INJECTIVE HULLS OF MODULES

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A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

October, 1966

DOCTOR OF PHILOSOPHY (1966)
(Mathematics)

McMASTER UNIVERSITY,
Hamilton, Ontario.

TITLE: Injective Hulls of Modules

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SUPERVISOR: Professor B. Banaschewski

NUMBER OF PAGES: viii, 59

SCOPE AND CONTENTS:

By an injective hull of a module M over any ring we mean a minimal injective extension module $E \supseteq M$. Our main objective in this thesis is to find an explicit description of injective hulls of modules in some special cases and to study their properties.

ACKNOWLEDGEMENTS

I deem it a great privilege to acknowledge my indebtedness to Professor B. Banaschewski for his guidance and help throughout my stay at the McMaster University.

My sincere thanks are due to Mr. H. P. Doctor for many valuable discussions on Boolean rings.

I am also grateful to Mrs. Maureen von Lieres for the prompt typing of my thesis in spite of her busy schedule.

For financial assistance I would like to express my grateful thanks to McMaster University.

Last, but not the least, I am most thankful to the Authorities of the Banaras Hindu University, India, for enabling me to complete my research work here.

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INTRODUCTION

The concept of injectivity first arose in connection with abelian groups when in 1940, R. Baer defined and studied "Abelian subgroups that are direct summands of every containing abelian group" [8]. Modules over arbitrary rings with the above property were later on called injective and several equivalent conditions characterizing these were obtained [9]. A major step in the theory of injective modules was achieved when B. Eckmann and A. Schopf proved that for every module M over a ring, there exist injective hulls, i.e. essential and injective extensions, and any two of these are isomorphic over M. However, the proofs showing the existence of these hulls use maximality arguments and as such they do not provide an explicit method of construction. It is, therefore, natural that attempts be made to describe injective hulls of modules by means of explicit constructions. In some cases this has already been done. For Z/Zp where Z is the ring of integers and p a prime number, one knows that $Z[p^{-1}]/Zp$ is an injective hull. In a recent paper [2] B. Banaschewski has proved that if J is a non-zero proper ideal of a Dedekind domain R, then $\bigcup J^{-k}/J$ is an R-injective hull of R/J, which generalizes one of his previous results for principal ideal domains [3]. B. Brainerd and J. Lambek have proved [7] that a complete Boolean ring is injective as a module over itself, and this fact suggests that one might succeed in obtaining injective hulls of ideals and quotients of a Boolean ring by suitably defining their completions. Another clue concerning the construction of an injective hull of a semi-simple module over a regular ring, is provided by the fact that all simple modules over regular rings are injective [21].

On the basis of the above suggestions we examine in this thesis methods of constructing injective hulls of modules in some special cases and give characterizations of these in suitable terms. In addition to these we study some properties of injective modules. A brief synopsis of the material included here is given below.

In Chapter 0, we collect together basic theorems and definitions which we utilize in later chapters. In particular we list the properties of semi-simple modules, injective modules, rings of quotients, Noetherian rings and Dedekind domains.

Chapter I deals with semi-primary modules. E. Matlis has proved [18] that over a left Noetherian ring every injective module is a direct sum of indecomposable injective submodules. Here we give characterizations for injective modules over left Noetherian rings to be semi-primary. For such modules we define the injective length and show that any two of these are isomorphic if and only if their injective lengths are the same. In the special case of a torsion module over a Dedekind domain, some properties of its injective hull are obtained.

Chapter II is concerned with the study of injective hulls of modules over a Boolean ring with unit. Here we give a description of the injective hulls of cyclic modules as well as of the ideals considered as modules over the ring. The main tool in this construction is the concept of Boolean completion. In the case of normal ideals it is proved that the injective hull of a quotient of the ring by a normal ideal is the quotient of their respective injective hulls.

In Chapter III we consider semi-simple modules over commutative regular rings and show that each monotypic component is injective. We also prove a topological lemma that if A and B are any two non-void subsets of a T_1-space with the property that each one of their non-void subsets has an isolated point, then $A \cup B$ itself has this property. These facts are used in order to find an explicit description of an injective hull of a semi-simple module. The last theorem in this chapter gives a characterization for a semi-simple module to be injective.

In Chapter IV we study the inheritance properties of R-injective modules E as the ring R is changed into a suitably related ring S and E is made into an S-module. It turns out that this device of changing rings is convenient in proving some interesting properties about injective hulls. In particular we generalize the fact that $Z(p^{\infty})$ is isomorphic to each of its non-zero homomorphic images, to indecomposable injective modules over rings more general than Dedekind domains. In the last theorem we consider two rings R and S suitably related and obtain an injective hull of a module over R from one over S and use this to show that if R is a Dedekind domain, $P \subsetneq R$ a proper prime ideal, then the R/P^k- injective hull of R/P is R/P^k.

CHAPTER 0

PRELIMINARIES

This chapter is essentially a collection of all the basic theorems and definitions which will be needed in the ensuing chapters.

1. Modules and Homomorphisms

Let R be a ring with unit 1.

Definition 1. A left R-module (or a left module over R) is an additive abelian group together with a mapping $m : R \times M \longrightarrow M$ such that for all $a, b \in R$, $x, y \in M$, the following conditions are satisfied:

(1) $m(a, x + y) = m(a, x) + m(a, y)$

(2) $m(ab, x) = m(a, m(b, x))$

(3) $m(a + b, x) = m(a, x) + m(b, x)$

(4) $m(1, x) = x$

We usually write $a\,x$ for $m(a, x)$ and call the operation m, the scalar multiplication. Right modules can similarly be defined. We shall deal only with left R-modules and hence call these simply R-modules or even modules if the reference is clear.

A submodule N of a module M is an additive subgroup such that $RN \subseteq N$.

Definition 2. Let M and M' be two modules. An additive group homomorphism $f : M \to M'$ is called a module homomorphism if $f(ax) = af(x)$ for all $a \in R$, $x \in M$. A one-to-one, onto homomorphism is called an isomorphism.

Definition 3. If $\{M_i\}_{i \in I}$ is a family of submodules of a module M, then $\sum_{i \in I} M_i$ which consists of all possible finite sums of elements from the various modules M_i, is clearly a submodule of M. We call $\sum_{i \in I} M_i$ the sum of the submodules M_i. It is also the smallest of all submodules of M containing the M_i.

Definition 4. A family of submodules $\{M_i\}_{i \in I}$ of a module M, is said to be free if $M_i \cap \sum_{i \neq j \in I} M_j = 0$ for all $i \in I$.

A module M is said to be a direct sum of submodules $\{M\}_{i \in I}$ if and only if

(1) $\{M_i\}_{i \in I}$ is a free family of submodules of M.

(2) $M = \sum_{i \in I} M_i$.

We shall express this by writing $M = \sum_{i \in I} M_i(\text{dir})$. A submodule N of M is called a direct summand of M if there exists a submodule N' of M such that $M = N + N'$ (dir).

Direct product and external sum.

Let $\{M_i\}_{i \in I}$ be a family of modules.

<u>Definition 5.</u> We consider all functions $f : I \to \bigcup_{i \in I} M_i$ and define $\prod_{i \in I} M_i = \{ f : I \to \bigcup M_i \mid f(i) \in M_i \text{ for all } i \in I \}$ then with respect to the addition and scalar multiplication defined by

$$\left.\begin{aligned} (f + g)(i) &= f(i) + g(i) \\ (af)(i) &= af(i) \end{aligned}\right\} \text{ for all } a \in R,\ i \in I,$$

it is clearly a module. We call $\prod_{i \in I} M_i$ the <u>direct product</u> of the family $\{M_i\}_{i \in I}$.

Now define $\bigoplus_{i \in I} M_i = \{ f \in \prod_{i \in I} M_i \mid f(i) = 0$ for all but finitely many $i \in I \}$ then $\bigoplus_{i \in I} M_i$ is a submodule of $\prod_{i \in I} M_i$ and it is called the <u>external sum</u> of modules M_i.

It can be shown that if $\{M_i\}_{i \in I}$ is a free family of submodules of a module M, then $\sum_{i \in I}' M_i \cong \bigoplus_{i \in I} M_i$.

<u>Definition 6.</u> If N is a submodule of a module M, then $M/N = \{ x + N \mid x \in M \}$ with scalar multiplication defined by $r(x + N) = rx + N$ is a left module and it is called the <u>quotient module</u> of M by the submodule N. The mapping $\nu : M \to M/N$ by $\nu(x) = x + N$ is called the <u>natural homomorphism</u>.

<u>Definition 7.</u> A module M is called <u>simple</u> if M is non-zero and contains no proper submodules. A module M is <u>indecomposable</u> if it is non-zero and its only direct summands are 0 and M.

<u>Definition 8.</u> A non-zero module M is said to be <u>semi-simple</u> if it is expressible as a sum of simple submodules.

The following theorem characterizes semi-simple modules:

Theorem 1. For a module M, the following three conditions are equivalent:

(1) M is semi-simple.

(2) M is a direct sum of simple submodules.

(3) Every submodule of M is a direct summand.

By Zorn's lemma, every free set of simple submodules of a semi-simple module is contained in a maximal free set of simple submodules. Moreover, a free set $\mathfrak{A}$ of simple submodules of a semi-simple module M is a maximal such set if and only if $M = \sum_{S \in \mathfrak{A}} S$.

Theorem 2. Any two maximal free sets of simple submodules of a semi-simple module have the same cardinality.

In view of the above theorem we have the following:

Definition 9. The length of a semi-simple module M is the cardinality of any maximal free set of simple submodules of M. We will denote the length of M by k(M).

Definition 10. Let Ω denote the set of maximal left ideals of R and M a semi-simple module. For any simple submodule $S \subseteq M$, there exists $P \in \Omega$ such that $S \cong R/P$. Let T be the set of isomorphism classes of $\{R/P \mid P \in \Omega\}$ For each $t \in T$, choose $S_t \in t$ and define $M_t = \sum_{S_t \cong S \subseteq M} S$ (S simple). Then M_t is called the t-monotypic component of M. Note that $M_t = 0$ if and only if there is no simple submodule $S \subseteq M$ with $S \cong S_t$.

If R is commutative then for each $t \in T$, there exists exactly one $P \in \Omega$ such that $t = \{R/P\}$. In this case we put $M_P = M_t$ and call M_P, the R/P - monotypic component of M.

Theorem 3. For a semi-simple module M, we have $M = \sum_{t \in T} M_t$ (dir). It follows that $M \cong \bigoplus_{t \in T} M_t$. Let M be a semi-simple module. For each $t \in T$, let $k_t(M) = k(M_t)$. Thus we can associate with M, a family $(k_t(M))_{t \in T}$ of cardinal numbers.

Theorem 4. Two semi-simple modules M and N are isomorphic if and only if $(k_t(M))_{t \in T} = (k_t(N))_{t \in T}$.

We now consider an arbitrary module M. The set L(M) of all submodules of M is partially ordered by $\leq$ and is a complete lattice with sum as join and intersection as meet.

Theorem 5. Let N be a submodule of a module M. Then the lattice of submodules of M/N is isomorphic to the lattice of submodules of M which contain N. Furthermore we have $M/L \cong (M/N)/(L/N)$ whenever $M \supseteq L \supseteq N$.

Definition 11. Let M be a module. For $x \in M$ define $O(x) = \{ r \in R \mid r\,x = 0 \}$, then O(x) is a left ideal of R and is called the order-ideal of x. It follows that $R\,x \cong R/O(x)$.

A module M is called a torsion module if $O(x) \neq 0$ for every $x \in M$.

Definition 12. For a module M, the socle of M denoted by S(M) is the sum of all simple submodules of M. It is also called the semi-simple part of M.

2. Rings of Quotients

Let R be a commutative ring with unit 1 and $S \subseteq R$ a multiplicatively closed subset.

Definition 13. For an ideal $J \subseteq R$, define $J_S = \{x \in R \mid$ there exists $c \in S$ with $c\,x \in J\}$ then J_S is called the S-component of J.

Definition 14. Let S be a multiplicatively closed subset of R such that $0 \notin S$. Let N be the S-component of the zero ideal and $\nu : R \to R/N$, the natural homomorphism. Then $\nu(S)$ is a multiplicatively closed subset of R/N and has no zero divisors in R/N. Let $R_S = \{(\nu(r)/\nu(s)) \mid r \in R,\ s \in S\}$, then R_S is called the generalized ring of quotients of R with respect to S.

Special Case: If $P \subsetneq R$ is a proper prime ideal then $S = R \setminus P$ is a multiplicatively closed subset of R with $0 \notin S$. In this case it is customary to denote R_S by R_P. The following hold in this case:

(1) $\nu(r) \in \nu(P)$ if and only if $r \in P$.

(2) $R_P\,\nu(P) \cap \nu(R) = \nu(P)$.

(3) R_P is a local ring with $R_P\,\nu(P)$ as its unique maximal ideal.

3. Noetherian Rings and Dedekind Domains.

Definition 15. A ring R is called a left Noetherian ring if any one of the following equivalent conditions holds:

(1) Every left ideal in R is finitely generated.

(2) R satisfies the ascending chain condition for its left ideals.

(3) Every non-void set of left ideals of R contains a maximal member.

Definition 16. An integral domain R is said to be a Dedekind domain if and only if

(1) R is a Noetherian ring

(2) Every proper prime ideal in R is maximal

(3) R is integrally closed in its quotient field K;

that is if $\alpha \in K$ is a zero of a monic polynomial belonging to $R[x]$, then $\alpha \in R$.

Definition 17. Let R be a Dedekind domain with K its quotient field. A fractionary ideal in K is a non-zero finitely generated R-submodule of K. For any ideal A in R, define A^{-1} by $A^{-1} = \{\alpha \in K \mid \alpha A \subseteq R\}$.

Theorem 6. Every proper ideal A in a Dedekind domain is a product of prime ideals and this decomposition is unique apart from the order in which the factors appear. Conversely an integral domain in which every proper ideal is a product of prime ideals, is a Dedekind domain.

Theorem 7. If K is the quotient field of a Dedekind domain R, then the fractionary ideals in K form an abelian multiplicative group with identity element R and in which the inverse of A is A^{-1}.

<u>Definition 18</u>. The ideals $A_1, A_2, \ldots, A_n$ in a Dedekind domain R are called <u>pairwise comaximal</u> if $A_i \neq R$ and $A_i + A_j = R$ for all $i, j = 1, \ldots, n$, with $i \neq j$.

<u>Theorem 8</u>. If $A_1, A_2, \ldots, A_n$ are pairwise comaximal ideals in R, then $R/A_1 A_2 \ldots A_n = \bigoplus_{k=1}^{n} R/A_k$.

<u>Corollary</u>. Let M be a torsion module over a Dedekind domain R. If $x \in M$ with $O(x) = P_1^{k_1} \ldots P_n^{k_n}$, all P_i distinct, then $Rx \cong R/P_1^{k_1} \oplus \cdots \oplus R/P_n^{k_n}$.

<u>Theorem 9</u>. For every proper prime ideal P in a Dedekind domain R, the ring of quotients R_P is a principal ideal ring with $R_P P$ as its unique maximal ideal.

4. <u>Injective Modules</u>

In this section we will consider an arbitrary but fixed ring R with unit 1. All modules will be left R-modules and all homomorphisms will be R-module homomorphisms.

<u>Definition 19</u>. A module M is said to be <u>injective</u> if for any two modules A and B with $A \subseteq B$ and any homomorphism $f : A \to M$, there exists a homomorphism $\bar{f} : B \to M$ which extends f.

<u>Definition 20</u>. A module M is called an <u>essential extension</u> of a module N if and only if N is a submodule of M and every non-zero submodule of M has a non-zero intersection with N.

Thus in order to show that $M \supseteq N$ is an essential extension of N, it is sufficient to prove that $Rx \cap N \neq 0$ for all non-zero $x \in M$. If $M \supseteq N$ is an essential extension, we will say that N is a <u>large</u> submodule of M.

<u>Remark</u>. Sometimes an additive group may be considered with module structures over different rings. In such a case we will use R-injective module or R-essential extension if the notions of injectivity or largeness refer to the considered R-module structure of the group. About essential extensions we note the following:

(1) A union of a directed set of essential extensions of M is itself an essential extension.

(2) If $A \subseteq B \subseteq C$ such that A is large in B, B is large in C, then A is large in C.

<u>Theorem 10</u>. For a module M, the following conditions are equivalent:

(1) M is injective

(2) M has no proper essential extension.

(3) M is a direct summand of every module which contains it

(4) For each left ideal $I \subseteq R$ and each homomorphism $f : I \to M$ (I being considered as an R-module), there exists an element $x \in M$ such that $f(a) = ax$ for each $a \in I$.

<u>Theorem 11</u>. A direct product of modules is injective if and only if each factor is injective.

<u>Corollary</u>. A finite direct sum of injective modules is injective if and only if each summand is injective.

Theorem 12. Every module can be imbedded in an injective module.

Definition 21. A module H is said to be an injective hull of a module M if H is an injective module containing M and is minimal with respect to this property.

Theorem 13. (Eckmann and Schopf [13]). Let M be a module. Then

(1) Any injective module containing M, contains an injective hull of M.

(2) A module $H \supseteq M$ is an injective hull of M if and only if H is a maximal essential extension of M.

(3) If $\varphi : M \to M'$ is an R-module isomorphism and H and H' are injective hulls of M and M' respectively, then φ can be extended to an isomorphism $\bar{\varphi} : H \to H'$.

Remark. In view of the above theorem any two injective hulls of a module M are isomorphic with respect to an isomorphism which maps M identically. We will use the notation $E = H(M)$ to express that E is an injective hull of M. Where no ambiguity can arise, we let H(M) stand for an arbitrary injective hull of M.

Theorem 14. [18] For a module M, the following conditions are equivalent:

(1) H(M) is indecomposable

(2) H(M) is an injective hull of each of its non-zero submodules

(3) M contains no non-zero submodules S and T such that $S \cap T = 0$.

Theorem 15. [18] Let $M = \sum_{i \in I} M_i$ (dir). Then

(1) If $I = \{1, 2, \ldots, n\}$ is finite, then $H(M) \cong \bigoplus_{i=1}^{n} H(M_i)$

(2) If R is left Noetherian, then $H(M) \cong \bigoplus_{i \in I} H(M_i)$.

The isomorphism, in either case, is given by extending the natural isomorphism $M \to \oplus M_i$.

Theorem 18. [18] Over a left Noetherian ring every injective module is a direct sum of indecomposable injective submodules.

Theorem 19. [2] If R is a Dedekind domain and $J \subsetneq R$ is any non-zero proper ideal of R, then $J^*/J = H(R/J)$ where $J^* = \sum J^{-k} = \bigcup J^{-k}$.

CHAPTER I

SEMI-PRIMARY MODULES

This chapter is devoted to the study of injective semi-primary modules over left Noetherian rings. We obtain a characterization for an injective module to be semi-primary in terms of a maximal free set of indecomposable injective submodules. For such a module we introduce the concept of injective length and establish that this length is invariant under isomorphism. This enables us to give a description of the injective hull of a semi-primary module.

To begin with we consider R to be an arbitrary ring with unit 1. For an R-module M and $x \in M$, recall that $\mathcal{S}(M)$ denotes the socle of M and O(x) denotes the order ideal of x.

Before we begin our study about semi-primary modules in general, we would like to note the following facts about arbitrary modules.

Lemma 1. Let E be an essential extension of a module M. Then $\mathcal{S}(E) = \mathcal{S}(M)$.

Proof. For any simple submodule $\mathcal{S} \subseteq E$, we have $\mathcal{S} \cap M \neq 0$. Let x be any non-zero element in $\mathcal{S} \cap M$, then $0 \neq R x \subseteq \mathcal{S}$ implies that $\mathcal{S} = R x \subseteq M$ since $\mathcal{S}$ is simple. Hence, $\mathcal{S}(E) \subseteq \mathcal{S}(M)$. The reverse inclusion always holds, hence $\mathcal{S}(M) = \mathcal{S}(E)$.

Lemma 2. If M is any module such that $M = \sum M_i$ (dir), then $\mathcal{S}(M) = \sum \mathcal{S}(M_i)$ (dir).

Proof. Any simple submodule of M_i is also a simple submodule of M. Thus $S(M_i) \subseteq S(M)$, hence $\sum S(M_i) \subseteq S(M)$. To show the inclusion the other way round, let $S \subseteq M$ be any simple submodule. Then $\mathcal{S} = Rx$ with $0 \neq x \in M$. Now with respect to the direct sum decomposition $M = \sum M_i$, x can be expressed as $x = x_1 + \ldots + x_n$ with $0 \neq x_k \in M_{i_k}$. Consider the homomorphism $Rx \longrightarrow Rx_k$ by $rx \longrightarrow rx_k$. It is clearly non-zero and onto, hence one-to-one since $\mathcal{S}$ is simple. Thus $Rx \cong Rx_k$ and so $Rx_k \subseteq \mathcal{S}(M_{i_k})$. This implies that $x = \sum x_k \in \sum \mathcal{S}(M_{i_k}) \subseteq \sum \mathcal{S}(M_i)$. Hence $\mathcal{S} = Rx \subseteq \sum \mathcal{S}(M_i)$. It follows that $\mathcal{S}(M) \subseteq \sum \mathcal{S}(M_i)$. Thus $\mathcal{S}(M) = \sum \mathcal{S}(M_i)$.

Definition 1. A left R-module M is said to be semi-primary if and only if for any non-zero submodule A of M, $\mathcal{S}(A) \neq 0$. Thus for M to be semi-primary, it is sufficient that $\mathcal{S}(Rx) \neq 0$ for every non-zero element $x \in M$.

The following proposition characterizes semi-primary modules:

Proposition 1. For a left module M, the following conditions are equivalent:

(1) M is semi-primary.

(2) $\mathcal{S}(M)$ is a large submodule of M.

(3) For every non-zero element $x \in M$, there exists in the lattice of left ideals of R, a left ideal minimal above $O(x)$.

Proof. (1) implies (2): Let M be semi-primary. Then for any non-zero submodule A of M, $\mathcal{S}(A) \neq 0$. Hence $A \cap \mathcal{S}(M) \neq 0$ since $\mathcal{S}(A) \subseteq A \cap \mathcal{S}(M)$. This shows that $\mathcal{S}(M)$ is a large submodule of M.

(2) implies (3): Let x be a non-zero element in M. Then by (2), $Rx \cap \mathcal{S}(M) \neq 0$. Since $Rx \cong R/O(x)$, this implies that $R/O(x)$ contains a simple submodule $S/O(x)$ where S is a left ideal of R. Thus $O(x)$ is a left ideal maximal in S, that is to say, S is a left ideal minimal above $O(x)$.

(3) implies (1): Take any non-zero element $x \in M$. Then by (3), there exists a left ideal $S \subseteq R$ such that $O(x)$ is a left ideal maximal in S. Hence $S/O(x)$ is simple. Since $S/O(x) \subseteq R/O(x) \cong Rx$, one has $\mathcal{S}(Rx) \neq 0$. Thus M is semi-primary.

Remark 1. It follows from the Proposition that every semi-simple module is semi-primary.

Corollary 1. Every non-zero submodule and every essential extension of a semi-primary module M is semi-primary.

<u>Proof</u>. Consider any non-zero submodule A of M. If B is a non-zero submodule of A, then $0 \neq \mathcal{S}(B) \subseteq B \cap \mathcal{S}(A)$ which shows that $\mathcal{S}(A)$ is a large submodule of A. Hence A is semi-primary.

Now, let $E \geq M$ be an essential extension of M. Take any non-zero submodule A of E. Then $A \cap M \neq 0$ implies $0 \neq \mathcal{S}(A \cap M) \leq A \cap \mathcal{S}(E)$. This shows that $\mathcal{S}(E)$ is large in E. Thus E is semi-primary.

<u>Corollary 2</u>. Torsion modules over a Dedekind domain are semi-primary.

<u>Proof</u>. Let R be a Dedekind domain and M a torsion module over R. Take any non-zero element $x \in M$. Then $O(x) \neq 0$ and $R/O(x)$ satisfies the descending chain condition [25]. Hence there exist ideals of R minimal above O(x). Therefore M is semi-primary by Proposition 1, (3).

For the next theorem we need the following facts about semi-primary modules:

<u>Lemma 3</u>. A semi-primary injective module I is indecomposable if and only if $\mathcal{S}(I)$ is simple.

<u>Proof</u>. Let I be indecomposable. Since I is semi-primary injective, by Proposition 1, I is an injective hull of its socle $\mathcal{S}(I)$. Suppose $\mathcal{S}(I)$ is not simple. Then there exists a simple submodule $S \subset \mathcal{S}(I)$. Since $\mathcal{S}(I)$ is semi-simple, $\mathcal{S}(I) = S + T$ (dir) for some non-zero submodule T of S(I). This implies that I = H(S(I) = H(S) + H(T) (dir) contrary to the hypothesis that I is indecomposable.

Hence S(I) is simple. Conversely if S(I) is simple, then I = H (S(I)) is indecomposable [18].

Remark 2. Since a simple module is semi-primary, the above Lemma implies that indecomposable injective semi-primary modules are precisely the injective hulls of simple modules.

Remark 3. Over a commutative regular ring, semi-primary indecomposable injective modules are exactly the simple ones since every simple module over a regular ring is injective [21].

Lemma 4. Let C = A + B (dir) where A and B are injective semi-primary. Then C is injective semi-primary.

Proof. Here A = H (S(A)), B = H(S(B)) since A and B are injective semi-primary. Thus C = A + B = H(S(A)) + H(S(B)) (dir). By Lemma 2, we have S(C) = S(A) + S(B) which implies that H(S(A)) + H(S(B)) is an injective hull of S(C). Hence C is injective semi-primary by Proposition 1.

Lemma 5. The union of a directed set of semi-primary modules is semi-primary.

Proof. Let $\{A_j \mid j \in J\}$ be a directed set of semi-primary modules. Then $\cup A_j$ is a module. Take any non-zero element $x \in \cup A_j$. Then Rx is a non-zero submodule of A_j for some $j \in J$. Hence $S(Rx) \neq 0$. This proves that $\cup A_j$ is semi-primary.

From now on we will consider R to be a left Noetherian ring.

In the following theorem we give a characterization of those injective modules which are semi-primary.

<u>Theorem 1</u>. An injective module E is semi-primary if and only if it is a direct sum of indecomposable injective semi-primary modules.

<u>Proof</u>. Let $E = \sum_{I \in \mathfrak{M}} I$ (dir) where $\mathfrak{M}$ is a set of indecomposable injective semi-primary submodules of E. Denote by $\mathcal{F}$, the collection of all finite sums of members of $\mathfrak{M}$. By Lemma 4, every member in $\mathcal{F}$ is semi-primary. Also if A and B are in $\mathcal{F}$, then A + B which contains both A and B, belongs to $\mathcal{F}$. Hence $\mathcal{F}$ is a directed set of semi-primary submodules of E. From Lemma 5, it follows that $\cup\mathcal{F}$ is semi-primary. Since $\cup\mathcal{F}$ contains $\bigcup_{I \in \mathfrak{M}} I$, it contains E. Hence $E = \cup\mathcal{F}$ is semi-primary.

To prove the converse, let E be semi-primary. Since E is an injective module over a left Noetherian ring, E is a direct sum of indecomposable injective submodules each of which is semi-primary by Corollary 1 of Proposition 1 [18]. This completes the proof of the theorem.

<u>Proposition 2</u>. Let $\mathfrak{M}$ be a set of indecomposable injective submodules of an injective semi-primary module E. Then $E = \sum_{I \in \mathfrak{M}} I$ (dir) if and only if $\mathfrak{M}$ is a maximal free set of indecomposable injective submodules of E.

Proof. Suppose first that $\mathfrak{M}$ is a maximal free set of indecomposable injective submodules of E. Let $E' = \sum_{I \in \mathfrak{M}} I$. Each summand I being injective, E' is an injective submodule of E [18]. Hence $E = E' + E''$ (dir) for some submodule $E'' \subseteq E$. Therefore E" is injective semi-primary. Assume $E'' \neq 0$. Then $S(E'') \neq 0$ and hence E" contains a simple submodule A. It follows that any $H = H(A)$ contained in E" is indecomposable injective semi-primary and this implies that $\mathfrak{M} \cup \{H\}$ is free contrary to the maximality of $\mathfrak{M}$. Hence $E'' = 0$ and so $E = E'$.

On the other hand if $E = \sum_{I \in \mathfrak{M}} I$ (dir), then $\mathfrak{M}$ is a free set and hence, is contained in a maximal free set $\mathfrak{A}$ of indecomposable injective submodules of E. Assume $\mathfrak{A} \neq \mathfrak{M}$, then there exists a non-zero $I \in \mathfrak{A}$, $I \notin \mathfrak{M}$. This implies that

$$0 = I \cap \sum_{I \neq A \in \mathfrak{A}} A \supseteq I \cap \sum_{A \in \mathfrak{M}} A = I \cap E = I,$$

a contradiction. Thus $\mathfrak{A} = \mathfrak{M}$.

Remark 4. For any free set $\mathfrak{M}$ of indecomposable injective submodules of a semi-primary module E, let $\mathfrak{M}_S = \{S(I) \mid I \in \mathfrak{M}\}$. Then $\mathfrak{M}$ is maximal free if and only if $\mathfrak{M}_S$ is a maximal free set of simple submodules of E.

Proof. Let $\mathfrak{M}$ be maximal free. Then for any $I \in \mathfrak{M}$, $S(I) \cap \sum_{I \neq I' \in \mathfrak{M}} S(I') \subseteq I \cap \sum I' = 0$ implies that $\mathfrak{M}_S$ is a free set of simple submodules. Suppose that $\mathfrak{M}_S$ is not maximal such. Then there exists a maximal free set $\mathfrak{A}$ of simple submodules of E, properly containing $\mathfrak{M}_S$. Let $A \in \mathfrak{A}$, $A \notin \mathfrak{M}_S$. Then for any $H = H(A) \subseteq E$, $A = S(H) \subseteq \sum_{I \in \mathfrak{M}} S(I)$ by Lemma 2, and hence $A = 0$ since $A \cap \sum_{I \in \mathfrak{M}} S(I) = 0$, a contradiction. This proves that $\mathfrak{M}_S$ is maximal free. Conversely if $\mathfrak{M}_S$ is maximal free then $\mathfrak{M}$ is maximal free since otherwise $\mathfrak{M} \subset \mathfrak{N}$ would mean $\mathfrak{M}_S \subset \mathfrak{N}_S$, a contradiction.

Corollary. Let $\mathfrak{M}$ be a free set of indecomposable injective submodules of a semi-primary injective module E. Then $S(E) = \sum_{I \in \mathfrak{M}} S(I)$ if and only if $\mathfrak{M}$ is maximal.

Proof. Let $\mathfrak{M}$ be a maximal free set of indecomposable injective submodules of E. Then by Proposition 2, $E = \sum_{I \in \mathfrak{M}} I$ (dir). Hence by Lemma 2, we have $S(E) = \sum_{I \in \mathfrak{M}} S(I)$. Conversely let $S(E) = \sum_{I \in \mathfrak{M}} S(I)$. This shows that $\mathfrak{M}_S$ is a maximal free set of simple submodules of S(E), and hence $\mathfrak{M}$ itself is maximal by Remark 4.

As before, we will denote by Ω, the set of maximal left ideals of R and by T, the set of isomorphism classes of the R/P with $P \in \Omega$. For each $t \in T$, pick $S_t \in t$ and let $I_t = H(S_t)$. We then have the following:

Remark 5. If I is semi-primary injective, then $I \cong I_t$ if and only if $S(I) \cong S_t$.

Proof. Let $\varphi : I \to I_t$ be an isomorphism. Then φ restricted to S(I) is a non-zero homomorphism and hence $\varphi(S(I))$ is simple. Since $\varphi(S(I))$ is a large submodule of I_t, $S_t \cap \varphi(S(I)) \neq 0$. Hence $S_t = \varphi(S(I)$. Conversely $S(I) \cong S_t$ implies $I = H(S(I)) \cong H(S_t) = I_t$.

Now let E be an injective semi-primary module and $\mathcal{M}$ a maximal free set of indecomposable injective submodules of E. Then by Corollary, Proposition 2, $S(E) = \sum_{I \in \mathcal{M}} S(I)$. Hence $S(E)_t = \sum S(I) \quad (S_t \cong S(I),\ I \in \mathcal{M})$. For each $t \in T$, define

$$E_{t,\mathcal{M}} = \sum_{S(I) \cong S_t,\, I \in \mathcal{M}} I .$$

It follows from Remark 5 that

$$E_{t,\mathcal{M}} = \sum_{I_t \cong I \in \mathcal{M}} I .$$

<u>Theorem 2</u>. If $\mathcal{M}$ and $\mathcal{N}$ are any two maximal free sets of indecomposable injective submodules of a semi-primary injective module E, then there exists an automorphism φ of E such that $\varphi(E_{t,\mathcal{M}}) = E_{t,\mathcal{N}}$ for all $t \in T$, mapping $S(E)$ identically.

<u>Proof</u>. Here for each $t \in T$, $S(E_{t,\mathcal{M}}) = \sum_{I_t \cong I \in \mathcal{M}} S(I) = \sum_{S(I) \cong S_t,\ I \in \mathcal{M}} S(I) = S(E)_t$ by Remark 5. Since $S(E)_t$ is independent of $\mathcal{M}$, this gives $S(E_{t,\mathcal{M}}) = S(E_{t,\mathcal{N}})$. Hence $E_{t,\mathcal{M}}$ and $E_{t,\mathcal{N}}$ are injective hulls of $S(E_{t,\mathcal{M}})$. It follows by [11] that there exists an isomorphism $\varphi_t : E_{t,\mathcal{M}} \to E_{t,\mathcal{N}}$ mapping $S(E_{t,\mathcal{M}})$ identically. Now $E = \sum_{t \in T} E_{t,\mathcal{M}}$ (dir) allows us to define $\varphi : E \longrightarrow E$ such that φ restricted to $E_{t,\mathcal{M}}$ is φ_t. Thus φ is an automorphism of E. Moreover since φ maps each $S(E)_t$ identically and $S(E) = \sum S(E)_t$, φ maps $S(E)$ identically. The proof of the theorem is thus complete.

<u>Injective length of a semi-primary injective module</u>

Let E be a semi-primary injective module. For $t \in T$, put $\iota_t(E) = k(S(E)_t)$.

Definition 2. The family $(\iota_t(E))_{t \in T}$ of cardinal numbers associated with an injective semi-primary module E, is called the injective length of E. We will denote this by $\iota(E)$. By the proof of Theorem 2, $\iota_t(E) = k(S(E_{t,\mathcal{M}}))$ where $\mathcal{M}$ is any maximal free set of indecomposable injective submodules of E. Also $S(E)_t \cong \iota_t(E) \odot S_t$ (this notation stands for the external sum of $\iota_t(E)$ copies of S_t), hence $E_{t,\mathcal{M}} \cong \iota_t(E) \odot I_t$.

The fact that the injective length completely characterizes isomorphic injective semi-primary modules is given by the following:

Theorem 3. Two injective semi-primary modules E and E' are isomorphic if and only if they have the same injective length.

Proof. Let $\mathcal{M}$ and $\mathcal{M}'$ be any two maximal free sets of indecomposable injective submodules of E and E' respectively. Suppose first that $\iota(E) = \iota(E')$, then $\iota_t(E) = \iota_t(E')$ for each $t \in T$. Hence

$$E = \sum_t E_{t,\mathcal{M}}(\text{dir}) \cong \bigoplus \iota_t(E) \odot I_t = \bigoplus \iota_t(E') \odot I_t$$

$$\cong \sum_t E'_{t,\mathcal{M}'}(\text{dir}) = E'.$$

Conversely if E is isomorphic to E', then $S(E) \cong S(E')$. Hence we have $\iota_t(E) = k(S(E)_t) = k(S(E')_t) = \iota_t(E')$ for each $t \in T$. This shows that $\iota(E) = \iota(E')$.

We are now in a position to give a description of the injective hull of a semi-primary module in terms of the injective length of the former in the following:

Theorem 4. Let E be an injective hull of a semi-primary module M. Then $\iota_t(E) = k\,(S(M)_t)$ for all $t \in T$.

Proof. By Lemma 1, $S(E) = S(M)$, hence here:
$S(M)_t = S(E)_t \cong \iota_t(E) \odot S_t$. Thus $k\,(S(M)_t) = \iota_t(E)$.

For the rest of the chapter we will consider R to be a Dedekind domain and M a torsion module over R.

The following proposition gives a description of the elements in S(M).

Proposition 3. Let x be a non-zero element in M. Then $x \in S(M)$ if and only if O(x) has no square factor.

Proof. Let $O(x) = P_1 P_2 \dots P_n$ where the factors P_i are distinct prime ideals. Then $Rx \cong R/O(x) \cong R/P_1 \oplus \dots \oplus R/P_n$. Hence, x belongs to a sum of simple submodules of M and thus $x \in S(M)$. Conversely if $x \in S(M)$ then we can express x in the form $x = x_1 + x_2 + \dots + x_n$ with Rx_i simple submodules of M so that $O(x_i)$ is a maximal ideal of R for each i. If x_i and x_j have the same order ideal P, then $P = O\,(x_i + x_j)$ since $P \subseteq R$ is a maximal ideal with $P \subseteq O\,(x_i + x_j)$. Collecting together all those x 's which have the same order ideal we can write $x = y_1 + \dots + y_m$ with $0 \neq y_k$, $O(y_k) = P_k$ and $P_\ell \neq P_k$ if $\ell \neq k$. It follows that $P_1 P_2 \dots P_m \subseteq O(x)$. On the other hand if $r \in O\,(x)$, then $ry_k = 0$ for each k whence $r \in P_1 \cap P_2 \cap \dots \cap P_m$. Thus $O(x) = P_1 P_2 \dots P_m$ is square free since the factors are all distinct.

Corollary 1. For any $P \in \Omega$ if $M_P = \{ x \in M \mid$ For some integer k, $P^k = O(x)\}$ denotes the P-primary component of M, then $S(M)_P = S(M_P)$.

Proof. Take a non-zero element $x \in S(M_P)$, then $O(x) = P^k$ is square free by Proposition 3, hence $O(x) = P$ and so $x \in S(M)_P$. On the other hand, if $x \in S(M)_P$, then $O(x)$ is square free since $x \in S(M)$. Also $O(x) = P^k$ for some integer k, since $x \in S(M)_P$. Hence $O(x) = P$ and thus $x \in S(M_P)$.

Corollary 2. If $M = Rx_1 + \dots + Rx_n$ (dir), then $k(S(M)_P)$ is the cardinality of the set of $i = 1, \dots, n$ such that $O(x_i) \subseteq P$.

Proof. We have $M_P = (Rx_1)_P + \dots + (Rx_n)_P$ (dir). Hence $S(M_P) = S((Rx_1)_P) + \dots + S((Rx_n)_P)$ (dir) by Lemma 2. Take any $x \in M$. If $O(x) \nsubseteq P$, then $O(x) = P_1^{k_1} \dots P_n^{k_n}$ where each prime ideal P_i is distinct from P. Hence $Rx \cong R/O(x) \cong R/P_1^{k_1} \oplus \dots \oplus R/P_n^{k_n}$ implies that $(Rx)_P = 0$. If $O(x) \subseteq P$, then $O(x)$ can be expressed $O(x) = P^k P_1^{k_1} \dots P_m^{k_m}$ which gives $(Rx)_P \cong R/P^k$. Now, since R/P^k has $P/P^k, \dots, P^{k-1}/P^k$ as its only proper ideals, $(Rx)_P$ contains only one simple submodule isomorphic to $P^{k-1}/P^k \cong R/P$. Hence $S((Rx)_P)$ is simple. Thus $S((Rx)_P)$ is simple or 0, depending on whether $O(x) \subseteq P$ or $O(x) \nsubseteq P$. In particular $k(S(Rx_i)_P) = 1$ or 0 according as $O(x_i) \subseteq P$ or $O(x_i) \nsubseteq P$ for $i = 1, 2, \dots, n$. This gives $k(S(M)_P) = |\{ i \in \{1, 2, \dots, n\} \mid O(x_i) \subseteq P\}|$. By Corollary 1, then the proof is complete.

Remark 6. From Corollary 1 and Theorem 4, it follows that if $E = H(M)$, then the number of indecomposable injective summands in a direct sum decomposition of E corresponding to any $P \in \Omega$, is the length of the socle of M_P.

Proposition 4. Let M be finitely generated. Then $k(S(M))$ is finite and $H(M) \cong \bigoplus_{i=1}^{n} P_i^*/P_i$ where $n = k(S(M))$, $P_i \subseteq R$ proper prime ideals and $P^* = \bigcup_k P^{-k}$.

Proof. Since R is Noetherian and M is finitely generated, M is a Noetherian module. Hence $S(M)$ is also Noetherian. The fact that $S(M)$ is semi-simple then implies that $S(M)$ satisfies both the chain conditions for its submodules. Hence $S(M) = A_1 + A_2 + \ldots + A_n$ (dir) with each A_i simple and therefore isomorphic to R/P_i where $P_i \in \Omega$. Thus we have $H(M) = H(S(M)) \cong \bigoplus_{i=1}^{n} H(R/P_i)$. By [2], $H(R/P_i) = P_i^*/P_i$ and so the proof is complete.

CHAPTER II

MODULES OVER BOOLEAN RINGS

In this chapter we intend to study the injective hulls of modules over a Boolean ring R. We establish that the injective hull of a cyclic module over R is obtained by completion in a sense to be made precise below. We also prove that the injective hull of an ideal considered as a module over the ring, is its completion. Finally, we show that the injective hull of the quotient of the ring by a normal ideal is the quotient of their injective hulls.

Definition 1. A ring R is said to be a Boolean ring if $r^2 = r$ for every $r \in R$. From the definition it follows that a Boolean ring is a commutative ring of characteristic 2.

In this chapter R will always denote a Boolean ring with unit e.

Divisibility relation in R

Let '$\leqslant$' be defined in R by $r \leqslant s$ if and only if $rs = r$. It can be easily checked that the relation $\leqslant$ is a partial ordering on R. We call this relation the divisibility relation of R in view of the fact that $r \leqslant s$ if and only if $r = ts$ for some $t \in R$. We also note that $r \leqslant s$ implies that $ar \leqslant as$ for any $a \in R$ since $ar\,as = ars = ar$. It might be mentioned that with this relation, R is a Boolean lattice where meet, join and complement are respectively given by $x \wedge y = xy$, $x \vee y = x + y + xy$ and $\sim x = x + e$. On the other hand, as is well known, any Boolean lattice can be made into a Boolean ring by defining $x + y = (x \wedge (\sim y)) \vee (y \wedge (\sim x))$ and $xy = x \wedge y$

For any subset $T \subseteq R$, the symbol $\bigvee T$ will denote the least upper bound of elements in T. This supremum need not always exist in R.

<u>Definition 2</u>. R is said to be <u>complete</u> if and only if $\bigvee T$ exists in R for all $T \subseteq R$.

<u>Definition 3</u>. A Boolean ring S is called a <u>Boolean completion</u> of R if and only if

(1) S contains R as a subring

(2) S is complete

(3) $s \in S$ implies $s = \bigvee \{ r \in R \mid r \leqslant s \}$

<u>Definition 4</u>. An ideal $J \subseteq R$ is a <u>complete ideal</u> of R if and only if $\bigvee A \in J$ whenever $A \subseteq J$ such that $\bigvee A$ exists in R.

An ideal $J \subseteq R$ is said to be <u>complete as a Boolean ring</u> if and only if for all $A \subseteq J$, $\overset{J}{\bigvee} A$ exists in J. Here it should be noted that a subring of R need not be a unitary subring.

We state without proof the following lemma which is a well known fact about Boolean lattices [14].

Lemma 1. Let $A \subseteq R$ such that $\bigvee A$ exists in R. Then for any $b \in R$, $b \cdot \bigvee A$ exists in R and $b \cdot \bigvee A = \bigvee \{ba \mid a \in A\}$.

Lemma 2. Let R be complete and $J \subseteq R$ an ideal which is complete as a Boolean ring, then R/J is complete.

Proof. We first show that J is complete as a Boolean ring if and only if it is a principal ideal. Now suppose that J is complete. Then $a_o = \overset{J}{\bigvee} J \in J$, hence $Ra_o \subseteq J$. Since for every element $a \in J$, $a \leqslant a_o$, we have $J \subseteq R a_o$. Therefore $J = Ra_o$. Conversely if $J = Ra_o$, then $a_o \in J$ and $a \leqslant a_o$ for all $a \in J$. This shows that for any $T \leq J$, $\overset{R}{\bigvee} T \leqslant a_o \in J$, hence $\overset{R}{\bigvee} T \in J$. Thus $\overset{J}{\bigvee} T$ $(= \overset{R}{\bigvee} T)$ exists. Therefore J is complete as a Boolean ring.

Let a_o be the unit in J and consider the mapping $\varphi : R \rightarrow [a_o, e]$ defined by $\varphi(r) = r \vee a_o$ where $[a_o, e]$ is considered with the ring structure it has as a Boolean lattice. In this, the addition $\boxplus$ is given in terms of the addition of R by $r \boxplus s = r + s + a_o$, while the multiplication coincides with that of R. We want to show that φ is a ring homomorphism. Now $(r + s) \vee a_o = r + s + ra_o + sa_o + a_o = \varphi(r) + \varphi(s) + a_o = \varphi(r) \boxplus \varphi(s)$, hence φ is additive. Also $rs \vee a_o = (r \vee a_o)(s \vee a_o)$ by the distributivity of the lattice structure of R. Hence φ is a ring homomorphism. It is onto since $x \in [a_o, e]$ implies $x = a_o \vee x = \varphi(x)$. Finally, $\varphi(r) = a_o$ iff $r \leqslant a_o$, i.e. $r \in J$, which shows $J = \mathrm{Ker}(\varphi)$.

Hence R/J is isomorphic to $[a_o, e]$ which is a complete ring. Therefore R/J is complete.

<u>Lemma 3.</u> Suppose S is a Boolean ring and also an R-module. If $I \subseteq R$ is an ideal and $f : I \to S$ is an R-module homomorphism such that $\bigvee f(I)$ exists in S, then there exists an element $c \in S$ such that $f(x) = xc$ for all $x \in I$.

<u>Proof.</u> Let $c = \bigvee f(I)$. Take $x \in I$, then $f(x) \leqslant c$. Hence $f(x) = f(x^2) = xf(x) \leqslant xc$. On the other hand, $xc = x.\bigvee\{f(y) \mid y \in I\} = \bigvee\{x \cdot f(y) \mid y \in I\}$ (by Lemma 1)

$= \bigvee\{f\ (xy) \mid y \in I\}$. Now, for each $y \in I$, $f(xy) = yf(x) \leqslant f(x)$, hence $xc = \bigvee\{f(xy) \mid y \in I\} \leqslant f(x)$. This implies that $xc = f(x)$ for all $x \in I$.

The injective hull of a cyclic R-module is given by the following:

<u>Theorem 1.</u> If $J \subseteq R$ is an ideal and $Q \geqslant R/J$ a Boolean completion of R/J as a Boolean ring, then Q can be made into an R-module and as such it is an R-injective hull of R/J.

<u>Proof.</u> Let $\nu : R \to R/J$ be the natural homomorphism, then Q being an R/J-module can be made into an R-module by $rx = \nu(r) \cdot x$ where $r \in R$, $x \in Q$. Let $I \leq R$ be any ideal and $f : I \to Q$ any R-module homomorphism. Since $c = \bigvee f(I)$ exists in Q, by Lemma 3, $f(x) = xc$ for all $x \in I$. The fact that R has a unit, then shows that Q is R-injective.

To prove that Q is an R-essential extension of R/J, consider any non-zero element $x \in Q$. Since Q is a Boolean completion of R we have $0 \neq x = \bigvee\{y \in R/J \mid y \leq x\}$. This implies that there exists a non-zero element $y \in R/J$ with $y \leq x$. Hence we have $0 \neq y = yx = \nu(r)\, x = rx$ for $r \in \nu^{-1}(y)$. Therefore $0 \neq y \in Rx \cap R/J$. This proves that R/J is a large submodule of Q. Thus Q is an R-injective hull of R/J.

<u>Corollary 1</u>. R/J is R-injective if and only if it is complete as a Boolean ring.

<u>Proof</u>. If R/J is injective, then by the theorem Q = R/J and hence R/J is complete. On the other hand if R/J is complete, then R/J = Q and hence R-injective.

<u>Corollary 2</u>. R is injective if and only if it is complete. [7].

<u>Proof</u>. Take J = 0 in Corollary 1.

<u>Remark 1</u>. Since any Boolean completion of R as a Boolean ring is an injective hull of R, it follows that such a completion is unique upto an isomorphism over R[7].

<u>Remark 2</u>. Corollary 2 should not be confused with the analogous, but different theorem that R is injective in the category of Boolean rings and unitary ring homomorphisms iff it is complete [14].

<u>Corollary 3.</u> Any simple R-module is injective.

<u>Proof</u>. Any simple module is isomorphic to R/P for some maximal ideal $P \subseteq R$ and $R/P \cong Z/Z2$, as is well known. But it is finite, hence complete and therefore injective.

Remark 3. Corollary 3 also follows from [21] since Boolean rings are commutative regular rings.

We now find an injective hull of an ideal of R considered as an R-module in the next:

Theorem 2. Let $I \subseteq R$ be an ideal, $\bar{R}$ a Boolean completion of R as a Boolean ring and $\bar{I} = \{\bigvee S \mid S \subseteq I\}$. Then $\bar{I}$ is a complete ideal of $\bar{R}$ and as an R-module, it is an injective hull of I.

Proof. Let $\bar{a} \in \bar{I}$, then $\bar{a}$ is the join of some elements in I and hence also the join of all its lower bounds in I. Thus

(1) $\bar{a} \in \bar{I}$ implies $\bar{a} = \bigvee\{a \in I \mid a \leqslant \bar{a}\}$

Now let $S = \{s_\alpha \mid \alpha \in A\} \subseteq \bar{I}$, $\bar{s} = \bigvee S$. We want to show that $\bar{s} \in \bar{I}$. We have $\bar{s} = \bigvee_\alpha s_\alpha = \bigvee_\alpha \bigvee_\beta t_{\alpha\beta}$ with $t_{\alpha\beta} \in I$. Hence $\bar{s} = \bigvee_\beta \bigcup_\alpha t_{\alpha\beta} \in \bar{I}$. We have thus proved that

(2) $S \subseteq \bar{I}$ implies $\bigvee S \in \bar{I}$

Take any two elements $\bar{a} = \bigvee_{a \in A} a$, $\bar{b} = \bigvee_{b \in B} b$ in $\bar{I}$ with $A \subseteq I$, $B \subseteq I$. Let $C = \{ab \mid a \in A, b \in B\}$. Then since $a \leqslant \bar{a}$, $b \leqslant \bar{b}$, one has $ab \leqslant \bar{a}\bar{b}$ and therefore $C \leqslant \bar{a}\bar{b}$. If $\bar{d} \geqslant C$, then $\bar{d} \geqslant \bigvee_{a \in A} ab$ for each $b \in B$ which implies that $\bar{d} \geqslant \bigvee_{b \in B} b\bar{a} = \bar{a} \cdot \bigvee_{b \in B} b = \bar{a}\bar{b}$. Therefore $\bar{a}\bar{b} = \bigvee C$. But $C \subseteq I$, hence $\bar{a}\bar{b} \in \bar{I}$. We have thus shown that

(3) $\bar{a}, \bar{b}$ in $\bar{I}$ implies $\bar{a}\bar{b} \in \bar{I}$.

Now take $\bar{r} \in \bar{R}$, $\bar{a} \in \bar{I}$ with $\bar{r} = \bigvee_{r \leq \bar{r}} r$ and $\bar{a} = \bigvee_{a \leq \bar{a}} a$, then $\bar{r}\bar{a} = \bigvee\{ra \mid r \leqslant \bar{r}, a \leqslant \bar{a}\} \in \bar{I}$ since each $ra \in I$. This shows that

(4) $\bar{r} \in \bar{R}$, $\bar{a} \in \bar{I}$ implies $\bar{r}\,\bar{a} \in \bar{I}$.

In order to prove that $\bar{I}$ is closed under addition, consider any two elements $\bar{a}$, $\bar{b}$ in $\bar{I}$. Let $\bar{x}_o = \bigvee I$, then $\bar{x}_o \in \bar{I}$ in view of (2), and $\bar{x} = \bigvee_{x \leqslant \bar{x}} x$ implies $\bar{x}_o\,\bar{x} = \bigvee \bar{x}_o x = \bigvee x = \bar{x}$ for any $\bar{x} \in \bar{I}$. Let 1 be the unit in $\bar{R}$, then $\bar{a} + \bar{b} = \big(\bar{a}\,(\bar{b} + 1) \vee \bar{b}\,(\bar{a} + 1)\big) = (\bar{a}\bar{b} + \bar{a}) \vee (\bar{b}\bar{a} + \bar{b}) = (\bar{a}\bar{b} + \bar{a}\bar{x}_o) \vee (\bar{b}\bar{a} + \bar{b}\,\bar{x}_o) = (\bar{a}(\bar{b} + \bar{x}_o) \vee \bar{b}(\bar{a} + \bar{x}_o))$.

We now claim that if $x \in R$, then $x \leqslant \bar{b} + \bar{x}_o$ if and only if $\bar{b}x = 0$ and $x \leqslant \bar{x}_o$. To prove this, first suppose that $\bar{b}x = 0$, $x \leqslant \bar{x}_o$. This implies $x\,(\bar{b} + \bar{x}_o) = x\,\bar{x}_o = x$, hence we get $x \leqslant \bar{b} + \bar{x}_o$. Conversely if $x \leqslant \bar{b} + \bar{x}_o$, then $x(\bar{b} + \bar{x}_o) = x$. Therefore $\bar{b}x = \bar{b}x(\bar{b} + \bar{x}_o) = \bar{b}x + \bar{b}\,\bar{x}_o\,x = \bar{b}x + \bar{b}x = 0$ since $\bar{b}\,\bar{x}_o = \bar{b}$. Now, from the given condition $x \leqslant \bar{b} + \bar{x}_o$ we have $x \leqslant \bar{b} + \bar{x}_o = \bar{b}\,\bar{x}_o + \bar{x}_o = (\bar{b} + \bar{x}_o)\,\bar{x}_o \leqslant \bar{x}_o$. This proves the claim.

Since $\bar{b} + \bar{x}_o \in \bar{R}$ we can write $\bar{b} + \bar{x}_o = \bigvee\{x \in R \mid x \leqslant \bar{b} + \bar{x}_o\} = \bigvee\{x \in R \mid \bar{b}\,x = 0,\ x \leqslant \bar{x}_o$ in view of the preceding claim. We thus have $\bar{b} + \bar{x}_o = \bigvee X$ where $X = \{x \in R \mid \bar{b}\,x = 0,\ x \leqslant \bar{x}_o\}$. Take $x \in X$ then $x = x\,\bar{x}_o \in \bar{I}$ since $\bar{I}$ is closed with respect to multiplication by elements in $\bar{R}$, in particular by elements in R in view of (4). Thus $X \subseteq \bar{I}$. It follows from (2) that $\bar{b} + \bar{x}_o = \bigvee X \in \bar{I}$. Replacing $\bar{b}$ by $\bar{a}$ we get $\bar{a} + \bar{x}_o \in \bar{I}$. Hence $\bar{a} + \bar{b} = (\bar{a}\,(\bar{b} + \bar{x}_o) \vee \bar{b}\,(\bar{a} + \bar{x}_o)) \in \bar{I}$ by using (2), (3) and (4). Thus

(5) $\bar{a}$, $\bar{b}$ in $\bar{I}$ implies $\bar{a} + \bar{b} \in \bar{I}$

As a consequence of (2), (3), (4) and (5), it follows that $\overline{I}$ is an ideal of R, is complete as a Boolean ring and is an R-module.

To show that $\overline{I}$ is R-injective, let $J \subseteq R$ be any ideal and $f : J \to \overline{I}$, any R-module homomorphism, then since $\bigvee f(J)$ exists in $\overline{I}$, by Lemma 3, there exists $\overline{C} \in \overline{I}$ such that $f(x) = x\overline{C}$ for all $x \in J$. This proves the R-injectivity of $\overline{I}$.

Now, let $\overline{x}$ be any non-zero element in $\overline{I}$, $\overline{x} = \bigvee\{x \in I \mid x \leqslant \overline{x}\}$. Then there exists $x \in I$ such that $0 \neq x \leqslant \overline{x}$. Therefore, $0 \neq x = x\,\overline{x} \in R\overline{x} \cap I$. This shows that $\overline{I}$ is an R-essential extension of I and hence an injective hull of I as an R-module.

<u>Lemma 4</u>. Let S be any commutative ring with unit and $I \subseteq S$ an ideal such that I is a Boolean ring (unit not assumed in I). If I is S-injective, then I has a unit and is I-injective.

<u>Proof</u>. Let $g : I \to I$ be the identity mapping. Then $g(sa) = sa = sg(a)$ for every $s \in S$, $a \in I$. Hence g is an S-homomorphism. Since I is injective as an S-module, this implies that there exists $a_o \in I$ such that $a = g(a) = aa_o$ for all $a \in I$. Therefore a_o is the unit in I. Now, let $J \subseteq I$ be an I-ideal, then $ax = ax^2 = (ax)\,x \in J$ for any $a \in S$, $x \in J$, hence J is an S-ideal. Moreover any I-homomorphism $f : J \to I$ is an S-homomorphism since $f(sb) = f(sb^2) = bsf(b) = sf(b^2) = sf(b)$ for any $b \in J$, $s \in S$. As a consequence of S-injectivity of I and the fact that S has a unit, there exists $r_o \in I$ such that $f(b) = br_o$ for all $b \in J$. Since I contains unit, this implies that I is I-injective.

Theorem 3. An ideal $I \subseteq R$ is injective as an R-module if and only if I is complete as a Boolean ring.

Proof. Let $\overline{I} = \left\{ \overset{\overline{R}}{\bigvee} A \mid A \subseteq I \right\}$ where $\overline{R}$ is a Boolean completion of R as a Boolean ring. Now if I is complete then $I = \overline{I}$, hence by Theorem 2, I is R-injective. Conversely if I is R-injective, then by Lemma 4, I is I-injective. Hence by Corollary 2 of Theorem 1, I is complete as a Boolean ring.

Remark 4. For R, the following statements are equivalent:

(1) Each ideal $I \subseteq R$ is complete as a Boolean ring.

(2) R is semi-simple.

(3) R is finite.

Proof. (1) implies (2): Suppose that every ideal in R is complete as a Boolean ring, then by Theorem 3, it is injective as an R-module. This implies that any ideal $I \subseteq R$ is a direct summand of R. Hence R is semi-simple.

(2) implies (3): If R is semi-simple, then it has minimality condition for its ideals and hence R is a direct sum of finitely many simple components A_i, $i = 1, 2, \ldots, n$. Since each A_i is isomorphic to R/P_i where P_i is a maximal ideal of R, A_i has only two elements. Hence R is finite.

(3) implies (1): If R is finite then each ideal $I \subseteq R$ is finite and therefore complete as a Boolean ring.

Definition 5. For $A \subseteq R$, define $M_i A = \left\{ x \in R \mid x \leq A \right\}$ and $M_a A = \left\{ x \in R \mid x \geq A \right\}$. An ideal $J \subseteq R$ is called normal if and only if $M_i M_a J = J$. Since for any ideal J, $J \subseteq M_i M_a J$ always holds, the condition $M_\iota M_a J \subseteq J$ is sufficient to ensure that an ideal $J \subseteq R$ is a normal ideal.

Let $\bar{R}$ be the Boolean completion of R as a Boolean ring. It was shown in Theorem 2 that if $J \subseteq R$ is any ideal, then $\bar{J} = \left\{ \bigvee^{\bar{R}} I \,\middle|\, I \subseteq J \right\}$ is a complete ideal of $\bar{R}$.

Lemma 5. An ideal $J \subseteq R$ is normal if and only if $\bar{J} \cap R = J$.

Proof. Let $c = \bigvee^{\bar{R}} J$. Then $\bar{J} = \bar{R}c$. Suppose that J is a normal ideal of R. Take any element $r \in \bar{J} \cap R$. Then $r \leqslant c \leqslant M_a J$. This implies that $r \in M_\iota M_a J = J$, hence $\bar{J} \cap R \subseteq J \subseteq \bar{J} \cap R$. Thus $J = \bar{J} \cap R$. Conversely, let the ideal $J \subseteq R$ satisfy the condition $\bar{J} \cap R = J$. Since $c \in \bar{R}$, we can express c as $c = \bigwedge_{c \leqslant b \in R} b$. Let $B = \{ b \in R \mid c \leqslant b \}$. It follows that $\bar{J} = \bar{R}c = \bigcap_{b \in B} \bar{R}b$. By the given condition we therefore have $J = R \cap \bigcap_{b \in B} \bar{R}b$. We want to show that $J = \bigcap_{b \in B} Rb$. Clearly $\bigcap_{b \in B} Rb \subseteq J$. To show the inclusion in the other direction, let $x \in J$ be any element. Then, since $x \leqslant c$ we get $x \leqslant b$ for all $b \in B$. Hence $x \in \bigcap_{b \in B} Rb$. This implies $J = \bigcap_{b \in B} Rb$. It suffices to prove that $M_\iota M_a J \subseteq J$. Take $y \in M_\iota M_a J$, then $y \leqslant t$ for all $t \in R$ with $t \geqslant J$. Since every $b \in B$ satisfies the condition $b \in R$, $b \geqslant c \geqslant J$, we have $y \leqslant b$ for all $b \in B$. Hence $y \in \bigcap_{b \in B} Rb = J$. This proves that $J \subseteq R$ is a normal ideal.

Theorem 4. If $J \subseteq R$ is a normal ideal then $\bar{R}/\bar{J}$ is an R-injective hull of R/J.

<u>Proof</u>. Consider the mapping $f : R/J \to \overline{R}/\overline{J}$ by $f(r + J) = r + \overline{J}$. It is clearly an R-module homomorphism and is one-to-one since by Lemma 5, $\overline{J} \cap R = J$. Thus $\overline{R}/\overline{J}$ contains an isomorphic copy of R/J.

To prove that $\overline{R}/\overline{J}$ is an essential extension of R/J, let $\overline{r} + \overline{J}$ be a non-zero element in $\overline{R}/\overline{J}$. Hence $\overline{r} = \bigvee\{r \in R \mid r \leqslant \overline{r}\} \notin \overline{J}$. This implies that there exists $r \in R$ with $r \leqslant \overline{r}$ such that $r \notin J$, hence $r \notin \overline{J}$. Thus we have $0 \neq r + \overline{J} = r\,(\overline{r} + \overline{J})$ which shows that $f(R/J) \cap R\,(\overline{r}+\overline{J}) \neq 0$. Therefore R/J is an essential extension of f(R/J). Consequently R/J is a large submodule of $\overline{R}/\overline{J}$.

In order to show the injectivity of $\overline{R}/\overline{J}$, we observe that $\overline{R}/\overline{J}$ is a complete Boolean ring and hence for any R-homomorphism φ from an ideal $I \subseteq R$ to $\overline{R}/\overline{J}$, $\bigvee\varphi(I)$ exists in $\overline{R}/\overline{J}$. We can therefore apply Lemma 3 to get the desired R-injectivity of $\overline{R}/\overline{J}$. This completes the proof of the theorem.

<u>Corollary</u>. If $J \subseteq R$ is a complete ideal, then $\overline{R}/\overline{J}$ is an R-injective hull of R/J.

<u>Proof</u>. Take $\overline{a} \in \overline{J} \cap R$. Then $\overline{a} = \bigvee S$ with $S \subseteq J$. Since J is a complete ideal of R and $\overline{a} = \bigvee S \in R$ we have $\overline{a} \in J$. Hence $\overline{J} \cap R \subseteq J$ and so J is a normal ideal of R. The corollary then follows immediately from the theorem.

CHAPTER III

SEMI-SIMPLE MODULES OVER REGULAR RINGS

Our main objective in this chapter is to provide an explicit description of the injective hull of a semi-simple module over a commutative regular ring. For this purpose we first show that every monotypic component of the module is injective. This fact together with some properties of isolated points in the Zarisky topology of the maximal ideal space of the ring, then lead to the desired construction of the injective hull. Finally a necessary and sufficient condition for the module to be injective is obtained.

Definition 1. A ring R is called (von Neumann) regular if for every $a \in R$, there exists an element $x \in R$ such that $axa = a$. This condition reduces to $a^2 x = a$ if R is commutative. A Boolean ring is an example of a commutative regular ring. It is well known [21] that a commutative ring R with unit is regular if and only if every simple R-module is injective.

Throughout this chapter we shall consider R to be a commutative regular ring with unit 1. Let Ω denote the set of maximal ideals of R. For each $a \in R$, define Ω_a by $\Omega_a = \{P \in \Omega \mid a \notin P\}$.

It follows that $\Omega_a \cap \Omega_b = \Omega_{ab}$. Thus Ω can be made into a topological space with $\{\Omega_a \mid a \in R\}$ as the system of its basic open sets. This topology of Ω is called the Zarisky topology. Ω is clearly a T_1-space since if P and Q are any two distinct points in Ω, then there exists $a \in P - Q$ which implies that Ω_a is a neighbourhood of Q not containing P.

Definition 2. For a semi-simple M, the support of M, to be denoted by supp (M), is the set of all those maximal ideals $P \in \Omega$, for which the associated monotypic component is non-zero.

In what follows M will denote a semi-simple module with supp (M) = S. For any $P \in S$, the R/P - monotypic component of M will be denoted by M_P. As usual, for any function f, the symbol supp(f) will denote the set of all elements x in domain (f) for which $f(x) \neq 0$.

Theorem 1. Every monotypic component of M is injective.

Proof. For any $P \in S$, consider the monotypic component M_P of M. Let α be the length of M_P and T a set with $|T| = \alpha$. Then $M_P \cong \alpha \odot R/P = E$. Let $\pi = \{f \mid f : T \to R/P\}$. Now each factor R/P of π being injective, π is injective and therefore there exists an $H(E) \subseteq \pi$. Without loss of generality we can take α to be an infinite cardinal. Assume E is not injective. Then $E \subset H(E) \subseteq \pi$. Take any element $f \in H(E) - E$. Since H(E) is an essential extension of E, one has $Rf \cap E \neq 0$ and this implies

$0 \neq rf \in E$ for some $r \in R \setminus P$. As R/P is a field and $f(t) \neq 0$ for infinitely many $t \in T$, we have $rf(t) = (r + P)\, f(t) \neq 0$ for infinitely many $t \in T$. But this contradicts the fact that $rf \in E$. Hence E is injective.

Remark 1. $\prod_{P \in S} M_P$ is injective since each factor M_P is injective.

Definition 3. Let X be any topological space and $A \subseteq X$. An element $x \in A$ is called an isolated point of A if there exists a neighbourhood U of x such that $U \cap A = \{x\}$, i.e. if $\{x\}$ is an open set in the relative topology of A. A subset $A \subseteq X$ is said to be discrete if every $x \in A$ is an isolated point of A.

Lemma 1. Let $f \in \prod_{P \in S} M_P$ and $a \in R$ such that $0 \neq af \in \bigoplus_{P \in S} M_P$, then every element in supp (af) is an isolated point of supp(f).

Proof. Let $\text{supp}(af) = \{P_1, P_2, \ldots, P_n\}$ where $P_i \neq P_j$ if $i \neq j$. This implies that there exist elements $a_i \in P_i \setminus P_1$ $i = 2, \ldots, n$. Put $b = a\, a_2\, a_3 \ldots a_n$. Then $b \notin P_1$ and b belongs to every P in supp (f) with $P \neq P_1$. Hence $\Omega_b \cap \text{supp}(f) = \{P_1\}$. This shows that P_1 is an isolated point of supp (f). Similar argument shows that $P_2, \ldots, P_n$ are also isolated points of supp (f).

Remark 2. It follows from the above Lemma that the support of any non-zero element in an essential extension of $\bigoplus_{P \in S} M_P$ contains an isolated point.

<u>Lemma 2</u>. Let E be a proper essential extension of $\bigoplus_{P \in S} M_P$. Then for any $f \in E \setminus \bigoplus M_P$, supp (f) contains infinitely many isolated points.

<u>Proof</u>. Since E is an essential extension of $\bigoplus M_P$ and $0 \neq f \in E$, we can find an element $a \in R$ such that $0 \neq af \in \bigoplus M_P$. Let supp (af) = $\{P_1, P_2, \ldots P_n\}$. By Lemma 1, each P_i is an isolated point of supp (f). Choose an element $Q \in$ Supp (f) $\setminus$ Supp (af). As $P_i \nsubseteq Q$, there exist elements $r_i \in P_i \setminus Q$ for $i = 1, 2, \ldots, n$. Then $r = r_1 r_2 \ldots r_n \in (P_1 \cap P_2 \cap \ldots \cap P_n) \setminus Q$. This shows that $0 \neq rf \in E$. Moreover no element in supp (af) belongs to supp (rf). Since for some $s \in R$, $0 \neq srf \in \bigoplus M_P$, we can apply Lemma 1 to show that the elements in supp (srf) are isolated points of supp (f) and they are all distinct from $P_1, P_2, \ldots, P_n$. Now supp (f) being infinite we can find an element in supp (f) $\setminus$ (Supp (af) $\cup$ supp (srf)) which will give rise to another set of finitely many isolated points of supp (f), each being different from the ones obtained before. Proceeding thus we get infinitely many isolated points of supp (f).

If R is Noetherian then R is semi-simple and every module is injective [18]. In order to describe the injective hull of a semi-simple module over a general regular ring, we need the following topological fact about T_1-spaces:

<u>Lemma 3</u>. In any T_1 space X, if A and B are non-void subsets such that A as well as every non-void subset of B has an isolated point, then there exists an isolated point in $A \cup B$.

Proof. Let p be an isolated point of A. Then there exists an open neighbourhood U of p such that $U \cap A = \{ p \}$. Since $U \cap (A \cup (B \cap \complement U)) = U \cap A$, we conclude that p is also an isolated point of $A \cup (B \cap \complement U))$. If $B \cap U$ is empty, then p is an isolated point of $A \cup B$ and so the Lemma holds. We have therefore to consider only the case when $B \cap U$ is non-void. By hypothesis $B \cap U$ contains an isolated point q which can be assumed to be distinct from p without any loss in generality. This assumption, together with the fact that X is T_1 implies that $\complement\{ p \}$ is an open set containing q. Now q being an isolated point of $B \cap U$, we have $V \cap B \cap U = \{ q \}$ for some neighbourhood V of q. Thus we obtain $U \cap V \cap \complement\{p\} \cap (A \cup B) = U \cap V \cap \complement\{p\} \cap B = \{q\} \cap \complement\{p\} = \{ q \}$. Since $U \cap V \cap \complement\{p\}$ is a neighbourhood of q, the above relation implies that q is an isolated point of $A \cup B$.

From Lemma 3, we immediately have the following:

Corollary 1. Let B be a discrete subset of a T_1-space X and $A \subseteq X$ with an isolated point. Then $A \cup B$ has an isolated point.

Corollary 2. If A and B are subsets of a T_1-space X with the property that each of their non-void subsets has an isolated point, then $A \cup B$ has the same property.

Proof. Let $Y \subseteq A \cup B$ be any non-void subset. Then $Y = A_1 \cup B_1$ where $A_1 = A \cap Y$, $B_1 = B \cap Y$. Without loss of generality we can assume that A_1 and B_1 are both non-empty. Then by the Lemma, Y has an isolated point.

Remark 3. (1) Let $A = \bigcup A_i$ where each A_i is without an isolated point. Then A has no isolated point since if p were an isolated point in A, then $p \in A_i$ for some i, would imply that p is an isolated point of A_i contrary to the hypothesis.

(2) If A has no isolated point then ΓA (Closure of A) also has no isolated point since if we assume that $p \in \Gamma A$ is an isolated point in ΓA with $V \cap \Gamma(A) = \{p\}$ for some neighbourhood V of p, then $p \in \Gamma A \cap \complement A$ implies the existence of an element $q \in V \cap A \subseteq V \cap \Gamma A$ with q distinct from p, which gives a contradiction.

Theorem 2. Let M be a semi-simple module over a regular ring R with supp (M) = S, then $H = \left\{ f \in \prod_{P \in S} M_P \mid \text{Every non-void subset of supp } (f) \text{ has an isolated point} \right\}$ is an injective hull of M.

Proof. Let f, g be any two elements in H, then since supp $(f + g) \subseteq$ supp $(f) \cup$ supp (g), we have $f + g \in H$ by Corollary 2 of Lemma 3. Now if $a \in R$, $f \in H$, then supp $(af) = \Omega_a \cap$ supp (f) implies that $af \in H$. Hence H is an R-module and it contains $\bigoplus_{P \in S} M_P$ since every non-void subset of a finite set is discrete. Now let $0 \neq f \in H$, then supp (f) is non-empty and hence contains an isolated point P so that for some $a \in R$, supp $(af) = \Omega_a \cap$ supp $(f) = \{P\}$. Thus $0 \neq af \in \bigoplus M_P$. Hence H is an essential extension of $\bigoplus M_P$.

As to the injectivity of H assume by way of contradiction that H has a proper essential extension E. Then $H \subset E \subseteq \prod M_P$.

Take $f \in E$, $f \notin H$. Then there exists a non-void subset of supp (f) without isolated points. Denote by X, the union of all those subsets of supp (f) which have no isolated points. By Remark 4, X has no isolated point. Let $Y = \text{supp}(f) \cap \complement X$. Then Y is non-void since by Lemma 1, supp (f) contains an isolated point which cannot belong to X. Thus Supp $(f) = X \cup Y$ is a decomposition of supp (f) into disjoint non-empty subsets X and Y. Moreover every non-void subset of Y contains an isolated point for otherwise it will have to be contained in X which is not possible. Now, for any subset $A \subseteq \text{supp}(f)$, define f_A to be the function such that

$$f_A(P) = \begin{cases} f(P) & \text{if } P \in A \\ 0 & \text{if } P \in S \cap \complement A \end{cases}$$

We can then write $f = f_X + f_Y$. Since supp $(f_Y) = Y$, one has $f_Y \in H$ and hence from $f_X = f - f_Y$, it follows that $f_X \in E$. The fact that $f_X \neq 0$ and E is an essential extension of $\bigoplus M_P$, then implies by Lemma 1 that $X = \text{Supp}(f_X)$ has an isolated point. We thus arrive at a contradiction. Hence H is injective. This completes the proof.

<u>Corollary 1.</u> $\prod_{P \in S} M_P$ is an injective hull of M if and only if every non-void subset of supp (M) has an isolated point. In particular if S is discrete in Ω, then $\prod M_P \cong H(M)$.

<u>Proof</u>. If S has the property that each of its non-void subsets contains an isolated point, then for every $f \in \prod M_p$, supp (f) has the same property. Hence, by the Theorem $\prod M_p = H(\bigoplus M_p)$. On the other hand let $\prod M_p = H(\bigoplus M_p)$. Suppose that some non-empty subset $A \subseteq S$ has no isolated point. Then A must be an infinite set. We can find a function $f \in \prod M_p$ with supp (f) = A. Then $f \notin \bigoplus M_p$ and hence $f \neq 0$. Since $\prod M_p$ is an essential extension of $\bigoplus M_p$, by Lemma 1, supp (f) has an isolated point contrary to the assumption that A has no isolated point. Hence every non-void subset of S has an isolated point. The last part of the corollary follows immediately from the fact that every element in a discrete set has an isolated point.

<u>Corollary 2</u>. If S contains only principal ideals, then $\prod M_p = H(\bigoplus M_p)$.

<u>Proof</u>. Take Ra and Rb two different principal ideals in S. Then $a \notin Rb$ since if $a = rb$ then $Ra \subseteq Rb$ will imply $Ra = Rb$ a contradiction. Now R being a regular ring, one can find an element x in R such that $a = a^2x$. Since $0 = a(1 - ax)$ belongs to every P in S, $1 - ax \notin Ra$ and $1 - ax$ belongs to every P in S distinct from Ra. Hence $\Omega_{1-ax} \cap S = \{Ra\}$. Thus every element in S is an isolated point which shows by Corollary 1 that $\prod M_p = H(\bigoplus M_p)$.

<u>Corollary 3</u>. There exist semi-simple modules over a regular ring, which are not injective.

Proof. Let R_o be the two-element Boolean ring $\{o, e_o\}$, I an infinite index set and $R = \{ f \mid f : I \to R_o \}$, then R is a complete Boolean ring and hence a commutative regular ring. For each $\alpha \in I$ define P_α by $P_\alpha = \{ f \in R \mid f(\alpha) = 0 \}$. It is easily seen that P_α is a maximal ideal of R. Let $M = \bigoplus R/P_\alpha$. Then M is a semi-simple module with supp (M) = $\{ P_\alpha \mid \alpha \in I \}$. Take any element $P_{\alpha_o} \in$ Supp (M) and define f by

$$f(\alpha) = \begin{cases} e_o & \text{if } \alpha = \alpha_o \\ 0 & \text{if } \alpha \neq \alpha_o \end{cases}$$

then $f \in R \setminus P_{\alpha_o}$ and $f \in P_\beta$ for all $\beta \in I$ with $\beta \neq \alpha_o$. Thus $\Omega_f \cap \text{Supp}(M) = \{ P_{\alpha_o} \}$. This implies that supp (M) is discrete and hence $\prod R/P_\alpha = H(\bigoplus R/P_\alpha)$. The fact that I is infinite then shows that $\bigoplus R/P_\alpha$ is not injective.

Corollary 4. If $S = A \cup D_1 \cup D_2 \cup \ldots \cup D_n$ where A has an isolated point and $D_1, D_2, \ldots, D_n$ are discrete sets then $\prod M_p \cong H(M)$.

Proof. It follows immediately from Lemma 3 and Corollary 1.

In Corollary 3, we have a concrete example showing that not every semi-simple module is injective. It is therefore worthwhile to ask under what conditions a semi-simple module is injective. The following theorem gives a characterisation for the injectivity of a semi-simple module.

<u>Theorem 3</u>. A semi-simple module M is injective if and only if its support S has only finite discrete subsets.

<u>Proof</u>. Let M be injective. Assume that $D \subseteq S$ is an infinite discrete subset. We can find $f \in \prod M_p$ with supp f = D. Since D is infinite, $f \notin \bigoplus M_p$. But the fact that supp (f) is discrete implies by Theorem 2, that $f \in H(\bigoplus M_p) = \bigoplus M_p$ and so we get a contradiction. Hence S contains only finite discrete subsets.

Conversely suppose that S has only finite discrete subsets. Assume that M is not injective. Then $\bigoplus M_p$ has a proper essential extension E inside $\prod M_p$. Hence for any $f \in E \setminus \bigoplus M_p$, supp (f) contains an infinite discrete subset by Lemma 2. This contradiction then proves that M is injective.

CHAPTER IV

CHANGE OF RINGS

This chapter is concerned with the study of the inheritance properties of R-injective hulls E of a module M by changing the ground ring R into a suitably related ring S and making E into an S-module. This device of changing rings is used to obtain a generalization of the known fact about $Z(p^{\infty})$ that it is isomorphic to any one of its quotients by a proper subgroup. In the case when R is a homomorphic image of S, we show how to obtain an R-injective hull of a module from its S-injective hull.

Let R be a commutative ring with unit 1. For any non-zero R-module M, we define A(M) by $A(M) = \{ s \in R \mid f_s : x \to sx$ is an automorphism of M $\}$. Since for any two elements s and t in A(M), $f_{st} = f_s f_t$ is an automorphism of M, A(M) is a multiplicative monoid with 1 as its unit. Moreover $0 \notin A(M)$ because $x \to 0.x$ is not one-to-one. We can therefore form the generalized ring of quotients of R with respect to A(M). As usual, this ring of quotients will be denoted by $R_{A(M)}$. Let $N = \{ r \in R \mid$ there exists $s \in A(M)$ with $sr = 0 \}$ be the A(M) - component of the zero ideal, then $R_{A(M)} = \left\{ \nu(r)/\nu(s) \mid r \in R,\ s \in A(M) \right\}$ where $\nu : R \longrightarrow R/N$ is the natural homomorphism. Now for any element $\nu(r)/\nu(s) \in R_{A(M)}$ and $x \in M$, define $(\nu(r)/\nu(s))x = f_s^{-1}(rx)$. In order to show that M is an $R_{A(M)}$ - module with respect to this definition, it is enough to check that $\nu(r) = 0$ implies $f_s^{-1}(rx) = 0$. Let us suppose that $\nu(r) = 0$, then there exists $s' \in A(M)$ such that $s'r = 0$. Hence $0 = s'rx$. This implies $rx = 0$ since $f_{s'}$ is one-to-one. Therefore, $f_s^{-1}(rx) = 0$ since $s \in A(M)$. Thus M is an $R_{A(M)}$ - module. This fact extends as follows to an R-injective hull E of M:

<u>Theorem 1</u>. By extending the $R_{A(M)}$ - module structure of M, E can be made into an $R_{A(M)}$ - module and as such it is an $R_{A(M)}$ -injective hull of M.

<u>Proof</u>. To prove the first part of the theorem it is sufficient to show that $A(M) \subseteq A(E)$. Let $s \in A(M)$ and take any non-zero element $x \in E$; then for some element $r \in R$, $0 \neq rx \in M$. Since f_s is one-to-one on M, this implies $srx \neq 0$, hence $sx \neq 0$. Thus f_s is one-to-one on E. Therefore E is isomorphic to $f_s(E)$. This shows that $f_s(E)$ is R-injective. Since $E \supseteq f_s(E) \supseteq f_s(M) = M$, one has $E = f_s(E)$. Hence f_s is an automorphism of E and so $s \in A(M)$. This proves $A(E) = A(M)$. One can therefore define for $\nu(r)/\nu(s) \in R_{A(M)}$, $x \in E$, $(\nu(r)/\nu(s))\, x = f_s^{-1}(rx)$ which makes E into an $R_{A(M)}$-module by extending the $R_{A(M)}$ module structure of M.

Let $0 \neq x \in E$, then $0 \neq rx \in M$ for some $r \in R$, hence $0 \neq (\nu(r)/\nu(1))x = rx \in R_{A(M)}\, x \cap M$ by the definition of E as $R_{A(M)}$ - module. This shows that E is an $R_{A(M)}$ -essential extension of M. To prove the injectivity of E, let $F \supseteq E$ be any $R_{A(M)}$-essential extension of E. Since any $R_{A(M)}$ - module can be made into an R-module by defining $\nu(r)x = rx$, E is an R-submodule of F. Hence by the R-injectivity of E, we have F = E + H (dir) for some R-submodule $H \subseteq F$. Suppose $H \neq 0$ and let $0 \neq x \in H$. Then $R_{A(M)}\, x \cap E \neq 0$. Hence there exists a non-zero element $(\nu(r)/\nu(s))\, x \in E$ which implies that $0 \neq \nu(r)\, x \in E$. But in F, $\nu(r)\, x = rx$. Hence $0 \neq rx \in E \cap H$ which is a contradiction. Therefore F = E and thus E has no proper $R_{A(M)}$ - essential extension and the proof is complete.

Remark 1. If $P \subseteq R$ is a maximal ideal and M = R/P, then $A(M) = R \setminus P$ and hence $R_{A(M)} = R_P$.

Proof. If $s \in A(M)$, then $s(1 + P) \neq 0$ and this implies $s \in R \setminus P$. Hence $A(M) \subseteq R \setminus P$. On the other hand if $s \in R \setminus P$ then f_s is one-to-one since $s(r + P) = 0$ implies $sr \in P$ whence $r \in P$. Moreover since R/P is a field, s + P is invertible and hence any r + P in R/P can be expressed as $r + P = s\,(rs' + P)$ where $s' + P = (s + P)^{-1}$. This shows that f_s is onto and hence an automorphism of M. Thus $R \setminus P \subseteq A(M)$. Hence $A(M) = R \setminus P$.

<u>Remark 2</u>. Let $P \subseteq R$ be a maximal ideal, N, the $R \setminus P$-component of the zero ideal and $\nu : R \to R/N$, the natural homomorphism, then R/P is isomorphic to $R_P/R_P\nu(P)$.

<u>Proof</u>. Consider $\varphi : R/P \to R_P/R_P\nu(P)$ by $\varphi(r + P) = \nu(r) + R_P\nu(P)$. Since $\nu(r) \in R_P\nu(P) \cap \nu(R) = \nu(P)$ implies $r \in P$, φ is one-to-one. Now take any element $\nu(r)/\nu(s) + R_P\nu(P)$ in $R_P/R_P\nu(P)$. Since $s \in R \setminus P$ we have $R = Rs + P$, hence $1 = r_o s + p$ with $r_o \in R$, $p \in P$. This gives $\nu(1) = \nu(s)\nu(r_o) + \nu(p)$ and therefore $\nu(r)/\nu(s) = \nu(rr_o) + \nu(rp)/\nu(s)$. Since the last term is in $R_P\nu(P)$, one has $\nu(r)/\nu(s) + R_P\nu(P) = \varphi(rr_o + P)$. This shows that φ is an epimorphism and hence an isomorphism.

From Theorem 1 and these remarks, one immediately has the following:

<u>Corollary</u>. If R is a commutative ring with unit, $P \subseteq R$ a maximal ideal, E an R-injective hull of R/P and E' an R_P-injective hull of $R_P/R_P\nu(P)$, then E as R_P-module is isomorphic to E'.

<u>Remark 3</u>. If R is an integral domain, then N = 0, ν is the identity mapping of R, $R_P\nu(P) = R_PP$ and $R_P = \{(r/s) \mid r \in R, s \in R \setminus P\}$. Thus in this case we have E isomorphic to the R_P-injective hull of R_P/R_PP.

For Z, the ring of integers and p a prime number, $Z(p^\infty) = Z[p^{-1}]/Zp$ is the injective hull of Z/Zp and hence it is indecomposable [2, 18]. It is well known that $Z(p^{\infty})$ is isomorphic to each of its non-zero homomorphic images [15]. The following theorem generalizes this fact about $Z(p^{\infty})$ to indecomposable injective modules over rings more general than the ring of integers which will include Dedekind domains as a special case.

Theorem 2. Let R be an integral domain, $P \subseteq R$ a maximal ideal such that R_P is a principal ideal ring. Then the injective hull of R/P is isomorphic to any of its quotients by a proper submodule [23].

Proof. Here $R_P P = R_P \pi$ for some $\pi \in R_P$, and $R_P/R_P\pi$ has $E = R_P[\pi^{-1}]/R_P\pi$ as its injective hull [3] where $R_P[\pi^{-1}]$ is generated by π^{-1} as ring extension of R_P in the quotient field of R. By the Corollary of Theorem 1, it suffices to consider this R-module E.

We first show that every R-submodule of E is also an R_P-submodule which will imply that the R_P-submodules are the same as the R-submodules. For this, it is sufficient to prove that if $S \subseteq E$ is any R-submodule of E and $s_o \in R \setminus P$, then $(1/s_o)\, S \subseteq S$. Now, $R_P[\pi^{-1}] = \bigcup_{k \geq 0} R_P \pi^{-k}$ implies that any element in S is of the form $x = (a/s)\, \pi^{-k} + R_P\pi$ where $a \in R$, $s \in R \setminus P$ and k is an integer. From $R = Rs_o + P^{k+1}$ we get $1 = s_o t + u$ with $t \in R$, $u \in P^{k+1}$ and, therefore, $(1/s_o)\, x = tx + (1/s_o)\, ux =$ $tx + (u/s_o)\, ((a/s)\pi^{-k} + R_P\pi) = tx \in S$. Hence $(1/s_o)\, S \subseteq S$ and we can talk about the submodules of E without reference to R or R_P.

Next we wish to show that every submodule of E is of the form $R_p\,\pi^{-n}/R_p\pi$. Now $E = R_p[\pi^{-1}]/R_p\pi$ implies that the lattice of all submodules of E is isomorphic to the lattice of R_p-submodules of $R_p[\pi^{-1}]$ which contain $R_p\pi$. Hence any submodule of E corresponds to exactly one fractional ideal S of R_p with $R_p\pi \subseteq S \subseteq R_p[\pi^{-1}] = \bigcup_{k \geq 0} R_p\,\pi^{-k}$. Let $S_k = S \cap R_p\,\pi^{-k}$, then $R_p\,\pi \subseteq S_k \subseteq R_p\,\pi^{-k}$ which implies that $R_p\,\pi^{k+1} \subseteq S_k\,\pi^{k} \subseteq R_p$. By the fact that R_p is a principal ideal ring, $S_k\,\pi^{k} = R_p\,u$ for some $u \in R_p$. Hence $\pi^{k+1} \in R_p\,u$ implies that $u \mid \pi^{k+1}$. But π is irreducible, hence u is some power of π with index $\leq k + 1$. Thus $u = \pi^{\ell_k}$ for some ℓ_k with $0 \leq \ell_k \leq k + 1$ and $S_k\,\pi^{k} = R_p\,\pi^{\ell_k}$ with $0 < \ell_k < k + 1$. Therefore $S_k = R_k{}^{\ell_k - k}$. If S corresponds to a proper submodule of E, then $S \subset R_p[\pi^{-1}]$ and since $S = \bigcup_{k \geq 0} S_k$ and the S_k's form an ascending sequence, one has $S = R_p\pi^{-n}$ for some integer n. Thus every proper submodule of E is of the form $R_p\,\pi^{-n}/R_p\pi$ and any quotient of E by such a submodule may be expressed as $R_p[\pi^{-1}]/R_p\pi^{-n}$.

Now if we compose the homomorphism $x \rightarrow \pi^{-(n+1)}x$ of $R_p[\pi^{-1}]$ into itself with the natural homomorphism $y \longrightarrow y + R_p\,\pi^{-n}$ from $R_p[\pi^{-1}]$ to $R_p[\pi^{-1}]/R_p\,\pi^{-n}$ we get an epimorphism whose kernal is $R_p\pi$. This shows that E is isomorphic to $R_p[\pi^{-1}]/R_p\pi^{-n}$ and the proof of the theorem is complete.

If R is a Dedekind domain then each proper prime ideal P of R is maximal and R_p is a principal ideal ring; therefore, Theorem 2 then applies to any R/P. Since torsion modules over R are semi-primary and hence the indecomposable injective torsion modules are injective hulls of simple modules, it follows from the above that the indecomposable injective torsion modules over a Dedekind domain all have the property that they are isomorphic to any of their non-zero homomorphic images. In the following, we provide an example which shows that this is not the case for indecomposable injective modules in general. For this, we first require:

Lemma 1. Let R be a commutative ring with unit 1, $P \subseteq R$ a proper prime ideal. If M is an R-module with the property that for no $x \in M$, $O(x) = P$, then the same holds for any essential extension of M.

Proof. Let $E \supseteq M$ be an essential extension of M. Suppose there eixsts $z \in E$ with $O(z) = P$, then since P is proper, $z \neq 0$. Hence $Rz \cap M \neq 0$ and therefore there exists $r \in R$ with $0 \neq rz \in M$, hence $O(rz) \neq P$. Since for every $u \in O(rz)$ we have $ur \in O(z) = P$, it follows that $O(rz)\, r \subseteq P$. Now $Prz = 0$ implies $P \subseteq O(rz)$; hence $P \subset O(rz)$. Take $s \in O(rz) \setminus P$, then $sr \in O(rz)\, r \subseteq P$. This implies $r \in P$ since P is a prime ideal and $s \notin P$. Hence $rz = 0$ and we arrive at a contradiction. This proves that E has no element whose order ideal is P.

Corollary. If R is a commutative ring with unit, and P and P' are proper prime ideals of R, then H(R/P) is isomorphic to H(R/P') if and only if P = P'.

<u>Proof</u>. If $P = P'$ then $H(R/P)$ and $H(R/P')$ are isomorphic trivially. Let $H(R/P) \cong H(R/P')$ and suppose that $P \neq P'$. Now $H(R/P)$ is an injective hull of R/P and hence an essential extension of R/P. Since R/P has the property that each of its non-zero elements has order ideal $P \neq P'$, it follows from Lemma 1 that there is no element in $H(R/P')$ with order ideal P' which is not true since every non-zero element in R/P' has order ideal P'. This contradiction shows that $P = P'$.

The next proposition will give the desired example where an indecomposable injective module has a quotient module which is neither zero nor isomorphic to itself.

<u>Proposition</u>. Let R be a commutative ring with unit, P a non-zero, non-maximal prime ideal in R and E an injective hull of R/P. Then E has a quotient module which is neither 0 nor isomorphic to E.

<u>Proof</u>. Since P is a non-zero non-maximal ideal, there exists a maximal ideal M such that $0 \subset P \subset M \subset R$ and so we have $E \supseteq R/P \supset M/P \neq 0$. Hence $E/(M/P) \neq 0$. We will show that E is not isomorphic to $E/(M/P)$. Assume the contrary. Then $E/(M/P)$ is indecomposable injective and contains $(R/P)/(M/P)$ which is isomorphic to the injective hull of R/M. This implies that R/M and R/P have isomorphic injective hulls. Hence by corollary, Lemma 1, $P = M$, a contradiction. Thus the quotient module $E/(M/P)$ is neither zero nor isomorphic to E.

We will now consider two rings R and S with unit and a unitary ring epimorphism $\varphi: S \to R$.

Remark 4. Any left module M over R can be made into a left S-module in a natural way by defining $sx = \varphi(s)x$ for $s \in S$, $x \in M$. We will denote this S-module by ${}_{\varphi}M$. Now for a left module M over S, define $M' = \{ x \in M \mid \operatorname{Ker}(\varphi) \cdot x = 0 \}$. Clearly M' is an S-submodule of M and it can be made into an R-module by setting $rx = sy$ where $r \in R$, $x \in M'$ and $s \in \varphi^{-1}(r)$. This R-module will be denoted by M_o. Let $\mathcal{M}(R)$ and $\mathcal{M}(S)$ denote the categories of left modules over R and S respectively. Define the functors $F_{\varphi}: \mathcal{M}(R) \to \mathcal{M}(S)$ and $G_{\varphi}: \mathcal{M}(S) \to \mathcal{M}(R)$ by $F_{\varphi}(M) = {}_{\varphi}M$ and $G_{\varphi}(M) = M_o$. Then it is easy to check that $G_{\varphi} \cdot F_{\varphi}$ is the identity on $\mathcal{M}(R)$ and $F_{\varphi} \cdot G_{\varphi}$ acts identically on those $M \in \mathcal{M}(S)$ for which $M' = M$.

We prove the following:

Lemma 2. F_{φ} carries essential extensions into essential extensions and G_{φ} carries injectives into injectives.

Proof. Let $E \supseteq M$ be an essential extension of M in $\mathcal{M}(R)$. We will show that $F_{\varphi}(E)$ is an essential extension of $F_{\varphi}(M)$ in $\mathcal{M}(S)$. Let $0 \neq x \in {}_{\varphi}E$. Since E is an R-essential extension of M, we have $0 \neq rx \in M$ for some $r \in R$ and hence $0 \neq \varphi(s)x \in M$ where $s \in \varphi^{-1}(r)$. This implies $0 \neq sx \in {}_{\varphi}M$. Thus $Sx \cap {}_{\varphi}M \neq 0$. Therefore $F_{\varphi}(E)$ is an essential extention of $F_{\varphi}(M)$ in $\mathcal{M}(S)$.

To prove the second part of the Lemma, let E be injective in $\mathcal{M}(S)$. Suppose $G_\varphi(E)$ has a proper essential extension A in $\mathcal{M}(R)$, then as shown above $F_\varphi A \supsetneq F_\varphi G_\varphi(E)$ is an essential extension in $\mathcal{M}(S)$. But E is injective in $\mathcal{M}(S)$ and contains $F_\varphi G_\varphi(E)$, hence there exists a B isomorphic to $F_\varphi A$ over $F_\varphi G_\varphi(E)$ in E. Hence $(\text{Ker}\,\varphi) B = 0$ and therefore $B = F_\varphi G_\varphi(E)$ which implies that $F_\varphi A = F_\varphi G_\varphi(E)$. Since $G_\varphi F_\varphi$ is an identity on $\mathcal{M}(R)$, we have $A = G_\varphi(E)$. Hence $G_\varphi(E)$ is injective in $\mathcal{M}(R)$ and so the proof of the Lemma is complete.

The relation between injective hulls of a module in the categories $\mathcal{M}(S)$ and $\mathcal{M}(R)$ is given by the following:

<u>Theorem 3</u>. Let $M \in \mathcal{M}(R)$, E an injective hull of $F_\varphi(M)$ in $\mathcal{M}(S)$. Then $G_\varphi(E)$ is an injective hull of M in $\mathcal{M}(R)$ [1].

<u>Proof</u>. By Lemma 2, $G_\varphi(E)$ is injective in $\mathcal{M}(R)$. Moreover $G_\varphi(E)$ contains $G_\varphi(F_\varphi(M)) = M$. We have therefore only to show that $G_\varphi(E)$ is an R-essential extension of M. Let $0 \neq x \in G_\varphi(E)$. Since E is an S-essential extension of $F_\varphi(M)$, there exists $s \in S$, $s \notin \text{Ker}\,(\varphi)$ such that $0 \neq sx \in F_\varphi(M)$. From the definition of $G_\varphi(E)$ as R-module, it follows that $0 \neq \varphi(s)x \in M$. Hence $Rx \cap M \neq 0$. Thus $G_\varphi(E)$ is an R-essential extension of M as desired.

<u>Corollary 1</u>. Let $J \subseteq R$ be a left ideal, M an R-module with $JM = 0$ and E an R-injective hull of M. Then $A = \{ x \in E \mid Jx = 0 \}$ made into an R/J-module is an R/J-injective hull of M.

Proof. Here φ is the natural homomorphism $R \to R/J$ with $\text{Ker}(\varphi) = J$. Hence by Theorem 3, $G_\varphi E = A$ is an R/J-injective hull of M.

Corollary 2. If R is a Dedekind domain, $P \subseteq R$ a proper prime ideal, then R/P^k is an R/P^k-injective hull of R/P.

Proof. Let $P^* = \bigcup_k P^{-k}$. Then $E = P^*/P$ is an R-injective hull of R/P [2]. Now $R/P^k \cong P^{-(k-1)}/P = \{ x + P \in E \mid x \in P^{-(k-1)}$ i.e. $P^{k-1} x \subseteq R \} = \{ y \in E \mid P^k y = 0 \}$. Hence by Corollary 1, R/P^k is an R/P^k-injective hull of R/P.

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