TESTING GROUP EFFECTS IN EXPERIMENTAL DESIGN FROM TYPE I CENSORED NORMAL SAMPLES

TESTING GROUP EFFECTS IN EXPERIMENTAL DESIGN FROM TYPE I CENSORED NORMAL SAMPLES

by

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

April 1975

©

MASTER OF SCIENCE
(Statistics)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Testing group effects in experimental design from type I censored normal samples

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NUMBER OF PAGES: (vi), 64

ABSTRACT

The estimators of the mean, standard deviation and group effects for one-way- classification experimental designs are obtained from type I censored samples. The bias and the variances and covariances of these estimators are evaluated. A test statistic is proposed for testing a linear contrast of the group-effects. Two numerical examples are presented.

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my sincere appreciation of Dr. Moti Lal Tiku for the excellent guidance and kind encouragement during the past two years.

Many thanks to Miss Linda Westfall for typing this manuscript.

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CHAPTER 1

INTRODUCTION

In this thesis, the problem dealt with concerns the estimation of parameters and analyses for a one-way-classification experimental design where the samples in the experimental-blocks are subject to type I censoring (i.e. the samples come from truncated normal distributions).

A censored sample may be described as one in which either a known or unknown number of the observations are not available for inference purposes. The censoring may occur at either the lower/upper extreme or at both extremes. There are two types of censoring which may be referred to as type I censoring or type II censoring. A type II censored sample is defined in Chapter 2. A type I censored sample occurs due to curtailment of the observations either below and/or above some predetermined fixed point(s).

Estimation based on type I censored samples, although not a recent problem, has not been considered in any great detail so far.

As early as 1908, Karl Pearson and Alice Lee used the method of moments in order to estimate the mean and standard deviation, μ and σ, in $N(\mu,\sigma)$ from a type I censored sample. These moment-estimators m' and σ' are solutions of the equations

$$(a) \quad m' = x_o' - h'\sigma'$$

$$(b)\quad \sigma' = \frac{1}{n}\sum_{i=1}^{n} x \cdot \psi_2$$

$$(c)\quad \frac{n\Sigma x^2 - (\Sigma x)^2}{(\Sigma x)^2} = \psi_1 \ ,$$

where

x_0' is the point of truncation (on the left),

$$h' = \frac{x_0' - m'}{\sigma'} \ ,$$

$$\psi_2 = \frac{I_0}{I_1} \ ,$$

$$\psi_1 = \frac{2I_0I_2}{I_1^2} - 1 \ ;$$

$$I_1 = I_{-1} - h'I_0 \ ,$$

$$2I_2 = I_0 - h'I_1 \ ,$$

$$I_0 = \frac{1}{\sqrt{2\pi}} \int_{h'}^{\infty} e^{-t^2/2} dt, \qquad I_{-1} = dI_0/dh' \ .$$

Tables for both of the moment functions ψ_1 and ψ_2 at intervals of 0.1 in h' are given by Pearson (1914, 31).[11] Using these tables, the computations go as follows:

(1) Evaluate the left side in (c) above from the sample data. Enter the table of ψ_1 with this value and obtain h' by inverse interpolation.

(2) Using the above value of h' as the argument, read ψ_2 from

the above mentioned table and substitute this in equation (b) to obtain σ'.

(3) With both σ' and h' determined, use equation (a) to obtain m'.

The difficulties with these estimators are apparent, (i) they are not available if the truncation occurs at both the sides, (ii) computations are involved, (iii) the functions ψ_1 and ψ_2 are tabulated at too wide intervals and contain too few significant digits to allow sufficient accuracy in the computed values of m' and σ'.

Fisher (1931)[7], using the "Maximum Likelihood" method of estimation, demonstrated that the ML estimators in case of truncation only at the left-end are identical to those obtained by using moments. He used a function of h', ξ, which he called an I_n function defined as follows,

$$I_n(h') = \frac{1}{\sqrt{2\pi}} \int_{h'}^{\infty} \frac{(t-h')^n}{n!} e^{-t^2/2} dt \ ;$$

$$(n+1)\, I_{n+1} + h' I_n - I_{n-1} = 0; \qquad n > -1 \ .$$

$$\frac{dI_n}{dh'} = - I_{n-1} \ .$$

His equations based on the I_n functions are:

$$\sigma' = \frac{1}{n} \sum x \left(\frac{I_0}{I_1}\right)$$

and

$$\frac{n\Sigma x^2}{(\Sigma x)^2} = \frac{2I_0 I_2}{I_1^2} \ .$$

When σ' and h' are determined, Fisher's equation for obtaining m' is the same as that given by Pearson and Lee above. Although the application of Fisher's results are almost identical to that of Pearson-Lee, a higher degree of accuracy is obtained. The computation of these estimates is also very laborious and involved.

Cohen (1950)[2], extending Pearson's and Fisher's work, worked out equations similar to above for obtaining estimators of μ and σ in case of truncation at both ends, left and right. Cohen's equations are much more complicated to work with and he suggests an iteration-procedure (called Newton-Raphson method) to solve these equations for m' and σ'.

Halperin (1952)[3] also employed maximum likelihood equations to estimate μ and σ from type I censored samples in which curtailment occurs below/above some fixed point. He gives a chart and suggests a new iterative procedure, other than Cohen's (1950)[2] method, which he called modified Newton-Raphson method. In 1955, Cohen considered the problem of type I censored-sample-estimation in the context of life-testing and response-time studies. In this paper he derives maximum likelihood estimates of the population mean and standard deviation and obtains their asymptotic variances. An iteration technique is still required to solve the maximum likelihood equations. Cohen and Woodward (1953)[6] computed special tables to facilitate this iterative-procedure of computation. Cohen (1963)[5] also considered the case of progressively censored samples in the context of life and dosage-response experiments. By progressive censoring, we mean that

some though not all of the sample observations are eliminated from further consideration at various stages of the experiment. In his paper, Cohen derives maximum likelihood estimates of the parameters for the normal and the exponential distributions when the samples are subject to progressive censoring, by using Newton's method of iteration to solve the maximum likelihood equations. Shah and Jaiswal (1964)[13] estimated the mean and standard deviation for doubly truncated normal samples by using method of moments. They obtained the estimates of μ and σ by using the first four sample moments, and point out that these estimates are consistent and of relatively high efficiency as compared with the maximum likelihood estimates.

Tiku (1967, 68)[14,16] proposed a simplification to the maximum likelihood equations and called the resulting estimators "modified maximum likelihood estimators". These estimators are simple and explicit functions of sample observations and are easy to compute. Besides, these estimators are asymptotically unbiased and efficient, and for small n are only a little less efficient than the maximum likelihood estimators. Tiku (1967)[15] also obtained modified maximum likelihood estimators of μ and σ from progressively censored samples.

In this thesis, Tiku's method is extended to obtain estimates of the unknown parameters in one-way classification experimental designs when the group-observations come from truncated normal distributions. The variances and covariances of these estimators are obtained and a statistic for testing a linear contrast of group-effects is proposed.

CHAPTER 2

ESTIMATING THE PARAMETERS OF TRUNCATED NORMAL CENSORED SAMPLES

Tiku [1967,[14] 1973][17] has estimated the parameters from Type II censored samples for a single normal population and for a number of normal populations relevant in Experimental Designs (one-way classification). He has also estimated the parameters for a single truncated normal distribution (Type I censoring). The purpose of this thesis is to extend this work further in order to be able to obtain estimates for the parameters from Type I censored samples in one-way classification experimental designs. Before persuing this, let me discuss the differences between the two types of censoring, TYPE I and TYPE II censoring.

2.1.1 Type II Censoring

In a type II censored sample, the experiment is terminated by the experimenter himself, i.e. "life-testing" experiments.

Say a manufacturer is producing certain electronic components. The experimenter may wish to determine the average life-span of these components. He starts with testing 100 components and records their life-to-failure:

Component:	1	2	3	4	. . .	50	51	52	53	. . .	100
Lasting Time:	t_1	t_2	t_3	t_4	. . .	t_{50}	t_{51}	t_{52}	t_{53}	. . .	t_{100} .

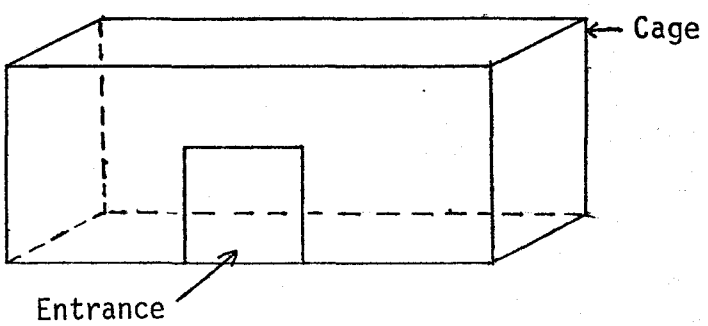
The experimenter may have to terminate the experiment at "t_{50}" due to lack of time, etc. Thus, fifty of the components, say, will still be operational when he decides that he must terminate the experiment and use the data results that he has by that period of time. Thus, we have a type II censored sample. As a result, the data generated by this experiment occur as ordered observations and the largest observations are not available.

Note: This is an example of type II censoring, the censoring has occurred on the right hand side only. We could also have experiments consisting of censoring on the left only, or on both sides.

2.1.2 Type I Censoring

In a type I censored sample, the experiment is not terminated by the experimenter at any specific time; but, the experimental conditions inherently do not allow all elements in the population to be included in experimentation.

Suppose one wished to experiment with monkeys who enter a cage to be fed over a period of time. We could have the following situation:



Apparently, only the monkeys which happen to be smaller than the cage entrance can enter and feed since those greater than the size of the cage entrance are unable to enter. Thus, we have a full sample (i.e. n = total number of monkeys that went inside the cage). However, this does not represent the whole range of the population of monkeys. Such situations give rise to type I censored sample where the experimental conditions exclude a certain proportion of the feasible observations.

A practical example of one-sided truncation is as follows:

(a) One-Sided Truncation: Type I Censoring

Suppose that for the management of a plant to ensure meeting a maximum weight specification of twelve ounces on a certain radio component for an aircraft installation, all production of this component is weighted and those units which exceed the specified maximum weight are discarded. For a random sample of fifty units selected from the accepted production, say,

(i) the sample mean $\bar{u} = \sum_{i=1}^{n} u_i/n = 9.63$ ounces,

(ii) the sample variance $s^2 = \sum_{i=1}^{n} (u_i-\bar{u})^2/(n-1) = 1.1480$ ounces.

Assuming the underlying distribution to be truncated normal, $(\frac{C}{\sqrt{2\pi}\sigma}) \exp\{-\frac{1}{2}(u-\mu)^2/\sigma^2\}$, one might like to estimate the mean μ and variance σ^2. (Note here that the range of u is $(-\infty, 12)$, not $(-\infty, \infty)$ as it would have been in the absence of truncation).

Note: Since the range of u has been truncated from $(-\infty, \infty)$

to $(-\infty, 12)$, we should not use $\bar{u}$ and s^2 as estimates since they are not unbiased and certainly not efficient. Thus, we would like to find more efficient and possibly unbiased estimates of μ and σ.

(b) Double Truncation: Type I Censoring

In this type of censoring (double truncation), we have restrictions on the lower bound as well as the upper bound of our variable. Considering the last example in this case, we can suppose that the management of the plant wishes to ensure meeting a minimum weight specification of 6.5 ounces and a maximum weight specification of 12 ounces on a certain radio component for an aircraft installation. All production of this component is weighed and those units which are less than the specified minimum and those units which exceed the specified maximum weight are discarded. Again, for a random sample of 50 units selected from the accepted production, say,

(i) the sample mean, $\bar{u} = 9.83$ ounces,

(ii) the sample variance, $s^2 = 0.9525$.

Assuming the underlying distribution to be truncated normal, one again wants to estimate its mean μ and variance σ^2. (Note that in this case the range of u is (6.5, 12.0); i.e., restricted lower and upper bounds).

Since the range of u has now been truncated from $(-\infty, \infty)$ to (6.5, 12.0), we again cannot use the above values of $\bar{u}$ and s^2 as our estimates since they are not unbiased and not efficient. So we want to find better estimates than $\bar{u}$ and s^2. But how conveniently do we

do this? A convenient method of obtaining efficient estimates was developed by Tiku [1968].[16]

Before extending Tiku's [1968][16] work to obtain estimates of the parameters from type I censored samples (truncated normals) in one-way classification experimental designs, I will present a review of this technique for obtaining the estimates μ and σ from a single type I censored sample.

2.2.1 Estimating the Parameters of the Truncated Normal Distribution (Type I Censoring) for a Single Normal Population

For the normal distribution we have the well known p.d.f.,

$$p(u) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{-\frac{1}{2}(u-\mu)^2/\sigma^2} \quad ; -\infty \leq u \leq \infty \tag{1}$$

The p.d.f. of the normal variate u defined over the truncated range $u' \leq u \leq u''$ is given by:

$$f(u) = \frac{\left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{-\frac{1}{2}(u-\mu)^2/\sigma^2}}{P(u'') - P(u')} ; \; u' \leq u \leq u''$$

$$= \frac{p(u)}{P(u'') - P(u')} ; \; u' \leq u \leq u'' \tag{2}$$

where, (i) $p(u) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{-\frac{1}{2}(u-\mu)^2/\sigma^2}$

(ii) $P(u'') = \int_{-\infty}^{u''} \left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{-\frac{1}{2}(u-\mu)^2/\sigma^2} du$

(iii) $P(u') = \int_{-\infty}^{u'} \left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{-\frac{1}{2}(u-\mu)^2/\sigma^2} du$.

We assume u' and u" are known, and clearly u'<u".

Now the problem at hand is to estimate μ and σ given a random sample $(u_1, u_2, u_3, \ldots, u_n)$ from (2).

2.2.2 The Maximum Likelihood Estimators

To obtain the maximum likelihood estimators, the likelihood function calculated from a type I censored sample is

$$\prod_{i=1}^{n} f(u_i;\mu,\sigma) = \prod_{i=1}^{n} \left\{\frac{p(u)}{P(u")-P(u')}\right\} = \frac{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n} z_i^2}}{\{P(z")-P(z')\}^n} \qquad (3)$$

where, $z_i = (u_i-\mu)/\sigma$,

$z" = (u"-\mu)/\sigma$,

and $z' = (u'-\mu)/\sigma$.

The log-likelihood function is

$$L = \text{constant} - n\{\log\sigma + \log[P(z")-P(z')]\} - \frac{1}{2}\sum_{i=1}^{n} z_i^2 \qquad (4)$$

The likelihood equations are

$$\frac{\partial L}{\partial \mu} = \left(\frac{n}{\sigma}\right)\left\{\frac{1}{n}\sum_{i=1}^{n} z_i + g_2(z") - g_1(z')\right\} \qquad (5)$$

$$\frac{\partial L}{\partial \sigma} = \left(\frac{n}{\sigma}\right)\left\{-1 + \frac{1}{n}\sum_{i=1}^{n} z_i^2 + z"g_2(z") - z'g_1(z')\right\} \qquad (6)$$

Here,

$$g_1(z') = p(z')/\{P(z")-P(z')\}$$

and $$g_2(z") = p(z")/\{P(z")-P(z')\} \quad .$$

The maximum likelihood estimators are the solutions of the equations

$$\frac{\partial L}{\partial \mu} = 0 \text{ and } \frac{\partial L}{\partial \sigma} = 0 \quad . \tag{7}$$

However these equations have no explicit solutions and are very difficult to solve even by iteration. To circumvent this difficulty, Tiku [1968][16] devised the following "modified" maximum likelihood equations by using the fact that the functions $g_1(z)$ and $g_2(z)$ may be closely approximated by linear functions of the z's.

2.2.3 Approximation to the Maximum Likelihood Equations

Tiku [1968][16] suggested that linear approximations replace the functions $g_1(z)$ and $g_2(z)$, that is,

$$g_1(z') \simeq \alpha_1 + \beta_1 z' \ , \tag{8}$$

$$g_2(z'') \simeq \alpha_2 + \beta_2 z'' \ . \tag{9}$$

The motivation behind this is that over small intervals of z, $g_1(z)$ and $g_2(z)$ are closely approximated by linear functions.

We want to locate intervals which possibly include z' and z". Now, for large n, z' is likely to be included or covered by the interval $[(u'-h_1)/\sigma, (u'-k_1)/\sigma]$ if we choose,

$$h_1 = \bar{u} + s/\sqrt{n} \text{ and } k_1 = \bar{u} - s/\sqrt{n} \tag{10}$$

where

$$\bar{u} = \sum_{i=1}^{n} u_i/n, \text{ (sample mean)}$$

$$s^2 = \sum_{i=1}^{n} (u_i-\bar{u})^2/(n-1), \text{ (sample variance)} .$$

To obtain α_1 and β_1 in the linear approximation $g_1(z') \simeq \alpha_1 + \beta_1 z'$, we solve the equations,

$$\begin{aligned} g_1(h_1) &\simeq \alpha_1 + \beta_1[(u'-h_1)/\sigma] \\ g_1(k_1) &\simeq \alpha_1 + \beta_1[(u'-k_1)/\sigma] \end{aligned} \tag{11}$$

where,

$$\begin{aligned} g_1(h_1) &= p((u'-h_1)/\sigma)/\{P((u''-h_1)/\sigma) - P((u'-h_1)/\sigma)\} \\ g_1(k_1) &= p((u'-k_1)/\sigma)/\{P((u''-k_1)/\sigma) - P((u'-k_1)/\sigma)\} . \end{aligned}$$

Thus,

$$\beta_1 = \frac{g_1(h_1) - g_1(k_1)}{\left(\frac{u'-h_1}{\sigma}\right) - \left(\frac{u'-k_1}{\sigma}\right)} , \tag{12}$$

and

$$\alpha_1 = g_1(h_1) - \beta_1\left(\frac{u'-k_1}{\sigma}\right) . \tag{13}$$

Similarly, we obtain α_2 and β_2 in the approximation $g_2(z'') \simeq \alpha_2 + \beta_2 z''$, that is

$$\beta_2 = \frac{g_2(h_2) - g_2(k_2)}{\left(\frac{u''-h_2}{\sigma}\right) - \left(\frac{u''-k_2}{\sigma}\right)} , \tag{14}$$

$$\alpha_2 = (g_2(h_2) - \beta_2\left(\frac{u''-h_2}{\sigma}\right). \tag{15}$$

Note: (a) If the truncation is symmetrical; i.e. $u''-\mu = -(u'-\mu)$, then $\alpha_1 = \alpha_2$ and $\beta_2 = -\beta_1$.

(b) If there is no truncation on the left, $\alpha_1=\beta_1=0$.

(c) If there is no truncation on the right, $\alpha_2=\beta_2=0$.

2.2.4 The "Modified" Maximum Likelihood Estimators

Replacing $g_1(z')$ and $g_2(z'')$ by the linear functions $\alpha_1 + \beta_1 z'$ and $\alpha_2 + \beta_2 z''$, respectively (as described in section 2.2.3), one obtains the "modified" log-likelihood equations

$$\frac{\partial L}{\partial \mu} \simeq \frac{\partial L^*}{\partial \mu} = \left(\frac{n}{\sigma}\right)\left\{\frac{1}{n}\sum_{i=1}^{n} z_i + (\alpha_2+\beta_2 z'') - (\alpha_1+\beta_1 z')\right\} \tag{16}$$

$$\frac{\partial L}{\partial \sigma} \simeq \frac{\partial L^*}{\partial \sigma} = \left(\frac{n}{\sigma}\right)\left\{-1 + \frac{1}{n}\sum_{i=1}^{n} z_i^2 + z''(\alpha_2+\beta_2 z'') - z'(\alpha_1+\beta_1 z')\right\} \tag{17}$$

Now, equating (16) to zero we obtain

$$\mu_{TN} = \left\{\frac{(\bar{u}+\beta_2 u''-\beta_1 u') + (\alpha_2-\alpha_1)\sigma}{(1+\beta_2-\beta_1)}\right\} = A + B\sigma, \text{ say,} \tag{18}$$

and,

if μ is known, σ_{TN} is the positive root of the equation

$$\begin{aligned}\sigma^2 - \{\alpha_2(u''-\mu) - \alpha_1(u'-\mu)\}\sigma \\ - \left\{\frac{1}{n}\sum_{i=1}^{n}(u_i-\mu)^2 + \beta_2(u''-\mu)^2 - \beta_1(u'-\mu)^2\right\} = 0 .\end{aligned} \tag{19}$$

If μ is not known, we replace μ by (18) and work out equation (17) again. Then σ_{TN} is the positive root of the equation

$$\begin{aligned}\sigma^2 - \{(\alpha_2 u''-\alpha_1 u') - (1+\beta_2-\beta_1)AB\}\sigma \\ - \left\{\frac{1}{n}\sum_{i=1}^{n} u_i^2 + \beta_2 u''^2 - \beta_1 u'^2 - (1+\beta_2-\beta_1)A^2\right\} = 0\end{aligned} \tag{20}$$

Note that (19) and (20) have only one positive root.

2.2.5 Variances and Covariances of μ_{TN} and σ_{TN}

Note:

$$\sum_{i=1}^{n} E(z_i)/n = \int_{z'}^{z''} zh(z)dz = g_1(z') - g_2(z'')$$
$$\sum_{i=1}^{n} E(z_i^2)/n = \int_{z'}^{z''} z^2 h(z)dz = 1 + z'g_1(z') - z''g_2(z''). \tag{21}$$

Now the asymptotic variances and covariances in μ_{TN} and σ_{TN} are given by the elements of the matrix,

$$V = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix}^{-1} = \begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} \simeq \left. \begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} \right|_{\mu=\mu_{TN},\sigma=\sigma_{TN}}, \tag{22}$$

where

$$J_{11} = -E\left(\frac{\partial^2 L^*}{\partial \mu^2}\right) \simeq -E\left(\frac{\partial^2 L^*}{\partial \mu_{TN}^2}\right) = \left(\frac{n}{\sigma^2}\right)\{1-\beta_1+\beta_2\} \tag{23}$$

$$J_{12} = J_{21} = -E\left(\frac{\partial^2 L^*}{\partial \mu \partial \sigma}\right) \simeq -E\left(\frac{\partial^2 L^*}{\partial \mu_{TN} \partial \sigma_{TN}}\right)$$
$$= \left(\frac{n}{\sigma^2}\right)\{2[g_1(z')-g_2(z'')] - (\alpha_1+2\beta_1 z') + (\alpha_2+2\beta_2 z'')\} \tag{24}$$

$$J_{22} = -E\left(\frac{\partial^2 L^*}{\partial \sigma^2}\right) \simeq -E\left(\frac{\partial^2 L^*}{\partial \sigma_{TN}^2}\right)$$
$$= \left(\frac{n}{\sigma^2}\right)\{2 + 3[z'g_1(z')-z''g_2(z'')] + z''(2\alpha_2+3\beta_2 z'') - z'(2\alpha_1+3\beta_1 z')\} \tag{25}$$

Using (8) and (9), (24) and (25) can be simplified further as follows:

$$J_{12} = J_{21} \simeq \left(\frac{n}{\sigma^2}\right)\{\alpha_1-\alpha_2\} \tag{26}$$

$$J_{22} \simeq (\frac{n}{\sigma^2})\{2+\alpha_1 z' - \alpha_2 z''\} \tag{27}$$

2.2.6 Bias in μ_{TN} and σ_{TN}

From (18) we have,

$$E(\mu_{TN}/\sigma) = \mu + \left[\frac{\sigma\{\frac{1}{n}\sum_{i=1}^{n} E(z_i)+(\alpha_2+\beta_2 z'')-(\alpha_1+\beta_1 z')\}}{(1+\beta_2-\beta_1)}\right] = \mu + \{\frac{E(\frac{\partial L^*}{\partial \mu})}{R^2(\mu)}\} \tag{28}$$

where,

(i) $E(\frac{\partial L^*}{\partial \mu}) = (\frac{n}{\sigma})\{\sum_{i=1}^{n} E(z_i)/n + (\alpha_2+\beta_2 z'') - (\alpha_1+\beta_1 z')\}$

(ii) $R^2(\mu) = -E(\frac{\partial^2 L^*}{\partial \mu^2}) = (\frac{n}{\sigma^2})\{1-\beta_1+\beta_2\}$.

Thus, the conditional bias (for known σ) in μ_{TN} is given by,

$$\frac{E(\frac{\partial L^*}{\partial \mu})}{R^2(\mu)} \simeq B_1 = \{\frac{E(\frac{\partial L^*}{\partial \mu})}{R^2(\mu)}\}_{\mu = \mu_{TN},\ \sigma = \sigma_{TN}} \tag{29}$$

The bias in σ_{TN} is difficult to obtain but from the Taylor expansion of $(\frac{\partial L^*}{\partial \sigma})$, the approximate conditional bias (for known μ) in σ_{TN} is

$$\frac{E(\frac{\partial L^*}{\partial \sigma})}{R^2(\sigma)} \simeq B_2 = \{\frac{E(\frac{\partial L^*}{\partial \sigma})}{R^2(\sigma)}\}_{\mu = \mu_{TN},\ \sigma = \sigma_{TN}} \tag{30}$$

Now, from (8), (9), (16), (17), and (21) it follows that

$$(\frac{\sigma}{n})E\{\frac{\partial L^*}{\partial \mu}\} = \{g_1(z')-(\alpha_1+\beta_1 z')\} - \{g_2(z'')-(\alpha_2+\beta_2 z'')\} \simeq 0 \tag{31}$$

and

$$(\tfrac{\sigma}{n})E\{\frac{\partial L^*}{\partial \sigma}\} = z'\{g_1(z')-(\alpha_1+\beta_1 z')\} - z''\{g_2(z'')-(\alpha_2+\beta_2 z'')\} \simeq 0 \ . \quad (32)$$

So it follows from (29), (30), (31) and (32) that the conditional bias in μ_{TN} and σ_{TN} is proportional to the error in the linear approximations (8) and (9) and thus should be very small, at least for large samples.

With this much detailed discussion of the Estimation of the Parameters of a single truncated Normal, I will now extend this work further to estimate the parameters for the truncated normals (type I censoring) in one-way-classification experimental designs. I evaluate the variances, convariances, and bias of these estimators and propose a "test statistic" to test a linear combination of group effects.

Incidently, note that μ_{TN} and σ_{TN} are asymptotically efficient. This is because of the following:

$$\lim_{n\to\infty} \beta_1 = \{\frac{d}{dz}[g_1(z)]\}_{z=z'} = \{\frac{d}{dz}[\frac{p(z)}{P(z'')-P(z')}]\}_{z=z'}$$

$$= \{\frac{-zp(z)}{P(z'')-P(z')}\}_{z=z'} = \frac{-z'p(z')}{P(z'')-P(z')} \ ,$$

and

$$\alpha_1 = g_1(z') - \beta_1 z' = \frac{p(z')}{P(z'')-P(z')} - \beta_1 z' \ .$$

Therefore,

$$\lim_{n\to\infty}\{\alpha_1+\beta_1 z'\} = \frac{p(z')}{P(z'')-P(z')} + \frac{z'^2 p(z')}{P(z'')-P(z')} - \frac{z'^2 p(z')}{P(z'')-P(z')}$$

$$= \frac{p(z')}{P(z'')-(P(z')} = g_1(z') \ .$$

Similarly,

$$\lim_{n\to\infty}\{\alpha_2+\beta_2 z''\} = g_2(z'') \quad .$$

The maximum likelihood and modified maximum likelihood equations are therefore asymptotically identical.

CHAPTER 3

ESTIMATING THE PARAMETERS AND TESTING GROUP EFFECTS FROM TYPE I CENSORED NORMAL SAMPLES IN EXPERIMENTAL DESIGN

3.1.1 Model Description

Assume that a one-way classification experimental design has been performed with k independent groups of observations with n observations in each group. Consider the mathematical model

$$u_{ij} = \mu + g_i + \varepsilon_{ij};\ i = 1, 2, \ldots, k \qquad j = 1, 2, \ldots, n \tag{33}$$

where ε_{ij}'s are independently and identically distributed as truncated normals

$$\frac{(\frac{1}{\sqrt{2\pi}\sigma})\ \exp\{-\frac{1}{2}(\varepsilon_{ij}{}^2)/\sigma^2\}}{P(u_i") - P(u_i')},\ u_i' \leq \varepsilon_{ij} \leq u_i";\tag{34}$$

u_i' and $u_i"$ are assumed known, and clearly $u' < u"$.

3.2.1 The Maximum Likelihood Estimators

The likelihood function for the ith group is

$$h(u_{ij};\mu,\sigma,g_i) = \prod_{j=1}^{n} f(u_{ij};\mu,\sigma,g_i) = \frac{(\frac{1}{\sqrt{2\pi}\sigma})^n \exp\{-\frac{1}{2}\sum_{j=1}^{n} z_{ij}{}^2\}}{P(z_i") - P(z_i')} \tag{35}$$

where,

$$z_{ij} = (u_{ij}-\mu-g_i)/\sigma,$$

$$z_i" = (u_i"-\mu-g_i)/\sigma,$$

and $$z_i' = (u_i' - \mu - g_i)/\sigma \ .$$

Now, with the assumption that the type I censored samples available from the k groups are independent, the likelihood function for all k groups is

$$\prod_{i=1}^{k} h(u_{ij};\mu,\sigma,g_i) = \prod_{i=1}^{k} \{\frac{(\frac{1}{\sqrt{2\pi}\sigma})^n \exp[-\frac{1}{2}\sum_{j=1}^{n} z_{ij}^2]}{P(z_i'') - P(z_i')}\} \tag{36}$$

The log-likelihood function L is

$$L = \sum_{i=1}^{k} \{\text{constant} - n[\log\sigma + \log(P(z_i'') - P(z_i'))] - \frac{1}{2}\sum_{j=1}^{n} z_{ij}^2\} \tag{37}$$

Now the likelihood equations are

$$\frac{\partial L}{\partial \mu} = (\frac{n}{\sigma}) \sum_{i=1}^{k} \{\sum_{j=1}^{n} z_{ij}/n + g_2(z_i'') - g_1(z_i')\} \tag{38}$$

$$\frac{\partial L}{\partial \sigma} = (\frac{n}{\sigma}) \sum_{i=1}^{k} \{-1 + \sum_{j=1}^{n} z_{ij}^2/n - z_i' g_1(z_i') + z_i'' g_2(z_i'')\} \tag{39}$$

$$\frac{\partial L}{\partial g_i} = (\frac{n}{\sigma})\{\sum_{j=1}^{n} z_{ij}/n - g_1(z_i') + g_2(z_i'')\} \qquad i = 1, 2, \ldots, k \tag{40}$$

where,

$$g_1(z_i') = p(z_i')/\{P(z_i'') - P(z_i')\}$$

and $$g_2(z_i'') = p(z_i'')/\{P(z_i'') - P(z_i')\} \ .$$

The maximum likelihood estimators are the solutions of the equations

$$\frac{\partial L}{\partial \mu} = 0, \ \frac{\partial L}{\partial \sigma} = 0 \text{ and } \frac{\partial L}{\partial g_i} = 0 \qquad i = 1, 2, \ldots, k \ . \tag{41}$$

However, these equations have no explicit solutions and are very difficult to solve even by iteration. In order to circumvent this difficulty, I will construct "modified" maximum likelihood equations, in the same fashion as Tiku [1968][16]. This makes use of the fact that the functions $g_1(z_i')$ and $g_2(z_i'')$ can be closely approximated by linear functions.

3.2.2 Approximation to the Maximum Likelihood Equations

We let linear approximations replace $g_1(z_i')$ and $g_2(z_i'')$ in the above maximum likelihood equations. That is, $g_1(z_i')$ is replaced by

$$g_1(z_i') \simeq \alpha_{1i} + \beta_{1i} z_i' \tag{42}$$

and $g_2(z_i'')$ is replaced by

$$g_2(z_i'') \simeq \alpha_{2i} + \beta_{2i} z_i'' \quad . \tag{43}$$

To choose values for α_{1i}, α_{2i}, β_{1i}, β_{2i}; recall that,

$$z_i' = (u_i' - \mu - g_i)/\sigma,$$

$$z_i'' = (u_i'' - \mu - g_i)/\sigma,$$

$$g_1(z_i') = p(z_i')/\{P(z_i'') - P(z_i')\},$$

and $$g_2(z_i'') = p(z_i'')/\{P(z_i'') - P(z_i')\}$$

where $z_i'' > z_i'$.

Now, we require values of h_{1i} and k_{1i} such that $g_1(z_i')$ is adequately represented. At least for large n, z_i' is likely to be included in

the interval $[(u_i'-h_{1i})/\sigma,\ (u_i'-k_{1i})/\sigma]$ if we choose

$$h_{1i} = \bar{u}_i + s_i/\sqrt{n} \quad \text{and} \quad k_{1i} = \bar{u}_i - s_i/\sqrt{n} \tag{44}$$

where,

$$\bar{u}_i = \sum_{j=1}^{n} u_{ij}/n, \text{ (sample mean of the ith group)}$$

and $$s_i^2 = \sum_{j=1}^{n} (u_{ij}-\bar{u}_i)^2/(n-1), \text{ (sample variance of the ith group)} .$$

In order to obtain α_{1i} and β_{1i} in the linear approximation (42), we solve the equations,

$$\begin{aligned} g_1(h_{1i}) &\simeq \alpha_{1i} + \beta_{1i}\,[(u_i'-h_{1i})/\sigma] \\ g_1(k_{1i}) &\simeq \alpha_{1i} + \beta_{1i}\,[(u_i'-k_{1i})/\sigma] \end{aligned} \tag{45}$$

where,

$$g_1(h_{1i}) = p((u_i'-h_{1i})/\sigma)/\{P((u_i''-h_{1i})/\sigma) - P((u_i'-h_{1i})/\sigma)\}$$

and $$g_1(k_{1i}) = p((u_i'-k_{1i})/\sigma)/\{P((u_i''-k_{1i})/\sigma) - P((u_i'-k_{1i})/\sigma)\} .$$

Thus,

$$\beta_{1i} = -\frac{g_1(h_{1i}) - g_1(k_{1i})}{\left(\frac{u_i'-h_{1i}}{\sigma}\right) - \left(\frac{u_i'-k_{1i}}{\sigma}\right)} \tag{46}$$

and

$$\alpha_{1i} = g_1(h_{1i}) - \beta_{1i}\left(\frac{u_i'-h_{1i}}{\sigma}\right) \tag{47}$$

Similarly, we obtain α_{2i} and β_{2i} in the approximation (43), that is

$$\beta_{2i} = \frac{g_2(h_{2i}) - g_2(k_{2i})}{(\frac{u_i''-h_{2i}}{\sigma}) - (\frac{u_i''-k_{2i}}{\sigma})} \tag{48}$$

and $$\alpha_{2i} = g_2(h_{2i}) - \beta_{2i}(\frac{u_i''-h_{2i}}{\sigma}) \tag{49}$$

Note: (a) For symmetrical truncation; i.e. $u_i'' - \mu - g_i = -(u_i'-\mu-g_i)$, $\alpha_{1i} = \alpha_{2i}$ and $\beta_{2i} = -\beta_{1i}$

(b) If there is no truncation on the left, $\alpha_{1i} = \beta_{1i} = 0$

(c) If there is no truncation on the right, $\alpha_{2i} = \beta_{2i} = 0$.

3.3.1 The "Modified" Maximum Likelihood Estimators

The "modified" maximum likelihood equations are

$$\frac{\partial L}{\partial \mu} \simeq \frac{\partial L'}{\partial \mu} = (\frac{n}{\sigma}) \sum_{i=1}^{k} \{ \sum_{j=1}^{n} z_{ij}/n + (\alpha_{2i}+\beta_{2i}z_i'') - (\alpha_{1i}+\beta_{1i}z_i')\} \tag{50}$$

$$\frac{\partial L}{\partial \sigma} \simeq \frac{\partial L'}{\partial \sigma} = (\frac{n}{\sigma}) \sum_{i=1}^{k} \{-1 + \sum_{j=1}^{n} z_{ij}^2/n + z_i''(\alpha_{2i}+\beta_{2i}z_i'') - z_i'(\alpha_{1i}+\beta_{1i}z_i')\} \tag{51}$$

$$\frac{\partial L}{\partial g_i} \simeq \frac{\partial L'}{\partial g_i} = (\frac{n}{\sigma})\{ \sum_{j=1}^{n} z_{ij}/n + (\alpha_{2i}+\beta_{2i}z_i'') - (\alpha_{1i}+\beta_{1i}z_i')\}_{i = 1, 2, \ldots, k} \tag{52}$$

Now equating (50) to zero, we obtain

$$\hat{\mu}_{TNE} = \frac{\sum_{i=1}^{k} \{\bar{u}_i - \beta_{1i} u_i' + \beta_{2i} u_i'' + (\alpha_{2i} - \alpha_{1i})\sigma - (1 - \beta_{1i} + \beta_{2i}) g_i\}}{\sum_{i=1}^{k} (1 - \beta_{1i} + \beta_{2i})} .$$

Assuming the restriction in the linear model (33)

$$\sum_{i=1}^{k} (1 - \beta_{1i} + \beta_{2i}) g_i = 0 , \tag{53}$$

we obtain the following estimator

$$\mu_{TNE} = \sum_{i=1}^{k} d_i K_i / d \tag{54}$$

Equating (52) to zero, we obtain

$$\begin{aligned} g_{i_{TNE}} &= \frac{\{\bar{u}_i - \beta_{1i} u_i' + \beta_{2i} u_i'' + (\alpha_{2i} - \alpha_{1i})\sigma\}}{(1 - \beta_{1i} + \beta_{2i})} - \mu_{TNE} \\ &= K_i - \mu_{TNE} \qquad\qquad i = 1, 2, \ldots, k \end{aligned} \tag{55}$$

and setting (51) to zero we find that σ_{TNE} is given by

$$\sigma_{TNE} = \frac{B + \sqrt{B^2 + 4AC}}{2A} \tag{56}$$

where,

$$d_i = (1 - \beta_{1i} + \beta_{2i}) \tag{56a}$$

$$d = \sum_{i=1}^{k} d_i \quad \text{and} \quad K_i = \frac{\{\bar{u}_i - \beta_{1i} u_i' + \beta_{2i} u_i'' + (\alpha_{2i} - \alpha_{1i})\sigma\}}{d_i} \tag{56b}$$

and

$$A = \sum_{i=1}^{k} 1 = k, \tag{56c}$$

$$B = \sum_{i=1}^{k} \{(\alpha_{2i}u_i''-\alpha_{1i}u_i') - d_iA_iB_i\} \tag{56d}$$

$$C = \sum_{i=1}^{k} \{\frac{1}{n}\sum_{j=1}^{n} u_{ij}^2 - \beta_{1i}u_i'^2 + \beta_{2i}u_i''^2 - d_iA_i^2\} \tag{56e}$$

and

$$A_i = \{\frac{\bar{u}_i-\beta_{1i}u_i'+\beta_{2i}u_i''}{d_i}\},\ B_i = \{\frac{\alpha_{2i}-\alpha_{1i}}{d_i}\} \quad . \tag{56f}$$

3.3.2 Variances and Covariances of μ_{TNE}, $g_{i_{TNE}}$, and σ_{TNE}

Noting that

$$\sum_{j=1}^{n} E(z_{ij})/n = \int_{z'}^{z''} zh(z)dz = g_1(z_i') - g_2(z_i'')$$

$$\sum_{j=1}^{n} E(z_{ij}^2)/n = \int_{z'}^{z''} z^2h(z)dz = 1 + z_i'g_1(z_i') - z_i''g_2(z_i''), \tag{57}$$

from (50), (51), and (52), we obtain the asymptotic variances and covariances in μ_{TNE}, $g_{i_{TNE}}$ and σ_{TNE} given by the elements of the matrix

$$V = \begin{vmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix}^{-1} = \begin{vmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix} \simeq \begin{vmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{vmatrix} \tag{58}$$

$\mu=\mu_{TNE}, g_i=g_{i_{TNE}}, \sigma=\sigma_{TNE}$

where

$$J_{11} = -E\left(\frac{\partial^2 L'}{\partial \mu^2}\right) \simeq -E\left(\frac{\partial^2 L'}{\partial \mu^2}_{TNE}\right) = \left(\frac{n}{\sigma^2}\right) \sum_{i=1}^{k} \{1-\beta_{1i}+\beta_{2i}\} = \frac{nd}{\sigma^2} \qquad (59)$$

$$J_{12} = J_{21} = -E\left(\frac{\partial^2 L'}{\partial \mu \partial g_i}\right) \simeq -E\left(\frac{\partial^2 L'}{\partial \mu_{TNE} \partial g_i}_{TNE}\right) = \left(\frac{n}{\sigma^2}\right)\{1-\beta_{1i}+\beta_{2i}\} = \frac{nd_i}{\sigma^2} \qquad (60)$$

$$J_{13} = -E\left(\frac{\partial^2 L'}{\partial \mu \partial \sigma}\right) \simeq -E\left(\frac{\partial^2 L'}{\partial \mu_{TNE} \partial \sigma_{TNE}}\right)$$

$$= \left(\frac{n}{\sigma^2}\right) \sum_{i=1}^{k} \{2[g_1(z_i')-g_2(z_i")] - (\alpha_{1i}+2\beta_{1i}z_i') + (\alpha_{2i}+2\beta_{2i}z_i")\} \qquad (61)$$

$$J_{22} = -E\left(\frac{\partial^2 L'}{\partial g_i^2}\right) \simeq -E\left(\frac{\partial^2 L'}{\partial g_i^2}_{TNE}\right) = \left(\frac{n}{\sigma^2}\right)\{1-\beta_{1i}+\beta_{2i}\} = \frac{nd_i}{\sigma^2} \qquad (62)$$

$$J_{23} = J_{32} = -E\left(\frac{\partial^2 L'}{\partial g_i \partial \sigma}\right) \simeq -E\left(\frac{\partial^2 L'}{\partial g_i}_{TNE} \partial \sigma_{TNE}\right)$$

$$= \left(\frac{n}{\sigma^2}\right)\{2[g_1(z_i')-g_2(z_i")] - (\alpha_{1i}+2\beta_{1i}z_i') + (\alpha_{2i}+2\beta_{2i}z_i")\} \qquad (63)$$

$$J_{33} = -E\left(\frac{\partial^2 L'}{\partial \sigma^2}\right) \simeq -E\left(\frac{\partial^2 L'}{\partial \sigma^2}_{TNE}\right)$$

$$= \left(\frac{n}{\sigma^2}\right) \sum_{i=1}^{k} \{2 + 3[z_i'g_1(z_i') - z_i"g_2(z_i")] - z_i'(2\alpha_{1i}+3\beta_{1i}z_i') + z_i"(2\alpha_{2i}+3\beta_{2i}z_i")\} \qquad (64)$$

Now, using (42) and (43) in (61), (63) and (64), we obtain the following expressions

$$J_{13} \simeq (\frac{n}{\sigma^2})\{\sum_{i=1}^{k}(\alpha_{1i}-\alpha_{2i})\} \tag{65}$$

$$J_{23} \simeq (\frac{n}{\sigma^2})\{\alpha_{1i}-\alpha_{2i}\} \qquad i = 1, 2, \ldots, k \tag{66}$$

$$J_{33} \simeq (\frac{n}{\sigma^2})\sum_{i=1}^{k}\{2 + \alpha_{1i}z_i' - \alpha_{2i}z_i''\} \tag{67}$$

3.3.3 Bias in μ_{TNE}, $g_{i_{TNE}}$, and σ_{TNE}

Assuming the restriction (53), from (54) we obtain

$$E(\mu_{TNE}/g_i,\sigma) = E(\mu) + \sigma\{\sum_{i=1}^{k}[\sum_{j=1}^{n}E(z_{ij})/n + (\alpha_{2i}+\beta_{2i}z_i'') - (\alpha_{1i}+\beta_{1i}z_i')]\}/d \tag{68}$$

$$= \mu + \{\frac{E(\frac{\partial L'}{\partial \mu})}{R^2(\mu)}\}$$

where $R^2(\mu) = -E(\frac{\partial^2 L'}{\partial \mu^2})$ is given by (59).

Therefore, the conditional bias in μ_{TNE} is

$$\{\frac{E(\frac{\partial L'}{\partial \mu})}{R^2(\mu)}\} \simeq B_1 = \{\frac{E(\frac{\partial L'}{\partial \mu})}{R^2(\mu)}\}_{\mu=\mu_{TNE},\, g_i=g_{i_{TNE}},\, \sigma=\sigma_{TNE}} \tag{69}$$

From (55) we have

$$E(g_{i_{TNE}}/\mu,\sigma) = E(g_i) + \sigma\{\sum_{j=1}^{n} E(z_{ij})/n + (\alpha_{2i}+\beta_{2i}z_i'') - (\alpha_{1i}+\beta_{1i}z_i')\}/d_i \qquad (70)$$

$$= g_i + \{\frac{E(\frac{\partial L'}{\partial g_i})}{R^2(g_i)}\} \qquad i = 1, 2, \ldots, k$$

where $R^2(g_i) = -E(\frac{\partial^2 L'}{\partial g_i^2})$ is given by (62).

The conditional bias in $g_{i_{TNE}}$ is therefore

$$\{\frac{E(\frac{\partial L'}{\partial g_i})}{R^2(g_i)}\} \simeq B_{2i} = \{\frac{E(\frac{\partial L'}{\partial g_i})}{R^2(g_i)}\}_{\mu=\mu_{TNE}, g_i=g_{i_{TNE}}, \sigma=\sigma_{TNE}} \qquad (71)$$

The bias in σ_{TNE} is difficult to obtain but from Taylor expansion of $(\frac{\partial L'}{\partial \sigma})$ the approximate conditional bias in σ_{TNE} is

$$\{\frac{E(\frac{\partial L'}{\partial \sigma})}{R^2(\sigma)}\} \simeq B_3 = \{\frac{E(\frac{\partial L'}{\partial \sigma})}{R^2(\sigma)}\}_{\mu=\mu_{TNE}, g_i=g_{i_{TNE}}, \sigma=\sigma_{TNE}} \qquad (72)$$

where $R^2(\sigma) = -E(\frac{\partial^2 L'}{\partial \sigma^2})$ is given by (64).

Note: If follows from (42), (43), (50), (51), (52) and (57) that

$$(\frac{\sigma}{n})E(\frac{\partial L'}{\partial \mu}) = \sum_{i=1}^{k}\{[g_1(z_i')-(\alpha_{1i}+\beta_{1i}z_i')] - [g_2(z_i'')+(\alpha_{2i}+\beta_{2i}z_i'')]\} \qquad (73)$$

$$\simeq 0$$

$$(\tfrac{\sigma}{n})E(\frac{\partial L'}{\partial g_i}) = \{[g_1(z_i')-(\alpha_{1i}+\beta_{1i}z_i')] - [g_2(z_i")-(\alpha_{2i}+\beta_{2i}z_i")]\}$$
$$i = 1, 2, \ldots, k \qquad (74)$$
$$\simeq 0$$

and

$$(\tfrac{\sigma}{n})E(\frac{\partial L'}{\partial \sigma}) = \sum_{i=1}^{k}\{z_i'[g_1(z_i')-(\alpha_{1i}+\beta_{1i}z_i')] - z_i"[g_2(z_i")-(\alpha_{2i}+\beta_{2i}z_i")]\} \qquad (75)$$
$$\simeq 0 \ .$$

So it follows from (69), (71), (72), (73), (74) and (75) that the conditional bias in μ_{TNE}, $g_{i_{TNE}}$, and σ_{TNE} is directly proportional to the error in (42) and (43) and thus should be very small, at least for large n (i.e. large samples).

3.3.4 Asymptotic Efficiency of the Estimators

For n tending to infinity (i.e. asymptotically), the estimators μ_{TNE}, σ_{TNE} and $g_{i_{TNE}}$ are efficient. This can be shown as follows:

$$\lim_{n\to\infty} \beta_{1i} = \{\frac{d}{dz}[g_1(z_i)]\}_{z_i=z_i'} = \{\frac{d}{dz}[\frac{p(z_i)}{P(z_i")-P(z_i')}]\}_{z_i=z_i'}$$
$$= \{\frac{-z_i\ p(z_i)}{P(z_i")-P(z_i')}\}_{z_i=z_i'} = \frac{-z_i'p(z_i')}{P(z_i")-P(z_i')} \qquad (76)$$

and

$$\alpha_{1i} = g_1(z_i') - \beta_{1i}z_i' \quad . \qquad (77)$$

Therefore,

$$\lim_{n\to\infty}(\alpha_{1i}+\beta_{1i}z_i') = g_1(z_i') + \frac{z_i'^2 p(z_i')}{P(z_i'')-P(z_i')} - \frac{z_i'^2 p(z_i')}{P(z_i'')-P(z_i')} = g_1(z_i') \quad . \tag{78}$$

Similarly,

$$\lim_{n\to\infty}(\alpha_{2i}+\beta_{2i}z_i'') = g_2(z_i'') \quad . \tag{79}$$

Due to the above properties, the estimators (54), (55), and (56) are asymptotically identical with the ML estimators and are therefore asymptotically unbiased and efficient.

3.4.1 A "Test Statistic" for Testing Linear Contrasts

Since $g_{i_{TNE}} = K_i - \mu_{TNE}$, therefore

$$\mathrm{Var}(K_i|\mu,\sigma) = \mathrm{Var}(g_{i_{TNE}}|\mu,\sigma)$$

[i.e. the conditional variance of g_i for a given μ and σ].

Thus, asymptotically (n tending to infinity),

$$\begin{aligned} n\,\mathrm{Var}(K_i|\mu,\sigma) &= n\{-[\frac{1}{E(\frac{\partial^2 L'}{\partial g_i^2})}]\} \\ &= \frac{\sigma^2}{d_i} \end{aligned} \tag{80}$$

where $-[\frac{1}{E(\frac{\partial^2 L'}{\partial g_i^2})}]$ is the Cramer - Rao "minimum variance bound" for the $\mathrm{Var}(g_{i_{TNE}}|\mu,\sigma)$.

For small n, we have the approximation

$$\mathrm{Var}(K_i) \simeq \frac{\sigma^2}{nd_i},\ i = 1, 2, \ldots, k\ .$$

Now,

$$\begin{aligned} \mathrm{Var}(g_{i_{TNE}}|\sigma) &\simeq \mathrm{Var}(K_i - \mu_{TNE}) = \mathrm{Var}(K_i) - \mathrm{Var}(\mu_{TNE}) \\ &= \frac{\sigma^2}{nd_i} - \frac{\sigma^2}{nd} = \frac{\sigma^2}{n}(\frac{1}{d_i} - \frac{1}{d})\ , \end{aligned} \tag{81}$$

since $\mathrm{Cov}(K_i, \mu_{TNE}|\sigma) = \frac{\sigma^2}{nd}$.

Also ,

$$\begin{aligned}\mathrm{Cov}(\mu_{TNE}, g_{i_{TNE}}|\sigma) &= \mathrm{Var}(\mu_{TNE}) + \mathrm{Var}(g_{i_{TNE}}) - \mathrm{Var}(K_i) \\ &= \frac{\sigma^2}{nd} + \frac{\sigma^2}{nd_i} - \frac{\sigma^2}{nd} - \frac{\sigma^2}{nd_i} \\ &= 0 \ .\end{aligned}$$

To obtain the exact small sample variance of σ_{TNE} is extremely difficult. However, asymptotically (n tending to infinity) the conditional variance of σ_{TNE} is given by

$$n\ \mathrm{Var}(\sigma_{TNE}|\mu, g_i) = n\{\frac{-1}{E(\frac{\partial^2 L'}{\partial \sigma^2})}\}$$

$$= \sigma^2 / \sum_{i=1}^{k} \{2+3[z_i' g_1(z_i') - z_i'' g_2(z_i'')] - z_i'(2\alpha_{1i}+3\beta_{1i}z_i') + z_i''(2\alpha_{2i}+3\beta_{2i}z_i'')\} \ . \qquad (82)$$

Note:

$$E(\frac{\partial^2 L'}{\partial g_i^2}) = -\frac{n}{\sigma^2}\{1 - \beta_{1i} + \beta_{2i}\} = -(\frac{nd_i}{\sigma^2}) \ .$$

Therefore,

$$E(\frac{\partial^3 L'}{\partial g_i^3}) = 0$$

$$\vdots$$

$$E(\frac{\partial^r L'}{\partial g_i^r}) = 0, \quad r \geq 3 \ .$$

Since derivatives $E(\frac{\partial^r L'}{\partial g_i^r})$ are zero for all $r \geq 3$, one would

expect from Bartlett's results [1953][1] that for large n and hence large nd, $g_{i_{TNE}}$ is approximately normally distributed (see also Tiku [1967][14], 1973][17]). Therefore for large n, the linear contrast $\sum_{i=1}^{k} \ell_i g_{i_{TNE}}$ would be expected to be approximately normally distributed with mean $E\{\sum_{i=1}^{k} \ell_i g_{i_{TNE}}\} = \sum_{i=1}^{k} \{E(\ell_i g_{i_{TNE}})\} = \sum_{i=1}^{k} \ell_i g_i$, and variance

$$\mathrm{Var}\{\sum_{i=1}^{k} (\ell_i g_{i_{TNE}})\} = \sum_{i=1}^{k} \{\mathrm{Var}(\ell_i g_{i_{TNE}})\} = \sum_{i=1}^{k} \{\ell_i^2 \mathrm{Var}(g_{i_{TNE}})\}$$

$$= \sum_{i=1}^{k} \{\ell_i^2 [(\frac{\sigma^2_{TNE}}{n})(\frac{1}{d_i} - \frac{1}{d})]\} .$$

Because of the above results, it seemed quite reasonable to propose the test statistic

$$t = \frac{\sum_{i=1}^{k} \ell_i g_{i_{TNE}}}{\sqrt{\sum_{i=1}^{k} \ell_i^2 [\frac{\sigma^2_{TNE}}{n} (\frac{1}{d_i} - \frac{1}{d})]}}$$

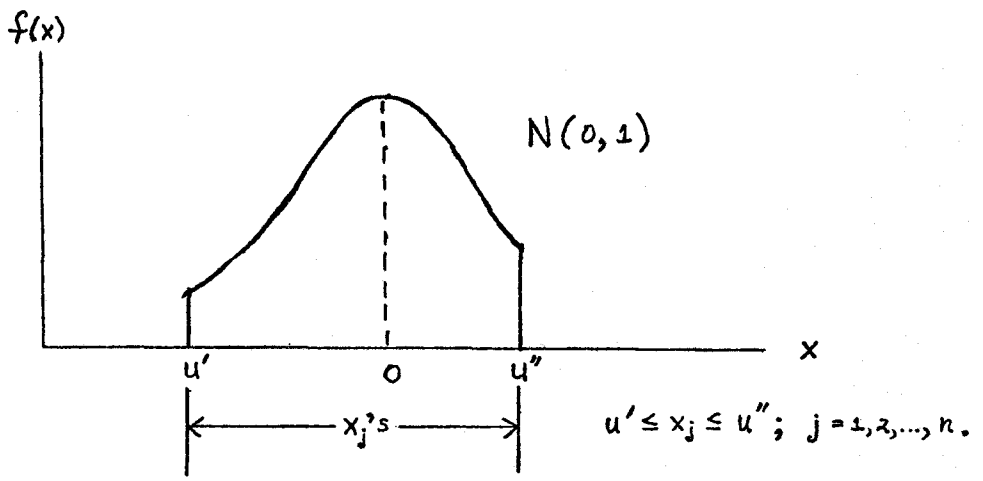
$$= \frac{\sum_{i=1}^{k} \ell_i g_{i_{TNE}}}{\sigma_{TNE} \sqrt{\frac{1}{n} \sum_{i=1}^{k} \ell_i^2 (\frac{1}{d_i} - \frac{1}{d})}} \qquad (83)$$

for testing the null hypothesis H_0: $g_1 = g_2 = \ldots = g_k$. Note that under H_0, $\sum_{i=1}^{k} \ell_i g_i = 0$ since $\sum_{i=1}^{k} \ell_i = 0$. For large n, the statistic t would be N(0,1). However, a Monte Carlo investigation revealed that the distribution of t does not approach normality even for sample sizes are large as 100. The percentage points of t have, therefore, to be simulated in practice. We have documented and included in this

thesis a Computer Programme to do that (see Appendix I).

3.4.2 Some Examples and Results

Using the computer programme referred to in the previous section, data were generated from a truncated normal, in the range $u' < x < u''$. For such a sample of size n, we have the observations $x_1, x_2, \ldots, x_n$, and $u' \leq x_j \leq u''$, for $j = 1, 2, \ldots, n$. So we have the following situation:



Example 1:

Suppose that the treatments T_1, T_2, and T_3 have been used in three such identical experiments (described above), each with n = 20, and the values of u' and u" for the three experiments given by $u_1' = u_2' = u_3' = P^{-1}(0.1)$ and $u_1'' = u_2'' = u_3'' = P^{-1}(0.8)$. So we have data for the three treatments given by the following three groups of twenty random observations generated from the truncated normal distri-

bution with $u_i' = P^{-1}(0.1)$ and $u_i'' = P^{-1}(0.8)$, i = 1, 2, 3.

T_1	T_2	T_3
.0128	.6701	- .2971
- .6027	- .5814	- .0929
.3503	- .1020	- .4941
- .3552	- .7006	.2242
.4700	- .2845	- .9574
.2890	.0273	- .4530
-1.0596	- .6332	- .4668
- .5068	.5464	- .1126
- .7308	- .6627	.3138
- .2564	- .1354	.5841
- .1508	.3174	- .7646
.2647	.5897	.6268
- .6963	.7312	-1.1717
- .7820	.5510	.0247
- .2647	- .3735	-1.0911
.4473	.4351	-1.0056
.4739	.4785	- .1377
- .1898	.2059	.8074
- .9388	.6589	.0547
- .6163	.3370	.7425

We assume model (33) for the above data and let g_i be the effect due to treatment T_i, i = 1, 2, 3. Here, we have the following:

	T_1	T_2	T_3
q_{1i}	.1000	.1000	.1000
q_{2i}	.2000	.2000	.2000
u_i'	-1.2817	-1.2817	-1.2817
u_i''	.8415	.8415	.8415
$\bar{u}_i$	- .2421	.1038	- .1833
α_{1i}	.2579	.0808	.3807
α_{2i}	.2278	.4971	.4275
β_{1i}	.0992	.0253	.1604
β_{2i}	- .0852	- .2342	- .1876
d_i	.8157	.7405	.6520

	T_1	T_2	T_3
A_i	- .2289	- .0824	- .2079
B_i	- .0369	.5622	.0718

Substituting these values in equations (54) to (56f), we get

$$A = 3$$
$$B = 1.9287$$
$$C = .0371$$
$$d = .8157 + .7405 + .6520 = 2.2082 \quad .$$

Therefore,

$$\sigma_{TNE} = \{1.9287 + \sqrt{(1.9287)^2 + 4.0(3.0)(.0371)}\}/6.0$$
$$= .6616$$
$$\mu_{TNE} = - .0438 \qquad \text{(Theoretic value is - .18)}$$
$$g_{1_{TNE}} = - .2095$$
$$g_{2_{TNE}} = .3333 \qquad \text{(Assumed values of } g_i\text{'s are zero)}$$
$$g_{3_{TNE}} = - .1165$$

The variances and covariances calculated from equations (59) to (67) are as follows:

(i) Var - Cov Matrix for μ_{TNE}, $g_{1_{TNE}}$, and σ_{TNE}

$$\begin{bmatrix} .01648 & - .01657 & .00230 \\ - .01657 & .04350 & - .00255 \\ .00230 & .00255 & .00691 \end{bmatrix}$$

(ii) Var - Cov Matrix for μ_{TNE}, $g_{2_{TNE}}$, and σ_{TNE}

$$\begin{bmatrix} .01491 & -.01487 & .00008 \\ -.01487 & .04662 & .00390 \\ .00008 & .00390 & .00708 \end{bmatrix}$$

(iii) Var - Cov Matrix for μ_{TNE}, $g_{3_{TNE}}$, and σ_{TNE}

$$\begin{bmatrix} .01448 & -.01436 & .00168 \\ -.01436 & .04785 & -.00120 \\ .00168 & -.00120 & .00679 \end{bmatrix}$$

From equations (76) to (79), the asymptotic values of the α_{ij}'s and β_{ij}'s were determined and are given as follows:

	T_1	T_2	T_3
$\lim_{n\to\infty}[\alpha_{1i}]$	.4570	.2001	.4128
$\lim_{n\to\infty}[\alpha_{2i}]$	.4258	.6054	.4691
$\lim_{n\to\infty}[\beta_{1i}]$	.2080	.0716	.1807
$\lim_{n\to\infty}[\beta_{2i}]$	-.1885	-.2978	-.2157

Example 2:

Suppose that we have the same experimental design structure as in Example 1 using three treatments T_1, T_2, and T_3 with truncation points $u_1' = u_2' = u_3' = P^{-1}(0.1)$ and $u_1'' = u_2'' = u_3'' = P^{-1}(0.8)$. However, this time, each treatment group is composed of 100 observa-

tions (i.e. n = 100). Thus, we have data for the three treatments given by three groups of 100 random observations generated from the truncated normal distribution with $u_i' = P^{-1}(0.1)$ and $u_i'' = P^{-1}(0.8)$, i = 1, 2, 3. For simplicity, the actual data point values will be ommitted here.

In the same manner as the previous example, we assume model (33) for this data and let g_i be the effect due to treatment T_i, i = 1, 2, 3. In this case, we have the following results:

	T_1	T_2	T_3
q_{1i}	.1000	.1000	.1000
q_{2i}	.2000	.2000	.2000
u_i'	-1.2817	-1.2817	-1.2817
u_i''	.8415	.8415	.8415
$\bar{u}_i$	- .1439	- .0974	- .1826
α_{1i}	.2638	.2610	.2771
α_{2i}	.3677	.4248	.3286
β_{1i}	.1030	.1014	.1097
β_{2i}	- .1572	- .1902	- .1360
d_i	.7398	.7085	.7543
A_i	- .1948	- .1799	- .2075
B_i	.1403	.2312	.0683

Substituting these values in equations (54) to (56f), we get

$$A = 3$$
$$B = 2.0316$$
$$C = .0409$$
$$d = .7398 + .7085 + .7543 = 2.2026$$

Therefore,

$$\sigma_{TNE} = \{2.0316 + \sqrt{(2.0316)^2 + 4.0(3.0)(.0409)}\}/6.0$$

$$= .6968$$

$$\mu_{TNE} = -.0934 \quad \text{(Theoretic value is } -.18\text{)}$$

$$g_{1_{TNE}} = -.0036$$

$$g_{2_{TNE}} = .0746 \quad \text{(Assumed values of } g_i \text{ are zero)}$$

$$g_{3_{TNE}} = -.0666$$

The variances and covariances calculated from equations (59) to (67) are as follows:

(i) Var - Cov Matrix for μ_{TNE}, $g_{1_{TNE}}$, and σ_{TNE}

$$\begin{bmatrix} .00335 & -.00332 & .00023 \\ -.00332 & .00988 & -.00001 \\ .00023 & -.00001 & .00157 \end{bmatrix}$$

(ii) Var - Cov Matrix for μ_{TNE}, $g_{2_{TNE}}$, and σ_{TNE}

$$\begin{bmatrix} .00327 & -.00323 & .00016 \\ -.00323 & .01013 & .00020 \\ .00016 & .00020 & .00157 \end{bmatrix}$$

(iii) Var - Cov Matrix for μ_{TNE}, $g_{3_{TNE}}$, and σ_{TNE}

$$\begin{bmatrix} .00340 & -.00339 & .00029 \\ -.00339 & .00980 & -.00018 \\ .00029 & -.00018 & .00157 \end{bmatrix}$$

From equations (76) to (79), the asymptotic values of the α_{ij}'s and β_{ij}'s are given as follows:

	T_1	T_2	T_3
$\lim_{n\to\infty} [\alpha_{1i}]$	.4221	.3861	.4502
$\lim_{n\to\infty} [\alpha_{2i}]$	.5231	.5487	.4999
$\lim_{n\to\infty} [\beta_{1i}]$	.1845	.1633	.2018
$\lim_{n\to\infty} [\beta_{2i}]$	- .2504	- .2683	- .2344

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APPENDIX I

LISTING OF THE FORTRAN PROGRAM

Note:

Three external subroutines used in the program and the installations which produced them are:

Subroutine	Publication
FRANDN	McMaster Internal Publication (MILIS), McMaster University Computing Centre, Hamilton, Ontario.
NDTR NDTRI	IBM System/360 Scientific Subroutine Package (SSP), (360A-CM-03X) Version III, *Programmer's Mannual*, IBM Technical Publications Dept., 112 East Post Rd., White Plains, N.Y.

```
C.....................................................................
C
C
C       MAIN PROGRAM
C
C       PURPOSE
C         THIS PROGRAM GENERATES N TRUNCATED NORMAL VARIATES FOR
C         SPECIFIED UPPER AND LOWER BOUNDS OF THE N(0,1) DISTRIBUTION.
C         WITH THESE VARIATES GENERATED FOR A SPECIFIED K NUMBER OF
C         SAMPLES, THE PROGRAM THEN CALCULATES ALL THE NECESSARY
C         QUANTITIES IN ORDER TO OBTAIN THE ESTIMATED MEAN, S.D., AND
C         TREATMENT EFFECTS FOR ANY ONE-WAY-CLASSIFICATION EXPERIMENT-
C         AL DESIGN WITH TYPE I CENSORING(I.E. TRUNCATED NORMAL)
C         APPLIED TO IT.  THE PROGRAM ALSO CALCULATES THE VARIANCE-
C         COVARIANCE MATRICES FOR THE MEAN, STANDARD DEVIATION AND
C         EACH TREATMENT EFFECT AS WELL AS THE ASYMPTOTIC VALUES FOR
C         THE ALFA(1I)'S, ALFA(2I)'S, BETA(1I)'S AND BETA(2I)'S WHICH
C         ARE USED IN THE LINEAR APPROXIMATION TO THE MAXIMUM LIKELI-
C         HOOD EQUATIONS.  FINALLY, THE PROGRAM SIMULATES THE PERCENT-
C         AGE POINTS OF THE STATISTIC T PROPOSED IN SECTION 3.4.1 OF
C         THIS THESIS.
C
C       SUBROUTINE AND FUNCTION SUBPROGRAMS REQUIRED
C         ASYMPT      - INTERNAL SUBROUTINE SUBPROGRAM
C         DENORMC     - INTERNAL FUNCTION SUBPROGRAM
C         FRANDN      - EXTERNAL SUBROUTINE SUBPROGRAM
C         GINV        - INTERNAL SUBROUTINE SUBPROGRAM
C         KOUNT       - INTERNAL SUBROUTINE SUBPROGRAM
C         NDTR        - EXTERNAL SUBROUTINE SUBPROGRAM
C         NDTRI       - EXTERNAL SUBROUTINE SUBPROGRAM
C         P           - INTERNAL FUNCTION STATEMENT
C         VARCOV      - INTERNAL SUBROUTINE SUBPROGRAM
C         WRITIT      - INTERNAL SUBROUTINE SUBPROGRAM
C
C       SOURCE LANGUAGE
C         FORTRAN
C
C       ACCURACY
C         13 OR 14 SIGNIFICANT FIGURES.  HOWEVER, VALUES ARE ALL
C         ROUNDED TO 8 DECIMAL PLACES OR LESS FOR PRINT OUT PURPOSES.
C
C
C                                  ..........
```

```
C
C       DESCRIPTION OF EXTERNAL ROUTINES EMPLOYED
C
C       SUBROUTINE FRANDN
C
C       PURPOSE
C         TO GENERATE A STRING OF PSEUDO-RANDOM FLOATING POINT
C         NUMBERS WHICH ARE UNIFORMLY DISTRIBUTED IN THE INTEGRAL (0,1)
C
C       CALL AND ARGUMENT LIST
C         CALL FRANDN(A,N,M) WHERE
C         A IS THE ARRAY IN WHICH THE GENERATED NUMBERS WILL BE
C         STORED, STARTING AT A(1)
C         N IS THE NUMBER OF ELEMENTS TO BE GENERATED
C         M IS AN INTEGER USED TO START THE STRING IF IT IS NON-ZERO
C         AND AN INDICATOR THAT THE STRING IS TO BE CONTINUED FROM THE
C         LAST VALUE IF M IS ZERO.
C
C       METHOD
C         MULTIPLY THE LAST NUMBER BY A GIVEN CONSTANT, THE RESULT
C         BEING TRUNCATED MOD 2.
C
C       MISC. COMMENTS
C         EVERY TIME THIS ROUTINE IS LOADED FROM THE LIBRARY, A
C         CONSTANT INITIAL VALUE IS ASSUMED, SO THAT STARTING WITH M=0
C         WILL ALWAYS GENERATE THE SAME SEQUENCE OF NUMBERS.  USUALLY
C         THE FIRST CALL SHOULD HAVE M ≠ 0, AND THE OTHERS M=0.
C
C       STORAGE USED
C         34 OCTAL WORDS
C
C       SUBROUTINES CALLED
C         NONE
C
C       TYPICAL TIMINGS
C         .8 SEC. TO GENERATE 5000
C
C                                  ..........
C
C       SUBROUTINE NDTR
C
C       PURPOSE
C         COMPUTES Y = P(X) = PROBABILITY THAT THE RANDOM VARIABLE
C         U, DISTRIBUTED NORMALLY(0,1), IS LESS THAN OR EQUAL TO X.
C         F(X), THE ORDINATE OF THE NORMAL DENSITY AT X, IS ALSO COM-
```

```
C          PUTED.
C        USAGE
C          CALL NDTR(X,P,D)
C
C        DESCRIPTION OF PARAMETERS
C          X - INPUT SCALAR FOR WHICH P(X) IS COMPUTED.
C          P - OUTPUT PROBABILITY.
C          D - OUTPUT DENSITY.
C
C        REMARKS
C          MAXIMUM ERROR IS 0.0000007.
C
C        SUBROUTINES AND SUBPROGRAMS REQUIRED
C          NONE
C
C        METHOD
C          BASED ON APPROXIMATIONS IN C. HASTINGS, APPROXIMATIONS FOR
C          DIGITAL COMPUTERS, PRINCETON UNIV. PRESS, PRINCETON, N.J.,
C          1955.  SEE EQUATION 26.2.17, HANDBOOK OF MATHEMATICAL
C          FUNCTIONS, ABRAMOWITZ AND STEGUN, DOVER PUBLICATIONS, INC.,
C          NEW YORK.
C*********************************************************************
      SUBROUTINE NDTR(X,P,D)
C
      AX=ABS(X)
      T=1.0/(1.0+.2316419*AX)
      D=0.3989423*EXP(-X*X/2.0)
      P=1.0 - D*T*((((1.330274*T - 1.821256)*T + 1.781478)*T -
     *   0.3565638)*T + 0.3193815)
      IF(X) 1,2,2
    1 P=1.0-P
    2 RETURN
      END
C
C                                     **********
C
C        SUBROUTINE NDTRI
C
C        PURPOSE
C          COMPUTES X = P**(-1)(Y), THE ARGUMENT X SUCH THAT Y= P(X) =
C          THE PROBABILITY THAT THE RANDOM VARIABLE U, DISTRIBUTED
C          NORMALLY(0,1), IS LESS THAN OR EQUAL TO X.  F(X), THE
C          ORDINATE OF THE NORMAL DENSITY, AT X, IS ALSO COMPUTED.
```

```
C
C       USAGE
C         CALL NDTRI(P,X,D,IER)
C
C       DESCRIPTION OF PARAMETERS
C         P   - INPUT PROBABILITY.
C         X   - OUTPUT ARGUMENT SUCH THAT P = Y = THE PROBABILITY THAT
C                 U, THE RANDOM VARIABLE, IS LESS THAN OR EQUAL TO X.
C         D   - OUTPUT DENSITY, F(X).
C         IER - OUTPUT ERROR CODE
C               = -1 IF  P IS NOT IN THE INTERVAL (0,1), INCLUSIVE.
C                  X=0=.99999E+74 IN THIS CASE
C               = 0 IF THERE IS NO ERROR.  SEE REMARKS, BELOW.
C
C       REMARKS
C         MAXIMUM ERROR IS 0.00045.
C         IF P = 0, X IS SET TO -(10)**74.  D IS SET TO 0.
C         IF P = 1, X IS SET TO  (10)**74.  D IS SET TO 0.
C
C       SUBROUTINES AND SUBPROGRAMS REQUIRED
C         NONE
C
C       METHOD
C         BASED ON APPROXIMATIONS IN C. HASTINGS, APPROXIMATIONS FOR
C         DIGITAL COMPUTERS, PRINCETON UNIV. PRESS, PRINCETON, N.J.,
C         1955.  SEE EQUATION 26.2.23, HANDBOOK OF MATHEMATICAL
C         FUNCTIONS, ABRAMOWITZ AND STEGUN, DOVER PUBLICATIONS, INC.,
C         NEW YORK.
C.....................................................................
      SUBROUTINE NDTRI(P,X,D,IE)
C
      IE=0
      X=.99999E+74
      D=X
      IF(P)1,4,2
    1 IE=-1
      GO TO 12
    2 IF (P-1.0)7,5,1
    4 X=-.99999E+74
    5 D=0.0
      GO TO 12
C
```

```
      7 D=P
        IF(D-0.5)9,9,8
      8 D=1.0-D
      9 T2=ALOG(1./(D*D))
        T=SQRT(T2)
        X=T-(2.515517+0.802853*T+0.010328*T2)/(1.0+1.432788*T+0.189269*T2
     1    +0.001308*T*T2)
        IF(P-0.5)10,10,11
     10 X=-X
     11 D=0.3989423*EXP(-X*X/2.0)
     12 RETURN
        END
C
C*****************************************************************
C
C*****************************************************************
C
C       DESCRIPTION OF ARRAYS USED
C
C       NOTE:  I = 1,...,K WHERE K IS THE NUMBER OF GROUPS
C          Q1(I)      - PROPORTION OF CENSORING ON THE LEFT SIDE FOR THE
C                       ITH GROUP
C          Q2(I)      - PROPORTION OF CENSORING ON THE RIGHT SIDE FOR THE
C                       ITH GROUP
C          U1(I)      - U1(I) = P**-1(Q1(I))
C          U2(I)      - U2(I) = P**-1(Q2(I))
C          XBAR(I)    - MEAN OF THE OBSERVATIONS FOR THE ITH GROUP
C          SISQ(I)    - VARIANCE OF THE OBSERVATIONS FOR THE ITH GROUP
C          H(I)       - (MEAN + S.E.) FOR THE ITH GROUP
C          K1(I)      - (MEAN - S.E.) FOR THE ITH GROUP
C          SSQ(I)     - SUM OF THE SQUARES OF THE OBSERVATIONS FOR THE
C                       ITH GROUP
C          A1(I)      - G1(H1I), GIVEN IN EQN. (46) OF THE THESIS
C          A2(I)      - G1(K1I), GIVEN IN EQN. (46) OF THE THESIS
C          B1(I)      - BETA(1I), GIVEN BY EQN. (46) OF THE THESIS
C          B2(I)      - BETA(2I), GIVEN BY EQN. (48) OF THE THESIS
C          ALFA1(I)   - ALFA(1I), GIVEN BY EQN. (47) OF THE THESIS
C          ALFA2(I)   - ALFA(2I), GIVEN BY EQN. (49) IN THE THESIS
C          C1(I)      - G2(H1I), SIMILAR TO G1(H1I) USING ALFA(2I) AND
C                       BETA(2I)
C          C2(I)      - G2(K1I), SIMILAR TO G1(K1I) USING ALFA(2I) AND
C                       BETA(2I)
C          D1(I)      - GIVEN BY EQN. (56A) IN THE THESIS
C          L1(I)      - (ALFA(2I) - ALFA(1I)) FOR THE ITH GROUP
```

```
C           KI(I)      - GIVEN BY EQN. (56F) IN THE THESIS
C           GI(I)      - THE ITH TREATMENT EFFECT
C           Z1(I)      - GIVEN BY EQN. (35) OF THE THESIS
C           Z2(I)      - GIVEN BY EQN. (35) OF THE THESIS
C           X(I)       - GENERATED TRUNCATED NORMAL VARIATES SUCH THAT
C                        ( U1(I) ≤ X ≤ U2(I) ) FOR THE ITH GROUP
C           P11(I)     - P11(I) = P(U1(I)) FOR THE ITH GROUP
C           P22(I)     - P22(I) = P(U2(I)) FOR THE ITH GROUP
C           T(J)       - TEST STATISTIC FOR THE JTH EXPERIMENTAL DESIGN
C                        GIVEN BY EQN. (83) IN THE THESIS
C           M(I)       - L(I) TERMS IN EQN. (83) IN THE THESIS (I.E.
C                        COEFFICIENTS FOR THE LINEAR CONTRAST TESTING)
C
C*************************************************************************
C
      REAL KI1,KI2,LI,KI
      DIMENSION Q1(10),Q2(10),U1(10),U2(10),XBAR(10),SISQ(10),H(10),
     - KI1(10),SSQ(10),A1(10),A2(10),B1(10),B2(10),ALFA1(10),ALFA2(10),
     -                                  C1(10),C2(10),DI(10),LI(10),
     - KI(10)
      DIMENSION GI(10),Z1(10),Z2(10)
      DIMENSION X(600)
      DIMENSION P11(10),P22(10)
      COMMON T(10,100)
      DIMENSION H2(10),KI2(10)
      EQUIVALENCE(NQ,QN),(H(1),H2(1)),(KI1(1),KI2(1))
      REAL M(10),MK
C
C FUNCTION TO COMPUTE THE NORMAL P.D.F. VALUE FOR A GIVEN EX
      P(EX)=1.0/SQRT(2.0*4.0*ATAN(1.0))*EXP(-0.5*EX**2)
C
C INITIALIZE THE SUM AND SUM OF SQUARES VARIABLES FOR THE TEST STATISTIC
C TO ZERO
      SUMT=SUMTSQ=0.0
C
C READ THE NUMBER OF GROUPS IN THE EXPERIMENTAL DESIGN AND THE PROPOR-
C TION OF CENSORING ON THE LEFT AND ON THE RIGHT FOR EACH OF THE ITH
C GROUPS
      READ(5,9) NQ,((Q1(I),Q2(I)),I=1,NQ)
    9 FORMAT(I5,15F5.3)
C
      WRITE(6,11) ((Q1(I),Q2(I)),I=1,NQ)
   11  ORMAT(*1      Q1(I)      Q2(I)*/(1X,2F10.4))
C
C DETERMINE THE VALUES OF U1(I) AND U2(I) FOR EACH OF THE ITH GROUPS
```

```
C USING THE RESPECTIVE Q1(I) AND Q2(I) VALUES
      DO 20 I=1,NQ
      CALL NDTRI(Q1(I),U1(I),C,IER)
      QT=1.0-Q2(I)
      CALL NDTRI(QT,U2(I),C,IER)
   20 CONTINUE
C
      WRITE(6,25) ((Q1(I),U1(I),Q2(I),U2(I)),I=1,NQ)
   25 FORMAT(*0Q1,U1,Q2,U2 */(2(5X,2F10.5)))
C
C
C READ THE NUMBER OF EXPERIMENTAL DESIGNS AND THE NUMBER OF OBSERVATIONS
C PER ITH GROUP IN THE DESIGN
      READ(5,30) NN,N,NK
   30 FORMAT(16I5)
      WRITE(6,31) NN,N,NK
   31 FORMAT(///1X,16I5)
C
C READ THE COEFFICIENTS TO BE USED IN THE TEST STATISTIC FOR TESTING
C ON LINEAR CONTRASTS.  THESE VALUES SHOULD SUM TO ZERO .
      READ(5,10) (M(I),I=1,NQ)
   10 FORMAT(16F5.2)
C
C
C NOW GENERATE LP RANDOM NUMBERS TO BE STORED IN X SUCH THAT
C ( U1(I) ≤ X ≤ U2(I) ) FOR THE ITH GROUP. (I.E. THESE RANDOM VARIATES ARE
C GENERATED FROM THE TRUNCATED NORMAL DISTRIBUTION WITH UPPER AND LOWER
C BOUNDS U2(I) AND U1(I) FOR THE ITH GROUP RESPECTIVELY
      CALL FRANDN(X,1,1)
      LP=N*NQ
      DO 1001 K=1,NQ
      CALL NDTR(U1(K),P11(K),D)
      CALL NDTR(U2(K),P22(K),D)
 1001 CONTINUE
      DO 400 IJK=1,NN
      CALL FRANDN(X,LP,0)
      DO 40 I=1,LP
      XJ=FLOAT(I)/FLOAT(N)+0.998
      J=INT(XJ)
      PRBVAL=X(I)*(P22(J)-P11(J))+P11(J)
      CALL NDTRI(PRBVAL,X(I),C,IER)
   40 CONTINUE
      WRITE(6,1999)(X(I),I=1,LP)
 1999 FORMAT(1H0,10F10.4)
C
```

```
C
      DO 60 I=1,NQ
C
C  CALCULATE THE MEAN AND VARIANCE FOR EACH OF THE ITH GROUPS IN THE
C  EXPERIMENTAL DESIGN
      SUM=SUMSQ=0.0
      J1=I*N-N+1
      J2=I*N
      DO 50 I2=J1,J2
      SUM=SUM+X(I2)
   50 SUMSQ=SUMSQ+X(I2)**2
      SSQ(I)=SUMSQ
      XBAR(I)=SUM/FLOAT(N)
      SUMSQ=SUMSQ-SUM**2/FLOAT(N)
      SISQ(I)=SUMSQ/FLOAT(N-1)
      SISQ(I)=SQRT(SISQ(I))
C
C
C  DETERMINE THE VALUES OF ALFA(1I) AND ALFA(2I) FOR THE ITH GROUP BY
C  USING EQNS. (46) AND (47) IN SECTION 3.2.2 OF THE THESIS
      TEMP=SISQ(I)/SQRT(FLOAT(N))
      H(I)=XBAR(I)+TEMP
      KI1(I)=XBAR(I)-TEMP
      DENOM=DENOMC(U2(I),U1(I),H(I),SISQ(I))
      A1(I)=P((U1(I)-H(I))/SISQ(I))/DENOM
      DENOM=DENOMC(U2(I),U1(I),KI1(I),SISQ(I))
      A2(I)=P((U1(I)-KI1(I))/SISQ(I))/DENOM
      B1(I)=(A1(I)-A2(I))*SISQ(I)/(KI1(I)-H(I))
      ALFA1(I)=A1(I)-B1(I)*(U1(I)-H(I))/SISQ(I)
C
C
C  DETERMINE THE VALUES OF ALFA(2I) AND BETA(2I) FOR THE ITH GROUP BY
C  USING EQNS. (48) AND (49) IN SECTION 3.2.2 OF THE THESIS
      DENOM=DENOMC(U2(I),U1(I),H2(I),SISQ(I))
      C1(I)=P((U2(I)-H2(I))/SISQ(I))/DENOM
      DENOM=DENOMC(U2(I),U1(I),KI2(I),SISQ(I))
      C2(I)=P((U2(I)-KI2(I))/SISQ(I))/DENOM
      B2(I)=(C1(I)-C2(I))*SISQ(I)/(KI2(I)-H2(I))
      ALFA2(I)=C1(I)-B2(I)*(U2(I)-H2(I))/SISQ(I)
C
C
C  COMPUTE D(I) FOR THE ITH GROUP GIVEN BY EQN. (56A) IN SECTION 3.3.1 OF
C  THE THESIS
      DI(I)=1.0+B2(I)-B1(I)
C
```

```
C COMPUTE ALFA(2I) - ALFA(1I) FOR THE ITH GROUP
      LI(I)=ALFA2(I)-ALFA1(I)
C
C CALCULATE K(I) GIVEN BY EQN.(56F) IN SECTION 3.3.1 OF THE THESIS
      KI(I)=(XBAR(I)+B2(I)*U2(I)-B1(I)*U1(I))/DI(I)
C
   60 CONTINUE
C
C
C
C PRINT OUT ALL THE INFORMATION COMPUTED ABOVE
      XNAME=6HQ1
      WRITE(6,1111)XNAME,(Q1(I),I=1,NQ)
 1111 FORMAT(1H0,A6,10F10.4)
      XNAME=6HQ2
      WRITE(6,1111)XNAME,(Q2(I),I=1,NQ)
      XNAME=6HU1
      WRITE(6,1111)XNAME,(U1(I),I=1,NQ)
      XNAME=6HU2
      WRITE(6,1111)XNAME,(U2(I),I=1,NQ)
      XNAME=6HXBAR
      WRITE(6,1111)XNAME,(XBAR(I),I=1,NQ)
      XNAME=6HSISQ
      WRITE(6,1111)XNAME,(SISQ(I),I=1,NQ)
      XNAME=6HHI
      WRITE(6,1111)XNAME,(H(I),I=1,NQ)
      XNAME=6HKI
      WRITE(6,1111)XNAME,(KI1(I),I=1,NQ)
      XNAME=6HSSQ
      WRITE(6,1111)XNAME,(SSQ(I),I=1,NQ)
      XNAME=6HA1
      WRITE(6,1111)XNAME,(A1(I),I=1,NQ)
      XNAME=6HA2
      WRITE(6,1111)XNAME,(A2(I),I=1,NQ)
      XNAME=6HB1
      WRITE(6,1111)XNAME,(B1(I),I=1,NQ)
      XNAME=6HB2
      WRITE(6,1111)XNAME,(B2(I),I=1,NQ)
      XNAME=6HALFA1
      WRITE(6,1111)XNAME,(ALFA1(I),I=1,NQ)
      XNAME=6HALFA2
      WRITE(6,1111)XNAME,(ALFA2(I),I=1,NQ)
      XNAME=6HC1
      WRITE(6,1111)XNAME,(C1(I),I=1,NQ)
      XNAME=6HC2
```

```
      WRITE(6,1111)XNAME,(C2(I),I=1,NQ)
      XNAME=6HDI
      WRITE(6,1111)XNAME,(DI(I),I=1,NQ)
      XNAME=6HLI
      WRITE(6,1111)XNAME,(LI(I),I=1,NQ)
      XNAME=6HKI
      WRITE(6,1111)XNAME,(KI(I),I=1,NQ)
C
C
C
C  SUM THE D(I) TERMS COMPUTED ABOVE FOR THE I GROUPS
       D=0.0
       DO 80 I=1,NQ
   80  D=D+DI(I)
C
C  CALCULATE THE MODIFIED MAXIMUM LIKELIHOOD ESTIMATOR OF THE STANDARD
C  DEVIATION FOR THE EXPERIMENTAL DESIGN USING EQN. (56) IN SECTION 3.3.1
C  OF THE THESIS
       F=0.0
       DO 90 I=1,NQ
       F=F+ALFA2(I)*U2(I)-ALFA1(I)*U1(I)-LI(I)*KI(I)
       E=E+SSQ(I)/FLOAT(N)+B2(I)*U2(I)**2-B1(I)*U1(I)**2-DI(I)*KI(I)**2
   90  CONTINUE
       F=F/FLOAT(NQ)
       E=E/FLOAT(NQ)
      XNAME=6HF
      WRITE(6,72)XNAME,F
   72  FORMAT(1X,A6,6E20.4)
      XNAME=6HE
      WRITE(6,72)XNAME,E
C
       SIGMA=(F+SQRT(F**2+4.0*E))/2.0
      XNAME=6HSIGMA
      WRITE(6,72)XNAME,SIGMA
C
C
C  COMPUTE THE MODIFIED MAXIMUM LIKELIHOOD ESTIMATOR OF THE MEAN FOR THE
C  EXPERIMENTAL DESIGN USING EQN. (54) IN THE THESIS
      SUMXM=0.0
      DO 1191 I=1,NQ
      SUMXM=SUMXM+ DI(I)*KI(I) + LI(I)*SIGMA
 1191 CONTINUE
      XMNTNE=SUMXM/D
      XNAME=6HMEAN
      WRITE(6,72)XNAME,XMNTNE
```

```
C
C
C  COMPUTE THE TREATMENT EFFECT FOR EACH OF THE ITH GROUPS IN THE EXPER-
C  IMENTAL DESIGN USING EQN. (55) IN SECTION 3.3.1 OF THE THESIS
       DO 1192 I=1,NQ
       GI(I)=KI(I) + ((LI(I)*SIGMA)/DI(I)) - XMNTNE
 1192  CONTINUE
       XNAME=6HGI
       WRITE(6,777)XNAME,(GI(I),I=1,NQ)
  777  FORMAT(1HC,A6,10F10.4)
C
C
C  CALL ON SUBROUTINE VARCOV TO COMPUTE THE VARIANCES-COVARIANCES OF
C  THESE MODIFIED MAXIMUM LIKELIHOOD ESTIMATORS(ABOVE) USING THE
C  THEORETICAL RESULTS IN SECTION 3.3.2 OF THE THESIS
       CALL VARCOV(N,D,SIGMA,NQ,ALFA1,ALFA2,U1,U2,XMNTNE,GI,DI,Z1,Z2)
C
C  CALL ON SUBROUTINE ASYMPT TO COMPUTE THE ASYMPTOTIC VALUES OF ALFA(1I)
C  ALFA(2I), BETA(1I), AND BETA(2I), I = 1,...,K WHERE K IS THE NUMBER OF
C  TREATMENT GROUPS IN THE EXPERIMENTAL DESIGN.  THESE VALUES ARE CALCU-
C  LATED USING THE THEORETICAL RESULTS OF SECTION 3.3.4 IN THE THESIS
       CALL ASYMPT(NQ,Z1,Z2)
C
C  COMPUTE THE TEST STATISTIC PROPOSED IN SECTION 3.4.1, EQN. (83) OFTHE
C  THESIS.  THIS TESTS ON LINEAR CONTRASTS FOR THE TREATMENT EFFECTS.
        MK=0.0
       DO 95 I=1,NQ
    95 MK=MK+M(I)*KI(I)
       SUM3=0.0
       SUM=0.0
       DO 100 I=1,NQ
       SUM3=SUM3+M(I)*LI(I)/DI(I)
  100  SUM=SUM+     (M(I)**2*(1.0/DI(I)-1.0/D))
       SUM1=SQRT(SUM/FLOAT(N))
       SUM21=SUM3/SUM1
       SUM=SQRT(SUM)
       T(IJK)=MK/(SIGMA/SQRT(FLOAT(N))*SUM)
      T(IJK)=T(IJK)+SUM21
      WRITE(6,110)IJK,T(IJK)
  110  FORMAT(////*  IJK=*,I5,*  T(IJK)=*,E20.4)
C
C  OBTAIN THE SUM AND SUM OF SQUARES OF THE TEST STATISTICS COMPUTED
       SUMT=SUMT+T(IJK)
       SUMTSQ=SUMTSQ+T(IJK)**2
C
```

```
  400  CONTINUE
C
CCC
C  COMPUTE THE MEAN AND VARIANCE OF THE TEST STATISTICS CALCULATED
       XNAME=6HSUMT
       WRITE(6,72)  XNAME,SUMT
       XNAME=6HSUMTSQ
       WRITE(6,72)  XNAME,SUMTSQ
       TMEAN=SUMT/FLOAT(NN)
       TVAR=(SUMTSQ-SUMT**2/FLOAT(NN))/FLOAT(NN-1)
       XNAME=6HTMEAN
       WRITE(6,72)  XNAME,TMEAN
       TMEAN=TMEAN-SUM21
       WRITE(6,72)  XNAME,TMEAN
       XNAME=6HTVAR
       WRITE(6,72)  XNAME,TVAR
C
CC CALL ON SUBROUTINE KOUNT TO PERFORM A MONTE CARLO INVESTIGATION ON
CCC THESE TEST STATISTICS.  HERE ONE OBTAINS THE MONTE CARLO PROPORTIONS
C  OF THE VALUES OF THE TEST STATISTIC T FOR SPECIFIED PERCENTAGE POINTS
       CALL KOUNT(T,NN,LN)
C
CC
       STOP
       END
C
CCCCC
CCCCC     SUBROUTINE VARCOV(N,D,SIGMA,NQ,ALFA1,ALFA2,U1,U2,XMNTNE,GI,DI,
CCCCC    *Z1,Z2)
CCCCC*****************************************************************
CCCCC         PURPOSE
CCCCC           TO OBTAIN THE VARIANCES AND COVARIANCES OF THE MAXIMUM LIKE-
CCCCC           LIHOOD ESTIMATORS
CCCCC
CCCCC         METHOD
CCCCC           THE THEORETICAL RESULTS OF SECTION 3.3.2 IN THE THESIS ARE
CCCCC           EMPLOYED
CCCCC
CCCCC         INPUT PARAMETERS
CCCCC           N           - NUMBER OF OBSERVATIONS IN EACH TREATMENT GROUP
C               D           - SUM OF THE D(I) TERMS FOR THE TREATMENT GROUPS
```

```
C          SIGMA      - MODIFIED MAXIMUM LIKELIHOOD ESTIMATOR OF THE
C                       STANDARD DEVIATION FOR THE EXPERIMENTAL DESIGN
C          NQ         - NUMBER OF TREATMENT GROUPS IN THE EXPERIMENTAL
C                       DESIGN
C          ALFA1(I)   - ALFA(1I), GIVEN BY EQN. (47) OF THE THESIS
C          ALFA2(I)   - ALFA(2I), GIVEN BY EQN. (49) IN THE THESIS
C          U1(I)      - U1(I) = P**-1(Q1(I)) WHERE Q1(I) IS THE PROPORTION
C                       OF CENSORING ON THE LEFT FOR THE ITH GROUP
C          U2(I)      - U2(I) = P**-1(Q2(I)) WHERE Q2(I) IS THE PROPORTION
C                       OF CENSORING ON THE RIGHT FOR THE ITH GROUP
C          XMNTNE     - THE MODIFIED MAXIMUM LIKELIHOOD ESTIMATOR FOR THE
C                       MEAN IN THE EXPERIMENTAL DESIGN
C          GI(I)      - THE MODIFIED MAXIMUM LIKELIHOOD ESTIMATORS OF THE
C                       TREATMENT EFFECTS IN THE EXPERIMENTAL DESIGN
C          DI(I)      - GIVEN BY EQN. (56A) IN THE THESIS
C
C        OUTPUT PARAMETERS
C          Z1(I)      - GIVEN BY EQN. (35) OF THE THESIS
C          Z2(I)      - GIVEN BY EQN. (35) OF THE THESIS
C*************************************************************************
C
C
      SUBROUTINE VARCOV(N,D,SIGMA,NQ,ALFA1,ALFA2,U1,U2,XMNTNE,GI,DI,
     *Z1,Z2)
      DIMENSION ALFA1(10),ALFA2(10),U1(1),U2(1),GI(1),DI(1),XJ(9),
     *IROW(10),ICOL(10),IWORK(10),JWORK(10),VARC(9),Z1(10),Z2(10),B(10)
      XN=FLOAT(N)
      SUMALF=0.0
      SUMSIG=0.0
      X11=(XN*D)/(SIGMA**2)
      DO 1 I=1,NQ
      SUMALF=SUMALF+(ALFA1(I) - ALFA2(I))
      Z1(I)=(U1(I) - XMNTNE - GI(I))/SIGMA
      Z2(I) = (U2(I) - XMNTNE - GI(I))/SIGMA
      SUMSIG = SUMSIG + (2.0 + (ALFA1(I)*Z1(I)) - (ALFA2(I)*Z2(I)))
    1 CONTINUE
      XNAME = 6HZ1
      WRITE(6,5)XNAME,(Z1(I),I=1,NQ)
    5 FORMAT(1H0,A6,10F10.4)
      XNAME = 6HZ2
      WRITE(6,5)XNAME,(Z2(I),I=1,NQ)
      XJ13 = (XN/SIGMA**2)*SUMALF
      XJ33 = (XN/SIGMA**2)*SUMSIG
```

```
      DO 2 I=1,NQ
      XJ(1) = XJ11
      XJ(2) = XJ(4) = (XN/SIGMA**2)*DI(I)
      XJ(3) = XJ(7) = XJ13
      XJ(5) = (XN/SIGMA**2)*DI(I)
      XJ(6) = XJ(8) = (XN/SIGMA**2)*(ALFA1(I) - ALFA2(I))
      XJ(9) = XJ33
      WRITE(6,3)I
    3 FORMAT(1H0,5X,*J-MATRIX FOR MEAN,VARIANCE AND G*,I1)
      CALL WRITIT(3,3,XJ)
      CALL GINV(XJ,3,VARC,3,3,IRANK,IROW,ICOL,IWORK,JWORK)
      WRITE(6,4)I
    4 FORMAT(1H0,5X,*VAR-COV MATRIX FOR MEAN, VARIANCE AND G*,I1)
      CALL WRITIT(3,3,VARC)
    2 CONTINUE
      RETURN
      END
C
C
C     SUBROUTINE ASYMPT(NQ,Z1,Z2)
C
C*****************************************************************
C
C     PURPOSE:
C       TO OBTAIN THE ASYMPTOTIC VALUES OF ALFA(1I), ALFA(2I),
C       BETA(1I), BETA(2I),
C       BETA(1I) AND BETA(2I), I = 1,...,K WHERE K IS THE NUMBER OF
C       TREATMENT GROUPS IN THE EXPERIMENTAL DESIGN
C
C     METHOD
C       THE THEORETICAL RESULTS IN SECTION 3.3.4 OF THIS THESIS ARE
C       EMPLOYED
C
C     INPUT PARAMETERS
C       NQ          - NUMBER OF TREATMENT GROUPS IN THE EXPERIMENTAL
C                     DESIGN
C       Z1(I)       - GIVEN BY EQN. (35) IN THE THESIS
C       Z2(I)       - GIVEN BY EQN. (35) IN THE THESIS
C
C*****************************************************************
C
C
      SUBROUTINE ASYMPT(NQ,Z1,Z2)
      DIMENSION Z1(10),Z2(10),ALPHA1(10),ALPHA2(10),BETA1(10),BETA2(10)
```

```
      DO 1 I=1,NQ
      CALL NDTR(Z1(I),PZ1,D)
      CALL NDTR(Z2(I),PZ2,D)
      WRITE(6,2)I,PZ1,I,PZ2
    2 FORMAT(1H0,*PZ1*,I1,* = *,F10.4,3X,*PZ2*,I1,* = *,F10.4)
      DENOM = PZ2 - PZ1
      XNUMER = (1.0/SQRT(2.0*4.0*ATAN(1.0)))*EXP(-0.5*Z1(I)**2))
      XNUM = -Z1(I)*XNUMER
      BETA1(I) = XNUM/DENOM
      ALPHA1(I) = (XNUMER/DENOM) - BETA1(I)*Z1(I)
      XNUMER = (1.0/SQRT(2.0*4.0*ATAN(1.0)))*EXP(-0.5*Z2(I)**2))
      XNUM = -Z2(I)*XNUMER
      BETA2(I) = XNUM/DENOM
      ALPHA2(I) = (XNUMER/DENOM) - BETA2(I)*Z2(I)
    1 CONTINUE
      WRITE(6,3)
    3 FORMAT(1H0,* ASYMPTOTIC VALUES OF ALPHAS AND BETAS*)
      XNAME = 6HALFA1
      WRITE(6,4)XNAME,(ALPHA1(I),I=1,NQ)
    4 FORMAT(1X,A6,10F10.4)
      XNAME = 6HALFA2
      WRITE(6,4)XNAME,(ALPHA2(I),I=1,NQ)
      XNAME = 6HBETA1
      WRITE(6,4)XNAME,(BETA1(I),I=1,NQ)
      XNAME = 6HBETA2
      WRITE(6,4)XNAME,(BETA2(I),I=1,NQ)
      RETURN
      END
C
C     SUBROUTINE GINV (A,B,G,M,N,IRANK,IROW,ICOL,IWORK,JWORK)
C
C     ..................................................................
C
C        SUBROUTINE GINV
C
C        PURPOSE
C           TO COMPUTE A GENERALISED INVERSE OF A REAL GENERAL MATRIX,
C           I.E. FOR A GIVEN MATRIX A, FIND A MATRIX G SUCH THAT AGA=A
C
C        USAGE
C           CALL GINV (A,B,G,M,N,IRANK,IROW,ICOL,IWORK,JWORK)
C
C        DESCRIPTION OF PARAMETERS
C           INPUT
```

```
C          A      - M BY N MATRIX
C          M      - NUMBER OF ROWS IN A AND COLUMNS IN G
C          N      - NUMBER OF COLUMNS IN A AND ROWS IN G
C       OUTPUT
C          G      - N BY M GENERALISED INVERSE MATRIX
C          IRANK  - RANK OF MATRIX A
C          IROW   - VECTOR OF SIZE M WITH SUBSCRIPTS OF BASIC
C                   ROWS OF MATRIX A
C          ICOL   - VECTOR OF SIZE N WITH SUBSCRIPTS OF BASIC
C                   COLUMNS OF MATRIX A
C       WORK
C          B      - VECTOR OF SIZE M*N
C          IWORK  - VECTOR OF SIZE IRANK
C          JWORK  - VECTOR OF SIZE IRANK
C
C    METHOD
C       THE S.S.P. ROUTINE MFGR IS CALLED TO DETERMINE THE RANK OF
C       MATRIX A AND ALSO A NON-SINGULAR MINOR B OF A OF SIZE
C       IRANK BY IRANK. B IS INVERTED USING S.S.P. ROUTINE MINV.
C       IN A, EACH ELEMENT OF B IS REPLACED BY THE CORRESPONDING
C       ELEMENT OF THE TRANSPOSE OF THE INVERSE OF B, AND ALL OTHER
C       ELEMENTS OF A ARE SET TO ZERO. THE TRANSPOSE OF THIS
C       RESULTING MATRIX IS THE GENERALISED INVERSE OF MATRIX A.
C
C       REFERENCES
C          SEARLE,S.R., LINEAR MODELS, CHAPTER 1, WILEY SERIES IN
C          PROBABILITY AND MATHEMATICAL STATISTICS, 1971.
C
C    REMARKS
C       THE RANK OF MATRIX A AS DETERMINED BY THE S.S.P. ROUTINE
C       MFGR IS NOT GUARANTEED FOR ALL CASES AND SO SHOULD BE
C       MONITORED FOR ACCURACY. NOTE THAT A IS NOT DESTROYED IN
C       COMPUTATION. ALSO THE MATRIX A MUST BE IN COMPRESSED FORM,
C       I.E. MUST NOT BE CONTAINED INSIDE A LARGER MATRIX. THE
C       S.S.P. ROUTINE ARRAY CAN BE CALLED TO CONVERT BETWEEN
C       SINGLE DIMENSION COMPRESSED STORAGE AND DOUBLE DIMENSION
C       STORAGE.
C
C    SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C       MFGR
C       MINV
C
C    SOURCE LANGUAGE
C       FORTRAN
C
```

```
C          ACCURACY
C             13 OR 14 SIGNIFICANT FIGURES WITH TOLERANCE EPS = 1.E-14.
C             EPS SHOULD BE APPROXIMATELY 10.0**(-D), WHERE D IS THE
C             NUMBER OF SIGNIFICANT FIGURES CARRIED.
C
C          STORAGE REQUIREMENTS
C             CDC 6400 -    166 OCTAL WORDS
C
C          TIMING
C             SIZE OF MATRIX A      TIME IN SEC.
C                  5 X  5              .010
C                 10 X 10              .059
C                 30 X 30             1.151
C                 50 X 50             4.911
C
C          AUTHOR                          CHECKED BY
C             I.D.MALLOCH, DEC 1971           I.D.MALLOCH, DEC 1971
C
CCC     ..................................................................
        DIMENSION A(1),B(1),G(1),IROW(1),ICOL(1),IWORK(1),JWORK(1)
C
C   CHECK DIMENSIONS
        IF(M.LT.1.OR.N.LT.1) GO TO 90
        EPS=1.E-14
C   COPY MATRIX A TO B
        NM=N*M
        DO 10 I=1,NM
     10 B(I)=A(I)
C   FIND RANK , BASIC ROWS, BASIC COLS OF B
        CALL MFGR(B,M,N,EPS,IRANK,IROW,ICOL)
C   SAVE NONSINGULAR MINOR IN B
        DO 20 I=1,IRANK
        DO 20 J=1,IRANK
        IA=IROW(I)+(ICOL(J)-1)*M
        IB=I+(J-1)*IRANK
     20 B(IB)=A(IA)
C   FIND INVERSE OF B
        CALL MINV(B,IRANK,D,IWORK,JWORK)
C   CLEAR G MATRIX
        DO 30 I=1,NM
     30 G(I)=0.
C   INSERT APPROPRIATE ELEMENTS OF B INTO G FOR GENERALISED INVERSE
        DO 40 I=1,IRANK
        DO 40 J=1,IRANK
```

```
      IB=J+(I-1)*IRANK
      IG=ICOL(J)+(IROW(I)-1)*N
   40 G(IG)=B(IB)
      GO TO 100
   90 IRANK=-1
  100 RETURN
      END
C
C
      SUBROUTINE WRITIT(M,N,A)
      DIMENSION A(1)
C     ***************************************************************
C     *                                                             *
C     * THIS SUBROUTINE PRINTS OUT ANY MATRIX IN ROW SEQUENCED INDICES.  *
C     * THIS IS NECESSARY SINCE ALL MATRICES ARE HANDLED BY COLUMN  *
C     * SEQUENCED INDICES, BUT TO WRITE OUT THE MATRIX ROW-WISE WE MUST  *
C     * TRANSFORM THE ELEMENTS TO BE WRITTEN ROW SEQUENTIALLY.      *
C     *                                                             *
C     * INPUT PARAMETERS-                                           *
C     *                                                             *
C     * A- INPUT MATRIX ( COLUMN SEQUENCED)                         *
C     *                                                             *
C     * M- NUMBER OF ROWS OF THE MATRIX A                           *
C     *                                                             *
C     * N- NUMBER OF COLUMNS OF THE MATRIX A                        *
C     *                                                             *
C     ***************************************************************
      K = (N-1)*M
      L1 = 1
      L2 = K+1
      DO 1 I = 1,M
      WRITE(6,2)(A(J),J=L1,L2,M)
    2 FORMAT(1H0,/,5(F20.8,3X),//5X,5(F20.8,3X),//10X,5(F20.8,3X),//15X,
     *5(F20.8,3X))
      L1 = L1+1
      L2 = L2+1
    1 CONTINUE
      RETURN
      END
C
C
```

```
C       SUBROUTINE KOUNT(R,NN,LN)
C
C*****************************************************************
C
C       PURPOSE:
C         TO COUNT THE NUMBER OF TEST STATISTICS T WITHIN SPECIFIED
C         INTERVALS BETWEEN -3.0 AND +3.0 AND TO COMPUTE THE PROBABILI-
C         TIES AT EACH OF THE SPECIFIED LEVELS
C
C       INPUT PARAMETERS
C         R(1)        - TEST STATISTICS COMPUTED FROM THE EXPERIMENTAL
C                       DESIGNS
C         NN          - NUMBER OF EXPERIMENTAL DESIGNS ANALYZED
C         LN          - DETERMINED BY SUBROUTINE VALUES
C
C*****************************************************************
C
C
C
      SUBROUTINE KOUNT(R,NN,LN)
      DIMENSION R(1)
      COMMON/LEVELS/AL(1000)
      DIMENSION K(1000),PROB(1000)
      CALL VALUES(LN)
      NN1=NN
       N1=1
    5 CONTINUE
      DO 1 I=1,LN
    1 K(I)=0
      DO 205 J=N1,NN1
      DO 200 I=1,LN
      IF(R(J).LT.AL(I)) K(I)=K(I)+1
  200 CONTINUE
  205 CONTINUE
      DO 210 I=1,LN
  210 PROB(I)=FLOAT(K(I))/FLOAT(NN)
      WRITE(6,220) ((AL(I),K(I),PROB(I)),I=1,LN)
  220 FORMAT(1H1,///////////
     -           20X,* COUNTING FOR RATIOS LESS THEN SPECIFIED LEVELS */
     -           20X,* ---------------------------------------------- *//
     - 22X,* SPECIFIED LEVEL     COUNT     PROBABILITIES */
     - 22X,* ---------------     -----     ------------- * ///
     - (26X,F10.4,I9,7X,F10.6))
C
      RETURN
```

```
      END
C
C
C       SUBROUTINE VALUES(LN)
C
C*****************************************************************
C
C       PURPOSE:
C         TO DETERMINE THE NUMBER OF SPECIFIED LEVELS BETWEEN -3.0 AND
C         +3.0 TO ALLOCATE THE COMPUTED TEST STATISTICS TO
C
C*****************************************************************
C
C
      SUBROUTINE VALUES(LN)
C
C AL WILL CONTAIN THE VALUES OF THE SPECIFIED LEVELS TO TEST AGAINST
C
      COMMON/LEVELS/AL(1000)
C
C READ IN THE LOWER BOUND AND UPPER BOUND ON THE LEVELS (I.E. -3.0 AND
C +3.0 RESPECTIVELY)  AS WELL AS THE INCREMENTING VALUE (I.E. 0.1 IS
C SUFFICIENT IN MOST CASES)
    5 READ(5,10) SMALL,BIG,FRA,NEXT
   10 FORMAT(3F10.4,I1)
      LN=1
      STORE=SMALL
C
C FILL THE ARRAY AL WITH THE VALUES OF THE SPECIED LEVELS TO BE USED AND
C DETERMINE THE NUMBER OF LEVELS TO BE USED (I.E. LN)
   20 AL(LN)=STORE
      STORE=STORE+FRA
      IF(STORE.GT.(BIG+.0000000001)) RETURN
      LN=LN+1
      GO TO 20
      END
C
C       FUNCTION DENOMC(X1,X2,X3,X4)
C
C*****************************************************************
C
C       PURPOSE:
C         TO COMPUTE THE DENOMINATOR OF G1(H1I), G1(K1I), G2(H1I), AND
```

```
C          G2(K1I) GIVEN IN EQN. (45) OF THE THESIS
CC
CC*************************************************************************
CC
CC
C
      FUNCTION DENOMC(X1,X2,X3,X4)
C
C
C***   X1 FOR FIRST U
C***   X2 FOR SECOND U
C***   X3 FOR H OR K
C***   X4 FOR SISQ
C
C
      P2I=(X2-X3)/X4
      P1I=(X1-X3)/X4
      CALL NDTR(P1I,P1,D)
      CALL NDTR(P2I,P2,D)
      DENOMC=P1-P2
C     WRITE(6,10) X1,X2,X3,X4,P1I,P1,P2I,P2,DENOMC
   10 FORMAT(///25H ** FROM DENOM CALCULATOR  ,5E20.4/25X,5E20.4)
      RETURN
      END
```