QF-1 RINGS

QF-1 RINGS

By

PETER KWOK WA TO, B.SC.

A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

May 1973

MASTER OF SCIENCE (1973)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario.

TITLE: QF-1 Rings

AUTHOR: Peter Kwok Wa To, B.Sc. (Acadia University)

SUPERVISOR: Professor B. J. Mueller

NUMBER OF PAGES: vii, 51

SCOPE AND CONTENT: A ring R is said to be QF-1 if every finitely generated faithfull R-module has the double centralizer property (or is balanced). A neccessary and sufficient condition for an artinian ring to be QF-1 is given. The class of QF-1 rings properly contains the class of QF rings and this is shown by an example. Several constructions of modules which are not balanced are collected. Finally, the structure of artinian local QF-1 rings which are finitely generated over their centers is gotten. This is a generalization of theorems of Floyd, and, Fuller and Dickson.

ACKNOWLEDGEMENT

The author would like to express his sincere appreciation to Dr. B.J. Mueller, without whose valuable assistance and advice this work would not have been possible.

TABLE OF CONTENTS.

PREFACE

R.M. Thrall and C.J. Nesbitt [10] proved that every faithful module over a quasi-Frobenius (QF) ring is balanced (i.e. has the double centralizer property). Later Thrall [9] gave an example, which is presented in Chapter II, of a ring over which every faithful finitely generated module is balanced, but which is not QF. He called the above class of rings QF-1 rings and posed the problem of classifying them in terms of their structure.

We shall first give, in Chapter I, a result of Morita [7] which relates the concepts of generation and co-generation to that of the double centralizer property and establishes a necessary and sufficient condition for an artinian ring to be QF-1.

In Chapter III, we have collected several constructions of modules which are not balanced. These constructions are employed in the proofs of several lemmas which lead eventually to establish the main theorem in Chapter IV which describes the structure of certain local QF-1 rings and can be considered as a partial solution to Thrall's problem. These results are due to V. Dlab and C.M. Ringel [3].

PRELIMINARIES

We shall denote by R an associative ring with identity and ${}_RM$ or M_R a left or right unital R-module respectively. If M is an R-module, then the (Jacobson) radical of M, denoted by radM, is the intersection of all the maximal submodules of M. The (Jacobson) radical of a ring R is defined as the radical of the R-module ${}_RR$ (or equivalently R_R). The dual notion of the radical is the socle. Denoted by SocM, the socle of an R-module M is the sum of all the minimal submodules of M. Hence if we consider the ring R as a left (right) R-module, we get the concept of the left (right) socle of R. The left and the right socle of R are two-sided ideals of R.

R is called a local ring if R satisfies one of the following equivalent statement:-

(1) R/radR is a division ring.

(2) R has exactly one maximal one-sided ideal.

(3) All non-units of R are contained in a proper ideal.

R is right perfect if and only if R/W is semisimple and every non-zero left module has a non-zero socle. By a perfect ring we shall mean a ring which is both left and right perfect. If M has a composition series, denote by $\partial(M)$ its length. In case R is a two-sided artinian ring, we can speak of the left length $\partial({}_RR)$ and right length $\partial(R_R)$ of R.

Let S be a subset of R. The left annihilator l(S) of S is defined as $l(S) = \{ a \in R : aS = 0 \}$, whereas the right annihilator $r(S)$ is given by $r(S) = \{ a \in R : Sa = 0 \}$. A two-sided artinian ring R is called a quasi-Frobenius (QF) ring if

$$l(r(L)) = L \quad \text{and} \quad r(l(J)) = J$$

for every left ideal L and right ideal J in R. An artinian ring R is QF if, and only if $_R R$ is injective.

Every left R-module M is a right C-module, where C is the endomorphism ring of $_R M$ (also called the centralizer of $_R M$). The double centralizer D is the endomorphism ring of M_C. The map $\phi : R \longrightarrow D$, defined by $\phi(a) = a_L$ where $a_L(m) = am$ for every $a \in R$ and $m \in M$, is a ring homomorphism. If this homomorphism is surjective, then M is said to have the double centralizer property, or to be balanced.

In [10] R.M. Thrall and C.J. Nesbitt showed that every faithful module over a QF ring has the double centralizer property. R.M. Thrall later introduced the concept of QF-1 rings which is a generalization of QF rings. R is said to be a left (right) QF-1 ring if every finitely generated faithful left (right) R-module has the double centralizer property. He also defined a QF-3 ring to be an artinian ring R which has a unique minimal faithful module. A faithful R-module M is said to be a minimal faithful module if the deletion of any nonzero direct summand of M leaves a nonfaithful module.

Let U and V be R-modules. U is said to generate V if

$$V = \sum \{ \operatorname{Im} \alpha : \alpha \in \operatorname{Hom}_R(U,V) \} \quad ,$$

or equivalently, if V is isomorphic to a factor module of a direct sum of copies of U. Furthermore, U is said to co-generate V if

$$\bigcap \left\{ \ker\phi : \phi \in \mathrm{Hom}_R(V,U) \right\} = 0 ,$$

or equivalently, if V is isomorphic to a submodule of a product of copies of U.

CHAPTER I

DOUBLE CENTRALIZER PROPERTY AND ARTINIAN QF-1 RINGS

We first prove the following theorem (K. Morita [7], theorem 1.1) which relates the notion of generation and co-generation to that of the double centralizer property.

THEOREM I.1. Let R be a left artinian ring and U a faithful finitely generated left R-module having the double centralizer property. Let V be an indecomposable finitely generated left R-module and $W = U \oplus V$. Then the following statements are equivalent:

(1) W has the double centralizer property.

(2) U generates or co-generates V.

PROOF. (1) $\Longrightarrow$ (2)

Denote the R-endomorphism ring of W by C, and assume that U neither generates nor co-generates V. Let $\lambda : U \oplus V \longrightarrow U$ and $\mu : U \oplus V \longrightarrow V$ be the natural projections. Then we have

$$B = \mathrm{Hom}_R(U,U) = \lambda C \lambda$$

and

$$D = \mathrm{Hom}_R(V,V) = \mu C \mu \ .$$

Write

$$V_1 = \sum \{\mathrm{Im}\,\alpha : \alpha \in \mathrm{Hom}_R(U,V)\}$$

and

$$V_o = \bigcap \{\mathrm{Ker}\,\gamma : \gamma \in \mathrm{Hom}_R(V,U)\} \ .$$

Then with D as the right operator domain of V we obtain the bimodules ${}_AV_{1\,D}$ and ${}_AV_{0\,D}$. Now since V is finitely generated and indecomposable as a left R-module and by Fitting's Lemma , D is a local ring with nilpotent radical N. If we set

$$V_1^* = V_1 + VN,$$

V_1^* becomes an R-D bimodule. We note that $V_1^* \neq V$. For suppose otherwise, let

$$V = V_1 + VN.$$

Then there exists $k \geqq 1$ such that

$$V = V_1 + VN^k$$

and

$$V_1 + VN^{k+1} \subsetneqq V.$$

Multiplying by N^k, we get

$$VN^k = V_1N^k + VN^{2k}.$$

Substituting this into the former equation we have

$$\begin{aligned} V &= V_1 + V_1N^k + VN^{2k} \subseteqq V_1 + VN^{2k} \\ &\subseteqq V_1 \quad VN^{k+1} \subsetneqq V, \end{aligned}$$

since $2k \geqq k+1$. This is a contradiction and therefore we can conclude that

$$V_1^* \neq V.$$

It follows that V/V_1^* is a non-zero R-$\overline{D}$ bimodule, where $\overline{D} = D/N$ is a skewfield.

Define a submodule

$$V_o^* = \{ v \in V_o \mid vN = 0 \}$$

of the non-zero left module V_o. Now V_o^* is a non-zero R-$\overline{D}$ bimodule.

To see that V_o^* is non-zero, suppose $N^k = 0$ for some positive integer k. Since $V_o \neq 0$, there exists $m \leqq k$ such that

$$V_o N^{m-1} \neq 0 \text{ and } V_o N^m = 0.$$

This means that $V_o N^{m-1} \neq 0$ belongs to V_o^*.

Hence considering V / V_1^* and V_o^* as right $\overline{D}$ vector spaces, there is a non-zero $\overline{D}$-homomorphism from V/V_1^* into V_o^*. It follows that there exists a non-zero D-homomorphism $\varphi : V \longrightarrow V$ such that

(a) $\varphi(v) = 0$ for $v \in V_1^*$,

(b) $\varphi(v)$ V_o^* for $v \in V$, and

(c) $\varphi(v_1) = v_o \neq 0$ for some $v_1 \in V$ and $v_o \in V_o^*$.

Now if we set

$$\phi(w) = \varphi(w\mu) \in V_o^* \text{ for } w \in W,$$

where μ is the natural projection of W onto V, ϕ becomes a C-endomorphism of W. To see this, it is sufficient to show that

$$\phi(wc) = \phi(w)c \text{ for } c \in C,$$

that is, to show for $u \in U$, $c \in C$ and $v \in V$

(i) $\phi(uc) = \phi(u)c = 0$,

(ii) $\phi(vc) = \phi(v)c$,

because if (i) and (ii) hold , we have

$$\begin{aligned}\phi(wc) &= \phi(w\lambda c + w\mu c) = \phi(w\lambda c) + \phi(w\mu c)\\ &= \phi(w\mu)c = (\phi(w\mu) + \phi(w\lambda))c\\ &= \phi(w\mu + w\lambda)c = \phi(w)c.\end{aligned}$$

To show (i) we note that since $u\mu = 0$, $\phi(u) = \varphi(u\mu) = 0$.

Moreover $\phi(uc) = \varphi(uc\mu) = 0$ since $uc\mu \in V_1 \subseteq V_1^* = V_1 + VN$ and because of (a).

Now $\phi(vc) = \varphi(vc\mu) = \varphi(v\mu c\mu)$. Note that $\mu c\mu \in \mu C\mu = D$ and is a D-homomorphism and so

$$\begin{aligned}\phi(vc) &= \varphi(v\mu c\mu) \\ &= \varphi(v)\mu c\mu .\end{aligned}$$

Furthermore, note that

$$\varphi(v) \in V_o^* \subseteq V_o .$$

Now

$$\varphi(v)c\lambda = \varphi(v)\mu c\lambda = 0$$

follows from the definition of V_o and by considering $\mu c\lambda$ as an R-homomorphism of V into U. Therefore

$$\begin{aligned}\phi(v)c &= \varphi(v\mu)c = \varphi(v)c \\ &= \varphi(v)\mu c(\lambda + \mu) \\ &= \varphi(v)\mu c\lambda + \varphi(v)\mu c\mu \\ &= \varphi(v)\mu c\mu .\end{aligned}$$

Hence $\phi(vc) = \phi(v)c$ and we have proved that ϕ is a C-endomorphism of W.

To complete the proof of (1) $\Longrightarrow$ (2), we have to show that $\phi \in End_c(W)$ cannot be obtained from a left multiplication by an element of R. Suppose there exists an $a \in R$ such that $\phi(w) = aw$ for every $w \in W$. Since from (c), $\phi(v_1) = \varphi(v_1) = v_o \neq 0$ for some $v_1 \in V$ and $v_o \in V_o^*$, we see that $a \neq 0$. Moreover since U is faithful, there is $u \in U$ such that $au \neq 0$, but $\phi(u) = 0$ and we have arrived at a contradiction.

(2) $\Longrightarrow$ (1).

Let $\phi \in \mathrm{End}_C(W)$. Then the restriction $\phi|_U$ of ϕ on U belongs to $\mathrm{End}_B(U)$ because

$$\phi(u) = \phi(u\lambda) = \phi(u)\lambda$$

where $\lambda : W \longrightarrow U$ is required as an element of C. Since U has the double centralizer property, there exists an $a \in R$ such that

$$\phi(u) = au \text{ for all } u \in U.$$

Let us set

$$\phi'(w) = \phi(w) - aw$$

for $w \in W$. Then $\phi'(w) = 0$ if $w \in U$. Our proof would be complete if we can show that

$$\phi'(v) = 0 \text{ , for } v \in V$$

also. We have two cases to consider.

<u>Case I</u> :- $V = V_1$, i.e. U generates V.

For every $v \in V$, there exists a finite number of $u_i \in U$, $c_i \in C$, $(i = 1,\ldots,n)$ such that

$$v = \sum_{i=1}^{n} u_i c_i \text{ .}$$

Hence

$$\phi'(v) = \sum_{i=1}^{n} \phi'(u_i)c_i = 0.$$

<u>Case II</u> :- $V_o = 0$, i.e. U cogenerates V.

Suppose there exists $v \in V$ such that $\phi'(v) \neq 0$. Now since $V_o = \cap \{ \ker \gamma \mid \gamma \in \mathrm{Hom}_R(V,U) \} = 0$, for such v, there exist $\psi : V \longrightarrow U$ such that $\phi'(v) \notin \mathrm{Ker}\,\psi$. Hence

$$0 \neq \phi'(v)\mu\psi$$
$$= \phi'(v\mu\psi) = 0,$$

since $v\mu\psi \in U$ and $\mu\psi \in C$. Due to this contradiction, we can concclude

$$\phi(w) = aw \quad .$$

for all $w \in W$. The proof of Theorem I.1 is completed.

Before we proceed to establishing a necessary and sufficient condition for a left artinian ring to be QF-1, we remark here that statement (2) in Theorem I.1 is equivalent to the following :

(3) there exists a positive integer n such that V is R-isomorphic either to a quotient module of $U^{(n)}$ (the direct sum of n copies of U) or to an R-submodule of $U^{(n)}$.

In order to check this equivalence, let U either generate or co-generate V. Consider the case where

$$V = \sum \left\{ \operatorname{Im} \alpha : \alpha \in \operatorname{Hom}_R(U,V) \right\} \quad .$$

Let

$$\{ v_1, \ldots , v_n \}$$

be a generating set of V. Then for each i (i = i, ... ,n) $v_i = \sum_{k=1}^{n} \phi_k(u_{ki}^*)$ for some $\phi_i \in \operatorname{Hom}_r(U,V)$ and $u_{ki}^* \in U$. Now for any $(u_1, \ldots, u_n) \in U^{(n)}$, let us set

$$\phi(u_1, \ldots, u_n) = \sum_{k=1}^{n} \phi_k(u_k).$$

If $V \ni v = \sum_{i=1}^{n} a_i v_i$, then

$$v = \sum_{k=1}^{n} \phi_k\Big(\sum_{i=1}^{n} a_i u_{ik}^* \Big) = \phi\Big(\sum_{i=1}^{n} a_i u_{i1}^*, \ldots, \sum_{i=1}^{n} a_i u_{in}^* \Big)$$

Hence $\phi : U^{(n)} \longrightarrow V$ is an epimorphism and V is therefore isomorphic to a quotient module of $U^{(n)}$.

Consider the case where $\bigcap \left\{ \ker \phi : \phi \in \operatorname{Hom}_R(V,U) \right\} = 0$.

Then, because R is artinian, there exists a finite number of R-homomorphisms $\phi_i \in \mathrm{Hom}_R(V,U)$, i = 1,...,n such that

$$\bigcap_{i=1}^{n} \ker \phi_i = 0.$$

If we set

$$\phi(v) = (\phi_1(v), \ldots, \phi_n(v))$$

for $v \in V$, then ϕ is an R-homomorphism of V into $U^{(n)}$ with

$$\ker \phi = \bigcap_{i=1}^{n} \ker \phi_i = 0.$$

Therefore V is R-isomorphic to a submodule of $U^{(n)}$. Hence we have proved (2) $\Longrightarrow$ (3) and the converse is obvious.

We are now ready to prove the following theorem (K. Morita [7] theorem 1.2).

THEOREM I.2. Let R be an artinian ring and $\{U_\alpha\}$ the totality of isomorphism types of minimal faithful finitely generated left R-modules. In order that R be a QF-1 ring it is necessary and sufficient that R satisfy the following two conditions :

(1) U_α has the double centralizer property for every α .

(2) For any indecomposable finitely generated left R-module V and for each α , U_α generates or co-generates V.

PROOF. Suppose that R is a QF-1 ring, then (1) is satisfied trivially. Let V be any indecomposable finitely generated left R-module, then $U_\alpha \oplus V$ is a faithful finitely generated left R-module.

Hence $U_\alpha \oplus V$ has the double centralizer property and by Theorem I.1 we obtain (2).

To prove the converse, let W by any finitely generated faithful left R-module. We may express W as a direct sum of a finite number of indecomposable submodules V_i, that is,

$$W = \sum_{i \in B} \oplus V_i ,$$

B finite. Consider the collection

$$C = \{ B_j \mid j \in J \}$$

of subsets of B such that $\sum_{i \in B_j} \oplus V_i$ is a faithful left R-module.

There is a B_k in C with the smallest cardinality. Then

$$P = \sum_{i \in B_k} \oplus V_i$$

is a minimal faithful left R-module, since if otherwise, then

$$P = \sum_{i \in B_k} \oplus V_i = P' \oplus P''$$

with P' faithful and P'' non-zero. Now the Krull-Schmidt Theorem tells us that for some proper subset B_k^* of B_k ,

$$P' \cong \sum_{i \in B_k^*} \oplus V_i$$

and this contradicts our choice of B_k. Hence P is isomorphic to a U_α for some α . Now we can write

$$W = P \oplus \sum_{i \in B - B_k = B'} \oplus V_i \cong U_\alpha \oplus \sum_{i \in B'} \oplus V_i$$

where B' is a finite index set. We can write $\sum_{i \in B'} \oplus V_i$ as $\sum_{i=1}^{n} \oplus V_i$

for some positive integer n. Then by Theorem I.1, $U_\alpha \oplus V_1$ has the double centralizer property. Similarly we can conclude the same for $U_\alpha \oplus V_1$ and V_2. Applying Theorem I.1 repeatedly, we would have, finally, that W has the double centralizer property. This completes the proof of our theorem.

As an immediate consequence of Theorem I.2, we have the following Corollary (K. Morita [7], Theorem 1.3).

Corollary I.3. Let R be a left artinian QF-3 ring and U its unique minimal faithful finitely generated left module. Then the following two conditions are necessary and sufficient in order that R be a QF-1 ring;

(1) U has the double centralizer property.

(2) For any finitely generated indecomposable left R-module V, U generates or co-generates V.

CHAPTER II

AN EXAMPLE OF A QF-1 RING WHICH IS NOT QF

The following is an example of an artinian ring which is QF-1 but not QF, showing that the class of QF-1 rings properly contains the class of QF rings.

Let A be the subalgebra of the full matrix ring $(K)_4$ over a field K consisting of all elements of the form

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ a_4 & a_2 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & a_5 & a_3 \end{pmatrix}$$

where $a_i \in K$, $i = 1, \ldots, 5$. The elements

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are indecomposable orthogonal idempotents of A with sum equal to the identity element. Then the indecomposable projective left ideals of A are Ae, Af and Ag.

Writing h = e + f, we claim that the left A-module Ah is injective. We first show that Ae is injective. The K-basis of Ae is given by $b_1 = e$ and b_2 , where

$$b_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Let us denote the K-dual of Ae by Ae^* (= $Hom_K(Ae,K)$) and let $\phi \in Ae^*$. Then the action of ϕ on Ae is described by its action on the basis elements b_1 and b_2. Hence ϕ is represented by a pair $(k,p) \in K \oplus K$ where

$$\phi(b_1) = k \text{ and } \phi(b_2) = p.$$

In order to determine the right A-module structure of Ae^*, we consider the following. Let $r \in A$, then

$$(\phi r)(b_1) = \phi(rb_1) = \phi \begin{pmatrix} a_1 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \phi(b_1)a_1 + \phi(b_2)a_4$$

$$= ka_1 + pa_4 .$$

Moreover

$$(\phi r)(b_2) = \phi(rb_2) = \phi \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \phi(b_2)a_2 = pa_2 .$$

Hence the right A-module structure of Ae^* is defined by

$$\phi r = (ka_1 + pa_4 , pa_2).$$

Setting

$$\chi(\phi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ k & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for $\phi \in Ae^*$, then $\chi : Ae^* \longrightarrow fA$ is an isomorphism. Indeed, for $r \in A$,

$$\begin{aligned} \chi(\phi r) &= \chi(ka_1 + pa_4 , pa_2) \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ ka_1 + pa_4 & pa_2 & 0 & 0 \\ 0 & 0 & pa_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \chi(\phi)r . \end{aligned}$$

Hence χ is an A-homomorphism and that χ is bijective is quite obvious. Using a similar procedure, we can deduce that $Af^* \cong gA$. Whence, equivalently, we have proved that $gA^* \cong Af$ and $fA^* \cong Ae$. Now since gA and fA are projective right A-modules, their duals are injective left A-modules. Therefore $Ae \oplus Af = Ah$ is an injective left A-module.

Now Ah is contained in every finitely generated faithful left A-module as a direct summand. Let ${}_AX$ be a finitely generated faithful left A-module. Then the intersection of the left annihilators of elements of X is 0. Since A is artinian, there is a finite number of elements of X such that

$$\bigcap_{i=1}^{n} l(x_i) = 0.$$

Hence A is embedded into a finite direct sum of copies of X by

$$a \longmapsto (ax_i)_{i=1}^{n} ,$$

$a \in A$. Expressing X as a finite direct sum of indecomposable submodules we can write

$$Ah \subseteq A \hookrightarrow \bigoplus_{i=1}^{n} \left(\bigoplus_{k=1}^{m} X_{i_k} \right) .$$

Now since Ah is injective, there exists a suitable left A-module Y such that

$$Ah \oplus Y = \bigoplus_{i=1}^{n} \left(\bigoplus_{k=1}^{m} X_{i_k} \right).$$

Since $Ae \neq Af$, it follows from the Krull Schmidt theorem that Ae and Af are isomorphic to two distinct indecomposable direct summand of X. Hence we have proven that for any finitely generated faithful left A-module X,

$$X = Ae \oplus Af \oplus X_1 \oplus \ldots \oplus X_s$$

where X_i are indecomposable submodules of X.

Now we make use of Theorem I.1 to show that X is balanced, that is, A is a QF-1 ring. To this end, we have to check the following:

(1) Ah has the double centralizer property, and

(2) For each indecomposable finitely generated left A-module M, Ah either generates or co-generates M.

Proof of (1) :-

The elements of A

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and } e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

are a basis of Ah over K. Then for each $a \in A$, we have

$$ae_k = \sum_{i=1}^{4} x_{ik} e_i ,$$

$k = 1,\ldots,4$. Hence the correspondence

$$a \longmapsto (x_{ik})$$

gives a matrix representation of A. It turns out, by direct computation, that the matrix (x_{ik}) is identically equal to a, for every $a \in A$.

Now the centralizer $\mathrm{End}_A(Ah)$ of Ah consists precisely of those matrices (y_{ij}), $i,j=1, \ldots,4$, which commute with all (x_{ij}). Hence

$$(y_{ij})\, e_1 = e_1\, (y_{ij})$$

or

$$\begin{pmatrix} y_{11} & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 \\ y_{31} & 0 & 0 & 0 \\ y_{41} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies that y_{11} is arbitrary and

$$y_{i1} = y_{1i} = 0 , \; i = 2,3,4.$$

Proceeding in the same manner with the elements e_2, e_3 and e_4 we see finally that $\mathrm{End}_A(Ah)$ consists precisely of the matrices of the form

$$(y_{ij}) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & z & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & y \end{pmatrix}$$

where x, y and $z \in K$.

Furthermore, the double centralizer of Ah consists of those matrices (z_{ij}) which commute with all the (y_{ij}) . It turns out, after going through a similar procedure as above, that A coincides with the double centralizer of Ah. Hence we have proven that Ah is balanced.

Proof of (2) :-

A is a semiperfect ring because it is artinian. Let

$$0 \longrightarrow K \longrightarrow P \xrightarrow{t} M \longrightarrow 0$$

be a projective cover for M, where K = ker t. Now there exists a homomorphism f which makes the following diagram commutative,

$$\begin{array}{ccc} A^{(n)} & \xrightarrow{\quad u \quad} & M \\ & \underset{\exists f}{\searrow} \quad \nearrow & \\ & P & \end{array}$$

where

$$(a_i)_{i=1}^{n} \longmapsto^{u} \sum_{i=1}^{n} a_i m_i$$

and $\{m_1, \ldots, m_n\}$ is a basis of M. Since t is co-essential, and u is onto, f is necessarily onto. Therefore, since P is projective, f splits, that is,

$$A^{(n)} = \ker f \oplus Y$$

where Y ≅ P. Hence P is finitely generated. This, together with the fact that A is semiperfect and P is projective implies that P is isomorphic to a direct sum of finitely many left A-modules each isomorphic to some indecomposable left ideal of A. Therefore

$$P = \oplus_{\text{finite}} Ae_i \ , \ e_i = e, f \text{ or } g.$$

Since P is a projective cover, K is small and hence

$$K \subseteq \text{rad } P = \oplus Je_i$$

where

$$J = \text{rad}A = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_5 & 0 \end{pmatrix} : a_4, a_5 \in K \right\} .$$

Now $Jg = 0$ and hence

$$K \subseteq \oplus Je_i \ , \ e_i = e \text{ or } f.$$

It follows immediately that

$$\begin{aligned} M &\cong P/K \\ &\cong \oplus Ae_i/K \ , \quad e_i = e, f \text{ or } g \\ &\cong \oplus Ae_i/K \oplus (\oplus_{\text{finite}} Ag) \ , \quad e_i = e \text{ or } f. \end{aligned}$$

Since M is indecomposable, either

(i) $M \cong \Sigma \oplus Ae_i/K$, $e_i = e$ or f, or

(ii) $M \cong Ag$.

In the first instance, M is generated by $Ae \oplus Af$ and in the latter,

$$Ag \cong Jf \subseteq Af$$

and hence M is co-generated by $Ae \oplus Af$. This completes the proof of (2) and consequently Theorem I.1 applies, proving that X is balanced. Therefore A is a QF-1 ring.

Finally we show that A is not a QF ring by observing that A is not self injective. Since Ag is isomorphic to Jf which is properly contained in the indecomposable left A-module Af, Ag is not injective which implies that ${}_{A}A$ is not injective and hence not QF.

CHAPTER III

CONSTRUCTIONS OF NON-BALANCED MODULES

We are going to present seven different ways of constructing modules which are not balanced. These constructions are essential for four theorems on certain QF-1 rings. The theorems, in turn, lead us to establishing a characterization of certain QF-1 rings. From here on we shall denote left ideals, two-sided ideals and left R-modules respectively by the capital letters U, I and M (with appropriate subscripts).

CONSTRUCTION I. Let R be a local ring with a minimal right ideal. Let U_1, U_2 be two non-zero isomorphic left ideals and I_1, I_2 be two two-sided ideals of R such that

$$U_i \subseteq I_i \ (i = 1,2) \text{ and } I_1 \cap I_2 = 0.$$

Then there is a finitely generated faithful left R-module which is not balanced.

PROOF. We construct the module M in question as follows. Let $t : U_1 \longrightarrow U_2$ be an isomorphism and set

$$D = \{ (d, -dt) : d \in U_1 \} .$$

Then

$$M = R \oplus R / D$$

is a finitely generated and faithful left R-module. The endomorphisms of M can be lifted to those endomorphisms of the left R-module $R \oplus R$ which map D into D. Let

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

be the matrix representation of such an endomorphism of $R \oplus R$ where the $\phi_{ij} \in \mathrm{End}_R(R)$ are right multiplications by elements of R. Consider

$$(d, -dt)\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = (d\phi_{11} - (dt)\phi_{21}\ ,\ d\phi_{12} - (dt)\phi_{22})$$

with $(d, -dt) \in D$. Hence

$$d\phi_{1i} - (dt)\phi_{2i} \in U_i \quad (i = 1, 2) \qquad \text{———} \qquad (1)$$

and we claim that both ϕ_{21} and ϕ_{12} belong to the radical of R which is denoted by W. For, suppose that ϕ_{21} is a unit, then from (1),

$$d\phi_{11}\phi_{21}^{-1} - dt \in U_1\phi_{21}^{-1}$$

and hence

$$U_2 = U_1 t \subseteq U_1\phi_{21}^{-1} + U_1\phi_{11}\phi_{21}^{-1} \subseteq I_1 ,$$

a contradiction. Similarly, if ϕ_{12} does not belong to W, then

$$\begin{aligned} U_1 &\subseteq U_2\phi_{12}^{-1} + (U_1 t)\phi_{22}\phi_{12}^{-1} \\ &= U_2\phi_{12}^{-1} + U_2\phi_{22}\phi_{12}^{-1} \subseteq I_2 , \end{aligned}$$

another contradiction.

We then construct a homomorphism $f : R \oplus R \longrightarrow R \oplus R$ which commutes with all

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

mapping D into D. Thus, f will induce an element of the double centralizer D(M) of M. Let us define f by

$$f(x, y) = (zx, 0)$$

where z is a non-zero element of the right socle and $(x, y) \in R \oplus R$. Now since $U_1 \subseteq W$ and $zW = 0$, $f(D) = 0$. Moreover

$$(f(x, y))\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = (zx\phi_{11}, zx\phi_{12})$$

$$= (zx\phi_{11}, 0)$$

because $\phi_{12} \in W$. Similarly,

$$f \quad (x, y)\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = (zx\phi_{11} + zy\phi_{21} , 0)$$

$$= (zx\phi_{11} , 0)$$

and thus f induces an element of D(M).

Now suppose that f is induced by an element $r \in R$, that is,

$$f(x, y) - (rx, ry) \in D,$$

for all $(x,y) \in R \oplus R$. Hence if $(x, y) = (0, 1)$, we get $(0 , r) \in D$ and it follows immediately that $r = 0$. But if $(x, y) = (1, 0)$ we have $(z, 0) \in D$ which is a contradiction. Therefore f is not induced by a left multiplication by an element of R, that is, M is not balanced.

We shall see later, in the proof of Lemma IV.1, that Construction I implies, that a perfect local QF-1 ring has a unique minimal two-sided ideal. Construction II will be used in the proof of Construction III which deals with a situation similar to that of the previous one.

CONSTRUCTION II. Let R be a local ring with the radical W. Let M_1 and M_2 satisfy $M_i f \subseteq WM_j$, for every homomorphism $f : M_i \longrightarrow M_j$ with $i \neq j$. Moreover, let, M_2 be faithful, and suppose that the annihilator of M_1 does not contain the right socle $\mathrm{Soc}R_R$ of R. Then

$$M = M_1 \oplus M_2$$

is not balanced.

PROOF: Let the matrices

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

represent the elements of $C = \mathrm{End}_R(M)$, where $\phi_{ij} : M_i \longrightarrow M_j$. Let $z \in \mathrm{Soc}R_R - \mathrm{Ann}(M_1)$, where $\mathrm{Ann}(M_1)$ denotes the annihilator of M_1, and define an additive homomorphism $f : M \longrightarrow M$ by

$$f(m_1, m_2) = (zm_1, 0)$$

for all $(m_1, m_2) \in M = M_1 \oplus M_2$.

Now f so defined actually belongs to $\mathrm{End}_C(M)$. To see this, consider

$$(f(m_1, m_2)) \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = (zm_1, 0) \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

$$= (zm_1\phi_{11}, zm_1\phi_{12}) = (zm_1\phi_{11}, 0)$$

since $zm_1\phi_{12} = 0$ because $m_i\phi_{ij} \in WM_j$ for $i \neq j$. By similar consideration, we have

$$f\ (m_1,m_2)\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = f((m_1\phi_{11} + m_2\phi_{21}, m_1\phi_{12} + m_2\phi_{22}))$$

$$= (zm_1\phi_{11} + zm_2\phi_{21}, 0)$$

$$= (zm_1\phi_{11}, 0)$$

$$= (f(m_1, m_2)) \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} .$$

To conclude the construction, suppose that there is a $r \in R$ such that

$$f(m_1, m_2) = (rm_1, rm_2) = (zm_1, 0).$$

Now since M_2 is faithful by assumption, we have that $r = 0$. Furthermore, since $z \notin \mathrm{Ann}(M_1)$, there is $m_1' \in M_1$ such that $zm_1' \neq 0$ and thus

$$f(m_1', 0) = (zm_1', 0) \neq (rm_1', 0) = (0, 0).$$

Therefore f is not induced by left multiplication, ie. M is not balanced.

Our next construction deals with a situation similar to that of Construction I. We shall make use of Constructions I, III and IV to prove Lemma IV.1.

CONSTRUCTION III. Let R be a local ring and U_1, U_2 be two non-zero left ideals and I_1 a two-sided ideal of R such that

$$U_1 \subseteq I_1 \quad \text{and} \quad I_1 \cap U_2 = 0.$$

Furthermore, let U_2 contain no non-zero two-sided ideal of R and let $\mathrm{Soc}R_R \nsubseteq U_1$. Then there is a finitely generated faithful left R- module which is not balanced.

PROOF. Since we are going to apply Construction II, let

$$M_i = R/U_i, \ i = 1, 2 \text{ and } M = M_1 \oplus M_2.$$

Now M_2 is faithful and since $\mathrm{Ann}(M_1) \subseteq U_1$, $\mathrm{Soc}R_R \nsubseteq \mathrm{Ann}(M_1)$. We next consider the morphisms between M_1 and M_2. Every homomorphism $f_1 : M_1 \longrightarrow M_2$ can be lifted to an element of $\mathrm{End}_R(R)$ which maps U_1 into U_2. It follows that there is a right multiplication by an element $r_1 \in R$ with $U_1 r_1 \subseteq U_2$. Since $U_1 \subseteq I_1$, we have

$$U_1 r_1 \subseteq I_1 \cap U_2 = 0 .$$

Necessarily $r_1 \in W$, otherwise we would have $U_1 = 0$. Hence

$$M_1 f_1 \subseteq WM_2.$$

In a similar manner, every $f_2 : M_2 \longrightarrow M_1$ can be lifted to a right multiplication by some $r_2 \in R$ with $U_2 r_2 \subseteq U_1$. Again $r_2 \in W$, otherwise

$$U_2 \subseteq U_1 r_2^{-1} \subseteq I_1$$

is a contradiction. Hence

$$M_2 f_2 \subseteq WM_1 .$$

The assumptions of Construction II are now satisfied.

CONSTRUCTION IV. Let R be a left artinian local ring. Let $U \subseteq \mathrm{Soc}R_R$ be a non-zero left ideal containing no non-zero two-sided ideal. Let r be a unit of R such that $Ur \not\subseteq U$ and $\mathrm{Soc}R_R \not\subseteq U + Ur$. Then there is a finitely generated faithful left R-module which is not balanced.

Proof. If we set

$$M = R/U,$$

then M is a finitely generated faithful left R-module. Every element $f \in \mathrm{End}_R(M) = C$ can be lifted to a right multiplication by an element $a_f \in R$ which satisfy

$$Ua_f \subseteq U.$$

Let us denote the radicals of R, M and C respectively by W, $T = W/U$ and W'. Now

$$W' = \left\{ f \in C : a_f \in W \right\}$$

and hence C is local and $MW' \subseteq T$. Moreover T is a C-submodule of M and M/T is a completely reducible right C-module. Writing

$$\bar{x} = x + U \in M$$

for every $x \in R$, we claim that

$$M/T = (\bar{1} + T)C \oplus (\bar{r} + T)C \oplus N$$

for a suitable right C-submodule of M/T. To see this, suppose that the right C-modules $(\bar{1} + T)C$ and $(\bar{r} + T)\,C$ have a non-trivial intersection, that is, there is some $f \in C$ with

$$(\bar{1} + T)f \in (\bar{r} + T)C \text{ and } \bar{1}f \notin T.$$

Hence $\bar{1}f - \bar{r} \in T$ and we can lift f to a right multiplication by an element a_f of R satisfying $Ua_f \subseteq U$. It follows that $1a_f - r \in W$. Now since $U \subseteq \mathrm{Soc}R_R$, we have

$$Ua_f - Ur \in UW = 0.$$

Hence $Ua_f = Ur \subseteq U$, contradicting the fact that a_f induces the endomorphism f of M.

Take a non-zero element $z \in \mathrm{Soc}R_R - (U + Ur)$ and observe that $\bar{z} \in \mathrm{Soc}M_C$ because $\bar{z}\, W' = 0$. Now we are ready to construct an element g of the double centralizer D(M) of M and eventually show that g is not induced by left multiplication. We first define a C-homomorphism $g' : (M/T)_C \longrightarrow \mathrm{Soc}M_C$ by setting

$$g'(\bar{1} + T) = \bar{0}, \quad g'(\bar{r} + T) = \bar{z}$$

and

$$g'(\bar{n} + T) = 0 \text{ for } \bar{n} + T \in N.$$

If $p : M_C \longrightarrow (M/T)_C$ and $m : \mathrm{Soc}M_C \longrightarrow M_C$ are the respective projection and injection, then we set $g \in D(M_C)$ to be

$$g = mg'p$$

and it is obvious that g has the required properties.

Now suppose that there exists $s \in R$ such that $s\bar{x} = g(\bar{x})$ for all $\bar{x} \in M$. By the definition of g, we see that $s\bar{1} = \bar{0}$ which implies $s \in U$. On the other hand $s\bar{r} = \bar{z}$ implies $z \in sr + U$. Therefore we have $z \in Ur + U$ which contradicts our choice of z and hence M is not balanced.

The following result will be needed in Construction VI.

CONSTRUCTION V. Let M be an indecomposable left R-module of finite length. Assume that M possesses a proper submodule and a quotient both isomorphic to a faithful left R-module N. Moreover, let M_C have a non-trivial socle and a non-trivial radical, where $C = \mathrm{End}_R(M)$. Then M is not balanced.

PROOF. By Fitting's lemma, C is a local ring with a nilpotent radical W'. If $f : N \longrightarrow M$ is an embedding, we shall show that $Nf \subseteq MW'$. Let $p : M \longrightarrow N$ be an epimorphism, then $pf \in C$. Since Nf is a proper submodule of M, pf is not invertible and hence belong to the set of non-units W'. It follows that $Nf = Mpf \subseteq MW'$.

Now since M_C has non-trivial socle and radical, there exists a non-zero C-homomorphism

$$\phi' : M/MW' \longrightarrow \mathrm{Soc}(M_C).$$

Denoting the embedding by $f' : \mathrm{Soc}(M_C) \longrightarrow M_C$ and the canonical epimorphism by $p' : M_C \longrightarrow M/MW'$, we see that the C-homomorphism

$$\phi = f'\phi'p'$$

belong to $\mathrm{End}_C(M)$.

Finally we show that ϕ is not induced by right multiplication. Assuming that $\phi(m) = rm$ for all $m \in M$ and a suitable $r \in R$. Since $\phi \neq 0$, $r \neq 0$ and

$$rNf \subseteq r(MW') = \phi(MW') = 0$$

contradicts the fact that N is faithful and hence M is not balanced.

In order to simplify the presentation of Constructions VI and VII, it is necessary for us to establish the following lemma.

LEMMA III.1. Let R be a local ring with the radical W. Let x, y and z be elements of R such that

$$x \neq 0,\ xW = 0,\ Wy = 0,$$

$$z \notin Rx + yR \text{ and } y \notin zW.$$

Then

$$M = (R \oplus R)/D\ ,$$

where $D = \{(by, -bz + ax) : a,b \in R\}$, is a faithful indecomposable left R-module. Moreover if

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$c_{ij} \in R$, represents an endomorphism of the left R-module $R \oplus R$ which maps D into D, then $c_{21} \in W$.

PROOF. Let

$$T = (W \oplus R)/D \subseteq M.$$

Firstly, we show that if $m \in T$, then $Ann_l(m) \neq 0$. For $m = (w,r) + D$, with $w \in W$, $r \in R$, if $r \in W$ then $SocR_R \subseteq Ann_l(m)$. If $r \notin W$, consider

$$xr^{-1}m = xr^{-1}((w,r) + D)$$

$$= (xr^{-1}w,\ x) + D.$$

Since $r^{-1}W \subseteq W$ and $xW = 0$, we have

$$xr^{-1}m = (0,x) + D = D$$

and hence $0 \neq xr^{-1} \in \mathrm{Ann}_1(m)$. Conversely if $\mathrm{Ann}_1(m) \neq 0$ for $m = (r_1, r_2) + D \in M$, then

$$u(r_1, r_2) = (by, -bx + ax) \in D \text{ ——— } (1)$$

for some $0 \neq u$, b and a of R and we are going to show that $m \in T$. Suppose that $r_1 \notin W$ and from (1) we get

$$u = byr_1^{-1} \quad \text{and} \quad ur_2 = byr_1^{-1}r_2 = -bz + ax \text{ —— } (2)$$

Now since $Wy = 0$ and $0 \neq u = byr_1^{-1}$, we have that b is a unit. Therefore from (2),

$$z = (b^{-1}a)x + y(-r_1^{-1}r_2) \quad \in \quad Rx + yR$$

which is a contradiction and we conclude that $r_1 \in W$ or $m \in T$. The above considerations thus allow us to characterize T in the following way :

$$T = \{ m : m \in M \text{ with } \mathrm{Ann}_1(m) \neq 0 \} \quad .$$

Now $m_o = (1,0) + D \notin T$. For supposing the contrary and say $um_o = 0$ for some $0 \neq u \in R$, then

$$(u,0) = (by, -bz + ax)$$

for some b, $a \in R$. Since $Wy = 0$ and $0 \neq u = by$, it follows that b is a unit, But then

$$-bz + ax = 0 \quad \text{or} \quad z = b^{-1}ax \in Rx$$

is a contradiction.

In order to prove that M is indecomposable we assume the contrary. Since $(M/\mathrm{radM}) \leq 2$, M is the direct sum of two local modules. This follows from the fact that if, say,

$$M = M_1 \oplus M_2 \oplus M_3 ,$$

then

$$M/\mathrm{rad}M = M_1/\mathrm{rad}M_1 \oplus M_2/\mathrm{rad}M_2 \oplus M_3/\mathrm{rad}M_3$$

and since M is finitely generated, each $M_i/\mathrm{rad}M_i$ $(i = 1, 2,3)$ is non-zero. Hence $\partial(M/\mathrm{rad}M) > 2$ is a contradiction and therefore we can write

$$M = Rp \oplus Rq$$

for suitable $p,q \in M$. Hence $m_o = r_1p + r_2q$ for some $r_1, r_2 \in R$, and since, as remarked, $m_o \notin T$ or $\mathrm{Ann}_1(m_o) = 0$, we can assume that $\mathrm{Ann}_1(r_1p) = 0$. It follows that $Rm_o \cap Rq = 0$; otherwise, for some $s, t \in R$, $sm_o = tq$. Then

$$s(m_o) = s(r_1p + r_2q) = sr_1p + sr_2q = tq,$$

or, equivalently,

$$sr_1p = -sr_2q + tq = (-sr_2 + t)q \in Rq \text{ ——— } (1)$$

and s=0 because $Rp \cap Rq = 0$. Moreover, r_1 is a unit because if $r_1 \in W$ we have

$$\mathrm{Soc}R_R \subseteq \mathrm{Ann}_1(r_1) \subseteq \mathrm{Ann}_1(r_1p),$$

contrary to the fact that $\mathrm{Ann}_1(r_1p) = 0$. Therefore (1) gives

$$p = r_1^{-1}m_o - r_1^{-1}r_2q,$$

and we can conclude that $M = Rm_o \oplus Rq$.

Let $M = Rm_o \oplus Rq \xrightarrow{k} Rm_o$ be the canonical epimorphism where $m_ok = m_o$. Define $\varphi : R \longrightarrow M$ by $1\varphi = (0,1) + D$. Now since $(y, -z) \in D$ or $(y,0) - (0,z) \in D$ we see that

$$z\varphi = (0,z) + D = (y,0) + D = ym_o.$$

Furthermore, define $\emptyset : R \longrightarrow Rm_o$ by $1\emptyset = m_o$ and $\emptyset$ is easily seen to be an isomorphism. Consider the composite homomorphism

$$k\emptyset^{-1} : R \longrightarrow M \xrightarrow{k} Rm_o \xrightarrow{\emptyset^{-1}} R$$

and its action on z;

$$\begin{aligned} z\varphi k\emptyset^{-1} &= ((y,0) + D)\, k\emptyset^{-1} \\ &= (y(m_o))\, k\emptyset^{-1} = y((m_o)k)\emptyset^{-1} \\ &= y(m_o\emptyset^{-1}) = y. \end{aligned}$$

Since $\varphi k\emptyset^{-1}$ is induced by a right multiplication, for a suitable $r \in R$ we obtain $y = zr$. But this is impossible because if $r \in W$, then $y \in zW$ which contradicts the assumption on y. On the other hand if $r \notin W$, then $z = yr^{-1} \in yR$, another contradiction. Therefore M must be indecomposable.

Finally, let

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

be an endomorphism of $_R(R \oplus R)$ mapping D into D. Now $(o,x) \in D$ and hence

$$(0,x) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = (xc_{21}, xc_{22})$$

$$= (by, -bz + ax) \in D$$

for suitable $b,a \in R$, that is, $xc_{21} = by$ and $xc_{22} = -by + ax$. Supposing that $c_{21} \notin W$, we have, from the above that,

$$0 \neq x = byc_{21}^{-1}$$

which means that $b \notin W$. Hence

$$z = b^{-1}ax - b^{-1}xc_{22}$$

and after substituting, the above becomes

$$x = b^{-1}ax + y(-c_{21}^{-1}c_{22}) \in Rx + yR.$$

Hence we arrive at a contradiction and we conclude that $c_{21} \in W$. The proof of our lemma is completed.

We shall examine the double centralizer of an indecomposable module, specifically the left R-module M of the previous lemma in Constructions VI and VII which will be used in the next chapter.

CONSTRUCTION VI. Let R be a left artinian local ring. Let

$$S = \mathrm{Soc}R_R \cap \mathrm{Soc}_R R.$$

Furthermore, let $x, y \in S$ such that Rx and Ry are not two-sided ideals and

$$S \neq Rx + yR.$$

Then there exists a finitely generated faithful left R-module which is not balanced.

PROOF. We begin by observing that, in view of Construction II, we may assume that $R/Rx \cong R/Ry$. To see this, suppose that the finitely generated faithful left R-modules R/Rx and R/Ry are not isomorphism. Let $\phi : R/Rx \longrightarrow R/Ry$ be an epimorphism, then

$$R/Ry \cong (R/Rx) / \ker\phi.$$

Hence

$$\partial_1(R/Ry) = \partial_1(R/Rx) - \partial_1(\ker \phi).$$

Now Ry and Rx are simple because $Rx \cong R/I$ where R/I is semisimple.

This implies that $W(R/I) \subseteq I$ and hence $W \subseteq I$. Necessarily, $W = I$, and therefore $Rx \cong R/W$ is simple since R is local. The same goes for Ry and we have

$$\partial_1(R/Rx) = \partial_1(R/Ry).$$

It follows that $\partial_1(\ker \phi) = 0$ and so $\ker \phi = 0$. Therefore an epimorphism between R/Rx and R/Ry is necessarily an isomorphism. In the case where the homomorphism is not epimorphic, then

$$\phi(R/Rx) = R\phi(1 + Rx) \subsetneq R/Ry.$$

Now $\phi(1 + Rx)$ must belong to rad(R/Ry) because, if not, there is $r \in R$ such that

$$r(\phi(1 + Rx)) = 1 + Ry$$

and $R(\phi(1 + Rx)) = R/Ry$. To satisfy the rest of the assumption of Construction II, suppose that $SocR_R \subseteq Ann_1(R/Rx)$, that is,

$$SocR_R(R/Rx) \subseteq Rx.$$

Hence $S \subseteq SocR_R \subseteq SocR_R.R \subseteq Rx$. Therefore we have $Rx = S$ since $Rx \subseteq S$, a contradiction. Whence

$$SocR_R \not\subseteq Ann_1(R/Rx)$$

and Construction II can be applied and we are through. Hence, in what follows, we assume that $R/Rx \cong R/Ry$.

Let $z \in S - (Rx + yR)$ and consider the finitely generated faithful left R-module $M = (R \oplus R)/D$ of lemma III.1. Since M is an indecomposable R-module of finite length (Lemma III.1), the centralizer C of M is a local ring with a nil radical W'. Moreover, if

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$c_{ij} \in R$, represents an endomorphism ϕ of the left R-module $R \oplus R$ mapping D into D, then $c_{21} \in W$. We shall show that also $c_{11} \in W$ and $c_{22} \in W$ provided that ϕ is nilpotent. Say $\phi^n = 0$ and taking $t \in \mathrm{Soc}R_R$, we have

$$(0,t) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = (tc_{21}, tc_{22}) = (0, tc_{22}).$$

Now since

$$(0,t) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^n = (0, tc_{22}^n) \in D,$$

we have

$$(0, tc_{22}^n) = (by, -bz + ax) \in D,$$

for suitable $b, a \in R$, and formulating we obtain

$$by = 0 \text{ and } tc_{22}^n = -bz + ax. \qquad \text{———} \qquad (1)$$

Since $Wy = 0$, $b \in W$ and hence $bz = 0$ and (1) becomes $tc_{22}^n = ax$. Now if $c_{22} \notin W$, $t = axc_{22}^{-n}$ and letting $t = x$, $x = axc_{22}^{-n} \in Rx$. Since $Wx = 0$, $a \notin W$ and so

$$a^{-1}x = xc_{22}^{-n} \in Rx.$$

Taking $t = z$, we get $z = azc_{22}^{-n} \in Rx$ which contradicts the choice of z, and hence we conclude that $c_{22} \in W$.

Now consider

$$(y, -z) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = (yc_{11}, yc_{12})$$

$$= (by, -bz + ax) \in D$$

for suitable $b, a \in R$ where $(y, -z) \in D$. Suppose that $c_{11} \notin W$. Since

$yc_{11} = by$ or $y = byc_{11}^{-1}$, which implies that $b \notin W$. Now

$$yc_{12} = -bz + ax,$$

or equivalently,

$$z = (b^{-1}a)x + y(-c_{11}^{-1}c_{12})$$

which again contradicts the fact that $z \in S - (Rx + yR)$. Therefore $c_{11} \in W$.

Consider the right C-module $M = (R \oplus R)/D$. We shall show that

$$\mathrm{Soc}M_C \neq 0 \quad \text{and} \quad \mathrm{rad}(M_C) \neq M_C.$$

In order to show the first inequality we first note that $zW = 0$ and $(0,z) \notin D$. Therefore $(0,z) + D = m$ is a non-zero element of M such that $mW = 0$. Indeed, every $\phi \in W'$ is nilpotent and if ϕ is represented by $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

$$m\phi = (0,z) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} + D = (zc_{21}, zc_{22}) + D$$

which is zero in M since $c_{21} \in W$ and $c_{22} \in W$.

To show that $\mathrm{rad}(M_C) \neq M_C$, we note that the radical W' of C is the set

$$W' = \{ \phi \in C : r_\phi \in W \}$$

Hence $\mathrm{rad}(M_C) \subseteq MW' \subseteq W \oplus R/D$ which is not equal to M because $m_o = (1,0) + D \in M$ is not contained in $(W \oplus R)/D$ (cf. proof of Lemma III.1).

Now R/Rx is a faithful left R-module and the map $h : R \longrightarrow Ry \oplus R/D$ defined by

$$h(1) = (0,1) + D$$

is surjective with Rx as its kernel. To see this, let $(ry, r') + D \in (Ry \oplus R)/D$, then

$$\begin{aligned}(ry, r') + D &= (ry, rz) + (0, -rz + r') + D \\ &= (-rz + r')\,(0,1) + D \\ &= (-rz + r')\,h(1)\end{aligned}$$

since $(ry, rz) \in D$. Moreover h has Rx as its kernel because $(0,rx) \in D$ for all $rx \in Rx$. Therefore $R/Rx \cong Ry \oplus R/D$.

The map $g : M \longrightarrow (R \oplus R)/Ry \oplus R$ defined by

$$g((1,1) + D) = (1,0) + (Ry \oplus R)$$

is surjective with kernel $Ry \oplus R/D = K$, i.e.,

$$M/K \cong (R\oplus R) \,/\, (Ry\oplus R).$$

For, let $(r,r') + (Ry \oplus R) \in (R\oplus R)/(Ry\oplus R)$, $r \notin Ry$, $r,r' \in R$, then

$$\begin{aligned}(r,r') + (Ry \oplus R) &= (r,0) + (0,r') + (Ry \oplus R) \\ &= (r,0) + (Ry \oplus R) \\ &= r(g((1,1) + D)).\end{aligned}$$

Let $(ry, r') + D \in (Ry \oplus R)/D$ where $ry \in Ry$, $r' \in R$, then

$$\begin{aligned}g((ry, r') + D) &= (ry, r')\,(g((1,1) + D)) \\ &= (ry, r')\,((1,0) + (Ry \oplus R)) \\ &= (ry, 0) + (Ry \oplus R) = \bar{0}.\end{aligned}$$

Hence

$$M/K \cong (R \oplus R)/(Ry \oplus R) \cong R/Ry \cong R/Rx.$$

We are now in a position to apply Construction V by taking R/Rx to be the faithful left R-module N. Hence the proof is completed.

CONSTRUCTION VII. Let R be a left artinian local ring. Let x be a non-zero element of $S = \mathrm{Soc}R_R \cap \mathrm{Soc}_R R$ such that Rx is a two-sided ideal. Furthermore, let y and z be two elements of R such that

$$y \notin Rx, \quad Wy = 0, \quad yW \subseteq Rx$$

and

$$z \notin Rx + yR, \quad Wz + zW \subseteq Rx.$$

Then there exists a finitely generated faithful left R-module which is not balanced.

PROOF. We shall show that the finitely generated faithful left R-module $M = (R \oplus R)/D$ of Lemma III.1 is the required non-balanced left R-module, where

$$D = \left\{ (by, -bz + ax) : b, a \in R \right\} .$$

Let $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ represent an endomorphism of the left R-module $R \oplus R$ mapping D into D and let the induced endomorphism ϕ of M be nilpotent, say $\phi^n = 0$ for some positive integer n. We claim that under this condition, all c_{ij} $(i,j = 1,2)$ belong to W. It was shown in Lemma III.1 that $c_{21} \in W$ and hence to show the rest consider

$$(x,0) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = (xc_{11}, xc_{12})$$

where $(x, 0) \notin D$ since $x \notin Ry$. Since Rx is a two-sided ideal, $xc_{12} \in Rx$, that is, $(0, xc_{12}) \in D$ and $(xc_{11}, xc_{12}) + D = (xc_{11}, 0) + D$. Now by induction, we obtain

$$(x,\ 0) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^n = (xc_{11}^n,\ xc_{12}^n) + D$$

$$= (xc_{11}^n,\ 0) + D$$

where $(0,\ xc_{12}^n) \in D$. Since $\phi^n = 0$, $(x,0)$ is being mapped into D and we have $(xc_{11}^n\ ,\ 0) \in D$ which in turn means that $xc_{11}^n = by$ for a suitable $b \in R$. Now if c_{11} is a unit, $xc_{11}^n \neq 0$ and hence $by \neq 0$. Since $Wy = 0$, b does not belong to W and $y = b^{-1}xc_{11}^n \in Rx$ is a contradiction. We conclude that $c_{11} \in W$.

We show next that $c_{22} \in W$. Consider, for arbitrary $u_k \in R$

$$(u_kx,\ yc_{22}^k) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = (u_kxc_{11} + yc_{22}^kc_{21},\ u_kxc_{12} + yc_{22}^{k+1})$$

$$= (yc_{22}^kc_{21},\ u_kxc_{12} + yc_{22}^{k+1}).$$

Since $c_{21} \in W$, $c_{22}^kc_{21} \in W$ and since $yW \subseteq Rx$

$$yc_{22}^kc_{21} = u_{k+1}x$$

for a suitable $u_{k+1} \in R$. Therefore

$$(u_kx,\ yc_{22}^k) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} + D = (u_{k+1}x,\ u_kxc_{12} + yc_{22}^{k+1}) + D.$$

Now if $u_k = 0$ and by induction, we have

$$(0,\ y) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^n + D = (u_nx,\ yc_{22}^n) + D.$$

Since $\phi^n = 0$, $(0, y)$ is mapped into D and so

$$(u_n x, yc_{22}^n) = (by, -bz + ax)$$

for suitable b and a $\in$ R. It follows that

$$u_n x = by \quad \text{and} \quad yc_{22}^n = -bz + ax,$$

and hence b $\in$ W, for otherwise $y = b^{-1}u_n x \in Rx$ which is a contradiction. Hence

$$yc_{22}^n = -bz + ax \subset Wz + Rx = Rx$$

and consequently $c_{22} \in W$.

Finally, for $(y, -z) \in D$

$$(y,-z) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = (yc_{11}-zc_{21}, yc_{12}-zc_{22})$$

$$= (by, -bz + ax) \in D$$

or equivalently,

$$yc_{11}-zc_{21} = by \text{ and } yc_{12}-zc_{22} = -bz + ax$$

for suitable b and a $\in$ R. Since c_{11} and $c_{21} \in W$, yc_{11} and $zc_{21} \in Rx$. This yields by $\in$ Rx and so b $\in$ W. Now we have

$$yc_{12}-zc_{22} = -bz + ax \in Wz + Rx = Rx$$

and $yc_{12} \in Rx$ because $-zc_{22} \in zW \subseteq Rx$. Therefore $c_{12} \in W$.

Since the left R-module M is indecomposable of finite length, its centralizer C is a local ring with a nil radical W'. Now we note that $(x,0) + D$ belongs to SocM_C. If $\phi \in W'$, then $\phi^n = 0$ for some n and if we let $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ be the endomorphism of $R \oplus R$ which induces

ϕ, then

$$(x, 0) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = (xc_{11}, xc_{12}) = (0, 0).$$

Hence $\mathrm{Soc}M_C \neq 0$.

We shall construct an element f of the double centralizer of M which cannot be induced by left multiplication. Since all c_{ij} belong to W when the induced endomorphism f of M belongs to W',

$$X = (W \oplus W)/D \subseteq MW'.$$

Hence if $M_C \xrightarrow{p} M_C/X$ is the canonical epimorphism and m: $\mathrm{Soc}M_C \longrightarrow M_C$ the embedding, we can define ϕ by setting

$$\phi = M_C \xrightarrow{p} M_C/X \xrightarrow{\phi'} \mathrm{Soc}M_C \xrightarrow{m} M_C$$

where

$$\phi'((0, 1) + X) = (x, 0) + D.$$

Now suppose that there is $r \in R$ such that

$$\phi((0, 1) + D) = r((0, 1) + D)$$

or

$$(x, 0) + D = (0, r) + D,$$

then

$$(x,0) - (0,r) \in D \text{ or } (x,-r) \in D.$$

Since by assumption $y \notin Rx$ and $Wy = 0$, $Rx \cap Ry = 0$. For if there is $0 \neq r \in Rr \cap Ry$, then $r = r'x = r''y$ with units r and $r'' \in R$. It follows that

$$(r'')^{-1}r'x = y$$

and so $y \in Rx$ which is a contradiction. Now if $(x, -r) \in D$, then $x \in Ry$ and hence from the above remark $x = 0$. This contradiction allows us to conclude that M is not balanced and the proof of the construction is completed.

CHAPTER IV

ARTINIAN LOCAL QF-1 RINGS FINITELY GENERATED OVER THEIR CENTERS ARE QF

This chapter is devoted to presenting the assertion stated as the title as a consequence of the preceeding Constructions. It can be considered as a partial solution to Thrall's problem. This is a generalization of theorems of D.R. Floyd [6], and that of Dickson and Fuller [5]. The following results will be needed.

LEMMA IV.1. Let R be a local right perfect left QF-1 ring with a minimal right ideal. Then R has a unique minimal two-sided ideal I and, moreover,

(i) $\partial_1(I) = 1$ and I is the left socle of R, or

(ii) $\partial_1(I) = 1$ and I is the right socle of R, or

(iii) $\partial_1(I) = 2$ and I is both the left and right socle of R.

PROOF. Case 1:

Assume that there is a two-sided ideal $I \subseteq \mathrm{Soc}R_R$ with $\partial_1(I) = 1$. In this case, we claim that I is the unique minimal two-sided ideal and that I is the left or the right socle of R. To check that I is the unique minimal two-sided ideal, we let I_2 be any non-zero two-sided ideal of R. The hypothesis of Construction I would be satisfied if we let $I = I_1$, $U_1 = I_1$ and U_2 be any simple left-submodule of I_2. Note that U_2 exists because R is right perfect so that every non-zero left module has a non-zero socle. Now since R is local, it has only one isomorphism type of simple left ideals and we have U_1 isomorphic to U_2. Consider $I_1 \cap I_2$ which is a two-sided ideal. If $I_1 \cap I_2 = 0$, the hypothesis of Construction I is satisfied contradicting the fact that R is QF-1. Hence $I_1 \cap I_2 \neq 0$ and since I_1 is left simple, $I_1 \cap I_2 = I_1$, i.e. $I_1 \subseteq I_2$. This means that I_1 is contained in every two-sided ideal of R.

To check the rest of the claim - that I is the left or the right socle of R, we make use of Construction III. Suppose that I is neither the left nor the right socle of R, take a minimal left ideal U_2 which is not contained in I and let $U_1 = I$. The assumptions of Construction III are satisfied and we obtain a contradiction.

<u>Case 2</u> : Suppose that there exists no two-sided ideal of left length 1 in $SocR_R$ and denote by I the left socle of S. Then $\partial_1(I) \geq 2$. Now I is a unique minimal two-sided ideal and is the left socle of R; for, otherwise, we can apply again Construction I or Construction III to obtain a contradiction as above. It remains to show that I is the right socle of R and, to this end, we make use of Construction IV. Supposing the contrary, say, $SocR_R \not\subseteq I$. Let U be a non-zero minimal left ideal contained in S, since R is right perfect. Now since U annihilates the non-units, there a unit $r \in R$ such that $Ur \not\subseteq U$. Since I is the left socle of $SocR_R$, $I \subseteq SocR_R$ and it follows that

$$SocR_R \not\subseteq U + Ur \subseteq I$$

for otherwise we would contradict our assumption that $I \neq \mathrm{Soc}R_R$. Now we can apply Construction IV to obtain a contradiction and consequently we conclude that

$$I = \mathrm{Soc}R_R.$$

Furthermore by using Construction IV again we easily check that

$$\partial_1(I) = \partial_1(S) \leq 2.$$

The proof of Lemma IV.1 is completed.

We proceed to examine further the left and right socles of the local left QF-1 rings. In view of the following lemma, the third case of Lemma Iv.1 will be eliminated for rings which are finitely generated over their centers.

LEMMA IV.2. Let R be a left artinian left QF-1 local ring. Then for any two non-zero

$$x, y \in S = \mathrm{Soc}R_R \cap \mathrm{Soc}_R R,$$

the following holds,

$$Rx + yR = S.$$

PROOF. From Lemma IV.1, S is a minimal two-sided ideal and $\partial_1(S) \leq 2$. When $\partial_1(S) = 1$, Rx is a non-zero left ideal contained in S and hence S = Rx. Now from a remark on page 31 (proof of Construction IV) Rx is simple. Therefore when $\partial_1(S) = 2$, Rx cannot be a two-sided ideal contained in S, and the same goes for Ry. Now the equality

$$Rx + yR = S$$

must hold, or else we can apply Construction VI to obtain a contradiction.

Our next lemma modifies the result of Lemma IV.1, where a perfect left QF-1 local ring was shown to possess a unique minimal two-sided ideal which was the left or the right socle.

LEMMA IV.3. Let R be a left artinian left QF-1 local ring. Assume that R is finitely generated over its center. Then the unique minimal two-sided ideal is both a minimal left ideal and a minimal right ideal.

PROOF. Let I be the unique minimal two-sided ideal. Now

$$WI = IW = 0$$

where $W = \operatorname{rad} R$, because I belongs to both the left and the right socle of R. If K stands for the center of R, we see that $(K + W)/W$ is contained in the center of the division ring R/W. Form the quotient field F of $(K + W)/W$ and consider it as subring of R/W. In view of the R/W - R/W - bimodule structure of I, it follows that R/W is finitely generated over K. Hence we can consider R/W as a finite-dimensional vector space over the quotient field F.

Now $\dim({}_F R/W) = \dim(R/W_F)$ because $(K + W)/W$ is contained in the center of R/W. It follows that

$$\dim({}_F R/W).\dim({}_{R/W} I) = \dim({}_F I)$$

$$= \dim(I_F)$$

$$= \dim(I_{R/W}) \cdot \dim(R/W_F)$$

implies

$$\dim({}_{R/W}I) = \dim(I_{R/W}).$$

We would be through if

$$\partial_1(I) = \dim({}_{R/W}I) = 1.$$

Hence, because of Lemma IV.1, we only have to show that $\dim({}_{R/W}I) \neq 2$. Say $\dim({}_{R/W}I) = 2$ and applying lemma IV.2, then

$$Rx + xR = I$$

for all $0 \neq x \in I$. Whence, if $n = \dim({}_F R/W)$ and $0 \neq x \in I$,

$$\begin{aligned} 2n &= \dim({}_{R/W}I) \cdot \dim({}_F R/W) \\ &= \dim({}_F I) \\ &= \dim({}_F(Rx + xR)) \\ &= \dim_F Rx + \dim_F xR - \dim_F(Rx \cap xR) \\ &= \quad n \quad + \quad n \quad - \dim_F(Rx \cap xR). \end{aligned}$$

This is a contradiction because, obviously, $\dim_F(Rx \cap xR) \neq 0$. Hence

$$\dim({}_{R/W}I) = \dim(I_{R/W}) = 1.$$

In order to prove the main theorem in this chapter, we need still another result.

Lemma IV.4. Let R be a left artinian left QF-1 local ring. Then the left socle of R is the unique minimal two-sided ideal.

PROOF. Write the left and the right socles of R as S_l and S_r

respectively. Assume that S_l is not the unique minimal two-sided ideal of R. In view of Lemma IV.1, it follows that S_r is properly contained in S_l and that $\partial_l(S_r) = 1$. We want to show that the intersection S of the left and the right socles of R/S_r is contained in S_l/S_r. So let $0 \neq x \in S_r$, then since

$$\partial_l(S_r) = 1, \; Rx = S_r$$

and since S_r is a two-sided ideal, so is Rx. Now let $y + S_r$ be a non-zero element of the right socle of S_l/S_r, that is,

$$y \notin S_r = Rx, \quad Wy = 0$$

and

$$yW \subseteq S_r = Rx.$$

Let z be an arbitrary element such that $z + S_r$ belongs to S. Then

$$Wz + zW \subseteq S_r = Rx.$$

Now the assumptions of Construction VII are satisfied for our choice of x, y and z except for the condition that

$$z \notin Rx + yR.$$

Since R by assumption is a left QF-1 ring, Construction VII implies that z is necessarily contained in $Rx + yR \subseteq S_l$ and hence

$$S \subseteq S_l/S_r.$$

Now, if $W^n \neq 0$ and $W^{n+1} = 0$, then since S_r is the unique minimal two-sided ideal (proof of Lemma IV.1), $S_r = W^n$. W^{n-1}/W^n is contained in the intersection of the left and the right socles of R/W^n because W^{n-1}/W^n annihilates W/W^n on both sides. This implies that

$$W^{n-1} \subseteq S_1.$$

Hence,

$$W^n = W.W^{n-1} \subseteq W.S_1 = 0,$$

contradicting our hypothesis. Since the assumption that S_1 was not the unique minimal two-sided ideal leads to this contradiction, we have proven the lemma.

Finally, we are in a position to prove our main assertion.

THEOREM IV.5. Let R be a two-sided artinian local ring finitely generated over its center. Then R is a QF-1 ring if, and only if it is QF.

PROOF. ($\Longleftarrow$)
This is trivial.

($\Longrightarrow$) We first apply Lemma IV.4 both to the right and the left of R. Then

$$\mathrm{Soc}R_R = I = \mathrm{Soc}_R R$$

is the unique minimal two-sided ideal of R. Now by Lemma IV.3, I is both a minimal left ideal and a minimal right ideal.

Let U be a non-zero left ideal of R and let U' be a maximal left subideal of U. We shall denote the left and the right annihilators of U by $l(U)$ and $r(U)$ respectively. If $a \in R$, then the left ideal

$$Ua \cong U/U \cap l(a).$$

If, in particular, $a \in r(U')$, then

$$l(a) \supseteq l(r(U')) \supseteq U',$$

and consequently,

$$U \cap l(a) \supseteq U \cap l(r(U')) \supseteq U \cap U' = U' .$$

Hence Ua is either left simple or equal to 0 because U' is maximal in U. This implies that, since $a \in r(U')$,

$$Ur(U') \subseteq \mathrm{Soc} R_R = \mathrm{Soc}_R R = I$$

but because I is (both left and right) simple we actually have $Ur(U')$ equal to I or 0. Taking $b \in U - U'$, we obtain

$$Rb + U' = U.$$

Moreover, by the preceeding remark

$$br(U') = I \text{ or } 0.$$

It follows that

$$br(U') = r(U')/r(U') \cap r(b)$$

is right simple or equal to 0. However,

$$r(U') \cap r(b) = r(U' + Rb) = r(U)$$

and hence

$$br(U') = r(U')/r(U)$$

is right simple or equal to 0.

Now consider a composition series of left ideals of length n,

$$U_o = 0 \subseteq U_1 \subseteq U_2 \subseteq \dots \subseteq U_n = R.$$

Correspondingly, we obtain a new series of right ideals given by,

$$r(U_o) = R \supseteq r(U_1) \supseteq r(U_2) \supseteq \ldots \supseteq r(R) = 0$$

having $r(U_i)/r(U_{i+1})$ $(i = 0, \ldots , n-1)$ either right simple or 0 in view of the above consideration. This means that

$$\partial_r(R) \leq \partial_l(R).$$

However, the inequality of the other direction can be seen in the same manner, had we started with right ideals instead of left ideals. Hence

$$\partial_r(R) = \partial_l(R)$$

and $r(U_i)/r(U_{i+1})$, $i = 0, \ldots , n-1$, is right simple for every i. Now, using the same argument, we have that

$$l(r(U_i))/l(r(U_{i-1})),\ i = 1,\ldots,n,$$

is simple for every i and

$$l(r(0)) = 0 \subseteq l(r(U_1)) \subseteq l(r(U_2)) \subseteq \ldots \subseteq l(r(U_{n-1})) \subseteq R = lr(R)$$

is a composition series of length n. Comparing the series with the following

$$U_o = 0 \subseteq U_1 \subseteq U_2 \subseteq \ldots \subseteq U_{n-1} \subseteq U_n = R,$$

we see that necessarily

$$l(r(U_{n-1}) = U_{n-1}$$

since they are both maximal in R and $U_{n-1} \subseteq l(r(U_{n-1}))$. Applying the same argument on the corresponding left ideals down the two series, we obtain

$$l(r(U_i)) = U_i$$

for every $i = 0, \ldots, n$. Analogously, for any right ideal J,

$$r(l(J)) = J.$$

Therefore R is a QF-ring and the proof of the theorem is completed.

We have collected the theorems of K.R. Fuller and S.E. Dickson in [5] as well as that of D.R. Floyd in [6] as immediate consequences of Theorem IV.5.

COROLLARY IV.6. (K.R Fuller and S.E. Dickson)

Let R be a commutative artinian ring. Then R is QF-1 if, and only if, it if QF.

PROOF. R is clearly finitely generated over itself. Moreover R is a finite product of commutative artinian local rings and since the property of being QF-1 is preserved under finite products, this corollary follows from Theorem IV.5.

COROLLARY IV.7. (D.R. Floyd)

A commutative finite dimensional algebra over a field is QF-1 if, and only if, it is QF.

PROOF. Since every finite dimensional algebra is an artinian ring, this corollary follows from the previous one.

BIBLIOGRAPHY

(1) E.A. Behrens, Ring Theory, Academic Press, Inc., 1972.

(2) V.P. Camillo, Balanced rings and a problem of Thrall. Trans. Amer. Math. Soc., Vol. 149 (1970) pp. 143 - 153.

(3) C.W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras. Interscience, New York, 1962.

(4) V. Dlab and C.M. Ringel, Rings with the double centralizer property. To appear in J. Algebra.

(5) S.E. Dickson and K.R. Fuller, Commutative QF-1 artinian rings are QF. Proc. Amer. Math. Soc. 24 (1970), 667 - 670.

(6) D.R. Floyd, On QF-1 algebra, Pacific J. Math. 27 (1968), 81 - 94.

(7) J. Lambek, Lectures on Rings and Modules, Blaisdell Publishing Company, 1966.

(8) K. Morita, On algebras for which every faithful representation is its own second commutator. Math. Z. 59 (1958), 429 - 434.

(9) R.M. Thrall, Some generalizations of QF algebras, Trans. Amer. Math. Soc. 64 (1948), 173 - 183.

(10) R.M. Thrall and C.J. Nesbitt, Some ring theorems with applications to modular representations, Ann. Math. 47 (1946) 551 - 567.