SYMMETRICALLY POSITIVE DEFINITE FUNCTIONS

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By

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SCOPE AND CONTENTS: In this thesis we study the representation theorems for evenly positive definite functions on Euclidean spaces. A generalization of the concept of evenness on R^n to a concept of symmetry on any locally compact abelian group is given. In addition, a result analogous to the Weil-Povzner-Raikov Theorem is obtained for the representation of symmetrically positive definite functions on locally compact abelian groups.

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Chapter 1

Preliminary

Introduction

1.1 A complex-valued function f defined on an arbitrary group G is called positive definite (abbreviated as p.d.) if the inequality

$$\sum_{i,j=1}^{n} f(x_j^{-1}x_i)\,\xi_i\,\overline{\xi_j} \geq 0$$

holds for every choice of complex numbers $\xi_1,\dots,\xi_n$ and $x_1,\dots,x_n$ in G. For the case where G is a locally compact abelian group, Weil [23], Povzner [16] and Raikov [17] proved that if f is a continuous p.d. function on G, then there is a positive bounded measure μ on $\hat{G}$, the dual group of G, such that

$$f(x) = \int_{\hat{G}} [x,\hat{x}]\, d\mu(\hat{x})$$

where $[x,\hat{x}]$ denotes the value of the character $\hat{x}$ at the point x. This generalizes theorems of Herglotz [10] (G=Z, the integers) and Bochner [2,3] (G=R, the real numbers).

1.2 For any locally compact abelian group G, written additively, there is another notion of positive-definiteness. Let F be a set of complex-valued functions on G. A complex-valued function f on G is called positive definite for F if the integral

$$\int_G \int_G f(x-y)\,\varphi(x)\overline{\varphi(y)}\,dxdy$$

exists as a Lebesgue integral and is non-negative for every $\varphi \in F$, where dx denotes integration with respect to Haar measure on G. The class of all functions which are p.d. for F will be denoted by P(F). Clearly $F_1 \subseteq F_2$ implies that $P(F_2) \subseteq P(F_1)$. Let us denote by $L^p(G)$ the ordinary L^p space with respect to Haar measure on G, by $L^p_c(G)$ the set of all functions in $L^p(G)$ with compact support, and by $C_c(G)$ the set of all continuous functions with compact support on G. It turns out that $P(L^1(G))$ is identical, up to sets of measure zero, with the class of ordinary continuous p.d. functions (see Naimark [15, §30, Theorems III and IV]). However $P(C_c)$ is a much more extensive class of functions. Hewitt and Ross [11], Edwards [5] and Rickert [18] have given constructions of functions on non-discrete locally compact groups which are in $P(C_c)$ but not in L^∞, and therefore not almost everywhere equal to the ordinary p.d. functions. Cooper [4] and Stewart [21] proved that $P(C_c) = P(L^p_c)$ for every $p \geq 2$ and that every $f \in P(C_c)$ is the Fourier-Stieltjes transform (in a suitable summability sense) of a positive measure, possibly unbounded, on $\hat{G}$ (Cooper had proved the result for G = R. The general result was proved by Stewart).

We are interested in the theory of p.d. functions because they play a very important role in the abstract theory of harmonic analysis on groups (see e.g. Loomis[14] and Rudin [19]) and in the theory of unitary representations of locally compact groups (see e.g. Gelfand and Raikow [6]). For a historical survey on p.d. functions, the reader is recommended to read the article by Stewart [22] .

Aim of the thesis

1.3 Let us denote by R^n the n-dimensional Euclidean space, and by $C_c^\infty(R^n)$ the space of all infinitely differentiable functions with compact support on R^n. A complex-valued function f on R^n is called even if the equality

$$f(\pm x_1, \dots, \pm x_n) = f(x_1, \dots, x_n)$$

holds for every combination of signs. Let F be a set of complex-valued functions on R^n, and E the set of all even functions in F. A complex-valued function f on R^n is called evenly positive definite for F (abbreviated as e.p.d. for F) if f is an even function which is p.d. for E. We denote by $P_e(F)$ the class of functions which are e.p.d. for F.

A Bochner-type theorem, which gave a description of all continuous functions in the class $P_e(C_c^\infty(R))$, was obtained by Krein. He proved that every such function has the form

$$f(x) = \int_0^\infty \cos\lambda x \, d\mu_1(\lambda) + \int_0^\infty \cosh\lambda x \, d\mu_2(\lambda),$$

where μ_1 and μ_2 are positive measures, μ_1 is finite, and μ_2 is such that the second integral converges (see Gelfand and Vilenkin [9, p.197]). Gelfand and Vilenkin [9, Chapter II, Sec. 5] had obtained representation theorems for all generalized functions which are e.p.d. Our aims are twofold. We first wish to obtain and to extend the representation theorems for e.p.d. functions in Gelfand and Vilenkin. Secondly, for any locally compact abelian group G, we wish to define a concept of symmetry on G that would generalize the concept of evenness on R^n, and to obtain representation theorems for functions in P(F) which would reflect the kind of symmetry that the class F might possess.

Evenly positive definite functions are considered in Chapter 2, where the results of Gelfand and Vilenkin are extended. A concept of symmetry is introduced in Chapter 3. Symmetrically positive definite functions, which is a generalization of the concept of evenly positive definite functions, is considered in Chapter 4. Most material in Chapters 3 and 4 is new, but enough hints to these results are obtained from Gelfand and Vilenkin (especially Sec. 5.4 in Chapter II [9]).

Notations and Terminologies

1.4. Throughout this paper, G will denote a locally compact abelian group, and $\hat{G}$ its dual group.

If $x \in G$ and $\hat{x} \in \hat{G}$, we shall write $[x, \hat{x}]$ for the value of the character $\hat{x}$ at the point x, and $[\hat{x}, x] = [-x, \hat{x}]$. We shall denote integration with respect to the Harr measures on G and $\hat{G}$ by dx and $d\hat{x}$, respectively. The Fourier transform of a function $\varphi \in L^1(G)$, denoted by $\hat{\varphi}$, is defined by

$$(1) \qquad \hat{\varphi}(\hat{x}) = \int_G [-x, \hat{x}]\, \varphi(x)\,dx \quad .$$

We denote by $M(G)$ the set of all bounded regular complex-valued measures on G, and by $A(\hat{G})$ the set $\{\hat{\varphi} \,;\, \varphi \in L^1(G)\}$. We denote by $\tilde{\varphi}$ the function $\tilde{\varphi}(x) = \overline{\varphi(-x)}$ for all complex-valued function φ defined on G. For any pair of measurable functions φ and ψ on G, we define their <u>convolution</u> $\varphi*\psi$ by

$$(2) \qquad (\varphi*\psi)(x) = \int_G \varphi(x-y)\,\psi(y)\,dy$$

provided that

$$(3) \qquad \int_G |\varphi(x-y)\psi(y)|\,dy < \infty \quad .$$

We have the following.

<u>Theorem</u> (see Rudin [19, p 4])

(a) If (3) holds for some $x \in G$, then $(\varphi*\psi)(x) = (\psi*\varphi)(x)$.

(b) If $\varphi \in L^1(G)$ and $\psi \in L^\infty(G)$, then $\varphi*\psi$ is bounded and uniformly continuous.

(c) If φ and ψ are in $C_c(G)$, with compact supports A and B, then the support of $\varphi*\psi$ lies in $A+B$, where

$$A+B = \{\, a+b;\ a \in A,\ b \in B \,\}, \quad \text{so} \quad \varphi*\psi \in C_c(G) \ .$$

(d) If φ and ψ are in $L^1(G)$, then (3) holds for almost all $x \in G$, $\varphi * \psi \in L^1(G)$, and the inequality

$$\|\varphi * \psi\|_1 \leq \|\varphi\|_1 \|\psi\|_1$$

holds.

(e) If φ, ψ, ξ are in $L^1(G)$, then $(\varphi * \psi) * \xi = \varphi * (\psi * \xi)$.

Simple Properties of the class P(F)

1.5 Theorem Let $f \in P(F)$, where F has the property that for any compact subset K of G, F contains a bounded function with compact support which is strictly positive on K. Then f is locally summable.

Proof: Let K be a compact subset of G, and let φ in F be a bounded function with compact support which is strictly positive on $(K+K) \cup K$. The function $\varphi * \tilde{\varphi}$ is then a continuous function with compact support and is strictly positive on K. Therefore, if $m = \inf\{\varphi * \tilde{\varphi}(x);\ x \in K\}$, then $m > 0$. But $f \in P(F)$ implies that the integral

$$\int_G f(x)\, \varphi * \tilde{\varphi}(x)\, dx = \int_G \int_G f(x-y)\, \varphi(x)\, \overline{\varphi(y)}\, dx dy$$

exists as Lebesgue integral, therefore

$$m \int_K |f(x)|\, dx \leq \int_K |f(x)\, \varphi * \tilde{\varphi}(x)|\, dx \leq \int_G |f(x)\, \varphi * \tilde{\varphi}(x)|\, dx < \infty$$

and hence f is summable over K. Thus f is locally summable. ∎

1.6 **Theorem:** Let $f \in P(F)$, where F is a linear space of complex-valued functions on G. Then the integral

(1) $$K(\varphi,\psi) = \int_G \int_G f(x-y)\,\varphi(x)\overline{\psi(y)}\,dxdy$$

exists as Lebesgue integral for every $\varphi, \psi \in F$, and the inequality

(2) $$|K(\varphi,\psi)|^2 \leq K(\varphi,\varphi)\,K(\psi,\psi)$$

holds for all $\varphi, \psi \in F$.

Proof: The integrals $K(\varphi,\varphi)$, $K(\psi,\psi)$ and $K(\varphi+\lambda\psi, \varphi+\lambda\psi)$ exist for all $\varphi, \psi \in F$ and every $\lambda \in \mathbb{C}$, the field of complex numbers, because $f \in P(F)$. Since

(3) $$K(\varphi+\lambda\psi, \varphi+\lambda\psi) = K(\varphi,\varphi) + K(\lambda\psi,\lambda\psi) + \int_G \int_G f(x-y)\,[\lambda\psi(x)\overline{\varphi(y)} + \varphi(x)\overline{\lambda\psi(y)}]\,dxdy$$

therefore the integral

(4) $$\int_G \int_G f(x-y)\,[\lambda\psi(x)\overline{\varphi(y)} + \varphi(x)\overline{\lambda}\,\overline{\psi(y)}]\,dxdy$$

exists as Lebesgue integral for any $\varphi, \psi \in F$, $\lambda \in \mathbb{C}$.

With $\lambda = 1$ and $\lambda = i$, we obtain from (4) that the integrals

$$\int_G \int_G f(x-y)\,[\psi(x)\overline{\varphi(y)} + \varphi(x)\overline{\psi(y)}]\,dxdy$$

$$\int_G \int_G f(x-y)\,[\psi(x)\overline{\varphi(y)} - \varphi(x)\overline{\psi(y)}]\,dxdy$$

exist as Lebesgue integrals, and, consequently, the integrals $K(\varphi,\psi)$ and $K(\psi,\varphi)$ exist as Lebesgue integrals. Now if we put $p = K(\varphi,\varphi)$, $q = K(\psi,\psi)$, $r = K(\varphi,\psi)$ and $s = K(\psi,\varphi)$ into (3) we obtain the inequality

(5) $$p + \overline{\lambda}r + \lambda s + |\lambda|^2 q = K(\varphi+\lambda\psi, \varphi+\lambda\psi) \geq 0$$

for every $\lambda \in \mathbb{C}$. Here we have used the fact that $f \in P(F)$. With $\lambda = 1$ and $\lambda = i$, we see from (5) that both $s+r$ and $i(s-r)$ are real. Hence $r = \bar{s}$, i.e. $K(\varphi, \psi) = \overline{K(\psi, \varphi)}$. Thus $K(\varphi, \psi)$ is a positive hermitian form on F, and therefore, inequality (2) holds. ∎

<u>A Factorization Theorem for Banach Algebras</u>.

<u>1.7</u> A vector space A over the complex field is a <u>commutative algebra</u> if a multiplication is defined in A which satisfies the usual commutative associative and distributive laws. If a norm is defined in a commutative algebra A which makes A into a Banach Space, and if the inequality $\|xy\| \leq \|x\| \, \|y\|$ holds for all $x, y \in A$, then A is a <u>commutative Banach Algebra</u>. A net $\{\alpha_\nu\}$ of elements in A is called a <u>bounded approximate unit</u> in A if $\{\|\alpha_\nu\|\}$ is a bounded set of strictly positive numbers and $\lim_\nu \alpha_\nu a = a$ for all $a \in A$.

<u>1.8</u> <u>Theorem</u> (Hewitt and Ross [12, Sec. 33.12] or Simon [20]). There is a net $\{\alpha_\nu\}$ in $C_c(G)$ such that the set $\{|\alpha_\nu(x)|; x \in G, \text{for all } \nu\}$ is bounded, $\alpha_\nu(x) \to 1$ uniformly on compacts, $\hat{\alpha}_\nu \geq 0$, $\hat{\alpha}_\nu \in L^1(\hat{G})$, $\|\hat{\alpha}_\nu\|_1 \leq 1$ and $\hat{\alpha}_\nu * f \to f$ in L^1 norm for every $f \in L^1(\hat{G})$.

<u>1.9</u> <u>Theorem</u> (<u>P. J. Cohen</u>) (Hewitt and Ross [12, Sec. 32.26]). Let A be a commutative Banach Algebra with a bounded approximate unit. Then every $z \in A$ is of the form $z = xy$ for some $x, y \in A$.

Some Theorems from Harmonic Analysis on Euclidean Spaces

1.10 Notations Let us denote by $\mathbb{C}^n$ the n-dimensional complex space with the usual inner product (,). Points of $\mathbb{C}^n$ will be denoted by $z = (z_1, \dots, z_n)$, where $z_k \in \mathbb{C}$. If $z_k = x_k + iy_k$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, then we write $z = x + iy$. The vectors $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are the real and imaginary parts of z, respectively; R^n will be thought of as the set of all $z \in \mathbb{C}^n$ with $\operatorname{Im} z = 0$. The term multi-index denotes an ordered n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers α_i. With each multi-index α, each $c = (c_1, \dots, c_n) \in R^n$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we adopt the following notations

(1) $|\alpha| = \alpha_1 + \dots + \alpha_n$

(2) $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$

(3) $c \geq 0$ means $c_k \geq 0$ for $1 \leq k \leq n$

(4) $c > 0$ means $c_k > 0$ for $1 \leq k \leq n$

(5) $cz = \sum_{k=1}^{n} c_k z_k$

(6) $cz^2 = \sum_{k=1}^{n} c_k z_k^2$

(7) $c\|z\| = \sum_{k=1}^{n} c_k |z_k|$

(8) $\|cz\| = \sum_{k=1}^{n} |c_k z_k|$

1.11 Let us give $C_c^\infty(R^n)$ the topology usual for the theory of distributions, i.e., $\varphi_m \to 0$ in $C_c^\infty(R^n) \iff$ the supports of all φ_m's lie in a common compact set, and φ_m and all its derivatives converge uniformly to 0. If T is a distribution, i.e., a continuous linear functional on $C_c^\infty(R^n)$, then T is called positive definite if $T(\varphi * \tilde{\varphi}) \geqslant 0$ for all $\varphi \in C_c^\infty$. Schwartz has extended the theory of p.d. functions to distributions via the following Bochner-type theorem.

1.12 **Theorem** (see Gelfand and Vilenkin [9, Chapter II, Sec. 3.3 Theorem 3]). If T is a positive definite distribution, then T is the Fourier transform of a positive tempered measure μ, i.e.,

$$T(\varphi) = \int_{R^n} \hat{\varphi}(x)\,d\mu(x)$$

where $\hat{\varphi}$ is the Fourier transform of φ, and for some multi-index $\alpha \geqslant 0$,

$$\int_{R^n} \frac{d\mu(x)}{(1+|x_1|^2)^{\alpha_1}\cdots(1+|x_n|^2)^{\alpha_n}} < \infty .$$

1.13 If D is an open set in $\mathbb{C}^n$, and if f is a continuous complex-valued function in D, then f is said to be holomorphic in D if it is holomorphic in each variable separately. A function that is holomorphic in all of $\mathbb{C}^n$ is said to be entire. We denote by Z(n) the space of all entire functions φ such that for any multi-index α, the inequality

$$|z^\alpha \varphi(z)| \leqslant C_\alpha \exp(a\|y\|), \qquad z = x + iy$$

holds for some constants a and C_α. We denote by Y(n) the space of all entire functions φ satisfying inequalities of the form

$$|\varphi(x+iy)| \leqslant K \exp(-ax^2 + by^2) .$$

<u>1.14</u> <u>Theorem</u> (see Gelfand and Shilov [7, Chapter II, 1])

If $\varphi \in C_c^\infty(R^n)$, and if we define $\hat{\varphi}$, the Fourier transform of φ, by

$$\hat{\varphi}(z) = \int_{R^n} \varphi(t)e^{-i(t,z)}\,dt \qquad (z \in \mathbb{C}^n),$$

then $\hat{\varphi} \in Z(n)$. Conversely, for any $\Phi \in Z(n)$, there is a φ in $C_c^\infty(R^n)$ such that $\hat{\varphi} = \Phi$. Furthermore, if $\{\varphi_m\}$ is a sequence in C_c^∞ such that $\varphi_m \longrightarrow \varphi$ in the topology of C_c^∞, then $\hat{\varphi}_m \longrightarrow \hat{\varphi}$ uniformly on compact subsets of $\mathbb{C}^n$, and every $\hat{\varphi}_m$ satisfies an inequality of the form

$$|z^\alpha \hat{\varphi}_m(z)| \leq c\,\exp(a\|y\|), \qquad z = x + iy$$

where the constants c and a are independent of m.

<u>1.15</u> <u>Theorem</u> (Gelfand and Shilov [8, Chapter IV, Sec. 6.2])

If $\varphi \in Y(n)$, and if we define $\hat{\varphi}$, the Fourier transform of φ, by

$$\hat{\varphi}(z) = \int_{R^n} \varphi(t)e^{-i(t,z)}\,dt,$$

then $\hat{\varphi} \in Y(n)$. Conversely, for every $\Phi \in Y(n)$, there is a $\varphi \in Y(n)$ such that $\hat{\varphi} = \Phi$. Furthermore, if $\{\Phi_m\}$ is a sequence in $Y(n)$ which converges to a function Φ in $Y(n)$ uniformly on compact subsets of $\mathbb{C}^n$, such that every Φ_m satisfies an inequality of the form

$$|\Phi_m(x+iy)| \leq K\exp(-ax^2 + by^2)$$

where the constants K, a, b do not depend upon m, then there is a sequence $\{\varphi_m\}$ in $Y(n)$ such that $\hat{\varphi}_m = \Phi_m$, $\varphi_m \to \varphi$ uniformly on compacts, where φ is the function in $Y(n)$ such that $\hat{\varphi} = \Phi$, and the inequality

$$|\varphi_m(x+iy)| \leq K' \exp(-a'x^2 + b'y^2)$$

holds for every m, where the constants K', a' and b' are independent of m.

Chapter 2

Evenly Positive Definite Functions on Euclidean Spaces

Introduction

2.1 In this chapter we will develop the representation theorems for evenly positive definite functions on R^n. Most of the material is covered in Gelfand and Vilenkin [9, Chapter II, Sec 5]. The author has obtained and extended the results in the above treatise to various $P_e(F)$ classes. In particular, results similar to the Cooper-Stewart Theorem (see Sec 1.2) and the Weil-Povzner-Raikov Theorem (see Sec 1.1) are obtained (Theorem 2.7 and Theorem 2.18, resp.). In the meantime, we will need the following auxiliary theorems.

2.2 Theorem (see Gelfand and Vilenkin [9, pp216-217])
Let ψ be an entire function of a single variable of order $\frac{1}{2}$ and finite type (i.e., it satisfies an inequality of the form $|\psi(z)| \leq C \exp(a|z|^{\frac{1}{2}})$) which assumes positive values on the real axis. Then ψ has the form $\psi = \varphi\bar{\varphi}$ where φ is an entire function of order $\frac{1}{2}$ and finite type, and $\bar{\varphi}$ is the function defined by $\bar{\varphi}(z) = \overline{\varphi(\bar{z})}$.

2.3 Theorem (Bernstein) (see Achieser [1, pp137-139])
Let φ be an entire function of a single variable which is of exponential type (i.e. it satisfies an inequality of the form $|\varphi(z)| \leq C \exp(\sigma|z|)$), and which satisfies an inequality of the form $\sup\{ |\varphi(x)| ; -\infty < x < \infty \} = M < \infty$.
Then φ also satisfies an inequality of the form

$$|\varphi(x+iy)| \leq M \exp(\sigma|y|)$$

where σ is such that the inequality

$$|\varphi(z)| \leq C \exp(\sigma|z|)$$

holds for some C.

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2.4 Lemma Let θ be an even function in Z(1) which assumes positive values on the set M consisting of the real and pure imaginary axes. Then θ has the form $\theta = \alpha\bar{\alpha}$ where α is some even function in Z(1).
Proof: (Gelfand and Vilenkin) The function ψ, defined by $\psi(z) = \theta(\sqrt{z})$ (this function is well defined because θ is even) is obviously an entire function which is positive on the real axis and has order $\frac{1}{2}$ and finite type. Theorem 2.2 enables us to write ψ in the form $\psi = \varphi\bar{\varphi}$, where φ is an entire function of order $\frac{1}{2}$ and finite type. Put $\alpha(z) = \varphi(z^2)$. Since

$$\theta(z) = \psi(z^2) = \varphi(z^2)\bar{\varphi}(z^2) = \alpha(z)\bar{\alpha}(z),$$

then for the proof of the assertion, it suffices to show that $\alpha \in Z(1)$.

By construction α is of exponential type and therefore the same is true of the functions $z^{2k}\alpha(z)$. But for real values of z, this last function is bounded, since $|x^{2k}\alpha(x)|^2 = |x^{4k}\ \theta(z)|$ and $z^{4k}\ \theta(z)$ is bounded on the real axis in view of the fact that $\theta \in Z(1)$. Application of Theorem 2.3 proves that $\alpha \in Z(1)$. ∎

<u>2.5</u> <u>Lemma</u> Let K be a linear space of complex-valued functions, and H a linear space of complex-valued functions which is closed under complex conjugation, and satisfies the condition that for any $\varphi \in H$ there exists $\psi \in K$ such that $|\varphi(x)| \leq \psi(x)$. Then any positive linear functional on K can be extended to a positive linear functional on H.

<u>Proof:</u> For the case where H is a space of real-valued functions, see Gelfand and Vilenkin [9, p219 Theorem 3], and since H is assumed to be closed under complex conjugation, we can extend the functional from the real-valued functions in H to all of H. ∎

<u>2.6</u> We remark that there is an even, positive function φ in $C_c^\infty(R^n)$ such that its Fourier transform $\hat{\varphi}$ is positive on $\mathbb{C}^n$; in fact, we can take the function $\psi * \tilde{\psi}$, where ψ is any even positive function in $C_c^\infty(R^n)$. Sometimes there is an advantage in having the transform $\hat{\varphi}$ strictly positive on the whole of $\mathbb{C}^n$, and this can be guaranteed in the following way. Let ψ be an even positive function in C_c^∞ such that $\hat{\psi}$ is positive on $\mathbb{C}^n$. Then $\hat{\psi}$ is an entire function of n-complex variables, and therefore the

set of zeros of $\hat{\psi}$ is closed and nowhere dense. Thus the function ψ^2 has $\hat{\psi} * \tilde{\hat{\psi}}$ for its Fourier transform, and the convolution is never 0 since the integrand $\hat{\psi}(z-\eta)\hat{\psi}(\eta)$ is strictly positive on a set of positive measure.

Let φ be a positive even function in $C_c^\infty(R)$ such that $\int_{-\infty}^{\infty}\varphi(t)dt=1$. For every real number $\alpha \geqslant 1$ let $\varphi_\alpha(t) = \alpha\varphi(\alpha t)$. $\Phi_\alpha = \hat{\varphi}_\alpha$ will be the summability function for the following integral representation of evenly positive definite functions.

<u>2.7</u> <u>Theorem</u> If $f \in P_e(C_c^\infty(R))$, there are even positive measures μ_1 and μ_2 on R such that

$$(1) \quad f(s) = \lim_{\alpha\to\infty} \int_{-\infty}^{\infty} e^{isx}\,\Phi_\alpha(x)\,d\mu_1(x) + \int_{-\infty}^{\infty} e^{sy}\,d\mu_2(y)$$

where the limit exists uniformly on any compact set on which f is continuous, and exists in L^1 over any compact subset of R. Furthermore, μ_1 must satisfy $\mu_1(x+C) \longrightarrow 0$ as $|x| \longrightarrow \infty$, where C is any compact subset of R, and μ_2 must be such that the integral

$$\int_{-\infty}^{\infty} e^{ay}\,d\mu_2(y)$$

converges for all $a \geqslant 0$.

<u>Proof:</u> Let M be the subset of $\mathbb{C}$ consisting of the real and imaginary axes, and K the set $\{\varphi|M;\ \varphi \in Z(1),\ \varphi \text{ even}\}$. Then, in view of Theorem 1.14 and the uniqueness Theorem on Fourier transforms, we see that to every $\Psi \in K$, there is a unique even function $\psi \in C_c^\infty(R)$ such that $\hat{\psi}|M = \Psi$. Therefore we can define a linear functional T on K by setting

$$(2) \qquad T(\Psi) = \int_{-\infty}^{\infty} f(t)\psi(t)\, dt$$

(the integral on the right hand side of (2) exists since f is locally summable by Theorem 1.5). Applying Lemma 2.4 and following the same argument as before, we see that every positive function Φ in K (i.e., Φ satisfies the condition that $\Phi(z) \geqslant 0$ for all $z \in M$) is of the form

$$\Phi = \widehat{\psi * \tilde{\psi}} \,|\, M$$

for some even function $\psi \in C_c^\infty(R)$. But since $f \in P_e(C_c^\infty(R))$, we can conclude that T is positive.

Let H be the space of all functions of the form Ψg where $\Psi \in K$ and $g \in C_o(M)$, the space of all continuous functions on M which tend to 0 as $|z| \to \infty$ ($C_o(M)$ is taken with the usual topology). We can extend T to be a positive linear functional on H by virtue of Lemma 2.5 since every function in H can be majorized by a suitable function in K. We may suppose, without loss of generality, that $T(\Psi g) = 0$ if g is an odd function.

We now associate with every positive function Φ in K a functional T_Φ on $C_o(M)$ by setting

$$T_\Phi(g) = T(\Phi g) \qquad (g \in C_o(M))$$

Then T_Φ is a positive linear functional on $C_o(M)$ since $T_\Phi(g) \geqslant 0$ for every positive function $g \in C_o(M)$. Hence by Riesz-Markov-Kakutani-theorem, there is a positive measure ν_Φ on M such that

$$T_\Phi(g) = \int_M g(z)\, d\nu_\Phi(z) \qquad (g \in C_o(M))$$

since $T_\Phi(g) = 0$ for odd functions g, the measure ν_Φ is even. If we write $d\mu_\Phi(z) = \dfrac{d\nu_\Phi(z)}{\Phi(z)}$, we obtain the equality

$$(3)\qquad T(\Phi g) = \int_M \Phi g \, d\mu_\Phi$$

for every positive function $\Phi \in K$ and every $g \in C_o(M)$. Let Ψ be a function in K with $\Psi(z) > 0$ for all $z \in M$ (such a Ψ exists by Sec. 2.6). Then the equality

$$\int_M \Phi g d\mu_\Phi = T(\Phi g) = T(\Psi \frac{\Phi}{\Psi} g) = \int_M \Phi g d\mu_\Psi$$

holds for every positive $\Phi \in K$ and every $g \in C_c(M)$, the space of all continuous functions with compact support on M. Thus the measure μ_Φ in (3) is independent of the choice of Φ, and we will denote it simply by μ. Let μ_1 and μ_2 be the restrictions of μ to the real and imaginary axes, respectively. Then μ_1 and μ_2 are even positive measures on R, and the equality

$$(4)\qquad T(\theta) = \int_{-\infty}^{\infty} \theta(x) d\mu_1(x) + \int_{-\infty}^{\infty} \theta(iy) d\mu_2(y)$$

holds for all θ of the form $\theta = \Phi g$, where Φ is a positive function in K, and $g \in C_o(M)$. But any positive function $\theta \in K$ can be written in this form by setting, for example

$$\theta(z) = \theta(z)(1+z^4)\frac{1}{(1+z^4)}$$

(it is obvious that $\theta(z)(1 + z^4)$ is positive on M and belongs to K, and $\dfrac{1}{(1 + z^4)}$ is in $C_o(M)$). Therefore (4) holds for all positive functions θ in K, and hence the equality

$$(5)\qquad \int_{-\infty}^{\infty} f(t)\varphi(t)dt = \int_{-\infty}^{\infty} \hat{\varphi}(x) d\mu_1(x) + \int_{-\infty}^{\infty} \hat{\varphi}(iy) d\mu_2(y)$$

holds for all functions φ of the form $\varphi = \psi * \tilde{\psi}$, where ψ is an even function in $C_c^\infty(R)$.

We wish to show that (5) continues to hold for all even functions φ in $C_c^\infty(R)$. We first show that the integral

$$(6) \qquad \int_{-\infty}^{\infty} e^{ay}\, d\mu_2(y)$$

converges for all $a \geqslant 0$. In fact, for any $a > 0$, let φ_a be the function which is equal to $\frac{2}{a} - \frac{2|t|}{a^2}$ for $|t| < a$ and is equal to 0 for $|t| \geqslant a$. Then $\hat{\varphi}_a(z) = 2\left\{\frac{\sin \frac{az}{2}}{\frac{az}{2}}\right\}^2$ and $\hat{\varphi}_a(iy) = 2\left(\frac{\sinh \frac{ay}{2}}{\frac{ay}{2}}\right)^2$.

Hence $\hat{\varphi}_a(iy) \geqslant 1$ for all $y \in R$, and $|\hat{\varphi}_a(iy)|^2 = \frac{4}{a^4 y^4}\left[e^{2ay} - 4e^{ay} + 6 - 4e^{-ay} + e^{-2ay}\right]$. Therefore there is a compact set K_1 such that $\frac{2}{a^4 y^4} e^{2ay} < |\hat{\varphi}_a(iy)|^2$ for all y not in K_1, and there is a compact set K_2 such that $e^{ay} < \frac{2}{a^4 y^4} e^{2ay}$ for all y not in K_2. Let $C_1 = K_1 \cup K_2$ then C_1 is compact and $e^{ay} < |\hat{\varphi}_a(iy)|^2$ for all y not in C_1. Now let $\{\varphi_m\}$ be a sequence of even functions in $C_c^\infty(R)$ with supports in a common compact set C_2 and such that $\varphi_m \longrightarrow \varphi_a$ uniformly. Then

$$\lim_m \int_{-\infty}^{\infty} f(t)\,\varphi_m * \tilde{\varphi}_m(t)\, dt = \int_{-\infty}^{\infty} f(t)\,\varphi_a * \tilde{\varphi}_a(t)\, dt \quad .$$

since $|\hat{\varphi}_a(iy)-\hat{\varphi}_m(iy)| \leq \int_{C_2} |\varphi_a(t)-\varphi_m(t)| e^{ty}dt$

$$\leq \sup\{e^{ty}; t\in C_2\} \int_{C_2} |\varphi_a(t)-\varphi_m(t)|dt \to 0 \text{ as } m \to \infty$$

therefore $\hat{\varphi}_m(iy) \to \hat{\varphi}_a(iy)$ for all $y\in R$ as $m\to\infty$,

and hence by Fatou's Lemma,

$$\int_{-\infty}^{\infty} |\hat{\varphi}_a(iy)|^2 d\mu_2(y) \leq \liminf_m \int_{-\infty}^{\infty} |\hat{\varphi}_m(iy)|^2 d\mu_2(y)$$

$$\leq \liminf_m \left\{ \int_{-\infty}^{\infty} |\hat{\varphi}_m(x)|^2 d\mu_1(x) + \int_{-\infty}^{\infty} |\hat{\varphi}_m(iy)|^2 d\mu_2(y) \right\}$$

$$= \liminf_m \int_{-\infty}^{\infty} f(t)\, \varphi_m * \tilde{\varphi}_m(t)dt = \int_{-\infty}^{\infty} f(t)\, \varphi_a * \tilde{\varphi}_a(t)dt < \infty .$$

Thus the integral

$$(7) \qquad \int_{-\infty}^{\infty} |\hat{\varphi}_a(iy)|^2 d\mu_2(y)$$

converges for all $a > 0$. But since $|\hat{\varphi}_a(iy)| \geq 1$ for all $y\in R$, (7) implies that μ_2 is finite. Moreover since $e^{ay} < |\hat{\varphi}_a(iy)|^2$ for all $y \notin C_1$, therefore

$$\int_{-\infty}^{\infty} e^{ay} d\mu_2(y) = \int_{C_1} e^{ay} d\mu_2(y) + \int_{R\setminus C_1} e^{ay} d\mu_2(y)$$

$$\leq \sup\left\{e^{ay};\, y\in C_1\right\} \int_{C_1} d\mu_2(y) + \int_{R\setminus C_1} |\hat{\varphi}_a(iy)|^2 d\mu_2(y) < \infty .$$

Thus the integral (6) converges for all $a \geq 0$.

We are now able to show that equality (5) also holds for even functions φ in $C_c^\infty(R)$, i.e., we have to show that the equality

$$(8) \qquad \int_{-\infty}^{\infty} f(t)\varphi(t)dt = \int_{-\infty}^{\infty} \hat{\varphi}(x)d\mu_1(x) + \int_{-\infty}^{\infty} \hat{\varphi}(iy)d\mu_2(y)$$

holds for all even functions φ in $C_c^\infty(R)$. In fact, if φ is an even function in $C_c^\infty(R)$, then $\hat{\varphi}$ satisfies an inequality of the form

$$|\hat{\varphi}(x+iy)| \leqslant C \exp(a|y|).$$

Hence there are constants A and b such that $|\hat{\varphi}(z)| < A(2+\cos bz)$ on M. Now let β be an even positive function in $C_c^\infty(R)$ such that $\int_{-\infty}^{\infty} \beta(t)\,dt = 1$, and $\hat{\beta}$ is positive on $\mathbb{C}$, and set $\beta_m(t) = m\,\beta(mt)$ for all integers $m > 0$. Let $\{\varphi_m\}$ be the sequence in $C_c^\infty(R)$ defined by $\varphi_m = \varphi * \beta_m$. Then the equality

$$(9) \qquad \lim_m \int_{-\infty}^{\infty} f(t)\varphi_m(t)\,dt = \int_{-\infty}^{\infty} f(t)\varphi(t)\,dt$$

holds, and moreover each $\hat{\varphi}_m$ can be represented in the form

$$\hat{\varphi}_m(z) = A(2+\cos bz)(1+z^4)\hat{\beta}_m(z)\,\frac{\hat{\varphi}(z)}{A(2+\cos bz)(1+z^4)} = \theta_m(z)g(z)$$

where $\theta_m(z) = A(2+\cos bz)(1+z^4)\hat{\beta}_m(z)$ is a positive function in K and $g(z) = \dfrac{\hat{\varphi}(z)}{A(2+\cos bz)(1+z^4)} \in C_o(M)$.

Thus the equality

$$(10) \qquad T(\hat{\varphi}_m) = \int_{-\infty}^{\infty} \hat{\varphi}_m(x)\,d\mu_1(x) + \int_{-\infty}^{\infty} \hat{\varphi}_m(iy)\,d\mu_2(y)$$

holds for every m. Now since the function $|\hat{\varphi}|$ is a positive function in K, the integral

$$\int_{-\infty}^{\infty} |\hat{\varphi}(x)|\,d\mu_1(x)$$

exists by virtue of the fact that equality (4) holds for the function $|\hat{\varphi}|$.

Using the fact that $|\hat{\varphi}_m(x)| \leqslant |\hat{\varphi}(x)|$ for all $x \in R$, we see that the equality

$$(11) \qquad \lim_m \int_{-\infty}^{\infty} \hat{\varphi}_m(x) d\mu_1(x) = \int_{-\infty}^{\infty} \hat{\varphi}(x) d\mu_1(x)$$

holds by dominated convergence. But since each $\hat{\varphi}_m$ satisfies an inequality of the form

$$|\hat{\varphi}_m(x + iy)| \leqslant K_1 \exp(a_1|y|)$$

where the constants K_1 and a_1 are independent of m, we can conclude that

$$(12) \qquad \lim_m \int_{-\infty}^{\infty} \hat{\varphi}_m(iy) d\mu_2(y) = \int_{-\infty}^{\infty} \hat{\varphi}(iy) d\mu_2(y)$$

by dominated convergence because the integral $\int_{-\infty}^{\infty} \exp(a_1|y|) d\mu_2(y)$ exists for all $a_1 \geqslant 0$. Equalities (9), (10), (11) and (12) prove that

$$\int_{-\infty}^{\infty} f(t)\varphi(t)dt = \int_{-\infty}^{\infty} \hat{\varphi}(x) d\mu_1(x) + \int_{-\infty}^{\infty} \hat{\varphi}(iy) d\mu_2(y).$$

By the evenness of both the function f and the measures μ_1 and μ_2, it is obvious that (8) also holds for odd functions φ in $C_c^{\infty}(R)$ (both sides being zero). Since every $\varphi \in C_c^{\infty}(R)$ is the sum of an even function and an odd function in $C_c^{\infty}(R)$, therefore (8) holds for every $\varphi \in C_c^{\infty}(R)$. In particular, if we apply this equality to the functions $\varphi_\alpha(t + s)$, we obtain

$$(13) \qquad \int_{-\infty}^{\infty} f(-t)\, \varphi_\alpha(t + s) dt = \int_{-\infty}^{\infty} \Phi_\alpha(x) e^{isx} d\mu_1(x) + \int_{-\infty}^{\infty} \Phi_\alpha(iy) e^{sy} d\mu_2(y) .$$

We note that Φ_1 satisfies an inequality of the form

$$|\Phi_1(x+iy)| \leq K_2 \exp(a_2|y|)$$

for some constants K_2 and a_2, hence the inequality

$$|\Phi_\alpha(iy)e^{sy}| \leq |\Phi_1(\frac{iy}{\alpha})e^{sy}| \leq K_2 \exp((\frac{a_2}{\alpha}+s)|y|)$$
$$\leq K_2 \exp((a_2+s)|y|)$$

holds for all $\alpha \geq 1$. Moreover since $\Phi_\alpha(iy)e^{sy} \to e^{sy}$ uniformly on compact subsets of R as $\alpha \to \infty$, we can conclude that

$$(14) \quad \lim_{\alpha\to\infty} \int_{-\infty}^{\infty} \Phi_\alpha(iy)e^{sy}d\mu_2(y) = \int_{-\infty}^{\infty} e^{sy}d\mu_2(y)$$

by dominated convergence. The integral on the left hand side of (13) is $(\varphi_\alpha * f)(s)$ which converges to $f(s)$ in the manner of the statement of the theorem. Hence (using (14))

$$f(s) = \lim_{\alpha\to\infty} \int_{-\infty}^{\infty} e^{isx}\Phi_\alpha(x)d\mu_1(x) + \int_{-\infty}^{\infty} e^{sy}d\mu_2(y)$$

where the limit exists in the manner of the statement of the theorem.

We now proceed to prove that μ_1 satisfies the condition of the theorem. Let C be any compact subset of R. Let β be an even positive function in C_0^∞ such that $\hat{\beta}$ is positive on $\mathbb{C}$ (such a β exists by Sec. 2.6). For every $m>0$ let $\beta_m(t) = m\beta(mt)$. Then $\hat{\beta}_m$ is positive on $\mathbb{C}$ and $\hat{\beta}_m \to 1$ uniformly on compacts. Hence there is an m_0 such that $\hat{\beta}_{m_0}(x) > \frac{1}{2}$ for all $x \in C$. Then $0 \leq \mu_1(x+C)$

$$\leq 2\int_{-\infty}^{\infty} \hat{\beta}_{m_0}(-x+y)d\mu_1(y) = 2\int_{-\infty}^{\infty} f(t)\beta_{m_0}(t)e^{ixt}dt - 2\int_{-\infty}^{\infty} \hat{\beta}_{m_0}(iy-x)d\mu_2(y)$$
$$\leq 2\int_{-\infty}^{\infty} f(t)\beta_{m_0}(t)e^{ixt}dt .$$

Since β_{m_0} has compact support, the right hand side of the above inequality is the Fourier transform of a function in $L^1(R)$, and thus converges to 0 as $|x| \to \infty$ by the well known Riemann-Lebesgue Lemma. Hence $\mu_1(x + C) \to 0$, and thus finishes the proof of the theorem. ∎

Let h be the function which is equal to

$$\left\{ \int_{-1}^{1} \exp\left(-\frac{1}{1-t^2}\right) dt \right\}^{-1} \exp\left(-\frac{1}{1-t^2}\right) \quad \text{for } |t| < 1 \text{ and is equal to } 0$$

for $|t| \geqslant 1$. For every real number $\alpha \geqslant 1$, let $h_\alpha(t) = \alpha h(\alpha t)$. Then every h_α is infinitely differentiable with support on the interval $\left[-\frac{1}{\alpha}, \frac{1}{\alpha}\right]$, and $\| h_\alpha \|_1 = 1$. Suppose f is locally summable. Then the convolution $f*h_\alpha$ is defined for every $\alpha \geqslant 1$, and the inequality

$$(1) \quad \left| f*h_\alpha(t) - f(t) \right| = \left| \int_{-\infty}^{\infty} h_\alpha(s) f(t+s)\, ds - \int_{-\infty}^{\infty} h_\alpha(s) f(t)\, ds \right|$$

$$\leqslant \int_{-\frac{1}{\alpha}}^{\frac{1}{\alpha}} \alpha h(\alpha s) \left| f(s+t) - f(t) \right| ds \leqslant \|h\|_\infty \int_{-1}^{1} \left| f\left(\frac{u}{\alpha} + t\right) - f(t) \right| du$$

holds for every $\alpha \geqslant 1$. A point t is said to be a Lebesque point of a locally summable function f if and only if the average occurring in the last term of (1) converges to 0. A standard theorem asserts that almost every point t in R is a Lebesque point for a given locally summable function f. It is obvious that a point of continuity of f is a Lebesque point. Thus (1) implies that $f*h_\alpha$ converges to f at every Lebesque point of f. Therefore if we consider this special summability function $H_\alpha = \widehat{h_\alpha}$ for the integral representation in Theorem 2.7, we can strengthen the mode of convergence in Theorem 2.7 via the following.

Corollary: If $f \in P_e(C_c^\infty(R))$, there are even positive measures μ_1 and μ_2 on R such that

$$f(s) = \lim_{\alpha \to \infty} \int_{-\infty}^{\infty} e^{isx} H_\alpha(x) d\mu_1(x) + \int_{-\infty}^{\infty} e^{sy} d\mu_2(y)$$

at every Lebesque point of f, and hence almost everywhere, and in particular at every point of continuity of f. Furthermore, μ_1 must satisfy $\mu_1(x + C) \to 0$ as $|x| \to \infty$, where C is any compact subset of R, and μ_2 must be such that the integral $\int_{-\infty}^{\infty} e^{ay} d\mu_2(y)$ converges for all $a \geqslant 0$.

2.8 Theorem (Krein) Let f be a continuous function in the class $P_e(C_c^\infty(R))$. Then f has the form

$$(1) \quad f(x) = \int_0^\infty \cos\lambda x \, d\mu_1(\lambda) + \int_0^\infty \cosh\lambda x \, d\mu_2(\lambda)$$

where μ_1 and μ_2 are positive measures, μ_1 is finite, and μ_2 is such that the second integral converges. Conversely, if two positive measures μ_1 and μ_2 satisfy the latter conditions, then (1) defines a continuous function in $P_e(C_c^\infty(R))$.

Proof: We start with the first part of the theorem. We know from Theorem 2.7 that there are even measures μ_1', μ_2' on R such that

$$(2) \quad f(x) = \lim_{\alpha \to \infty} \int_{-\infty}^{\infty} e^{i\lambda x} \Phi_\alpha(x) \, d\mu_1'(\lambda) + \int_{-\infty}^{\infty} e^{\lambda x} d\mu_2'(\lambda)$$

where the limit exists uniformly on compacts since f is continuous. Since $\Phi_\alpha \to 1$ uniformly on compact subset C of R ,

$$\mu_1'(C) = \lim_{\alpha\to\infty} \int_C \Phi_\alpha(\lambda)\, d\mu_1'(\lambda) \leq \lim_{\alpha\to\infty} \int_{-\infty}^{\infty} \Phi_\alpha(\lambda) d\mu_1'(\lambda)$$

$$= f(0) - \int_{-\infty}^{\infty} d\mu_2'(\lambda) < \infty$$

The bound is independent of C, therefore μ_1' is finite, and hence the function h defined by

$$h(x) = \int_{-\infty}^{\infty} e^{i\lambda x} d\mu_1'(\lambda)$$

is continuous and thus

$$\lim_{\alpha\to\infty} \int_{-\infty}^{\infty} e^{i\lambda x} \Phi_\alpha(\lambda) d\mu_1'(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} d\mu_1'(\lambda)$$

holds for all $x \in R$. If μ_1 and μ_2 are the restrictions of $2\mu_1'$ and $2\mu_2'$ onto the interval $[0, \infty)$, respectively, then (2) becomes

$$(3) \qquad f(x) = \int_0^{\infty} \cos \lambda x \, d\mu_1(\lambda) + \int_0^{\infty} \cosh \lambda x \, d\mu_2(\lambda)$$

where μ_1 and μ_2 obviously satisfy the conditions of the theorem.

Now we proceed to the proof of the converse part of the theorem. It suffices to show that $f \in P_e(C_c^\infty(R))$ since continuity of f comes easily from the conditions on μ_1 and μ_2. Let μ_1' and μ_2' be the even positive measures defined on R such that their restrictions onto the interval $[0, \infty)$ are $\frac{1}{2}\mu_1$ and $\frac{1}{2}\mu_2$, respectively. Then for any even function $\varphi \in C_c^\infty(R)$, we can apply Fubini's theorem to obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\varphi(x)\,\overline{\varphi(y)}\, dxdy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \cos \lambda(x-y) d\mu_1(\lambda) + \int_0^{\infty} \cosh \lambda(x-y) d\mu_2(\lambda)\right) \varphi(x)\overline{\varphi(y)}\, dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-i\lambda(x-y)}d\mu_1'(\lambda) + \int_{-\infty}^{\infty} e^{\lambda(x-y)}\, d\mu_2'(\lambda)\right) \varphi(x)\overline{\varphi(y)}dxdy$$

$$= \int_{-\infty}^{\infty} |\hat{\varphi}(\lambda)|^2\, d\mu_1'(\lambda) + \int_{-\infty}^{\infty} |\hat{\varphi}(i\lambda)|^2\, d\mu_2'(\lambda) \geq 0 \quad ,$$

because μ_1 and μ_2 are positive. Thus we have completed the proof of this assertion. ∎

Remark

2.9 We remark that the measures μ_1 and μ_2 in Theorems 2.7 and 2.8 are not uniquely defined. An example on the non-uniqueness of these measures can be found in Gelfand and Vilenkin [9, pp226-228]. Note that we have only proved the integral representation theorem for the case of a single variable, and the result is not true for the case of several variables. However, if f also satisfies the growth condition that the integral $\int_0^{\infty}\exp(-cx^2)f(x)dx$ converges for all $c>0$, then the integral representation for f is unique. We will show that the result is also true for the case of several variables if a similar growth condition is imposed on f.

2.10 Notations We adopt freely the notations introduced in Sec.1.10 and Sec. 1.13. We denote by $G(n)$ the set of all functions f on R^n such that the integral

$$\int_{R^n} \exp(-cx^2)f(x)dx$$

converges for all $c>0$.

The Class $G(n) \cap P_e\,(C_c^\infty(R^n))$

2.11 Lemma For any $c = (c_1,\ldots\ldots,c_n) > 0$, and any $\psi \in C_c^\infty(R^n)$, the function Φ, defined by $\Phi(z) = \exp(-cz^2)\hat{\psi}(z^2)$, is an even function in $Y(n)$ (for definition of $Y(n)$ see Sec. 1.13). Furthermore, if $\{\psi_m\}$ is a sequence in $C_c^\infty(R^n)$ such that $\psi_m \longrightarrow \psi$ in the topology of C_c^∞, then there is a sequence $\{\varphi_m\}$ in $Y(n)$ such that $\hat{\varphi}_m(z) = \exp(-cz^2)\hat{\psi}_m(z^2)$, $\varphi_m \rightarrow \varphi$ uniformly on compact subsets of C^n, where φ is the function in $Y(n)$ such that $\hat{\varphi}(z) = \exp(-cz^2)\hat{\psi}(z^2)$, and the following inequality

$$|\varphi_m(x + iy)| \leq K\exp(-ax^2 + by^2)$$

holds for every m, where the constants K, a, b do not depend upon m.

Proof: For any $\psi \in C_c^\infty$, by Theorem 1.14, there are constants C and d such that $|\hat{\psi}(x + iy)| \leq C\exp(d\,\|y\|)$

and therefore

$$|\exp(-cz^2)\hat{\psi}(z^2)| \leq C\exp(-c\mathrm{Re}z^2 + d\,\|\mathrm{Im}z^2\|)$$

holds for all $c > 0$.

For any d and c there is an $r = (r_1,\ldots\ldots,r_n)$, $r_k > 0$, such that $c_k + r_k > (d_k^2 + r_k^2)^{\frac{1}{2}}$, $1 \leq k \leq n$. We set

$$d' = \left((d_1^2 + r_1^2)^{\frac{1}{2}},\ \ldots\ldots,\ (d_n^2 + r_n^2)^{\frac{1}{2}}\right)$$

and $\quad c' = (c_1 + r_1,\ \ldots\ldots\ldots,\ c_n + r_n)$.

Since $\quad r\mathrm{Re}z^2 + d\,\|\mathrm{Im}z^2\| \leq d'(\mathrm{Re}z^2 + \|\mathrm{Im}z^2\|) \leq d'\,\|z\|^2$

therefore

$$
\begin{aligned}
|\exp(-cz^2)\hat{\psi}(z^2)| &\leq C\exp(-c\mathrm{Re}z^2 + d\|\mathrm{Im}z^2\|) \\
&= C\exp(-c'\mathrm{Re}z^2 + r\mathrm{Re}z^2 + d\|\mathrm{Im}z^2\|) \\
&\leq C\exp(-c'\mathrm{Re}z^2 + d'\|z\|^2) \\
&= C\exp(-(c'-d')x^2 + (c'+d')y^2)
\end{aligned}
$$

Since, in view of the choice of r, $c' > d'$, the function $\exp(-cz^2)\hat{\psi}(z^2)$ belongs to $Y(n)$.

If $\{\psi_m\}$ is a sequence in C_c^∞ such that $\psi_m \longrightarrow \psi$ in the topology of C_c^∞, then Theorem 1.14 shows that $\exp(-cz^2)\hat{\psi}_m(z^2) \longrightarrow \exp(-cz^2)\hat{\psi}(z^2)$ uniformly on compacts, and the inequality

$$|\hat{\psi}_m(x+iy)| \leq C\exp(d\|y\|)$$

holds for every m, where the constants C and d do not depend upon m. By following the same argument as before, we see that every $\exp(-cz^2)\hat{\psi}_m(z^2)$ satisfies an inequality of the form

$$|\exp(-cz^2)\hat{\psi}_m(z^2)| \leq C'\exp(-a'x^2 + e'y^2).$$

By Theorem 1.15, there is a sequence $\{\varphi_m\}$ in $Y(n)$ such that $\hat{\varphi}_m(z) = \exp(-cz^2)\hat{\psi}_m(z^2)$, $\varphi_m \longrightarrow \varphi$ uniformly on compacts, where φ is a function in $Y(n)$ such that $\hat{\varphi}(z) = \exp(-cz^2)\hat{\psi}(z^2)$, and there are constants a and b, not depending upon m, such that

$$|\varphi_m(x+iy)| \leq k\exp(-ax^2 + by^2)$$

holds for every m. ∎

2.12 **Lemma** If $0<b<2c$; i.e. $0<b_k<2c_k$ $(1\leq k\leq n)$, then there is a sequence $\{\psi_m\}$ in C_c^∞ such that

(1) $\hat{\psi}_m(x)\geqslant 0$ for all $x\in R^n$

(2) $\exp(-cz^2)\hat{\psi}_m(z^2)\longrightarrow\exp(-bz^2)$ uniformly on compacts

(3) the inequality

$$|\exp(-cz^2)\hat{\psi}_m(z^2)|\leq K\exp(-ax^2+by^2)$$

holds for every m, where the constants K, a,b do not depend upon m

(4) for real values $x\in R^n$, the inequality

$$|\exp(-cx)\hat{\psi}_m(x)|\leqslant K_1\exp(-hx)$$

holds for every m, where the constants K_1, h do not depend upon m, and $0<h<2c$.

__Proof:__ (Gelfand and Vilenkin) Take any function $\alpha\in Z(n)$ such that $\alpha(0)=1$ and $|\alpha(x+iy)|\leqslant C\exp(r\|y\|)$

where r satisfies the inequality $0<r<\frac{1}{2}(c-\|b-c\|)$. We set

$$(5)\qquad \theta_m(z)=\alpha(\tfrac{z}{m})\bar{\alpha}(\tfrac{z}{m})\left[\sum_{|k|=0}^{m}\frac{(c-b)^k z^k}{2^k k!}\right]^2 .$$

Then each $\theta_m\in Z(n)$, because it is the product of the function $\alpha(\frac{z}{m})\bar{\alpha}(\frac{z}{m})\in Z(n)$ and a polynomial, and hence by Theorem 1.14 there is a sequence $\{\psi_m\}$ in C_c^∞ such that $\hat{\psi}_m=\theta_m$. We shall prove that the sequence $\{\psi_m\}$ satisfies the conditions of the Lemma. The expression within the square bracket of (5) is the partial sum of the Taylor series for $\exp(\frac{1}{2}(c-b)z)$ and therefore converges to $\exp(\frac{1}{2}(c-b)z)$ uniformly on compacts as $m\rightarrow\infty$. At the same time, the function $\alpha(\frac{z}{m})\bar{\alpha}(\frac{z}{m})$ converges to $\alpha(0)=1$ uniformly on compacts as $m\rightarrow\infty$. Therefore $\exp(-cz^2)\hat{\psi}_m(z^2)\longrightarrow\exp(-bz^2)$ uniformly on compacts. Moreover

$$|\exp(-cz)\hat{\psi}_m(z)| \leq |\exp(-cz)\alpha(\tfrac{z}{m})\bar{\alpha}(\tfrac{z}{m})| \exp(\|(c-b)z\|)$$
$$\leq C^2\exp(-cx + \frac{2r\|y\|}{m} + \|(c-b)z\|)$$
$$\leq C^2\exp(-cx + s\|z\|)$$

where $s = \|c-b\| + 2r < c$ in view of the choice of r.

Hence $|\exp(-cz^2)\hat{\psi}_m(z^2)| \leq C^2 \exp(-(c-s)x^2 + (c+s)y^2)$

and therefore condition (3) holds.

Now since the functions $\alpha(\frac{z}{m})$ have a common bound on the reals, the inequality

$$(6) \quad |\exp(-cx)\hat{\psi}_m(x)| \leq K_1 \exp(-cx + \|(c-b)x\|)$$

holds with K_1 independent of m. In expanded form the expression

$-cx + \|(c-b)x\|$ is $\displaystyle -\sum_{k=1}^{n}\left[c_k x_k + \|(c_k - b_k)x_k\|\right]$

$$= -\sum_{k=1}^{n} x_k\left[c_k - |b_k - c_k| \operatorname{sign} x_k\right].$$

But in view of the inequality $0 < b_k < 2c_k \quad (1 \leq k \leq n)$

we have $0 < c_k - |b_k - c_k| \operatorname{sign} x_k < 2c_k$. Set $h_k = c_k - |b_k - c_k| \operatorname{sign} x_k$, then $0 < h < 2c$ and (6) becomes

$$|\exp(-cx)\hat{\psi}_m(x)| \leq K_1\exp(-hx)$$

which proves condition (4) ∎

2.13 **Lemma** For any $a = (a_1, \ldots, a_n) > 0$ and any φ in C_c^∞, there exist $c = (c_1, \ldots, c_n) > 0$ and a sequence $\{\psi_m\}$ in $C_c^\infty(R^n)$ such that

(1) $\exp(-cz^2)\hat{\psi}_m(z^2) \longrightarrow \exp(-az^2)\hat{\varphi}(z)$

uniformly on compact subsets of $\mathbb{C}^n$ as $m \longrightarrow \infty$

(2) the inequality

$$| \exp(-cz^2)\hat{\psi}_m(z^2) | \leq K\exp(-dx^2 + ey^2)$$

holds for every m, where the constants K, d, e do not depend upon m

(3) for real values x in R^n, the inequality

$$| \exp(-cx)\hat{\psi}_m(x) | \leq K \exp(-hx) \quad , \qquad h > 0$$

holds for every m, where the constants K, h are independent of m.

Proof: In view of Theorem 1.14, $\hat{\varphi}$ satisfies an inequality of the form $|\hat{\varphi}(x + iy)| \leq L \exp(b' \| y \|)$ for some constants L and b', and therefore

$$\begin{aligned} |\exp(-az^2)\hat{\varphi}(z)| &\leq L|\exp(-az^2 + b'\| y \|)| \\ &\leq L|\exp(-az^2 + \| b' \| + b'y^2)| \\ &= L_1\exp(-ax^2 + by^2) \\ &= L_1\exp(-\tfrac{1}{2}(a+b)z^2 + \tfrac{1}{2}(b-a)\| z^2 \|) \end{aligned}$$

where $L_1 = L\exp(\| b' \|)$ and $b = a + b'$.

The function $\hat{\varphi}(\sqrt{z})$ (this function is well defined since $\hat{\varphi}$ is even) is obviously an entire function. We take $c > a$ and let $\rho_m(z)$ be the m^{th} partial sum of the Taylor series for the entire function $\exp((c-a)z)\hat{\varphi}(\sqrt{z})$. Let $\alpha \in Z(n)$ be such that $\alpha(0) = 1$ and $|\alpha(x + iy)| \leq M \exp(r\| y \|)$, where $0 < r < \frac{1}{2}a$, and let $\Psi_m(z) = \alpha(\frac{z}{m})\rho_m(z)$. Then $\Psi_m \in Z(n)$, and by Theorem 1.14, there is a sequence $\{\psi_m\}$ in C_c^∞

such that $\hat{\varphi}_m = \Psi_m$. It is obvious that condition (1) holds, and

$$|\exp((c-a)z)\,\hat{\varphi}(\sqrt{z})| \leq L_1 \exp(cx - \tfrac{1}{2}(a+b)x + \tfrac{1}{2}(b-a)\|z\|)$$
$$\leq L_1 \exp((c-a)\|z\|).$$

Hence for all $b_1 > c-a$, one has

$$(4) \qquad |\rho_m(z)| \leq K_1 \exp(b_1\|z\|)$$

where K_1 does not depend upon m. In particular it holds for $b_1 = c - \frac{1}{2}a$ and hence

$$(5) \qquad |\exp(-cz)\hat{\varphi}_m(z)| \leq \exp(-cz)|\alpha(\tfrac{z}{m})||\rho_m(z)|$$
$$\leq L_1K_1 \exp(-cx + \frac{r\|y\|}{m} + (c - \tfrac{1}{2}a)\|z\|)$$
$$\leq N \exp(-cx + s\|z\|)$$

where $s = c - \frac{a}{4}$. Obviously $0 < s < c$, and (5) is equivalent to condition (2). For real values $x \in R^n$, (5) is equivalent to condition (3). ∎

2.14 **Theorem** If $f \in G(n) \cap P_e(C_c^\infty(R^n))$ (i.e., f is a function in the class $P_e(C_c^\infty)$ such that $\int \exp(-ct^2)f(t)dt$ converges for all $c > 0$), there is a uniquely defined even positive measure μ, concentrated on the set M of points $z = (z_1, \ldots, z_n)$, each of whose coordinates z_k is either real or pure imaginary, such that the integral

$$(1) \qquad \int_M \exp(-cz^2)d\mu(z)$$

converges for all $c > 0$, and the equality

$$(2) \qquad f(t) = \int_M e^{i(t,z)}\, d\mu(z)$$

holds for almost all $t \in R^n$.

Conversely, if μ is an even positive measure on M such that the integral (1) converges for all $c > 0$, then the integral

$$\int_M e^{i(t,z)}\, d\mu(z)$$

converges for almost all $t \in R^n$, and the function f, which is defined almost everywhere by (2), is a function in the class $G(n) \cap P_e(C_c^\infty)$.

Proof: We start with the second part of the theorem. Suppose μ is an even positive measure on M such that the integral (1) converges for all $c > 0$. In view of Fubini's Theorem, we see that

$$\int_{R^n} \exp(-ct^2) \int_M e^{i(t,z)}\, d\mu(z)dt = \int_M \int_{R^n} \exp(-ct^2)e^{i(t,z)}\, dt\, d\mu(z) =$$

$$= \frac{\pi^{\frac{n}{2}}}{\sqrt{c_1 \cdots c_n}} \int_M \exp\left(-\frac{1}{4c}\, z^2\right) d\mu(z) \quad < \quad \infty$$

for all $c = (c_1,\dots,c_n) > 0$, where $\frac{1}{4c} = (\frac{1}{4c_1},\dots,\frac{1}{4c_n})$.

Hence the integral $\int_M e^{i(t,z)}\, d\mu(z)$ converges for almost all $t \in R^n$, and the function f, which is almost everywhere defined by (2), is in the class G(n). Now for any even function $\varphi \in C_c^\infty$, we can apply Fubini's Theorem to obtain the inequality

$$\int_{R^n}\int_{R^n} f(t-s)\varphi(t)\overline{\varphi(s)}\,dt\,ds = \int_M |\hat{\varphi}(z)|^2 d\mu(z) \geqslant 0$$

Here we have used the fact that μ is positive. Therefore f is also in the class $P_e(C_c^\infty)$, and hence we finish the proof of the second part of the theorem.

We now proceed to the proof of the direct assertion of the theorem. Since the integral (1) converges for all $c > 0$, we can apply the dominated convergence theorem to prove that $f \in P_e(Y(n))$ (for the definition of Y(n) see Sec 1.13). In view of Theorem 1.15, we can define a linear functional T on Y(n) by setting

$$(3) \qquad T(\Phi) = \int_{R^n} f(t)\varphi(t)\,dt \qquad (\Phi \in Y(n))$$

where φ is the function in Y(n) such that $\hat{\varphi} = \Phi$. Then for every $c=(c_1,\dots c_n) > 0$, the functional T_c defined by

$$(4) \qquad T_c(\psi) = T(\exp(-cz^2)\hat{\psi}(z^2)) \qquad (\psi \in C_c^\infty)$$

exists by virtue of Lemma 2.11. We first show that T_c is continuous on C_c^∞. Indeed, if $\psi_m \longrightarrow \psi$ in the topology of C_c^∞, let $\{\varphi_m\}$ be the sequence constructed in Lemma 2.11 such that $\varphi_m \longrightarrow \varphi$ uniformly on compacts, where φ is the function in Y(n) such that
$\hat{\varphi}(z) = \exp(-cz^2)\hat{\psi}(z^2)$, $\hat{\varphi}_m(z) = \exp(-cz^2)\hat{\psi}_m(z^2)$, and the inequality

$$|\varphi_m(x+iy)| \leqslant K \exp(-ax^2 + by^2)$$

holds for every m, where K, a, b are independent of m. We can apply the dominated convergence theorem to show that

$$\lim_m \int_{R^n} f(t)\varphi_m(t)dt = \int_{R^n} f(t)\varphi(t)\,dt$$

and hence $T_c(\psi) = T(\exp(-cz^2)\hat{\psi}(z^2)) = \int_{R^n} f(t)\varphi(t)\,dt =$

$$= \lim_m \int_{R^n} f(t)\varphi_m(t)\,dt = \lim_m T(\exp(-cz^2)\hat{\psi}_m(z^2)) = \lim_m T_c(\psi_m)$$

showing that T_c is continuous on C_c^∞. Since $f \in P_e(Y(n))$, the inequality

$$T_c(\psi * \tilde{\psi}) = T(\exp(-cz^2)\hat{\psi}(z^2)\overline{\hat{\psi}}(z^2)) = T\left(e^{-\frac{1}{2}z^2}\hat{\psi}(z^2)\,\overline{e^{-\frac{1}{2}\bar{z}^2}\hat{\psi}(\bar{z}^2)}\right) \geqslant 0$$

holds for all $\psi \in C_c^\infty(R^n)$. Therefore T_c is a positive definite distribution, and by Theorem 1.12, there is a positive tempered measure σ_c on R^n such that

$$T(\exp(-cz^2)\hat{\psi}(z^2)) = \int_{R^n} \hat{\psi}(t)d\sigma_c(t)$$

holds for all $\psi \in C_c^\infty$. Setting $d\nu_c(t) = e^{ct}d\sigma_c(t)$, we obtain for every $c > 0$ a positive measure ν_c such that

$$(5) \qquad T(\exp(-cz^2)\hat{\psi}(z^2)) = \int_{R^n} \exp(-ct)\hat{\psi}(t)\,d\nu_c(t)$$

for all $\psi \in C_c^\infty$. Thus we have proven that the equality

$$(6) \qquad T(\theta(z^2)) = \int_{R^n} \theta(t)\,d\nu_c(t)$$

holds for all entire functions θ of the form $\theta(z) = \exp(-cz)\hat{\psi}(z)$, where $\psi \in C_c^\infty$.

We wish to show that the measures ν_c are independent of the choice of c. We first show that the integral

$$(7) \qquad \int_{R^n} \exp(-bt)\,d\nu_c(t)$$

converges for $0<b<2c$. Indeed, if $\{\psi_m\}$ is the sequence of functions constructed in Lemma 2.12, then the inequality

$$|\exp(-cz^2)\,\hat{\psi}_m(z^2)| \leq K\,\exp(-ax^2 + by^2)$$

holds for every m, where the constants K, a, b do not depend upon m. Therefore we can apply Theorem 1.15 and the dominated convergence theorem to obtain

$$\lim_m T(\exp(-cz^2)\,\hat{\psi}_m(z^2)) = T(\exp(-bz^2)).$$

Hence by Fatou's Lemma (since $\hat{\psi}_m(t) \geq 0$ for all t),

$$\int_{R^n} \exp(-bt)\,d\nu_c(t) \leq \liminf_m \int_{R^n} \exp(-ct)\,\hat{\psi}_m(t)\,d\nu_c(t) =$$

$$= \liminf_m T(\exp(-cz^2)\,\hat{\psi}_m(z^2)) = T(\exp(-bz^2)) < \infty,$$

i.e., the integral (7) converges. We next show that (6) continues to hold for all functions θ of the $\theta(z) = \exp(-bz)\hat{\psi}(z)$, where $0<b<2c$ and $\psi \in C_c^\infty$. In fact, if $\{\psi_m\}$ is the sequence constructed in Lemma 2.12, then $\exp(-cz)\hat{\psi}_m(z)\hat{\psi}(z) \longrightarrow \exp(-bz)\hat{\psi}(z)$ uniformly on compacts as $m \to \infty$. In view of condition (3) of Lemma 2.12, we can follow the same argument as in the proof of Lemma 2.11 to conclude that the sequence $\left\{ \exp(-cz^2)\hat{\varphi}_m(z^2)\hat{\psi}(z^2) \right\}$ is uniformly dominated by a function of the form $K\exp(-ax^2 + by^2)$. Applications of Theorem 1.15 and the dominated convergence theorem show that

$$(8) \qquad \lim_m T(\exp(-cz^2)\hat{\varphi}_m(z^2)\hat{\psi}(z^2)) = T(\exp(-bz^2)\hat{\psi}(z^2)) \ .$$

Since the function $\hat{\psi}$ is bounded on R^n, by condition (4) of Lemm 2.12, the inequality

$$\left| \exp(-ct)\hat{\varphi}_m(t)\hat{\psi}(t) \right| \leq K_1\exp(-ht)$$

holds for every m, where the constants K_1 and h do not depend upon m, and $0<h<2c$, and hence

$$\lim_m \int_{R^n} \exp(-ct)\hat{\varphi}_m(t)\hat{\psi}(t)d\nu_c(t) = \int_{R^n}\exp(-bt)\hat{\psi}(t)d\nu_c(t)$$

by virtue of dominated convergence theorem. Therefore we obtain

$$T(\exp(-bz^2)\hat{\psi}(z^2)) = \int_{R^n}\exp(-bt)\hat{\psi}(t)d\nu_c(t)$$

by virtue of (8). Comparing the above equality with (5), we see that

$$\int_{R^n} \exp(-bt)\hat{\psi}(t)d\nu_c(t) = \int_{R^n} \exp(-bt)\hat{\psi}(t)d\nu_b(t)$$

for every $\psi \in C_c^\infty$. Hence $\nu_b = \nu_c$ since the set of all $\hat{\psi}$ is dense in $C_0(R^n)$.

Thus we have proven that $\nu_b = \nu_c$ if $0< b < 2c$. Now for any $c,d >0$, there exists at least one $b > 0$ such that $b< 2c$ and $b< 2d$, and hence $\nu_c = \nu_b = \nu_d$. Therefore the measures ν_c are independent of the choice of c, and we shall denote the common value of the measures ν_c by ν.

So far we have proven that the equality

$$(9) \qquad T(\theta(z^2)) = \int_{R^n} \theta(t)\,d\nu(t)$$

holds for all entire functions θ of the form $\theta(z) = \exp(-cz)\hat{\psi}(z)$, where $c > 0$ and $\psi \in C_c^\infty$. We shall now show that (9) continues to hold for all functions θ of the form $\theta(z) = \exp(-bz)\hat{\varphi}(\sqrt{z})$, where $b > 0$ and φ is an even function in C_c^∞. In fact, the sequence $\{\psi_m\}$ in Lemma 2.13 satisfies the inequality

$$|\exp(-cz^2)\hat{\psi}_m(z^2)| \leq K\exp(-dx^2 + ey^2)$$

where the constants K, d, e do not depend upon m, therefore we can apply Theorem 1.15 and the dominated convergence theorem to prove that

$$(10) \qquad \lim_m T(\exp(-cz^2)\hat{\psi}_m(z^2)) = T(\exp(-bz^2)\hat{\varphi}(z))$$

But the sequence $\{\psi_m\}$ also satisfies the inequality

$$|\exp(-cx)\hat{\psi}_m(x)| \leq K\exp(-hx) \quad , h > 0$$

for real values $x \in R^n$, where the constants K and h are independent of m, we see that $\lim_m \int_{R^n} \exp(-cx)\hat{\psi}_m(x)\,d\nu(x) = \int_{R^n} \exp(-bx)\hat{\varphi}(\sqrt{x})\,d\nu(x)$

by dominated convergence, and hence by (10) we obtain

$$T(\exp(-bz^2)\hat{\varphi}(z)) = \int_{R^n} \exp(-bx)\hat{\varphi}(\sqrt{x})\,d\nu(x),$$

i.e., (9) holds for functions θ of the form $\theta(z) = \exp(-bz^2)\hat{\varphi}(z)$, $b > 0$, φ is an even function in C_c^∞.

Now let μ be the even positive measure concentrated on the set M, as defined in the statement of the theorem, such that

$$\int_{R^n} \hat{\varphi}(t)\,d\nu(t) = \int_M \hat{\varphi}(z^2)\,d\mu(z)$$

for all $\varphi \in C_c^\infty$. From the properties of ν, we see that the integral

$$(11) \qquad \int_M \exp(-cz^2)\,d\mu(z)$$

converges for all $c > 0$, and the equality

$$(12) \qquad T(\theta) = \int_M \theta(z)\,d\mu(z)$$

holds for all θ of the form $\theta(z) = \exp(-bz^2)\hat{\varphi}(z)$, where $b > 0$ and φ is an even function in C_c^∞.

By the converse part of the theorem, we see that the integral

$$\int_M e^{i(t,z)}\,d\mu(z)$$

exists for almost all $t \in R^n$, and that the function g, defined by

$$g(t) = \int_M e^{i(t,z)}\,d\mu(z) \qquad \text{a.e.}$$

is locally in $L^1(R^n)$. Let $\{G_m\}$ be the Gauss kernel, i.e.,

$$G_m(x) = \left(\frac{m}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{mx^2}{2}\right) \quad ,$$

then by virtue of the fact that

$$\widehat{G_m * \varphi}\,(z) = \exp\left(-\frac{z^2}{m}\right)\hat{\varphi}\,(z) \quad ,$$

we can apply (12) to obtain

$$(13) \qquad \int_{R^n} f(t) G_m * \varphi(t)\,dt = \int_M \widehat{G_m * \varphi}(z)\,d\mu(z)$$

for all $m > 0$ and all even functions $\varphi \in C_c^\infty$. Applying Fubini's Theorem to the integrals in (13), we can prove that the equality

$$\int_{R^n} (f * G_m)(t)\varphi(t)\,dt = \int_{R^n} (g * G_m)(t)\varphi(t)\,dt$$

holds for all $m > 0$, and all even functions $\varphi \in C_c^\infty$. Since $f * G_m \longrightarrow f$ and $g * G_m \longrightarrow g$ on compact subsets of R^n in L^1 norms, therefore the equality

$$(14) \qquad \int_{R^n} f(t)\varphi(t)\,dt = \int_{R^n} g(t)\varphi(t)\,dt$$

holds for all even functions $\varphi \in C_c^\infty$.

Now for all $\varphi \in C_c^\infty(R^n)$, let $\mathring{\varphi}$ be the function defined by

$$\mathring{\varphi}(x_1, \ldots, x_n) = \frac{1}{2^n} \sum \varphi(\pm x_1, \ldots, \pm x_n)$$

where the summation is taken over all permutations of the signs. Then $\mathring{\varphi}$ is an even function in C_c^∞, and by the evenness of both f and g, we see that

$$\int f\varphi = \int f\mathring{\varphi} = \int g\mathring{\varphi} = \int g\varphi$$

and hence (14) holds for all $\varphi \in C_c^\infty$. The fact that both f and g are locally in $L^1(R^n)$ shows that $f = g$ locally almost everywhere in R^n. But since R^n is a countable union of compact subsets, we can conclude that $f = g$ almost everywhere, and the proof of the theorem is completed. ∎

<u>2.15</u> <u>Theorem</u> Let f be a continuous function in the class $P_e(C_c^\infty(R^n))$ such that the integral

$$\int_{R^n} \exp(-ct^2) f(t)\,dt$$

converges for all $c > 0$, then there is an even positive measure μ on the set M of points, each of whose coordinates is either real or pure imaginary, such that

$$(1) \qquad f(t) = \int_M e^{i(t,z)}\, d\mu(z)$$

The measure μ is such that the integral

$$(2) \qquad \int_M \exp(cy^2)\, d\mu(z)$$

converges for all $c > 0$. Conversely, if the positive even measure μ is such that the integral (2) converges for all $c > 0$, then (1) defines a continuous function in the class $G(n) \cap P_e(C_c^\infty(R^n))$.

<u>Proof:</u> The second (converse) part of the theorem is obvious. For the direct assertion of the theorem, it suffices to prove that the measure μ defined in Theorem 2.14 is finite. Let G_m be the Gauss kernel, i.e.,

$$G_m(x) = \left(\frac{m}{2}\right)^{\frac{n}{2}} \exp\left(-\frac{mx^2}{2}\right) \qquad , \ m > 0$$

then by Theorem 1.14, we obtain

$$(3)\quad \int_{R^n} f(t)G_m(t)dt = \int_{R^n} G_m(t)\int_M e^{-i(t,z)}d\mu(z)dt$$

The integral on the left hand side of (3) converges by the growth restriction on f, and we can apply Fubini's Theorem to obtain

$$(4)\quad \int_{R^n} f(t)G_m(t)dt = \int_M \widehat{G}_m(z)d\mu(z)$$

Since $\widehat{G}_m \longrightarrow 1$ uniformly on compacts, therefore for any compact subset C of M,

$$\begin{aligned}\mu(C) &= \int_C d\mu(z)\\ &= \lim_m \int_C \widehat{G}_m(z)d\mu(z)\\ &\leq \lim_m \int_M \widehat{G}_m(z)d\mu(z)\\ &= \lim_m (G_m * f)(0)\\ &= f(0)\ .\end{aligned}$$

Since the bound is independent of C, μ is finite. ∎

Remarks on the Class $P_e(F)$

2.16 We know from Cooper [4] and Stewart [21] that $P(C_c^\infty(R^n)) = P(C_o(R^n)) = P(L_c^p(R^n))$ for every $p \geq 2$. We remark that the corresponding theorem also holds for evenly positive definite functions, i.e., $P_e(C_c^\infty(R^n)) = P_e(C_o(R^n)) = P_e(L_c^p(R^n))$ for every $p \geq 2$. In fact, the following inclusion always holds

$$P_e(L_c^p(R^n)) \subseteq P_e(C_o(R^n)) \subseteq P_e(C_c^\infty(R^n)) \quad .$$

But if $f \in P_e(C_o^\infty(R^n))$, then f is locally summable by virtue of Theorem 1.5. Hence the integral

$$(1) \quad \int_{R^n} \int_{R^n} f(x-y)\varphi(x)\overline{\varphi(y)}dxdy$$

exists as Lebesgue integral for every even function φ in $L_c^p(R^n)$ $(p \geq 2)$. Since the set of all even functions in $C_c^\infty(R^n)$ is dense in the set of all even functions in L_c^p, there is a sequence of even function $\{\varphi_m\}$ in $C_o^\infty(R^n)$ such that $|\varphi_m|$ increases to $|\varphi|$ and hence, by dominated convergence theorem,

$$\int_{R^n} \int_{R^n} f(x-y)\varphi(x)\overline{\varphi(y)}dxdy = \lim_m \int_{R^n}\int_{R^n} f(x-y)\,\varphi_m(x)\overline{\varphi_m(y)}dxdy \geq 0.$$

Thus f is e.p.d. for $L_c^p(R^n)$.

We remark that the class $P_e(L^1(R^n))$ is a much more restricted class of functions than the class $P_e(C_c^\infty(R^n))$. In the next section we will show that every function in the class $P_e(L^1(R^n))$ is essentially bounded, and consequently, they are exactly all those even functions in the class $P(L^1(R^n))$.

The Class $Pe(L^1(R^n))$

2.17 **Lemma** If $f \in Pe(L^1(R^n))$, then $f \in L^\infty(R^n)$.

Proof: Let E be the set of all even functions in $L^1(R^n)$, and $\{\varphi_\alpha\}$ the sequence constructed in Sec 2.6. Then E is obviously a Banach subalgebra of the commutative Banach algebra $L^1(R^n)$, and $\{\varphi_\alpha\}$ is a bounded approximate unit of E. In view of Theorems 1.6 and 1.9, we see that the integral

$$(1) \qquad \int_{R^n} |f(t)\varphi(t)|\, dt$$

converges for every even function $\varphi \in L^1(R^n)$. Now for every function $\psi \in L^1(R^n)$, let ψ' be the function defined by

$$\psi'(x \ ,\dots,x \ ) = \frac{1}{2^n} \sum |\psi(\pm x_1,\dots, \pm x_n)|$$

where the summation is taken over all possible combinations of signs. Then for $\psi \in L^1(R^n)$, the equality

$$\int_{R^n} |f(t)\psi(t)|\, dt = \int_{R^n} |f(t)\psi'(t)|\, dt$$

holds by virtue of the fact that f is even. Since ψ' is an even function in $L^1(R^n)$, therefore the integral (1) converges for all $\varphi \in L^1(R^n)$. Thus $f \in L^\infty(R^n)$ (see Hewitt and Stromberg [13, p348 Theorem 20.15]). ∎

__2.18__ __Theorem__ If $f \in P_e(L^1(R^n))$, there is a unique even positive measure $\mu \in M(R^n)$ such that

$$f(t) = \int_{R^n} e^{i(t,x)}\, d\mu(x)$$

for almost all $t \in R^n$.

__Proof:__ We know from Lemma 2.17 that $f \in L^\infty(R^n)$, and therefore the integral

$$\int_{R^n} \exp(-ct^2) f(t)\,dt$$

converges for all $c > 0$. Hence by Theorem 2.14, there is a unique even positive measure μ on the set M of all points $z = (z_1,\ldots,z_n)$, each of whose coordinate z_k is either real or pure imaginary, such that

$$(1) \quad f(t) = \int_M e^{i(t,z)}\, d\mu(z)$$

for almost all $t \in R^n$. We first show that the measure μ is finite. In fact, if $\{\varphi_\alpha\}$ is the summability kernel defined in Sec 2.6, then we obtain from (1) that

$$(\varphi_\alpha * f)(t) = \int_M \Phi_\alpha(z) e^{i(t,z)}\, d\mu(z)$$

Here we have used Fubini's Theorem. Since $\varphi_\alpha * f \to f$ in L^∞ norm, the function $g = \lim_{\alpha\to\infty} \varphi_\alpha * f$ is a bounded continuous function which is equal to f almost everywhere. Now since $\Phi_\alpha(z) \to 1$ uniformly on compacts, therefore if C is a compact subset of $\mathbb{C}^n$, then

$$\mu(C) = \lim_{\alpha\to\infty} \int_C \Phi_\alpha(z)\,d\mu(z) \leq \lim_{\alpha\to\infty} \int_M \Phi_\alpha(z)\,d\mu(z) = \lim_{\alpha\to\infty} (\varphi_\alpha * f)(0) = g(0).$$

Since the bound is independent of the choice of C, μ is finite.

Now let μ_1 and μ_2 be the restriction of μ onto the real and imaginary axes, respectively, such that $\mu_2(\{0\}) = 0$. We obtain from (1) that

$$(2) \qquad f(t) = \int_{R^n} e^{i(t,x)}\, d\mu_1(x) + \int_{R^n} e^{(t,y)}\, d\mu_2(y)$$

for almost all $t \in R^n$. Since μ_1 and μ_n are finite, and $f \in L^\infty$, the function h, defined by $h(t) = \int_{R^n} e^{(t,y)}\, d\mu_2(y)$ is bounded and continuous. Let $m = (m,\dots,m) \in R^n$, then the inequality

$$h(m) = \int_{R^n} e^{(m,|y|)} d\mu_2(y) \geqslant \int_{R^n} (m,|y|) d\mu_2(y) =$$

$$= m \int_{R^n} (|y_1| + \dots + |y_n|) d\mu_2(y)$$

holds for arbitrary large values of m, and thus $\mu_2 = 0$ since h is bounded. Therefore $\mu = \mu_1$ and

$$f(t) = \int_{R^n} e^{i(t,x)}\, d\mu(x)$$

for almost all $t \in R^n$. ∎

Chapter 3

Symmetry on Groups

Definition of Symmetry on Groups

3.1 Let G be a locally compact abelian group. We denote by Aut G the group of all continuous automorphisms on G, where group operations in Aut G is function composition and is written multiplicatively. A subgroup Γ of Aut G is called a symmetry on G if Γ is finite. We first note the following invariant property of Γ.

3.2 Theorem Let Γ be a symmetry on G. Then the equality

$$(1) \qquad \int_G \varphi(\gamma x)\,dx = \int_G \varphi(x)\,dx$$

holds for every $\varphi \in C_c(G)$ and every $\gamma \in \Gamma$.

Proof: Since γ is a continuous automorphism on G, the integral

$$J(\varphi) = \int_G \varphi(\gamma x)\,dx \qquad (\varphi \in C_c(G))$$

exists and is translation invariant. Therefore there is a positive real number $D(\gamma)$ such that

$$\int_G \varphi(\gamma x)\,dx = D(\gamma)\int_G \varphi(x)\,dx$$

holds for all $\varphi \in C_c(G)$. Clearly $D(e) = 1$ if e is the identity automorphism of G. We show that $D(\gamma_1\gamma_2) = D(\gamma_1)D(\gamma_2)$. Indeed,

$$D(\gamma_1\gamma_2)\int_G \varphi(x)\,dx = \int_G \varphi(\gamma_1\gamma_2 x)\,dx = \int_G (\varphi\circ\gamma_1)(\gamma_2 x)\,dx =$$

$$= D(\gamma_2)\int_G \varphi(\gamma_1 x)\,dx = D(\gamma_1)D(\gamma_2)\int_G \varphi(x)\,dx$$

holds for all $\varphi \in C_c(G)$. Consequently, each $D(\gamma)$ must satisfy the equation $(D(\gamma))^n = 1$ where n is the order of Γ. The fact that $D(\gamma) \geq 0$ forces $D(\gamma) = 1$, and hence (1) holds.

3.3 Let Γ be a symmetry on G. For any $\gamma \in \Gamma$, it is natural to look at its adjoint γ^*, which is a function defined from $\hat{G}$ into $\hat{G}$ by

$$[x, \gamma^*\hat{x}] = [\gamma x, \hat{x}] \qquad (x \in G,\ \hat{x} \in \hat{G}).$$

It is clear that γ^* is an automorphism of $\hat{G}$. We wish to show that γ^* is continuous. In fact, the sets

$$V_{K,\varepsilon} = \{ \hat{x} \in \hat{G};\ |[x,\hat{x}] - 1| < \varepsilon \text{ for all } x \in K \}$$

with K a compact subset of G and $\varepsilon > 0$ form a neighbourhood basis of 0 in $\hat{G}$. Since $\gamma^*\hat{x} \in V_{K,\varepsilon}$ for all $\hat{x} \in V_{\gamma^{-1}K,\varepsilon}$, where $\gamma^{-1}K = \{ \gamma^{-1}x;\ x \in K \}$ which is compact whenever K is, we can conclude that γ^* is continuous. Therefore the group $\Gamma^* = \{\gamma^*;\ \gamma \in \Gamma\}$ is a symmetry on G, and we shall call this the adjoint symmetry of Γ on G. Throughout this paper, Γ will denote a symmetry on G, and Γ^* its adjoint symmetry on $\hat{G}$.

Examples

3.4 The trivial symmetry I Let $I = \{ e \}$ be the trivial subgroup of Aut G. Then it is obvious that I is a symmetry on G. We shall call this the trivial symmetry on G, and we can identify any locally compact abelian group with a symmetric group with trivial symmetry.

3.5 The even symmetries E(n) If $G = R^n$, let E(n) be the group of all linear transformations γ in R^n such that mat γ is an $n \times n$ diagonal matrix with diagonal entries taken from the set $\{-1, 1\}$. It is clear that E(n) is a symmetry on R^n, and we call this the even symmetry on

R^n. It turns out that E(n) is equivalent to the ordinary concept of evenness in R^n.

Symmetric functions and measures

3.6 Definitions: A complex-valued function φ on G is said to be Γ-symmetric if $\varphi(\gamma x) = \varphi(x)$ for all $\gamma \in \Gamma$ and for all $x \in G$. A measure μ on G is said to be Γ-symmetric if for all $\varphi \in C_c(G)$ the following equality

$$(1) \qquad \int_G \varphi(\gamma x) d\mu(x) = \int_G \varphi(x) d\mu(x)$$

holds for all $\gamma \in \Gamma$. Thus any constant function on G is Γ-symmetric, and the Haar measure on G is also Γ-symmetric by virtue of Theorem 3.2.

A subset M of G is said to be Γ-invariant if $\gamma M \subseteq M$ for all $\gamma \in \Gamma$, where $\gamma M = \{ \gamma x ; x \in M \}$.

For any complex-valued function φ on G, we define the Γ-mean of φ, denoted by $\mathring{\varphi}$, by

$$(2) \qquad \mathring{\varphi} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \varphi \circ \gamma ,$$

where $|\Gamma|$ = order of Γ.

Let X be a set of complex-valued functions on G. We define the Γ-part of X, denoted by ΓX, by

$$\Gamma X = \{ \varphi \in X ; \varphi \text{ is } \Gamma\text{-symmetric} \} .$$

3.7 It would be desirable if we could single out some function spaces on G that would reflect the symmetry that is on G. Throughout this paper, we are interested in all those function spaces which are also topological linear spaces, and we make the following definition.

Let X be a topological linear space of complex-valued functions on G. Then X is called a <u>Γ-symmetric function space</u> on G if the operator T_γ, $T_\gamma(\varphi) = \varphi\circ\gamma$, is a continuous linear operator on X for all $\gamma \in \Gamma$. The following theorem is valid.

<u>3.8</u> <u>Theorem</u> Let X be a Γ-symmetric function space on G. Then the operator P, defined from X onto ΓX by $P(\varphi) = \overset{\circ}{\varphi}$, is a continuous projection from X onto ΓX.

<u>Proof</u>: Obvious.

<u>3.9</u> <u>Corollary</u> Let X and Y be Γ-symmetric function spaces on G such that X is dense in Y. Then ΓX is dense in ΓY.

<u>Proof</u>: Obvious.

<u>3.10</u> <u>Theorem</u> (a) The spaces $L^p(G)$, $L^p_c(G)$ $(1 \le p \le \infty)$, $C_0(G)$, $C_c(G)$ and $C(G)$ are Γ-symmetric.

(b) The space $A(\hat{G})$ is Γ^*-symmetric and $\overset{\circ}{\hat{\varphi}} = \hat{\overset{\circ}{\varphi}}$ for all $\varphi \in L^1(G)$.

<u>Proof</u>: (a) The proof is easy. For the spaces L^p, L^p_c $(1 \le p \le \infty)$ use the symmetric property of the Haar measure on G. For the other spaces, use the fact that every $\gamma \in \Gamma$ is continuous.

(b) It is obvious that $\hat{f}\circ\gamma^* = (f\circ\gamma^{-1})^\wedge \in A(\hat{G})$, and $\|\hat{f}\circ\gamma^*\|_\infty = \|\hat{f}\|_\infty$, hence $A(\hat{G})$ is Γ^*-symmetric. Moreover for every $\varphi \in L^1(G)$,

$$\overset{\circ}{\hat{\varphi}} = \frac{1}{|\Gamma^*|}\sum_{\gamma^*\in\Gamma^*}\hat{\varphi}\circ\gamma^* = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}(\varphi\circ\gamma^{-1})^\wedge = \Big(\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}(\varphi\circ\gamma^{-1})\Big)^\wedge =$$

$$= \hat{\overset{\circ}{\varphi}}\,.$$

∎

Symmetric Linear Functionals

3.11 Let X be a Γ-symmetric function space on G. It is natural to look at all those linear functionals on X that would preserve the symmetries that exist on both X and G. Accordingly, a linear functional T on X is said to be Γ-symmetric if $T(\varphi \circ \gamma) = T(\varphi)$ for every $\varphi \in X$ and every $\gamma \in \Gamma$. It is clear that the above equality is equivalent to the equality

$$T(\overset{\circ}{\varphi}) = T(\varphi)$$

for all $\varphi \in X$.

To give an example of such a symmetric linear functional, we take the case $X = C_c(G)$ and define a linear functional T on X by

$$T(\varphi) = \int_G \varphi(x)dx \qquad (\varphi \in X).$$

Then T is a Γ-symmetric linear functional by virtue of Theorem 3.2.

It is clear that every Γ-symmetric(positive, continuous,resp.) linear functional on X is a (positive, continuous, resp.) linear functional on ΓX. The following theorem shows that the converse of this is also true.

3.12 Theorem Let X be a Γ-symmetric function space on G. Then every (positive, continuous, resp.) linear functional T on ΓX can be extended uniquely to a Γ-symmetric (positive, continuous, resp.) linear functional T_1 on X.

Proof: Define a linear functional T_1 on X by

$$T_1(\varphi) = T(\overset{\circ}{\varphi}) \qquad (\varphi \in X)$$

Then T_1 is uniquely defined, because if T_2 is another Γ-symmetric extension of T, then

$$T_2(\varphi) = T_2(\overset{\circ}{\varphi}) = T(\overset{\circ}{\varphi}) = T_1(\varphi)$$

holds for all $\varphi \in X$, and hence $T_1 = T_2$. The rest of the theorem is obvious. ∎

Combining the above theorem with the Riesz-Markov-Kakutani Theorem (see e.g. Rudin [19, p266]), we obtain the following symmetric form of the Riesz-Markov-Kakutani Theorem.

3.13 Theorem

(a) To each continuous linear functional T on $\Gamma C_0(G)$, there corresponds a unique Γ-symmetric measure $\mu \in M(G)$ such that

$$T_1(\varphi) = \int_G \varphi \, d\mu \qquad (\varphi \in C_0(G))$$

where T_1 is the Γ-symmetric extension of T to $C_0(G)$.

(b) To each positive linear functional T on $\Gamma C_c(G)$, there corresponds a unique Γ-symmetric regular non-negative measure μ on G such that

$$T_1(\varphi) = \int_G \varphi \, d\mu \qquad (\varphi \in C_c(G))$$

where T_1 is the Γ-symmetric extension of T to $C_c(G)$.

Chapter 4

Symmetrically Positive Definite Functions

Introduction

4.1 Let X be a Γ-symmetric function space on G. A complex-valued function f on G is called Γ-positive-definite for X if f is Γ-symmetric and $f \in P(\Gamma X)$. The class of all functions which are Γ-positive-definite for X will be denoted by $P_\Gamma(X)$.

Remarks

4.2 If G is a locally compact abelian group, then we can identify G as a group with the trivial symmetry I (see Sec.3.4). In this case, the class P(X) coincides with the class $P_I(X)$.

4.3 If $G = R^n$ and $\Gamma = E(n)$ (see Sec 3.5), then the class $P_\Gamma(X)$ coincides with the class $P_e(X)$.

4.4 By Theorem 3.10(a), we know that the spaces $L^p(G)$, $L^p_c(G)$ $(1 \leq p \leq \infty)$, and $C_c(G)$ are Γ-symmetric function spaces on G. Hence it is meaningful to consider the classes $P_\Gamma(L^p)$, $P_\Gamma(L^p_c)$ and $P_\Gamma(C_c)$.

The Class $P_\Gamma(L^1)$

4.5 Lemma If $f \in P_\Gamma(L^1)$, then $f \in L^\infty$.

Proof: Let $\{\alpha_U\}$ be a bounded approximate unit of L^1 which exists by virtue of Theorem 1.8, and set $h_U = \mathring{\alpha}_U$ where $\mathring{\alpha}_U = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \alpha_U \circ \gamma$ Then $\| h_U\|_1 = \|\mathring{\alpha}_U\| \leq \|\alpha_U\|$ by virtue of Theorem 3.8. But since

$$h_U * \varphi = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}(\alpha_U \circ \gamma) * \varphi = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}(\alpha_U * \varphi)\circ\gamma = (\alpha_U * \varphi)^\circ \to \mathring{\varphi} = \varphi$$

for every $\varphi \in \Gamma L^1$, therefore $\{h_U\}$ is a bounded approximate unit in the Banach algebra ΓL^1 which is a Banach subalgebra of the commutative Banach algebra L^1. In view of Theorem 1.9 and 1.6, we see that the integral

(1) $$\int_G |f(x)\varphi(x)|\, dx$$

exists for every $\varphi \in \Gamma L^1$. For any $\psi \in L^1$, let ψ' be the function defined by

$$\psi'(x) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}|\psi(\gamma x)|.$$

Then by the symmetric property of f, the equality

$$\int_G |f(x)\psi(x)|\, dx = \int_G |f(x)\psi'(x)|\, dx$$

holds for every $\psi \in L^1$. Since $\psi' \in \Gamma L^1$, the integral on the right hand side of the above equality converges, and hence the integral

$$\int_G |f(x)\psi(x)|\, dx$$

exists for all $\psi \in L^1$. Thus $f \in L^\infty$ (see Hewitt and Stromberg [13, p348 Theorem 20.15]).

<u>4.6</u> <u>Theorem</u> If $f \in P_\Gamma(L^1)$, there is a unique positive Γ^*-symmetric measure $\mu \in M(\hat{G})$ such that

$$f(x) = \int_{\hat{G}} [x, \hat{x}] d\mu(\hat{x})$$

for almost all $x \in G$.

<u>Proof</u>: In view of Lemma 4.5, $f \in L^\infty$ and therefore the functional T defined by

$$(1) \qquad T(\varphi) = \int_G f(x)\varphi(x)dx \qquad (\varphi \in L^1)$$

is a continuous linear functional on L^1. If $(\varphi, \psi) = T(\varphi * \tilde{\psi}) =$
$= \int_G \int_G f(x-y)\varphi(x)\overline{\psi(y)}dxdy$, then the inequality

$$(2) \qquad |(\varphi, \psi)|^2 \leq (\varphi, \varphi)(\psi, \psi)$$

holds for all $\varphi, \psi \in \Gamma L^1$ by virtue of Theorem 1.6. Let $\{h_U\}$ with $\|h_U\|_1 \leq 1$ be an approximate unit in ΓL^1 which exists by the argument in the proof of Lemma 4.5. Then, since T is continuous, for every $\varphi \in \Gamma L^1$ and any $\varepsilon > 0$ there is a U such that

$$|T(\varphi)| \leq |T(\tilde{\varphi} * h_U)| + \varepsilon$$

and together with (2), we obtain

$$\begin{aligned} |T(\varphi)| &\leq |T(\varphi * \tilde{\varphi})|^{\frac{1}{2}} |T(h_U * \tilde{h}_U)|^{\frac{1}{2}} + \varepsilon \\ &\leq |T(\varphi * \tilde{\varphi})|^{\frac{1}{2}} \{ \|f\|_\infty \|h_U\|^2 \}^{\frac{1}{2}} + \varepsilon \\ &\leq \|f\|_\infty^{\frac{1}{2}} |T(\varphi * \tilde{\varphi})|^{\frac{1}{2}} + \varepsilon \quad . \end{aligned}$$

But this holds for all $\varepsilon > 0$, therefore the inequality

$$(3) \qquad |T(\varphi)| \leq \|f\|_\infty^{\frac{1}{2}} |T(\varphi * \tilde{\varphi})|^{\frac{1}{2}}$$

holds for all $\varphi \in \Gamma L^1$. Setting $\lambda = \varphi * \tilde{\varphi}$, $\lambda^2 = \lambda * \lambda$, $\lambda^n = \lambda^{n-1} * \lambda$,

(3) gives

$$|T(\varphi)|^2 \leq \|f\|_\infty\, T(\lambda) \leq \|f\|_\infty^{1+\frac{1}{2}} \left[T(\lambda^2)\right]^{\frac{1}{2}} \leq \cdots$$

$$\leq \|f\|_\infty^{2-2^{-n}} \left[\, T(\lambda^{2^n}) \,\right]^{2^{-n}} \leq \|f\|_\infty^{2-2^{-(n-1)}} \|\lambda^{2^n}\|_1^{2^{-n}}$$

$$\rightarrow \|f\|_\infty^2 \, \|\hat{\lambda}\|_\infty \qquad \text{as } n \rightarrow \infty$$

by the spectral radius theorem (see Rudin [19, Appendix D6 and Theorem 1.2.2]). This implies that $|T(\varphi)|^2 \leq \|f\|_\infty^2 \|\hat{\lambda}\|_\infty = \|f\|_\infty^2 \|\hat{\varphi}\|_\infty^2$, i.e., $|T(\varphi)| \leq \|f\|_\infty \|\hat{\varphi}\|_\infty$. Thus the functional T_1, defined by

$$T_1(\hat{\varphi}) = T(\varphi) = \int_G f(x)\varphi(x)dx$$

is a bounded linear functional on $\Gamma A(\hat{G})$. Here we have made use of the fact that $\{\hat{\varphi}\,;\ \varphi \in \Gamma L^1\} = \Gamma A(\hat{G})$ proved in Theorem 3.10. Moreover, T_1 is positive since

$$T_1(|\hat{\varphi}|^2) = \int_G \int_G f(x-y)\,\varphi(x)\,\overline{\varphi(y)}\,dxdy \geq 0 \text{ for all } \varphi \in \Gamma L^1.$$

Therefore T_1 is a bounded positive linear functional on $\Gamma A(\hat{G})$. But since $A(\hat{G})$ is dense in $C_0(\hat{G})$, we conclude that $\Gamma A(\hat{G})$ is dense in $\Gamma C_0(\hat{G})$ by virtue of Corollary 3.9, and hence T_1 can be extended to a positive bounded linear functional on $\Gamma C_0(\hat{G})$. The functional T_2 defined by

$$T_2(\hat{\varphi}) = \int_G f(x)\varphi(x)dx \qquad (\varphi \in L^1)$$

is obviously the Γ^*-symmetric extension of T_1 to $A(\hat{G})$, hence in view of the symmetric form of Riesz-Markov-Kakutani Theorem (Theorem 3.13), there is a unique positive Γ^*-symmetric measure $\mu \in M(\hat{G})$ such that

$$T_2(\hat{\varphi}) = \int_{\hat{G}} \hat{\varphi}(-\hat{x})\,d\mu(\hat{x}) \qquad (\hat{\varphi} \in A(\hat{G})).$$

Then for every $\varphi \in L^1$, we can apply Fubini Theorem to obtain

$$\int_G f(x)\varphi(x)dx = T_2(\hat{\varphi}) = \int_{\hat{G}} \hat{\varphi}(-\hat{x})d\mu(x) = \int_G \varphi(x)\left(\int_{\hat{G}}[x,\hat{x}]d\mu(\hat{x})\right)dx$$

and hence

$$f(x) = \int_{\hat{G}} [x,\hat{x}]\, d\mu(\hat{x})$$

for almost all $x \in G$. ∎

<u>4.7</u> <u>Theorem</u> Let μ be a positive Γ*-symmetric measure in $M(\hat{G})$. Then the function f, defined almost everywhere by

$$f(x) = \int_{\hat{G}} [x,\hat{x}]\, d\mu(\hat{x})$$

is in the class $P_\Gamma(L^1)$.

<u>Proof</u>: Obvious. ∎

The Class $P_\Gamma(L^1_c)$

4.8 Lemma: If $f \in P_\Gamma(L^1_c)$, then f is locally in L^∞.

Proof: It is obvious that ΓL^1_c is a Banach subalgebra of the Banach algebra ΓL^1. Let $\{h_U\}$ be a bounded approximate unit of ΓL^1 which exists by the proof of Lemma 4.5. Let $\{\alpha_\beta\}$ be the net in $C_c(G)$ such that $|\alpha_\beta(x)| \leq 1$, $\alpha_\beta(x) \longrightarrow 1$ uniformly on compacts. Such a net existsby virtue of Theorem 1.8. Then $\{\overset{\circ}{\alpha}_\beta\}$ is a net in $\Gamma C_c(G)$ such that $|\overset{\circ}{\alpha}_\beta(x)| \leq 1$, $\overset{\circ}{\alpha}_\beta(x) \to 1$ uniformly on compacts. Now let us define a net $\{g_{(U,\beta)}\}$ by setting $g_{(U,\beta)}(x) = \overset{\circ}{\alpha}_\beta(x) h_U(x)$, where we direct the pairs (U,β) by setting $(U_1, \beta_1) \leq (U_2, \beta_2)$ iff $U_1 \leq U_2$ or $U_1 = U_2$ and $\beta_1 \leq \beta_2$. Then $g_{(U,\beta)} \in \Gamma L^1_c$, $g_{(U,\beta)} * \varphi \to \varphi$ and $\| g_{(U,\beta)} \| \leq \| h_U \|$, i.e., $\{ g_{(U,\beta)} \}$ is a bounded approximate unit in ΓL^1_c. In view of Theorems 1.6 and 1.9, we see that the integral

$$(1) \qquad \int_G |f(x)\varphi(x)|\, dx$$

exists for every $\varphi \in \Gamma L^1_c$. For any $\psi \in L^1_c$, let ψ' be the function defined by

$$\psi'(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |\psi(\gamma x)| .$$

Then by the symmetric property of f, the equality

$$\int_G |f(x)\psi(x)|\, dx = \int_G |f(x)\psi'(x)|\, dx$$

holds for every $\psi \in L_c^1$. Since $\psi' \in \Gamma L_c^1$, the integral on the right hand side of the above equality converges, and hence the integral

$$\int_G |f(x)\psi(x)|\,dx$$

exists for all $\psi \in L_c^1$. Thus f is locally in L^∞ (see Hewitt and Stromberg [13, p348 Theorem 20.15]). ∎

4.9 Theorem If $f \in P_\Gamma(L_c^1)$, then $f \in L^\infty$.

Proof: Let $K(\varphi,\psi) = \int_G \int_G f(x-y)\varphi(x)\overline{\psi(y)}\,dxdy$. Then by Theorem 1.6 the inequality

$$(1) \qquad |K(\varphi,\psi)|^2 \leq K(\varphi,\varphi)\, K(\psi,\psi)$$

holds for all $\varphi, \psi \in \Gamma L_c^1$. Let U and V be compact neighbourhoods of 0 in G such that $V-V \subseteq U$. Let a and b be arbitrary elements in G, and let $\varphi, \psi \in \Gamma L_c^1$ be such that φ is zero outside the set a+V and ψ is zero outside the set b+V. Then $\varphi * \tilde{\varphi}$ and $\psi * \tilde{\psi}$ are zero outside the side V-V, which is compact. Hence

$$\begin{aligned}
K(\varphi,\varphi) &= \int_G f(x)\,dx \int_G \varphi(x+y)\overline{\varphi(y)}\,dy \\
&= \int_{V-V} f(x)\,dx \int_G \varphi(x+y)\overline{\varphi(y)}\,dy \\
&\leq c_V \int_{V-V} \left(\int_G \varphi(x+y)\overline{\varphi(y)}\,dy\right)dx \\
&= c_V \int_G \int_G \varphi(x+y)\overline{\varphi(y)}\,dxdy \\
&= c_V \|\varphi * \tilde{\varphi}\|_1 \leq c_V \|\varphi\|_1^2
\end{aligned}$$

where $C_V = \text{ess sup}\{|f(x)| \; ; x \in V-V\} < \infty$.

Since f is locally in L^∞ by virtue of Lemma 4.8. Then we obtain from (1) that

$$\left| \int_{b+V} \overline{\psi(y)}dy \int_{a+V} f(x-y)\,\varphi(x)dx \right| \leq C_V \,\|\varphi\|_1 \,\|\psi\|_1 \quad .$$

But since the set a-b+V-V is compact, we can conclude that

$$\text{ess sup}\{\, |f(x)| \; ; x \in a-b+V-V\} \leq C_V \quad .$$

Since a and B are arbitrary, we obtain $|f(x)| \leq C_V$ for almost all $x \in G$, and hence $f \in L^\infty$. ∎

<u>4.10</u> <u>Theorem</u> $P_\Gamma(L^1_c) = P_\Gamma(L^1)$

<u>Proof:</u> Obvious by virtue of Theorem 4.9 and dominated convergence Theorem. ∎

Concluding Remarks

4.11 We remark that the relation

$$P_{\Gamma}(C_c) = P_{\Gamma}(L_c^p) \quad \text{for } 2 \leq p \leq \infty$$

also holds. A proof can be obtained by following the argument of Sec 2.16. In this chapter we have only obtained representation theorems for the classes $P_{\Gamma}(L^1)$ and $P_{\Gamma}(L_c^1)$. It would be interesting if we could find integral representation theorems for the class $P_{\Gamma}(C_c)$ for, if this is done, we will have a generalization of Krein's theorem for evenly positive definite functions.

REFERENCES

1. Achieser, N.I. Theory of Approximation. New York, Ungar, 1956.

2. Bochner, S. Vorlesungen über Fouriersche Integrale. Leipzig, Akademische Verlagsgessellschaft,1932.

3. Bochner, S. "Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse", Math. Ann., 108 (1933) 378-410. [References 2 and 3 were translated by M. Tenenbaum and H. Pollard as: "Lectures on Fourier Integrals" (with the author's supplement on Monotonic Functions, Stieltjes Integrals and Harmonic Analysis), Annals of Mathematical Studies, No. 42, Princeton University Press, 1959].

4. Cooper, J. L. B. "Positive definite functions of a real variable", Proc. London Math. Soc. (3), 10 (1960) 53-66.

5. Edwards, R.E. "Unbounded integrally positive definite functions", Studia Math., 33 (1969) 185-191.

6. Gelfand, I.M. and R.A. Raikov. "Irreducible unitary representations of locally bicompact groups", (in Russian), Mat. Sbornik N.S., 13 (1943) 301-316. English trans; Amer. Soc. Trans., Ser. 2, 36 (1964) 1-15.

7. Gelfand, I. M. and G. E. Shilov. Generalized Functions. Vol. 1. New York, Academic Press, 1964.

8. Gelfand, I. M. and G. E. Shilov. Generalized Functions. Vol. 2. New York, Academic Press, 1964.

9. Gelfand, I. M. and N. Vilenkin. Generalized Functions. Vol. 4. Applications of Harmonic Analysis. New York, Academic Press, 1964.

10. Herglotz, G. "Über Potenzreihen mit Positiven reelen Teil im Einheitskreis", Leipziger Berichte, 63 (1911), 501-511.

11. Hewitt, E. and K. A. Ross. "Integrally Positive-definite functions on groups", Studia Math., 31 (1968) 145-151.

12. Hewitt, E. and K. A. Ross. Abstract Harmonic Analysis. Vol. II, New York, Springer-Verlag, 1970.

13. Hewitt, E. and K. Stromberg. Real and Abstract Analysis. Berlin and New York, Springer-Verlag, 1963-1970.

14. Loomis, L. An Introduction to Abstract Harmonic Analysis. Princeton, N.J., Van Nostrand, 1953.

15. Naimark, M. A. Normed Rings. Groningen, Netherlands, Erven P. Noordhoff, Ltd., 1960 (original Russian edition, 1955).

16. Povzner, A. "Über pesitive Funktionen auf einer Abelschen Gruppe," Dokl. Akad. Nauk SSSR, 28 (1940) 294-295.

17. Raikov, D. A. "Positive definite functions on commutative groups with an invariant measure", Dokl. Akad. Nauk SSSR, 28 (1940) 296-300.

18. Rickert, N. W. "A note on positive definite functions", Studia Math., 36 (1970) 223-225.

19. Rudin, W. Fourier Analysis on Groups. New York, Interscience, 1962.

20. Simons, A. B. "Cesáro summability on groups: characterization and inversion of Fourier transforms", Proc. Internat. Sympos. on Function Algebras, Tulane University, 1965. Chicago, Illinois, Scott-Foresman, 1966. 208-215.

21. Stewart, J. "Unbounded positive definite functions", Canadian Journal of Mathematics, 21 (1969) 1309-1318.

22. Stewart, J. Positive definite functions and generalizations, an historical survey. McMaster Mathematical Report No. 65, 1973.

23. Weil, A. "L'intêgration dans les groupes topologique et ses applications", Actualités Sci. Indust., No. 869. Paris, Hermann, 1940; revised, 1951.