#### GAIN SATURATION AND PULSE SOLUTIONS

FOR A CO2-HE T.E.A. LASER

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#### ABSTRACT

The saturation behaviour of various gain models for the CO<sub>2</sub>-HE T.E.A. laser was investigated. The corresponding laser pulse shapes for an oscillating cavity were generated by computer and compared to experimentally observed output pulses. Good agreement is obtained for low gain using a simple two-level model but at high gain, vibrational coupling between levels must be included.

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#### 1.0 INTRODUCTION

The objective of this paper is to investigate the saturation behaviour of two different gain models for the CO<sub>2</sub>-He T.E.A. laser. The laser rate equations are solved numerically to obtain the time dependent buildup of the photon density within the cavity. The photon pulse shape so obtained can then be compared with actual photographs of experimentally generated T.E.A. laser pulses.

The work done in this paper draws upon the recent studies by Garside et al (1,2) of the unsaturated gain buildup and by Lyon et al (3) on the dynamic pulse buildup in T.E.A. lasers. The approach taken in our investigation of gain saturation is to begin in section 2.0 with a simple gain model. This consists of the 00°l and 10°O vibrational levels which contain the J = 19, and J = 20 rotational laser levels respectively for the 10.6  $\mu$ CO<sub>2</sub> transition. With this model, which incorporates rotational coupling but neglects vibrational coupling, we obtain an understanding of when and how the pulse solutions deviate from the experimental laser pulses. In section 3.0 a "thermal" gain model is developed to account for certain vibrational effects which become important at high laser intensities. Lastly, the important conclusions obtained from studying these gain models under dynamic situations are given in section 4.0.

#### 2.0 THE TWO LEVEL GAIN MODEL

The number of vibrational transitions to the laser levels of the 10.6  $\mu$  CO<sub>2</sub> emission is large. To write differential equations for all paths would be hopelessly complex and little "feel" for the interplay of coupling processes could be obtained.

The unsaturated gain buildup in T.E.A. laser systems has been studied by Garside et al (1,2). Their results indicate that the shape of the gain curve can be reasonably well approximated by a two level model with each level decaying exponentially as a result of collisional relaxation processes. Such a system is shown in Fig. 1 using standard rotational and vibrational notation (Ref. 4).



#### Fig. 1 Simple Two Level Gain Model

The unsaturated gain may be represented by the expression:

(1) 
$$g(t) = g_0(e^{-t/\tau_2} - e^{-t/\tau_1})$$

Here  $g_0$  represents the maximum attainable gain and is achieved only when  $\tau_1 = 0$ . This form for the time-dependent gain is appropriate for the situation in which the upper and lower laser levels are equally excited at t = 0. Using experimental data the relaxation times  $\tau_2$  and  $\tau_1$  were found to be about 12 µsec and 1.2 µsec respectively for a 3% CO<sub>2</sub>, 97% He, atmospheric pressure gas mix (Ref. 5). The fact that the model fits the low signal gain so well tends to imply that the numerous rotationalrotational and vibrational-vibrational couplings achieve amongst themselves a set of quasi-equilibrium distribution states.

The low signal photon density pulse buildup can be described by a single rate equation in conjunction with equation (1) provided that the stimulated emission rate does not produce gain saturation. Just how high a photon density is allowed in the cavity depends on the magnitude of the Einstein coefficient for stimulated emission B, in relation to the relaxation times  $\tau_2$  and  $\tau_1$ . In other words, the value of photon density that results in sufficient stimulated emission to perturb the exponential decay of the levels defines the point at which saturation effects enter in. Beyond this point the nature of the internal couplings to other rotational and vibrational levels becomes important. By solving for the pulse buildup at higher and higher laser intensities the deviations from experimental pulses should reveal the effects of this internal structure.

## 2.1 RATE EQUATIONS

The following rate equations describe the dynamic behaviour in the simple two level gain-medium model:

(a)	$\frac{d\rho}{dt} = W(n_2 - n_1)\rho - \frac{\rho}{\tau_c}$
(b)	$\frac{dn_2}{dt} = - W(n_2 - n_1)\rho - \frac{n_2}{\tau_2}$
(c)	$\frac{dn_1}{dt} = W(n_2 - n_1)\rho - \frac{n_1}{\tau_1}$

The symbols denote:

ρ	The density of photons
<sup>n</sup> 2	The density of CO2 molecules in the 00°1
	vibrational level
nl	The density of CO2 molecules in the 10°0
	vibrational level
τ2	The effective relaxation time for the 00°1,
	J=19 rotational level
τι	The effective relaxation time for the 10°0,
	J=20 rotational level
Loss	The loss per pass of the cavity
L	The cavity length
đ	The active discharge length

 $\tau_{c} = \frac{L}{c(Loss)}$  The cavity relaxation time for photon loss

 $\gamma_{\mathbf{R}}$ 

The ratio of molecules in the rotational laser level to those in the vibrational level

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B The basic integrated rotational Einstein coefficient for stimulated emission
 B<sub>eff</sub>=γ<sub>R</sub>B The effective vibrational Einstein coefficient for stimulated emission
 W=B<sub>eff</sub> d/L The Einstein coefficient with the appropriate filling factor for the cavity.

It must be noted that for the development of equations in this paper the assumption is made that spatially homogeneous inversion exists throughout the cavity. To incorporate the rotational coupling in equations (2) we assume that the coupling rate is so fast that a population density distribution is maintained at the background translational gas temperature. This allows the population densities in the rotational laser levels, for the 10.6  $\mu$  transition, to be written as a fixed fraction  $\gamma_{\rm p}$ of the vibrational level population densities to which they be-This can only be achieved if the stimulated emission rate long. remains below the rotational coupling rate. Also the background gas temperature is assumed to be constant during the laser pulse buildup. Actually this will not be entirely correct, depending upon how close the photon pulse is to the discharge current pulse. However calculations given in Appendix A indicate that the effect on  $\gamma_R$  and thus on  $B_{EFF}$  is relatively minor for "reasonable" translational temperature variations.

The effect of vibrational coupling on the B coefficient and thus on saturation behaviour is much more difficult to incorporate into a two level system. Even when we assume that the vibrational coupling is fast enough to maintain Boltzmann temperatures for the upper and lower system of levels we cannot simply modify  $B_{eff}$  by another population factor. The reason for this is that the temperature is dynamic and thus varies with time as levels are depopulated by collisions and stimulated emission. The vibrational coupling can be incorporated by adding more levels to the model, thereby increasing the number of population rate equations. This is considered in more detail in section 3.0.

#### 2.2 NORMALIZED RATE EQUATIONS

For convenience in showing explicit parameter dependence of the rate equations, the following normalized variables were used.

(3)

- (a)  $T = \frac{L}{c}$  Scale length or time for one pass of the cavity (b)  $\rho_f$  - Photon density used as starting
  - value for the numerical integration (Ref. 5)
- (c)  $n_{Th} = \frac{1}{W\tau_c}$  The C.W. threshold inversion defined by the loss per pass
- (d)  $t' = \frac{t}{\pi}$  Normalized time

(e)  $\rho' = \frac{\rho}{\rho_f}$  - Normalized photon density

(f) 
$$n'_{2} = \frac{n_{2}}{n_{Th}}$$
 - Normalized population density for  
the upper vibrational laser level  
(g)  $n'_{1} = \frac{n_{1}}{n_{Th}}$  - Normalized population density for  
the lower vibrational laser level  
(h)  $n' = (\frac{n_{2}-n_{1}}{n_{Th}})$  - Normalized inversion or gain to  
loss ratio.

The following are the normalized rate equations:

(4)  
(a) 
$$\frac{d\rho'}{dt'} = Loss(n'-1)\rho'$$
  
(b)  $\frac{dn'_2}{dt'} = -(\frac{\rho_f W L}{c})\rho'n' - (\frac{L}{c\tau_2})n'_2$   
(c)  $\frac{dn'_1}{dt'} = (\frac{\rho_f W L}{c})\rho'n' - (\frac{L}{c\tau_1})n'_1$   
(d)  $n' = n'_2 - n'_1$ .

Typical parameters used:

Loss 
$$\approx .020$$
 (Ref. 5)  
 $\rho_f \approx 5 \times 10^{11} \text{ cm}^{-3}$   
 $B_{eff} \approx 4 \times 10^{-9} \text{ cm}^3 \text{sec}^{-1}$   
L  $\approx 160 \text{ cm}$   
 $d \approx 80 \text{ cm}$   
 $\tau_1 \approx 1.2 \text{ }\mu \text{sec}$   
 $\tau_2 \approx 12 \text{ }\mu \text{sec}$ .

#### 2.3 SOLUTIONS AND COMPARISONS

Equations (4) were solved numerically using the Hamming's predictor-modifier-corrector integration procedure (Ref. 6 & Appendix B). The integration was effectively started at a finite photon density  $\rho_f$ , which was chosen small enough so as not to seriously perturb the vibrational level populations via stimulated emission. This allowed computation of the initial population density values using the simple relationships:

(5)  
(a) 
$$n'_{2} = n'_{0} e^{-t_{d}/\tau_{2}}$$
  
(b)  $n'_{1} = n'_{0} e^{-t_{d}/\tau_{1}}$   
(c)  $n'_{0} = \frac{g_{0}}{Loss}$ 

Here  $n_2$  and  $n_1$  are the normalized population densities of the upper and lower levels respectively and  $n_0$  is the normalized inversion corresponding to the maximum attainable gain  $g_0$ . The experimental variable  $t_d$  (Ref. 1), which measures the time to reach a photon density of  $\rho_f$ , was used as the starting time for the numerical solution.

Figures 2 and 3 show the pulse solutions for low and high gain along with the appropriate experimental pulse shapes (Ref. 5). As was expected the low gain pulse is matched very well to experiment but significant deviations show up with high gain. It was stated previously that at high gain (i.e. high intensity) the effects of vibrational coupling on gain saturation behaviour and thus on pulse shape should become

important. Just how significant is revealed by the long fall time of the computer generated pulse compared to its experimental counterpart. This is an indication that the upper laser level and its coupled partners are being depleted more than the present simple model indicates. The problem is that as the stimulated emission rate becomes large compared to  $n_1/\tau_1$  then the two laser levels in this model become almost isolated. Thus every molecule which returns to the lower laser level due to stimulated emission effectively reduces the population inversion by two molecules. If just prior to the rising edge of the pulse we assume N molecules reside in the upper laser level and neglect the number in the lower laser level, then at the pulse peak approximately N/2 molecules will lie in both levels. This means that a sizeable inversion can re-establish itself as the lower level decays rapidly with the result that stimulated emission keeps the pulse tail from falling fast enough.



Fig. 2 Low gain pulses (≈ 7.5% per pass) generated from the simple two level model and from experiment. The agreement is good for both rise and fall regions of the pulse.



Fig. 3 High gain pulses (~ 28% per pass) generated from the simple two level gain model and from experiment. While the initial rising shape of the pulses match, the peak and tail regions show significant deviations.

#### 3.0 THE "THERMAL" GAIN MODEL

The simplest two level gain model has served a purpose in showing the importance of vibrational coupling in determining the gain saturation behaviour. The problem now is to develop a model which incorporates these vibrational effects. Following Lyon et al (3) we have combined the Fermi coupled symmetric-stretch  $(v_1)$  and bending  $(v_2)$  modes of  $CO_2$  into one mode with appropriate degeneracies for the levels. Harmonic spacing approximations of 960°K and 3365°K were used for the combined symmetric-bend mode and the separate asymmetric stretch  $(v_3)$  mode respectively. This simplified structure given in Fig. 4 allows for only three levels in each system. The reason for this, as will become evident later, is to allow solving explicitly for the temperature as a function of energy and population densities.

The key to understanding the operation of this model is the concept of a Boltzmann distribution of molecules in excited states. The intramode vibration-vibration (V-V) processes within both the lower and upper systems tend to drive their population densities to Boltzmann distributions which would exist if these systems were isolated with the appropriate molecular numbers and vibrational energy. This defines a sort of quasi-equilibrium vibrational temperature



Fig. 4 The simplified energy level structure for the CO<sub>2</sub> molecule. The degeneracy factors are indicated in brackets for each level. Harmonic spacing is used for both upper and lower systems. which is a convenient variable in tracing the dynamic behaviour of these levels.

We assume that the upper and lower energy level systems decay with characteristic times  $\tau_{TA}$  and  $\tau_{TB}$  respectively. The  $\tau_{TA}$  depends mainly on the intermode vibration-vibration (V-V) processes from the upper system via mixed modes to the lower system (Ref. 7). The  $\tau_{TB}$  involves relaxation via vibrational-translational (V-T) processes between the lower system and the ambient gas (ground state). The ambient gas temperature defines a Boltzmann distribution towards which the lower system and ground state are driven with a decay time of about 1 µsec. The equilibrium populations for the upper system at this temperature are negligibly small compared to those after discharge excitation and can be neglected.

An important assumption of the model is that the driving rates to quasi-equilibrium in the V-V and V-T processes are taken to be proportional to the deviation from the equilibrium population values based on energy and number in the system at a particular instant of time.

#### 3.1 RATE EQUATIONS

The following equations describe the behaviour of the "thermal" gain model.

(6) (a) 
$$\frac{d\rho}{dt} = W(n_4 - \frac{n_2}{4})\rho - \frac{\rho}{\tau_c}$$

(b) 
$$\frac{dn_1}{dt} = -\frac{(n_1 - VE_1)}{\tau_{VB}} - \frac{(n_1 - TE_1)}{\tau_{TB}}$$

(c) 
$$\frac{dn_2}{dt} = -\frac{(n_2 - VE_2)}{\tau_{VB}} - \frac{(n_2 - TE_1)}{\tau_{TB}} + W(n_4 - \frac{n_2}{4})\rho$$
  
(d)  $\frac{dn_3}{dt} = -\frac{(n_3 - VE_3)}{\tau_{VB}} - \frac{(n_3 - TE_3)}{\tau_{TB}}$ 

(e) 
$$\frac{dn_4}{dt} = \frac{(n_4 - VE_4)}{\tau_{VA}} - \frac{n_4}{\tau_{TA}} - W(n_4 - \frac{n_2}{4})\rho$$

(f) 
$$\frac{dn_5}{dt} = -\frac{(n_5 - VE_5)}{\tau_{VA}} - \frac{n_5}{\tau_{TA}}$$

(g) 
$$\frac{dn_6}{dt} = -\frac{(n_6 - VE_6)}{\tau_{VA}} - \frac{n_6}{\tau_{TA}}$$

The symbol meanings are as follows:

W	- The effective stimulated emission coefficient (includes rotational population factor $\gamma_R$ and filling factor d/L)
ρ	- The photon density
<sup>n</sup> 1' <sup>n</sup> 2' <sup>n</sup> 3	- The vibrational population densities for levels in the lower system
<sup>n</sup> 4' <sup>n</sup> 5' <sup>n</sup> 6	- The vibrational population densities for the levels in the upper system
<sup>VE</sup> 1, <sup>VE</sup> 2, <sup>VE</sup> 3	- The vibrational quasi-equilibrium population densities calculated from total energy and number in the lower (B) system.
<sup>VE</sup> 4, <sup>VE</sup> 5, <sup>VE</sup> 6	- The same as above but for the upper (A) system The calculation assumes a Boltzmann distri- bution in both cases.
<sup>TE</sup> 1' <sup>TE</sup> 2' <sup>TE</sup> 3	<ul> <li>The vibrational quasi-equilibrium population densities in the (B) system and ground state assuming a Boltzmann distribution and a trans- lational temperature of 400°K.</li> </ul>

<sup>T</sup> VA' <sup>T</sup> VB	-	The relaxation times for the intramode V-V collisional processes in the (A) and (B) systems respectively.
<sup>τ</sup> TA	-	The relaxation time for the (A) system via intermode V-V collisional processes.
ттв	-	The relaxation time for the (B) system via V-T collisional processes.

To numerically integrate these equations the quasiequilibrium population density variables VE<sub>i</sub> and TE<sub>i</sub> must be known at each instant of time. To do this the temperature as a function of total energy and number must be known. The following expressions are appropriate to the "thermal" model (see Appendix C).

For the upper system (A):

$$T = \frac{T_A}{-ln\left[\frac{1}{2(3-c)}\left[c-2+\sqrt{-3c^2+12c-8}\right]\right]}$$
$$T_A = 3365^{\circ}K$$
$$c = \frac{Total \ energy \ in \ (A) \ system}{(Energy \ spacing) \ (Total \ number \ in \ (A) \ system)}$$

The variable c is computed from occupation number densities obtained from the step by step numerical integration of equations (6).

For the lower system (B):

$$T = \frac{T_{B}}{-\ln \left[\frac{1}{(9-3c)}\left[(c-2)+\sqrt{2} \cdot \sqrt{4c-2.5-c^{2}}\right]\right]}$$
$$T_{B} = 960^{\circ}K$$

c = (Total energy in (B) system) (Energy spacing)(Total number in (B) system) .

#### 3.2 SOLUTIONS AND COMPARISONS

The major difficulty in relating the pulse shapes generated by computer to experimental pulses is the initial conditions. We must know what appropriate starting values to use for the vibrational and translational temperatures. Of importance to both is the time separation between the current discharge and laser pulses. The following temperatures were chosen for a 3%  $CO_2$ , 97% He mixture:

> T (Upper system) = 5000°K(Ref. 3) T (Lower system) = 1000°KT (Translational) = 400°K.

The relaxation times were taken to be:

 $\tau_{TA} = 12 \ \mu s$  (Ref. 5)  $\tau_{TB} = 1.2 \ \mu s$  $\tau_{VB} = \tau_{VA} = 50 \ \text{nsec.}$ 

The procedure for obtaining the initial population densities for the levels is to choose equal numbers in both upper and lower laser levels and use the vibrational temperatures to define the rest. This approach is experimentally justified by the fact that at high gain there appears to be no initial instantaneous gain in  $CO_2$ -He T.E.A. lasers. Knowing these initial occupation numbers and the time  $t_d$  to reach a photon density value of  $\rho_f$  completes the initial conditions necessary to begin the numerical integration. A computer generated high gain pulse is shown in Fig. 5 along with the experimental oscilloscope pulse trace. The falling portions match much better than for the case of the simple two level gain model. However the fact that the deviation is still sizeable indicates that either more levels are vibrationally coupled or that a better choice of initial vibrational temperatures is required. An experimental determination of these temperatures for high gains would help clear up this uncertainty.



Fig. 5 High gain pulses (28%/pass) generated from the "thermal" gain model and from experiment. The fall rate of the computer solution is closer to reality than was obtained from the simple two level model.

The effect of including vibrational coupling in the lower system is to help drain the lower laser level during the photon pulse buildup. In the upper system the coupled levels act as reservoirs to feed the upper laser level. Consequently a greater peak photon density is achieved while at the same time leaving less potential inversion in the upper laser level. At high gains and high pulse intensities this allows the laser pulse to die with a relaxation time which is close to the cavity ringing time  $\tau_c$ .

#### 4.0 CONCLUSIONS

The unsaturated gain behaviour in a CO<sub>2</sub>-He T.E.A. laser has been investigated through the use of computer and experimentally generated photon pulses. It was found that a simple two level gain model with rotational coupling incorporated was sufficient to fit pulses for low gain conditions. However at higher gains and intensities the fall times of the computer solutions were found to be much longer than experimentally observed. A more complicated "thermal" model accounting for vibrational coupling between levels was developed. This involved driving 6 vibrational levels towards dynamic Boltzmann distributions with a perturbation type driving force. The photon pulses for this gain model were of higher peak power and had fall times closer to experimental reality.

The model differs from that developed by Lyon et al (3) in the number of levels used. Twelve levels in total were used to describe the gain and pulse behaviour; four of which were symmetric-bend levels used to absorb energy from the asymmetric stretch mode. Since the time constant for decay via this process is relatively long (  $10-20 \ \mu sec$ ) and most high intensity pulses peak at a few microseconds after the discharge pulse, then the process is not expected to affect the lower system greatly. Its effect may be significant in describing the unsaturated gain behaviour.

The work presented in this paper has indicated that both rotational and vibrational coupling are important for understanding the dynamic behaviour of the CO<sub>2</sub>-He T.E.A. laser. More experimental studies are needed to determine the magnitudes of vibrational and translational temperatures and how they behave after initiation of the discharge pulse. In addition we require accurate data on the vibrational coupling rates and cavity loss as well as the parametric behaviour of the gain.

The parameters used in both gain models were the same and no attempt was made to make accurate fits due to lack of knowledge about the above parameters. Further work would involve a more accurate measurement of experimental parameters.

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#### REFERENCES

- 1. Garside, B.K., Ballik, E.A., Reid, J., J. Appl. Phys., vol. <u>43</u>, No. 5, 2387 (1972).
- Reid, J., Garside, B.K., Ballik, E. A., IEEE J. Quantum Elect., vol. 8, No. 5, 449 (1972).
- Lyon, D.A., M.I.T. QPR. (Electrodynamics of Media), vol. <u>4</u>, No. 103, 51 (1972).
- Hertzberg, G., "Molecular Spectra and Molecular Structure.
   II. Infrared and Raman Spectra of Polyatomic Molecules",
   Van Nostrand, Princeton (1960).
- 5. Reid, J., Garside, B.K., Private communication.
- Ralston, A., Wilf, H.S., "Mathematical Methods for Digital Computers", Wiley, New York/London (1960).
- Moore, C.B., Wood, R.E., Hu, B.L., Yardley, J.T.,
   J. Chem. Physics, vol. <u>46</u>, 227 (1966).
- 8. Robinson, A.M., D.R.E.V. Document TN-1989/71.
- 9. Gerry, E.T., Appl. Physics Letters, vol. 8, 227 (1966).
- 10. Chang, T.Y., Opt. Commun., vol. 2, 77 (1970).

APPENDIX A

# COMPILATION OF PARAMETERS PERTINENT TO THE CO2-He T.E.A. LASER

# (1) SPONTANEOUS TRANSITION PROBABILITY (A21)

Robinson (8) gives the following formulae to compute the spontaneous transition probability.

$$A_{21} = \frac{64\pi^4}{3h\lambda_0^3} \frac{|R_{12}|^2 s_J F_J}{g_2}$$

 $|R_{12}|^2$  - vibrational contribution to the transition moment  $S_J$  - rotational contribution to the transition moment  $F_J$  - interaction factor between vibration and rotation.

For the 00°l (J=19) to 10°0 (J=20) transition we have the following data.

 $R_{12} = (3.77 \pm .03) \times 10^{-20} \text{ esu}$   $F_J = 1 + .0022J \text{ for P branch}$   $S_J = J$   $g_2 = 2J' + 1$  J' = 19 = upper rotational laser level J = 20 = lower rotational laser level

The computed value for  $A_{21}$  of .200 sec<sup>-1</sup> gives a spontaneous relaxation time  $\tau_{21} = \frac{1}{A_{21}}$  of about 5 sec. This compares favourably with the experimentally determined value of 4.7±.5 sec obtained by Gerry (9)

## (2) FREQUENCY WIDTH FOR THE LINESHAPE FUNCTION $(\Delta v_{T})$

For pressure broadening Robinson gives for the half width at half maximum (HWHM) or  $\Delta v_L$  the following expression.

$$\Delta v_{\rm L} = \frac{22.79(3.585 - .0299 \text{J})(300)^{n/2}}{T^{(n+1)/2}} \left[ P_{\rm CO_2} + \frac{P_{\rm He}}{1.69} \right]$$

n = 0 for a  $T^{-1/2}$  dependence n = 1 for a  $T^{-1}$  dependence.

For our particular transition and mixture (2.8% CO<sub>2</sub>, 97.2% He) we have:

 $P_{CO_2} = 21.28 \text{ torr}$   $P_{He} = 738.72 \text{ torr}$   $T = 400^{\circ}K$ J = 20

This gives:  $\Delta v_{T} = [1.56, 1.35]$  GHz.

The greatest uncertainty in  $\Delta v_{\rm L}$  comes from not knowing the temperature dependence accurately enough. From this estimate of  $\Delta v_{\rm L}$  comes the Lorentz lineshape function weighting factor at  $\lambda_{\rm O} = 10.6 \ \mu$ .

$$g(v_0) = \frac{1}{\pi \Delta v_L}$$

Thus

$$g(v_0) \simeq [2.4 \times 10^{-10}, 2.0 \times 10^{-10}]$$
 sec.

For the 10.6  $\mu$  transition we have the following relations:

$$(2J+1)B_{12} = (2J'+1)B_{21}$$
 (Ref. 7)  
41  $B_{12} = 39 B_{21}$ 

A reasonable approximation is to make  $B_{12} = B_{21}$ . Also  $(B_{21} \text{ at } \lambda_0) = \frac{\lambda_0^3}{8\pi\hbar\tau_{21}}$ 

$$...B_{21} \simeq 1.43 \times 10^{15} \text{ cm/gm}$$

To obtain the integrated Einstein coefficient for stimulated emission at line centre we use the formula:

$$B = hv_0 B_{21} g(v_0) = \frac{g(v_0) \lambda_0^2 c}{8\pi\tau_{21}}$$
$$\approx [6.3, 5.5] \times 10^{-8} \text{ cm}^3 \text{.sec}^{-1}$$

# (4) ROTATIONAL POPULATION FACTOR $(\gamma_R)$

The distribution function for population densities amongst rotational levels within a vibration level is given approximately by the expression (Ref. 4):

$$\gamma_{R} = \frac{N_{J}}{N_{V}} = \frac{2hcB(2J+1)}{kT} e^{-BJ(J+1)hc/kT}$$

For the 00°l vibrational level: J = 19  $B = .3871404 \text{ cm}^{-1}$  (Ref. 10) For the 10°0 vibrational level: J = 20 $B = .390188 \text{ cm}^{-1}$  (Ref. 10)

(a) At T = 300°K: 
$$\frac{N_{19}}{N_{00}\circ 1} \simeq .0715$$
,  $\frac{N_{20}}{N_{10}\circ 0} \simeq .0699$ 

Thus  $\gamma_R \simeq .070$  can be taken as an average value for these two ratios.

(b) At T = 400°K: 
$$\frac{N_{19}}{N_{00°1}} \approx .0640$$
,  $\frac{N_{20}}{N_{10°0}} \approx .0638$ 

Thus  $\gamma_R \simeq .064$  can be taken as the average.

From the above computations we see that  $\gamma_{\rm R}$  is not very sensitive to "reasonable" temperature variations.

# (5) EFFECTIVE EINSTEIN COEFFICIENT FOR STIMULATED EMISSION (B<sub>EFF</sub>)

The integrated Einstein coefficient B, previously developed, was based on the assumption that the J' = 19 and J = 20 rotational laser levels were isolated. In reality they are coupled to other rotational levels within the particular vibrational level to which they belong (i.e. 00°l or 10°0). Since rotational-rotational and rotational-translational transfer mechanisms are expected to be much faster than V-V, V-T and stimulated emission rates at atmospheric pressure then it is reasonable to assume rotational and translational temperature equality. Thus assuming that this temperature remains relatively constant during the laser pulse then  $\gamma_R$  becomes constant. The stimulated rate term appearing in the rate equations can now be written in terms of vibrational population densities providing B is replaced by  $B_{eff} = \gamma_B B$ . Looking at equations (4) (b),(c) in section 2.2 of this paper one can see that since  $B_{eff} << B$  then the ability to perturb the unsaturated gain behaviour at a given photon density  $\rho$  is reduced.

At T = 400°K 
$$B_{eff} \simeq [4.5 \times 10^{-9}, 3.5 \times 10^{-9}] \text{ cm}^3 \text{ sec}^{-1}$$

#### (6) PEAK INVERSION

The internal power has been measured (Ref. 5) for high gain conditions of 30 kv applied voltage and a .01 µF capacitor. This yields a peak unsaturated gain of about 30% per pass (Ref. 1). The maximum internal power measured lies in the range [5,.5] M watts. An approximate estimate of the peak inversion can now be made.

Let

 $p_{max} \simeq 10^{6}$  watts (peak power)  $\rho_{max} \simeq \frac{n_{max}}{2}$  (reasonable at high gains)

 $A \simeq .33 \text{ cm}^2$  (cross sectional area of TEM<sub>00</sub> mode).

As a first approximation we may say:

$$n_{\max} \simeq \frac{2p_{\max}L}{c A h v_{d}} \simeq 2 \times 10^{16} cm^{-3}$$

Here  $n_{max}$  is the maximum inversion,  $p_{max}$  is the maximum pulse power, c is the speed of light, A is the cross sectional area of the cavity mode, h is Planck's constant and  $v_0$  is the laser transition frequency. The factor L/d is the ratio of cavity length to active discharge length.

As a comparison we can use the estimated peak gain to get a second value for peak inversion.

$$n_{max} = \left(\frac{c}{B_{eff}d}\right) g_{max}$$
  
d ~ 80 cm,  $B_{eff} \simeq 4 \times 10^{-9} \text{ cm}^3 \text{ sec}^{-1}$ ,  $g_{max} \simeq .3$ 

Thus

$$n_{max} \simeq 3 \times 10^{16} \text{ cm}^{-3}$$
.

# (7) <u>DENSITY OF CO<sub>2</sub> MOLECULES (N<sub>T</sub>)</u>

We assume that the discharge excites the gas mixture at constant volume so that a temperature correction factor is not needed in the following calculation.

For a 2.8%  $CO_2$ , 97.2% He mixture at atmospheric pressure and 400°K the density of  $CO_2$  molecules is:

 $N_{T} = \frac{(\text{Avogadro's number})(\text{CO}_{2} \text{ molecular fraction})}{(2.24 \times 10^{4} \text{ cm}^{3})}$   $\approx 7.5 \times 10^{17} \text{ CO}_{2} \text{ molecules/cm}^{3}.$ 

## APPENDIX B

## MODIFICATIONS TO THE I.B.M. S.S.P

SUBROUTINE "HPCG"

Due to the large magnitude ranges of the population and photon densities it was concluded that a weighted relative error criterion for changing step size was needed. The stability of the Hamming's predictor-modifier-corrector method (Ref. 6) was desired but unfortunately the S.S.P. HPCG subroutine uses a weighted absolute truncation error criterion.

The following modifications were made to HPCG:

16 DELT = DELT+AUX(15,I)\*ABS(1.-AUX(4,I)/Y(I)) HPCG 1690
209 DELT = DELT+AUX(15,I)\*ABS(AUX(16,I)/Y(I)) HPCG 2930

The input variable PRMT(4) now becomes the upper relative error bound. It should be noted that it is the user's responsibility to make sure that Y(I) does not become exactly zero. APPENDIX C

# TEMPERATURE FORMULAE FOR THE "THERMAL" GAIN MODEL

Consider the upper system (A) consisting of three asymmetric stretch levels as shown in Fig. 6.



Fig. 6

$$c = \frac{E_{tot}}{\Delta EN_{tot}} = \frac{(n_4 + 2n_5 + 3n_6)}{(n_4 + n_5 + n_6)}$$
$$= \frac{\frac{1 + 2e}{P_A + 3e} - 2T_A + 3e}{1 + e^{T_A + 3e} - 2T_A + 2E_A + 3E_A}.$$

Solving the quadratic yields:

$$T = \frac{T_A}{-\ln \left[\frac{1}{2(3-c)} \left[c-2+\sqrt{-3c^2+12c-8}\right]\right]}$$

 $T_{A} = 3365^{\circ}K$ 

The lower system (B) consists of the lowest bending and symmetric stretching levels. Degeneracy factors account for levels with small energy separations (Fig. 7).



$$c = \frac{E_{tot}}{\Delta EN_{tot}} = \frac{n_1 + 2n_2 + 3n_3}{n_1 + n_2 + n_3}$$
$$= \frac{1 + 4e^{-T_B/T} - 2T_B/T}{1 + 2e^{-T_B/T} + 3e^{-2T_B/T}}.$$

Solving the above for T gives:

$$T = \frac{T_B}{-\ln \left[\frac{1}{(9-3c)}\right] \left[(c-2) + \sqrt{2} \cdot \sqrt{4c-2.5-c^2}\right]}$$

.7.
## INTERNAL STRUCTURE OF T.E.A. LASER

#### DISCHARGES

by

Terence Tricker, B.Eng.

PART B: MCMASTER INDUSTRIAL PROJECT

A project \* report submitted in partial fulfillment of the requirements

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#### ABSTRACT

Recent developments in high power, high pressure gas lasers constitute a major breakthrough in laser technology. The principal problem in preventing the realization of their full potential is the lack of stability of such discharges. It is imperative to understand why their internal structure causes them to degenerate into arcs. We solve for the stationary internal structure of such discharges and find that it is dominated by the electron flux obtainable at the cathode surface and the form of the ionization coefficient.

## RESUME

Les progrès récents accomplis dans l'étude des lasers à giz à haute puissance et à haute pression constituent une solution soudaine à la technologie des lasers. Le problème principal qui empêche ces lasers de se réaliser pleinement réside dans le manque de stabilité de ces genres de décharges. Il est absolument essentiel de comprendre les raisons pour lesquelles leur structure interne cause la décharge à dégénérer en arc. En cherchant une solution pour la structure interne stationaire de ces décharges, nous trouvons que cette décharge est dominée par le flux d'électrons obtenus à la surface de la cathode et aussi à la forme du coefficient d'ionization.

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## 1.0 INTRODUCTION

The objective of this paper is to study the internal structure of high pressure gas discharges. The understanding of such structure is basic to the control of arcing phenomena so characteristic of these laser systems. To solve the arcing problem completely requires a full temporal solution which is, needless to say, quite difficult. It was decided instead to solve for the stationary homogeneous discharge in order to gain some insight into the processes involved.

Previous work on discharges (Ref. 1) has dealt mainly with the low pressure regime where diffusion and wall effects become important. At atmospheric pressure, diffusion effects are reduced to the point where the electrons and ions do not see the wall. For such discharges the equilibrating mechanism is recombination (Ref. 2) in the main volume. It is important to note that the recombination coefficient  $\alpha$  is relatively insensitive to the local electric field but the ionization coefficient Z is extremely dependent. In the main body it is the balancing of the ionization and recombination rates that allows equilibrum to establish.

The electrical characteristics of discharges are matched to the electrodes through a sheath (Ref. 1). At these boundaries the electron and ion fluxes readjust to form a region of high electric field. At the cathode surface, secondary emission of electrons is assumed to occur through ion bombardment or from external sources such as ultraviolet irradiation or electron gun bombardment. Little data is available on how processes such as photon bombardment from the main body or metastable bombardment effect the electron emission at the surface. Summarizing, we have an internal structure which

consists of the sheaths and a homogeneous discharge far from the electrodes where recombination and avalanche mechanisms balance.

In section 2.0 of this paper the equations describing the electric field and the ion and electron fluxes and densities are formulated and put in dimensionless form. Section 3.0 deals with the solution of these equations on a hybrid computer, the techniques used, and the problems encountered. In section 4.0 a discussion of the physical behaviour based on results of these solutions is presented.

## 2.0 FORMULATION OF EQUATIONS

The equations for the discharge model to be developed in this section are those proposed by T.S. Brown (2). The basic assumption for the mathematical formulation of this problem is the validity of the magneto-hydro-dynamic equations (M.H.D.). The discharge is treated as a mixture of an electron, ion and neutral (background gas) fluid. This requires in particular that the mean free path for electrons be much less than any characteristic gradient lengths such as those associated with the temperature, electric field, fluxes etc. It must also be much less than the physical dimensions of the enclosure for the gas. The following equations also assume that the variables are time independent and that the discharge is one dimensional.

#### 2.1 BASIC EQUATIONS

#### CONSERVATION OF PARTICLES

(1) 
$$\pm \frac{\partial \Gamma_{\pm}}{\partial x} = Z n_{\pm} - \alpha n_{\pm} n_{\pm} + S$$

(2)  $\int_{\pm}^{1} = n_{\pm} v_{\pm}$ 

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## CONSERVATION OF MOMENTUM

(3)  $v_{+} = \mu_{+} E$ 

## POISSON'S EQUATION

(4)  $\frac{\partial E}{\partial x} = \frac{4}{\epsilon_0} \left( n_+ - n_- \right)$ 

All quantities are scalars and the origin is assumed to be at the anode. The subscripts ± refer to ions and electrons respectively. The other symbols denote:

Е	electric field
ą	electron's charge
ε O	dielectric constant
n	particle density
α	recombination coefficient
Z	ionization rate
μ	particle's mobility
S	external source
v	drift velocity
γ	Second Townsend coefficient.

#### 2.2 BOUNDARY CONDITIONS

CATHODE:

The electron flux  $\Gamma$  at the surface of the cathode comes from secondary emission due to ion bombardment and, possibly, an external source  $\Gamma_{v}$ .

(5)  $\Gamma_{-} = \gamma \Gamma_{+} + \Gamma_{-x}$ 

The efficiency of electron emission (2ND Townsend coefficient) is denoted by  $\gamma.$ 

ANODE: At the anode there is no ion emission so that:

(6)  $\Gamma_{+} = 0$ 

There is also another integral condition since the voltage between anode and cathode is specified. However we assume that the typical sheath thickness is so small compared to anode-cathode separation that the bulk of the discharge is in equilibrium. This means that the two sheaths can be solved separately subject to the condition that variables go to their asymptotic values at infinity. Using these values the equations can be written in normalized variables which tend to unity "far away". Knowing a typical electrode separation  $x_o$  and applied voltage  $V_o$ , then a field  $E_o$  can be defined for the discharge. This, then, defines the equilibrium recombination coefficient  $\alpha_o$ and ionization coefficient  $Z_o$ . Possible scale lengths to be used include the avalanche length  $L_{\pm} = v_{\pm}$  and the shielding length  $L_s = \frac{\varepsilon_o}{q} \frac{E_o}{n_E}$ . The importance of the internal field distortion due to charge separation requires the use of  $L_c$ .

## 2.3 NORMALIZED SHEATH EQUATIONS

The normalized equations become (Ref. 2):

(7) 
$$\frac{\partial E'}{\partial x'} = n_{+}' - n_{-}'$$

(8) (a) 
$$\frac{\epsilon}{I_o} \frac{\partial \Gamma_+}{\partial x'} = \left(\frac{Z_o}{\alpha_o n_E}\right) Z n_- - \alpha' n_+ n_-' + \frac{S_o}{\alpha_o n_E}$$

(b) 
$$-\frac{1}{I_o}\frac{\partial \Gamma}{\partial x'} = \left(\frac{Z_o}{\alpha_o n_E}\right)Zn' - \alpha n_+n' + \frac{S}{\alpha_o n_E^2}$$

(9) 
$$\Gamma'_{\pm} = n_{\pm}' E'$$

## 2.4 SYMBOLS AND TYPICAL VALUES

The symbolism and typical values are as follows (Ref. 1,2)

(10)

(a) 
$$E_{o}$$
 EQUILIBRIUM ELECTRIC FIELD  $10^{4}V \text{ cm}^{-1}$   
(b)  $Z_{o} = Z_{o}(E_{o})$  IONIZATION RATE AT  $E_{o}$   $10^{8} \text{ sec}^{-1}$   
(c)  $\alpha_{o} = \alpha_{o}(E_{o})$  RECOMBINATION RATE AT  $E_{o}$   $10^{-7}\text{ cm}^{3}\text{ sec}^{-1}$   
(d)  $\varepsilon = \mu_{+}$  IMMOBILITY FACTOR  $10^{-2}$   
(e)  $n_{o} = \frac{Z_{o}}{\frac{C_{o}}{\alpha_{o}}}$  EQUILIBRIUM PARTICLE DENSITY  $10^{15} \text{ cm}^{-3}$   
(f)  $S^{1} = \frac{S}{Z_{no}}$  NORMALIZED VOLUME SOURCE

(g) 
$$n_E = \frac{n_o}{2} (1 + \sqrt{1 + 4S}^{+})$$
 EQUILIBRIUM PARTICLE DENSITY  
(h)  $I_o = \frac{\varepsilon}{\alpha} \frac{\alpha}{\mu_- q}$  DIMENSIONLESS RECOMBINATION  $10^{-4}$   
(i)  $L_s = \frac{\varepsilon}{\alpha} \frac{E}{q} = \frac{x}{x'}$  SCALE LENGTH  $10^{-3}$  cm  
(j)  $Z' = Z/Z_o$  NORMALIZED IONIZATION COEFFICIENT  
(k)  $\alpha' = \alpha/\alpha_o$  NORMALIZED RECOMBINATION COEFFICIENT  
(l)  $n_{\pm}' = \frac{n_{\pm}}{n_E}$  NORMALIZED PARTICLE DENSITY  
(m)  $E' = \frac{E}{E_o}$  NORMALIZED ELECTRIC FIELD  
(n)  $\Gamma_{\pm}' = \frac{\Gamma_{\pm}}{\Gamma_{\pm}}$  NORMALIZED FLUX.

## 2.5 NORMALIZED BOUNDARY CONDITIONS

The boundary conditions can be written in terms of these normalized variables:

# CATHODE:

 $\Gamma_{-}' = \epsilon \nabla \Gamma_{+}' + \Gamma_{-x}'$ (11) (a)

(b) 
$$\Gamma'_{x} = \frac{\Gamma_{x}}{\mu_{e}n_{e}E_{o}} = \frac{\Gamma_{-x}}{\Gamma_{o}}$$

Furthermore, Equations (8) a,b express the continuity of current throughout the discharge.

$$(12) \qquad \in \Gamma_{+}' + \Gamma_{-}' = 1 + \epsilon$$

Equations (11) and (12) yield the constraint on the electron flux at the cathode surface:

(13) 
$$\Gamma_{-}' = \frac{\gamma(1+\epsilon) + \Gamma_{-\chi}}{(1+\gamma)}$$

At the anode surface:

O

(14) 
$$\Gamma_+' =$$

#### 2.6 TRANSFORMED NORMALIZED EQUATIONS

For convenience in finding asymptotic solutions and for analog programming it was found useful to define new variables X, Y (Ref. 2) as: (15)  $E' = \mathscr{A}$ (16)  $\Gamma_{-}' = \mathscr{A}$ Using these variables the following equations for the anode result: (17)  $\underline{\epsilon} \quad \frac{\partial X}{\partial x} = \mathscr{A} \left(1 - \mathscr{A}\right) (1 + \epsilon)$ 

	6	0X			-X/2		Y	\ -X	· ····································
(18)	- <u></u> I	° <u>9 ×</u> , 9 ×,	kater Long	$\frac{2\varepsilon}{1+\sqrt{1+4s'}}$	Z'e	+ 01 (	e - (1+E)	)e +	$\frac{45 \text{ e}}{\left(1 + \sqrt{1 + 45'}\right)^2}$

The cathode equations are obtained by simply changing the sign of the derivatives. Equations (17) and (18) constitute the description of the internal structure of a stationary recombination controlled discharge. Other relevent formulas relating the computed variables X and Y to n\_', and n ' the ion and electron densities respectively are:

(19)  $n'_{+} = \pm \left(1 + \epsilon - \epsilon^{Y}\right) \epsilon^{-X/2}$ 

(20)  $h' = e^{\frac{y}{2}}$ 

It should be noted that in formulating all the previous equations the diffusion effects due to temperature and density gradients have been neglected along with "overshoot" effects (convective acceleration) due to high electric field gradients. Magnetic field and electron pressure terms have also been dropped from the momentum equation. For a detailed discussion on the justification for dropping these complicating terms see T.S. Brown Ref. (2).

2.7 PARAMETER DEPENDENCE ON ELECTRIC FIELD

In order to solve equations (17) and (18) for a given E and S' the the functional dependence of  $\alpha$ ' and Z' is needed. RECOMBINATION COEFFICIENT ( $\alpha$ ')

The recombination coefficient has the form (Ref. 2)

(21)  $\alpha' = (\varepsilon')^b$ ,  $b \sim \bot$ 

The above form for  $\alpha$ ' has been deduced from extrapolated experimental data on the N<sub>2</sub><sup>+</sup>ion (Ref. 3) and is assumed to hold for T.E.A. laser mixes. For  $1 \le E' \le 10$  then  $\alpha'$  can be taken as unity for all practical purposes.

#### IONIZATION COEFFICIENT (Z')

The form of the ionization coefficient Z' is known experimentally (Ref. 4) for a number of  $CO_2$  TEA laser mixes. For a 5%  $CO_2$ , 5%  $N_2$ , 90% He mixture a power law can be fitted accurately to the data (Ref. 2). For a pressure of 760 torr, and room temperature the range of validity for E is  $4 \times 10^3 \le 10^4 \text{ v/cm}$  which corresponds to a 3 orders of magnitude change in Z'. The power law fit can be written as:

(22) 
$$Z' = (E')^n$$

Unfortunately Carmichael (4) gives no data for the ionization coefficient behaviour for high E/P ratios. However numerous past experimental studies (Ref. 5) on pure gases have shown that a curve of the form given in FIG. 1 is representative.



FIG. 1 Variation of 1st Townsend Coefficient with electric field.

Here  $\frac{\alpha_t}{p}$  is the number of ion pairs created per cm travel per torr. Von Engel (5) fits regions of data for  $\alpha_t$  with the analytic expression:

(23) 
$$\frac{\alpha' t}{P} = A \mathcal{R}$$

This accounts for region (1) but not for the drop in region (2). A useful analytic form which allows for the latter region is:

(24) 
$$\frac{\alpha'_{t}}{P} = \frac{A \cdot e}{E}$$

The proportionality factor between  $\frac{\alpha_t}{P}$  and  $\frac{Z}{P}$  is the electron drift velocity v =  $\mu E$ .

Carmichael (4) chooses to fit equation (24) to his data yielding for the ionization coefficient Z (ion pairs/sec.) the form:

(25) 
$$\frac{Z}{P} = A \mathcal{L}$$

For a 5/5/90 mix 
$$A = 1.29 \times 10^7 \text{ sec.}^{-1} \text{ torr}^{-1}$$
,  $B = 53 \frac{v}{\text{cm-torr}}$ 

The other extreme would be to neglect the fall off in region (2) and use Equation (23) to give the expression:

(26) 
$$\frac{Z}{P} = AE \mathscr{L}$$

In summary the three types of analytic approximations to Z'(E') in normalized variables are:

(27) (a) 
$$Z' = (E')^n = Exp(\frac{n}{2}X)$$

(b) 
$$Z = E \cdot E \times P \left[ \frac{BP}{E_o} \left( 1 - \frac{L}{E} \right) \right] = E \times P \left[ \frac{X}{2} + \frac{BP}{E_o} \left( 1 - \frac{X/2}{2} \right) \right]$$

(c) 
$$Z' = ExP\left[\frac{BP}{E_o}\left(1-\frac{L}{E'}\right)\right] = ExP\left[\frac{BP}{E_o}\left(1-\frac{X/2}{2}\right)\right]$$

All three forms of Z' have their regions of applicability and must be fitted to data pertinent to those regions. Our concern with the dependence of Z' on the electric field will be understandable when we see that it is of critical importance in determining the sheath structure. MOBILITY:

The electron mobility  $\mu_{-}$  and ion mobility  $\mu_{+}$  have been assumed independent of electric field.

## 2.8 ASYMPTOTIC SOLUTIONS

In the application of reverse integration (SECTION 3.6) for the cathode case, the linearized or asymptotic solutions become important. The linearized avalanche and recombination coefficients will have the forms:

(28) (a) 
$$Z' = \varrho = 1 + C X$$
, C is a constant  
(b)  $\alpha' = \varrho = 1 + D X$ , D is a constant.

It can be shown that the asymptotic forms of Equations (17) & (18), with derivative signs reversed, yield a second order homogeneous equation for X:

(29) (a) 
$$\frac{\partial^{2} x}{\partial x^{\prime 2}} + A \frac{\partial x}{\partial x^{\prime}} + B x = 0$$
  
(b) 
$$A = -\frac{I_{o}}{\epsilon} \left(1 - \frac{45' \epsilon}{(1 + \sqrt{1 + 45'})^{2}}\right)$$
  
(c) 
$$B = -\frac{2I_{o}}{\epsilon} (1 + \epsilon) \left[\frac{2C - 1}{(1 + \sqrt{1 + 45'})} + (1 - D)\right]$$

Where typically: 
$$I_0 \sim 10^{-4}$$
,  $\epsilon \sim 10^{-2}$ ,  $D \sim 0$ 

The variable Y is related to X through the relation:

(30) 
$$Y = \frac{\epsilon}{2(1+\epsilon)} \frac{\partial x}{\partial x'}$$

In the following development the origin for the asymptotic equations is "far" from the cathode surface.

The general solution of Equation (29) (a) is:

(31) 
$$X = A_1 e^{k_1 X'} + A_2 e^{k_2 X'}$$

The constants  $A_1, k_1, A_2, k_2$  are determined by initial conditions  $\frac{\partial x}{\partial x'}\Big|_{x'=0}^{\prime}$  and  $X_o = X(x'=0)$ . The decay constants are the roots of the auxiliary equation:

2 %

(32) 
$$k^2 + Ak + B = 0$$

In order to constrain the solution to a simple exponential decay we must satisfy the equation:

(33) 
$$\frac{\partial X}{\partial x'}\Big|_{x'=0} = k, X_{o}$$

The parameter  $k_1$  is the negative root of Equation (32). Using Equation (33) the required relationship linking the initial starting values Y<sub>o</sub> and X<sub>o</sub> becomes:

## 3.0 SOLUTION OF THE SHEATH EQUATIONS

Assuming the power law dependence for Z' (n=7) and no volume source terms (S'=0), the following differential equations can be obtained from Equations (17) and (18).

3.1 ANODE EQUATIONS

$$(35) \qquad \underline{1} \quad \underline{\partial \chi} = 1.01 \quad \underline{e}^{-\chi} \left( 1 - \underline{e}^{\Upsilon} \right)$$

$$3\chi \qquad -\chi \left( \chi \right)$$

$$(36) - 100 \frac{\partial Y}{\partial x'} = .01 e + e (-1.01 + e)$$

The boundary conditions become:

(37) (a) 
$$Y_{o} = Y(x'=o) = ln(1+\epsilon) = ln(1.01) \sim .01$$

(b) 
$$Y(x'=\infty) = X(x'=\infty) = 0$$

## 3.2 CATHODE EQUATIONS

If the origin is placed at the cathode then the signs of the derivatives are changed in Equations  $(35) \notin (36)$ . That is:

(38) 
$$-\frac{1}{200} \frac{\partial x}{\partial x^{1}} = 1.01 \frac{e}{2} \left(1-\frac{Y}{2}\right)$$
  
(39) 
$$100 \frac{\partial Y}{\partial x^{1}} = .01 \frac{3x}{e} + \frac{-x}{e} \left(-1.01 + \frac{Y}{e}\right)$$

The boundary conditions are:

(40) (a) 
$$Y_o = Y(x'=o) = lm \left[ \frac{1.0(3 + \Gamma_{-x})}{(1 + 3)} \right]$$
  
(b)  $Y(x'=o) = X(x'=o) = 0$ 

Here  $\gamma$  is the secondary emission coefficient for electrons due to ion bombardment of the cathode and  $r_x$  is the normalized externally applied surface electron flux.

Other important relationships are:

(41)

(a)

$$n_{-}^{\prime} = \mathcal{A}$$

- (b)  $n_{+}' = \frac{1.01 \varrho}{01 \, \varrho}$
- (c)  $E' = e^{X/2}$
- (d)  $\Gamma' = e^{Y}$

## 3.3 PHASE PLANE SOLUTION

One approach to solving a non linear system such as this one is to obtain the phase plane solution in rough form. While it does not give the exact distance dependence of the variables it allows an estimate of the initial value,  $X_0$ , for satisfying the boundary conditions. Also the sensitivity and behaviour of solutions other than the desirable physical solution can be shown.

The phase diagram of the equations, (Ref. 2), relavent to both cathode and anode is shown in FIG. 2. The dotted lines indicate possible anode and cathode solutions satisfying the boundary condition  $X(\infty) = Y(\infty) = 0$ .





The curve in quadrant (1) is the only physically meaningful solution for the anode case since the quadrant (3) solution implies ion emission from the surface. Quadrants (2) and (4) are both physically meaningful cathode solutions. The former corresponds to an enhanced electric field or "positive sheath"while the latter implies that the externally applied surface flux  $\Gamma_{-x}$  has caused a depressed electric field or "negative sheath" The solutions obtained in this paper deal with only the positive sheath case. The splaying of the phase plane curves about the desired curve near the origin indicates that small errors in choosing the initial  $X_0$  will cause large deviations from the desired solution. This is especially true of the cathode solution with

no externally applied surface flux. Problems encountered with sensitivity eventually led to the adoption of a reverse integration scheme.

## 3.4 ANODE SOLUTION BY HYBRID COMPUTER

The anode case was solved on an EAI 8845 hybrid computer with the analog section performing the integration and the digital performing the control, the sampling of the solution and its subsequent storage and plotting. MAGNITUDE SCALING (Ref. 6):

In order to program differential equations on an analog computer it is necessary to scale the variables so that the output of amplifiers involved in the circuit never exceeds one machine unit or 100 volts. Also the full dynamic range of the computer (1 part in  $10^4$ ) should be employed to reduce loss of accuracy. This means that the maximum values of variables that could occur during the integration should be normalized to one machine unit or less.

There is also one other magnitude scaling problem in our case and this involves the exponential function generator. Since an amplifier is used, the maximum value becomes one machine unit and we are limited to producing the curve  $e^{-10u}$ . The variable u is normally required to lie between 0 and 1 unless hard zero diode limiting is employed (Ref. 7). To generate a curve of the form  $e^{6u}$  we generate  $e^{-6(1-u)}$  and adjust the equation coefficients to absorb a factor of  $e^{6}$ . From the phase plane and asymptotic solution it was found that the required  $X_0$  should be close to 1.5 (Ref. 2). The maximum value for Y is .01 and occurs at the anode surface. From these maximum values the computer variables U and V were defined to be:

(43) 
$$V = 50Y$$

Also in Equations (35) & (36), the fact that Y is so small means that e<sup>Y</sup> can be expanded as 1+Y for all practical purposes. The contribution of the next expansion term lies outside the accuracy range of analog components. This linear expansion also by-passes the situation of subtracting two almost equal numbers in Equation (35).

Taking into account all the magnitude scaling and the linear expansion of the exponential we obtain the equations actually programmed on the analog computer.

(44) 
$$\frac{\partial U}{\partial x'} = -2.02 \text{ e}^{-2U} \text{ V}$$
  
 $\frac{\partial V}{\partial x'} = -2.017 \text{ e}^{-6(1-U)} -.005(2V-1) \text{ e}^{-2U}$   
(45)  $\frac{\partial V}{\partial x'} = -2.017 \text{ e}^{-0.005(2V-1)} \text{ e}^{-2U}$ 

The analog circuit for the anode equations is shown in FIG. 3 in which standard analog component schematics (Ref. 6) are used. In the initial set up stage and debugging of the circuit it was found useful to control the



FIG. 3 Analog circuitry for solving the anode equations.

repetitive generation of the solution through the hardware "REP-OP" timer (Refs. 6,7). Remembering that time represents distance on the analog computer, it was observed that using a 1 second integration time constant made it necessary to integrate for 20 seconds in order to satisfy the boundary conditions. Integration for such a long time introduces problems in linearity for the integrator capacitors. The solution was speeded up by a factor of 100 by choosing a .01 second integration time constant which effectively increased the input gain to the integrators. Now instead of requiring a storage scope the solution could be swept every 200 milli-seconds on a regular scope.

The potentiometer representing U<sub>o</sub> was varied until both U and V were close to zero "far away" from the anode. The solution however was very sensitive to this initial condition pot setting. Varying its value by 1 part in 10,000 was sufficient to splay the solution considerably from the desired one. Near the critical point the amplifier noise introduced at the start of the integration caused repetitive cycles of the solution to lie within a band.

This is shown in FIG. 4 with the desired solution represented by the dotted line. A similar situation occurred for the V curve.



FIG. 4 Splaying of anode solutions due to noise.

Because of this splaying on to non desired solutions many component amplifiers were sent into saturation. To reduce amplifier recovery time it was necessary to diode limit their outputs as shown in FIG. 3.

At this point a FORTRAN IV program was written using hybrid linkage routines (Ref. 8) to enable digital control of the initial condition - operate cycle as well as performing optimization and sampling. Since noise generated all possible solutions in the splayed region it was then simply a matter of sampling the solution and choosing the best. Within the digital computer, U and V were converted to X and Y and using Equations (41) a,b,c the desired normalized ion density  $(n_+')$ , normalized electron density  $(n_-')$  and normalized electric field (E')were generated. These latter variables could then be displayed on the X-Y plotter and storage scope (Ref. 9) or hard copied on line printer or magnetic tape.

For high quality plotting the data was fed into an EAI 640 computer program that produced a plotting tape for the EAI 430 incremental plotter. In addition to plotting the curve the plotter has a printing head to type titles and scale markings. Such plots of the analog generated anode variables are shown in FIGS. 5,6, and 7 for a power law (n=7) Z' dependence on electric field.



FIG. 5 The electric field in the anode sheath assuming a power law dependence for the ionization coefficient (i.e. Z'=(E')<sup>7</sup>). The equilibrium density is represented by the dotted line at 1.0.



FIG. 6 The electron density in the anode sheath assuming a power law dependence for the ionization coefficient (i.e. Z'=(E')<sup>7</sup>). The equilibrium density is represented by the dotted line at 1.0.



FIG. 7 The ion density in the anode sheath assuming a power law dependence for the ionization coefficient (i.e. Z'=(E')<sup>7</sup>). The equilibrium density is represented by the dotted line at 1.0.

#### 3.5 CATHODE SOLUTION - ITERATIVE FORWARD INTEGRATION

Initially an analog solution was attempted for the cathode case with no externally applied surface flux ( $\Gamma_{-x}$ '=0) and a 2nd Townsend coefficient of 0.1 . The boundary conditions were:

(46) 
$$Y(x'=0) = Y_0 = lm\left(\frac{1+\epsilon}{1+\frac{1}{2}}\right) = lm\left(\frac{1\cdot01}{11}\right) \sim -2.4$$

From phase plane and asymptotic solutions it was found (Ref. 2) that X<sub>o</sub> should be about 4.04. The fact that avalanching involves an exponential dependence on X resulted in enormous gradient variations in the Y derivative. As a result it was found that only 1 normalized distance unit of the solution could be generated by the analog because of sensitivity and dynamic range problems. Much of the dynamic range problem centered on generating the exponential X dependence of the ionization coefficient in Equation (39). In order to generate this term over the full sheath length the integration was stopped by putting the computer into "HOLD". Then the computer variables and the ionization term were rescaled and the solution continued from that point. However the sensitivity to the noise introduced at the beginning of integration plus loss of significant figures during the solution, plus errors introduced by stopping and "HOLDING", hopelessly masked the solution behaviour beyond 1 distance unit.

At this point it was deemed necessary to resort to digital numerical integration techniques on the Xerox SIGMA 7 and EAI 640 computers. A fourth order double precision Runga Kutta method proposed by GILL (10) was employed. The analog value for  $X_0$  obtained previously and accurate to 4 decimal places

was used as a starting value. The digital approach removed the dynamic range problem but the extreme sensitivity to  $X_0$  remained. One required a very small step size in the independent variable x' to reduce truncation errors and needed fine tuning in the 8th or 9th decimal place of  $X_0$  to get a "good" solution. This iterative approach to the correct solution was very tedious and sensitive to initial conditions and step size.

## 3.6 CATHODE SOLUTION - NON-ITERATIVE REVERSE INTEGRATION

From the phase plane sketch in FIG. 2 one can see that all solutions tend to funnel in on the desired physical cathode solution for large negative values of Y (i.e. small electron flux values). When starting in this closely packed branch region even very tiny fluctuations can cause large deviations in the splaying region around X = Y = 0. It seemed logical that the equations would behave in a more stable manner if integrated in the reverse direction since any small errors made would tend to balance out and bring you back to the physically meaningful curve. In addition, using the asymptotic solutions to define the initial conditions, the need to iterate is removed.

Using the general expressions for the asymptotic equations developed in section 2.8, (Equations (29) - (34)), with C = 3.5, D = 0, and S' = 0, we obtain the relations:

(47) (a) 
$$\frac{\partial^2 X}{\partial x'^2} - .01 \frac{\partial X}{\partial x'} - .0808 X = 0$$
  
(b)  $k_1 = \frac{.01 - \sqrt{.3233}}{2} \sim .279$   
(c)  $X_0 = \frac{202 Y_0}{k_1}$ 

The above constraints force the solution to be a simple exponential decay,  $EXP(k_1x')$ . The value of  $Y_0$  and thus  $X_0$  must be chosen small enough to ensure that these asymptotic solutions are describing the behaviour in this region. To apply reverse integration the independent variable x' is replaced by -x' in Equations (38) and (39). Note that although the new equations become formally the same as the anode equations the behaviour is different because Y is negative. The solutions obtained by reverse integration were checked against the sensitive forward integration and found to be the same. Also the solutions are relatively insensitive to pertubations in initial conditions and truncation errors as predicted.

Another point to be considered with this integration procedure is the fact that there is no obvious origin. Where one places the cathode electrode depends on what the surface coefficient  $\gamma$  is and what external electron flux sources  $\Gamma'_{x}$  are applied. The curves for the different variables can be  $\Gamma'_{x}$  considered universal for a given recombination ( $\alpha$ ') and avalanching (Z') dependence. For the purpose of referencing, an arbitrary origin was chosen to be the point at which the asymptotic solution generated backwards produced a doubling in E'. The resulting computer plots of the cathode variables appear in FIGS. 8, 9, 10 and 11 for a power 1aw, (n=7), Z' dependence on electric field.

Using equations (17)  $\xi$  (18) it is a simple matter to extend the solution generation to include cases where volume sources (S') and other forms of ionization dependence Z'(E') become appropriate. The cathode solutions for two cases of special interest to T.E.A. laser discharge research are given in



FIG. 8 The electric field in the cathode region assuming a power law dependence for the ionization coefficient (i.e. Z'=(E')). The equilibrium density is represented by the dotted line at 1.0. The origin is defined as the point where E'=2, based on asymptotic solutions.



FIG. 9 The electron density in the cathode region assuming a power law dependence for the ionization coefficient (i.e. Z'=(E')<sup>7</sup>). The equilibrium density is represented by the dotted line at 1.0. The origin is defined as the point where E'=2, based on asymptotic solutions.



FIG. 10 The ion density in the cathode region assuming a power law dependence for the ionization coefficient (i.e. Z'=(E')<sup>7</sup>). The equilibrium density is represented by the dotted line at 1.0. The origin is defined as the point where E'=2, based on asymptotic solutions.



FIG. 11 The electron flux in the cathode region assuming a power law dependence for the ionization coefficient (i.e. Z'=(E')<sup>7</sup>). The equilibrium density is represented by the dotted line at 1.0. The origin is defined as the point where E=2 based on asymptotic solutions.

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Appendix A. These correspond to the electron gun assisted discharge system and the "double-discharge" system which operate at E/p ratios of about 4 and 8 respectively. For these solutions, a Z' dependence defined by equation (27)(b), with A=1.29  $\times 10^7$ , B=53, was used.

#### 4.0 DISCUSSION

The most obvious conclusion that can be drawn from the computer solutions is that the sheaths are much more violent in the region of the cathode than for the anode. Near the cathode, ions tend to collect while secondary electrons produced are swept out leaving a large positive space charge layer and thus a large enhanced electric field. The key to sheath formation near the cathode is the immobility factor  $\varepsilon = \mu_{\perp}$ 

The anode curves generated are independent of the electrode material. Here we assume that secondary electrons which are emitted by electron bombardment do not seriously affect the sheath. The boundary condition is dependent only on  $\varepsilon$  which is a property of the gas and is usually close to .01 .The solution has been generated by an analog iterative method but could have been produced from asymptotic solutions using reverse integration.

The cathode solutions are universal for a given Z'(E'). That is for a given gas mix the curves apply to all electrode materials and all applied surface fluxes. Where one places the cathode boundary depends on the 2nd Townsend coefficient ( $\gamma$ ) and what externally applied surface flux ( $\Gamma_{-}'$ ) is used. An investigation into the effects of these two parameters on the cathode position relative to the curves was made. Rearranging the boundary condition Equation (13) yields  $\gamma$  as a function of the computer generated

 $\Gamma$  and chosen parameter  $\Gamma$  '. The equation becomes:

(48)

$$\gamma = \underline{\Gamma' - \Gamma'},$$

$$(1 + \epsilon) - \overline{\Gamma'}$$
FIG. 12 shows the behaviour of  $\gamma = \gamma (x', \Gamma'_{x})$  and reveals some interesting results. The curves have been generated for a range of  $\gamma$ between 0 and about 1 because most common laser materials have relatively poor secondary emission coefficients. With no externally applied surface flux a variation of  $\gamma$  over the range shown yields little displacement along the curves in FIGS. 8,9,10. That is the electric field varies little from going from the poorest electron emitter to a 1 electron - 1 ion emitting surface. Of course the electron flux at the surface, which follows  $\gamma$  closely, varies greatly in magnitude.

When external surface flux is applied the ability to reduce the electric field of the sheath depends on the closeness of  $\Gamma_{x}$  to unity. The curves in FIG. 12 not only translate along the x' axis but also their slope decreases making the surface  $\gamma$  more important. In theory when  $\Gamma_{x}$  becomes equal to  $\Gamma_{x}$  then sheath disappears and the whole discharge is homogeneous. For larger values of surface flux the sheath will actually depress; that is the electric field E' will drop below the main discharge value E.

It must be stressed that the changing of the problem from the two sided boundary condition (ANODE-CATHODE) to a simple initial value problem (REVERSE INTEGRATION) requires that there exist a large equilibrium region whose field is equal to the applied field  $E_0$ . If one cannot achieve a reasonably depressed cathode surface flux corresponding to a given  $\gamma$  and  $\Gamma_x$  in a distance small compared to the electrode separation then the original problem reasserts itself.





The form of the ionization coefficient dependence on E'is of critical importance in determining the sheath length. The best way to see the effects is through a graphical qualitative presentation. We assume that there is no volume ionization present (S'=0). Using equations (17) & (18) it can be shown that the dominate derivative terms for most of the sheath structure are:

(48) 
$$\frac{\partial E'}{\partial X'} \sim -Z_o \frac{(I - \Gamma')}{E'}$$

(49) 
$$\frac{\partial \ln \Gamma}{\partial \chi'} \sim \frac{Z}{E'} \sim \alpha_{t}(E')$$

The equations are written in terms of true distance x since units of x' depend on  $E_o$ . We already know the experimental form for  $\alpha_t(E')$  shown in FIG. 1 and discussed in section 2.7. As mentioned before in this section, three types of analytic expressions can be made to represent  $\alpha_t$  for certain ranges of E. We wish to obtain a graphical "feel" for what happens when  $E_o$ is changed. FIGS. 13 & 14 give the qualitative dependence of  $\alpha_t$  and  $\ln(\Gamma_{-}')$ as a function of E'. It is where you start on this curve (i.e. what applied field  $E_o$  is chosen) that determines how quickly you can achieve a given surface flux from the asymptotic starting values. For most practical laser discharges,  $E_o$  lies within the power law approximation to  $\alpha_t$ . The fact that  $\alpha_t$  varies over three orders of magnitude when  $E_0$  decreases by a factor of 2.5 means that sheath lengths and peak electric fields are critically sensitive to the operating field.







FIG. 14 A sketch of the electron flux behaviour as a function of electric field.

One can divide up the  $\alpha_{\star}$  curve into four regions as follows:

- (1) Low field region
- (2) Operating region
- (3) Saturation region
- (4) Fall off region

In region (1) the electrons have so little energy that those ionizations that occur come from electrons very far in the tail of the semi-Maxwellian energy distribution curve. A stationary recombination controlled discharge operated in this region will be very weak and will tend to have long, low field sheaths.

In region (2) with fields typically from 2 x  $10^3$  V/<sub>cm</sub> to 2 x  $10^4$  V/<sub>cm</sub> we have high enough ionization rates to suitably excite T.E.A. laser mixtures. Here the Z or  $\alpha_t$  curves can be fitted to a power law in E'. If the sheath fields remain in this region then the computer curves in FIGS 8,9,10,11 are representative. Operating at the upper end will result in sheaths whose fields will enter region (3) and possibly (4). The sheaths become very short and violent in this case.

Present lasers do not operate with  $E_0$  in regions (3) and (4) but if they did presumably long high field sheaths would result.

The effects of operating at a lower discharge voltage and using an external ionization source can be seen by comparing the solutions in Appendix A

for E/p ratios of 4 and 8. The form of the ionization dependence was taken to be that of equation (27)(b). Note that these curves are plotted against normalized distance x' and not true distance x. One interesting point to note is the shape of the ion density curve. Approaching the cathode from "far away" we encounter an initial depression due to the electric field gradient. Eventually ionization begins to dominate to such an extent that a peak in the ion density occurs. Finally because the ions are now carrying almost all of the fixed total current but the electric field continues to rise, we enter a fall off region. Another useful observation is that the electron density tends to make a much sharper transition as the applied voltage is increased.

The addition of volume sources into the discharge results in an increase in the cathode sheath field. At least from a naive point of view one would think that the concomitant increase in the excitation rate of the neutral background gas would result in a faster "arcing time". However to compare the effects of volume terms on a laser discharge system we should compare on the basis of equal excitation rates. There will be a trade off between the decrease in the time to arcing due to the increased sheath field and the decreasing in the pumping time.

As a final point it should be noted that the solutions at an E/p ratio of 8 have sufficiently large convective acceleration terms to warrant questioning the validity of the model. This is especially true of the case with volume excitation.

## 5.0 CONCLUSIONS

The behaviour of a system of equations describing the internal structure of a plasma in a T.E.A. laser discharge was investigated. In addition to the use of conventional analog and digital computer techniques, a reverse integration, non-iterative method, which is based on the properties of an asymptotic solution was developed.

It was shown that the solution of the equations, which predict the electron and ion densities and electric field strength in the region near the cathode, could be expressed as universal curves. They apply for any cathode secondary emission factor or enhanced surface flux at the cathode.

Results show the electric field solution to be insensitive to the cathode secondary emission factor, but very sensitive to the enhanced cathode surface flux when this term becomes comparable to the flux in the equilibrium region of the discharge.

It was concluded that the internal structure or sheath depends greatly upon the shape of the ionization coefficient and the magnitude of the operating electric field in the main body of the discharge. The former is influenced by the component gas ratios in the mixture while the latter is defined by the applied voltage. Low applied voltages result in a weak discharge with long, low field sheaths while high voltages produce short violent sheaths. Very high voltages may result in a long, high field, sheath structure.

Stationary solutions corresponding to the electron gun assisted discharge and "double-discharge" systems were investigated. The addition of volume source excitation, while increasing the electric field in the sheath, also reduced the pumping time for equivalent excitation. It follows that an analysis of the stability of the stationary solutions is required before the beneficial influence of the external source can be known. The solutions developed in this paper provide a foundation for further investigation to obtain an understanding of the dynamic arcing problem.

For some discharges, probably not those best suited for pumping a T.E.A. laser, our model should be extended to include other physical effects such as convective acceleration. A final description of sheath dynamics would also have to account for three dimensional effects.

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## REFERENCES

- NASSER, E., "FUNDAMENTALS OF GASEOUS IONIZATION AND PLASMA ELECTRONICS". WILEY INTERSCIENCE (1971).
- 2. T.S. BROWN, PRIVATE COMMUNICATION.
- 3. BIONDI, M.A, CAN. J. CHEM. 47, 1711 (1969).
- 4. C.H.H. CARMICHAEL, REPORT 442, AUSTRALIAN DEFENCE SCIENTIFIC SERVICE, DEFENSE STANDARDS LABORATORIES.
- 5. VON ENGEL, A., "IONIZED GASES", OXFORD (1965).
- "HANDBOOK OF ANALOG COMPUTATION", ELECTRONIC ASSOCIATES, INC., PRINCETON, NEW JERSEY, PUBL. NO. 00800.0001-4, MAY 1971.
- "SYSTEMS REFERENCE HANDBOOK", ELECTRONIC ASSOCIATES, INC., PRINCETON, NEW JERSEY, PUBL. NO. 00 800.1155-1, MAY 1971.
- "HYBRID LINKAGE SUBROUTINES", ELECTRONIC ASSOCIATES, INC., PRINCETON, NEW JERSEY, PUBL. NO. 00 827.8611-3, MAY 1970.
- 9. J.F. BOULTER, PRIVATE COMMUNICATION.

10. J. NOEL, PRIVATE COMMUNICATION.

## APPENDIX A

Included in this appendix are the computer solutions for the electron gun (E/ $_{\rm p}$  = 4) and "double-discharge" (E/ $_{\rm p}$  = 8) systems. Each case is illustrated with and without volume source excitation. The ionization dependence was taken to be of the form Z/ $_{\rm p}$  = AE exp(-BP/ $_{\rm E}$ ) with A=1.29x10<sup>7</sup>, B=53. On each plot the equilibrium value is indicated by a dotted line. The origin is defined to be the point at which E'=2 based on asymptotic solutions.



FIG. 15 The variation of electric field in the cathode sheath for an E/p ratio of 4 with no volume sources (S'=0). This corresponds to field conditions for the electron gun assisted discharge system in which the gun is turned off.

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FIG. 16 The variation of electron density in the cathode sheath for an E/p ratio of 4 with no volume sources (S'=0). This corresponds to field conditions for the electron gun assisted discharge system in which the gun is turned off.



FIG. 17 The variation of ion density in the cathode sheath for an E/p ratio of 4 with no volume sources (S'=0). This corresponds to field conditions for the electron gun assisted discharge system in which the gun is turned off.



FIG. 18 The variation of electron flux in the cathode sheath for an E/p ratio of 4 with no volume sources (S'=0). This corresponds to field conditions for the electron gun assisted discharge system in which the gun is turned off.



FIG. 19 The variation of electric field in the cathode sheath for an E/p ratio of 4 with volume sources (S'=10<sup>3</sup>) added. This corresponds to a typical electron-gun assisted discharge system.



FIG. 20 The variation of electron density in the cathode sheath for an E/p ratio of 4 with volume sources (S'=10<sup>3</sup>) added. This corresponds to a typical electron-gun assisted discharge system.



FIG. 21 The variation of ion density in the cathode sheath for an E/p ratio of 4 with volume sources  $(S'=10^3)$  added. This corresponds to a typical electron-gun assisted discharge system.



FIG. 22 The variation of electron flux in the cathode sheath for an E/p ratio of 4 with volume sources (S'=10<sup>3</sup>) added. This corresponds to a typical electron-gun assisted discharge system.



FIG. 23 The variation of electric field in the cathode sheath for an E/p ratio of 8 with no volume sources (S'=0). This corresponds to field conditions present in a "double-discharge" type system with no volume excitation.



FIG. 24 The variation of electron density in the cathode sheath for an E/p ratio of 8 with no volume sources (S'=0). This corresponds to field conditions present in a "double-discharge" type system with no volume excitation.



FIG. 25 The variation of ion density in the cathode sheath for an E/p ratio of 8 with no volume sources (S'=0). This corresponds to field conditions present in a "double-discharge" type system with no volume excitation.



FIG. 26 The variation of electron flux in the cathode sheath for an E/p ratio of 8 with no volume sources (S'=0). This corresponds to field conditions present in a "double-discharge" type system with no volume excitation.



FIG. 27 The variation of electric field in the cathode sheath for an E/p ratio of 8 with volume sources (S'=10<sup>3</sup>) added. This corresponds to a "double-discharge" type system with volume excitation.



FIG. 28 The variation of electron density in the cathode sheath for an E/p ratio of 8 with volume sources (S'=10<sup>3</sup>) added. This corresponds to a "double-discharge" type system with volume excitation.



FIG. 29 The variation of ion density in the cathode sheath for an E/p ratio of 8 with volume sources  $(S'=10^3)$  added. This corresponds to a "double-discharge" type system with volume excitation.



FIG. 30 The variation of electron flux in the cathode sheath for an  $E_p$  ratio of 8 with volume sources (S'=10<sup>3</sup>) added. This corresponds to a "double-discharge" type system with volume excitation.

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