ROCKING THE INFLATIONARY BOAT

Rocking the Inflationary Boat: Looking at the sensitivity to initial conditions of solutions to novel inflationary scenarios

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Abstract

Inflation, the currently favoured solution to the grievous initial conditions problems of the Big Bang model of the universe, is a very general framework that can be constructed from any number of underlying theories. As inflation is meant to solve a problem of initial conditions, it is generally preferred that it not introduce its own initial conditions problem. The purpose of this thesis is to explore the sensitivity to initial conditions of solutions to two toy models of inflation. The models in and of themselves are not intended to explain inflation, but rather seek to begin to explore, in a controlled way, interesting properties that a full inflationary theory might have.

The first model is one with a single scalar inflaton, but two compact extra dimensions. We find this model has two inflationary solutions that can be well understood analytically. These solutions are power laws in time. One is found to be marginally insensitive to its initial conditions, and the other is found to be highly sensitive to its initial conditions. We also find a solution to this model that exhibits 4D quasi-de Sitter space, but is difficult to understand analytically, and its sensitivity to initial conditions is not yet well known.

The second model examines an n-scalar field Lagrangian that includes kinetic terms first-order in the derivatives of the fields (similar to certain ferromagnetic Lagrangians). It is found that this model can realize slow-roll inflation with arbitrarily steep potentials. A solution is constructed that can realize an exact de Sitter equation of state without saying anything about the slope of its potential. This solution is found to be marginally insensitive to its initial conditions for a certain range of parameters. Corrections from higher order terms in the Lagrangian are found to introduce a parameter space for which this solution is in fact highly insensitive to its initial conditions.

We therefore make progress in understanding higher-dimensional inflation, slow-roll inflation with steep potentials, and the sensitivity of solutions in both those cases to their initial conditions.

Acknowledgements

This work only exists thanks to important contributions from several individuals. First and foremost, Dr. Cliff Burgess supplied the ideas, knowhow, connections, and a truly inordinate amount of patience. He has my most sincere gratitude.

Furthermore, I owe many thanks to Mr. Jared Enns. He is responsible for innumerable checks, corrections, ideas, numerics, and above all, improving my functionality in both LaTeX and Mathematica. It was a privilege to work with him, and he has my thanks.

Finally, Dr. Subodh Patil was a key contributor to much of the work from which this thesis derives. It has been my pleasure to learn from him, and I am much obliged.

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Declaration of Academic Achievement

I, Peter Hayman, declare that this thesis titled, "Rocking the Inflationary Boat: Looking at the sensitivity to initial conditions of solutions to novel inflationary scenarios" and the work presented in it are my own. Significant contributions from collaborators are as follows:

- The idea for chapter 2 was due to Dr. Cliff Burgess, and the derivation of the field equations and the equivalent 4D description was first performed by Dr. Subodh Patil.
- The sensitivity to initial conditions of the slow-roll power-law solution in chapter 2 was first calculated by Mr. Jared Enns.
- The code to produce the plots in chapter 2 was written in collaboration with Mr. Jared Enns.
- The idea for chapter 3 was due to Dr. Cliff Burgess. The field equations were derived in collaboration with Mr. Jared Enns.
- The computations of Appendix B were performed independently and concurrently with Mr. Jared Enns, however the particular arrangement of the Christoffel symbols (B.1) is due to Mr. Jared Enns.

The contents of chapters 2 and 3 have been submitted for publication—[1] and [2], respectively.

Introduction

This thesis aims to study the sensitivity to initial conditions of solutions to two distinct inflationary toy models. The framework of inflation [3, 4] was introduced to solve problems of astonishingly precise initial conditions that inevitably arise in the standard Hot Big Bang model of cosmology, so it is important that a successful inflationary model not itself be plagued with a problem of initial conditions.

The Big Bang model is based on the observations that the universe (on the largest scales) is homogeneous, isotropic, and expanding. However, one can use the Big Bang model to compute the expected degree of homogeneity, isotropy, and flatness in the universe today given some set of initial conditions, and doing so, one finds that the universe today would have required an extremely precise set of initial conditions. Nowhere is this easier to see than with the Cosmic Microwave Background Radiation (CMBR). This is the bath of microwave radiation seen in all directions that is a relic of the epoch when the universe first became transparent to light (the physical region corresponding to the source of the radiation we are receiving today is referred to as the *surface of last scattering*), and it is homogeneous to one part in 10^5, after correcting for the Earth's motion. It can be calculated in the Big Bang model that every $\sim 2^\circ$ of the CMBR corresponds to an area on the surface of last scattering that should never have been in causal contact with the rest of the surface. In particular, this means that every region of the CMBR larger than a few full moons had to have started out with almost exactly the same initial conditions as every other such region.

It can be shown that this issue of initial conditions in the Big Bang model can be resolved if there had been an earlier, sufficiently long, period of accelerated expansion. Inflation is the simplest suggestion that that period of acceleration was an almost de Sitter geometry, which corresponds to a universe dominated by an approximately constant energy density. This is typically realized by introducing one or more scalar fields, arranged so that there is a period when the potential energy of (at least one of) the fields dominates over the kinetic energy, hence approximating a constant energy density (this is called *slow-roll*). Given the origin of the idea of inflation, it is important how sensitive any model realizing inflation is to its own initial conditions. It is all well and good to construct a model that can produce an inflationary epoch , but if that model demands an even more restrictive set of initial conditions than the Big Bang model alone, it hasn't solved anything. To that end, in this thesis we study two different toy models of inflation, and where possible analyze the sensitivity of their solutions to perturbations in the initial conditions.

The first model we study is the standard single-field inflaton in a universe with two

compact extra spatial dimensions. The main idea behind this toy model is to fully study an extra-dimensional scenario that is more complicated than the well-explored co-dimension 1 models [5, 6, 7, 8](i.e., those with a single extra dimension), but still simple enough to be analytically (and numerically) soluble. We compactify the extra dimensions onto a sphere for simplicity (since it only adds one more degree of freedom, the radius), but in principle one could also study more complicated geometries. This model has a couple of important features. First, we find solutions for the full 6D Einstein equations, rather than an easier, approximate 4D system. Secondly, and very importantly, we include a mechanism for stabilizing the extra dimensions (flux-compactification), and find solutions that achieve this stability. This is a key result because it is often neglected in higher dimensional models, but is a necessary feature, since we do not seem to live in a world with macroscopic extra dimensions.

Our analysis of this model is not exhaustive as there are many parameters to vary, however we do uncover two classes of solutions. First, if the extra-dimensional radius (which, from a 4D perspective, appears as an additional field, the *radion*), begins near its minimum, then it quickly settles and the inflaton undergoes a period of slow-roll. For this solution, we typically find the slow-roll parameters $\varepsilon \sim 0.009$ (roughly, a measure of the degree to which this geometry approximated de Sitter), and $\eta \sim 0.016$ (roughly, a measure of the rate of change of ε). These correspond to $n_s \sim 0.975$ (a measure of the scale invariance of the primordial scalar power spectrum), and $r \sim 0.15$ (a measure of the ratio of tensor to scalar modes in the CMBR polarization), so are in tension with data from the Planck satellite [9], assuming the usual perturbation analysis is applicable. This solution has the distinct advantage that it comes by definition with radius stabilization. It is also, however, difficult to understand analytically, and as a result, we do not study its sensitivity to initial conditions in detail.

The second class of solutions for this model are of a power-law form, so that the fields all scale as powers of time. This has the pleasant feature that $\eta = 0$ for free. It is also found that the duration of inflation (measured in e-foldings) is directly related to how much the extra dimensional radius inflates, $\mathcal{N}_e \propto 2\ln(b_f/b_0)$ (where $b(t)$ is the extra-dimensional scale-factor). This is interesting because it is a direct relationship between an intuitive 4D property, and a fundamental property of the extra-dimensions. We find analytically and numerically two examples of this power-law form. One solution we refer to as the Attractor solution, since it is found to be marginally insensitive to initial conditions. This Attractor solution is limited to $\varepsilon \geq 0.5$, so is excluded if the standard perturbation analysis is applicable. Nevertheless, it is of interest for the fact that the extra-dimensional radius can still be stabilized in this solution. The other example of power-law we find is referred to as the Slow-Roll solution, since it is possible to construct examples with arbitrary ε. This solution suffers from high sensitivity to initial conditions.

Our second toy model studies an extension to n-scalar field inflation where we use a kinetic energy that is dominated by terms of first-order in the derivatives of the fields

(we refer to this model as "magnon" inflation, since the Lagrangian is similar to one used to describe spin waves in a ferromagnet [10, 11]). This research was motivated by the Chromo-Natural model of inflation described by [12], and seeks to understand the reason why that model was able to achieve slow-roll inflation despite having steep potentials. Indeed, we find that their inclusion of first-order derivatives was the culprit, as a key result of ours is that field configurations which are orthogonal (in field space) to the gradient of the potential generically produce $\varepsilon = 0$ to first order. As an example, we explicitly construct such a model with two fields, and find conditions on its parameters necessary to ensure it is relatively insensitive to perturbations in the initial conditions. This analysis is performed both at the first- and second-order in the derivatives of the fields.

The rest of this thesis is laid out as follows. In chapter 1, we expand on the framework of inflation by detailing the phenomena it hopes to explain, the mathematics, the canonical example, and (briefly) how it could potentially be observed. In chapter 2, we explore the extra-dimensional model of inflation, detail its solutions, and study the sensitivity of each case to initial conditions. Finally, in chapter 3, we explore the magnon model, detail the canonical two-field example, and study its sensitivity to initial conditions, to both first and second order in the derivatives of the fields.

Throughout, we use the mostly plus signature for the metric, Weinberg's conventions for the curvature tensor, and units such that $\hbar = c = 1$.

Chapter 1

Inflation Background

As this thesis concerns the study of inflationary models, it is important to elaborate on the idea of inflation, and establish the vocabulary and context for the following chapters. We begin by describing the framework of large-scale cosmology and the Hot Big Bang model. This is followed by a detailed treatment of perhaps the most uneasy drawback to the Big Bang model, the near uniformity of the Universe despite the Big Bang prediction that it should consist mainly of causally disconnected patches. It is subsequently shown that an early period of accelerated expansion is capable of solving this problem by ensuring the universe is covered by a single causal patch. The special case of quasi-de Sitter space (inflation) is described as an example accelerating geometry, and the related parameters $\mathcal{N}_e$, ε, and η are introduced. Finally, the single field inflaton is introduced as the quintessential example of matter content that can give rise to quasi-de Sitter space, and it is used to demonstrate how an inflationary model can be connected to observation by the imprint quantum fluctuations of an inflaton field can leave on the CMBR.

1.1 Hot Big Bang Cosmology

Two observations have been the driving force for large-scale cosmology since its inception (see e.g., [13]). First, on the largest scales (typically larger than a gigaparsec), the spatial distribution of the Universe seems thoroughly isotropic and homogeneous. Second, we observe an almost linear relationship between the distance and recession velocity of objects in the universe (this is the Hubble law [13]), so that more distant objects appear to be receding from us faster than closer objects. The first observation tells us that we can describe the universe as $\mathbb{R} \times M^3$, where M^3 is any of the three maximally symmetric 3-spaces, S^3 (the 3-sphere, a closed geometry), $\mathbb{R}^2$ (Euclidean 3-space, a flat geometry), and H^3 (hyperbolic 3-space, an open geometry). The second observation suggests that the radius of curvature of M^3 is time-dependent (and increasing), so ultimately, the universe is described by the

metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right) \tag{1.1}$$

where $\mathrm{d}\Omega^2$ is the angular metric $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2$, $a(t)$ is the "scale-factor," and κ takes on the value 1, 0, or -1 for closed, flat, or open geometries (respectively). For the purposes of this thesis, we take a to be real, continuous, and positive-definite. Observations suggest that the Universe has a flat geometry, so from here on we take $\kappa = 1$. The spatial coordinates measure the physical separation of points at a fixed point in time where $a(t_0) = 1$. These spacial coordinates are called "co-moving" coordinates because they are permanent labels, whereas the physical separation between points varies with $a(t)$. Hubble's law can be extracted from this metric as follows. The recession velocity of a distant object is:

$$\begin{aligned} v := \frac{\mathrm{d}(D)}{\mathrm{d}t} = \frac{\mathrm{d}(a\Delta r)}{\mathrm{d}t} = \dot{a}\Delta r = HD, \\ \implies v \propto D, \end{aligned} \tag{1.2}$$

where D is the proper distance from us to some object in space, Δr is the corresponding co-moving distance from us to that object, and

$$H := \frac{\dot{a}}{a} \tag{1.3}$$

is the Hubble parameter. Equation (1.2) is the Hubble law.

The metric (1.1) is referred to as the Friedmann-Robertson-Walker (FRW) metric, after several of its discoverers. The field of large-scale cosmology is essentially the study of the dynamics of the scale factor $a(t)$, and how it reacts to, and interacts with, the matter content of the Universe. Speaking of matter content, given the symmetries of our problem, the only available stress-energy tensor is that of a fluid:

$$T^{\mu}{}_{\nu} = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}, \tag{1.4}$$

where ρ is the energy density, and p is the pressure of the matter-energy content of the Universe, and both only depend on time. With that, the Einstein equations read

$$3M_p^2 H^2 = \rho, \quad \text{and} \quad \frac{\ddot{a}}{a} + 2H^2 = \frac{1}{2M_p^2}(\rho - p), \tag{1.5}$$

The first of these equations is known as the Friedmann equation. Instead of the second equation, it is common to pair the Friedmann equation with the local conservation of energy $\nabla_\mu T^\mu{}_\nu = 0$ instead:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho a^3\right) + p\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\right) = 0. \tag{1.6}$$

In what follows, it is very useful to be able to relate the scale-factor to an observable quantity. As it happens, redshift is just such an observable quantity with a simple, direct relationship to $a(t)$. The redshift of light emitted from distant objects is simply the fractional change in wavelength of that light between emission and observation:

$$z := \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}. \tag{1.7}$$

In a static Universe, light travelling from one static object to another would display no redshift, but in a dynamic FRW universe, the physical size of the intermediate space changes between the time of emission and observation of a ray of light, so even for locally static emitters and observers, there is a redshift. If we define the co-moving wavelength λ_{co}, then the physical wavelengths at emission and observation are $\lambda_{\text{em}} = a(t_{\text{em}})\lambda_{\text{co}}$, and $\lambda_{\text{obs}} = a(t_{\text{obs}})\lambda_{\text{co}}$, so that we can cleanly write the redshift in terms of the scale-factor:

$$z = \frac{a(t_{\text{obs}})\lambda_{\text{co}} - a(t_{\text{em}})\lambda_{\text{co}}}{a(t_{\text{em}})\lambda_{\text{co}}} = \frac{a(t_{\text{obs}})}{a(t_{\text{em}})} - 1 \qquad \Longrightarrow \qquad 1 + z = \frac{a(t_{\text{obs}})}{a(t_{\text{em}})}. \tag{1.8}$$

We typically choose t_{obs} to be now, and $a(t_{\text{obs}})$ to be 1. What's important for the history of the Universe is t_{em}, so we will take that as our dynamic variable, and just call it t. Even then, (1.8) is typically difficult to invert, but we can at least use it to relate the differentials $\mathrm{d}t$ and $\mathrm{d}z$, since at least that way we can exchange them in integrals. Doing this, we find

$$\mathrm{d}z = -\frac{1}{a^2(t)}\dot{a}(t)\mathrm{d}t = -(1+z)H\mathrm{d}t. \tag{1.9}$$

Another observable quantity of importance is temperature. The history of the Universe can be fairly well segmented into epochs wherein the energy density of the universe is dominated by matter content with different equations of state. For instance, there is an

epoch for which the universe is dominated by relativistic matter (i.e., radiation) with equation of state $p = \rho/3$, and an epoch of non-relativistic matter domination with equation of state $p = 0$ (today, we have just begun an epoch of vacuum-energy domination, where $p = -\rho$). During these epochs, we consider the universe to be in thermal equilibrium, so we characterize the evolution of the universe by the black-body temperature of the universe. From the Stefan-Boltzmann law, this temperature scales as

$$T \propto \frac{1}{a} \tag{1.10}$$

when the Universe is radiation-dominated (this is a helpful relation in later calculations). This is also the origin of the term "hot" in the Hot Big Bang model. In an expanding universe, $1/a$ is a decreasing function of time, so that even if the universe is currently very cold today, it used to be very hot at early times.

To summarize, large-scale cosmology is essentially the study of the FRW metric (1.1), and how the scale-factor $a(t)$ evolves along with the energy-density and pressure of the matter-energy content of the Universe. The Hot Big Bang model describes the history of the universe in terms of an evolution through different epochs of matter content domination starting from an initial state with very small a, hence very high temperature. The scale-factor itself can be related to the more common, observational quantities redshift and temperature through (1.8) and (1.10), respectively. This model has been very successful, for instance in explaining the existence of the CMBR, Hubble's law, and the abundances of light elements. However, alone it does lead to a couple of suspicious predictions, the most grievous of which we describe next.

1.2 The Horizon Problem and Inflation

Using the FRW metric and our understanding of the Standard Model of particle physics, we are able to predict many things about the history and the current state of the Universe. Hubble's law is a direct consequence of the form of the metric, the cosmic microwave background is the relic radiation from the era when the universe cooled enough for neutral (transparent) hydrogen to form, etc. We can also compute the expected degree of homogeneity in the universe today, and this is where we encounter the first major disagreement between the standard Hot Big Bang model and observation. A calculation of the number of causally connected patches in the universe predicts an enormous number of causally *disconnected* patches. This is in disagreement with observation because as noted above, the universe on the largest scales is very homogeneous, which strongly suggests that the universe as a whole was at one time in causal contact (so that every region could come to an agreement on what density distribution and temperature to have). This issue is known as

the "Horizon Problem," and the resolution is to postulate an early period of accelerated expansion. Inflation is one particularly simple example of an accelerating geometry that can be described by the condition that $H \sim$ const.

The cosmic microwave background radiation gives us a two-dimensional snapshot of the Universe at a relatively well understood point in its history, and makes an excellent playground for us to use to perform this calculation. Every point on the surface of last scattering has a corresponding causal volume within which interactions involving the point in question could have occurred (essentially its past light-cone). The radius of this volume is computed from the null geodesic equation which reads (setting the origin at the point in question):

$$\Delta r_{\text{caus}} = \int_{t_0}^{t_{\text{rec}}} \frac{\mathrm{d}t'}{a(t')}. \tag{1.11}$$

(note that we are using the co-moving distance here. The corresponding physical distance at time t_* is $D = a(t_*)\Delta r_{\text{caus}}$, but since we wish to compare this to another distance measured at the same time, it will also be multiplied by $a(t_*)$, making the extra factor unnecessary).

Meanwhile, the co-moving distance from us to that point on the surface of last scattering is similarly given by:

$$\Delta r_{\text{CMB}} = \int_{t_{\text{rec}}}^{t_{\text{now}}} \frac{\mathrm{d}t'}{a(t')}. \tag{1.12}$$

In terms of the physically observable quantity redshift, we can use (1.9) to write these as:

$$\Delta r_{\text{caus}} = \int_{z_{\text{CMB}}}^{\infty} \frac{\mathrm{d}z}{H(z)} \qquad \text{and} \qquad \Delta r_{\text{CMB}} = \int_{0}^{z_{\text{CMB}}} \frac{\mathrm{d}z}{H(z)}, \tag{1.13}$$

where we have taken t_0 to correspond to very small a (i.e., approaching $a \sim 0$) so that $z_0 \to \infty$, and we again made the standard choice that $a(t_{\text{now}}) = 1$ so that $z_{\text{now}} = 0$. Non-relativistic matter has an energy-density that simply scales as the inverse of volume, while radiation has an extra factor of inverse length from its non-trivial momentum. This means that they respectively scale as

$$\rho_{\text{mat}} \propto a^{-3} = (1+z)^3 \qquad \text{and} \qquad \rho_{\text{rad}} \propto a^{-4} = (1+z)^4. \tag{1.14}$$

From the Friedmann equation $3M_p^2 H^2 = \rho$, this means that the integrand H^{-1} falls as $z^{-3/2}$ for matter, but z^{-2} for radiation, which means H^{-1} is dominated by the matter contribution for smaller z (neglecting the very recent transition to vacuum energy domination). It can

be calculated that radiation and matter energy densities are of the same order around $z \approx 3000$. Since the surface of last scattering is at $z_{\text{CMB}} \approx 1100$ (computable given known T_{CMB} today and T_{rec} the temperature of recombination), it is a reasonable approximation to use ρ_{mat} for both integrals. With this, we can compute

$$\Delta r_{\text{caus}} = \int_{z_{\text{CMB}}}^{\infty} \mathrm{d}z H_0^{-1}(1+z)^{-3/2} = -2H_0^{-1}\,(1+z)^{-1/2}\Big|_{1100}^{\infty} \approx \frac{2H_0^{-1}}{\sqrt{1100}}, \tag{1.15}$$

and

$$\Delta r_{\text{CMB}} = \int_{\infty}^{z_{\text{CMB}}} \mathrm{d}z H_0^{-1}(1+z)^{-3/2} = -2H_0^{-1}\,(1+z)^{-1/2}\Big|_{0}^{1100} \approx 2H_0^{-1}. \tag{1.16}$$

In other words, looking at the CMBR from Earth, a causally connected patch has an angular diameter of roughly

$$\theta \approx \frac{\Delta r_{\text{caus}}}{\Delta r_{\text{CMB}}} \approx \frac{2H_0^{-1}/\sqrt{1100}}{2H_0^{-1}} \sim 1.7^\circ. \tag{1.17}$$

Hence, from the Big Bang model alone, we would expect about every 1.7° of the CMBR to never have been in causal contact with the rest, so we should expect it to be replete with $\mathcal{O}(1)$ variations in temperature, when in reality it is measured to only vary by around one part in 10^5 (after accounting for our peculiar motion, which induces a dipole of order one part in 10^3). Moreover, the CMBR is almost a perfect blackbody (see figure 1.1 from the FIRAS experiment on the COBE satellite), which is a property that *strongly* suggests the Universe was in thermal equilibrium at the surface of last scattering—a thermal equilibrium that should only have been attainable within a single causal horizon. This is the gist of the "Horizon Problem."

There are two possible solutions to the Horizon Problem as presented. First, there could be no problem. We could just accept that an entire Universe that has never been in causal contact miraculously agreed on an entire blackbody spectrum. For anyone who finds that a bit too much to accept, however, there is the second solution: the postulate of at least one period of time in the early universe when the scale-factor had a positive acceleration. We now explore this latter solution in more detail.

The mathematical derivation is a bit technical, so it is relegated to appendix A, however the main point is shown schematically in figure 1.2. Conceptually, if the form of H^{-1} for known matter falls off too fast for our tastes, then we must propose an early period of new physics that sufficiently alters the form of H^{-1}. The result of appendix A is to add some precision to that statement. Assuming FRW geometry is valid even at early times, and assuming H^{-1} is a continuous function, it is found that a necessary condition for solving

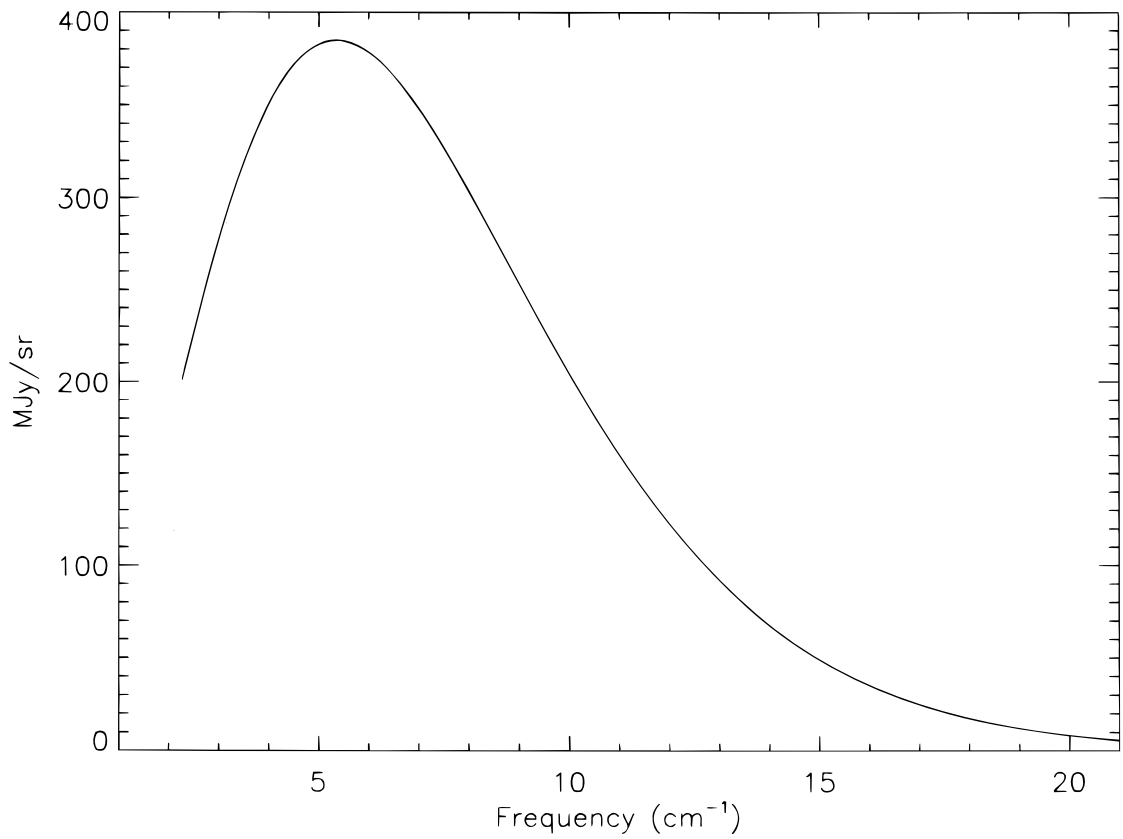

FIGURE 1.1: The CMBR blackbody curve as measured by the FIRAS experiment on the COBE satellite (reproduced from [14]). The data points and blackbody fit overlap exactly on this plot, with experimental error on the data points too small to be seen on this scale.

the Horizon Problem is for there to exist at least one sufficiently long period for which

$$\varepsilon := -\frac{\dot{H}}{H^2} \leq 1. \tag{A.4}$$

which is equivalent to the condition that

$$\ddot{a} > 0. \tag{1.18}$$

This is a general statement, and is satisfied by any number of conceivable solutions (such as the schematic example in figure 1.2). Inflation is a particular realization of this

condition that also satisfies

$$H \approx \text{const.} \tag{1.19}$$

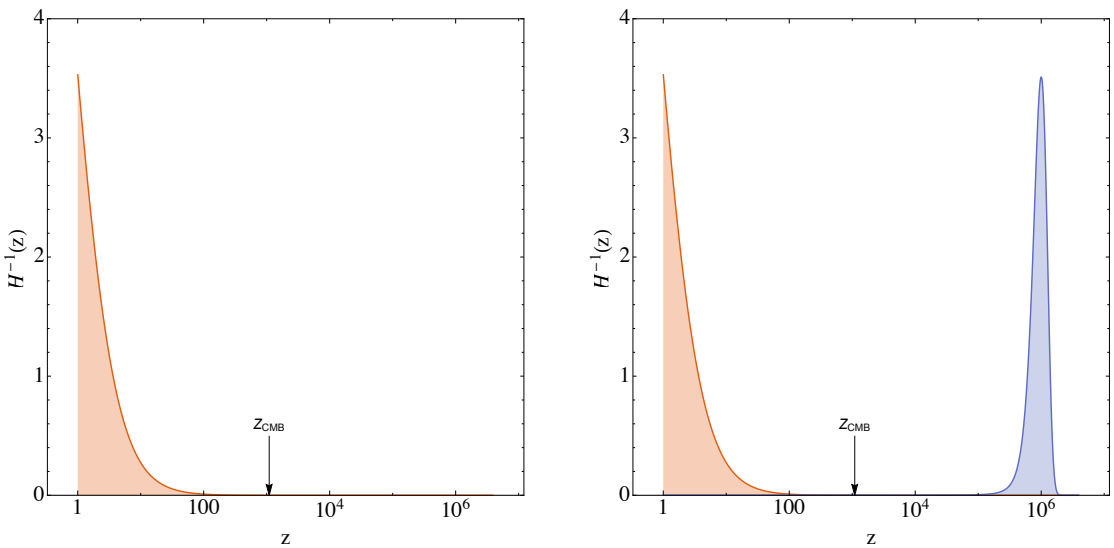

FIGURE 1.2: The Horizon Problem. *Left*: The prediction of the Big Bang model alone. The area under the curve (i.e., the shaded region) to the left of z_{CMB} is approximately the radius of the surface of last scattering, and the (invisible) area under the curve to the right of z_{CMB} is approximately the radius of a causal horizon on the surface of last scattering. *Right*: The Horizon Problem is solved by modifying the form of $H^{-1}(z)$ at large z so that the radius of a causal horizon on the surface of last scattering (the shaded area under the curve to the right of z_{CMB}) is roughly the same as the radius of the surface itself.

If $H \sim$ const., then $a \sim \exp\{Ht\}$. When these relations are exact, the geometry is known as de Sitter space, and when they are only approximate it is known as quasi-de Sitter space (the degree of approximation it requires depends on the problem to be solved). Quasi-de Sitter space is a very useful solution as it arises from an equation of state that is consistent with vacuum energy, $p = -\rho =$ const., and can be approximated in a dynamic way (we explore this in the next section).

In the quasi-de Sitter case, we can even make a general statement about how much of this inflation would need to occur. Suppose the transition to this period occurs at some z_* for which Hubble radius is H_*^{-1}. If inflation started at z_i, then the inflationary epoch would

contribute to Δr_{caus} an amount

$$\Delta r_{\text{inf}} \approx H_*^{-1}(z_i - z_*). \tag{1.20}$$

Assuming $z_* \gg z_{\text{eq}}$, the redshift of radiation-matter equality, we can find $H_*^{-1} \approx H_0^{-1}(1 + z_*)^{-2}$ (we will see shortly that this is a good approximation). Using this, we can write

$$\Delta r_{\text{inf}} \approx H_0^{-1} \frac{z_i - z_*}{(1 + z_*)^2}. \tag{1.21}$$

In order to solve the Horizon Problem, Δr_{inf} needs to be of order $\Delta r_{\text{CMB}} \sim H_0^{-1}$, which implies

$$\begin{aligned} \frac{z_i - z_*}{(1 + z_*)^2} &\sim 1, \\ \implies \frac{z_i}{z_*} &\sim z_*, \\ \frac{a_*}{a_i} &\sim \frac{a_0}{a_*}, \end{aligned} \tag{1.22}$$

where in the second and third lines, we used that $z_*, z_i \gg 1$, and $a_0 = 1$.

In an effort to preempt the large numbers that will soon arrive, we now define the quantity "e-foldings" $\mathcal{N}$ by $\mathrm{d}\mathcal{N} := \mathrm{d} \ln a = H \mathrm{d}t$. This way, we can write $a_*/a_i = \exp(\mathcal{N}_e)$, with $\mathcal{N}_e := \mathcal{N}_* - \mathcal{N}_i$. This way, we have that inflation must last long enough to satisfy the condition:

$$\mathcal{N}_e \sim \ln\left(\frac{a_0}{a_*}\right). \tag{1.23}$$

To get a sense of the size of this number, we need an estimate for a_*. First, we note that $z_{\text{eq}} \approx 3000$ is of order $z_{\text{CMB}} \approx 1000$, so it is at roughly the same temperature scale, $T_{\text{CMB}} \sim 10^{-1}$eV. Thanks to the Large Hadron Collider, and other wonderful experiments, we have a good understanding of physics below roughly the TeV scale. As a result, any new physics to drive inflation must have occurred when the Universe was at a temperature of at *least* a TeV, which means that $z_* \gg z_{\text{eq}}$, and it is not unreasonable to approximate the universe as radiation dominated from $z_{\text{now}} = 0$ to z_*. Moreover, in that regime, we have from (1.10) that $T \sim 1/a$. Taking the temperature of the universe today to be the temperature of the CMBR today, $T_0 \sim 10^{-4}$eV [9], we find the numerical estimate

$$\mathcal{N}_e > \ln\left(\frac{T_*}{T_0}\right) \sim \ln\left(\frac{10^{12}}{10^{-4}}\right) \sim 37. \tag{1.24}$$

Concisely, one of the greatest sources of unease about the highly successful Big Bang

Model is the Horizon Problem, that vast regions of clearly equilibrated space should never have been in causal contact. Any active solution to this problem must involve at least one period of time when the expansion of space was accelerating, i.e., $\ddot{a} > 0$, or equivalently, $\varepsilon < 1$. Inflation is one geometry that achieves this acceleration. It does so by approximating the de Sitter geometry defined by the condition $H \sim$ const., and physically realized in a Universe dominated by an approximately constant energy-density. In this scenario, it can be calculated that there needs to be a bare minimum of roughly $\mathcal{N}_e \sim 40$ e-foldings, defined by $\mathrm{d}\mathcal{N} := \mathrm{d}\ln a$. The task is now to construct a physical model that meets all of these requirements, and in the next section, we explore the simplest toy model that can be made to do so.

1.3 Canonical Example (Single Field Slow-Roll)

The basic premise of inflation is that an epoch when $H \sim$ const. would have an accelerating geometry, so could solve the Horizon Problem, but would also correspond to a predominantly constant energy-density. The importance of this particular form for the energy-density is that there are known means of producing such an equation of state. Obviously the simplest way is just to have a vacuum energy (aka a cosmological constant), but since the inflationary epoch needs to end, there needs to be a dynamic means of transitioning into a radiation-dominated epoch. The next simplest trick is for the energy-density to be tied up in the potential of some dynamical degree of freedom, so that in a period when this potential is much greater than the kinetic energy, the total energy-density *appears* constant. That the kinetic energy in this scenario must be negligible with respect to the potential energy is the origin of the term "Slow-Roll." Here we use a single scalar field to demonstrate this technique in practice.

Consider, as a toy model, a Universe containing only Einstein gravity (i.e., General Relativity) and a single scalar field ϕ. We impose that this model satisfy the symmetries of our universe (homogeneity and isotropy), so we make the *ansatz* that the metric is FRW, and the scalar field only depends on time. The action for this universe is:

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2\kappa}\mathcal{R} + \frac{1}{2}(\partial\phi)^2 + V(\phi)\right), \tag{1.25}$$

where g is the metric determinant, R is the Ricci curvature scalar $R = g^{\mu\nu}R_{\mu\nu}$, κ is the gravitational constant $\kappa = 1/M_p^2$, and $(\partial\phi)^2 := g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$. The relevant field equations for this model are the Friedmann equation

$$3M_p^2H^2 = \frac{1}{2}\dot{\phi}^2 + V, \tag{1.26}$$

and the Klein-Gordon equation

$$\Box\phi - V_{,\phi} = \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \tag{1.27}$$

where commas denote differentiation with respect to the following symbol (e.g., $V_{,\phi} := \partial_\phi V$), and the d'Alembertian operator $\Box$ is defined as $\Box := g^{\mu\nu}\nabla_\mu\nabla_\nu$. The parameter ε (typically referred to as a slow-roll parameter) is found to be:

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_p^2 H^2}. \tag{1.28}$$

Equation (1.28) is the mathematical expression for the quasi-de Sitter condition described above; if $\dot{\phi}$ is sufficiently small (i.e., $\dot{\phi}^2 \ll V(\phi)$), then $\varepsilon \ll 1$ is satisfied, and the geometry is almost de Sitter. We can do better than this though, and turn this into a more precise statement on the form of the potential for the scalar field.

If $\dot{\phi}$ is to be sufficiently small long enough for inflation to make its mark, then we should politely request that its fractional rate of change be small—i.e., $\left|\Delta\dot{\phi}\right| < \left|\dot{\phi}\right|$. We can also write this as:

$$\begin{aligned} \left|\Delta\dot{\phi}\right| = \left|\int_{t_i}^{t} \mathrm{d}t' \ddot{\phi}\right| &< \left|\dot{\phi}\right|, \\ &< \frac{\left|\dot{\phi}\right|}{\Delta t}\Delta t. \end{aligned} \tag{1.29}$$

It is then straightforward to show that

$$\left|\ddot{\phi}\right| < \frac{\left|\dot{\phi}\right|}{\Delta t} \approx \frac{\left|\dot{\phi}\right| H}{\mathcal{N}}. \tag{1.30}$$

We are interested in scales for which $0 < \mathcal{N} \sim \mathcal{O}(10)$, so throughout the slow-roll regime, we should have $\left|\ddot{\phi}\right| \ll H\left|\dot{\phi}\right|$. With these approximations in play, the equations of motion now look like:

$$3M_p^2 H^2 \approx V, \tag{1.31}$$

and

$$3H\dot{\phi} \approx -V_{,\phi}. \tag{1.32}$$

Equations (1.31) and (1.32) can then be substituted into $\varepsilon := -\dot{H}/H^2$ to find

$$\varepsilon \approx \frac{M_p^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1, \tag{1.33}$$

which imposes a fairly strict condition on the slope of the scalar potential. Furthermore, taking the derivative of (1.32), we find

$$\ddot{\phi} \approx -\frac{\dot{\phi}\, V_{,\phi\phi}}{3H}. \tag{1.34}$$

Substituting this into the condition (1.30) that $\left|\ddot{\phi}\right| \ll H\left|\dot{\phi}\right|$, we find an even more stringent condition:

$$|\eta_{\mathrm{V}}| := M_p^2\left|\frac{V_{,\phi\phi}}{V}\right| \ll 1. \tag{1.35}$$

η_{V} is another frequently-quoted slow-roll parameter. The subscript V is to distinguish this definition from another common definition derived by demanding ε remain small for sufficiently long. For situations when $|\varepsilon| \ll 1$, following similar steps to (1.29) and (1.30) brings one to define

$$|\eta| := \left|\frac{\dot{\varepsilon}}{H\varepsilon}\right| \ll 1. \tag{1.36}$$

We can therefore conclude that, given a suitably chosen potential (one that satisfies both conditions (1.33) and (1.35)), this toy model is capable of plunging a toy Universe into a temporary epoch of quasi-de Sitter space. There are many reasons why such a field-theoretic approach is attractive. For one, we have just demonstrated that it is technically feasible. Moreover, as already stated, an inflationary epoch would need to take place at temperatures higher than we have probed in the lab. In such high energy regimes, there are many theories of new physics that offer innumerable candidates (see e.g. [15]) for this scalar field (which, when quantized, is referred to as the *inflaton*), or possibly even fields of higher spin. Even more than as a playground for speculative models of high energy physics, a successful quantum field-theoretic mechanism for inflation would be exciting for the fact that it would be the only known case where quantum field theory in a curved background had had an effect on scales large enough to be observed (i.e., solving the Horizon Problem, as well as shaping the CMBR in ways we describe in the following section). We now conclude with a short foray into the observational predictions of our toy model.

1.4 Connection to Observations

The scalar field toy model studied in section 1.3 demonstrates how a field theory can explain the Horizon Problem, a classical observation. However, as noted above, inflation would have to take place at high energies, so such a field theoretic realization must have non-negligible quantum effects, and therein lies the potential for predictability in an inflationary theory. A quantum field is subject to quantum uncertainty, so the spatially homogeneous inflaton field $\phi(t)$ is subject to small perturbations $\delta\phi(t, \mathbf{x})$ that are *not* necessarily spatially homogeneous. This amounts to small variations in the energy density across space, which naturally corresponds to variations in the gravitational response—variations that can leave an imprint on the evolution of the Universe.

In fact, we already see fluctuations in the gravitational response to the content of the Universe. We see small clumps of matter distributed throughout the universe in the form of galaxies and large-scale structures, and we see temperature variations in the CMBR corresponding to density fluctuations (via the Sachs-Wolfe effect, where photons are redshifted when escaping from gravitational wells). It is a remarkable fact that inflation, introduced simply as a means of solving a causal conundrum, is also a natural candidate for explaining the origin of structure in our universe via inescapable quantum fluctuations.

The full analysis of these perturbations is highly detailed and can easily fill a textbook or two (for an excellent treatment, see [16]). It is also only tangentially related to this thesis, so we only highlight two key relations that connect the properties of the single-field inflaton model from section 1.3 to observations.

Scalar quantum fluctuations of the inflaton and the metric during inflation can be described in terms of the density-density correlation function:

$$\left\langle \delta(\mathbf{r}' + \mathbf{r})\delta(\mathbf{r}') \right\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P(k) e^{i\mathbf{k}\cdot\mathbf{r}/a}, \tag{1.37}$$

where the average is over $\mathbf{r}'$, and $P(k)$ is the power spectrum, a function to be determined by the physics of inflation. When the angular integral is carried out, this can be written

$$\left\langle \delta(\mathbf{r}' + \mathbf{r})\delta(\mathbf{r}') \right\rangle = \int \frac{\mathrm{d}k}{k} \Delta^2(k) \frac{\sin(kr/a)}{kr/a}, \tag{1.38}$$

where Δ^2 is called the dimensionless power spectrum. The deviation of this spectrum from scale-invariance is parameterized by the *spectral index* n_s:

$$\Delta^2(k) \sim k^{n_s - 1}, \tag{1.39}$$

so that $n_s = 1$ indicates exactly de Sitter space during inflation. It can be shown, with some work, that in the single-field slow-roll model above, this parameter is given by

$$n_s = 1 - 2\varepsilon - \eta. \tag{1.40}$$

The power spectrum (1.38) can be directly measured with surveys of large-scale structure (see e.g. the Sloan Digital Sky Survey [17]). The spectral index itself can be more precisely measured in the CMBR.

The CMBR has its own power-spectrum which is usually written as a Legendre series:

$$\left\langle \frac{\delta T}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \right\rangle = \frac{1}{4\pi} \sum (2l+1) C_l P_l(\cos(\theta)), \tag{1.41}$$

where the average is over all $\mathbf{n}, \mathbf{n}'$ satisfying $\mathbf{n} \cdot \mathbf{n}' = \cos(\theta)$. The coefficients C_l make up the CMBR power spectrum, and come equipped with a factor of $l^{n_s - 1}$, so the relative heights of peaks in the CMBR power spectrum can tell us about the slow-roll parameters through (1.40). The current measurement of the spectral index from the Planck satellite [9] is:

$$n_s = 0.968 \pm 0.006. \tag{1.42}$$

Secondly, information can be gleaned from the polarization of the CMBR. Light from the surface of last scattering can become polarized by matter and space on its long journey to us. The matter density perturbations caused by scalar perturbations during inflation can only polarize light in a curl-free way (so-called "E modes", referring to $\nabla \times E = 0$ from the static Maxwell equations). Meanwhile, tensor perturbations of the metric during inflation correspond to gravitational waves which can polarize light in a divergence-free way (so-called "B modes", referring to $\nabla \cdot B = 0$ from Maxwell's equations). So a measure of the ratio of these modes in the CMBR is a measure of the ratio of the tensor to scalar power spectra, which can be related to the slow-roll parameter ε.

$$r := \frac{\Delta_T^2(k)}{\Delta_S^2(k)} = 16\varepsilon. \tag{1.43}$$

The current bound on r from combined Planck, Keck Array, and BICEP2 data [18] is

$$r < 0.07 \quad \text{at 95\% CL.} \tag{1.44}$$

It should be clear then, that since quantum fluctuations provide a natural source for remnant perturbations in the Universe (e.g., large-scale structure), it is possible to measure, or at least constrain, properties of inflation with precision observations. See figure 1.3 for a plot of exactly such constraints prepared by the Planck collaboration.

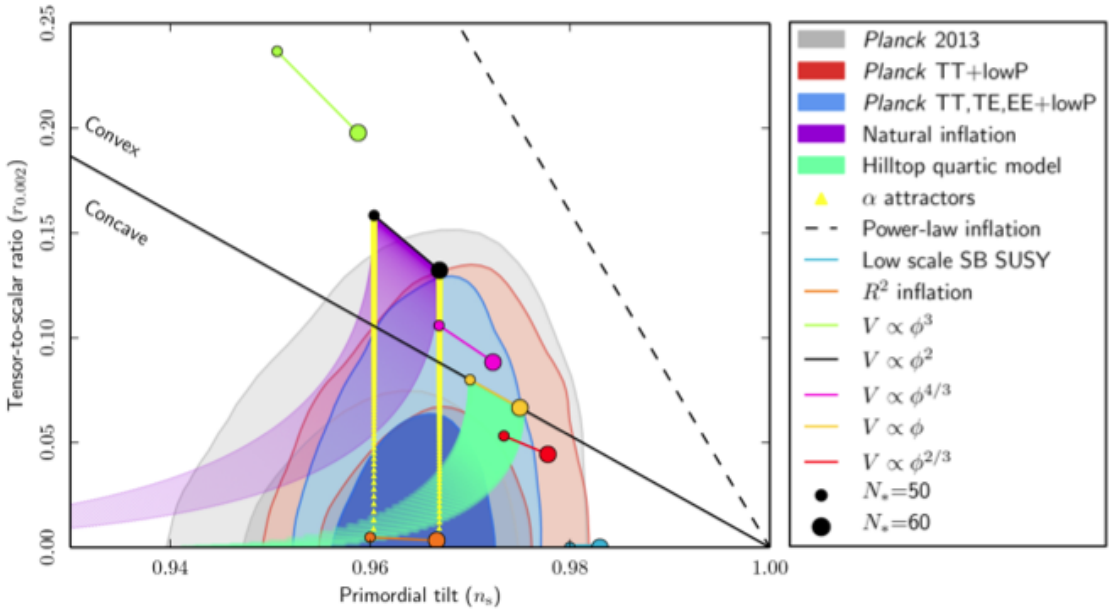

FIGURE 1.3: A plot showing inflationary predictions of several models as compared to observations. Reproduced from [9].

1.5 Summary

Thus we see that the highly successful Big Bang model of the evolution of the Universe is plagued with a problem of initial conditions. The Horizon problem that a universe full of causally disconnected regions manages to agree on a high degree of homogeneity either says that the universe began with an incredibly precise set of initial conditions, or that there exists an early period of new physics. Inflation is a particular class of candidates for the latter.

Inflation is characterized by an epoch when the geometry of space is quasi-de Sitter, and can be realized dynamically with a field theory. Something as simple a single scalar field can realize a temporary quasi-de Sitter equation of state $p \approx -\rho$ if its velocity is very small compared to its potential, so that it is approximately static (or "rolling slowly"). One can verify that one's model is indeed inflating by checking that it satisfies the bound (A.4), and one can check that it should last long enough with the bound (1.36). A successful inflationary model needs to last for safely longer than 40 e-foldings.

That inflation is a solution to an initial conditions problem means it is best if it not introduce its own initial conditions problem. Therefore in this thesis, we test and classify the sensitivity to initial conditions of our novel inflationary scenarios. The motivation and terminology thus laid out, we proceed with the meat of the thesis.

Chapter 2

High Definition Inflation

Extra dimensions have a number of motivating factors that make them worthy of study. A major contributor is that String Theory, a leading candidate for a theory of quantum gravity, predicts the existence of many extra dimensions [19]. It is also simply the case that extra dimensions up to $\sim 1\mu$m in size have not been excluded by experiment [20], and thus need to be considered when studying physics at higher energies. If indeed there exist extra dimensions, it is important to ask how they would behave during inflation. Indeed, perhaps it is even the case that inflation tells us why the extra dimensions are the size that they are today. Having said that, it is also typically very difficult to solve the field equations in a geometry with many extra dimensions. As the complexity of the metric increases, so too does the number and complexity of Einstein's equations. Here, we take the approach of working our way up the extra-dimensional ladder, one complication at a time.

The case of a single extra dimension has been explored in some detail in the literature [5, 6, 7, 8], so in this chapter, we construct a toy model with two extra dimensions, but compactify them on a sphere, for simplicity. In order to have a hope of stabilizing the extra dimensions at the end of inflation, we also include in our model a Maxwell field restricted to the extra dimensions, which acts as another energy-density that can compete with gravity. Lastly, we include a single scalar inflaton field to drive inflation.

In this chapter, we study this system numerically and analytically. We begin by deriving the full 6D Einstein equations, and subsequently develop a 4D system that is equivalent to the full 6D system. This allows for a more intuitive understanding of the dynamics as a whole, since we are 4D creatures, and tend to think with a 4D bias. The resulting system has a large parameter space. We perform an initial search numerically, and find two classes of solutions (however, our search is not exhaustive, and there may well be more classes of solutions yet to uncover). These classes are: 1) a "Cradle" scenario, where the extra-dimensional radius is trapped almost immediately, after which the inflaton goes through a period of familiar slow-roll, and 2) scaling solutions, where the fields scale as powers of time.

The properties of our solutions are as follows. The Cradle scenario typically yields $\varepsilon \sim 0.009$, and $\eta \sim 0.016$, so that $n_s \sim 0.975$ and $r \sim 0.15$, which is in tension with recent observations, if the standard perturbative analysis applies (we do not perform a full

6D perturbations analysis from scratch). On the bright side, this solution easily stabilizes the extra dimensions. The analytic description of this scenario is not well understood at this time, so a full study of its sensitivity to initial conditions is not known. Numerically, there does appear to be some range of acceptable initial conditions, but its relation to the parameters of the theory is unknown.

Finally, the scaling solutions have the properties that $\eta = 0$ exactly, and $\mathcal{N}_e \propto 2\ln(b_f/b_0)$, where $b(t)$ is the radius of the extra dimensions. Moreover, we find numerical examples of two particular scaling solutions, which demonstrate the extremes of sensitivity to initial conditions. On the one hand, we have one solution we refer to as the Attractor, which is marginally insensitive to initial conditions. It has the advantage that it is possible in that solution to stabilize the extra dimensions, however it also has the disadvantage that it is restricted to $\varepsilon \geq 0.5$. On the other hand, we have a solution we refer to as the Slow-Roll solution, which is highly sensitive to the choice of initial conditions. Its name reflects the fact that for this solution, ε is free, however it does also have the disadvantage that it is impossible to trap the extra dimensions in this solution.

In the end, we find for this toy model one clean example of a solution that is thoroughly generic and insensitive to initial conditions, one solution that is highly particular, and very sensitive to initial conditions, and one solution whose sensitivity to initial conditions is not fully known.

2.1 Action and Field Equations

We begin our study of extra-dimensional inflation with the "Einstein-Maxwell-Scalar" theory described above. The action for this theory is:

$$S = -\int \mathrm{d}^6x \sqrt{-g_{(6)}} \left(\frac{1}{2\kappa^2}\mathcal{R} + \frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}\partial_M\phi\,\partial^M\phi + V(\phi) \right), \tag{2.1}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$ is the Maxwell field strength tensor, while $g_{(6)}$ is the determinant of the 6D metric, and $\mathcal{R} = g^{MN}\mathcal{R}_{MN}$ is the 6D Ricci scalar.

For the inflaton potential, we choose

$$\begin{aligned} V(\phi) &= V_0\left(e^{-\beta_1\phi} - e^{-\beta_2\phi}\right) + \Lambda\,, \\ V_0,\ \beta_i &> 0\,, \end{aligned} \tag{2.2}$$

which is convenient when we integrate out the extra dimensions (when we do so, we perform field redefinitions on the extra-dimensional radius and the 4D metric to arrange for the equivalent 4D Lagrangian to have a standard form. Propagating those redefinitions is easiest if they are exponentials, so the end result is a Lagrangian populated with several

exponential terms). This potential is minimized at

$$\phi_\star = \frac{1}{\beta_2 - \beta_1} \ln\left(\frac{\beta_2}{\beta_1}\right) . \tag{2.3}$$

We fix the 6D cosmological constant Λ so that 4D space is Minkowski when (and if) the inflaton and the extra-dimensional radius are both stabilized (as they would have to be today).

We choose a FRW form for our metric:

$$\begin{aligned} \mathrm{d}\hat{s}^2 &= \hat{g}_{\mu\nu}\,\mathrm{d}\hat{x}^\mu \mathrm{d}\hat{x}^\nu + g_{mn}\,\mathrm{d}y^m \mathrm{d}y^n \\ &= -\mathrm{d}\hat{t}^2 + \hat{a}^2(\hat{t}\,)\,\delta_{ij}\,\mathrm{d}\hat{x}^i\,\mathrm{d}\hat{x}^j + b^2(\hat{t}\,)\gamma_{mn}(y)\,\mathrm{d}y^m \mathrm{d}y^n\,, \end{aligned} \tag{2.4}$$

where γ_{mn} represents the standard metric on the two-sphere, while $\hat{a}$ and b are the scale-factors for the regular 4D and the extra 2D, respectively. The hats are used to distinguish 6D coordinates from the 4D coordinates we use later. Lastly, we choose the scalar field to be independent of spatial coordinates.

Since there is only one linearly-independent two-form in two-dimensions, we can also write $F_{mn} = f\,\epsilon_{mn}$, where ϵ_{mn} is the Levi-Civita tensor on the extra-dimensional two-sphere. To be explicit, this is $\epsilon_{mn} = g_{(2)}\,\tilde{\epsilon}_{mn} = b^2\,\gamma\,\tilde{\epsilon}_{mn}$, where $\tilde{\epsilon}_{mn}$ is the simple Levi-Civita symbol in two dimensions. This action and these *ansätze* lead to the following equations of motion.

- ϕ *EOM*:

$$\Box\,\phi - \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = \phi'' + \Big(3\hat{H} + 2\mathcal{H}\Big)\phi' + \frac{\partial V}{\partial \phi} = 0\,, \tag{2.5}$$

- *Maxwell's Equations*:

$$\nabla_M F^{MN} = \partial_m f = 0\,, \tag{2.6}$$

- *Einstein Equations*:

$$\mathcal{G}_{MN} + \kappa^2 T_{MN} = 0\,, \tag{2.7}$$

where $\hat{H} := \hat{a}'/\hat{a}$, $\mathcal{H} := b'/b$ and primes denote $\mathrm{d}/\mathrm{d}\hat{t}$ (we use over-dots later for derivatives with respect to 4D time). $\mathcal{G}_{MN} = \mathcal{R}_{MN} - \frac{1}{2}\,\mathcal{R}\,g_{MN}$ is the Einstein tensor, and the stress-energy tensor is

$$T_{MN} = \partial_M\phi\,\partial_N\phi + F_{MP}F^P{}_N - g_{MN}\left(\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}\,\partial_M\phi\,\partial^M\phi + V(\phi)\right). \tag{2.8}$$

Given our *ansätze*, the Einstein equations take on the fairly simple form (see appendix B for details):

$$
\begin{aligned}
3\left(\frac{\hat{a}'}{\hat{a}}\right)^2 + \left(\frac{b'}{b}\right)^2 + 6\left(\frac{\hat{a}'b'}{\hat{a}b}\right) + \frac{1}{b^2} &= \kappa^2\left\{\frac{1}{2}\left[\left(\phi'\right)^2 + \frac{\mathfrak{f}^2}{b^4}\right] + V\right\} \\
2\left(\frac{\hat{a}''}{\hat{a}} + \frac{b''}{b}\right) + \left(\frac{\hat{a}'}{\hat{a}}\right)^2 + \left(\frac{b'}{b}\right)^2 + 4\left(\frac{\hat{a}'b'}{\hat{a}b}\right) + \frac{1}{b^2} &= \kappa^2\left\{\frac{1}{2}\left[-\left(\phi'\right)^2 + \frac{\mathfrak{f}^2}{b^4}\right] + V\right\} \\
\frac{b''}{b} + 3\left[\frac{\hat{a}''}{\hat{a}} + \left(\frac{\hat{a}'}{\hat{a}}\right)^2\right] + 3\left(\frac{\hat{a}'b'}{\hat{a}b}\right) &= \kappa^2\left\{-\frac{1}{2}\left[\left(\phi'\right)^2 + \frac{\mathfrak{f}^2}{b^4}\right] + V\right\}.
\end{aligned}
\tag{2.9}
$$

The Maxwell field also satisfies the Bianchi identity, as usual,

$$
\mathrm{d}F = \frac{\mathrm{d}}{\mathrm{d}\hat{t}}\left(fb^2\right) = 0\,. \tag{2.10}
$$

Finally, the requirement that the extra-dimensional flux be quantized (as would be necessary to stabilize the extra-dimensions) leads to:

$$
\begin{aligned}
\int_{S^2} F = 4\pi f b^2 &= \frac{2\pi n}{e}\,, \\
\implies f &= \frac{\mathfrak{f}}{b^2}\,,
\end{aligned}
\tag{2.11}
$$

where we have defined $\mathfrak{f} := n/2e$, with $n \in \mathbb{Z}$, and e is the EM field's coupling constant, and we have used Maxwell's equation (2.6), and the Bianchi identity (2.10).

2.2 4D Perspective

Since we mere mortals live down here in our lowly four dimensions, it is a good idea to explore this system from a 4D perspective. Fortunately, our *ansätze* make this relatively straightforward, as we can simply integrate out our extra dimensions. We also later show explicitly that our new 4D equations of motion are equivalent (by a change of coordinates) to the full 6D equations of motion, and therefore that our solutions fully capture the dynamics of the 6D theory, at least at the classical level (this is known as a *consistent truncation*. See [21]).

This reduction can be carried out in a few logical steps.

1. *Setup*: First, we need to write things in terms of the separate 4D and 2D components of our metric. Using equation (2.4), we can write the scalar curvature as:

$$\begin{aligned}\mathcal{R}_{(6)} := g_{MN}\mathcal{R}^{MN} &= g_{\mu\nu}\mathcal{R}^{\mu\nu} + g_{mn}\mathcal{R}^{mn}, \\ &= \hat{\mathcal{R}}_{(4)} + \frac{\mathcal{R}_{(2)}}{b^2} + 4\left(\frac{\hat{\Box} b}{b}\right) + 2\left(\frac{\hat{\partial} b}{b}\right)^2,\end{aligned} \tag{2.12}$$

Here, $\hat{\mathcal{R}}_{(4)}$ is the Ricci scalar built from $\hat{g}_{\mu\nu}$, $\mathcal{R}_{(2)}$ is the scalar built from γ_{mn}, and $\hat{\Box}$ and $\hat{\partial}$ only run over the 4D coordinates, so $\hat{\Box} := \hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu$, and $(\hat{\partial} b) := \hat{g}^{\mu\nu}\hat{\partial}_\mu b\,\hat{\partial}_\nu b$. For the sphere, $\mathcal{R}_{(2)} = -2$. We can also write the metric determinant $\sqrt{-g_{(6)}} = \sqrt{-\hat{g}_{(4)}}\,\sqrt{\gamma}\,b^2$.

2. *Integrate out the extra dimensions*: Now that we can see nothing depends on the extra-dimensional coordinates, we can explicitly perform the extra-dimensional integral.

$$\int_{S^2} \mathrm{d}^2 y \sqrt{\gamma} = 4\pi. \tag{2.13}$$

Thus the Einstein-Hilbert action (i.e., the gravitational action, with the Lagrangian $L = \kappa^{-2}\mathcal{R}$) becomes:

$$\begin{aligned}S_{\mathrm{EH}} &= -\int \mathrm{d}^4 x \sqrt{-\hat{g}_{(4)}}\frac{2\pi}{\kappa^2}\left(b^2\hat{\mathcal{R}}_{(4)} - 2 + 4b\hat{\Box} b + 2\left(\hat{\partial} b\right)^2\right), \\ &= -\int \mathrm{d}^4 x \sqrt{-\hat{g}_{(4)}}\frac{2\pi}{\kappa^2}\left(b^2\hat{\mathcal{R}}_{(4)} - 2 - 2\left(\hat{\partial} b\right)^2\right),\end{aligned} \tag{2.14}$$

where in the second line, we integrate by parts, and drop the surface term. The matter Lagrangian is simply multiplied by a factor of $4\pi b^2$.

3. *Re-scale*: It is usually helpful to view the system from the Einstein frame (i.e., the frame in which the Ricci scalar is not multiplied by a dynamic quantity in the Lagrangian). To do so, we make the field re-definition:

$$g_{\mu\nu} = e^{\psi/M_p}\hat{g}_{\mu\nu}, \tag{2.15}$$

where we have defined $M_p^2 := 4\pi b_\star^2/\kappa^2 = 4\pi b_\star^2 M_{(6)}^4$, and $b_\star = b(t_\star)$ is the size of the extra-dimensions today. In terms of this metric,

$$\sqrt{-\hat{g}_{(4)}} = \sqrt{-g_{(4)}}\, e^{-2\psi/M_p}, \text{ and}$$
$$\hat{\mathcal{R}}_{(4)} = e^{\psi/M_p} g^{\mu\nu}\left\{\mathcal{R}_{\mu\nu} - \frac{2}{M_p}\nabla_\mu\nabla_\nu\psi + \frac{3}{2M_p^2}\partial_\mu\psi\partial_\nu\psi\right\}.$$

4. *Make everything canonical*: To write the kinetic energy of the extra-dimensional radius in a canonical way, we perform the field re-definition:

$$b = b_\star e^{\psi/2M_p}, \tag{2.16}$$

so that

$$\begin{aligned}\left(\hat{\partial} b\right)^2 &= \frac{1}{4M_p^2} e^{\psi/M_p}\hat{g}^{\mu\nu}\hat{\partial}_\mu\psi\hat{\partial}_\nu\psi, \\ &= \frac{1}{4M_p^2} e^{2\psi/M_p} g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi.\end{aligned} \tag{2.17}$$

We call the field ψ the "radion," reflecting its origin as the radius of the extra dimensions.

With this, the Einstein-Hilbert action becomes:

$$\begin{aligned}S_{\text{EH}} = &-\int \mathrm{d}^4x\sqrt{-g_{(4)}}\, e^{-2\psi/M_p}\frac{2\pi}{\kappa^2}\left[\left(b_\star e^{\psi/2M_p}\right)^2\right. \\ &\times\left(e^{\psi/M_p} g^{\mu\nu}\left\{\mathcal{R}_{\mu\nu} - \frac{2}{M_p}\nabla_\mu\nabla_\nu\psi + \frac{3}{2M_p^2}\partial_\mu\psi\partial_\nu\psi\right\}\right) \\ &\left.-2 - 2\left(\frac{b_\star^2}{4M_p^2} e^{2\psi/M_p} g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi\right)\right], \\ = &-\int \mathrm{d}^4x\sqrt{-g_{(4)}}\left[\frac{M_p^2}{2}\mathcal{R} + \frac{1}{2}(\partial\psi)^2 - \frac{e^{-2\psi/M_p}}{b_\star^2}\right],\end{aligned} \tag{2.18}$$

where we make use of the definition of M_p^2, and in the last line, drop the contribution from the surface term $\Box\psi$. Now we can see the field ψ has a canonical kinetic term $0.5(\partial\psi)^2$.

For the matter contribution, integrating the extra dimensions means the matter Lagrangian receives an overall multiplicative factor of $4\pi b^2$. The field re-definition 2.15

means the kinetic term for the inflaton can be written:

$$
\begin{aligned}
4\pi b_\star^2 e^{\psi/M_p}\left(\hat{\partial}\phi\right)^2 &= 4\pi b_\star^2 e^{\psi/M_p} e^{-\psi/M_p}(\partial\phi)^2, \\
&= 4\pi b_\star^2\left(\hat{\partial}\phi\right)^2.
\end{aligned} \tag{2.19}
$$

Hence, the inflaton can be canonically normalized by defining $\varphi := \sqrt{4\pi} b_\star \phi$. At the end of the day, we are left with the 4D action:

$$
S = -\int \mathrm{d}^4x\sqrt{-g}\left\{\frac{M_p^2}{2}\mathcal{R}_{\mu\nu} + \frac{1}{2}(\partial\psi)^2 + \frac{1}{2}(\partial\varphi)^2 + W(\varphi,\psi)\right\}, \tag{2.20}
$$

where the effective potential $W(\varphi,\psi)$ is defined by

$$
W(\varphi,\psi) := 4\pi b_\star^2 e^{-\psi/M_p} U(\varphi) - \frac{M_p^2}{b_\star^2} e^{-2\psi/M_p} + \frac{2\pi \mathfrak{f}^2}{b_\star^2} e^{-3\psi/M_p}, \tag{2.21}
$$

with $U(\varphi) := V(\phi)$.

The action 2.20 can now be treated as any other 4D action, and continuing to assume homogeneity and isotropy (i.e., using an FRW form for $g_{\mu\nu}$, and taking $\varphi = \varphi(t)$, $\psi = \psi(t)$) we can obtain the following equations of motion:

$$
\begin{aligned}
&\ddot{\varphi} + 3H\dot{\varphi} + W_{,\varphi} = 0, \\
&\ddot{\psi} + 3H\dot{\psi} + W_{,\psi} = 0, \text{ and} \\
&\frac{\dot{\varphi}^2}{2} + \frac{\dot{\psi}^2}{2} + W = 3M_p^2 H^2\,,
\end{aligned} \tag{2.22}
$$

where $H := \dot{a}/a$, over-dots refer to derivatives with respect to t, and commas denote derivatives with respect to φ or ψ, as indicated. In appendix C, we show explicitly that a coordinate transformation relates these equations of motion (2.22) to the full 6D EOMs (2.5) and (2.9), and therefore that solutions to either set of equations are solutions to the other set.

This system (2.22) is now much more intuitive in that we can think of it as a 4D system with two scalar fields. However, it is still a very complicated system, and we do not have a complete classification of its solutions. Nevertheless, in our numerical searches, we are able to find and understand two classes of solutions. Next, we detail those solutions, and when possible, we perform an analysis of the sensitivity of the solution to variations in its initial conditions.

2.3 Solutions

In our numerical searches, we explore the system by manually searching small regions of parameter-space, and we find two recurring classes of solutions. First, in some cases when we start the radion, ψ, near its minimum, we see it minimize quickly, after which the inflaton, φ, engages in a long period of slow-roll. We refer to this solution as the "Cradle" scenario. Second, when we start the radion far from its minimum, we typically see a period where both fields are well-described by power-laws. Using scaling *ansätze* for our fields, we are able to show analytically that there are indeed power-law solutions to our system in certain regimes, including one that is difficult, but not impossible to find numerically. We detail these solutions below in the order just described, and conclude with an analysis of their sensitivity to initial conditions.

2.3.1 Cradle

If the radion starts very near to its minimum, we numerically see the system find a solution that appears to exhibit a period of 4D quasi-de Sitter space, and almost static extra dimensions (see figure 2.1 for an example). We call this solution the "Cradle" solution because the radion is nestled into its minimum like in a cradle (and so as to distinguish it from the scaling solution that also demonstrates slow-roll behaviour). To describe the inflationary portion of this system analytically, we first write ψ in terms of φ using the fact that it minimizes its potential. That is, we find $\psi_c(\varphi)$ such that $W_{,\psi}(\varphi, \psi_c) = 0$:

$$\frac{\psi_c}{M_p} = -\ln\Big(1 + \sqrt{1 - 6\pi b_\star^4 U(\varphi)/M_p^2}\Big) - \ln(2/3), \tag{2.23}$$

where we use $\Lambda = M_p^2/8\pi b_\star^4$ in order to enforce that the potential vanish when both ψ and φ are minimized, and we choose $\mathfrak{f}^2 = M_p^2/4\pi$ so that the minimum for ψ is at 0 when φ is also minimized. Note that ψ_c is only real for φ such that $U(\varphi) < M_p^2/6\pi b_\star^4$.

Thus we reduce the two-field problem to a (rather complicated) single-field problem. If indeed the segment is quasi-de Sitter, as we suspect, it should be possible to describe the inflaton as slowly rolling, so that the equations of motion are equivalent to those in chapter 1.3:

$$\dot{\varphi} \approx -\frac{W_{,\varphi}(\varphi, \psi_c(\varphi))}{3H} \qquad \text{and} \qquad 3M_p^2 H^2 \approx W(\varphi, \psi_c(\varphi)). \tag{2.24}$$

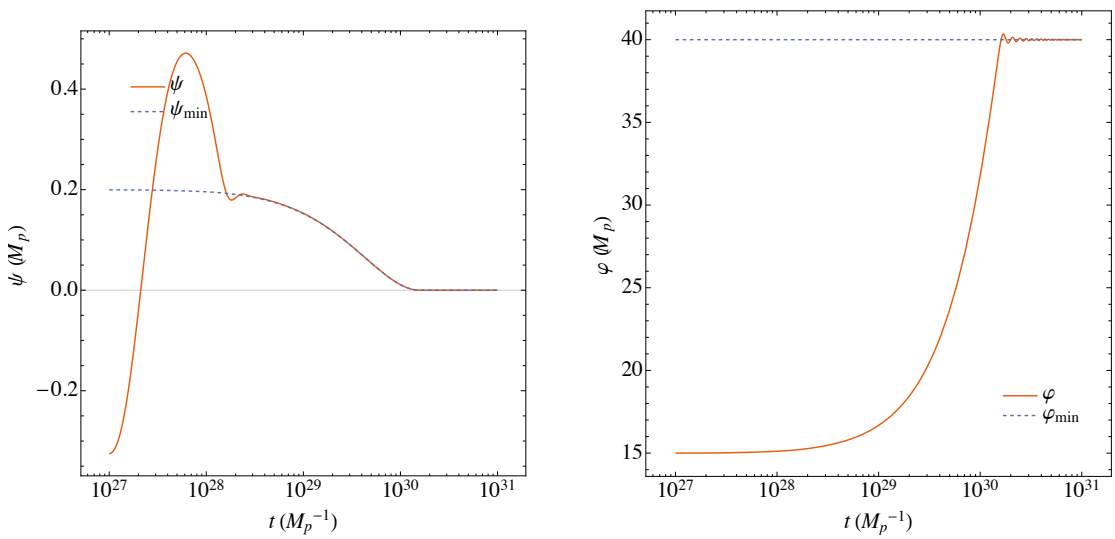

FIGURE 2.1: A numerical example of the Cradle scenario. The radion ψ is on the left, and the inflaton φ is on the right. The suspected slow-roll regime begins around $t \sim 10^{28} M_p^{-1}$, when ψ appears to settle. The parameters for this run are: $U_0 = 10^{-108}$, $b_* = 10^{27}$, $\beta_1/\sqrt{4pi}b_* = 38.15$, $\beta_2/\sqrt{4\pi}b_* = 41.97$, $\varphi_0 = 15$, $\psi_0 = -0.325$, $\dot{\varphi}_0 = \dot{\psi} = 0$, and $t_0 = 10^{27}$, all in units such that $M_p = 1$.

Following section 1.3, we can use these to compute ϵ.

$$\begin{aligned}\epsilon := -\frac{\dot{H}}{H^2} &= -\frac{1}{6M_p^2}\frac{W_{,\varphi}\dot{\varphi} + W_{,\psi}\dot{\psi}}{H^3} \\ &= -\frac{1}{6M_p^2}\frac{W_{,\varphi}\dot{\varphi}}{H^3} \\ &= \frac{M_p^2}{2}\left(\frac{W_{,\varphi}}{W}\right)^2 .\end{aligned} \tag{2.25}$$

exactly in analogy to equation (1.33).

These equations look easy to manage, but W is a very complicated function, so integrating the φ equation of motion is highly non-trivial, and at this stage, we have no analytic solution. We can, however, use the approximate Friedmann equation to write $H \approx \sqrt{W/3M_p^2}$. This relation is a good trademark of this slow-roll regime, so it can be used to verify that these approximations do describe the inflationary regime we observe in our numerical solution. An example showing this is exactly the case is given in figure 2.2.

It is usually found for this solution that $\varepsilon \sim 0.009$, and $\eta \sim 0.016$, corresponding in the

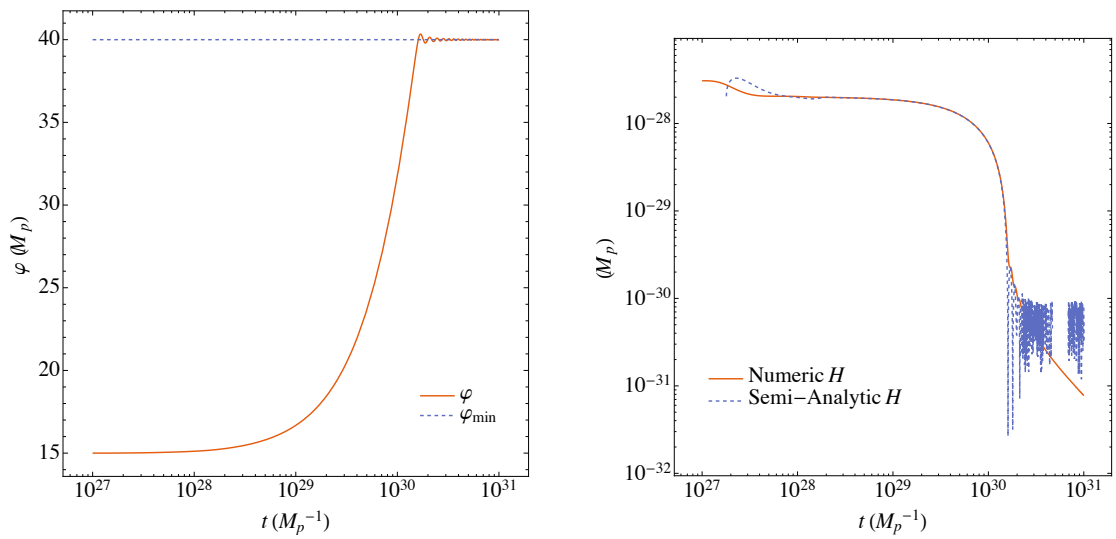

FIGURE 2.2: 4D Inflation in the Cradle scenario. *Left:* The inflaton field in the Cradle scenario is effectively the only degree of freedom. *Right:* The Hubble parameter in the Cradle scenario. The semi-analytic curve is a plot of $H \approx \sqrt{W/3M_p^2}$. The slow-roll inflationary regime occurs where the numeric and semi-analytic curves agree. The parameters for this run are the same as in figure 2.1.

usual perturbative analysis to $n_s \sim 0.975$, and $r \sim 0.15$, which is in tension with recent observations. However, this inflationary scenario is still not well understood, so it remains to be seen whether this is a generic prediction, or if there exists a parameter space that may yield acceptable values for these quantities. An in depth analysis is left for future work.

2.3.2 Power Laws

Besides the Cradle solutions, the other pattern we see frequently in our numerics is demonstrated in figure 2.3. That is, on a log-linear plot, we see long segments where both φ and ψ are straight lines. This suggests there exist solutions to this system for which the fields scale as powers of t, at least in certain regimes. Indeed, we find this is exactly the case.

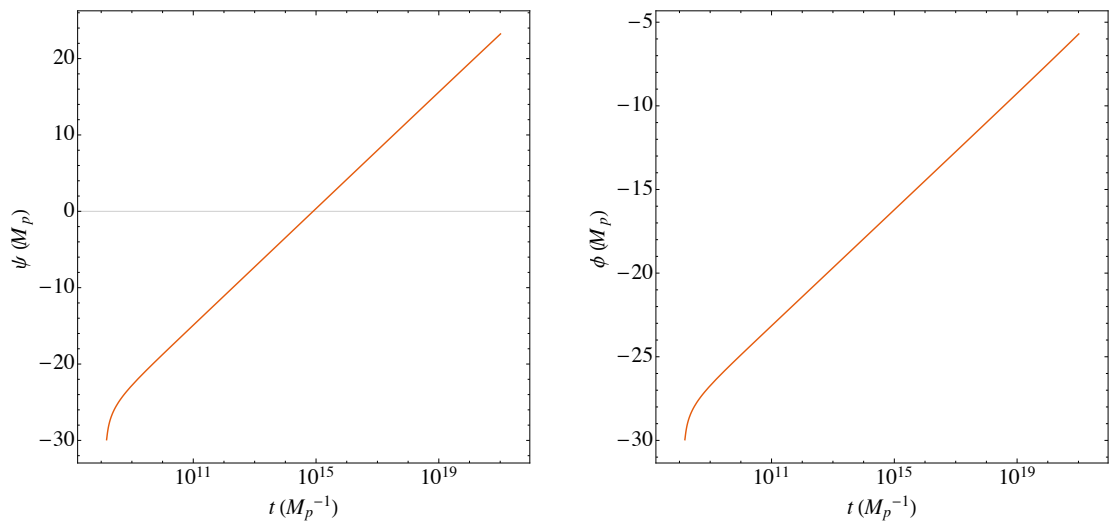

FIGURE 2.3: An example of suspected power-law behaviour in our numerical searches (the radion ψ is on the left, and the inflaton φ is on the right). Notice how the numerics immediately seek solutions with straight lines on these log-linear plots, strongly suggesting the existence of scaling solutions. The parameters for this run are: $U_0 = 5 \times 10^{-90}$, $b_* = 10^{28}$, $\beta_1/\sqrt{4\pi}b_* = 2.2$, $\beta_2/\sqrt{4\pi}b_* = 15.4$, $\varphi_0 = -30$, $\psi_0 = -30$, $\dot{\varphi}_0 = \dot{\psi} = 0$, and $t_0 = 10^8$, all in units such that $M_p = 1$.

To uncover these solutions, we first assume the following forms for the dynamical degrees of freedom in our problem:

$$\begin{aligned} \frac{\varphi}{M_p} &= \frac{\varphi_0}{M_p} + p_1 \ln\left(t/t_0\right), \\ \frac{\psi}{M_p} &= \frac{\psi_0}{M_p} + p_2 \ln\left(t/t_0\right), \text{and} \\ a &= a_0 \left(\frac{t}{t_0}\right)^{\alpha}, \end{aligned} \tag{2.26}$$

hence

$$H = \frac{\alpha}{t}. \tag{2.27}$$

It is worth pausing here to note that these *ansätze* alone, if satisfied, have interesting things to say. Most importantly, we can immediately see that

$$\epsilon := -\frac{\dot{H}}{H^2} = 1/\alpha \tag{2.28}$$

is constant, and less than 1 as long as $\alpha > 1$. Moreover, we can also easily calculate $\mathcal{N}_e$, and connect it with the motion of the fields:

$$\begin{aligned}\mathcal{N}_e := \ln\left(\frac{a_f}{a_0}\right) &= \int_{t_0}^{t_f} \mathrm{d}t\ H = \alpha \ln\left(\frac{t}{t_0}\right) \\ &= \frac{\alpha}{p_2}\left(\frac{\psi_f - \psi_0}{M_p}\right) = \frac{2\alpha}{p_2}\ln\left(\frac{b_f}{b_0}\right),\end{aligned} \tag{2.29}$$

where in the last equality, we write ψ in terms of the extra-dimensional radius. In this way, we can see that a power-law solution inherently has a direct connection between the amount by which our extra dimensions inflate, and the amount of time it takes to get them there. Incidentally, this allows for a very large possible value for $\mathcal{N}_e$, given that b could potentially range from the Planck scale all the way to the micron scale. However in practice, additional constraints, such as requiring energy-densities be sub-Planckian, limit this range to much stricter values.

Finally, with these *ansätze*, the equations of motion 2.22 become:

$$\begin{aligned}-\frac{p_1}{t^2} + \frac{p_1\alpha}{t^2} &= \frac{\lambda}{M_p^2}W^{(\varphi)}, \\ -\frac{p_2}{t^2} + \frac{p_2\alpha}{t^2} &= \frac{1}{M_p^2}W^{(\varphi)} - \frac{2}{M_p^2}W^{(c)} + \frac{3}{M_p^2}W^{(f)}, \quad \text{and} \\ \frac{p_1^2 + p_2^2 - 6\alpha^2}{t^2} &= -\frac{1}{M_p^2}W^{(\varphi)} + \frac{1}{M_p^2}W^{(c)} - \frac{1}{M_p^2}W^{(f)}.\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}W^{(\varphi)} &:= U_0\left(\frac{t}{t_0}\right)^{-\lambda p_1 - p_2}, \\ W^{(c)} &:= \frac{M_p^2}{b_\star^2}e^{-2\psi_0/M_p}\left(\frac{t}{t_0}\right)^{-2p_2}, \qquad \text{and} \\ W^{(f)} &:= \frac{2\pi \mathfrak{f}^2}{M_p^2 b_\star^2}e^{-3\psi_0/M_p}\left(\frac{t}{t_0}\right)^{-3p_2},\end{aligned} \tag{2.31}$$

and we have taken the inflaton potential to be

$$U(\varphi) \sim V_0 e^{-\lambda\varphi/M_p}, \tag{2.32}$$

which is generally a good approximation of (2.2) when $\lambda = \beta_1/\sqrt{4\pi}b_* > 0$ and $\varphi < 0$. We also define $U_0 := 4\pi b_\star^2 V_0 \exp(-(\lambda\varphi_0 + \psi_0)/M_p)$, for convenience.

It is found that these equations are consistent in two regimes, with different combinations of the terms $W^{(\varphi)}$, $W^{(c)}$, and $W^{(f)}$ dominating the potential on the RHS of each equation. An important point regarding this hierarchy of terms is that it also determines whether or not a minimum can exist for the radion. To see this, recall that we ask the total potential W to vanish at the minimum of both φ and ψ, and for that minimum to occur when $\psi = 0$. It can be shown that this then requires all the terms $W^{(i)}$ to be of approximately the same magnitude at that point. Since this minimum must occur *after* the inflationary regime, and since the flux and curvature terms $W^{(f)}$ and $W^{(c)}$ scale as different powers of $e^{-\psi}$, this imposes the condition that the radion can only be trapped if, throughout the inflationary regime, the hierarchy holds that

$$W^{(f)} > W^{(c)}. \tag{2.33}$$

With that in mind, we proceed to detail the two inflationary scaling solutions.

2.3.2.1 Attractor

First, if the potential is entirely dominated by the term containing the inflaton potential, we find a solution we label the "attractor." It is named so because (as is detailed in the next section) it is marginally insensitive to variations in its initial conditions. When the flux and curvature terms $W^{(f)}$ and $W^{(c)}$ in the potential are dropped, we find two sets of equations. Equating the powers of time tells us

$$p_1 = \alpha\lambda\,, \qquad \text{and} \qquad p_2 = \alpha\,, \tag{2.34}$$

while equating the coefficients of time yields

$$\alpha = \frac{2}{1+\lambda^2}\,, \qquad \text{and} \qquad t_0^2 = \frac{\left(5-\lambda^2\right)\alpha^2 M_p^2}{2U_0}\,. \tag{2.35}$$

If we insist that α and λ are real (which we do), we can see right away that this solution gives us an upper bound on α of 2, hence from (2.28), a lower bound on $\epsilon = 1/\alpha$ of $1/2$. As a result, this solution alone could not describe the cosmology we observe, assuming the usual perturbation analysis applies. Nevertheless, it is an important solution because of its attractor nature, so it is worth exploring. In figure 2.4, we show an example of this solution found in our numerics.

Note that since both the flux and curvature terms were neglected in this solution, their hierarchy is arbitrary, so (2.33) can certainly be satisfied. Despite that this is technically allowed, however, we are unable to produce a numerical example of this attractor scenario that also traps the radion. Numerically, we always see the radion overshoot its minimum, so it is likely that a bound exists related to the kinetic energy of the fields that makes

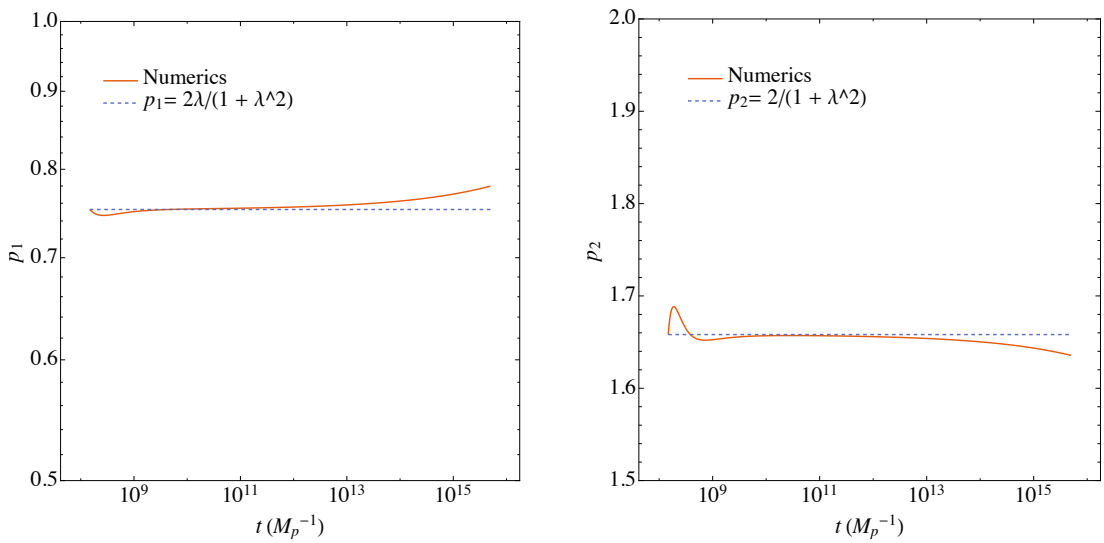

FIGURE 2.4: Plots of $t\dot{\varphi}$ (left) and $t\dot{\psi}$ (right) vs t. Reasonable agreement is seen between the numerics and equation (2.34), indicating we are reasonably seeing the attractor scaling solution. Note that the agreement begins to break down as the approximation that $U(\varphi)$ is a single exponential begins to fail. The parameters for this run are: $\varphi_0 = -18.6$, $\psi_0 = -30$, $\dot{\varphi}_0 = 10^{-9}$, $\dot{\psi} = 10^{-8}$, and all other parameters are the same as in figure 2.3 (again, in units such that $M_p = 1$).

trapping the radion very difficult in this scenario. An exact statement to that effect is not known at this time, and is relegated to future work.

2.3.2.2 Slow-Roll

Lastly, we find one more consistent solution in the regime where the flux term in the potential $W^{(f)}$ is much smaller than the other terms, which are roughly on par with each other. Note that this immediately violates the condition (2.33), so this inflationary solution is incapable of trapping the radion. This solution is termed "Slow-Roll," because we are able to use this solution to obtain slow-roll parameters consistent with observational data.

When the flux term is dropped, we again obtain two sets of equations. From the equality of the powers of time, we learn

$$p_1 = \frac{1}{\lambda}\,, \qquad \text{and} \qquad p_2 = 1\ , \tag{2.36}$$

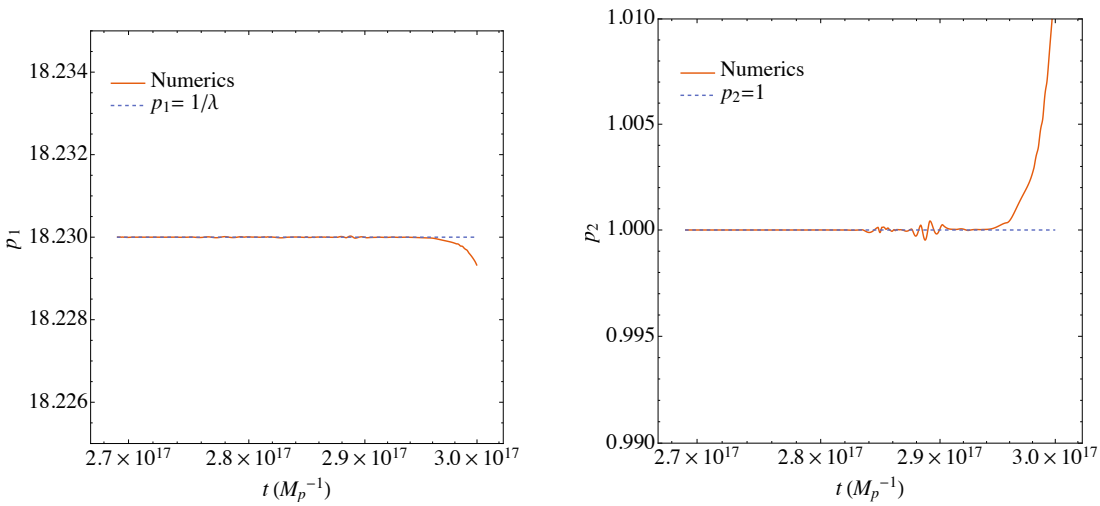

FIGURE 2.5: Plots of $t\dot{\varphi}$ (left) and $t\dot{\psi}$ (right) vs t. Reasonable agreement is seen between the numerics and equation (2.36), indicating we are reasonably seeing the slow-roll scaling solution. The large deviations at the end indicate the numerics transitioning to the attractor solution. The parameters for this run are: $U_0 = 10^{-101}$, $b_* = 10^{28}$, $\beta_1/\sqrt{4\pi}b_* = 18$, $\beta_2/\sqrt{4\pi}b_* = 36$, $\varphi_0 = -30$, $\psi_0 = -30$, $\dot{\varphi}_0 = 6 \times 10^{-18}$, $\dot{\psi} = 4 \times 10^{-17}$, and $t_0 = 10^{17}$, and all units are such that $M_p = 1$.

whereas the equality of the coefficients provides us with

$$U_0 = \frac{2M_p^2}{(1-\lambda^2)\, b_\star^2}\, e^{-2\psi_0/M_p}\,, \qquad t_0^2 = \frac{\left(\lambda^2+3\right) M_p^2}{2\lambda^4 U_0}, \qquad \text{and} \qquad \alpha = \frac{1+\lambda^2}{2\lambda^2}\,. \tag{2.37}$$

Here we can see that there is now no upper bound on α, so we are free to choose parameters such that ϵ and η are observationally satisfactory. In figure 2.5, we show an example of such a solution found in our numerics.

The cradle and power-law solutions are the sum of our analytical understanding of the system (2.22), thus far. There may be more analytic inflationary solutions that can be coaxed out, but that remains for future work. For now, we proceed to study the sensitivity of these solutions to their initial conditions, as much as possible.

2.4 Sensitivity to Initial Conditions

In chapter 1, we detail how inflation is intended to solve the problem that the Big Bang model alone requires an entire universe of causally disconnected regions to have started out with almost exactly the same initial conditions. Any successful inflationary model

will help to solve the problem by at least reducing it to a few initial conditions (e.g., the initial values and velocities of a few fields), however to really properly solve the issue, it is necessary for the inflationary model to be generic. Here we formalize that notion.

Consider an inflationary scenario governed by the fields χ^i, with an inflationary solution χ^i_*. Suppose at some arbitrary initial time t_0 the fields are offset from the inflationary solution by some small amount $\delta\chi^i(t_0)$. The evolution of the small perturbations can then be determined by linearizing the equations of motion in $\delta\chi^i$. In this model, we have a second-order ordinary differential equation for each perturbation (except for the Hubble parameter, which is determined entirely in terms of the other fields by the Friedmann equation), so we expect to find two linearly independent modes for each perturbation. The late-time behaviour of these modes determines the sensitivity of the solution to its initial conditions. Modes that decay to 0 are referred to as *stable*, and those that decay to a non-zero constant are labelled *marginally stable*. Any mode that increases with time is called *unstable*. Any solution for which all modes are stable is itself called an *stable*, or *insensitive to initial conditions*. If any mode is marginally stable, the solution is itself called *marginally stable*, or *marginally insensitive to initial conditions*. Finally, if any mode is unstable, the solution is termed *unstable*, or *sensitive to initial conditions*. These categorizations are shown schematically in figure.

For our system, the fields to be perturbed are[1]

$$\begin{aligned} \varphi &\to \varphi_* + \delta\varphi, \\ \psi &\to \psi_* + \delta\psi, \text{ and} \\ H &\to H_* + \delta H. \end{aligned} \tag{2.38}$$

Substituting these into the 4D equations of motion 2.22, and using the fact that the 0th-order fields solve those equations, we find in general:

$$\begin{aligned} \delta\ddot{\varphi} + 3(H_*\delta\dot{\varphi} + \delta H\dot{\varphi}_*) + W_{,\varphi\varphi}\Big|_{(\varphi_*,\psi_*)}\delta\varphi + W_{,\varphi\psi}\Big|_{(\varphi_*,\psi_*)}\delta\psi &= 0 \\ \delta\ddot{\psi} + 3\Big(H_*\delta\dot{\psi} + \delta H\dot{\psi}_*\Big) + W_{,\psi\varphi}\Big|_{(\varphi_*,\psi_*)}\delta\varphi + W_{,\psi\psi}\Big|_{(\varphi_*,\psi_*)}\delta\psi &= 0 \\ \dot{\varphi}_*\delta\dot{\varphi} + \dot{\psi}_*\delta\dot{\psi} + W_{,\varphi}\Big|_{(\varphi_*,\psi_*)}\delta\varphi + W_{,\psi}\Big|_{(\varphi_*,\psi_*)}\delta\psi &= 6M_p^2\, H_*\delta H \end{aligned} \tag{2.39}$$

From the Einstein constraint (the third equation in 2.39), it is clear that the perturbation in H is entirely determined in terms of the perturbations of the other fields, so we can

[1]Note that we have chosen to perturb H directly instead of the scale factor, since the scale factor only appears in our equations through H. Had we perturbed a directly instead, we would simply perform the same analysis for δH, but then also have to solve $\delta H = \delta\dot{a}/a_* - H_*\delta a/a_*$.

eliminate δH from our equations, and arrive at a system of two equations for the two functions $\delta\varphi$ and $\delta\psi$.

Finally, we perform this analysis as much as possible on the analytic solutions found for this system. Since we do not have an analytic expression for $\varphi(t)$ in the cradle solution, we do not know the time dependence of the background solutions so cannot solve for the perturbations. This analysis must be relegated to future work for the moment, and we proceed to study the well-understood scaling solutions.

2.4.1 Power Laws

As when deriving the solutions in section 2.3.2, we assume the approximate form 2.32 for the inflaton potential. Now, the perturbation equations read:

$$\begin{aligned}
\delta\ddot{\varphi} + 3\left(H_*\delta\dot{\varphi} + \delta H\dot{\varphi}_*\right) + \frac{\lambda U(t)}{M_p^2}\left(\lambda\delta\varphi + \delta\psi\right) &= 0 \\
\delta\ddot{\psi} + 3\left(H_*\delta\dot{\psi} + \dot{\psi}_*\delta H\right) + \left(\frac{U(t)}{M_p^2} - \frac{4}{b_\star^2}\,e^{-2\psi_*/M_p}\right)\delta\psi + \frac{\lambda U(t)}{M_p^2}\,\delta\varphi &= 0 \\
\dot{\varphi}_*\delta\dot{\varphi} + \dot{\psi}_*\delta\dot{\psi} - \frac{\lambda U(t)}{M_p}\,\delta\varphi + \left(\frac{2M_p}{b_\star^2}\,e^{-2\psi_*/M_p} - \frac{U(t)}{M_p}\right)\delta\psi &= 6H_*M_p^2\,\delta H
\end{aligned} \tag{2.40}$$

where $U(t) = 4\pi b_\star^2 V_0 \exp[-(\lambda\varphi_* + \psi_*)/M_p]$, and we drop the terms corresponding to contributions from the flux, since both of our scaling solutions require their absence.

Recalling the general form for our power-law solutions 2.26, and now substituting the solution for δH as noted above, we arrive at:

$$\begin{aligned}
&\delta\ddot{\varphi} + \left(3\alpha + \frac{p_1^2}{2\alpha}\right)\frac{\delta\dot{\varphi}}{t} + \frac{\lambda U_0}{M_p^2}\left(\lambda - \frac{p_1}{2\alpha}\right)\left(\frac{t}{t_0}\right)^{\zeta}\delta\varphi + \left(\frac{p_1p_2}{2\alpha}\right)\frac{\delta\dot{\psi}}{t} \\
&\qquad + \left[\frac{U_0}{M_p^2}\left(\lambda - \frac{p_1}{2\alpha}\right)\left(\frac{t}{t_0}\right)^{\zeta} + \frac{p_1}{\alpha b_\star^2}\,e^{-2\psi_0/M_p}\left(\frac{t}{t_0}\right)^{-2p_2}\right]\delta\psi = 0 \\
&\delta\ddot{\psi} + \left(3\alpha + \frac{p_2^2}{2\alpha}\right)\frac{\delta\dot{\psi}}{t} + \left[\frac{U_0}{M_p^2}\left(1 - \frac{p_2}{2\alpha}\right)\left(\frac{t}{t_0}\right)^{\zeta} + \frac{(p_2 - 4\alpha)}{\alpha b_\star^2}\,e^{-2\psi_0/M_p}\left(\frac{t}{t_0}\right)^{-2p_2}\right]\delta\psi \\
&\qquad + \left(\frac{p_1p_2}{2\alpha}\right)\frac{\delta\dot{\varphi}}{t} + \frac{\lambda U_0}{M_p^2}\left(1 - \frac{p_2}{2\alpha}\right)\left(\frac{t}{t_0}\right)^{\zeta}\delta\varphi = 0
\end{aligned} \tag{2.41}$$

where we define the quantity $\zeta := -\lambda p_1 - p_2$, for convenience.

2.4.1.1 Attractor

As described in section 2.3.2.1, we now also drop terms in the potential due to the curvature (i.e., the terms in 2.41 containing factors of $1/b_\star$). This is important because the solutions we are perturbing are formally solutions to the equation without the curvature so we need to be consistent.

Substituting the solutions 2.34 and 2.35 into 2.41, we find the equations greatly simplify:

$$\begin{aligned}
&\delta\ddot{\varphi} + \frac{6+\lambda^2}{1+\lambda^2}\frac{\delta\dot{\varphi}}{t} + \frac{\lambda^2(5-\lambda^2)}{(1+\lambda^2)^2}\frac{\delta\varphi}{t^2} + \frac{\lambda}{1+\lambda^2}\frac{\delta\dot{\psi}}{t} + \frac{\lambda(5-\lambda^2)}{(1+\lambda^2)^2}\frac{\delta\psi}{t^2} = 0 \\
&\delta\ddot{\psi} + \frac{7}{1+\lambda^2}\frac{\delta\dot{\psi}}{t} + \frac{5-\lambda^2}{(1+\lambda^2)^2}\frac{\delta\psi}{t^2} + \frac{\lambda}{1+\lambda^2}\frac{\delta\dot{\varphi}}{t} + \frac{\lambda(5-\lambda^2)}{(1+\lambda^2)^2}\frac{\delta\varphi}{t^2} = 0\,.
\end{aligned} \tag{2.42}$$

Being that these are two second-order equations for two functions, we expect four linearly-independent solutions. The equations themselves look very open to power-law solutions of their own, so we employ the *ansätze*

$$\delta\varphi = t^n, \qquad \text{and} \qquad \delta\psi = At^m\,. \tag{2.43}$$

which in turn yield

$$\begin{aligned}
&\left\{(\lambda^2+1)^2n^2 + 5(\lambda^2+1)n + (5-\lambda^2)\lambda^2\right\}t^n \\
&\qquad\qquad + A\lambda\left[(\lambda^2+1)m + (5-\lambda^2)\right]t^m = 0\,, \\
&\lambda\left[(\lambda^2+1)n + (5-\lambda^2)\right]t^n \\
&\qquad\qquad + A\left\{(\lambda^2+1)^2m^2 - (\lambda^2+1)(\lambda^2-6)m + (5-\lambda^2)\right\}t^m = 0\,.
\end{aligned} \tag{2.44}$$

If $n \neq m$, then the equations in square brackets, which are linear in n and m, would both need to vanish. However, those equations are identical, so it cannot be the case that they are both satisfied if $n \neq m$. Therefore, we are left searching for solutions with $n = m$. This leads to two quadratic equations for n:

$$(\lambda^2+1)^2n^2 + (\lambda^2+1)(5+A\lambda)n + \lambda(5-\lambda^2)(A+\lambda) = 0, \text{ and} \tag{2.45}$$

$$A(\lambda^2+1)^2n^2 + (\lambda^2+1)(\lambda - A\big(\lambda^2-6\big))n + (5-\lambda^2)(A+\lambda) = 0 \tag{2.46}$$

These equations need not be consistent, but certainly *could* be for some well-chosen value for A. Indeed, one solution is immediately evident. If $A = -\lambda$, then $n = 0$ solves both equations. Another solution can be similarly obtained by trying $A = 0$. In that case,

(2.46) reduces to:

$$n = \frac{\lambda^2 - 5}{\lambda^2 + 1}. \tag{2.47}$$

It is easy to verify that this also solves (2.45) when $A = 0$, so we have a second solution.

The third and fourth solutions can be found by taking $A = 1/\lambda$. With this choice, both equations reduce to the same form:

$$(\lambda^2 + 1)^2 n^2 + 6(\lambda^2 + 1)n + (5 - \lambda^2)(\lambda^2 + 1) = 0, \tag{2.48}$$

which has solutions

$$n = -1 \qquad \text{and} \qquad n = \frac{\lambda^2 - 5}{\lambda^2 + 1}. \tag{2.49}$$

Hence, we have found the four linearly-independent solutions to this system.

In summary, the linearly-independent solutions to (2.42) are:

$$(\delta\varphi, \delta\psi) = \begin{cases} (1, -\lambda), \\ (t^q, 0), \\ (0, t^q), \\ \left(t^{-1}, \frac{1}{\lambda} t^{-1}\right), \end{cases} \tag{2.50}$$

where $q := \left(\lambda^2 - 5\right)/(\lambda^2 + 1)$. Note that we have used a linear combination of the solution with $A = 0$ and $A = 1/\lambda$ to choose a more elucidating basis for the system.

Recall from (2.28) that for the power-law systems, $\varepsilon = 1/\alpha$, and for the attractor solution, $\alpha = 2/(1 + \lambda^2)$, so we have

$$\varepsilon = \frac{1 + \lambda^2}{2}. \tag{2.51}$$

It is also the case that we define λ to be non-negative, so for this solution to be inflating, $\varepsilon < 1$ implies $\lambda < 1$. Under that condition, q is always less than 0, so all modes decay with time or, at worst, remain constant. For this reason, we consider this solution *marginally stable*, and an attractor. See figure 2.7 for numerical examples demonstrating this behaviour.

Lastly, we perform the same analysis on the slow-roll scaling solution derived in section 2.3.2.2.

2.4.1.2 Slow-Roll

In contrast to the previous section, this time we do not drop any terms in (2.41) as the slow-roll solution formally solves the system including the curvature terms. Substituting in the results (2.36) and (2.37) of section 2.3.2.2, we find

$$\begin{aligned}
&\delta\ddot{\varphi} + \frac{3\lambda^4+8\lambda^2+3}{2\lambda^2(1+\lambda^2)}\frac{\delta\dot{\varphi}}{t} + \frac{3+\lambda^2}{2(1+\lambda^2)}\frac{\delta\varphi}{t^2} + \frac{\lambda}{1+\lambda^2}\frac{\delta\dot{\psi}}{t} + \frac{3+\lambda^2}{2\lambda^3(1+\lambda^2)}\frac{\delta\psi}{t^2} = 0,\\
&\delta\ddot{\psi} + \frac{5\lambda^4+6\lambda^2+3}{2\lambda^2(1+\lambda^2)}\frac{\delta\dot{\psi}}{t} + \frac{(\lambda^2+3)(\lambda^4+\lambda^2-1)}{2\lambda^4(1+\lambda^2)}\frac{\delta\psi}{t^2}\\
&\qquad\qquad + \frac{\lambda}{1+\lambda^2}\frac{\delta\dot{\varphi}}{t} + \frac{3+\lambda^2}{2\lambda^3(1+\lambda^2)}\frac{\delta\varphi}{t^2} = 0
\end{aligned} \tag{2.52}$$

Once again, this is a system of second-order ordinary differential equations for two functions, so we expect to find four linearly-independent solutions.

The system (2.52) also smells of power laws, so we proceed again with the *ansätze* (2.43), which we repeat here for convenience:

$$\delta\varphi = t^n, \qquad \text{and} \qquad \delta\psi = At^m\,. \tag{(2.43)}$$

Substituting (2.43) in (2.52), we find

$$\begin{aligned}
&\lambda\left\{2\lambda(\lambda^2+1)^2n^2 + (\lambda^4+6\lambda^2+3)n + (\lambda^2+3)\lambda^2\right\}t^n\\
&\qquad + A\lambda\left[2\lambda^4 m + (\lambda^2+3)\right]t^m = 0\,,\\
&\lambda\left[2\lambda^4 n + (\lambda^2+3)\right]t^n\\
&\qquad + A\left\{2\lambda^4(\lambda^2+1)m^2 + \lambda^2(3\lambda^4+4\lambda^2+3)m + \lambda^6+4\lambda^4-3\right\}t^m = 0\,.
\end{aligned} \tag{2.53}$$

Again, if $n \neq m$, then the equations in square brackets (i.e., those linear in n and m) would separately need to be satisfied, but since they are identical, they could not both be satisfied for $n \neq m$. Therefore, we must search for solutions where $n = m$. In that case, we get another set of quadratic equations for n.

$$2\lambda^3(\lambda^2+1)n^2 + \lambda(2A\lambda^3+\lambda^4+6\lambda^2+3)n + (\lambda^2+3)(A+\lambda^3) = 0\,, \tag{2.54}$$

$$\begin{aligned}
&2A\lambda^4(\lambda^2+1)n^2 + \lambda^2(A(3\lambda^4+4\lambda^2+3)+2\lambda^3)n\\
&\qquad + (\lambda^2+3)(A(\lambda^4+\lambda^2-1)+\lambda) = 0\,.
\end{aligned} \tag{2.55}$$

These also need not be consistent, but *could* be for appropriate forms for A.

It is a bit more difficult to see the solutions for this system, but with the benefit of hindsight, the process is relatively straightforward. There are two choices for A which will turn (2.54) and (2.55) into the same quadratic equation. First, if $A = \lambda$, both equations

reduce to:

$$2\lambda^2 n^2 + 3\big(\lambda^2+1\big)n + \lambda^2 + 3 = 0. \tag{2.56}$$

This equation has solutions

$$n = -1 \qquad \text{and} \qquad n = -\frac{\lambda^2+3}{2\lambda^2}. \tag{2.57}$$

Finally, if instead we try $A = -1/\lambda$, we find both (2.54) and (2.55) reduce to:

$$2\lambda^4 n^2 + \lambda^2\big(\lambda^2+3\big)n + \big(\lambda^2+3\big)\big(\lambda^2-1\big) = 0, \tag{2.58}$$

with solutions

$$n = \frac{1}{4\lambda^2}\Big\{-\big(\lambda^2+3\big) \pm \sqrt{(\lambda^2+3)(11-7\lambda^2)}\Big\} \tag{2.59}$$

To summarize, this system as a whole has the four linearly independent solutions

$$(\delta\varphi, \delta\psi) = \begin{cases} \big(t^{-1}, \lambda t^{-1}\big), \\ (t^r, \lambda t^r), \\ \big(t^{s_+}, -\frac{1}{\lambda}t^{s_+}\big), \\ \big(t^{s_-}, -\frac{1}{\lambda}t^{s_-}\big), \end{cases} \tag{2.60}$$

where

$$r := -\big(\lambda^2+3\big)/2\lambda^2, \qquad \text{and} \qquad s_\pm := \Big(-\big(\lambda^2+3\big) \pm \sqrt{(\lambda^2+3)(11-7\lambda^2)}\Big)/4\lambda^2. \tag{2.61}$$

Recall from section 2.3.2.2 that

$$\varepsilon = \frac{1}{\alpha} = \frac{2\lambda^2}{1+\lambda^2} \tag{2.62}$$

and λ is real and non-negative. In the inflationary regime, $\varepsilon < 1$, so we must have $0 < \lambda < 1$. With these restrictions, it is clear that r and s_- are always less than 0, hence those modes always decay in time. The s_+ mode, however, is much more interesting.

First, we note that $s_+(\lambda = 0) = \sqrt{33} - 3 \sim 2.74$, and $s_+(\lambda = 1) = 0$. Next, computing

$$\frac{\mathrm{d}s_+}{\mathrm{d}\lambda} = \frac{-28\lambda^3 - 20\lambda}{2\sqrt{(\lambda^2+3)(11-7\lambda^2)}} - 2\lambda, \tag{2.63}$$

and noting that the quantity in the denominator is real on the interval $0 < \lambda < 1$, we see

that s_+ is a strictly decreasing function of λ on the domain of inflationary values. Given the boundary values computed above, s_+ is a strictly positive function on the domain of interest. This means the s_+ mode is the *increasing* mode that makes this power-law solution *unstable*, or *highly* sensitive to its initial conditions. Given enough time, any perturbation that evolves with any component of this mode will grow without bound, and ultimately spoil the inflationary scenario. Numerical examples of this behaviour can be seen in figure 2.8.

Thus we can categorize the sensitivity to initial conditions of two extra-dimensional inflationary solutions. Finally, we take stock and reiterate the key lessons from this chapter.

2.5 Conclusion

There is a very real possibility that our universe has extra dimensions. They are a firm prediction of a leading candidate for a quantum theory of gravity, and they still have a safe parameter-space as yet un-probed by experiment. If indeed there was an early period of inflation in the history of our universe (as chapter 1 hopefully makes a strong case for), it is perfectly likely that it would have involved any extra dimensions. To study this possible connection, we construct a toy model with two extra dimensions, which expands upon the well-studied case of one extra dimension. We include a scalar field to drive inflation, and a Maxwell field in the hopes of stabilizing the extra dimensions. It remains to perform an exhaustive search for solutions to this model, however we begin the process with two classes of solutions.

First, we find a numerical solution demonstrating 4D quasi-de Sitter space, while trapping the extra-dimensional radius. In this example, the radion is trapped almost immediately, and is nestled in its potential minimum when the inflaton takes action, so this it is termed the "Cradle" solution. An exact analytical understanding of this solution thus far eludes the author, however it is clear that its inflationary regime is equivalent to a single-field slow-roll model with a very complicated potential. Future work may well uncover an elegant mathematical description of the time-dependence of this system. Our numerical trials show this scenario typically returns $n_s \sim 0.975$ and $r \sim 0.15$, so it is likely incompatible with observations.

Second, we find a class of solutions that scale as powers of time. We find two examples of this type of solution, both of which can be realized in the numerics. The first example turns up generically in numerical trials, and is compatible with a stabilized extra-dimensional radius, however suffers from an unacceptably large prediction for ε (assuming the usual perturbative analysis applies). The second example has a freely tunable ε, but is difficult to realize numerically, requiring very careful choices of initial conditions.

As inflation is intended to solve a problem of initial conditions, it is important to understand how an inflationary scenario itself depends on its own initial conditions. To that end, we analyze our scaling solutions for sensitivity to their initial conditions (we only study

the scaling solutions since we do not have a thorough understanding of the Cradle solution). It is found that the example that can trap the radion is an attractor, and marginally insensitive to its initial conditions, while the solution with that allows for acceptable values of ε is highly sensitive to its initial conditions.

To summarize, some headway is made into understanding inflationary models with more complicated extra-dimensional manifolds. In our model with two spherical extra dimensions, we are able to find some solutions that can stabilize the extra-dimensional radius, some that can allow for reasonable values of the slow-roll parameters, and some that are insensitive to initial conditions. As yet, the intersection of those three sets of solutions unfortunately remains empty, however that each set is individually populated is important. An extra-dimensional inflationary scenario needs to be able to stabilize the extra dimensions because we do not currently observe infinitely large extra dimensions. It is also obviously important that an extra-dimensional inflationary scenario allow slow-roll parameters that agree with observations. Finally, it is desirable for any inflationary scenario to be insensitive to its initial conditions. While it is true that a model requiring extremely precise initial field values is still an improvement over the standard Big Bang model, it would ultimately be another problem to solve, so it is ideal to avoid it altogether.

More work on the topic clearly needs to be done. This includes a proper understanding of the time-dependence in the Cradle scenario, a more exhaustive search of the model's parameter space, a full perturbations analysis, and studies of different forms for the inflaton potential.

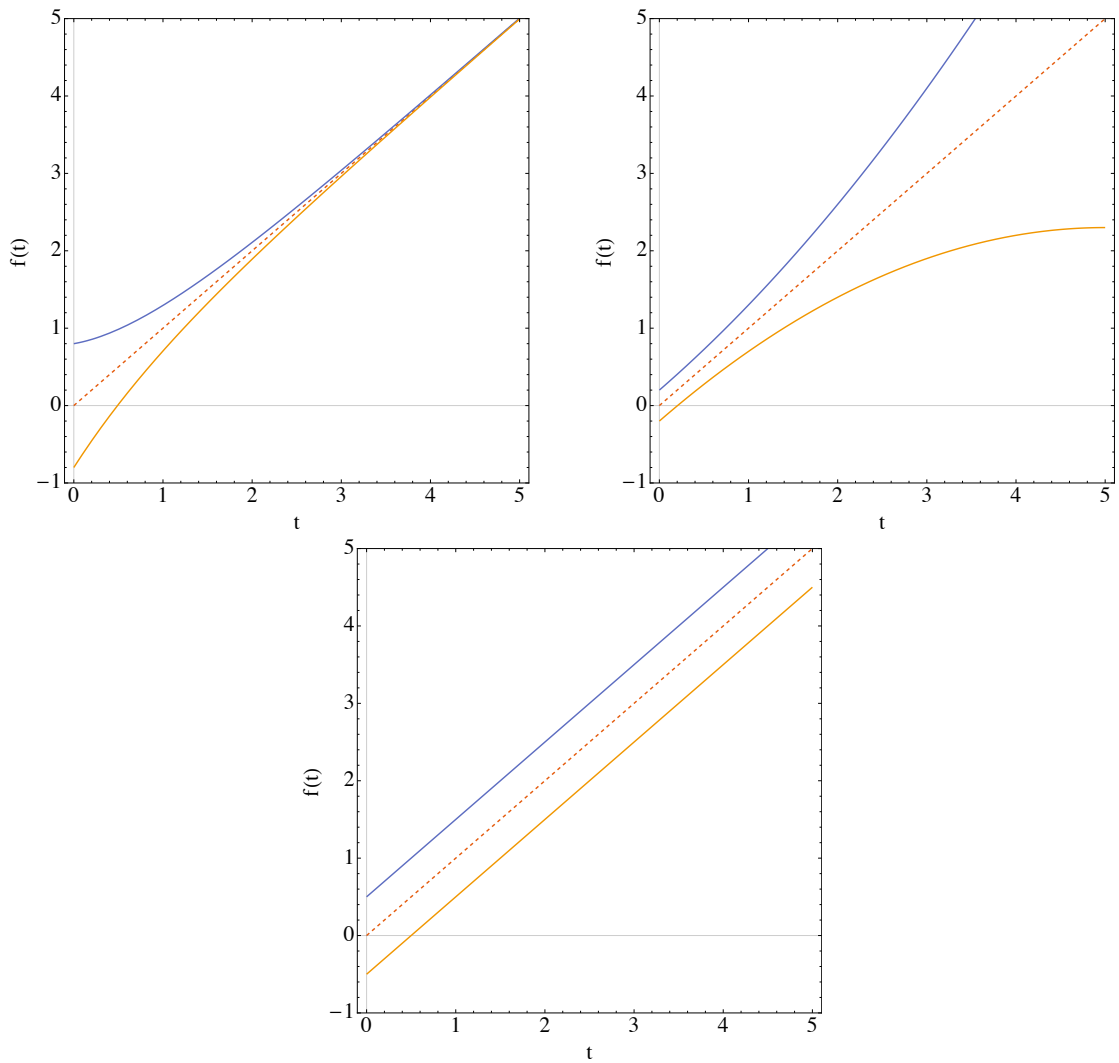

FIGURE 2.6: A schematic example of the classification of a solution's sensitivity to its initial conditions. The dashed line is the background solution, and the solid lines are the full solutions $f = f_* + \delta f$. The plot on the top left shows an example that is *insensitive* to its initial conditions, in that the perturbation δf falls off with time. The plot on the top right shows an example that is *sensitive* to its initial conditions, in that the perturbations δf diverge with time. The plot on the bottom shows an example that is marginally sensitive to its initial conditions. The function δf is a constant in time to lowest order, so more information is needed to determine the late-time behaviour of the solutions.

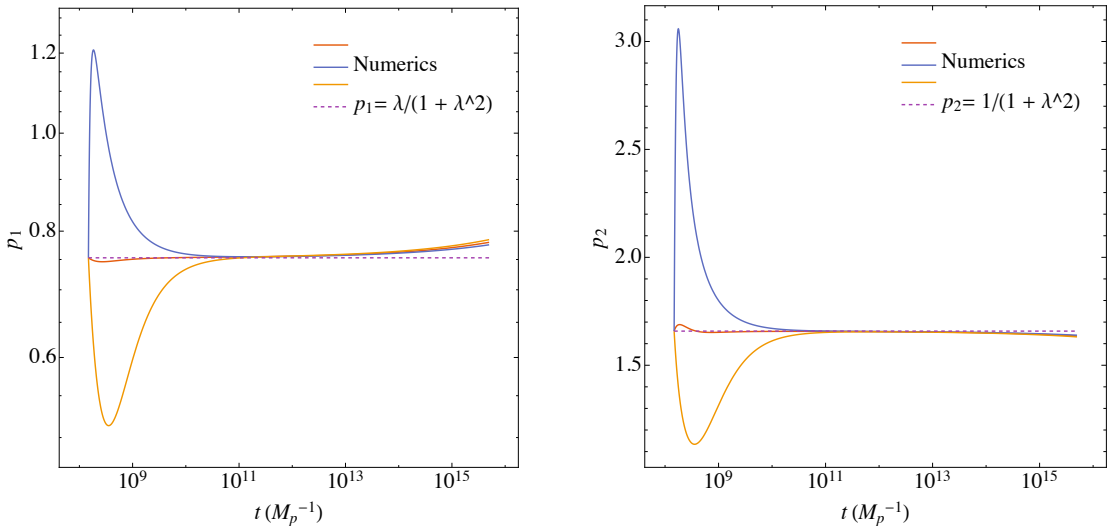

FIGURE 2.7: Plots of $t\dot{\varphi}$ (left) and $t\dot{\psi}$ (right) vs t for several similar initial conditions (solid lines are numerics, dashed lines are the analytic forms of p_1 and p_2 from equations (2.34) and (2.35)). Note that despite an assortment of initial conditions ((φ_0/M_p, ψ_0/M_p) = (-19.6,-31), (-18.6, -30), and (-17.6, -29)), each solution approaches roughly the same behaviour. Besides the initial conditions already noted, all parameters are the same as in figure 2.4

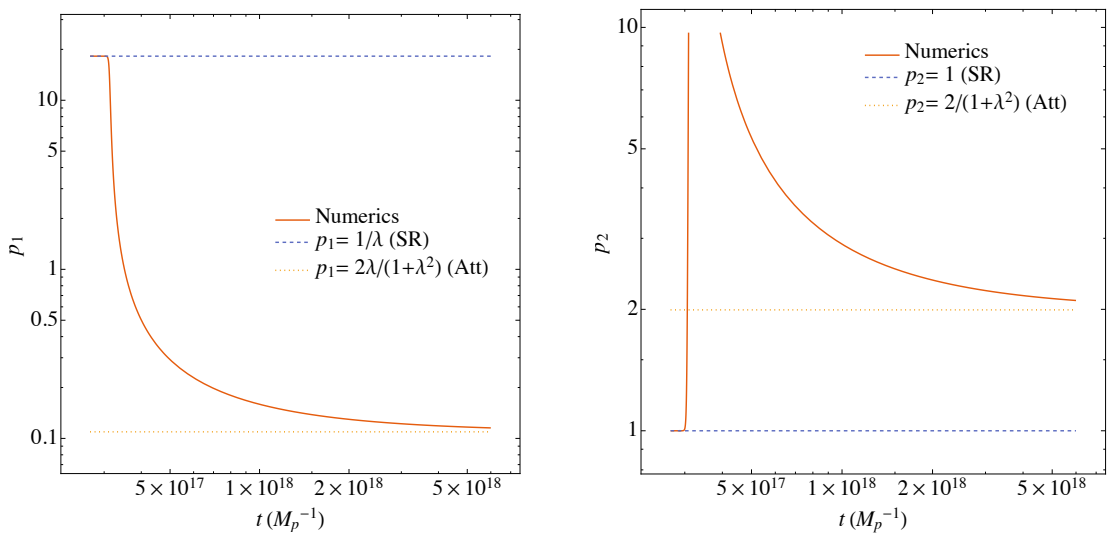

FIGURE 2.8: Plots of $t\dot{\varphi}$ (left) and $t\dot{\psi}$ (right) vs t for the slow-roll power-law solution. Figure 2.5 is a close up of the short initially flat regions on these plots. In contrast to that figure, here we plot the solution for long enough to see it transition into the attractor power-law solution. The parameters for this run are the same as in figure 2.5

Chapter 3

Magnon Inflation

And now for something completely different—slow-roll with a steep potential. As mentioned in chapter 1, slow-roll inflation driven by a single scalar field imposes rather stringent conditions on the slope of the inflaton's potential (equation (1.33)). This result can be generalized to the case of inflation with multiple scalar fields ϕ^a so that in general, slow-roll requires

$$\varepsilon \approx \frac{M_p^2}{2} \frac{\mathcal{G}^{ab} \partial_a \mathcal{V} \, \partial_b \mathcal{V}}{\mathcal{V}^2} \ll 1, \tag{3.1}$$

where $\mathcal{G}^{ab}$ is the inverse of the target-space metric defined by the kinetic terms

$$T = \frac{1}{2} \mathcal{G}_{ab} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b, \tag{3.2}$$

while $\mathcal{V}$ is the potential in terms of all the fields ϕ^a, and ∂_a refers to the partial derivative with respect to the field ϕ^a. Hence typically, slow-roll can only take place when fields are subject to very shallow potentials. Such a strict requirement can be a hindrance to model-builders seeking to incorporate inflation into a bigger picture, such as a UV completion to the Standard Model. As a result, theorists have been seeking means of escaping this bound.

A hint for a general technique for avoiding the bound (3.1) can be found in the Chromo-Natural model of inflation [12]. This model couples an axion to a collection of SU(2) scalars, and seeks to use their interaction as a means of keeping the axion rolling slowly, despite its own potential being relatively steep. The scalars are taken to be at their rotationally-invariant vacuum expectation value described by a field $\psi(t)$ (a choice which inherently breaks Lorentz invariance), and the axion is taken to be homogeneous $\chi = \chi(t)$, depending only on a cosmic time coordinate. The action for this configuration is:

$$S_{CN} = -\int \mathrm{d}^4x a^3 \left[\frac{3}{2} \frac{1}{a^2} \left(\frac{\partial(\psi a)}{\partial t} \right)^2 + \frac{1}{2} \dot{\chi}^2 - 3\tilde{g} \frac{\lambda}{f} \chi \frac{\psi^2}{a} \frac{\partial(\psi a)}{\partial t} + V(\psi, \chi) \right], \tag{3.3}$$

where

$$V(\psi, \chi) := -\frac{3}{2}\tilde{g}^2\psi^4 - \mu^4(1 + \cos(\chi/f)). \tag{3.4}$$

In the above, a is the scale-factor, $\tilde{g}$ is the SU(2) gauge coupling constant, f is the axion decay constant, and λ and μ are coupling constants for the ψ-χ interaction and the axion potential, respectively. This model finds the slow-roll parameter ε:

$$\varepsilon \approx \frac{3\tilde{g}^2\psi^4}{\mu^4(1 + \cos(\chi/f))} + \psi^2, \tag{3.5}$$

without a Planck mass in sight.

Equation (3.5) is an interesting result, and one might be led to wonder exactly what was so special about the action (3.3). One possibility is that since slow-roll is defined by the condition that the field velocities are small, it is terms linear in field derivatives, like the interaction $\sim \chi\psi^2\partial_t(\psi a)$, that actually dominate the dynamics. Indeed, such terms are not usually included when constructing an inflationary theory. Therefore in this chapter, we construct a toy model designed to study exactly those often-omitted interactions, and their general effects on multi-field inflation.

In our toy model, we take the general approach of studying a system of n scalar fields coupled to a fixed time-like vector-field U^μ. We construct a Lagrangian to first order in the derivatives of the fields, and find that indeed the condition (3.1) on ε is relaxed in general, with some field configurations even permitting it to vanish despite the presence of an arbitrarily steep potential. A two-field model that demonstrates this vanishing ε is also explicitly constructed. Following the theme of this thesis, we then proceed to study the sensitivity of this two-field model to its initial conditions, and find conditions on the parameters of the theory that allow it to be relatively insensitive to its initial conditions. Finally, we include corrections due to all possible second-order terms in the Lagrangian, and find conditions on those parameters that can make or break the sensitivity of the first-order solutions initial conditions.

3.1 Action and Field Equations

In order to include first-order derivatives in a general way, we construct our model with a system of scalar fields coupled to gravity and a vector-field U^μ whose value is fixed, presumably by some ultra-violet completion[1] (that is, we are describing a low-energy effective theory, and assume that the full theory that describes higher energies has an explanation for the fixed value of this vector-field). The action to first order in derivatives of the scalar

[1] In this respect, our model resembles the Einstein-Aether theories [22].

and U^μ fields is:

$$S_{\mathcal{M}} = -\int \mathrm{d}^4x\sqrt{-g}\,\left[\mathcal{V}(\phi) + \mathcal{A}_a(\phi)\,U^\mu\partial_\mu\phi^a + \xi\,(g_{\mu\nu}U^\mu U^\nu + 1)\right]. \tag{3.6}$$

where the last term is used to fix the value of the field U^μ to point along a time-like direction. Since this action is similar to that of spin-waves in a ferromagnet [10, 11], we label this model "Magnon" inflation.

Given the action (3.6), the equations of motion can be computed directly:

- *ξ-EOM*:

$$g_{\mu\nu}U^\mu U^\nu + 1 = 0\,, \tag{3.7}$$

- *U^μ-EOM*:

$$\mathcal{A}_a\,\partial_\mu\phi^a + 2\xi\,U_\mu = 0\,, \tag{3.8}$$

- *ϕ^a-EOM*:

$$-\partial_a\mathcal{V} - \mathcal{F}_{ab}\,U^\mu\partial_\mu\phi^b + \mathcal{A}_a\,\nabla\cdot U = 0\,, \tag{3.9}$$

where we have defined $\mathcal{F}_{ab} := \partial_a\mathcal{A}_b - \partial_b\mathcal{A}_a$. A convenient expression for ξ can be obtained by contracting the U^μ equation of motion (3.8) with U^μ and making use of the first equation:

$$2\xi = \mathcal{A}_a U^\mu\partial_\mu\phi^a. \tag{3.10}$$

If we look for homogeneous and isotropic solutions for the scalars (which we do), the U^μ equation of motion implies $U^0 = 1$ while $U^i = 0$, whenever $\xi \neq 0$. In looking for homogeneous and isotropic solutions, we assume the FRW form for the metric. That combined with our expression for U^μ means we have

$$\nabla_\mu U^\nu = \begin{cases} H, & \text{if } \mu,\nu \in \{1,2,3\}, \\ 0, & \text{else.} \end{cases} \tag{3.11}$$

so that $\nabla\cdot U = 3H$. This can be neatly written using (3.7) as

$$\nabla_\mu U^\nu = H(\delta^\mu_\nu + U_\mu U^\nu), \tag{3.12}$$

In the event that $\mathcal{F}$ is invertible, the ϕ equation of motion also yields the handy formula

$$\dot{\phi}^a := U^\mu\partial_\mu\phi^a = \tilde{\mathcal{F}}^{ab}[3H\mathcal{A}_b - \partial_b\mathcal{V}], \tag{3.13}$$

where $\tilde{\mathcal{F}}^{ab}$ is the inverse of $\mathcal{F}_{ab}$, and we have made the notational choice to represent the combination $U^\mu \nabla_\mu$ with an over-dot.

Finally, we also have the Einstein Equations. To use them, we need the Stress-Energy tensor:

$$\begin{aligned} T_{\mu\nu} &= g_{\mu\nu} L_{\mathcal{M}} - 2\frac{\delta L_{\mathcal{M}}}{\delta g^{\mu\nu}}, \\ &= -g_{\mu\nu}\left[\mathcal{V} + \mathcal{A}_a\,\dot{\phi}^a\right] - 2\xi U_\mu U_\nu, \\ &= -g_{\mu\nu}\mathcal{V} - \mathcal{A}_a\dot{\phi}^a[g_{\mu\nu} + U_\mu U_\nu], \end{aligned} \tag{3.14}$$

where $L_{\mathcal{M}}$ is the negative of the Lagrangian scalar in (3.6), and we have made use of the equations of motion to eliminate ξ and set the norm of U^μ to -1.

Immediately, we can see that our choice of a time-like U^μ and FRW metric make the stress-energy diagonal, and in order to satisfy homogeneity and isotropy, it must have the form of a perfect fluid, as in chapter 1:

$$T^\mu{}_\nu = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}, \tag{3.15}$$

The components can be calculated directly as

$$\rho = U^\mu U^\nu T_{\mu\nu} = \mathcal{V}, \qquad \text{and} \qquad p = N^\mu N^\nu T_{\mu\nu} = -\mathcal{V} - \mathcal{A}_a\dot{\phi}^a, \tag{3.16}$$

for some space-like unit vector N^μ (which is therefore orthogonal to U^μ). From here we can see that all that is needed to produce a de Sitter equation of state $p = -\rho$ is

$$\mathcal{A}_a\dot{\phi}^a = 0. \tag{3.17}$$

Already, this is a great success. In (3.17), we find that in our simple toy model, we can achieve an exact vacuum equation of state (so that $\varepsilon = 0$ identically) simply due to the *alignment* of $\dot{\phi}^a$ in field-space. Note that this condition makes no demands on the size of $\mathcal{V}$, so in a sense, we can already see that this model achieves the results it was designed for. That said, we can still find a more useful way to phrase this result.

Contracting (3.13) with $\mathcal{A}_a$, we can see one way to achieve the condition (3.17) would be if $\mathcal{A}_a$ were parallel to the gradient of $\mathcal{V}$, so that $\epsilon^{ab}\mathcal{A}_a\partial_b\mathcal{V} = 0$. More generally, with the stress-energy tensor above, we can compute the Friedmann equation

$$3M_p^2 H^2 = \mathcal{V} \tag{3.18}$$

so that explicitly, the slow-roll parameter ε is

$$\varepsilon := -\frac{\dot{H}}{H^2} = -\frac{\partial_a \mathcal{V} \dot{\phi}^a}{6 M_p^2 H^3} = \frac{3}{2} \frac{\tilde{\mathcal{F}}^{ab} \mathcal{A}_a \partial_b \mathcal{V}}{\mathcal{V}}. \tag{3.19}$$

This expression (3.19) should be compared with the classic multi-field result (3.1). Not only does our new expression have no Planck mass in sight, but it can be set to 0 solely with an appropriate alignment of the vectors $\mathcal{A}$ and $\partial\mathcal{V}$. In general then, this model clearly does loosen the restrictions on slow-roll with a steep potential, as desired.

Finally, it is important to note that this explains the results from the Chromo-Natural model above. Dropping the second-derivative terms in the action (3.3), and applying framework just described, we find that for the Chromo-Natural model,

$$\varepsilon \approx \frac{3\tilde{g}^2 \psi^4}{\mu^4 (1 + \cos(\chi/f))}, \tag{3.20}$$

so that to leading order, it is indeed the contributions of the first-derivative terms that control the size of ε.

Now that we understand some general properties of this model, it is worth constructing an explicit example of a system that can realize the condition (3.17), and in particular, studying the sensitivity to initial conditions of solutions to that system. For instance if it were the case that all solutions in this model required highly specific initial conditions to inflate, then we might have aided model-builders slightly, but ultimately have replaced one initial conditions problem with another.

3.2 Two-Field Solution

A non-trivial solution to (3.17) requires a minimum of two fields, so we construct our solution from exactly two fields. Let $\phi^1 = \psi$, and $\phi^2 = \chi$. We then make the following choices for $\mathcal{V}$ and $\mathcal{A}_a$:

$$\mathcal{V} = V(\psi), \qquad \text{and} \qquad \mathcal{A} = \mathcal{A}_1 \, \mathrm{d}\phi^1 = A(\chi) \, \mathrm{d}\psi. \tag{3.21}$$

which implies

$$\mathcal{F}_{ab} = f \epsilon_{ab}, \tag{3.22}$$

where $f := -A(\chi)$. With this choice, the gradient of the potential $\partial_a \mathcal{V}$ is parallel to the gauge potential $\mathcal{A}_a$, so we have

$$\tilde{\mathcal{F}}^{ab}\mathcal{A}_a\partial_b\mathcal{V} = -\frac{1}{f}\tilde{\epsilon}^{ab}\mathcal{A}_a\partial_b\mathcal{V} = 0, \tag{3.23}$$

and as a result, the slow-roll ε vanishes to first-order, for an *arbitrarily* steep potential $V(\psi)$. Finally, we may use the equations of motion (3.13) to determine:

$$\begin{aligned} \dot{\psi} &= 0, \text{ and} \\ \dot{\chi} &= \frac{V'(\psi) - 3HA(\chi)}{A'(\chi)}, \end{aligned} \tag{3.24}$$

as long as $A' \neq 0$.

The first equation in (3.24) clearly states that ψ is a constant in time. Using the Friedmann equation, $3M_p^2H^2 = \mathcal{V}$, this implies the Hubble parameter is also a constant in time. With those helpful facts, we can solve the second equation in (3.24) exactly for $A(t)$:

$$\begin{aligned} \dot{A} &= -3H\,A + V'(\psi), \\ \implies A(t) &= \left[A_0 - \frac{V'(\psi)}{3H}\right] e^{-3H(t-t_0)} + \frac{V'(\psi)}{3H}. \end{aligned} \tag{3.25}$$

where $A_0 = A(t = t_0)$. Note that in the above, V' and H are evaluated at the constant value of ψ, so that A does not depend on ψ.

From here, it is impossible to find a unique expression for χ, however its late-time behaviour can be inferred in the special case where $\mathcal{A}'$ does not approach 0. At late times, the exponential in A dies out, and $V' - 3HA \to 0$. As a result, χ approaches a constant value in the far future.

Now we have an example of a solution to our model that explicitly demonstrates its trademark feature—slow-roll with an arbitrarily steep potential. It just remains to explore the sensitivity of this solution to its initial conditions.

3.2.1 Sensitivity to Initial Conditions

In the same manner as section 2.4, we test the sensitivity of our two-field slow-roll solution to its initial conditions by applying some small perturbations to the fields in our problem, and studying their dynamics with the equations of motion. Hence we take

$$\phi^a \to \phi^a_* + \delta\phi^a \qquad \text{and} \qquad H \to H_* + \delta H, \tag{3.26}$$

where ϕ^a_* and H_* satisfy the first-order, unperturbed equations of motion. We could also perturb the fields U^μ and ξ, but doing so would have no effect on the fields of interest (the

ξ equation of motion along with (3.10) mean that expressions like $\nabla \cdot U = 3H$ still hold true).

The linearized equation of motion for ϕ^a is

$$\mathcal{F}_{ab,c}\,\dot{\phi}^b_* \delta\phi^c + \mathcal{F}_{ab}\,\delta\dot{\phi}^b + \mathcal{V}_{ac}\delta\phi^c - 3[H_*\mathcal{A}_{a,c}\delta\phi^c + \delta H \mathcal{A}_a] = 0. \tag{3.27}$$

where all appearances of $\mathcal{V}$, $\mathcal{F}$, and $\mathcal{A}$ are evaluated at the background solution. The Friedmann equation yields

$$\delta H = \frac{1}{6M_p^2 H_*} V_{,a}\delta\phi^a, \tag{3.28}$$

so that

$$\mathcal{F}_{ab}\,\delta\dot{\phi}^b = -\mathcal{M}_{ab}\,\delta\phi^b. \tag{3.29}$$

where

$$\mathcal{M}_{ab} := \mathcal{F}_{ac,b}\,\dot{\phi}^c_* + \mathcal{V}_{ab} - 3\left[H_*\,\mathcal{A}_{a,b} + \frac{1}{6M_p^2 H_*}\mathcal{A}_a\mathcal{V}_{,b}\right] \tag{3.30}$$

The expression (3.30) can be simplified greatly by making use of the background equation of motion for ϕ^a. Taking the derivative of the ϕ equation of motion (3.9) with respect to ϕ^b, we have

$$\mathcal{V}_{,ab} + \left(\mathcal{F}_{ac}\,\dot{\phi}^c_*\right)_{,a} - 3(H_*\mathcal{A}_a)_{,b} = 0. \tag{3.31}$$

The Friedmann equation also tells us that $H_{*,b} = \mathcal{V}_{,b}/6M_p^2 H_*$, so that altogether, $\mathcal{M}_{ab}$ takes on the clean form

$$\mathcal{M}_{ab} = -\mathcal{F}_{ac}\left(\dot{\phi}^c_*\right)_{,b}. \tag{3.32}$$

Next, for invertible $\mathcal{F}$ (as it is in our two-field example), we can simplify this further:

$$\delta\dot{\phi}^a = -\widetilde{\mathcal{M}}^a{}_b\delta\phi^b, \tag{3.33}$$

where we have defined

$$\widetilde{\mathcal{M}}^a{}_b := \tilde{\mathcal{F}}^{ac}\mathcal{M}_{cb} = \left[\tilde{\mathcal{F}}^{ac}(\mathcal{V}_{,c} - 3H_*\mathcal{A}_c)\right]_{,b}, \tag{3.34}$$

(using (3.13) to eliminate $\dot{\phi}^c_*$). In general, the solution to this system is

$$\delta\phi^a = \left[\mathcal{T}\exp\left(-\int_{t_0}^{t} \mathrm{d}\, t' \widetilde{\mathcal{M}}\right)\right]^a_{\ b} \delta\phi^b_0 \tag{3.35}$$

where $\mathcal{T}$ is the time-ordering operator, and $\delta\phi^b_0 := \delta\phi^b(t = t_0)$.

The late-time behaviour of the perturbations is therefore dictated by the sign of the eigenvalues of $\widetilde{\mathcal{M}}$. If they are all positive (negative), the solution is sensitive (insensitive) to its initial conditions. Vanishing eigenvalues corresponds to modes that are marginally stable at the first-order level, and whose stability is ultimately determined by the higher order corrections.

In our two-field example, we can explicitly evaluate $\widetilde{\mathcal{M}}$.

$$\widetilde{\mathcal{M}} = \begin{bmatrix} 0 & 0 \\ (3H' - V'')/A & \{(3HA - V')/A\}_{,\chi} \end{bmatrix}, \tag{3.36}$$

which has eigenvalues 0 and $\{(3HA - V')/A\}_{,\chi}$. Therefore, this solution has one mode which can be stable depending on the signs of A, H, and V, and one mode whose stability needs to be determined by other means.

It seems then that we cannot say one way or the other about the sensitivity to initial conditions of our explicit solution demonstrating slow-roll with an arbitrarily steep potential. It is important, however, to place the action (3.6) in a broader picture. Any model that employs these results, like Chromo-Natural inflation, is more than likely to also include higher derivative terms, such as the standard second-order kinetic terms. Therefore, we should take into account corrections to the equations of motion (3.7) to (3.9) due to all possible terms second-order in the derivatives of our fields. If we linearize our original two-field solution about these corrected equations of motion, we could well find something decisive to say about the marginally stable mode.

3.2.2 Second-Order Corrections

There are a great many possible terms second-order in the derivatives of our fields. As a result, the algebra becomes highly detailed very quickly, without being very illuminating, so it is almost entirely relegated to appendix D. The key points are as follows. The possible contributions to second order can be summarized with the following addition to the

Lagrangian:

$$
\begin{aligned}
-\Delta L := \frac{1}{2}\Big[\mathcal{G}_{ab}(\phi)\, g^{\mu\nu} + \mathcal{I}_{ab}(\phi)\, U^\mu U^\nu\Big]\, \partial_\mu\phi^a \partial_\nu\phi^b + \frac{1}{2}\Big[\mathcal{C}^{(1)}(\phi)\, \nabla^\mu U^\nu \nabla_\mu U_\nu \\
+ \mathcal{C}^{(2)}(\phi)(\nabla\cdot U)^2 + \mathcal{C}^{(3)}(\phi)\, \nabla_\nu U^\mu \nabla_\mu U^\nu + \mathcal{C}^{(4)}(\phi)\, U^\lambda \nabla_\lambda U^\mu U^\nu \nabla_\nu U_\mu\Big] \\
+ \mathcal{C}^{(5)}_a(\phi)\, U^\nu\, (\nabla_\nu U^\mu)\, \partial_\mu\phi^a + \mathcal{C}^{(6)}_a(\phi)\, U^\mu\, (\nabla\cdot U)\, \partial_\mu\phi^a. \qquad ((\text{D.1}))
\end{aligned}
$$

To second order, the aligned solution for U^μ as used above is still a solution. Lastly, the equation of motion for ϕ^a receives an additional term on the LHS:

$$
\Delta^{(\phi)} = \frac{\partial(\Delta L)}{\partial\phi^a} + \nabla_\mu\left\{\mathcal{G}_{ab}\nabla^\mu\phi^b + \mathcal{I}_{ab}\, U^\mu\dot{\phi}^b + \mathcal{C}^{(5)}_a\, \dot{U}^\mu + \mathcal{C}^{(6)}_a\, U^\mu \nabla\cdot U\right\}, \tag{3.37}
$$

and the Friedmann equation receives a modification of

$$
3M_p^2 H^2 = \mathcal{V} + \Delta\rho, \tag{3.38}
$$

where

$$
\Delta\rho := \frac{1}{2}\mathcal{Q}_{ab}\dot{\phi}^a\dot{\phi}^b - 3H\mathcal{C}^{(6)}_a\dot{\phi}^a - \frac{9}{2}H^2\Big(\mathcal{C}^{(1)} + 3\mathcal{C}^{(2)} + \mathcal{C}^{(3)}\Big) - 3\mathcal{C}^{(1)}\dot{H}, \qquad ((\text{D.16}))
$$

and $\mathcal{Q}_{ab} := \mathcal{G}_{ab} - \mathcal{I}_{ab}$ is the target-space metric.

In motivating this model, we used that derivatives in time should be costly during slow-roll, so the dynamics should be dominated by terms with fewer derivatives in time. In that spirit, we treat derivatives in time as small quantities when linearizing these equations, so the corrections (3.37) and (D.16), which are entirely second-order in time, can simply be evaluated at the background. This means that the linearized equations of motion including second-order contributions are simply

$$
\mathcal{F}_{ab}\, \delta\dot{\phi}^b = -\mathcal{M}_{ab}\, \delta\phi^b + \mathcal{J}_a, \tag{3.39}
$$

where

$$
\begin{aligned}
\mathcal{J}_a := -\mathcal{Q}_{ab}\left(\ddot{\phi}^b_* + 3H_*\, \dot{\phi}^b_* + \Gamma^b_{cd}\, \dot{\phi}^c_*\dot{\phi}^d_*\right) + 3H_*(\mathcal{C}^{(6)}_{a,b} - \mathcal{C}^{(6)}_{b,a})\dot{\phi}^b_* \\
- \frac{3H_*^2}{2}\left[(\mathcal{C}^{(1)} + 3\mathcal{C}^{(2)} + \mathcal{C}^{(3)})_{,a} - 6\,\mathcal{C}^{(6)}_a\right],
\end{aligned}
$$

and Γ^b_{cd} are the Christoffel symbols built from the metric, $\mathcal{Q}_{ab} := \mathcal{G}_{ab} - \mathcal{I}_{ab}$. That is:

$$
\Gamma^b_{cd} := \frac{1}{2}\mathcal{Q}^{be}\{\mathcal{Q}_{ec,d} + \mathcal{Q}_{ed,c} - \mathcal{Q}_{cd,e}\}. \tag{3.40}
$$

Again, in our two-field scenario, $\mathcal{F}$ is invertible, so we write:

$$\delta\dot{\phi}^a = -\widetilde{\mathcal{M}}^a{}_b\,\delta\phi^b + \tilde{\mathcal{J}}^a, \tag{3.41}$$

with

$$\tilde{\mathcal{J}}^a := \tilde{\mathcal{F}}^{ab}\mathcal{J}_b. \tag{3.42}$$

This is just the inhomogeneous version of equation (3.33), so the solution is a sum of the homogeneous solution with a particular solution,

$$\delta\phi^a = \left[\mathcal{T}\exp\left(-\int_{t_0}^{t}\mathrm{d}\,t'\widetilde{\mathcal{M}}\right)\right]^a{}_b\,\delta\phi_0^b + \left(\widetilde{\mathcal{M}}^{-1}\right)^a{}_b\tilde{\mathcal{J}}^a. \tag{3.43}$$

It is then the behaviour of this new term $\widetilde{\mathcal{M}}^{-1}\mathcal{J}$ that controls the late-time behaviour of the perturbations, and ultimately determines the sensitivity of our solution to its initial conditions. Of particular note, this term now assigns dynamics even to the zero-modes of $\widetilde{\mathcal{M}}$. In our two-field model, we saw that one of the perturbations' solutions could be made stable with an appropriate choice of A, H, and V, but not much could be said for the other mode. Now, the late-time behaviour of *both* modes is also affected by the extra $\mathcal{J}$ term, so a judicious choice of A, H, and V, as well as $\mathcal{Q}$ and the various $\mathcal{C}^{(i)}$ can ensure insensitivity to initial conditions. Therefore, we have shown that there exists parameter space not only for which the slow-roll parameter ε vanishes to first order, but also for which the solution is insensitive to its initial conditions, and could therefore be a conclusive solution to the Big Bang model's initial conditions problem.

3.3 Conclusion

It is typically the case that slow-roll inflation demands stringent constraints on the shape of the inflating scalar fields' potential (i.e., (3.1)). However some models, such as the Chromo-Natural inflation governed by the action (3.3), succeed in producing slow-roll despite having a surprisingly steep potential for the field driving their inflation. Based on the suspicion that their success is due to the inclusion of an interaction involving a first-order derivative, we construct a model explicitly governed by interactions first-order in derivatives of our fields. We find that such a model does indeed generically predict much weaker constraints on the form of the scalar fields' potential, and can in fact arrange for $\varepsilon = 0$ identically for arbitrarily steep potentials. We are also able to finger the single-derivative interaction as the culprit in the Chromo-Natural model's success by reproducing the leading order contribution to their expression for ε using our framework.

Finally, we explicitly construct a solution to our model that aligns the fields and potential such that ε vanishes to first order with no restriction on the size of the gradient of the potential. We find that such a model is marginally sensitive to its initial conditions if the Lagrangian only contains terms to first order in the derivatives of the fields. However, if the model has higher order terms, their contributions can in fact be used to nudge the solution in the direction of being safely insensitive to its initial conditions. Therefore, we successfully demonstrate a framework wherein the steepness of the potential driving inflation can be detached from the conditions to achieve slow-roll. Furthermore, we demonstrate that there exists at least one such solution in that framework, and show that it is possible to construct such a solution in a way such that it is insensitive to its initial conditions, and can therefore safely solve the Big Bang's initial conditions problems.

Conclusion

The idea of inflation is to solve the problem of highly specific initial conditions in the Big Bang model. Inflation itself is simply the idea of a temporary epoch of quasi-de Sitter space, of which there are any number of possible realizations. It is important, therefore, that any such implementation of inflation not itself be beholden to highly specific initial conditions, otherwise it would only be transferring the problem. In this thesis, we therefore study the sensitivity to initial conditions of solutions to two novel inflationary realizations.

First, we study a model of extra-dimensional inflation. This is motivated both by the fact that a leading theory of quantum gravity, String Theory, inevitably predicts the existence of extra dimensions, and by the fact that small enough extra dimensions are still experimentally viable. We construct our toy model with two dimensions, since the case of a single extra dimension has been well explored. We compactify our extra dimensions on a sphere, and include a Maxwell field over them in an effort to be able to stabilize them after inflation (which is necessary, but typically difficult to accomplish). We then drive inflation with a single scalar inflaton. Although we do not fully explore its parameter space, we find two classes of inflationary solutions to this model.

One class of solutions we refer to as the Cradle solution. It is achieved for a certain range of parameters when the extra-dimensional radius (the radion) is initially very close to its minimum. The radion then quickly minimizes, and the inflaton undergoes standard slow-roll inflation. While this class of solutions is conceptually straightforward, it is difficult to model mathematically, and we do not have an analytic description yet, so we are unable to study its sensitivity to initial conditions in any detail. Nevertheless, it is a class of solutions worth exploring in future work, since it does manifestly stabilize the extra dimensions. Moreover, it typically predicts values for the slow-roll parameters $\varepsilon \sim 0.009$ and $\eta \sim 0.016$ which correspond to $n_s \sim 0.975$ and $r \sim 0.15$, which are just in conflict with recent data. Future work should therefore include a comprehensive study of the parameter space for this class of solutions, as it may well turn out that there exist parameters that make acceptable predictions for these values.

The second class of solutions are of a form that scales as a power of time. In general, these solutions predict $\eta = 0$ identically, as well as an intuitive relation $\mathcal{N}_e \propto 2\ln(b_f/b_0)$, so that the duration of inflation is directly tied to the change in size of the extra dimensional radius $b(t)$ (which, for instance, means if b ranges from the Planck scale $\sim 10^{19}$ GeV to the Electroweak scale $\sim 10^2$GeV, say, then $\mathcal{N}_e \sim 70$ if the constant of proportionality is $\mathcal{O}(1)$).

In this class of solutions, we find two examples of inflation. The first is referred to as the Attractor solution, and has the advantages that it can be used to stabilize the extra dimensions, and is shown to be marginally insensitive to its initial conditions. It does, however, have the drawback that it predicts $\varepsilon \geq 0.5$, which implies $r \geq 8$, so is excluded by data as long as the usual perturbative analysis is applicable. The second inflationary example is referred to as the Slow-Roll solution because it has the advantage that it can predict a generic value for ε. This example suffers from the disadvantages that it is incapable of stabilizing the extra dimensions, and it is shown to be highly sensitive to its initial conditions.

Secondly, we study a toy model constructed explicitly with interactions first-order in derivatives of the fields. During slow-roll, it is expected that higher orders of derivatives come with a penalty, so it should be the lowest orders that dominate the dynamics. It is indeed found that this model greatly loosens the usual restrictions on the steepness of the inflaton fields potential, no longer requiring it be extremely shallow to allow slow-roll to take place. Moreover, it is found that an appropriate field configuration alone can yield $\varepsilon = 0$ while saying nothing of the magnitude of the slope of the potential. That is to say, this model successfully demonstrates an example of how to achieve slow-roll with an *arbitrarily* steep potential. We furthermore construct an explicit example of a solution that realizes this condition, and we show that there exists a region of parameter-space for which it is marginally insensitive to its initial conditions. Finally, we include corrections due to higher-order terms in the Lagrangian, and show that they can be used to find a set of parameters for which the example solution is highly insensitive to its initial conditions.

In the end, we are able to understand analytically three new inflationary solutions, and can show that one of them is generically highly sensitive to its initial conditions, one is marginally insensitive, and one can be made highly insensitive. While these are simply toy models, they may well describe more general features, for instance of some potential Stringy inflation in the one model, and of examples like Chromo-Natural inflation in the other. This thesis therefore makes some progress in understanding how successful some inflationary realizations could be at addressing the original motivation for the framework. Future work should involve properly understanding the cradle solution to the extra-dimensional model, as well as a full understanding of the perturbations in both toy models.

Appendix A

Horizon Problem Details

Mathematically, we arrived at the Horizon Problem by comparing the integral of H^{-1} to the left and right of a fixed point $z = z_{\text{CMB}}$, and noting that the integral to the left (from $z = 0$ to $z = z_{\text{CMB}}$) easily trumped the integral to the right (from $z = z_{\text{CMB}}$ to $z \to \infty$). Any reasonable solution to this problem therefore must modify the form of H^{-1} either for small or large z (one might propose a modification in both ranges, but that would imply that both integrals were wrong, which violates the firm physical principle that two wrongs make a right). However, it is also important that any solution maintain the successes of the current Big Bang model, which tracks the evolution of the universe very well through temperature ranges we mostly understand in the lab. This includes events such as the formation of neutral hydrogen (recombination, $z_{\text{CMB}} \sim 1100$), the radiation-matter crossover ($z_{\text{eq}} \sim 3000$, the value of which affects how much large structure we should see), and the formation of nuclei (big bang nucleosynthesis, $z \sim \mathcal{O}(10^{10})$). Clearly then, we should only consider augmenting H^{-1} for very large z. Figure 1.2 show schematically how some modification to H^{-1} at very large z can solve the problem as stated above.

At this point, there are two possible mathematical solutions. First, H^{-1} could maintain the same scaling with z, but suffer a wild discontinuity. It is, however, extremely difficult, if not impossible, to imagine the physical consequences of a discontinuous Hubble parameter, and we make no attempt to do so here. The second option is that there exists at least one transition to an epoch where $H^{-1} > H^{-1}_{\text{mat}} := H_0^{-1}(1+z)^{-3/2}$. In order for this to be a continuous transition, there must be some point z_* such that $H^{-1}(z_*) = H^{-1}_{\text{mat}}(z_*)$, after which the ratio $H^{-1}(z)/H^{-1}_{\text{mat}}(z)$ must increase. That is, a successful, continuous augmentation of the Hubble parameter must have some segment for which

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{H^{-1}}{H^{-1}_{\text{mat}}}\right) > 0, \\ \implies &\frac{\left(H^{-1}\right)'}{H^{-1}} > \frac{\left(H^{-1}_{\text{mat}}\right)'}{H^{-1}_{\text{mat}}} = -\frac{3}{2(1+z)}, \end{aligned} \tag{A.1}$$

where primes denote differentiation with respect to z.

We can actually do better than the strict inequality in (A.1). Consider the following form for H^{-1}:

$$H_{\text{test}}^{-1} := \frac{A}{(1+z)^{1-p}}, \qquad \text{for} \quad z_* \leq z < \infty, \tag{A.2}$$

where the constant of proportionality is fixed by continuity (so $A = (1+z_*)^{1-p}/H_*$). This test form contributes an insufficient finite amount to Δr_{caus} if $p < 0$. However, if $p \geq 0$, then the integral over H_{test}^{-1} diverges, so can easily be made large enough to solve the Horizon Problem. With this, we can now put a lower bound on the inequality (A.1):

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{H^{-1}}{H_{\text{test}}^{-1}}\right) \geq 0, \\ \implies &\frac{\left(H^{-1}\right)'}{H^{-1}} \geq \frac{\left(H_{\text{test}}^{-1}\right)'}{H_{\text{test}}^{-1}} \geq -\frac{1}{(1+z)}, \end{aligned} \tag{A.3}$$

So the robust, reasonable solution to the Horizon Problem can be summarized as the requirement that there exists some period of time in the distant past where H^{-1} satisfied the condition (A.3) for sufficiently long. Using (1.9) to exchange redshift for z in that expression leads immediately to the more common phrasing of this condition:

$$\varepsilon := -\frac{\dot{H}}{H^2} \leq 1. \tag{A.4}$$

where we have defined the inflationary parameter ε. Recalling that $H = \dot{a}/a$, it is easy to see that this inequality is only satisfied if $\ddot{a} > 0$.

Appendix B

HDI Einstein Equations

Here we explicitly compute the Einstein Equations for the system in chapter 2. Recall, the action is

$$S = -\int \mathrm{d}^6x \sqrt{-g_{(6)}}\left(\frac{1}{2\kappa^2}\mathcal{R} + \frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}\partial_M\phi\,\partial^M\phi + V(\phi)\right), \tag{2.1}$$

with $F = F_{mn}\mathrm{d}y^m\mathrm{d}y^n = f\epsilon_{mn}\mathrm{d}y^m\mathrm{d}y^n$ so that $F^2 = 2f^2$. We also have the FRW metric ansatz

$$\mathrm{d}\hat{s}^2 = -\mathrm{d}\hat{t}^2 + \hat{a}^2(\hat{t}\,)\,\delta_{ij}\,\mathrm{d}\hat{x}^i\,\mathrm{d}\hat{x}^j + b^2(\hat{t}\,)\gamma_{mn}(y)\,\mathrm{d}y^m\mathrm{d}y^n\,. \tag{2.4}$$

Given this metric, we compute the Christoffel symbols directly from

$$\Gamma^I_{JK} = \frac{1}{2}g^{IM}(g_{MJ,K} + g_{MK,J} - g_{JK,M})$$

so that

$$\begin{gathered}\Gamma^0_{ij} = \hat{a}\hat{a}'\delta_{ij}, \quad \Gamma^0_{44} = bb', \quad \Gamma^0_{55} = \Gamma^0_{44}\sin^2\theta \\ \Gamma^i_{0j} = \frac{\hat{a}'}{\hat{a}}\delta^i_j, \quad \Gamma^a_{0b} = \frac{b'}{b}\delta^a_b, \quad \Gamma^4_{55} = -\cos\theta\sin\theta, \quad \Gamma^5_{45} = \cot\theta\end{gathered} \tag{B.1}$$

where i, j run over the spacial 4-indices, and a, b run over the extra 2-indices. All other Christoffel symbols vanish. Using Weinberg's conventions, the Ricci tensor is

$$R_{MN} = R^Q{}_{MQN} = \partial_M\Gamma^Q_{NQ} - \partial_Q\Gamma^Q_{MN} + \Gamma^Q_{PM}\Gamma^P_{NQ} - \Gamma^Q_{PQ}\Gamma^P_{MN} \tag{B.2}$$

R_{MN} is Diagonal

To see that the Ricci tensor is diagonal, consider R_{MN} for $M \neq N$. Let us consider the expression term by term.

- $\partial_M \Gamma^Q_{NQ}$: From (B.1), the only non-zero Γ^Q_{NQ} are Γ^Q_{0Q} and Γ^5_{45} ($Q \neq 0$). The first depends only on $\hat{t}$, while the second depends only on θ. However, if $M \neq N$, then the derivative acting on the first Christoffel could not be with respect to $\hat{t}$, nor the second with respect to θ, therefore this term must vanish on the off-diagonals.

- $\partial_Q \Gamma^Q_{MN}$: From (B.1), the only non-zero Γ^Q_{MN} with $M \neq N$ are again Γ^Q_{0Q} and Γ^5_{45} ($Q \neq 0$). Again, $\partial_Q \Gamma^Q_{0Q}$ vanishes since $Q \neq 0$, and $\partial_5 \Gamma^5_{45}$ vanishes because there is no dependence on ϕ.

- $\Gamma^Q_{PM}\Gamma^P_{NQ}$: This term requires a bit more direct substitution. First, if $Q = 0$, then $P = M$ for the first Christoffel not to vanish. In that case, the second factor is Γ^M_{N0}, but this is only non-zero if $M = N$, so Q cannot be 0. The same argument says that P cannot be 0 either. Similarly, if Q is a non-compact spatial dimension (i.e., 1-3), then $P = Q$, and $M = N = 0$, so that one's out too. Finally, if Q is an extra-dimensional coordinate, then there is a non-zero option. If $M = 0$ and $N = 4$, then we have the non-zero $\Gamma^5_{50}\Gamma^5_{45} = b' \cot\theta / b$ (this term is of course the same for $M \leftrightarrow N$).

- $\Gamma^Q_{PQ}\Gamma^P_{MN}$: This term is easier to handle. The only Christoffels of the form Γ^Q_{PQ} either have $P = 0$ or $P = 4$. For the former, we again find the second factor vanishes unless $M = N$. For the latter, however, we can have the non-zero $\Gamma^5_{45}\Gamma^4_{04} = b' \cot\theta$ corresponding to $M = 0, N = 4$. This is precisely the same contribution as the previous term, and since they appear with a relative minus sign in the Ricci tensor, the tensor itself vanishes for $M \neq N$.

Einstein Tensor

We conclude then, that the Ricci tensor is diagonal, and hence so is the Einstein tensor $G^M{}_N := R^M{}_N - \frac{1}{2}\delta^M_N R$. To compute this, we need:

$$
\begin{aligned}
R := g^{MP}R_{PM} &= -R_{00} + \frac{1}{\hat{a}^2}\left(\sum R_{ii}\right) + \frac{1}{b^2}\left(\sum R_{aa}\right), \\
&= -3\frac{a''}{a} - 2\frac{b''}{b} + 3\left(-2\left(\frac{a'}{a}\right)^2 - \frac{a''}{a} - 2\frac{a'b'}{ab}\right) \\
&\quad + 2\left(-\left(\frac{b'}{b}\right)^2 - \frac{b''}{b} - \frac{1}{b^2} - 3\frac{a'b'}{ab}\right), \\
&= -6\frac{a''}{a} - 6\left(\frac{a'}{a}\right)^2 - 12\frac{a'b'}{ab} - 2\left(\frac{b'}{b}\right)^2 - 4\frac{b''}{b} - \frac{2}{b^2}
\end{aligned}
\tag{B.3}
$$

so that we can write:

$$
\begin{aligned}
G^0{}_0 &= 3\left(\frac{a'}{a}\right)^2 + 6\frac{a'b'}{ab} + \left(\frac{b'}{b}\right)^2 + \frac{1}{b^2}, \\
G^i{}_j &= \left[2\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + 4\frac{a'b'}{ab} + \left(\frac{b'}{b}\right)^2 + 2\frac{b''}{b} + \frac{1}{b^2}\right]\delta^i_j, \\
G^m{}_n &= \left[3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2 + 3\frac{a'b'}{ab} + \frac{b''}{b}\right]\delta^m_n.
\end{aligned}
\tag{B.4}
$$

Stress-Energy Tensor

Lastly, we need the Stress-Energy tensor $T^M{}_N = \delta^M_N L_{\mathcal{M}} - 2g^{MC}\delta L_{\mathcal{M}}/\delta g^{CN}$, where $L_{\mathcal{M}}$ is the matter Lagrangian density—in this case:

$$
-\mathcal{L}_{\mathcal{M}} = \frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}\,\partial_M\phi\,\partial^M\phi + V(\phi) \tag{B.5}
$$

from which:

$$
-\frac{\delta\mathcal{L}_{\mathcal{M}}}{\delta g^{CN}} = \frac{1}{2}F_{AC}\,F^A{}_N + \frac{1}{2}\partial_C\phi\partial_N\phi
$$

so that altogether

$$
\begin{aligned}
T^M{}_N &= F_A{}^M F^A{}_N + \partial^M\phi\partial_N\phi - \left[\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}\,\partial_M\phi\,\partial^M\phi + V(\phi)\right]\delta^M_N \\
&= f^2\delta^m_n + \partial^M\phi\partial_N\phi - \left[\frac{1}{2}f^2 + \frac{1}{2}\,\partial_M\phi\,\partial^M\phi + V(\phi)\right]\delta^M_N,
\end{aligned}
\tag{B.6}
$$

where the first term is only present if $M = m, N = n \in [4, 5]$. Once we make the ansatz that ϕ is homogeneous and isotropic, the Stress-Energy tensor becomes diagonal.

Einstein Equations

Finally, we piece it all together in the Einstein Equations $G^M{}_N = -\kappa^2 T^M{}_N$:

$$
\begin{aligned}
3\left(\frac{a'}{a}\right)^2 + 6\frac{a'b'}{ab} + \left(\frac{b'}{b}\right)^2 + \frac{1}{b^2} &= \kappa^2\left\{\frac{1}{2}\left[\dot{\phi}^2 + f^2\right] + V\right\}, \\
2\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + 4\frac{a'b'}{ab} + \left(\frac{b'}{b}\right)^2 + 2\frac{b''}{b} + \frac{1}{b^2} &= \kappa^2\left\{\frac{1}{2}\left[\dot{\phi}^2 - f^2\right] - V\right\}, \\
3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2 + 3\frac{a'b'}{ab} + \frac{b''}{b} &= \kappa^2\left\{\frac{1}{2}\left[\dot{\phi}^2 + f^2\right] - V\right\},
\end{aligned}
\tag{2.9}
$$

as seen in chapter 2.1.

Appendix C

Consistency of the Truncation

Here we explicitly demonstrate that a coordinate transformation is all that separates the 4D and the 6D equations of motion. Given that, solutions to the 4D equations are then just a coordinate transformation away from solutions to the 6D equations, so at least at the classical level, studying our 4D system is sufficient to understand the full 6D system.

First we use the changes of variables $a = \hat{a}\, e^{\psi/2M_p}$ and $t' = \mathrm{d}t/\mathrm{d}\hat{t} = e^{\psi/2M_p}$ to write

$$H = \frac{\dot{a}}{a} = \left(\hat{H} + \frac{\psi'}{2M_p}\right) e^{-\psi/2M_p} \qquad \text{and} \qquad \mathcal{H} = \frac{\psi'}{2M_p}\,. \tag{C.1}$$

Substituting these into (2.22) and undoing the field re-definition $\varphi = \sqrt{4\pi}\, b_\star\, \phi$ gives:

$$\begin{aligned}
-\frac{\psi'}{2M_p}\,\phi' + \phi'' + 3\left(\frac{\psi'}{2M_p} + \hat{H}\right)\phi' + \frac{\partial V}{\partial \phi} &= 0\,,\\
-\frac{\psi'}{2M_p}\,\psi' + \psi'' + 3\left(\frac{\psi'}{2M_p} + \hat{H}\right)\psi' - \frac{4\pi b_\star^2}{M_p}\,V(\phi)&\\
+\frac{2M_p}{b_\star^2}\,e^{-\psi/M_p} - \frac{6\pi \mathfrak{f}^2}{b_\star^2 M_p} e^{-2\psi/M_p} &= 0\,,\\
4\pi b_\star^2 \frac{(\phi')^2}{2} + \frac{(\psi')^2}{2} + 4\pi b_\star^2\, V(\phi) - \frac{M_p^2}{b_\star^2}\,e^{-\psi/M_p}&\\
+\frac{2\pi \mathfrak{f}^2}{b_\star^2}\,e^{-2\psi/M_p} &= 3M_p^2\left(\frac{\psi'}{2M_p} + \hat{H}\right)^2,
\end{aligned} \tag{C.2}$$

where we have also expanded the definition of $W(\psi, \varphi)$. Using $b = b_\star \exp(\psi/2M_p)$, we can get rid of ψ in favour of b:

$$\phi'' + \left(3\hat{H} + 2\mathcal{H}\right)\phi' + \frac{\partial V}{\partial \phi} = 0\,, \tag{C.3}$$

$$\frac{\psi''}{2M_p} + \left(3\hat{H} + 2\mathcal{H}\right)\mathcal{H} - \frac{4\pi b_\star^2}{2M_p^2}V(\phi) + \frac{1}{b^2} - \frac{3\pi b_\star^2}{M_p^2}\frac{\mathfrak{f}^2}{b^4} = 0\,, \tag{C.4}$$

and

$$\frac{4\pi b_\star^2}{M_p^2}\left\{\frac{1}{2}\left[\frac{(\phi')^2}{2} + \frac{\mathfrak{f}^2}{b^4}\right] + V(\phi)\right\} - \frac{1}{b^2} = 3\hat{H}^2 + 6\hat{H}\mathcal{H} + \mathcal{H}^2\,. \tag{C.5}$$

Now using the definition of M_p, we can write $\kappa^2 = 4\pi b_\star^2/M_p^2$, which shows that (C.3) is equivalent to the 6D inflaton equation of motion, (2.5), while (C.5) is equivalent to the 6D Friedmann equation (i.e., the first in (2.9)). Finally, using

$$\frac{\psi''}{2M_p} = \frac{b''}{b} - \mathcal{H}^2 \tag{C.6}$$

shows (C.4) becomes

$$\frac{b''}{b} + 3\hat{H}\mathcal{H} + \mathcal{H}^2 + \frac{1}{b^2} = \kappa^2\left(\frac{1}{2}V(\phi) + \frac{3}{4}\frac{\mathfrak{f}^2}{b^4}\right). \tag{C.7}$$

This is equivalent to $1/4$ times the first, plus $3/4$ times the second, minus $1/2$ times the third of the Einstein equations (2.9). Thus we recover all the information of the 6D equations of motion using our 4D reduction, and hence we are capturing the full dynamics of the system just by studying our 4D equivalent system. This is a *consistent truncation* [21] since our extra-dimensional manifold S^2 is homogeneous.

Appendix D

Second-Order Corrections to the Magnon Inflation Model

In order to include corrections to the first-order results, we can construct the matter Lagrangian to second-order by adding to the action

$$\Delta S = \int \mathrm{d}^4x\sqrt{-g}\,\Delta L,$$

where we have defined the second-order Lagrangian contribution

$$\begin{aligned}
-\Delta L := \frac{1}{2}\Big[\mathcal{G}_{ab}(\phi)\,g^{\mu\nu} + \mathcal{I}_{ab}(\phi)\,U^\mu U^\nu\Big]\,\partial_\mu\phi^a\partial_\nu\phi^b + \frac{1}{2}\Big[\mathcal{C}^{(1)}(\phi)\,\nabla^\mu U^\nu\nabla_\mu U_\nu \\
+ \mathcal{C}^{(2)}(\phi)(\nabla\cdot U)^2 + \mathcal{C}^{(3)}(\phi)\,\nabla_\nu U^\mu\nabla_\mu U^\nu + \mathcal{C}^{(4)}(\phi)\,U^\lambda\nabla_\lambda U^\mu U^\nu\nabla_\nu U_\mu\Big] \\
+ \mathcal{C}^{(5)}_a(\phi)\,U^\nu\,(\nabla_\nu U^\mu)\,\partial_\mu\phi^a + \mathcal{C}^{(6)}_a(\phi)\,U^\mu\,(\nabla\cdot U)\,\partial_\mu\phi^a. \qquad \text{(D.1)}
\end{aligned}$$

This correction includes all terms involving two powers of field derivatives, modulo integration by parts[1]. Given this modification of the action, we find the following corrections are added to the LHS of the equations of motion (3.7) to (3.9):

$$\begin{aligned}
\Delta^{(\xi)} &= 0, \\
\Delta^{(U)}_\mu &= \mathcal{I}_{ab}\,\partial_\mu\phi^a\dot{\phi}^b + \mathcal{C}^{(4)}\,\dot{U}^\nu\nabla_\mu U_\nu + \mathcal{C}^{(5)}_a\nabla_\mu U^\nu\nabla_\nu\phi^a \\
&\quad + \mathcal{C}^{(6)}_a(\nabla\cdot U)\nabla_\mu\phi^a - \nabla_\nu\Big\{\mathcal{C}^{(1)}\,\nabla^\nu U_\mu + \mathcal{C}^{(2)}\delta^\nu_\mu\nabla\cdot U + \mathcal{C}^{(3)}\nabla_\mu U^\nu \\
&\quad + \mathcal{C}^{(4)}U^\nu\dot{U}_\mu + \mathcal{C}^{(5)}_a\,U^\nu\nabla_\mu\phi^a + \mathcal{C}^{(6)}_a\delta^\nu_\mu\dot{\phi}^a\Big\}, \text{ and} \\
\Delta^{(\phi)} &= \frac{\partial(\Delta L)}{\partial\phi^a} + \nabla_\mu\left\{\mathcal{G}_{ab}\nabla^\mu\phi^b + \mathcal{I}_{ab}\,U^\mu\dot{\phi}^b + \mathcal{C}^{(5)}_a\,\dot{U}^\mu + \mathcal{C}^{(6)}_a\,U^\mu\nabla\cdot U\right\}, \qquad \text{(D.2)}
\end{aligned}$$

[1] With constant coefficients, the terms in the second square brackets are the major players in the Einstein-Aether theory [22] (whose authors deserve the credit for finding all those terms).

where the contribution $\Delta^{(X)}$ is added to the LHS of the X equation of motion.

Note that the ξ equation of motion is unchanged since there are no new terms involving ξ at the second order level (it is, after all, just a Lagrange multiplier). However, the correction to the U^μ equation of motion *does* mean that the value of ξ changes. To see this, contract the new equation of motion for U^μ with U^μ as before. Now, we have

$$\begin{aligned}2\xi = \mathcal{A}_a\dot{\phi}^a + \mathcal{I}_{ab}\dot{\phi}^a\dot{\phi}^b + \mathcal{C}^{(4)}\dot{U}^\nu\dot{U}_\nu + \mathcal{C}_a^{(5)}\dot{U}^\nu\nabla_\nu\phi^a \\ + \mathcal{C}_a^{(6)}(\nabla\cdot U)\dot{\phi}^a - U^\mu\nabla_\nu\Big\{\mathcal{C}^{(1)}\nabla^\nu U_\mu + \mathcal{C}^{(2)}\delta^\nu_\mu\nabla\cdot U + \mathcal{C}^{(3)}\nabla_\mu U^\nu \\ + \mathcal{C}^{(4)}U^\nu\dot{U}_\mu + \mathcal{C}_a^{(5)}U^\nu\nabla_\mu\phi^a + \mathcal{C}_a^{(6)}\delta^\nu_\mu\dot{\phi}^a\Big\}.\end{aligned}$$

One useful thing we can do with these abominable equations is to verify that the value of U^μ found before still holds, as long as we can assume the same homogeneous cosmic frame. In this frame, we had that the scalars only depended on a time coordinate t, and the metric took the FRW form (implying the helpful relation (3.12)). Under these assumptions, the correction $\Delta^{(U)}$ becomes

$$\begin{aligned}\Delta^{(U)}_\mu = \mathcal{I}_{ab}\,\partial_\mu\phi^a\dot{\phi}^b + \mathcal{C}_a^{(5)}\Big(\nabla_\mu\phi^a + U_\mu\dot{\phi}^a\Big) + 3\mathcal{C}_a^{(6)}H\nabla_\mu\phi^a \\ - \nabla_\nu\Big\{\mathcal{C}^{(1)}\nabla^\nu U_\mu + 3\mathcal{C}^{(2)}\delta^\nu_\mu H + \mathcal{C}^{(3)}\nabla_\mu U^\nu \\ + \mathcal{C}_a^{(5)}U^\nu\nabla_\mu\phi^a + \mathcal{C}_a^{(6)}\delta^\nu_\mu\dot{\phi}^a\Big\}.\end{aligned} \tag{D.3}$$

while ξ takes the value

$$\begin{aligned}2\xi = \mathcal{A}_a\dot{\phi}^a + \mathcal{I}_{ab}\,\dot{\phi}^a\dot{\phi}^b + 3\mathcal{C}_a^{(6)}H\dot{\phi}^a - U^\mu\nabla_\nu\Big\{\mathcal{C}^{(1)}\nabla^\nu U_\mu + 3\mathcal{C}^{(2)}\delta^\nu_\mu H \\ + \mathcal{C}^{(3)}\nabla_\mu U^\nu + \mathcal{C}_a^{(5)}U^\nu\nabla_\mu\phi^a + \mathcal{C}_a^{(6)}\delta^\nu_\mu\dot{\phi}^a\Big\}.\end{aligned} \tag{D.4}$$

Clearly, the choice $U^0 = 1,\ U^i = 0$ satisfies the first-order components of the full U^μ equation of motion, but it remains to be seen if the new contributions are satisfied. To do so, first we evaluate the expressions (D.3) and (D.4) using this ansatz.

$$\begin{aligned}\Delta^{(U)}_0\Big|_{U^0=1} &= \mathcal{I}_{ab}\,\dot{\phi}^a\dot{\phi}^b + 3\mathcal{C}_a^{(6)}H\dot{\phi}^a - \nabla_\nu\Big\{\Big(\mathcal{C}^{(1)} + \mathcal{C}^{(3)}\Big)H(\delta^\nu_0 - U^\nu) \\ &\quad + 3\mathcal{C}^{(2)}\delta^\nu_0 H + \mathcal{C}_a^{(5)}U^\nu\dot{\phi}^a + \mathcal{C}_a^{(6)}\delta^\nu_0\dot{\phi}^a\Big\}, \\ \Delta^{(U)}_i\Big|_{U^i=0} &= 0,\end{aligned}$$

while

$$2\Delta\xi\Big|_{U=(1,0,0,0)} = U^\mu \Delta_\mu^{(U)},$$
$$= \Delta_0^{(U)},$$

where we have defined $2\Delta\xi := 2\xi - \mathcal{A}_a\dot{\phi}^a$.

The second-order contributions to the U^μ equation of motion then become

$$2\Delta\xi U_0 + \Delta_0^{(U)} = -\Delta_0^{(U)} + \Delta_0^{(U)} = 0, \text{ while}$$
$$2\Delta\xi U_i + \Delta_i^{(U)} = 0 + 0 = 0.$$

Therefore, even including second-order contributions, U^μ has the simple solution $U^0 = 1,\ U^i = 0$ as long as there still exist homogeneous and isotropic solutions for the scalars and the metric. Finally, we must compute the corrections to the stress-energy tensor.

D.1 Corrections to the Stress-Energy

In order to compute the corrections to the stress-energy, we can write (D.1) in a more suggestive way:

$$-\Delta L = \frac{1}{2}(\mathcal{G}_{ab}g^{\mu\nu} + \mathcal{I}_{ab}U^\mu U^\nu)\partial_\mu\phi^a\partial_\nu\phi^b + \mathcal{C}_a^{(5)}\dot{U}^\mu\partial_\mu\phi^a$$
$$- \mathcal{C}_a^{(6)}(\nabla\cdot U)\dot{\phi}^a + \frac{1}{2}K^{\mu\nu}{}_{\alpha\beta}\nabla_\mu U^\alpha\nabla_\nu U^\beta, \tag{D.5}$$

where we have defined:

$$K^{\mu\nu}{}_{\alpha\beta} := \mathcal{C}^{(1)}g_{\alpha\beta}g^{\mu\nu} + \mathcal{C}^{(2)}\delta^\mu_\alpha\delta^\nu_\beta + \mathcal{C}^{(3)}\delta^\mu_\beta\delta^\nu_\alpha + \mathcal{C}^{(4)}g_{\alpha\beta}U^\mu U^\nu. \tag{D.6}$$

The variation can then proceed as usual.

$$\Delta T_{\mu\nu} = g_{\mu\nu}\Delta L - 2\frac{\delta\Delta L}{\delta g^{\mu\nu}},$$
$$= g_{\mu\nu}\Delta L + \mathcal{G}_{ab}\partial_\mu\phi^a\partial_\nu\phi^b + \mathcal{C}^{(1)}\Big(\nabla_\mu U^\lambda\nabla_\nu U_\lambda - \nabla_\lambda U_\mu\nabla^\lambda U_\nu\Big)$$
$$- \mathcal{C}^{(4)}\dot{U}_\mu\dot{U}_\nu - 2J^\lambda{}_\alpha\frac{\delta\nabla_\lambda U^\alpha}{\delta g^{\mu\nu}}, \tag{D.7}$$

where this time we defined

$$J^{\lambda}{}_{\alpha} := K^{\lambda\rho}{}_{\alpha\beta}\nabla_{\rho}U^{\beta} + \mathcal{C}_a^{(5)}U^{\lambda}\partial_{\alpha}\phi^a + \mathcal{C}_a^{(6)}\dot{\phi}^a\delta^{\lambda}_{\alpha}. \tag{D.8}$$

This just leaves the computation of:

$$\begin{aligned} J^{\lambda}{}_{\alpha}\,\delta(\nabla_{\lambda}U^{\alpha}) &= J^{\lambda}{}_{\alpha}U^{\beta}\,\delta\Gamma^{\alpha}_{\lambda\beta} \\ &= -\frac{1}{2}J^{\lambda}{}_{\alpha}U^{\beta}\left[g_{\gamma\lambda}\nabla_{\beta}(\delta g^{\gamma\alpha}) + g_{\gamma\beta}\nabla_{\lambda}(\delta g^{\gamma\alpha}) - g_{\lambda\gamma}g_{\beta\sigma}\nabla^{\alpha}(\delta g^{\gamma\sigma})\right]. \end{aligned} \tag{D.9}$$

After integrating by parts, this term is equivalent to

$$J^{\lambda}{}_{\alpha}\,\delta(\nabla_{\lambda}U^{\alpha}) \to -\frac{1}{2}\nabla_{\sigma}\left[J_{\gamma\alpha}U^{\sigma} + J^{\sigma}{}_{\alpha}U_{\gamma} - J_{\gamma}{}^{\sigma}U_{\alpha}\right]\delta g^{\gamma\alpha}. \tag{D.10}$$

Altogether then, the additional contributions to the stress-energy are

$$\begin{aligned} \Delta T_{\mu\nu} = g_{\mu\nu}\Delta L + \mathcal{G}_{ab}\partial_{\mu}\phi^a\partial_{\nu}\phi^b + \mathcal{C}^{(1)}\Big(\nabla_{\mu}U^{\lambda}\nabla_{\nu}U_{\lambda} - \nabla_{\lambda}U_{\mu}\nabla^{\lambda}U_{\nu}\Big) \\ - \mathcal{C}^{(4)}\dot{U}_{\mu}\dot{U}_{\nu} + \nabla_{\sigma}\left[J_{\mu\nu}U^{\sigma} + J^{\sigma}{}_{\nu}U_{\mu} - J_{\mu}{}^{\sigma}U_{\nu}\right]. \end{aligned} \tag{D.11}$$

Corrections to the Energy Density

With this information in hand, we can also compute the corrections to the energy density. Here it is very important to remember that the value of ξ is also modified by the higher order terms. For ease, we also evaluate this as the solution $U = (1, 0, 0, 0)$.

$$\begin{aligned} U^{\mu}U^{\nu}T_{\mu\nu} &= \rho + \Delta\rho \\ &= \mathcal{V} - 2\Delta\xi + U^{\mu}U^{\nu}\Delta T_{\mu\nu}. \end{aligned} \tag{D.12}$$

Separately (and using the assumptions of homogeneity and isotropy of the fields), we have:

$$\begin{aligned} -2\Delta\xi = -\mathcal{I}_{ab}\,\dot{\phi}^a\dot{\phi}^b + 3H\Big(\mathcal{C}_a^{(5)} - \mathcal{C}_a^{(6)} + \mathcal{C}_{,a}^{(2)}\Big)\dot{\phi}^a - 3H^2\Big(\mathcal{C}^{(1)} + \mathcal{C}^{(3)}\Big) \\ + 3\dot{H}\Big(\mathcal{C}^{(2)} - \mathcal{C}^{(1)}\Big) + \Big(\mathcal{C}_{a,b}^{(5)} + \mathcal{C}_{a,b}^{(6)}\Big)\dot{\phi}^a\dot{\phi}^b + \Big(\mathcal{C}_a^{(5)} + \mathcal{C}_a^{(6)}\Big)\ddot{\phi}^a, \end{aligned} \tag{D.13}$$

and

$$U^\mu U^\nu \Delta T_{\mu\nu} = \frac{1}{2}(\mathcal{G}_{ab} + \mathcal{I}_{ab})\dot{\phi}^a\dot{\phi}^b + 3\mathcal{C}^{(6)}H\dot{\phi}^a - \frac{3}{2}H^2\Big(\mathcal{C}^{(1)} + 3\mathcal{C}^{(2)} + \mathcal{C}^{(3)}\Big)$$
$$- 3H\mathcal{C}^{(2)}_{,a}\dot{\phi}^a - 3\mathcal{C}^{(2)}\Big(\dot{H} + 3H^2\Big) - \Big(\mathcal{C}^{(5)}_a\ddot{\phi}^a + 3H\mathcal{C}^{(5)}_a\dot{\phi}^a + \mathcal{C}^{(5)}_{a,b}\dot{\phi}^a\dot{\phi}^b\Big)$$
$$- \Big(\mathcal{C}^{(6)}_a\ddot{\phi}^a + 3H\mathcal{C}^{(6)}_a\dot{\phi}^a + \mathcal{C}^{(6)}_{a,b}\dot{\phi}^a\dot{\phi}^b\Big). \tag{D.14}$$

Altogether then,

$$U^\mu U^\nu T_{\mu\nu} = \rho + \Delta\rho, \tag{D.15}$$

with $\rho = \mathcal{V}$, and $\Delta\rho$ defined as

$$\Delta\rho := \frac{1}{2}\mathcal{Q}_{ab}\dot{\phi}^a\dot{\phi}^b - 3H\mathcal{C}^{(6)}_a\dot{\phi}^a - \frac{9}{2}H^2\Big(\mathcal{C}^{(1)} + 3\mathcal{C}^{(2)} + \mathcal{C}^{(3)}\Big) - 3\mathcal{C}^{(1)}\dot{H}, \tag{D.16}$$

and $\mathcal{Q}_{ab} := \mathcal{G}_{ab} - \mathcal{I}_{ab}$ is the target-space metric.

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