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Infinite discrete group actions

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Abstract

The nature of this paper is expository. The purpose is to present the fundamental material concerning actions of infinite discrete groups on the n-sphere and pseudo-Riemannian space forms $S^{p,q}$ based on the works of Gehring and Martin [6], [26] and Kulkarni [20], [21], [22] and provide appropriate examples. Actions on the n-sphere split $\mathbb{S}^n$ into ordinary and limit sets. Assuming, additionally, that a group acting on $\mathbb{S}^n$ has a certain convergence property, this thesis includes conditions for the existence of a homeomorphism between the limit set and the set of Freudenthal ends, as well as topological and quasiconformal conjugacy between convergence and Mobius groups. Actions on the $S^{p,q}$ are assumed to be properly discontinuous. Since $S^{p,q}$ is diffeomorphic to $\mathbb{S}^p \times \mathbb{R}^q$ and has the sphere $\mathbb{S}^{p+q}$ as a compactification, the work of Hambleton and Pedersen [9] gives conditions for the extension of discrete co-compact group actions on $\mathbb{S}^p \times \mathbb{R}^q$ to actions on $\mathbb{S}^{p+q}$. An example of such an extension is described.

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Contents

To my family

Chapter 1

Discrete Group Actions

1.1 Introduction

Let G be a group and X be a set. A group action of G on X is defined by the function $\phi : G \times X \to X$ which satisfies the following conditions for all $x \in X$:

- $\phi(e, x) = x$, where e is the identity element of G;
- $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $g, h \in G$.

The study of group actions is a significant part of group theory, and although mathematicians have spent many years investigating this topic, there are still many open problems. The purpose of this thesis is to give the fundamental material concerning actions of infinite discrete groups on the n-sphere and the pseudo-Riemannian space forms $S^{p,q}$ based on the works of Gehring and Martin [6], [26] and Kulkarni [20], [21], [22]. Actions on $S^{p,q}$ are assumed to be properly discontinuous, while actions on the n-sphere admit both ordinary and limit sets. The construction of $S^{p,q}$ space is the following: given $n + 1$ dimensional real space $\mathbb{R}^{n+1}$, and given a pair of nonnegative integers $(p + 1, q)$ with $p + q = n$, we define $S^{p,q}$ as the component of the quadric $\{v \in \mathbb{R}^{n+1} | Q(v) = 1\}$ containing $(1, 0...0)$, where

$$Q(x_1...x_{p+1}, y_1...y_q) = \sum_{i=1}^{p+1} {x_i}^2 - \sum_{j=1}^{q} {y_j}^2.$$

It is known that $S^{p,q} \cong O(p+1,q)/O(p,q)$, where $O(p+1,q)$ is a subgroup of the full group of Q-orthogonal transformations which preserve $S^{p,q}$, and $O(p,q)$ is its isotropy subgroup at $(1,0...0)$.

Remark. We assume $p \geq 0$ and $q \geq 1$, otherwise, $S^{p,q} = \emptyset$ when $p = -1$ and $S^{p,q} \cong \mathbb{S}^p$ when $q = 0$.

Note that if $x = (x_1...x_{p+1})$ and $y = (y_1...y_q)$, through the homeomorphism $(x,y) \to (\frac{x}{|x|}, y)$, we get $S^{p,q} \cong \mathbb{S}^p \times \mathbb{R}^q$. Theorem 4.1.1 proves that $\mathbb{S}^{p+q} \cong \mathbb{S}^p \times \mathbb{R}^q \cup \mathbb{S}^{q-1}$. Therefore, one might compactify properly discontinuous actions on $S^{p,q}$ to actions on $\mathbb{S}^{p+q}$ using conditions given by Hambleton and Pedersen [9].

Remark. $S^{p,q} \cong \mathbb{S}^p \times \mathbb{R}^q$ is a connected, locally compact, Hausdorff space.

An infinite discrete group G acting on the n-sphere is said to be a convergence group if any family of elements of G contains a subsequence $\{g_j\}$, for which there exist points $x, y \in \mathbb{S}^n$ such that $g_j \to x$, respectively ${g_j}^{-1} \to y$ locally uniformly on $\mathbb{S}^n \backslash \{y\}$, respectively $\mathbb{S}^n \backslash \{x\}$.

In Chapter 1 we give definitions used in the study of convergence group actions on the n-sphere. Moreover, we show properties of the limit set of convergence groups and give conditions under which the limit set and the set of Freudenthal ends might be homeomorphic.

In Chapter 2 we define quasiconformal groups of the n-sphere, and prove that they are, in fact, convergence groups. In addition, Chapter 2 includes an example given by Tukia in [32] as a partial answer to the question of Gehring and Palka, first raised in [7]:

Is every quasiconformal group G acting on the n-sphere quasiconformal conjugate to a Mobius group?

Tukia's example gives a negative response for $n \geq 3$. However, the statement is true when $n = 1$ and $n = 2$, see [12] [24] [31] [37].

Further, in Chapter 2 we give results concerning a topological conjugacy of discrete convergence groups of $\mathbb{S}^n$ to Mobius groups. Chapter 3 deals with conditions for the existence and non-existence of free, properly discontinuous and co-compact actions on $S^{p,q}$. Chapter 4 gives conditions for an extension of discrete co-compact group actions on $\mathbb{S}^p \times \mathbb{R}^q \cong S^{p,q}$ to actions on $\mathbb{S}^{p+q}$, and provides an appropriate example.

1.2 Notations and definitions

We are going to start by providing definitions and notations which will be used further in this paper. Most of the theorems and definitions in Chapter 1 are from Sections 1 and 2 of Martin [26].

Definition 1.2.1. *A subfamily F of $\mathrm{Homeo}(\mathbb{S}^n)$, the homeomorphism group of the n-sphere, has a* ***convergence property*** *if each infinite subset of F has a subsequence $\{f_j\}$ which satisfies one of the following properties:*
(i) there is an $f \in \mathrm{Homeo}(\mathbb{S}^n)$ such that $f_j \to f$ and ${f_j}^{-1} \to f^{-1}$ uniformly,
(ii) there are points $x, y \in \mathbb{S}^n$, not necessarily distinct, such that $f_j \to x$ locally uniformly in $\mathbb{S}^n - \{y\}$ and ${f_j}^{-1} \to y$ locally uniformly in $\mathbb{S}^n - \{x\}$.

Remark. Here, "$f_j \to x$ locally uniformly in $\mathbb{S}^n - \{y\}$" means that given an arbitrary point $z \in \mathbb{S}^n \backslash \{y\}$, there exists a neighborhood $V \subset \mathbb{S}^n \backslash \{y\}$ of z such that $f_j \to x$ uniformly in V.

Definition 1.2.2. *If a subgroup G of $\mathrm{Homeo}(\mathbb{S}^n)$ has the convergence property, we call it a* ***convergence group****.*

A good example of a convergence group acting on $\mathbb{S}^n$ is a quasiconformal subgroup of $\mathrm{Homeo}(\mathbb{S}^n)$; the proof is given in Chapter 2, or see pages 333-334 in [6]. A metric defined in $\mathbb{S}^n$ is induced by the chordal metric $\rho(x, y)$ for any $x, y \in \mathbb{S}^n$. For each $f \in \mathrm{Homeo}(\mathbb{S}^n)$ and $x \in \mathbb{S}^n$ define

$$\mathbb{U}(x, r) = \max(\rho(f(x), f(y)) : \rho(x, y) = r)$$

and

$$\mathbb{L}(x, r) = \min(\rho(f(x), f(y)) : \rho(x, y) = r).$$

Then a homeomorhism f of $\mathrm{Homeo}(\mathbb{S}^n)$ is said to be *K-quasiconformal*, if for each $x \in \mathbb{S}^n$

$$\limsup_{r \to 0} \frac{\mathbb{U}(x,r)}{\mathbb{L}(x,r)} \leq K$$

for a finite $K \geq 1$, and a subgroup G of $\mathrm{Homeo}(\mathbb{S}^n)$ is called a *quasiconformal group* if there exists a finite $K \geq 1$ such that each element of G is K-quasiconformal.

Definition 1.2.3. *A topological group G is* ***discrete*** *if its identity e is isolated, and then G is a* ***discrete convergence group*** *if it has a convergence property.*

Therefore, a discrete group never has the convergence property *(i)*, otherwise $f_j \to f$ implies $f_j f_{j+1}^{-1} \to e$ uniformly, which contradicts the definition of a discrete group.

Definition 1.2.4. *Let* Ω *be a region in* $\mathbb{S}^n$*, possibly* $\Omega = \mathbb{S}^n$*, with* $x \in \Omega$ *and G be a subgroup of* $\mathrm{Homeo}(\mathbb{S}^n)$*. We say G* ***acts discontinuously at*** x *if there is a neighbourhood* U *of* x *in* Ω *such that the set*

$$\zeta_G(U) = \{g \in G : g(U) \cap U \neq \emptyset\}$$

has finitely many elements. We say G ***acts discontinuously in*** Ω *if it acts discontinuously at all* $x \in \Omega$*.*

We say G ***acts properly*** *in* Ω *if, for every compact subset C in* Ω*, the set*

$$\zeta_G(C) = \{g \in G : g(C) \cap C \neq \emptyset\}$$

is compact in G.

We say G ***acts properly discontinuously*** *in* Ω *if, for every compact subset C in* Ω*, the set*

$$\zeta_G(C) = \{g \in G : g(C) \cap C \neq \emptyset\}$$

has finitely many elements.

Therefore, a properly discontinuous action takes place when the action is proper and G is discrete. The rest of this section we assume that G is an infinite discrete convergence group acting on $\mathbb{S}^n$.

Definition 1.2.5. *The set*

$$\Omega(G) = \{x \in \mathbb{S}^n : \text{G acts discontinuously at x}\}$$

is called an ***ordinary set*** *for* G*, or* ***a domain of discontinuity*** *of* G*, and the set*

$$\Lambda(G) = \mathbb{S}^n - \Omega(G)$$

is called a ***limit set*** *of* G*.*

Theorem 1.2.6. An infinite discrete convergence group G acts properly discontinuously on $\Omega(G)$.

The proof will be given later in Lemma 1.3.11.

Theorem 1.2.7. Both Ω and Λ are G-invariant.

Proof. It suffices to show that Ω is G-invariant. Suppose Ω is not G-invariant.

There exist $x \in \Omega$ and $g_0 \in G$ such that $y = g_0(x) \notin \Omega$. Choose an arbitrary neighborhood V of x in Ω. Since $y \notin \Omega$ and $g_0(V) = V_0$ is a neighborhood of y, we have $g(V_0) \cap V_0 \neq \emptyset$ for infinitely many $g \in G$. Alternatively, $g_0^{-1}g(V_0) \cap g_0^{-1}(V_0) \neq \emptyset$. Using $g_0(V) = V_0$ we get $g_0^{-1}gg_0(V) \cap V \neq \emptyset$ for infinitely many $g \in G$. Since the choice of V was arbitrary, this contradicts the assumption $x \in \Omega$. Therefore, $g(x) \in \Omega$, and Ω is G-invariant. ∎

Depending on the size of the limit set $\Lambda(G)$, convergence groups have the following classifications:

Definition 1.2.8. *An infinite discrete convergence group G is **elementary** if*

$$card(\Lambda(G)) \leq 2,$$

*otherwise we say G is **non-elementary**.*

Next definition demonstrates classifications of elements of an infinite discrete convergence group G:

Definition 1.2.9. *Let for each $g \in G$,*

$$ord(g) = min\{m > 0 | g^m = e\}, \quad fix(g) = \{x \in X | g(x) = x\}.$$

*Then g is **elliptic** if $ord(g) < \infty$, or g is **parabolic** if $ord(g) = \infty$ and card$(fix(g)) = 1$, or g is **hyperbolic** if $ord(g) = \infty$ and card$(fix(g)) = 2$.*

One may ask if it is possible to find an element $g \in G$ of infinite order which fixes more than two points. Next few steps will answer that question.

Clearly, if $g \in G$ is an element of the infinite order then a group $\langle g \rangle$ generates an infinite discrete convergence group.

Lemma 1.2.10. Let G be an infinite discrete convergence group, then card(fix(G))≤ 2.

Proof. Suppose that G fixes at least three distinct points, say x, y and z. Let $\{g_j\}$ be an infinite family of elements in G. Then, by the convergence property, we can find points $x_0, y_0 \in \mathbb{S}^n$ such that $g_j \to y_0$ locally uniformly in $\mathbb{S}^n \backslash \{x_0\}$. By relabelling if necessary, $x \neq x_0 \neq z$, and so by the triangle inequality:

$$\rho(x, y_0) \leq \lim_{j \to \infty} (\rho(x, g_j(x)) + \rho(g_j(x), y_0)) = \lim_{j \to \infty} \rho(g_j(x), y_0) = 0.$$

Therefore, $x = y_0$, and similarly $z = y_0$. $\Rightarrow x = z$, contradiction. ■

By this Lemma, since $g \in G$ has the infinite order, then $\langle g \rangle$ has at most two fixed points. Henceforth, g being the generator of $\langle g \rangle$ has at most two fixed points.

1.3 Limit set of an infinite discrete convergence group acting on $\mathbb{S}^n$

We assume throughout this section that G is an infinite discrete convergence group acting on $\mathbb{S}^n$. The convergence property for G implies next theorem, and the proof comes directly from the Definitions 1.2.1 and 1.2.3:

Theorem 1.3.1. [6, p. 335] For each infinite subfamily of G there exist a subsequence $\{g_j\}$ and points $x_0, y_0 \in \mathbb{S}^n$ such that

$$g_j \to y_0 \quad \text{and} \quad {g_j}^{-1} \to x_0$$

locally uniformly in $\mathbb{S}^n \backslash \{x_0\}$ and $\mathbb{S}^n \backslash \{y_0\}$, respectively.

Suppose there is a closed G-invariant set $E \subset \mathbb{S}^n$ with at least two points contained in. Then for each $x \in E \backslash \{x_0\}$, points $\{g_j(x)\}$ are in E and therefore $y_0 \in E$. Similarly, we prove that $x_0 \in E$. Hence, the following immediate consequence appears.

Corollary 1.3.2. Any G-invariant closed set $E \subset \mathbb{S}^n$ with $card(E) \geq 2$ contains points x_0 and y_0 given in 1.3.1.

Turning our attention to limit sets, we give the following important theorem which is slightly similar to Theorem 1.3.1:

Theorem 1.3.3. [6, p. 338-339] For each point $y_1 \in \Lambda(G)$ there exist a point $x_1 \in \Lambda(G)$ (not necessarily distinct) and a sequence $\{g_j\}$ in G such that

$$g_j \to y_1 \quad \text{and} \quad {g_j}^{-1} \to x_1$$

locally uniformly in $\mathbb{S}^n\backslash\{x_1\}$ and $\mathbb{S}^n\backslash\{y_1\}$, respectively.

Proof. Let $\{r_j\}$ be a sequence of positive numbers converging to zero, then for each $j = 1, 2, ...$, let $V_j = B_{y_1}(r_j)$ be a ball centered at y_1 of the radius r_j defined by the chordal metric ρ. Since G is not discontinuous at y_1, there is an infinite sequence $\{h_j\}$ in G such that for all j, $h_j(V_j) \cap V_j \neq \emptyset$. By Theorem 1.3.1 there is a subsequence $\{f_j\}$ of $\{h_j\}$, and points $x_0, y_0 \in \mathbb{S}^n$, such that

$$f_j \to y_0 \quad \text{and} \quad {f_j}^{-1} \to x_0$$

locally uniformly in $\mathbb{S}^n\backslash\{x_0\}$ and $\mathbb{S}^n\backslash\{y_0\}$, respectively.

If $y_1 = x_0$, we simply take $x_1 = y_0$ and $g_j = {f_j}^{-1}$. Otherwise, since $h_j(V_j) \cap V_j \neq \emptyset$, for each j there is a point $z_j \in V_j$ with $f_j(z_j) \in V_j$, and so $h_j(z_j) \to y_0$ by Theorem 1.3.1. Hence by the triangle inequality

$$\rho(y_0, y_1) \leq \lim_{j\to\infty} (\rho(h_j(z_j), y_0) + \rho(h_j(z_j), y_1)) = 0,$$

i.e. $y_0 = y_1$, and we take $x_1 = x_0$, $g_j = f_j$.

To show that $x_1 \in \Lambda(G)$, let U be a neighbourhood of x_1, and since ${g_j}^{-1}(x) \to x_1$ we get

$${g_j}^{-1}(x) \in {g_j}^{-1}(U) \cap U \neq \emptyset$$

for $x \in U\backslash\{y_1\}$ and large j. That means $x_1 \in \Lambda(G)$. ■

One feature to be noted is that $\Lambda(G)$ is never empty, when G is an infinite discrete convergence group. (see [6], Theorem 5.7):

Theorem 1.3.4. $\Lambda(G)$ is an empty set if and only if G is a finite group of elliptic elements.

In fact, $\Lambda(G)$ has either $1, 2$ points, or else $card(\Lambda(G)) \geq 3$ and $\Lambda(G)$ is a perfect set (by Martin in [26], Theorem 2.2 and Theorem 2.5):

Theorem 1.3.5. (i) $card(\Lambda(G)) = 1$ with $\Lambda(G) = \{x_0\}$ if and only if G is an infinite group consisting only of elliptic and parabolic elements which fix x_0.
(ii) $card(\Lambda(G)) = 2$ with $\Lambda(G) = \{x_0, y_0\}$ if and only if G is an infinite group consisting only of elliptic and loxodromic elements which either fix or interchange x_0 and y_0.
(iii) If $card(\Lambda(G)) \geq 3$ then $\Lambda(G)$ is a perfect set.

To show that $\Lambda(G)$ may have the cardinality ≤ 2, we give the following examples, cf. [6]:

Example 1.3.6. $card(\Lambda(G)) = 1$.

Let t be an arbitrary nonzero vector in $\mathbb{R}^n$, and let $p : \bar{\mathbb{R}}^n \to \mathbb{S}^n$ be a stereographic projection from the extended plane $\mathbb{R}^n \cup \{\infty\}$ to the n-sphere:

$$p(x) = e_{n+1} + \frac{2(x, -1)}{|x, -1|^2}, \quad x \in \bar{\mathbb{R}}^n.$$

Consider $g(x) = x + t$ and $h(x) = -x$, two self-homeomorphisms of $\bar{\mathbb{R}}^n$. The group $\langle g, h\rangle$ acting on $\bar{\mathbb{R}}^n$ contains only elliptic and parabolic elements which fix ∞, and so the limit set of $\langle g, h\rangle$ contains only ∞. Since p is a homeomorphism, the maps $\pi_1 = p \circ g \circ p^{-1}$ and $\pi_2 = p \circ h \circ p^{-1}$ are self-homeomorphisms of the n-sphere to itself. Therefore, we may conclude that the group $\langle \pi_1, \pi_2\rangle$ acting on $\mathbb{S}^n$ consists of elliptic and parabolic elements only, and $\Lambda(\langle \pi_1, \pi_2\rangle) = \{(0, ..., 0, 1)\}$, i.e. $card(\Lambda(G)) = 1$ for $G = \langle \pi_1, \pi_2\rangle$.

Example 1.3.7. $card(\Lambda(G)) = 2$.

Let $p : \bar{\mathbb{R}}^n \to \mathbb{S}^n$ be as in Example 1. Consider an infinite discrete group $\langle g, h\rangle$ acting on $\bar{\mathbb{R}}^n$, where $g(x) = 2x$ and $h(x) = \frac{1}{x}$. Note that $\langle g, h\rangle$ acting on $\bar{\mathbb{R}}^n$ contains only infinitely many loxodromic elements fixing 0 and ∞ and infinitely many elliptic elements of order 2 interchanging points 0 and ∞.
Therefore, the group $\langle p \circ g \circ p^{-1}, p \circ h \circ p^{-1}\rangle$ acting on $\mathbb{S}^n$ consists of infinitely many loxodromic elements fixing $(0, ..., 0, 1)$ and $(0, ..., 0, -1)$ and infinitely many elliptic elements of order 2 interchanging these points, i.e. $card(\Lambda(G)) = 2$ for $G = \langle p \circ g \circ p^{-1}, p \circ h \circ p^{-1}\rangle$.

Remark. Notice that definitions regarding a group G acting on $\bar{\mathbb{R}}^n$ come from Definitions 1.2.1-1.2.9 with $\mathbb{S}^n$ replaced by $\bar{\mathbb{R}}^n$.

Points fixed by parabolic and loxodromic elements have one remarkable property:

Theorem 1.3.8. [6, p.345] Let g be a parabolic and h be a loxodromic elements in G. Then

$$\lim_{j\to\infty} g^j = \lim_{j\to\infty} g^{-j} = fix(g)$$

locally uniformly in $\mathbb{S}^n \backslash \{fix(g)\}$, and

$$\lim_{j\to\infty} h^j = y_0 \quad \text{and} \quad \lim_{j\to\infty} h^{-j} = x_0$$

locally uniformly in $\mathbb{S}^n \backslash \{x_0\}$ and $\mathbb{S}^n \backslash \{y_0\}$, respectively. Here, $fix(h) = \{x_0, y_0\}$.

Kulkarni in [20], pages 255-256, first suggested a slightly different way to construct the limit set of a group G acting on a locally compact Hausdorff space X, e.g. $\mathbb{S}^n$, using the notion of cluster points:

Definition 1.3.9. *Let J be some infinite indexing set, and, for j running over J, let $\{S_j\}$ be a collection of subsets of $\mathbb{S}^n$. The **cluster point** of $\{S_j\}$ is a point $x \in \mathbb{S}^n$ such that for every neighborhood U of x,*

$$U \cap S_j \neq \emptyset$$

for infinitely many $j \in J$.

Definition 1.3.10. *Let $L_0(G)$ be the closure of the set of points with infinite isotropy group, $L_1(G)$ be the closure of the set of cluster points of $\{g(y)\}_{g\in G}$ for y running over $\mathbb{S}^n - L_0(G)$, and $L_2(G)$ be the closure of the set of cluster points of $\{g(K)\}_{g\in G}$ for K running over compact subsets of $\mathbb{S}^n - \{L_0(G) \cup L_1(G)\}$. Then, we call*

$$\Lambda = L_0(G) \cup L_1(G) \cup L_2(G)$$

*as a **limit set** of G.*

Lemma 1.3.11. The sets $L_j(G)$ are G-invariant for $j = 0, 1, 2$. Hence, both Λ and Ω are G-invariant.

Moreover, G acts properly discontinuously on the set $\Omega = \mathbb{S}^n - (L_0 \cup L_1 \cup L_2)$.

Proof. Let $x \in L_0(G)$ be a point with infinite isotropy group Γ, and let $g \in G$ be an arbitrary element. Then $g \circ \Gamma \circ g^{-1}$ is an infinite isotropy group of $g(x)$, so $L_0(G)$ is G-invariant. Similarly, we prove that $L_1(G)$ and $L_2(G)$ are G-invariant. Then Λ and Ω are G-invariant by Definition 1.3.10.

Let K be a compact subset in Ω. If $\{g \in G : gK \cap K \neq 0\}$ is infinite, then K contains a cluster point of $\{gK\}_{g \in G}$ and $K \cap L_2 \neq \emptyset$, which is a contradiction. Therefore, G acts properly discontinuously on Ω. ■

Corollary 1.3.12. The set Λ is closed, and the set Ω is open.

Corollary 1.3.13. Definitions 1.2.5 and 1.3.10 coincide.

Proof. Let Ω_1 and Λ_1 be from Definition 1.2.5, and Ω_2 and Λ_2 be from Definition 1.3.10. By the theorem above, G acts properly discontinuously on $\Omega_2 \Rightarrow \Omega_2 \subseteq \Omega_1$. Choose an arbitrary point $x \in \Omega_1$, then there is a compact neighborhood K of x in Ω_1 such that

$$\{g \in G : g(K) \cap K \neq \emptyset\}$$

has finitely many elements. $\Rightarrow x \notin L_0$.

Suppose $x \in L_1$, then we can find a point $x_0 \in K$ which is a cluster point of $\{g(y)\}_{g \in G}$ for some $y \in \mathbb{S}^n - L_0$. Since int(K) is a neighborhood of x_0, there exists an infinite family of distinct elements $\{g_j\}_{j \in \mathbb{N}}$ such that $g_j(y) \in K$ for all j. $\Rightarrow y \in {g_1}^{-1}(K)$. Hence, $g_j(y) \in g_j{g_1}^{-1}(K)$, and

$$\{g_j{g_1}^{-1} : j \in \mathbb{N}\} \subseteq \{g \in G : g(K) \cap K \neq \emptyset\}.$$

This contradicts the assumption that $x \in \Omega_1 \Rightarrow x \notin L_1$.

If $x \in L_2$, then we can find a point $x_0 \in K$ which is a cluster point of $\{g(C)\}_{g\in G}$ for some compact $C \subset \mathbb{S}^n - \{L_0 \cup L_1\}$. Since int($K$) is the neighborhood of x_0, there exists an infinite family of distinct elements $\{g_j\}_{j\in\mathbb{N}}$ such that $g_j(C) \cap K \neq \emptyset$ for all j. Λ_1 is G-invariant, so $C' = C - \Lambda_1 \neq \emptyset$, and $g_j(C') \cap K \neq \emptyset$ for all j. Notice that the family $F = \{g_j\}_{j\in\mathbb{N}}$ is infinite, so there is an infinite subfamily $\{g_{j'}\}$ and points $x', y' \in \Lambda_1$ such that $g_{j'} \to y'$ locally uniformly on $\mathbb{S}^n \backslash \{x'\}$. Therefore, if for $C' \subset \mathbb{S}^n \backslash \{x'\}$ we have $g_{j'}(C') \cap K \neq \emptyset$ for all j', then $y' \in K$. This is a contradiction since $y' \in \Lambda_1$ and $K \subset \Omega_1$. $\Rightarrow x \notin L_2$.

From the above, $\Omega_1 \subseteq \mathbb{S}^n - \{L_0 \cup L_1 \cup L_2\} = \Omega_2$. $\Rightarrow \Omega_1 = \Omega_2$. ■

A good example when $L_2 \backslash (L_0 \cup L_1) \neq \emptyset$ was given by Kulkarni in [20] for an action on the Euclidean plane, and we slightly change it to get an action on the sphere. Consider the Euclidean space $\bar{\mathbb{R}}^2$ and the group action generated by the map $g : (x, y) \to (2x, \frac{1}{2}y)$. We get $L_0 = \{\text{origin}, \infty\}$, $L_1 = \{\text{origin}, \infty\}$ and $L_2 = \{x - \text{axis}\}$. Let $p : \bar{\mathbb{R}}^2 \to \mathbb{S}^2$ be a stereographic projection. Then we get an action of $p \circ g \circ p^{-1}$ on $\mathbb{S}^2$ with $L_0 = \{\text{south and north poles}\}$, $L_1 = \{\text{south and north poles}\}$ and L_2 is a circle which is the intersection of $\mathbb{S}^2$ and the xz-plane.

Remark. Let F be an infinite subfamily of $\langle g \rangle$ acting on $\bar{\mathbb{R}}^2$. Each element of F is a power of g. Therefore, any infinite subsequence $\{g_j = g^{j_k}\}$ of F converges to ∞ as $j_k \to \infty$ on the complement of 0. However, as $j_k \to -\infty$, $g^{j_k} \to 0$ on the $x-$axis and $g^{j_k} \to \infty$ otherwise. Therefore, $\langle g \rangle$ does not have the convergence property.

Proposition 1.3.14. If G is an infinite discrete convergence group acting on $\mathbb{S}^n$, then $L_2 \subseteq L_0 \cup L_1$.

Proof. Suppose that $L_2 \not\subseteq L_0 \cup L_1$, then there exists $x \in L_2 \backslash (L_0 \cup L_1)$ which is a cluster point of $\{gK\}_{g\in G}$ for some compact K in $\mathbb{S}^n - (L_0 \cup L_1)$. Since $x \in \Lambda = L_0 \cup L_1 \cup L_2$, there exist a subsequence $\{g_j\}$ in G and a point $y \in \Lambda$ such that $g_j \to x$ locally unifromly on $\mathbb{S}^n \backslash \{y\}$.

Let $K - \{y\} \neq \emptyset$. For any $z \in K - \{y\}$ we have $g_j(z) \to x$, and so $x \in L_1$. This is a contradiction.

Let $K - \{y\} = \emptyset$, then x is a cluster point of $\{gK\}_{g \in G} = \{gy\}_{g \in G}$ for $y \in \mathbb{S}^n - L_0$. Therefore, $x \in L_1$ and we get a contradiction. ■

1.4 The end-compactification of Ω for $G \curvearrowright \mathbb{S}^n$

We remind that G is an infinite discrete convergence group acting on $\mathbb{S}^n$. In this section we show the connection between Λ and the set of ends which compactifies an ordinary set Ω, and their properties. Theory of end-compactifications was first established by Freudenthal in [4] and we start with constructions and fundamental definitions used in this theory.

Suppose S is an open connected subset of the n-sphere. Let $\{K_j\}_{j \in \mathbb{N}}$ be an increasing sequence of compact subsets of S, $K_j \subset \text{int}(K_{j+1})$, such that $S = \bigcup_{j=1}^{\infty} K_j$. A decreasing nested sequence of components $\{C_j \subset S \backslash \text{int}(K_j)\}_{j \in \mathbb{N}}$ is defined to be a *Freudenthal end*. We define the set of all Freudenthal ends by $\epsilon(S)$. Then the end-compactification of S is $S_E = S \cup \epsilon(S)$.

By Theorem 1 in Peschke [30], a compact Hausdorff space S_E and an inclusion map $S \to S_E$ must have following properties:

Property 1: S is homeomorphically embedded as a dense open subset of S_E;

Property 2: $S_E - S$ is totally disconnected;

Property 3: every map $\phi : S \to S'$ satisfying properties $1-2$ factors uniquely through S_E.

Throughout this section we assume that Ω is an open connected subset of $\mathbb{S}^n$, which automatically implies that $\Omega \neq \emptyset$.

First, we show examples of constructions of the Freudenthal ends.

Example 1.4.1. Let $n = 1$. Since Ω is an open connected subset of the unit circle, then Λ is a point, say x. Let $\{U_j\}$ be a decreasing nested sequence of connected neighborhoods of x in $\mathbb{S}^1$ converging to x. Note that each U_j is separated to two disjoint neighborhoods, say V_j and

W_j, by the point x. That is $U_j = V_j \cup \{x\} \cup W_j$. By this construction $\{K_j = \mathbb{S}^1 - U_j\}$ is an increasing sequence of compact subsets of Ω. Then each of $\{V_j\}$ and $\{W_j\}$ is a decreasing nested sequence of components which defines a Freudenthal end. Therefore, in this case, the set of ends has the cardinality 2.

Example 1.4.2. Let $n = 2$. Assume that the limit set Λ of $\mathbb{S}^n$ is a curve λ, which is an embedding of the closed unit interval. Let $\{U_j\}$ be a decreasing nested sequence of connected neighborhoods of λ converging to λ, that is $\{K_j = \mathbb{S}^n - U_j\}$ is an increasing sequence of compact subsets of Ω required for the construction of Freudenthal ends. Therefore, we have one component which constructs an end, and so the set $\epsilon(\Omega)$ consists of one element.

Theorem 1.4.3. Let $\bar{\Omega}$ be a closure of Ω, then $\bar{\Omega} = \mathbb{S}^n$.

Proof. Choose an arbitrary point $y_0 \in \mathbb{S}^n$. Assume that $y_0 \notin \Omega$, then $y_0 \in \Lambda$. We can find an infinite sequence $\{g_j\}$ in G and a point $x_0 \in \Lambda$ such that $g_j \to y_0$ locally uniformly in $\mathbb{S}^n \backslash \{x_0\}$. Since $\Omega \neq \emptyset$, this property, in fact, proves that y_0 is a limit point of Ω. Therefore, each point of $\mathbb{S}^n$ either belongs to Ω or is a limit point of Ω. ■

Corollary 1.4.4. $\partial\Omega = \Lambda$.

We define the equivalence relation R in $\mathbb{S}^n$ as follows: let x and y be arbitrary points in $\mathbb{S}^n$, then we say xRy if either $x = y$ or x and y belong to the same component of Λ. Then let Ω_1 be a quotient space $\Omega_1 = \bar{\Omega}/R$.

By Theorem 2.1 in Kulkarni [22], one might obtain the following result for an open connected subset Ω of $\mathbb{S}^n$:

Theorem 1.4.5. Let Ω_E and Ω_1 be defined as above. If $n > 1$, then Ω_E and Ω_1 are homeomorphic.

To see that it does not hold for $n = 1$, consider the following example. Let G be group acting on $\mathbb{S}^1$ with card(Λ)$= 1$. Then, the set $\Omega_E - \Omega$ has two points obtained by approaching the limit point from both sides, as it was proved in the example above. However, $\Omega_1 - \Omega$ has only one point which is actually the limit point of G. Therefore, Ω_E and Ω_1 are not homeomorphic.

To see the construction of the set of ends for $n > 1$, we use Theorem 4.1 from Kulkarni [20]:

Theorem 1.4.6. If Ω is a connected subset of $\mathbb{S}^n$, then $\partial\Omega = \Lambda$ has ≤ 2 components, or else it is a perfect set with uncountably many components.

Combining this result with Theorem 1.4.5, we get

Corollary 1.4.7. For $n > 1$, the set of ends $\epsilon(\Omega)$ has ≤ 2 points, or else it is a perfect set with uncountably many points.

We finish this section by proving that G is a Kleinian group. Kulkarni in [22] gives the following definition of Kleinian groups using the notion of G-accessible ends:

Definition 1.4.8. *[22, p.899] An element of $\epsilon(\Omega)$ is G-**accessible** if it is a cluster point of $\{gp\}_{g \in G}$ for some point $p \in \Omega$.*

*G is called **Kleinian** if it acts properly discontinuously on Ω and every element of $\epsilon(\Omega)$ is G-accessible.*

Remark. In the classical case, a group G acting on a compact space X is said to be a *Kleinian group* if the domain of discontinuity of G is nonempty. In our case, $X = \mathbb{S}^n$ and Ω is assumed to be an open connected subset of the n-sphere, and so $\Omega \neq \emptyset$. Therefore, G is a Kleinian group.

We prove this result using Definition 1.4.8.

Theorem 1.4.9. Given an infinite discrete convergence group G acting on $\mathbb{S}^n$. Given Ω a set of discontinuity of G. Assume that $\Omega_E = \mathbb{S}^n$. Then G is Kleinian.

Proof. Remind that Ω is an open connected dense subset of $\mathbb{S}^n$. Pick an arbitrary point $y \in \epsilon(\Omega)$. Then, there exists $\{K_j\}$ an increasing sequence of compact subsets of Ω, and a decreasing nested sequence of components $\{C_j \subset \Omega \backslash \text{int}(K_j)\}$ which define y as a Freudenthal end. Hence, we can choose points $y_j \in C_j$ for all j, such that $y_j \to y$. Since $\mathbb{S}^n$ is compact, $y \in \mathbb{S}^n$ and so $y \in \Lambda$. By Theorem 1.3.3, there exists an infinite sequence $\{g_j\}$ in G and a point $x \in \Lambda$ so that $g_j \to y$ locally uniformly in $\mathbb{S}^n \backslash x$. Therefore, for any point $z \in \Omega$, we have $g_j(z) \to y$, and so $y \in \epsilon(\Omega)$ is G-accessible. Since the choice of y was arbitrary, every element of $\epsilon(\Omega)$ is G-accessible. Combining this result with properly discontinuous action of G on Ω, we get that G is a Kleinian group. ■

Chapter 2

Infinite Groups acting on Spheres

This chapter presents properties of infinite discrete group actions on the n-sphere. Useful definitions for this chapter are :

Definition 2.0.1. *The Mobius transformation acting in $\bar{\mathbb{R}}^n$ is a finite composition of reflections. The Mobius group $\mathbb{M}(n)$ is the full group of Mobius transformations acting on $\bar{\mathbb{R}}^n$, and a Fuchsian group is a discrete subgroup of the group of Mobius transformations of $\bar{\mathbb{R}}^2$ with invariant open unit disk int(D^2), and an action on int(D^2) is properly discontinuous.*

In Section 2.2, one of theorems includes the notion of "restricted Fuchsian groups", which will denote the restriction of a Fuchsian group to $\mathbb{S}^1$. Sometimes we use the expression "Mobius group of the n-sphere" which defines a group of Mobius transformations preserving the n-sphere.

Definition 2.0.2. *A group G acting on $\mathbb{S}^1$ is said to be **quasisymmetric** if there exists a quasiconformal group $\hat{G}$ of the closed unit disk which extends G.*

2.1 Quasiconformal groups acting on $\mathbb{S}^n$

The main purpose of this section is to show that a discrete quasiconformal group acting on $\mathbb{S}^n$ is a convergence group. The detailed proof is given by Gehring and Martin in Theorem 3.2,

[6], and we give the sketch of their proof. Next part of this section contains results/answers to the question raised by Gehring and Palka in [7]:

Is every quasiconformal group G acting on the n-sphere quasiconformal conjugate to a Mobius group?

We remind that a group G, a subgroup of Hom($\mathbb{S}^n$), is *quasiconformal* if there is a finite $K \geq 1$ such that each element $g \in G$ is K-quasiconformal, i.e. for each $x \in \mathbb{S}^n$

$$\underset{r \to 0}{lim\,sup} \frac{\mathbb{U}(x,r)}{\mathbb{L}(x,r)} \leq K,$$

where

$$\mathbb{U}(x,r) = \max(\rho(f(x), f(y)) : \rho(x,y) = r),$$

$$\mathbb{L}(x,r) = \min(\rho(f(x), f(y)) : \rho(x,y) = r)$$

and ρ is a chordal metric defined in $\mathbb{S}^n$.

Theorem 2.1.1. [6] Let G be an infinite discrete quasiconformal group acting on $\mathbb{S}^n$. Then G has the convergence property.

Proof. Choose any infinite subfamily F of the group G. Suppose F is equicontinuous in $\mathbb{S}^n$. By Arzela-Ascoli theorem, there is a convergent subsequence $\{f_j\}$ in F, and $f_j \to f$ locally uniformly for some function f defined on the n-sphere. Using Theorems 21.1, 21.11 and 37.2 in [38], f is a K-quasiconformal element of Hom($\mathbb{S}^n$) $\Rightarrow f \in G$. This gives a contradiction since G is discrete and f should be isolated. Therefore, there is no infinite subfamily of equicontinuous functions in G.

If F is not equicontinuous in $\mathbb{S}^n$ then, by Lemma 3.1 in [6], there exists a point $x_0 \in \mathbb{S}^n$ and an infinite subfamily F_0 of F such that F_0 is equicontinuous in $\mathbb{S}^n \backslash x_0$. By Arzela-Ascoli theorem, there is a convergent subsequence $\{f_j\}$ in F_0, and $f_j \to f$ locally uniformly in $\mathbb{S}^n \backslash x_0$ for some function f on the n-sphere. Using the Theorems 21.1 and 37.2 in [38], f

is either a K-quasiconformal mapping of $\mathbb{S}^n \backslash x_0$ or a constant. Since G is discrete, f is a constant function $\Rightarrow f_j \to y_0$ locally uniformly in $\mathbb{S}^n \backslash x_0$.

From the above, $H = \{f^{-1} : f \in F\}$ has no infinite subfamily which is equicontinuous in $\mathbb{S}^n$, so there are points $x_1, y_1 \in \mathbb{S}^n$ and a subsequence $\{f_j^{-1}\}$ of H such that $f_j^{-1} \to y_1$ locally uniformly in $\mathbb{S}^n \backslash x_1$. It is not hard to show that $y_0 = x_1$ and $x_0 = y_1$, therefore $\{f_j\}$ is the desired sequence. ■

The natural way to construct a quasiconformal group acting on the n-sphere is to conjugate a Mobius group $\mathbb{M}$ of the n-sphere by a K-quasiconformal homeomorphism f of $\mathbb{S}^n$. The produced group $f \circ M \circ f^{-1}$ is a K^2-quasiconformal group of the n-sphere.

Gehring and Palka in [7] first raised a problem:

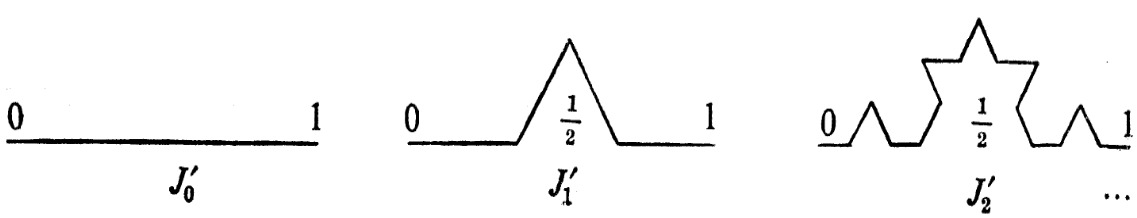
Is every quasiconformal group G acting on the n-sphere quasiconformal conjugate to a Mobius group?

Hinkkanen in [12] proved that the statement is true for elementary discrete quasisymmetric groups acting on the circle ($n = 1$), and the result was extended for all discrete quasisymmetric groups by Markovic in [24].

Sullivan in [31] and Tukia in [37] proved that the statement is true for $n = 2$.

However, the statement is no longer true for $n \geq 3$. Tukia in [32] gives an example of a quasiconformal group on $\mathbb{R}^n$ which is not quasiconformally conjugate to a Mobius group. Martin in [25] gives a discrete example with similar property based on the Tukia's work.

We show a way to construct the example of Tukia. Let $J_0', J_1', \ldots$ be arcs as below (pictures are due to [25]):



Then $\{J'_i\} \to J'$, where J' is a non-rectifiable quasiconformal arc. Let

$$J = \bigcup_{i \geq 0} 3^i(J' \cup (-J')).$$

There is a natural map $f' : [0,1] \to J'$, such that $f'(4^i x) = 3^i f'(x)$ for $i \geq 0$ and $0 \leq x \leq 4^i x \leq 1$. We extend f' to f by $f(\pm 4^i x) = \pm 3^i f'(x)$ for $x \in [0,1]$ and $i \geq 0$. Then, we get a homeomorphism $f : \mathbb{R} \to J$. One might prove that if $\alpha = \log_4 3$, there is $M \geq 1$ with

$$\frac{|x-y|^\alpha}{M} \leq |f(x) - f(y)| \leq M|x-y|^\alpha$$

for all $x, y \in \mathbb{R}$. Therefore, given for all $a, b, x \in \mathbb{R}$ satisfying $|a-x| \leq |b-x|$ we have:

$$|f(a) - f(x)| \leq M|a-x|^\alpha \leq M|b-x|^\alpha = M^2 \frac{|b-x|^\alpha}{M} \leq M^2|f(b) - f(x)|.$$

$\Rightarrow f$ is, so called, weakly-quasisymmetric with $H = M^2$. Tukia in [34] proved that the weakly-quasisymmetric f can be extended to a quasiconformal homeomorphism F' of $\mathbb{R}^2$. Let $F = F' \times id : \mathbb{R}^n \to \mathbb{R}^n$.

Let $a \in \mathbb{R}^n$. Define a map h_a of $\mathbb{R}^n$ by $h_a(z) = z + a$ for $z \in \mathbb{R}^n$. Let $G'_0 = \{h_a : a = (a_1, 0, a_3, ..., a_n), a_i \in \mathbb{R}\}$. Construct a new group $G_0 = FG'_0F^{-1}$.

Tukia in [32], page 157, proved that:

The group G_0 is a Lipschitz group of $\mathbb{R}^n$ such that for no quasiconformal homeomorphism h of $\mathbb{R}^n$ is the conjugation hG_0h^{-1} a Mobius group.

Remark. G_0 is a Lipschitz group on $\mathbb{R}^n$ if there is $L \geq 1$ such that for every $g \in G_0$ and for all $x, y \in \mathbb{R}^n$ we get

$$\frac{|x-y|}{L} \leq |g(x) - g(y)| \leq L|x-y|.$$

Therefore, for any $g \in G_0$ and arbitrary $x \in \mathbb{R}^n$

$$\underset{r\to 0}{lim\,sup}\frac{\max(|g(x)-g(y)|\colon |x-y|=r)}{\min(|g(x)-g(y)|\colon |x-y|=r)} \leq \frac{Lr}{r\backslash L} = L^2.$$

Hence, G_0 is a quasiconformal group, and G_0 is not a quasiconformal conjugate of a Mobius group.

Martin in Theorem 3.8, [25] proved that every discrete subgroup of maximal rank in the group G_0 is not quasiconformal conjugate of a Mobius group.

Remark. To apply these result on the n-sphere, we extend actions on $\mathbb{R}^n$ to actions on $\bar{\mathbb{R}}^n$ fixing ∞, and conjugate them by a stereographic projection.

2.2 Topological conjugacy for discrete convergence groups

Based on the previous section one may ask whether discrete convergence groups are topologically conjugate to Mobius groups in dimension n, or not. The answer is positive for certain groups acting on the unit circle by Theorem 6.B in Tukia, [35]:

Theorem 2.2.1. A discrete convergence group G of a circle is either topologically conjugate to a Fuchsian group or has a semi-triangle group of finite index.

Definition 2.2.2. *Given two elements f and g such that*

$$f^p = g^q = (f \circ g)^{-r}$$

for some $p, q, r > 1$, and $\langle f, g\rangle$ is a discrete nonelementary convergence group. Then the group $\langle f, g\rangle$ is called a semi-triangle group.

In [35], page 50, one can find some conditions when G does not have a semi-triangle group of finite index, and so is topologically conjugate to a Mobuis group. The few of these conditions are:

- G is torsionless,
- G is isomorphic to a Fuchsian group,
- G contains a parabolic element,
- $\Lambda(G) \neq \mathbb{S}^1$,
- G is infinitely generated.

Tukia stated that it is difficult to find a case when G is not topologically conjugate to a Fuchsian group, and it is likely to think about its non-existence. Independent works of Casson - Jungreis in [2] and Gabai in [5] proved that, in fact, all discrete convergence groups of the circle are topologically conjugate to a Fuchsian group. This, actually, disproves the existence of semi-triangle groups of a finite index in a discrete convergence groups of the unit circle. The original statement by Gabai in [5] is:

Theorem 2.2.3. [5, p. 395] G is a discrete convergence group of $\mathbb{S}^1$ if and only if G is conjugate in $\mathrm{Homeo}(\mathbb{S}^1)$ to the restriction of a Fuchsian group.

However, it is not true for all discrete convergence groups acting on $\mathbb{S}^n$ with $n \geq 2$ to be conjugate to a group of Mobius transformations. Here we provide our example for $n = 2$, and then we generalize it for higher dimensions n.

Example 2.2.4. Let G be a group acting on some Hausdorff compact set X, then G is non-elementary if the limit set of G has more than 2 points, where the limit set of G is defined as in Definition 1.2.5 with $\mathbb{S}^n$ replaced by X.

Let G be a nonelementary Fuchsian group acting on $D^2 = int(D^2) \cup \partial D^2$, where $\partial D^2 = \mathbb{S}^1$. By the definition, G acts on $int(D^2)$ properly discontinuously, and so leaves it invariant. Identify the boundary of the disk to a point x, then we get $D^2/\mathbb{S}^1 \cong \mathbb{S}^2$. Now, the group G induces an action by homeomorphisms on the quotient $\mathbb{S}^2$. By the construction, G acts properly discontinuously on the complement of the point x, thus G is a convergence group.

Then the stabilizer of $\{x\}$ is G. If G is conjugate to some Mobius group M, then M must fix a point corresponding to x. But the stabilizer of a point in a Mobius group is virtually abelian by Theorem 2.1 in [36], and this contradicts to our assumption that G is nonelementary (see Theorem 2.2.5).

Now we generalize this example to all $n > 2$. Let G be a nonelementary discrete subgroup of Mobius transformations of the closed n-dimensional ball $B^n = int(B^n) \cup \partial B^n$ acting properly discontinuously on $int(B^n)$. Identify the boundary of the ball to a point x, then $B^n/\mathbb{S}^{n-1} \cong \mathbb{S}^n$, and we get a convergence group G acting properly discontinuously on the complement of x. Continue the steps done above, and we get that G is not conjugate to a Mobius group.

Theorem 5.15 in [6] proves that any abelian discrete convergence group is elementary. Using this result we may get the following theorem for any virtually abelian discrete convergence group:

Theorem 2.2.5. Any virtually abelian convergence group G acting on $\mathbb{S}^n$ is elementary.

Proof. Since G is a virtually abelian group, there exists an abelian subgroup H of G of finite index. Clearly, $\Lambda(H) \subseteq \Lambda(G)$. We will prove that, in fact, $\Lambda(H) = \Lambda(G)$. This result is due Tukia in [33].

Pick any point $y_0 \in \Lambda(G)$. There exist an infinite sequence $\{g_j\} \in G$ and a point $x_0 \in \Lambda(G)$ such that $g_j \to y_0$ locally uniformly in $\mathbb{S}^n \backslash \{x_0\}$. Since the number of cosets is finite, we may pick a subsequence $\{g_j = h_j g\}$ with $\{h_j\} \in H$ and some $g \in G$. Then $h_j \to y_0$ locally uniformly in $\mathbb{S}^n \backslash \{g(x_0)\}$, and so y_0 is a limit point of H. Therefore, $\Lambda(G) \subseteq \Lambda(H)$ $\Rightarrow \Lambda(H) = \Lambda(G)$.

Now, since H is the abelian convergence group acting on $\mathbb{S}^n$, by Theorem 5.15 in [6], $\Lambda(H)$ is elementary, and so $\Lambda(G)$ is elementary. ∎

Definition 2.2.6. *A homeomorphism g acting on $\mathbb{S}^n$ is **standard** if g is topologically conjugate to a Mobius transfomation acting on $\bar{\mathbb{R}}^n$.*

From Chapter 1 we know that given a discrete convergence group G of the circle then an element g of G is either elliptic, parabolic or loxodromic. Clearly, g also generates a discrete convergence group of the circle. Therefore, the following theorem proves that all elements of a discrete convergence group of the circle are standard.

Theorem 2.2.7. [35, p.4-5] If g generates a convergence group of $\mathbb{S}^1$ then g is topologically conjugate to a Mobius transformation.

Proof. Let g be a loxodromic element acting on the circle, then g fixes two points, say x and y. If g is orientation preserving, then components of $\mathbb{S}^1 \backslash \{x, y\}$ are g-invariant. We may assume that there is a homeomorphism $\hat{h} : \mathbb{S}^1 \to \bar{\mathbb{R}}$ such that $h = \hat{h} g \hat{h}^{-1}$ is a self-homeomorphism of the extended real line with fixed $-\infty, 0$ and ∞. Since g has a convergence property we may assume that

$$h^k(x) \to 0 \quad \text{as} \quad k \to -\infty,$$

$$h^k(x) \to \infty \quad \text{as} \quad k \to \infty \quad \text{and} \quad x > 0,$$

$$h^k(x) \to -\infty \quad \text{as} \quad k \to \infty \quad \text{and} \quad x < 0.$$

Let $\tau(x) = 2x$ for all $x \in \mathbb{R}$. Pick any point $x_0 > 0$. Let $f_0 : [x_0, h(x_0)] \to [1, 2]$ be a homeomorphism such that $f_0(x_0) = 1$ and $f_0(h(x_0)) = 2$. Extend f_0 to a homeomorphism $f : (0, \infty) \to (0, \infty)$ by

$$f|_{[h^k(x_0), h^{k+1}(x_0)]} = \tau^k f_0 h^{-k} \quad \text{for each} \quad k \in \mathbb{Z}.$$

Choose any point x and assume that $x \in [h^k(x_0), h^{k+1}(x_0)]$ for some integer k. Then

$$fh(x) = f(h(x)) = \tau^{k+1} f_0 h^{-k-1}(h(x)) = \tau^{k+1} f_0 h^{-k}(x),$$

$$\tau f(x) = \tau \tau^k f_0 h^{-k}(x) = \tau^{k+1} f_0 h^{-k}(x).$$

As a result, $fh = \tau f$ on the interval $(0, \infty)$.

In order to get a conjugacy on $(-\infty, 0)$, let $f_0 : [h(-x_0), -x_0] \to [-2, -1]$ be a homeomorphism such that $f_0(-x_0) = -1$ and $f_0(h(-x_0)) = -2$. Doing similar extension of f_0 on the negative line,

$$f|_{h^{k+1}(-x_0), h^k(-x_0)]} = \tau^k f_0 h^{-k} \quad \text{for each} \quad k \in \mathbb{Z}.$$

Then similarly we get $fh = \tau f$ on the interval $(-\infty, 0)$.

To have a homeomorphism $f : \mathbb{R} \to \mathbb{R}$, we may assume $f(0) = 0$ and it suffices to prove that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0.$$

Given $\epsilon > 0$, to show that $\lim_{x \to 0^+} f(x) = 0$ we need to find a constant $\delta > 0$ such that $|f(x)| < \epsilon$ whenever $0 < x < \delta$. Choose an integer $N > 0$ so that $\frac{1}{2^k} < \epsilon$ for all $k \geq N$. Let $\delta = h^{-N}(x_0)$. Choose any positive $x < \delta$, then $x \in [h^k(x_0), h^{k+1}(x_0)]$ for some $k < -N$. Therefore, since $f_0(x)$ for $x \in [x_0, h(x_0)]$ is contained in the interval $[1, 2]$, we get

$$0 < f(x) = \tau^k f_0 h^{-k}(x) = \frac{f_0(h^{-k}(x))}{2^{|k|}} \leq \frac{1}{2^{|k+1|}} \leq \frac{1}{2^N} < \epsilon.$$

Similarly, we prove that $\lim_{x \to 0^-} f(x) = 0$.

Therefore, $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Since $h = f^{-1} \tau f = \hat{h} g \hat{h}^{-1}$ and both f and $\hat{h}$ are homeomorphisms, we get that g is conjugate to a Mobius transformation $x \mapsto 2x$.

If g is orientation reversing, we may assume that there is a homeomorphism $\hat{h} : \mathbb{S}^1 \to \bar{\mathbb{R}}$ such that $h = \hat{h} g \hat{h}^{-1}$ is a self homeomorphism of $\bar{\mathbb{R}}$ with fixed $-\infty, 0$ and ∞. The only difference from the orientation preserving case is that $h : (-\infty, 0] \to [0, \infty)$ and $h : [0, \infty) \to (-\infty, 0]$. For $-h$, similarly as above, we have that $-h$ is conjugate to $x \mapsto 2x$ Mobius transformation, and so one can easily prove that g is conjugate to $x \mapsto -2x$.

If g is a parabolic element, then g fixes one point. Using the same idea for constructions of homeomorphisms, one can prove that g is conjugate to $x \mapsto x + 1$ translation.

Let g be elliptic. Pick any point $x \in \mathbb{S}^1$. $A_x = \{g^k(x) : k \in \mathbb{Z}\}$ is finite. Then g permutes the components of $\mathbb{S}^1 \backslash A_x$, and so g is conjugate to an elliptic Mobius transform. ■

One might be interested if elements of discrete convergence groups of $\mathbb{S}^n$ are standard for $n \geq 2$. The answer to that question was given dy different authors, and we just mention their results.

- *Elliptic elements.*

  Brouwer - Kerekjarto - Eilenberg in [3] proved that each periodic element of Hom($\mathbb{S}^2$) is topologically conjugate to an orthogonal transformation.

  Affirmative solution of the Smith conjecture in [29] implies that periodic diffeomorphisms of the 3-sphere with nonempty fixed points set are conjugate to orthogonal transformations. Nothing is known for an empty fixed points set. However, there is a self-homeomorphism of the 3-sphere of period 2 constructed by Montgomery and Zippin in [28] with a wild knot set of fixed points which is not topologically conjugate to an orthogonal transformations.

  When $n \geq 4$, Giffen in [8] shows that there exist quasiconformal periodic orientation-preserving homeomorphisms of the n-sphere which are not conjugate to orthogonal transformations.

- *Parabolic elements.*

  The convergence property of parabolic elements in higher dimensions is known as Sperner's condition, and homeomorphisms of $\mathbb{S}^n$ satisfying Sperner's conditions, i.e. parabolic elements of a convergence group of $\mathbb{S}^n$ for $n > 2$, are called quasitranslations. Kerekjarto in [17] proved that quasitranslations of $\mathbb{S}^2$ are topologically conjugate to translation. However, they are not conjugate to translations in higher dimensions. The counter-examples are given by Kinoshita in [18] for dimension 3, and by Husch in [14] in higher dimensions.

- *Loxodromic elements.*

  The loxodromic homeomorphisms of $\mathbb{S}^n$ are called topological dilations. It has been proved that orientation-preserving topological dilations are conjugate to the standard dilations $x \to 2x$. The proof for $n = 2$ is given by Kerekjarto in [16], for $n = 3$ by Homma and Kinoshita in [13] and by Husch in [15] for all $n \geq 6$.

We finish this section with alternative definition of a discrete convergence group acting on $\mathbb{S}^n$ (including $n = 1$).

Definition 2.2.8. *Let* $T_n = \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{S}^n - \Delta$, *where* $\Delta = \{(x, y, z) : x = y \quad or \quad x = z \quad or \quad y = z\}$.

Given a group G of homeomorphisms of $\mathbb{S}^n$, then for all $\alpha = (x, y, z) \in T_n$ we define a map

$$\phi(g)(\alpha) = (g(x), g(y), g(z))$$

which is, in fact, a homeomorphism embedding G in Hom(T_n). Then we get the following definition which we can introduce as a theorem, see also [27].

Theorem 2.2.9. A subgroup G of Hom($\mathbb{S}^n$) is an infinite discrete convergence group if and only if $\phi(G)$ acts properly discontinuously on T_n.

Proof. **The "only if" part:** Let G be a discrete convergence group of $\mathbb{S}^n$, and suppose the action of G on T_n is not properly discontinuous. Then, one can find a compact subset K of T_n and an infinite sequence $\{g_j\}$ such that

$$\phi(g_j)(K) \cap K \neq 0 \quad \text{for all} \quad j.$$

Therefore, there exists a sequence $(x_j, y_j, z_j) \in K$ such that $\phi(g_j)(x_j, y_j, z_j) \in K$ for all j. Since K is compact, there exist convergent subsequences $x_j \to x, y_j \to y, z_j \to z$ such that $g_j(x_j) \to x'$, $g_j(y_j) \to y'$ and $g_j(z_j) \to z'$, and so $(x', y', z') \in K \subset T_n$.

On the other hand, the convergence property of G implies that for a given infinite sequence $\{g_j\}$ there exist points $a, b \in \mathbb{S}^n$ such that $g_j \to a$ locally uniformly in $\mathbb{S}^n \backslash \{b\}$. Since $(x_j, y_j, z_j) \to (x, y, z) \in T_n$, we may assume that $x \neq b \neq y$. Hence,

$$g_j(x_j) \to a \quad \text{and} \quad g_j(y_j) \to a \quad \Rightarrow \quad x' = a = y'.$$

This contradicts to $(x', y', z') \in T_n$, and so $\phi(G)$ acts properly discontinuously on T_n.

The "if" part: Let $\phi(G)$ act properly discontinuously on T_n. Let $\{g_j\}$ be an infinite sequence in G converging to some $g \in G$. Then for an arbitrary $(x, y, z) \in T_n$, the set $K = \Big((x, y, z), (g_j x, g_j y, g_j z), (gx, gy, gz)\Big)$ is compact in T_n and $g_j K \cap K \neq \emptyset$ for all j. This contradicts to properly discontinuous actions of $\phi(G)$ on T_n. Therefore, G is discrete.

Given an infinite family $F = \{g_j\}$ in G. Pick an arbitrary $(x, y, z) \in T_n$. There exists an infinite subsequence $F' = \{g_j\}$ of F such that $g_j x \to x_0$, $g_j y \to y_0$ and $g_j z \to z_0$. If all x_0, y_0 and z_0 are distinct then $(x_0, y_0, z_0) \in T_n$, and the properly discontinuous property fails for the compact set $K = \Big((x, y, z), (g_j x, g_j y, g_j z), (x_0, y_0, z_0)\Big)$. Hence, x_0, y_0 and z_0 are not all distinct, and we can assume that

$$g_j x \to a, \quad g_j y \to a \quad \text{and} \quad g_j z \to b$$

for some $a, b \in \mathbb{S}^n$. *We claim that $g_j \to a$ locally uniformly in $\mathbb{S}^n \backslash \{z\}$.*

Since for any point $w \in \mathbb{S}^n \backslash \{z\}$ there exists an infinite subsequence $\{g_j\}$ of F' such that $\{g_j w\}$ converges, we assume, for simplicity, that $F'(w) = \{g_j w\}$ converges for all $w \in \mathbb{S}^n \backslash \{z\}$.

Choose any point $w \in \mathbb{S}^n \backslash \{z\}$. If $g_j w \to c \notin \{a, b\}$ then the properly discontinuous property fails for the compact set $K = \Big((x, z, w), (g_j x, g_j z, g_j w), (a, b, c)\Big)$. Therefore, $g_j w \to \{a, b\}$.

- $a \neq b$.

  Suppose $g_j w \to b$. Take any $c \in \mathbb{S}^n \backslash \{a, b\}$. Let $u_j = g_j^{-1} c$, and by reducing to its subsequence we get that $u_j \to u$. Since x, y, z, w are all distinct, by relabelling if necessary, one might assume that $(x, z, u) \in T_n$. Then for the compact set $K = \Big((x, z, u_j), (x, z, u), (g_j x, g_j z, g_j u_j), (a, b, c)\Big) \subset T_n$ the properly discontinuous property fails. Hence $g_j w \to a$. The choice of w was arbitrary, so $g_j \to a$ pointwise in $\mathbb{S}^n \backslash \{z\}$.

  Since the n-sphere is locally compact space, the locally uniform convergence on $\mathbb{S}^n \backslash \{z\}$ is equivalent to uniform convergence on compact sets of $\mathbb{S}^n \backslash \{z\}$.

  Suppose that $g_j \to a$ convergence is not locally uniform on $\mathbb{S}^n \backslash \{z\}$, then there exists a compact set $K \subset \mathbb{S}^n \backslash \{z\}$ and a sequence $\{w_j\} \in K$ such that $g_j w_j \to c \neq a$. If $c \neq b$, the properly discontinuous property does not hold for the compact set $K' = \{(x, z, w_j), (g_j x, g_j z, g_j w_j), (a, b, c)\}$. Therefore, $g_j w_j \to b$. Using similar argument with $u_j = g_j^{-1} d$ for some $a \neq d \neq b$ as we did above, we would get a contradiction. Henceforth, the convergence is locally compact.

- $a = b$.

  Take any $c \in \mathbb{S}^n \backslash \{a\}$. Let $w_j = g_j^{-1} c$. The proof goes similarly to the previous case: first, one can show that the convergence is pointwise, and then prove that $g_j \to a$ locally uniformly on $\mathbb{S}^n \backslash \{a\}$ for both $j \to -\infty$ and $j \to \infty$.

$\blacksquare$

Chapter 3

Pseudo-Riemannian Space forms

In this chapter we consider an action of subgroups of $O(p+1,q)$ on the space $S^{p,q}$ defined in Chapter 1. Our main goal is to show conditions when

- infinite discrete subgroups Γ of $O(p+1,q)$ act properly discontinuously on $S^{p,q}$,
- and the orbit space $\Gamma\backslash S^{p,q}$ is compact.

We remind that Q is the quadratic form in $\mathbb{R}^{n+1}$ of type $(p+1,q)$ with $p+q=n$ and

$$Q(x_1,...,x_{p+1},y_1,...,y_q)=\sum_{j=1}^{p+1}x_j^2-\sum_{j=1}^{q}y_j^2.$$

Then $S^{p,q}$ is the component of $\{v\in\mathbb{R}^{n+1}:Q(v)=1\}$ containing $(1,0,...,0)$ and $O(p+1,q)$ is the group of Q-orthogonal transformations which preserve $S^{p,q}$. The Q-orthogonality is defined by preserving the bilinear form $b_q^p(s,t)$ for any $s,t\in S^{p,q}$, $s=(s_1,...,s_{n+1})$ and $t=(t_1,...,t_{n+1})$ where

$$b_q^p(s,t)=s_1t_1+...+s_{p+1}t_{p+1}-s_{p+2}t_{p+2}-...-s_{n+1}t_{n+1}.$$

That is $g\in O(p+1,q)$ if for any two points $s,t\in S^{p,q}$ we have

$$b_q^p(s,t)=b_q^p(gs,gt).$$

Further, for simplicity, $(x, y) \in S^{p,q}$ denotes a point $(x_1, ...x_{p+1}, y_1, ..., y_q) \in S^{p,q}$ with $x = (x_1, ..., x_{p+1})$ and $y = (y_1, ..., y_q)$.

Let all definitions used in this chapter be similar to definitions from Chapter 1, replacing $\mathbb{S}^n$ by $S^{p,q}$ and no confusion should occur.

3.1 Existence/non-existence of properly discontinuous actions on $S^{p,q}$

Kulkarni and Wolf answered to the question raised above:

are there infinite discrete subgroups Γ of $O(p+1, q)$ acting properly discontinuously on $S^{p,q}$?

Theorem 3.1.1. [39, p.78-79] There is no infinite subgroups of $O(p+1, q)$ acting properly discontinuously on $S^{p,q}$ when $p \geq q$.

Proof. Let Γ be an infinite subgroup of $O(p+1, q)$ acting properly discontinuously on $S^{p,q}$. Let S_p be a subspace of $S^{p,q}$ such that if $(x, y) \in S_p$ then $y_j = 0$ for $j = 1, ..., q$. For an arbitrary non-singular linear transformation g of $\mathbb{R}^{n+1}$ we have:

$$\dim(S_p + gS_p) = \dim(S_p) + \dim(gS_p) - \dim(S_p \cap gS_p).$$

The nature of g implies

$$\dim(S_p) = \dim(gS_p).$$

$\dim(\mathbb{R}^{n+1}) = n + 1 \Rightarrow \dim(S_p + gS_p) \leq n + 1$, and noting that $p \geq q$ we get

$$\dim(S_p \cap gS_p) \geq 2 \cdot \dim(S_p) - (n + 1) > 0.$$

That is gS_p meets S_p.

Since Γ is infinite and S_p is compact (because S_p is a kind of the sphere in $S^{p,q}$), there is a sequence $\{g_j\}$ of distinct elements of Γ and a sequence $\{s_j\}$ of points in S_p such that $\{g_j(s_j)\} \to s$ for some point $s \in S_p$. There is a convergent subsequence of $\{s_j\}$, which will be denoted as $\{s_j\}$ again, such that $\{s_j\} \to s' \in S_p$. $\Rightarrow$ there is an element $h \in \Gamma$ with $h(s') = s$, so $\{h^{-1}g_j(s_j)\} \to h^{-1}(s) = s'$. Thus, for each neighborhood U of s' there is an integer n such that $s_j \in U$ and $h^{-1}g_j(s_j) \in U$ for all $j > n$. As a result,

$$\{h^{-1}g_j\}_{j>n} \subset \{g \in \Gamma : gU \cap U \neq \emptyset\}$$

and, since all g_j are distinct, the set $\{h^{-1}g_j\}_{j>n}$ is infinite. This contradicts to the fact that the action of Γ on $S^{p,q}$ is properly discontinuous. $\Rightarrow \Gamma$ is not infinite. ∎

Remark. Assume $p > 1$ and let Γ be, in addition, a free action on $S^{p,q}$. The quotient map $\alpha : S^{p,q} \to \Gamma\backslash S^{p,q}$ is a regular cover, and since $S^{p,q} \cong \mathbb{S}^p \times \mathbb{R}^q$ we have, in fact, α is the universal cover for $p > 1$. Henceforth, Γ is a group of deck transformations of the covering and, so, is isomorphic to the fundamental group $\pi_1(\Gamma\backslash S^{p,q})$. Theorem 3.1.1 gives us the statement from [39]: $\Gamma\backslash S^{p,q}$ *has a finite fundamental group when the action is free and properly discontinuous and* $p \geq q$.

However, this is no longer true when $p < q$. Kulkarni in [21] proved the following theorem:

Theorem 3.1.2. [21, p. 27-28] If $p < q$ there exist infinite subgroups of $O(p+1, q)$ acting properly discontinuously on $S^{p,q}$.

The idea of the proof of this theorem is to find a subgroup G of $O(p+1, q)$ acting properly on $S^{p,q}$, and then to construct its discrete subgroup Γ which obviously will act properly discontinuously by the definition.

First, we introduce a useful lemma which is used as a simplification in the study of proper actions on $S^{p,q}$.

Lemma 3.1.3. [21, p.20-21] Let G be a subgroup of $O(p+1,q)$ and $\mathbb{R}^{n+1} = V_1 \oplus V_2$ be a Q-orthogonal decomposition into G-invariant proper subspaces. Let $S_j = V_j \cap S^{p,q}$ for $j = 1,2$. Then G acts properly on $S^{p,q}$ if and only if it acts properly on each S_j.

Proof. The "only if" part is obvious. So we restrict our attention to the "if" part.

Suppose G acts properly on each S_j. Let $Q_j = Q|_{V_j}$. Note that each $v \in \mathbb{R}^{n+1}$ is $v = v_1 + v_2$, where v_j is a projection of v in V_j. Therefore, $v = (x_1, ..., x_{p+1}, y_1, ..., y_q) \in \mathbb{R}^{n+1}$ is equal to the sum of $v_j = (x_{1,j}, ..., x_{p+1,j}, y_{1,j}, ..., y_{q,j}) \in V_j$ for $j = 1,2$. That implies

$$\begin{aligned} Q(v) &= Q(x_1, ..., x_{p+1}, y_1, ..., y_q) \\ &= \sum_{j=1}^{p+1} x_j^2 - \sum_{j=1}^{q} y_j^2 \\ &= \sum_{j=1}^{p+1} (x_{j,1} + x_{j,2})^2 - \sum_{j=1}^{q} (y_{j,1} + y_{j,2})^2 \\ &= \sum_{j=1}^{p+1} x_{j,1}^2 - \sum_{j=1}^{q} y_{j,1}^2 + \sum_{j=1}^{p+1} x_{j,2}^2 - \sum_{j=1}^{q} y_{j,2}^2 + 2\sum_{j=1}^{p+1} x_{j,1}x_{j,2} - 2\sum_{j=1}^{q} y_{j,1}y_{j,2} \end{aligned} \tag{3.1}$$

Let $Q_i(v_i) = \sum_{j=1}^{p+1} x_{j,i}^2 - \sum_{j=1}^{q} y_{j,i}^2$ and note that Q-orthogonal decomposition $\mathbb{R}^{n+1} = V_1 \oplus V_2$ gives $b_q^p(v_1, v_2) = \sum_{j=1}^{p+1} x_{j,1}x_{j,2} - \sum_{j=1}^{q} y_{j,1}y_{j,2} = 0$, so it follows $Q(v) = Q_1(v_1) + Q_2(v_2)$.

Choose any compact subset C of $S^{p,q}$. For each $j = 1,2$ define $C_j = \{v \in C : Q_j(v_j) \geq \frac{1}{2}\}$. Since $Q(v) = Q_1(v_1) + Q_2(v_2) = 1$ we get $C = C_1 \cup C_2$. Let g be an element of G such that $gv \in C$ for some $v \in C$, i.e. $g \in \zeta_G(C)$ (we remind that $\zeta_G(C)$ is given in Definition 1.2.4). Then $v \in C_j$ implies $gv \in C_j$. $\Rightarrow \zeta_G(C) = \zeta_G(C_1) \cup \zeta_G(C_2)$.

Let $p_j : \mathbb{R}^{n+1} \to V_j$ be the Q-orthogonal projection, then $p_j(C_j)$ is compact and $\zeta_G(C_j)$ is a closed subset of $\zeta_G(p_j(C_j))$. Since the action of G on S_j is proper, the set $\zeta_G(p_j(C_j))$ is compact $\Rightarrow \zeta_G(C_j)$ is compact. The result is true for both $j = 1$ and $j = 2$, and since $\zeta_G(C) = \zeta_G(C_1) \cup \zeta_G(C_2)$, we get compact $\zeta_G(C)$, i.e. proper action of G on $S^{p,q}$. ∎

Remark. All of $S^{p,q}$, V_1 and V_2 in Lemma 3.1.3 are G-invariant, so both S_1 and S_2 are G-invariant.

Using Lemma 3.1.3 one can prove that:

Theorem 3.1.4. [21, p. 27-28] If $p < q$ then there exists a subgroup G, $G \cong \mathbb{R}$, of $O(p+1, q)$ acting properly on $S^{p,q}$.

Proof. Let $\mathbb{R}^{n+1} = W \oplus \{\oplus_j V_j\}$ be an orthogonal direct sum, where $1 \leq j \leq p+1$, dim(V_j)= 2 and dim(W)= $q-p-1$. If $v = (x_1, ..., x_{p+1}, y_1, ..., y_q) \in \mathbb{R}^{n+1}$ and $v = w + \sum_{j=1}^{p+1} v_j$ with $w \in W$, $v_j \in V_j$, then $v_j = (0, ..., 0, x_j, 0, ..., 0, y_j, 0, ..., 0)$ (for simplicity, we say $v_j = (x_j, y_j)$) and $w = (0, ..., 0, y_{p+2}, ..., y_q)$. Therefore,

$$Q|_{V_j}(v) = x_j^2 - y_j^2 \quad \text{and} \quad Q|_W(v) = -y_{p+2}^2 - ... - y_q^2.$$

Rearrange the coordinates in v so that $v = (x_1, y_1, x_2, y_2, ..., x_{p+1}, y_{p+1}, y_{p+2}, ..., y_q)$, then we get $v^t A v = Q(v)$ for

$$A = \text{diag}(\underbrace{1, -1, ..., 1, -1}_{2p+2 \quad \text{entries}}, -1, ..., -1).$$

Let $x_j = \frac{a_j + b_j}{2}$ and $y_j = \frac{a_j - b_j}{2}$ be a change of variables for each v_j, then we get

$$Q|_{V_j}(v) = a_j b_j.$$

Let $\mathbb{R}_+^* \cong \mathbb{R}$ be a multiplicative group of positive reals. For each $t \in \mathbb{R}_+^*$ let

$$A_t = \text{diag}(\underbrace{t, \frac{1}{t}, ..., t, \frac{1}{t}}_{2p+2 \quad \text{entries}}, 1, ..., 1).$$

Then A_t acts trivially on W and throught the map $(a_j, b_j) \to (ta_j, \frac{b_j}{t})$ on V_j. This defines a subgroup $G = \{A_t : t \in \mathbb{R}_+^*\}$ of $O(p+1, q)$, $G \cong \mathbb{R}$, acting properly on $S^{p,q} \cap W = \emptyset$ and each $S_j = S^{p,q} \cap V_j$ for $j = 1, ..., p+1$. Using Lemma 3.1.3, this implies that G acts properly on $S^{p,q}$ ∎

Remark. The group G from Theorem 3.1.4 is not discrete. To obtain a discrete subgroup of G take

$$G' = \{A_t : t = 2^n \quad \text{for all} \quad n \in \mathbb{Z}\}.$$

Then, $G' \subset O(p+1,q)$ acts properly discontinuously on $S^{p,q}$.

In addition to Theorem 3.1.4, Kulkarni in [21], Thm.5.7(b) shows the following result:

Theorem 3.1.5. If $p+1 < q$ or $p+1 = q$ is even, then there exists a subgroup of G of $O(p+1,q)$, locally isomorphic to $SL(2,\mathbb{R})$, acting properly on $S^{p,q}$.

3.2 Properly discontinuous, free, co-compact actions on $S^{p,q}$

Definition 3.2.1. *Let Γ be a subgroup of $O(p+1,q)$ acting freely and properly discontinuously on $S^{p,q}$. Then the orbit space $\Gamma\backslash S^{p,q}$ is called a **space form of** Γ, or we simply call it as a **space form** so that no confusion should appear.*

We show conditions for the existence and non-existence of subgroups of $O(p+1,q)$ acting on $\mathbb{S}^n$ properly discontinuously with a compact space form, and their properties.

Definition 3.2.2. *Let Γ be a group acting freely and properly discontinuously on $S^{p,q}$. We denote $B\Gamma = K(\Gamma,1)$, where $K(\Gamma,1)$ is the Eilenberg-Maclane space with a contractible universal covering space and $\pi_1(K(\Gamma,1)) \cong \Gamma$.*

Let R be a commutative ring with $1 \neq 0$. Let $hd_R\Gamma$ be a homological dimension of Γ over R. Let $vhd_R\Gamma$ be a virtual homological dimension of Γ over R, that is $vhd_R\Gamma = d$ if $hd_R\Gamma' = d$ for some subgroup Γ' of Γ of finite index.

First, we give a homological restriction for the existence of compact space forms, see Theorem 2.1 by Kulkarni in [21]:

Theorem 3.2.3. Let G be a group acting freely and properly discontinuously on $S^{p,q}$. Let R be a commutative ring with $1 \neq 0$. Let $M = \Gamma \backslash S^{p,q}$. Suppose $vhd_R\Gamma < \infty$. Then $vhd_R\Gamma \leq q$, and the equality holds if and only if M is compact.

Example 3.2.4. As an example we can consider the action of $G \cong \mathbb{Z}^r$ on $\mathbb{S}^p \times \mathbb{R}^q \cong S^{p,q}$ for $r \leq q$. That is given a space $\mathbb{S}^p \times \mathbb{R}^q$, let $r \leq q$ and G be a group generated by r transformations $g_1, g_2, ..., g_r$ of $\mathbb{S}^p \times \mathbb{R}^q$ where each transformation g_j acts on $\mathbb{S}^p \times \mathbb{R}^q$ by $x \mapsto x+1$ translation of only one copy of $\mathbb{R}$. Notice that any two transfomations g_i and g_j with $i \neq j$ translate different copies of $\mathbb{R}$ in $\mathbb{S}^p \times \mathbb{R}^q$. So, we get an action of $\mathbb{Z}^r$ on $\mathbb{S}^p \times \mathbb{R}^q$ which is clearly free and properly discontinuous.

Notice that the r-dimesional torus $\mathbb{T}^r$ is $K(\mathbb{Z}^r, 1)$ with $H^r(\mathbb{T}^r, \mathbb{Z}) \neq 0$, and so $hdG < \infty$. Now, for $r < q$, the quotient $\mathbb{S}^p \times \mathbb{R}^q/\mathbb{Z}^r \cong \mathbb{S}^p \times \mathbb{T}^r \times \mathbb{R}^{q-r}$ is not compact. However, when $r = q$ we get $\mathbb{S}^p \times \mathbb{R}^q/\mathbb{Z}^q \cong \mathbb{S}^p \times \mathbb{T}^q$ which is compact.

More possible restrictions for the existence of the compact space forms were given by Kobayashi and Yoshino in [19], pages 619-620:

Conjecture: There exists an infinite discrete subgroup Γ of $O(p+1, q)$ acting freely and properly discontinuousy on $S^{p,q}$ with compact space form if and only if (p, q) pair has one of the following forms:

- $(p, q) = (r, 0)$ for any $r \in \mathbb{N}$;
- $(p, q) = (0, r)$ for any $r \in \mathbb{N}$;
- $(p, q) = (1, 2r)$ for any $r \in \mathbb{N}$;
- $(p, q) = (3, 4r)$ for any $r \in \mathbb{N}$;
- $(p, q) = (7, 8)$.

The best known results for the "only if" part is that there is no infinite discrete subgroups of $O(p+1, q)$ acting freely, properly discontinuously and co-compactly on $S^{p,q}$ when (p, q) pair satisfies to either $p \geq q > 0$, $p+1 = q =$ odd, or $pq =$ odd.

However, the "if" part of this Conjecture is true:

- $(p,q) = (r,0)$ with $r \in \mathbb{N}$.

  Then $S^{p,q} \cong \mathbb{S}^p$ is compact, and so is $\Gamma \backslash S^{p,q}$. That is because the projection map $S^{p,q} \to \Gamma \backslash S^{p,q}$ is continuous and surjective, the continuous image of compact space is compact.

- $(p,q) = (0,r)$ for any $r \in \mathbb{N}$.

  Then $S^{p,q} \cong \mathbb{R}^q$ is a Riemannian symmetric space and, by [1], it admits compact space form.

- $(p,q) = (1,2r)$ for any $r \in \mathbb{N}$.

  Then Q is a non-degenerate quadratic form in $\mathbb{R}^{2r+2}$ of type $(2,2r)$. Let W be a $\mathbb{C}$-vector space of dimension $r+1$ and H be a non-degenerate hermitian form of type $(1,r)$ for $\mathbb{C}$.

  If $w = (x_1 + ix_2, y_{11} + iy_{21}, ..., y_{1r} + iy_{2r}) \in W$, then

  $$H(w,w) = (x_1+ix_2)(x_1-ix_2)-(y_{11}+iy_{21})(y_{11}-iy_{21})-...-(y_{1r}+iy_{2r})(y_{1r}-iy_{2r}).$$

  Clearly,

  $$Re(H(w,w)) = x_1^2 + x_2^2 - \sum_{j=1}^{r}(y_{1j}^2 + y_{2j}^2) = Q(v,v)$$

  for $v = (x_1, x_2, y_{11}, y_{21}, ..., y_{1r}, y_{2r}) \in \mathbb{R}^{2r+2}$. Therefore, denoting the underlying real vector space of W as $W_{\mathbb{R}}$, we get that $(W_{\mathbb{R}}, Re(H))$ may be identified with $(\mathbb{R}^{2r+2}, Q)$.

  We define an H-isometry to be a bijection $f : \mathbb{R}^{2r+2} \to \mathbb{R}^{2r+2}$ with $H(f(v), f(v)) = H(v,v)$ for all $v \in \mathbb{R}^{2r+2}$. Let $U(H)$ be a group of H-isometries. Since $(W_{\mathbb{R}}, Re(H))$ may be identified with $(\mathbb{R}^{2r+2}, Q)$ and $U(H)$ acts on $\mathbb{R}^{2r+2}$ then $U(H) \subseteq O(2,2r)$.

Kulkarni in Theorem 6.1, [21] proves that there exist a discrete, co-compact torsion-free subgroup of $U(H)$ acting freely and properly discontinuously on $S^{2,2r}$ with compact space form.

Kobayashi and Yoshino in [19] also give a proof using the Lie theory.

- $(p, q) = (3, 4r)$ for any $r \in \mathbb{N}$.

  The proof is given by Kulkarni in Theorem 6.1, [21] which is similar to $(p, q) = (1, 2r)$ case using the real quaternions space instead of complex plane $\mathbb{C}$.

  Kobayashi and Yoshino in [19] also give a proof using the Lie theory.

- $(p, q) = (7, 8)$

  The proof is given by Kobayashi and Yoshino in [19].

3.3 Some properties of $O(2, 1)$ group

One of the results of Conjecture given above is that there is no discrete subgroups of $O(p + 1, q)$ acting freely, properly discontinuously and co-compactly on $S^{p,q}$ when $pq =$ odd. Hence, this statement holds for $p = q = 1$, and the next proposition might be considered as a particular corollary.

Proposition 3.3.1. Let $(p, q) = (1, 1)$. Let Γ be a group generated by two homeomorphisms of $\mathbb{S}^1 \times \mathbb{R}^1$, that is $\Gamma = \langle a, b\rangle$ where

- $a : \mathbb{S}^1 \times \mathbb{R}^1 \to \mathbb{S}^1 \times \mathbb{R}^1$ is a homeomorphism such that $a(x, y) = (-x, -y)$ for any $(x, y) \in \mathbb{S}^1 \times \mathbb{R}^1$. In other words, the map a acts on $\mathbb{S}^1$ as the antipodal map, and acts on $\mathbb{R}^1$ as a reflection with respect to a point 0.

- $b : \mathbb{S}^1 \times \mathbb{R}^1 \to \mathbb{S}^1 \times \mathbb{R}^1$ is a homeomorphism such that $b(x, y) = (-x, 2 - y)$ for any $(x, y) \in \mathbb{S}^1 \times \mathbb{R}^1$. In other words, the map b acts on $\mathbb{S}^1$ as the antipodal map, and acts on $\mathbb{R}^1$ as a reflection with respect to a point 1.

Then the action of Γ on $\mathbb{S}^1 \times \mathbb{R}^1$ is free and properly discontinuous, and $\Gamma \backslash \mathbb{S}^1 \times \mathbb{R}^1$ is compact. Moreover, since $S^{1,1} \cong \mathbb{S}^1 \times \mathbb{R}^1$, the group $\alpha^{-1} \circ \Gamma \circ \alpha$ acting on $S^{1,1}$ is not a subgroup of $O(2,1)$, where $\alpha : S^{1,1} \to \mathbb{S}^1 \times \mathbb{R}^1$ is a homeomorphism defined by $(x,y) \to \left(\frac{x}{|x|}, y\right)$.

Proof. Notice that both generators of Γ are of finite order 2. Therefore, any element $g \in \Gamma$ is a finite combination of elements a and b such that any two consecutive maps inside g are of different types, i.e. we define $g = a^{\alpha}ba...aba^{\beta}$ where α and β are either 0 or 1, and no two adjacent elements are equal. This construction of g is of *reduced* form, and the length of g is a number of a's and b's in the reduced form of g. Γ is clearly discrete since if $g_j \to g$ then g must be an element of the infinite length, which, in fact, is not an element of Γ.

First, we show that the action of Γ is free. Let g be an arbitrary element of Γ, and choose an arbitrary point $(x,y) \in \mathbb{S}^1 \times \mathbb{R}^1$. Suppose that the action of g starts with the element a, i.e.

$$g(x,y) = aba...ba(x,y) \quad \text{or} \quad g(x,y) = bab...ba(x,y).$$

$a(x,y) = (-x,-y)$ and $ba(x,y) = b(-x,-y) = (x, 2+y)$. Therefore, the action of a map the point x to $-x$, and the point x returns back to itself only after the action of type ba. Since $x \neq -x$ for any point $x \in \mathbb{S}^1$, $g(x,y) = (x,y)$ might happen only when $g = bab...ba$. However, each next action of b moves the point (x,y) further and further from itself in $\mathbb{R}^1$ line, that is $g(x,y) \neq (x,y)$ for all $g \in \Gamma$, $g = bab..ba$. Similarly we can prove this for all $g = aba...ab \in \Gamma$. The difference in these proofs is that in the first case the point (x,y) moves to the right in $\mathbb{R}^1$, while in the second case the point moves to the left. One can assume that $g \in \Gamma$ is of type $g = aba...ba$ or $g = bab...ab$. But, in that cases, g does not return the point $x \in \mathbb{S}^n$ to itself. So we conclude that the action of Γ is free.

It is not hard to see, that the action of Γ on $\mathbb{S}^1 \times \mathbb{R}^1$ is also properly discontinuous. This can be easily seen by the fact that b function moves each point further and further from itself in the y-coordinate.

Therefore, we get an infinite discrete group Γ acting freely and properly discontinuously on $\mathbb{S}^1 \times \mathbb{R}^1$. Note that $\Gamma \backslash \mathbb{S}^1 \times \mathbb{R}^1 \cong \mathbb{S}^1 \times [0,1]$ which is compact. This contradicts to the non-existence of the compact space form for $(p,q) = (1,1)$, therefore $\alpha^{-1} \circ \Gamma \circ \alpha$ is not a subgroup of $O(2,1)$, where α is a homeomorphism from $S^{1,1}$ to $\mathbb{S}^1 \times \mathbb{R}^1$ and so $\alpha^{-1} \circ \Gamma \circ \alpha$ is a homeomorphism of $S^{1,1}$.

The generators of $\Gamma' = \alpha^{-1} \circ \Gamma \circ \alpha$ are $a' = \alpha^{-1} \circ a \circ \alpha$ and $b' = \alpha^{-1} \circ b \circ \alpha$. Remind that

$$\alpha : S^{1,1} \to \mathbb{S}^1 \times \mathbb{R}^1,$$

$$(x,y) \mapsto \Big(\frac{x}{|x|}, y\Big).$$

Therefore,

$$\alpha^{-1} : \mathbb{S}^1 \times \mathbb{R}^1 \to S^{1,1},$$

$$(s,t) \mapsto (s\sqrt{1+t^2}, t).$$

As a result, a' acts on $(x,y) \in S^{1,1}$ in the following way:

$$(x,y) \mapsto \Big(\frac{x}{|x|}, y\Big) \mapsto \Big(-\frac{x}{|x|}, -y\Big) \mapsto \Big(-\frac{x}{|x|}\sqrt{1+y^2}, -y\Big),$$

and b' acts on $(x,y) \in S^{1,1}$ via

$$(x,y) \mapsto \Big(\frac{x}{|x|}, y\Big) \mapsto \Big(-\frac{x}{|x|}, 2-y\Big) \mapsto \Big(-\frac{x}{|x|}\sqrt{1+(2-y)^2}, 2-y\Big).$$

To see if a' and b' are elements of $O(2,1)$ we need to check whether they are Q-orthogonal. That is for any $z, z' \in S^{1,1}$ the bilinear form must be preserved, i.e. $b_1^1(z,z') = b_1^1(a'(z), a'(z')) = b_1^1(b'(z), b'(z'))$. Since $z = (x_1, x_2, x_3), z' = (y_1, y_2, y_3) \in \mathbb{R}^3$ we get

$$a'(x_1,x_2,x_3) = \Big(-\frac{x_1}{\sqrt{x_1^2+x_2^2}}\sqrt{1+x_3^2}, -\frac{x_2}{\sqrt{x_1^2+x_2^2}}\sqrt{1+x_3^2}, -x_3\Big)$$

$$b'(x_1,x_2,x_3) = \Big(-\frac{x_1}{\sqrt{x_1^2+x_2^2}}\sqrt{1+(2-x_3)^2}, -\frac{x_2}{\sqrt{x_1^2+x_2^2}}\sqrt{1+(2-x_3)^2}, 2-x_3\Big)$$

Therefore, the bilinear forms are:

$$b_1^1(z,z') = x_1y_1 + x_2y_2 - x_3y_3,$$

$$b_1^1(a'(z),a'(z')) = (x_1y_1+x_2y_2)\frac{\sqrt{1+x_3^2}\sqrt{1+y_3^2}}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} - x_3y_3$$

Since $x_1^2+x_2^2-x_3^2 = 1$ and $y_1^2+y_2^2-y_3^2 = 1$, we get

$$\frac{\sqrt{1+x_3^2}\sqrt{1+y_3^2}}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} = 1,$$

and so $b_1^1(a'(z),a'(z')) = b_1^1(z,z')$.

The bilinear form for b' is:

$$b_1^1(b'(z),b'(z')) = (x_1y_1+x_2y_2)\frac{\sqrt{1+(2-x_3)^2}\sqrt{1+(2-y_3)^2}}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} - (2-x_3)(2-y_3).$$

Let $z = (x_1,x_2,x_3) = (1,2,2)$ and $z' = (y_1,y_2,y_3) = (2,1,2)$, then $b_1^1(z,z') = 8 \neq \frac{4}{5} = b_1^1(b'(z),b'(z'))$. $\Rightarrow b' \notin O(2,1)$, and so Γ' is not a subgroup of $O(2,1)$. ■

One might be interested in the construction of elements of $O(2,1)$ group.

Each element of $O(2,1)$ is represented by some 3×3 matrix in GL$(3,\mathbb{R})$. Let $\lambda \in O(2,1)$ be an arbitrary element with corresponding matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

Pick a point $p = (x,y,z) \in S^{1,1}$. That is $x^2+y^2-z^2 = 1$. The action of λ on p moves the point to

$$\lambda(p) = Ap^t = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+iz \end{pmatrix}.$$

Since $\lambda(p) \in S^{1,1}$, then

$$(ax + by + cz)^2 + (dx + ey + fz)^2 - (gx + hy + iz)^2 = 1.$$

Expanding brackets we get:

$$(a^2 + d^2 - g^2)x^2 + (b^2 + e^2 - h^2)y^2 + (c^2 + f^2 - i^2)z^2 +$$

$$+2xy(ab + de - gh) + 2xz(ac + df - gi) + 2yz(bc + ef - hi) = 1.$$

Notice that points $(1, 0, 0), (0, 1, 0)$ and $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ are in $S^{1,1}$. The action of λ on these point preserves them in $S^{1,1}$, so substituting them instead of x, y and z in the previous expansion, we get:

$$a^2 + d^2 - g^2 = b^2 + e^2 - h^2 = 1 \quad \text{and} \quad ab + de - gh = 0.$$

Therefore, the general equation now is

$$x^2 + y^2 + (c^2 + f^2 - i^2)z^2 + 2xz(ac + df - gi) + 2yz(bc + ef - hi) = 1.$$

Since $x^2 + y^2 - z^2 = 1$, we get

$$(-c^2 - f^2 + i^2 - 1)z^2 = 2xz(ac + df - gi) + 2yz(bc + ef - hi).$$

Since $z \neq 0$ and the matrix A has constant real entries, we get the linear equation

$$(-c^2 - f^2 + i^2 - 1)z = 2x(ac + df - gi) + 2y(bc + ef - hi)$$

which must be true for all points $(x, y, z) \in S^{1,1}$. Therefore

$$c^2 + f^2 - i^2 = -1 \quad \text{and} \quad ac + df - gi = bc + ef - hi = 0.$$

One may ask if the bilinear form is preserved in this case. Suppose we given two points $p_1 = (x, y, z)$ and $p_2 = (u, v, w)$ in $S^{1,1}$. Then $b_1^1(p_1, p_2) = xu + yv - zw$. After the action of λ we get

$$b_1^1(\lambda(p_1), \lambda(p_2)) = (ax+by+cz)(au+bv+cw)+(dx+ey+fz)(du+ev+fw)-(gx+hy+iz)(gu+hv+iw).$$

By expanding the brackets we can see that a coefficient of each of xv, xw, yu, yw, zu, zv is either $(ab + de - gh), (ac + df - gi)$ or $bc + ef - hi$ which are zero. So, the bilinear form of $(\lambda(p_1), \lambda(p_2))$ is

$$b_1^1(\lambda(p_1), \lambda(p_2)) = (a^2+d^2-g^2)xu+(b^2+e^2-h^2)yv-(c^2+f^2-i^2)zw = xu+yv-zw.$$

As a result, $b_1^1(\lambda(p_1), \lambda(p_2)) = b_1^1(p_1, p_2)$.

Corollary 3.3.2. Any Q-orthogonal transformation preserving $S^{1,1}$ might be represented by matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

with real entries satisfying

$$a^2 + d^2 - g^2 = b^2 + e^2 - h^2 = i^2 - f^2 - c^2 = 1,$$

$$ab + de - gh = ac + df - gi = bc + ef - hi = 0.$$

Corollary 3.3.3. $O(2,1) = \{A \in GL(3, \mathbb{R}) : A^t Q A = Q\}$, where $Q = \mathrm{diag}(1, 1, -1)$.

Remark. In fact, this is true in general:

$$O(p+1,q)=\{A\in GL(p+q+1,\mathbb{R}):A^tQA=Q\},$$

where $Q=\text{diag}(\underbrace{1,...,1}_{p+1\quad\text{numbers}},-1,...,-1)$. Therefore, $det(A)=\pm 1$ for $A\in O(2,1)$.

Any point $(x,y,z)\in S^{1,1}$ satisfies $x^2+y^2-z^2=1$, and so $z^2-x^2-y^2=-1$ is also true. Rearrange the coordinates in $S^{1,1}$ such that $(z,x,y)\in S^{1,1}$ with $z^2-x^2-y^2=-1$. Then one might check

$$O(2,1)=\{A\in GL(3,\mathbb{R}):A^tQA=Q\},$$

where $Q=\text{diag}(1,-1,-1)$.

For such constructions of $S^{1,1}$ and $O(2,1)$ elements of $O(2,1)$ might be defined as follows:

Lemma 3.3.4. [23, p.124]

For each $A\in O(2,1)=\{A\in GL(3,\mathbb{R}):A^tQA=Q\quad\text{where}\quad Q=diag(1,-1,-1)\}$ with $det(A)=+1$, there exist $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ such that $\alpha\delta-\beta\gamma=1$ and

$$A=\begin{pmatrix}\frac{1}{2}(\alpha^2+\beta^2+\gamma^2+\delta^2) & \alpha\beta+\gamma\delta & \frac{1}{2}(\alpha^2-\beta^2+\gamma^2-\delta^2)\\ \alpha\gamma+\beta\delta & \alpha\delta+\beta\gamma & \alpha\gamma-\beta\delta\\ \frac{1}{2}(\alpha^2+\beta^2-\gamma^2-\delta^2) & \alpha\beta-\gamma\delta & \frac{1}{2}(\alpha^2-\beta^2-\gamma^2+\delta^2)\end{pmatrix}$$

Each element $A\in O(2,1)$ with $det(A)=-1$ is represented in similar form with opposite signed entries.

This lemma helps to prove the following results:

Theorem 3.3.5. There are infinitely many elements of $O(2,1)$ isomorphic to $\mathbb{Z}_2$.

Proof. Using Lemma 3.3.4 one may find subgroups of $O(2,1)$ of order 2. Let A be an element of $O(2,1)$ with $det(A) = +1$ and $A^2 = I$. Then row 2 - column 2 entry of $A^2 = I$ is

$$(\alpha\gamma + \beta\delta)(\alpha\beta + \gamma\delta) + (\alpha\delta + \gamma\beta)^2 + (\alpha\gamma - \beta\delta)(\alpha\beta - \gamma\delta) = 1.$$

After simplifications using $\beta\gamma = \alpha\delta - 1$ we get

$$(\alpha^2 + \delta^2)(\alpha\delta - 1) = -2\alpha\delta(\alpha\delta - 1)$$

which implies that

$$\text{either} \quad \alpha\delta = 1 \Rightarrow \beta\gamma = 0 \quad \text{or} \quad \alpha + \delta = 0. \tag{3.2}$$

The row 1 - column 1 entry of $A^2 = I$ implies that

$$\alpha^4 + \delta^4 + \alpha^2(\beta + \gamma)^2 + \delta^2(\beta + \gamma)^2 + 2\alpha\delta(\beta^2 + \gamma^2) + 2\beta^2\gamma^2 = 2.$$

The row 3 - column 3 entry of $A^2 = I$ implies that

$$\alpha^4 + \delta^4 - \alpha^2(\beta - \gamma)^2 - \delta^2(\beta - \gamma)^2 - 2\alpha\delta(\beta^2 + \gamma^2) + 2\beta^2\gamma^2 = 2.$$

The last two equations are both equal to 2, so we equalize them and simplify:

$$(\alpha + \delta)^2(\beta^2 + \gamma^2) = 0. \tag{3.3}$$

Equation 3.2 implies that either $\alpha\delta = 1$ or $\alpha + \delta = 0$ is true. Note that they can not be true simultaneously.

- $\alpha\delta = 1$.

Then $\alpha + \delta \neq 0$, and so by Equation 3.3 we have $\beta = \gamma = 0$. Hence,

$$A = \begin{pmatrix} \frac{1}{2}(\alpha^2 + \delta^2) & 0 & \frac{1}{2}(\alpha^2 - \delta^2) \\ 0 & \alpha\delta & 0 \\ \frac{1}{2}(\alpha^2 - \delta^2) & 0 & \frac{1}{2}(\alpha^2 + \delta^2) \end{pmatrix}$$

Since non-diagonal entries of $A^2 = I$ are equal to 0, we get that $\alpha^2 = \delta^2$. Therefore, $\alpha\delta = 1$ implies that $\alpha^2 = \delta^2 = 1$, and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $\alpha + \delta = 0$.

  Then $\alpha^2 = \delta^2$, and using this equation Lemma 3.3.4 implies that

$$A = \begin{pmatrix} \frac{1}{2}(2\alpha^2 + \beta^2 + \gamma^2) & \alpha(\beta - \gamma) & \frac{1}{2}(-\beta^2 + \gamma^2) \\ \alpha(\gamma - \beta) & -\alpha^2 + \beta\gamma & \alpha(\gamma + \beta) \\ \frac{1}{2}(\beta^2 - \gamma^2) & \alpha(\beta + \gamma) & \frac{1}{2}(2\alpha^2 - \beta^2 - \gamma^2) \end{pmatrix}$$

  with $\beta\gamma = -\alpha^2 - 1$. One might check that such matrix A is an element of $O(2,1)$ isomorphic to $\mathbb{Z}_2$.

  Since $\beta\gamma = -\alpha^2 - 1$ has infinitely many real solutions, there are infinitely many matrices $A \in O(2,1)$ isomorphic to $\mathbb{Z}_2$.

■

Theorem 3.3.6. There is no subgroup of $O(2,1)$, isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$, which acts properly discontinuously on $S^{1,1}$.

Proof. Let $a, b \in O(2,1)$ be elements generating a subgroup of $O(2,1)$ isomorhic to $\mathbb{Z}_2 * \mathbb{Z}_2$. Therefore, $a^2 = b^2 = e$ and ab is an element of $O(2,1)$ with the infinite order. For simplicity, we assume that $\gamma = ab$, and so ord(γ)$= \infty$. The proof might easily follow by Theorem 3.1.1 proved by Wolf. However, we use a different way to prove this result.

Recall that $(x, y, z) \in S^{1,1}$ satisfies $x^2 + y^2 - z^2 = 1$. Let $m \in \mathbb{Z}$ be an arbitrary integer. Consider the compact set $K = \{(x, y, z) \in S^{1,1} : -1 \leq z \leq 1\} \subset S^{1,1}$. Pick any nonzero point $\rho \in K$. There are two possible cases for $\gamma^m(\rho)$:

- the z-coordinate of $\gamma^m(\rho)$ is 0.

  Then $\gamma^m(\rho)$ is a point of the unit circle $\{(x, y, 0) \in S^{1,1} : x^2 + y^2 = 1\} \subset K$. $\Rightarrow$ $\gamma^m(\rho) \in K$. Since $\rho \in K$, we get

$$\gamma^m(K) \cap K \neq \emptyset.$$

- the z-coordinate of $\gamma^m(\rho)$ is nonzero.

  Then $\gamma^m(\rho) = (x_0, y_0, z_0) \in S^{1,1}$ with $z_0 \neq 0$. Notice that $\rho \in K$ implies $-\rho \in K$, and $\gamma^m(-\rho) = (-x_0, -y_0, -z_0)$. Since $\gamma^m \in O(2,1)$ and K is path-connected, then $\gamma^m(K)$ is also path-connected. Hence, there exists a path between points $\gamma^m(\rho)$ and $\gamma^m(-\rho)$ contained in $\gamma^m(K)$. Since z-coordinates of these points are opposite signed, by Intermediate Value theorem the path goes through the unit circle $\{(x, y, 0) \in S^{1,1} : x^2 + y^2 = 1\} \subset K$. Therefore,

$$\gamma^m(K) \cap K \neq \emptyset.$$

In both cases, $\gamma^m(K) \cap K \neq \emptyset$. Since the choice of m was arbitrary, this result is true for all $m \in \mathbb{Z}$. Therefore, $\langle a, b \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$ does not act properly discontinuously on $S^{1,1}$.

$\blacksquare$

Remark. Theorem 3.3.6 might be generalized for all subgroups of $O(p+1,1)$ isomorhic to $\mathbb{Z}_2 * \mathbb{Z}_2$. The proof will go the same way for the compact set $K = \{(x_1, ..., x_{p+1}, y_1) \in S^{p,1} : -1 \leq y_1 \leq 1\} \subset S^{1,1}$, and Intermediate Value theorem guarantees that

$$\gamma^m(K) \cap K \neq \emptyset$$

for all $m \in \mathbb{Z}$.

Chapter 4

Compactifying Infinite Group Actions

4.1 Extension of infinite group actions on $\mathbb{S}^p \times \mathbb{R}^q$ to actions on $\mathbb{S}^{p+q}$

Theorem 4.1.1. $\mathbb{S}^{p+q} \cong \mathbb{S}^p \times \mathbb{R}^q \cup \mathbb{S}^{q-1}$

Proof. Let $*$ denote the topological join. Then it is known that $\mathbb{S}^p * \mathbb{S}^{q-1} \cong \mathbb{S}^{p+q}$. A direct way to show this homeomorphism is to use the map $\mathbb{S}^p * \mathbb{S}^{q-1} \to \mathbb{S}^{p+q}$ given by

$$(x, t, y) \mapsto (x\cos(t\pi/2), y\sin(t\pi/2))$$

where $(x, t, y) \in \mathbb{S}^p \times [0, 1] \times \mathbb{S}^{q-1}$.

Remove a collapsed copy of $\mathbb{S}^{q-1}$ from the construction of the join $\mathbb{S}^p * \mathbb{S}^{q-1}$. Then we are left with a collapsed copy of $\mathbb{S}^p$ and open intervals, one for each point of this $\mathbb{S}^p$ copy. That is, given a point x in the collapsed copy of $\mathbb{S}^p$, we get the open interval $\{x\} \times (0, 1) \times \mathbb{S}^{q-1}$. Notice that each such open interval is homeomorphic to $\mathbb{S}^{q-1} \times (0, \infty)$, which together with the point $\{x\} \in \mathbb{S}^p$ gives $\mathbb{R}^q$ space.

Therefore, $\mathbb{S}^{p+q} \backslash \mathbb{S}^{q-1} \cong \mathbb{S}^p \times \mathbb{R}^q$. ∎

The connection given by Theorem 4.1.1 might be used for an extension of infinite discrete group actions on $\mathbb{S}^p \times \mathbb{R}^q$ to the actions on $\mathbb{S}^{p+q}$.

Hambleton and Pedersen in [9] give conditions under which infinite discrete co-compact groups acting properly discontinuously on $\mathbb{S}^p \times \mathbb{R}^q$ extend to actions on $\mathbb{S}^{p+q}$. To state criterions for that extension we need, first, to introduce some necessary definitions.

Let Γ be a group acting properly discontinuously on the space $\mathbb{S}^p \times \mathbb{R}^q$. Let Γ_0 be a torsion-free subgroup of Γ. Remind that $B\Gamma_0 = K(\Gamma_0, 1)$ is the Eilenberg-Maclane space with a contractible universal covering space $E\Gamma_0$.

Definition 4.1.2. *Given metric spaces X and Y. A metric space is **proper** if any closed metric ball is compact. A map $f : X \to Y$ is a **proper map** if the inverse image of any compact set is compact.*

*We call f to be **Lipschitz** if there exist constants $K > 0$ such that $d(f(x), f(x')) \leq K d(x, x')$ for all $x, x' \in X$.*

*Let X and Y be proper metric spaces. Let $f_0, f_1 : X \to Y$ be proper Lipschitz maps. f_0 and f_1 are **Lipschitz homotopy equivalent**, $f_0 \cong_{Lip} f_1$, if one can find a proper Lipschitz map $H : X \times \mathbb{R} \to Y \times \mathbb{R}$ and a continuous function $\phi : X \to [0, \infty)$ such that*

- *$H(x,t) = (h_t(x), t)$,*
- *$h_t(x) = f_1(x)$ if $t \geq \phi(x)$,*
- *$h_t(x) = f_0(x)$ if $t \leq 0$,*
- *there is a proper map $\varphi : \{(x,t) : 0 \leq t \leq \phi(x)\} \to Y$ such that the following diagram commutes for $H : \{(x,t) : 0 \leq t \leq \phi(x)\} \to Y \times \mathbb{R}$ and the projection $Y \times \mathbb{R} \to Y$*

$$\begin{tikzcd} \{(x,t) : 0 \leq t \leq \phi(x)\} \arrow[r] \arrow[dr] & Y \times \mathbb{R} \arrow[d] \\ & Y \end{tikzcd}$$

Proper metric spaces X and Y are ***Lipschitz homotopy equivalent****, $X \cong_{Lip} Y$, if there exist proper Lipschitz maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \cong_{Lip} id_X$ and $f \circ g \cong_{Lip} id_Y$.*

Let Γ be a group acting on a topological space X. We say that the topological action (X, Γ) is ***continuously controlled*** *at a Γ-invariant subset A of X if, given a compact subset K in $X - A$, for each neighborhood U of $x \in A$, one can find a neighborhood V of x in U such that whenever $gK \cap V \neq \emptyset$, for some $g \in \Gamma$, we get $gK \subset U$.*

Recall that Γ_0 is the torsion-free subgroup of Γ, and suppose Γ acts on $E\Gamma_0$ by isometries. Let $(\overline{E\Gamma}_0, \Gamma)$ be a Γ-equivariant compactification of $(E\Gamma_0, \Gamma)$. Here, $\overline{E\Gamma}_0$ is supposed to be a compact, contractible topological space containing $E\Gamma_0$ as a dense open subset. Then the action is called ***eventually small at infinity*** *if $(\overline{E\Gamma}_0, \Gamma)$ is continuously controlled at $\partial\overline{E\Gamma}_0 = \overline{E\Gamma}_0 - E\Gamma_0$.*

The group Γ is said to be ***eventually (α, k)-euclidean*** *if its virtual cohomological dimension $vcd(\Gamma)$ is finite and it has the torsion-free normal subgroup Γ_0 of finite index with compact $B\Gamma_0$, such that*

- *Γ acts on $E\Gamma_0$ by isometries extending the Γ_0 action properly discontinuously, co-compactly and with finite isotropy,*
- *$E\Gamma_0$ is Lipschitz homotopy equivalent to $\mathbb{R}^q$, $E\Gamma_0 \cong_{Lip} \mathbb{R}^q$,*
- *$E\Gamma_0$ has a Γ-equivariant compactification $(\overline{E\Gamma}_0, \Gamma) = (D^q, \Gamma)$ where the action is eventually small at infinity,*
- *the action of Γ restricted to the boundary of D^q is given by a homomorphism $\alpha : \Gamma \to \mathrm{Homeo}(\mathbb{S}^{q-1})$.*

The following theorem gives the conditions under which discrete co-compact group actions on $\mathbb{S}^p \times \mathbb{R}^q$ extend to actions on $\mathbb{S}^{p+q}$:

Theorem 4.1.3. [9, p.2] Let Γ be an eventually (α, k)-euclidean group. If Γ acts freely, properly discontinuously and co-compactly on $\mathbb{S}^p \times \mathbb{R}^q$ then one can find a compactification $(\mathbb{S}^{p+q}, \Gamma)$ such that

(i) there is a Γ-invariant linear subsphere $\mathbb{S}^{q-1}$ in $\mathbb{S}^p \times \mathbb{R}^q$,

(ii) the action on $\mathbb{S}^{p+q} \backslash \mathbb{S}^{q-1} = \mathbb{S}^p \times \mathbb{R}^q$ is topologically conjugate to the given action, and

(iii) the Γ action on $\mathbb{S}^{q-1}$ is given by α.

A good example was proposed by Hambleton and Pedersen in [9]. Consider a group $\Gamma = \mathbb{Z}^q \rtimes_\alpha G$ with a finite group G. The map $\alpha : G \to GL(q, \mathbb{Z})$ is a homomorphism, and for each $g \in G$ and $z \in \mathbb{Z}^q$

$$z^{\alpha(g)} = gzg^{-1}.$$

Suppose that the action is free, co-compact and properly discontinuous. The example of such action is given by [10] and [11]:

Theorem 4.1.4. [11, p.124] The group $\mathbb{Z}^r \rtimes_\alpha D_t$ acts freely, properly, and co-compactly on $\mathbb{S}^p \times \mathbb{R}^q$ if and only if $p = 3(mod 4)$, $r = q$, and α considered as a real representation has at least two $\mathbb{R}_-$ summands. Here, D_t is a finite dihedral group with an odd prime t.

In order to compactify Γ action using Theorem 4.1.3 one needs to check that the group $\Gamma = \mathbb{Z}^q \rtimes_\alpha G$ is, indeed, eventually (α, k)-euclidean:

- let $\Gamma_0 = \mathbb{Z}^q$ be a torsion-free subgroup of Γ. Then $B\Gamma_0 = \mathbb{T}^q$ is compact and $E\Gamma_0 = \mathbb{R}^q$;
- $vcd(\Gamma) = q < \infty$ where Γ_0 is a subgroup of Γ with a finite index;
- $E\Gamma_0$ is Lipschitz homotopy equivalent to $\mathbb{R}^q$, since $E\Gamma_0 = \mathbb{R}^q$;
- the action on $\mathbb{S}^{q-1}$ is given by a homomorhism $\hat{\alpha} : G \to O_q(\mathbb{R})$ which is similar to $\alpha : G \to GL(q, \mathbb{Z})$ as a real representation. Here $GL(q, \mathbb{Z})$ extends to $GL(q, \mathbb{R})$ which contains $O_q(\mathbb{R})$ as a subgroup;

- to show that Γ acts on $E\Gamma_0 = \mathbb{R}^q$ we need to check two axioms of group actions. Notice that any element of $\Gamma = \mathbb{Z}^q \rtimes_\alpha G$ is of type zg, where $z \in \mathbb{Z}^q$ is represented by an isometry (translation) of $\mathbb{R}^q$ using the map $\mathbb{Z}^q \to Isom(\mathbb{R}^q)$, and $g \in G$ is represented by an orthogonal transformation $\hat{\alpha}(g)$. Now, we check the axioms:

  (i) let e be the identity of Γ, then $e.x = x$ for all $x \in \mathbb{R}^q$,

  (ii) suppose $z_1g_1, z_2g_2 \in \Gamma$ with $z_j \in \mathbb{Z}^q$ and $g_j \in G$ for $j = 1, 2$. Then for all $x \in \mathbb{R}^q$ using the associativity of groups we have

$$z_1g_1(z_2g_2.x) = z_1g_1z_2{g_1}^{-1}g_1{z_2}^{-1}(z_2g_2.x) = z_1{z_2}^{\alpha(g_1)}(g_1{z_2}^{-1}z_2g_2.x)$$

$$= z_1{z_2}^{\alpha(g_1)}(g_1g_2.x) = z_1{z_2}^{\alpha(g_1)}g_1g_2.x = z_1g_1z_2{g_1}^{-1}g_1g_2.x = (z_1g_1z_2g_2).x$$

  Therefore, Γ acts on $E\Gamma_0$. Both $\mathbb{Z}^q$ and G are isometries on $E\Gamma_0 = \mathbb{R}^q$,so Γ acts on $E\Gamma_0$ by isometries. Since G is a finite group and $\mathbb{Z}^q$ acts on $\mathbb{R}^q$ by translations, the action of Γ on $\mathbb{R}^q$ is properly discontinuous, co-compact and with finite isotropy.

- let $\theta : \bar{\mathbb{R}}^q \to D^q$ be a homeomorphism. Given some $\gamma \in \Gamma = \mathbb{Z}^q \rtimes_\alpha G$, and pick any point $x \in \bar{\mathbb{R}}^q$. Since γ acts by an orthogonal transformation and a translation, and the map θ only changes the "scale", one might check that $\gamma\theta(x) = \theta\gamma(x)$, i.e. $(\bar{\mathbb{R}}^q, \Gamma) = (D^q, \Gamma)$ is a Γ-equivariant compactification.

  Pick any point $x \in \mathbb{S}^{q-1}$ and let U be an arbitrary neighborhood of x. Choose any compact subset K of $D^q \backslash \mathbb{S}^{q-1}$. Since $g \in G$ is represented by an orthogonal transformation, the image of K under Γ moves towards x only under translations generated by $\mathbb{Z}^q$. When the image moves closer to x, the size of K decreases radially. This, actually, guarantees the existence of a neighborhood V of x satisfying "eventually small at infinity" condition.

Bibliography

[1] Armand Borel. “Compact Clifford-Klein forms of symmetric spaces”. In: *Topology* 2 (1963), pp. 111–122. ISSN: 0040-9383.

[2] Andrew Casson and Douglas Jungreis. “Convergence groups and Seifert fibered 3-manifolds”. In: *Invent. Math.* 118.3 (1994), pp. 441–456. ISSN: 0020-9910. DOI: 10.1007/BF01231540. URL: http://dx.doi.org/10.1007/BF01231540.

[3] Samuel Eilenberg. “Sur les transformations périodiques de la surface du spheres’”. In: *Fund. Math.* 22 (1934), pp. 28–41.

[4] Hans Freudenthal. “Über die Enden topologischer Räume und Gruppen”. In: *Math. Z.* 33.1 (1931), pp. 692–713. ISSN: 0025-5874. DOI: 10.1007/BF01174375. URL: http://dx.doi.org/10.1007/BF01174375.

[5] David Gabai. “Convergence groups are Fuchsian groups”. In: *Bull. Amer. Math. Soc. (N.S.)* 25.2 (1991), pp. 395–402. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-1991-16082-9. URL: http://dx.doi.org/10.1090/S0273-0979-1991-16082-9.

[6] F. W. Gehring and G. J. Martin. “Discrete quasiconformal groups. I”. In: *Proc. London Math. Soc. (3)* 55.2 (1987), pp. 331–358. ISSN: 0024-6115. DOI: 10.1093/plms/s3-55_2.331. URL: http://dx.doi.org/10.1093/plms/s3-55_2.331.

[7] F. W. Gehring and B. P. Palka. “Quasiconformally homogeneous domains”. In: *J. Analyse Math.* 30 (1976), pp. 172–199. ISSN: 0021-7670.

[8] Charles H. Giffen. "The generalized Smith conjecture". In: *Amer. J. Math.* 88 (1966), pp. 187–198. ISSN: 0002-9327.

[9] Ian Hambleton and Erik K. Pedersen. "Compactifying infinite group actions". In: *Geometry and topology: Aarhus (1998)*. Vol. 258. Contemp. Math. Amer. Math. Soc., Providence, RI, 2000, pp. 203–211. DOI: 10.1090/conm/258/04065. URL: http://dx.doi.org/10.1090/conm/258/04065.

[10] Ian Hambleton and Erik K. Pedersen. "More examples of discrete co-compact group actions". In: *Algebraic topology: applications and new directions*. Vol. 620. Contemp. Math. Amer. Math. Soc., Providence, RI, 2014, pp. 133–143. DOI: 10.1090/conm/620/12363. URL: http://dx.doi.org/10.1090/conm/620/12363.

[11] Ian Hambleton and Erik Kjær Pedersen. "Bounded surgery and dihedral group actions on spheres". In: *J. Amer. Math. Soc.* 4.1 (1991), pp. 105–126. ISSN: 0894-0347. DOI: 10.2307/2939256. URL: http://dx.doi.org/10.2307/2939256.

[12] A. Hinkkanen. "Uniformly quasisymmetric groups". In: *Proc. London Math. Soc. (3)* 51.2 (1985), pp. 318–338. ISSN: 0024-6115. DOI: 10.1112/plms/s3-51.2.318. URL: http://dx.doi.org/10.1112/plms/s3-51.2.318.

[13] Tatsuo Homma and Shin'ichi Kinoshita. "On a topological characterization of the dilatation in E^3". In: *Osaka Math. J.* 6 (1954), pp. 135–143.

[14] L. S. Husch. "A homotopy theoretic characterization of the translation in E^n". In: *Compositio Math.* 24 (1972), pp. 55–61. ISSN: 0010-437X.

[15] L. S. Husch. "A topological characterization of the dilation in E^n". In: *Proc. Amer. Math. Soc.* 28 (1971), pp. 234–236. ISSN: 0002-9939.

[16] B. von Kerékjártó. "Topologische Charakterisierung der linearen Abbildungen". In: *Acta Litt. Sci. Szeged* 6 (1934), pp. 235–262.

[17] B. von Kerékjártó. "“Über die fixpunktfreien Abbildungen der Ebbene". In: *Acta Litt. Sci. Szeged* 6 (1934), pp. 226–234.

[18] S. Kinoshita. "On quasi-translations in 3-space". In: *Fund. Math.* 56 (1964), pp. 69–79. ISSN: 0016-2736.

[19] Toshiyuki Kobayashi and Taro Yoshino. "Compact Clifford-Klein forms of symmetric spaces—revisited". In: *Pure Appl. Math. Q.* 1.3, part 2 (2005), pp. 591–663. ISSN: 1558-8599. DOI: 10.4310/PAMQ.2005.v1.n3.a6. URL: http://dx.doi.org/10.4310/PAMQ.2005.v1.n3.a6.

[20] R. S. Kulkarni. "Groups with domains of discontinuity". In: *Math. Ann.* 237.3 (1978), pp. 253–272. ISSN: 0025-5831. DOI: 10.1007/BF01420180. URL: http://dx.doi.org/10.1007/BF01420180.

[21] Ravi S. Kulkarni. "Proper actions and pseudo-Riemannian space forms". In: *Adv. in Math.* 40.1 (1981), pp. 10–51. ISSN: 0001-8708. DOI: 10.1016/0001-8708(81)90031-1. URL: http://dx.doi.org/10.1016/0001-8708(81)90031-1.

[22] Ravindra S. Kulkarni. "Some topological aspects of Kleinian groups". In: *Amer. J. Math.* 100.5 (1978), pp. 897–911. ISSN: 0002-9327. DOI: 10.2307/2373952. URL: http://dx.doi.org/10.2307/2373952.

[23] Wilhelm Magnus. *Noneuclidean tesselations and their groups*. Pure and Applied Mathematics, Vol. 61. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974, pp. xiv+207.

[24] Vladimir Markovic. "Quasisymmetric groups". In: *J. Amer. Math. Soc.* 19.3 (2006), pp. 673–715. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-06-00518-2. URL: http://dx.doi.org/10.1090/S0894-0347-06-00518-2.

[25] Gaven J. Martin. "Discrete quasiconformal groups that are not the quasiconformal conjugates of Möbius groups". In: *Ann. Acad. Sci. Fenn. Ser. A I Math.* 11.2 (1986), pp. 179–202. ISSN: 0066-1953. DOI: 10.5186/aasfm.1986.1113. URL: http://dx.doi.org/10.5186/aasfm.1986.1113.

[26] Gaven J. Martin. "Infinite group actions on spheres". In: *Rev. Mat. Iberoamericana* 4.3-4 (1988), pp. 407–451. ISSN: 0213-2230. DOI: 10.4171/RMI/79. URL: http://dx.doi.org/10.4171/RMI/79.

[27] Benjamin Miesch. "On Cannon's Conjecture". In: *Master Thesis* (2013).

[28] Deane Montgomery and Leo Zippin. "Examples of transformation groups". In: *Proc. Amer. Math. Soc.* 5 (1954), pp. 460–465. ISSN: 0002-9939.

[29] J. W. Morgan and H. Bass. "The Smith conjecture". In: Academic Press, New York, 1984.

[30] Georg Peschke. "The theory of ends". In: *Nieuw Arch. Wisk. (4)* 8.1 (1990), pp. 1–12. ISSN: 0028-9825.

[31] Dennis Sullivan. "On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions". In: *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*. Vol. 97. Ann. of Math. Stud. Princeton Univ. Press, Princeton, N.J., 1981, pp. 465–496.

[32] Pekka Tukia. "A quasiconformal group not isomorphic to a Möbius group". In: *Ann. Acad. Sci. Fenn. Ser. A I Math.* 6.1 (1981), pp. 149–160. ISSN: 0066-1953. DOI: 10.5186/aasfm.1981.0625. URL: http://dx.doi.org/10.5186/aasfm.1981.0625.

[33] Pekka Tukia. "Convergence groups and Gromov's metric hyperbolic spaces". In: *New Zealand J. Math.* 23.2 (1994), pp. 157–187. ISSN: 1171-6096.

[34] Pekka Tukia. “Extension of quasisymmetric and Lipschitz embeddings of the real line into the plane”. In: *Ann. Acad. Sci. Fenn. Ser. A I Math.* 6.1 (1981), pp. 89–94. ISSN: 0066-1953. DOI: 10.5186/aasfm.1981.0624. URL: http://dx.doi.org/10.5186/aasfm.1981.0624.

[35] Pekka Tukia. “Homeomorphic conjugates of Fuchsian groups”. In: *J. Reine Angew. Math.* 391 (1988), pp. 1–54. ISSN: 0075-4102. DOI: 10.1515/crll.1988.391.1. URL: http://dx.doi.org/10.1515/crll.1988.391.1.

[36] Pekka Tukia. “On isomorphisms of geometrically finite Möbius groups”. In: *Inst. Hautes Études Sci. Publ. Math.* 61 (1985), pp. 171–214. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1985__61__171_0.

[37] Pekka Tukia. “On two-dimensional quasiconformal groups”. In: *Ann. Acad. Sci. Fenn. Ser. A I Math.* 5.1 (1980), pp. 73–78. ISSN: 0066-1953. DOI: 10.5186/aasfm.1980.0530. URL: http://dx.doi.org/10.5186/aasfm.1980.0530.

[38] Jussi Väisälä. *Lectures on n-dimensional quasiconformal mappings*. Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin-New York, 1971, pp. xiv+144.

[39] Joseph A. Wolf. “The Clifford-Klein space forms of indefinite metric”. In: *Ann. of Math. (2)* 75 (1962), pp. 77–80. ISSN: 0003-486X.