defined above. For any real number a, $(a - \varepsilon, a + \varepsilon)$ is an open set in $\mathbb{R}$ and then

$$T^{-1}(a - \varepsilon, a + \varepsilon) = \left\{ \phi \in L_q(G): \; T(\phi) \subset (a - \varepsilon, a + \varepsilon) \right\}$$

$$= \left\{ \phi \in L_q(G): \; a - \varepsilon < \int_G f^* * f(x)\, \phi(x)\, dx < a + \varepsilon \right\}$$

Taking $f^* * f(x) = f_j(x) \in L_p(G)$ for some j and $\int_G f^* * f(x)\, \phi_0(x)\, dx = a$, we see that $T^{-1}(a - \varepsilon, a + \varepsilon)$ is open in $L_q(G)$ in the weak*-topology of $L_q(G)$ and this proves T is continuous.

Remark: Continuity of T can also be established in a simpler way as follows:

$$|T(\phi)| \leq \|f^* * f\|_p \ \|\phi\|_q \quad \text{by Hölder's inequality.}$$

Hence T is bounded and therefore norm continuous and hence a fortiori weakly continuous.

Let $A = P(L_p) \cap L_q = \left\{ \phi \in L_q : \ T(\phi) = \iint_{GG} \phi(y^{-1}x)\overline{f(y)}\ f(x)dx\ dy \geq 0 \right\}$

for all $f \in L_p$.

We wish to show A is closed in the weak*-topology.

Consider a net $\{\phi_\alpha\}$ in A which converges in the weak*topology of $L_q(G)$ to $\phi \in L_q(G)$. We shall prove $\phi \in A$. The continuity of T implies $\{T\phi_\alpha\}$ converges to $T\phi$ in the weak*-topology. Consequently ϕ which belongs to $L_q(G)$ satisfies

$$\int_G f^* * f(x)\ \phi(x)\ dx \geq 0$$

since the last condition is valid for ϕ_α for each α by (i). Hence $\phi \in A$.

It is not difficult to see that $P(L_p) \cap L_q$ is, in fact, a convex weakly closed cone of $L_q(G)$.

Remark: M. A. Naimark [6] has proved that for any locally compact group $P(L_1) \cap L_\infty$ is a weakly closed set in L_∞. Our result is a generalization but only for compact groups.

It is well-known that a Banach lattice is quasi-order complete. Since the weak*topology is weaker than the norm topology on $L_q(G)$ every weakly closed cone is a closed cone. Since L_p-spaces $(1 < p < \infty)$ are reflexive and being Banach lattices, are quasi-order complete we have the following by Theorem 1.40 (T. Ando).

<u>Corollary 2.11</u>: $P(L_p) \cap L_q$ generates $L_q(G)$.

<u>Corollary 2.12</u>: For $p > 1$,

$$(P(L_p) \cap L_q)^* \text{ generates } L_p(G).$$

We recall from Theorem 2.10, that

$$P(L_p) \cap L_q = \left\{ \phi \in L_q : \iint_{GG} \phi(y^{-1}x)\, \overline{f(y)}\, f(x)\, dx\, dy \geq 0 \quad \forall f \in L_p \right\}.$$

As observed before $P(L_p) \cap L_q$ is a <u>cone</u> in $L_q(G)$ and by Theorem 2.10, it is a <u>weakly closed cone in the Banach space</u> $L_q(G)$.

We use theorem 2.10 to prove the following result:

<u>Corollary 2.13</u>: The set of all normalized functions in $L_q(G)$ which are positive definite for the class $L_p(G)$, is a weakly compact subset of $L_q(G)$.

<u>Proof</u>: By a normalized function $\phi \in L_q(G)$ we mean one such that $\|\phi\|_q = 1$. It may be noted here that such a function exists in view of the theorem: "There is a function $\phi \in L_p$ $(1 < p < \infty)$ such that $\|\phi\|_p = 1$, and $L(\phi) = \|L\|$, where L is an arbitrary bounded Linear functional on L_p different from 0".

Let $B = \left\{ \phi \in L_q : \|\phi\|_q = 1 \right\}$.

By the theorem of Alaoglu, the unit ball B in L_q is compact in the weak*-topology of L_q. Being a compact subset of a Hausdorff space, B is weakly closed.

Denoting $A = P(L_p) \cap L_q$, we observe $A \cap B$ is a closed subset of a compact set B and hence $A \cap B$, the set of all normalized functions in L_q which are positive definite for the class L_p, is compact in the weak*-topology of L_q.

CHAPTER 3

Positive definite functions for the class $L_p(G)$ over locally compact groups

We shall reserve ϕ for the function positive definite for F i.e. $\phi \in P(F)$.

Our aim is to prove the following:

Theorem 3.1: Let G be a locally compact group and let $\Delta^{-1/2} \phi \in L_1(G)$, where Δ is the modular function for G. Then if ϕ is positive definite for the class $L_1(G) \cap L_2(G)$, it is positive definite for the class $L_2(G)$.

Proof: It is well known that for $1 < p < \infty$, $L_1 \cap L_p$ is a dense subset of L_p. Hence $L_1 \cap L_2 \subset L_2$, implies

$$P(L_1 \cap L_2) \supset P(L_2).$$

Let $\phi \in P(L_1 \cap L_2)$ and suppose $f \in L_2$. There exists a sequence $\left\{f_n\right\}_1^\infty$ in $L_1 \cap L_2$ such that

$$\lim_{n \to \infty} \| f_n - f \|_2 = 0 \qquad \text{(i)}$$

Also,

$$\int_{G \times G} \left| \phi(y^{-1}x) \overline{f_n(y)} f_n(x) \right| d(x,y) < \infty$$

and

$$\int_{G \times G} \phi(y^{-1}x) \overline{f_n(y)} f_n(x) \, d(x,y) \geq 0$$

for each $f_n \in L_1 \cap L_2$. (ii)

We claim that the integral

$$\int_{G\times G} |\phi(y^{-1}x)\, \overline{f(y)}\, f(x)|\; d(x,y) < \infty \quad \forall\, f \in L_2$$

for, by Theorem 1.32 (iii) the function

$$\overline{f} * \phi(x) = \int_G \overline{f}(y)\, \phi(y^{-1}x)\; dy$$

exists and is finite for λ-almost all $x \in G$ and is a function in $L_2(G)$ for which

$$\|f * \phi\|_2 \leq \|f\|_2 \; \|\Delta^{-1/2}\phi\|_1 \; \ldots \qquad \text{(iii)}$$

By Hölder's inequality, the integral

$$\int_G \int_G |\overline{f(y)}|\; |\phi(y^{-1}x)|\; dy\; |f(x)|\; dx$$

is finite. By Fubini's theorem the integral

$$\int_G \int_G \phi(y^{-1}x)\; \overline{f(y)}\; f(x)\; dx\; dy$$

exists and is finite for all $f \in L_2$. Hence we can write

$$\int_{G\times G} \phi(y^{-1}x)\; \overline{f(y)}\; f(x)\; d(x,y) = \int_G (\overline{f} * \phi)\; f\; d\lambda$$

It remains to show that the above integral is non-negative. To see this, we have by (iii) and Hölder's inequality

$$\left| \int_G (\bar{f}_n * \emptyset)\, f_n \, d\lambda - \int_G (\bar{f} * \emptyset)\, f \, d\lambda \right|$$

$$\leq \int_G \left| \bar{f}_n * \emptyset \right| \left| f_n - f \right| d\lambda + \int_G \left| \bar{f}_n * \emptyset - \bar{f} * \emptyset \right| \left| f \right| d\lambda$$

$$\leq \| \bar{f}_n * \emptyset \|_2 \, \| f_n - f \|_2 + \| (\bar{f}_n - \bar{f}) * \emptyset \|_2 \, \| f \|_2$$

$$\leq \| f_n \|_2 \, \| \Delta^{-1/2} \emptyset \|_1 \, \| f_n - f \|_2 + \| f_n - f \|_2 \, \| \Delta^{-1/2} \emptyset \|_1 \, \| f \|_2$$

$$= (\| f_n \|_2 \, \| f_n - f \|_2 + \| f_n - f \|_2 \, \| f \|_2) \, \| \Delta^{-1/2} \emptyset \|_1$$

By (i) and (ii) on taking limits as $n \to \infty$ $\int_G (\bar{f} * \emptyset)\, f \, d\lambda$ is the limit of the non-negative sequence $\int_G (\bar{f}_n * \emptyset)\, f \, d\lambda$.

Hence we have shown

$\int_G (\bar{f} * \emptyset)\, f \, d\lambda \geq 0$ for all $f \in L_2$ and this implies $\emptyset \in P(L_2)$ which proves the theorem.

<u>Corollary 3.2</u>: Let G be a locally compact group and p a real number > 1 and q is conjugate to p. Let $\emptyset \in P(L_1 \cap L_p \cap L_q)$ be such that $\Delta^{-1/q} \emptyset \in L_1(G)$. Then

$$\emptyset \in P(L_p \cap L_q)$$

<u>Proof</u>: Let $f \in L_p \cap L_q$. It is always possible to construct a sequence $\{f_n\}$ of functions in $L_1 \cap L_p \cap L_q$ such that

$$\lim_{n\to\infty} \| f_n - f \|_p = \lim_{n\to\infty} \| f_n - f \|_q = 0$$

For example we may choose $\{f_n\}$ to be a sequence of simple functions which are dense in L_p.

Now essentially the same proof of Theorem 3.1 goes for this Corollary.

<u>Corollary 3.3</u>: If $\emptyset \varepsilon P(Coo)$ satisfies the condition $\Delta^{-1/q} \emptyset \varepsilon L_1(G)$, then

$$\emptyset \varepsilon P(L_p \cap L_q)$$

<u>Proof</u>: Let $f \varepsilon L_p \cap L_q$. We can find a sequence $\{f_n\}$ in Coo $\subset L_p \cap L_q$ such that

$$\lim_{n\to\infty} \| f_n - f \|_p = \lim_{n\to\infty} \| f_n - f \|_q = 0$$ and so on so forth.

<u>Corollary 3.4</u>: If the group G is compact then

$$P(Coo) \cap L_1 = P(L_p \cap L_q) \cap L_1$$

In particular $P(Coo) \cap L_1 = P(L_2) \cap L_1$.

<u>Corollary 3.4</u>: If $p \geq 2$ and G is compact we obtain by Corollary 3.4

$$P(Coo) \cap L_1 = P(L_p) \cap L_1$$ because $L_q \supset L_p$.

That is an L_1 function which is positive definite for the class Coo is in fact positive definite for the class L_p, $p \geq 2$, which may be compared with the Theorem 1.38 (James Stewart)i.e. the class of functions which are positive definite for the class L_c^p does not change as p varies over $(2, \infty)$.

J. L. B. Cooper [9] has remarked in his paper: Positive definite functions of a real variable (Proc. Lond. Math. Soc. (3) 10(1960)) that the class of functions positive definite for the class L_c^p does vary as p ranges over $(1, 2)$.

On the other hand Corollary 3.3 says the following:

In particular for such ϕ the class of functions positive definite for the class $L_p \cap L_q$ does not vary for any pair of conjugate real numbers p and q greater than 1.

Corollary 3.5: Suppose $\Delta^{-1/2} \phi \in L_1(G)$ and G locally compact. If $1 \leq p < 2 < q$, where p and q are conjugate real numbers, then, if $\phi \in P(Coo)$

$$\phi \in P(L_2)$$

Proof: By the Theorem 1.18, we have

$$Coo \subset L_p \cap L_q \subset L_2$$

Hence $P(Coo) \supset P(L_p \cap L_q) \supset P(L_2)$. As in Theorem 3.1, we can prove the assertion.

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