# MODIFIED SPHERICAL HARMONICS

# A MODIFIED SPHERICAL HARMONICS APPROACH TO SOLVING THE NEUTRON TRANSPORT EQUATION

By

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## Abstract

Another approach is adopted for deriving the moments equations in spherical geometry using a spherical harmonics expansion of the neutron transport equation over a variable range of the direction cosine. Because of complications and uncertainties in establishing boundary conditions for the equations, only the zero'th order equations are solved, in an idealized situation, in order that a feel for equations and boundary conditions may be obtained.

The equations are compared to equations given in a paper 'Directionally Discontinuous Harmonic Solutions of the Neutron Transport Equation in Spherical Geometry', by A. A. Harms and E. A. Attia. Analytical solutions for the zero'th order equations are given for equations developed there and to the equations developed in this paper. Numerical values are presented to give an idea of what accuracies might be expected. It is hoped that similar techniques can be used to solve the higher order equations analytically, and that appropriate boundary conditions can be found.

## Acknowledgements

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## Introduction

It is of interest to study the behaviour of neutrons in a specified volume within a nuclear reactor. To do so, an equation must be set down describing a neutron's interaction with its environment.

Let $\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)$ be the number of neutrons at position $\underset{\sim}{r}$, time t, travelling at speed $v(\underset{\sim}{r},t)$ that pass a unit area normal to $\underset{\sim}{\Omega}$. Neutrons at position $\underset{\sim}{r}$, moving at speed $v(\underset{\sim}{r},t)$ will travel $v\Delta t$ units in a time $\Delta t$ in the direction $\underset{\sim}{\Omega}$. The net change in $\psi$ per unit distance of travel over this time interval is

$$\frac{\psi(\underset{\sim}{r}+v(\underset{\sim}{r},t)\Delta t\underset{\sim}{\Omega},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t+\Delta t) - \psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)}{v(\underset{\sim}{r},t)\Delta t}$$

$$=\{[\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)+v(\underset{\sim}{r},t)\Delta t\underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla}\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)+\Delta t\frac{\partial\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)}{\partial t}$$

$$+ O(\Delta t^2)] - \psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)\}/v(\underset{\sim}{r},t)\Delta t$$

$$= \frac{1}{v(\underset{\sim}{r},t)}\frac{\partial\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t}{\partial t} + \underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla}\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t) + O(\Delta t) \quad . \qquad (1)$$

The instantaneous rate of change on $\psi$ at position $\underset{\sim}{r}$, time t is then

$$\frac{1}{v(\underset{\sim}{r},t)}\frac{\partial\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)}{\partial t} + \underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla}\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t) \quad . \qquad (2)$$

One thing that can happen in the length of time $\Delta t$ and distance $v\Delta t$ is that neutrons can interact with other nuclei (although not with other neutrons). This interaction can involve reactions such as (n,f), (n,$\gamma$), (n,n') etc. At time t, if $\Sigma(r)$ is the probability of a neutron interaction per unit distance, then removal of neutrons by interaction with other elements results in a change in $\psi$ that is

$$-\Sigma(\underset{\sim}{r})\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t); \tag{3}$$

the minus sign signifies removal. The other possible contribution in a time $\Delta t$ and distance $v\Delta t$ is that of other neutrons which are travelling at $v'(\underset{\sim}{r},t)\underset{\sim}{\Omega}'$, emerging from some interaction to travel at $v(\underset{\sim}{r},t)\underset{\sim}{\Omega}$. This contribution can be written

$$\int_{v'}\int_{\underset{\sim}{\Omega}'} \Sigma(\underset{\sim}{r})f(\underset{\sim}{r},v'(\underset{\sim}{r},t)\underset{\sim}{\Omega}' \rightarrow v(\underset{\sim}{r},t)\underset{\sim}{\Omega})\psi(\underset{\sim}{r},v'(\underset{\sim}{r},t)\underset{\sim}{\Omega}',t)\,d\Omega' dv' \tag{4}$$

where $f(\underset{\sim}{r},v'(\underset{\sim}{r},t)\underset{\sim}{\Omega}' \rightarrow v(\underset{\sim}{r},t)\underset{\sim}{\Omega})$ denotes the probability of instantaneous deflection to $v\underset{\sim}{\Omega}$. A source at the point considered will be denoted

$$S(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t) \tag{5}$$

and will have dimensions $n/cm^3$ sec. Then

$$\frac{1}{v}\frac{\partial\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)}{\partial t} + \underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla}\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t) =$$

$$-\Sigma(\underset{\sim}{r})\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\Omega,t) + \int_{v'}\int_{\underset{\sim}{\Omega}'} \Sigma(\underset{\sim}{r})f(\underset{\sim}{r},v'(\underset{\sim}{r},t)\underset{\sim}{\Omega}'\rightarrow v(\underset{\sim}{r},t)\underset{\sim}{\Omega})\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)\,d\Omega' dv'$$

$$+ S(\underset{\sim}{r},v(\underset{\sim}{r},t),t) \;. \tag{6}$$

This is known as the Boltzmann integrodifferential equation.

A convenient way to solve the Boltzmann integrodifferential equation is to assume the flux $\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t)$ can be expanded as

$$\psi(\underset{\sim}{r},v(\underset{\sim}{r},t)\underset{\sim}{\Omega},t) = \sum_{\ell} A_{\ell}(\underset{\sim}{r},v(\underset{\sim}{r},t),t)B_{\ell}(\underset{\sim}{\Omega}) \quad . \tag{7}$$

With such an expansion, it is necessary to require that either $\{A_{\ell}(\underset{\sim}{r},v(\underset{\sim}{r},t),t)\}$ or $\{B_{\ell}(\underset{\sim}{\Omega})\}$ form a complete set. The method of Spherical Harmonics assumes $\{B_{\ell}(\underset{\sim}{\Omega})\}$ is the complete set and specifies it as the set of spherical harmonics

$$P_{\ell}^{m}(\mu)e^{im\phi} \ , \ \mu = \cos\theta \ , \ 0 \leq \phi \leq 2\pi \ , \ -1 \leq \mu \leq +1, \tag{8}$$

where $\theta$ is the azimuthal angle, measured from the direction $\underset{\sim}{r}/|\underset{\sim}{r}|$ and $\phi$ is the polar angle. In cases of spherical or planar symmetry, dependence on $\phi$ is not needed and these functions reduce by a simple integration over $\phi$ to the set of Legendre polynomials of the first kind

$$P_{\ell}(\mu) \ , \quad \mu = \cos\theta \ , \quad -1 \leq \mu \leq +1 \tag{9}$$

which form a complete set on $\mu\varepsilon(-1,+1)$. When this symmetry exists, $\psi$ need only be a function of $|\underset{\sim}{r}|$ and the component of $\underset{\sim}{\Omega}$ in the direction $\frac{\underset{\sim}{r}}{|\underset{\sim}{r}|}$ ; denoted $\mu$. Now the expansion of $\psi$ looks like

$$\psi(r,\mu) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2}\, \psi_{\ell}(r)P_{\ell}(\mu), \tag{10}$$

the coefficients $\frac{2\ell+1}{2}$ being chosen so that

$$\psi_{\ell}(r) = \int_{-1}^{+1} \psi(r,\mu)P_{\ell}(\mu)d\mu \ . \qquad (11)$$

It has been tacitly assumed that $\psi(r,\mu)\varepsilon L_r^2(-1,+1)$ (the set of functions square integrable over $1 \leq \mu \leq +1$) and so

$$\lim_{N\to\infty} \int_{-1}^{+1} [\psi(r,\mu) = \sum_{\ell=0}^{N} \psi_{\ell}(r)P_{\ell}(\mu)]^2 d\mu = 0 \qquad (12)$$

or, the mean square deviation in $\mu$ of $\psi(r,\mu)$ approaches zero as N approaches infinity. Moments, $\psi_{\ell}(r)$, are found using Eq. 11 by multiplying Boltzmann's equation, Eq. 6, by $P_{\ell}(\mu)$ and integrating over $-1 \leq \mu \leq +1$ as follows.

This report considers only the case of isotropic scattering and so the function $f(\underset{\sim}{r},v'(\underset{\sim}{r},t)\underset{\sim}{\Omega}' \to v(\underset{\sim}{r},t)\underset{\sim}{\Omega})$ in Eq. 6 is just $\frac{1}{4\pi \text{ (steridians)}}$. The flux $\psi$ is not considered as a function of time and so

$$\frac{1}{v(\underset{\sim}{r})} \frac{\partial\psi(\underset{\sim}{r},v(\underset{\sim}{r})\underset{\sim}{\Omega})}{\partial t} = 0 \ . \qquad (13)$$

The directional derivative of $\psi$ in the direction $\underset{\sim}{\Omega}$, denoted $\underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla}\psi(\underset{\sim}{r},v(\underset{\sim}{r})\underset{\sim}{\Omega})$ can be reduced in the case of spherical symmetry following a derivation by Tait[1]. From figure 3,

$$\underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla}\psi = -\frac{d\psi}{d\rho} = -\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial\rho} - \frac{\partial\psi}{\partial\mu}\frac{\partial\mu}{\partial\rho} \ . \qquad (14)$$

Since

$$\Delta r = \cos(\pi-(\theta+\Delta\theta))\Delta\rho \simeq -\cos(\theta)\Delta\rho \qquad (15)$$

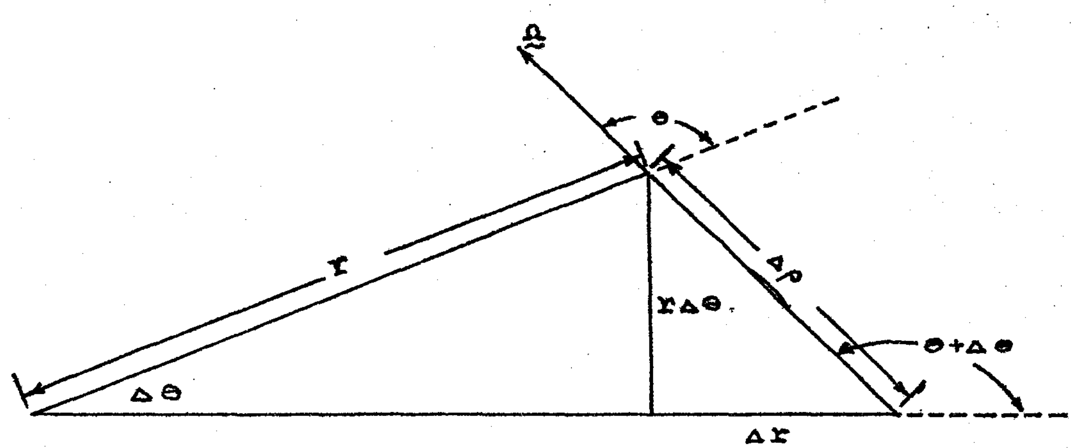


Figure 1 Directional derivative in spherical geometry

then

$$\frac{dr}{d\rho} = -\cos\theta = -\mu. \tag{16}$$

Also

$$r\Delta\theta \simeq \sin\theta\Delta\rho$$

$$\Rightarrow \frac{d\mu}{d\rho} = -\sin\theta\,\frac{d\theta}{d\rho} = -\,\frac{(1-\mu^2)}{r}\,. \tag{17}$$

Therefore

$$\underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla}\psi(r,\mu) = \mu\,\frac{\partial\psi(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\,\frac{\partial\psi(r,\mu)}{\partial\mu}\,. \tag{18}$$

Thus, the Boltzmann integrodifferential equation for a problem with spherical symmetry, no time dependence, constant velocity as well as no external sources reduces to

$$\mu\,\frac{\partial\psi(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\,\frac{\partial\psi(r,\mu)}{\partial\mu} + \Sigma\psi(r,\mu) = \frac{C\Sigma}{2}\int_{-1}^{+1}\psi(r,\mu)\,d\mu. \tag{19}$$

Multiplying this equation by $P_\ell(\mu)$ and integrating over $-1 \leq \mu \leq +1$ while making use of the recurrence relation for the Legendre polynomials

$$(1-\mu^2)\,\frac{dP_n(\mu)}{d\mu} = \frac{n(n+1)}{2n+1}\,(P_{n-1}(\mu) - P_{n+1}(\mu)) \tag{20}$$

gives the following equations for the moments:

$$(n+1)\left[\frac{d}{dr}+\frac{(n+2)}{r}\right]\psi_{n+1}(r)+n\left[\frac{d}{dr} - \frac{(n-1)}{r}\right]\psi_{n-1}(r)+(2n+1)\Sigma\psi_n(r) \tag{21}$$

$$= C\Sigma\psi_0(r)\,\delta_{n0}\,.$$

When $\psi$ is discontinuous in the variable $\mu$, difficulties in the form of slow convergence of the expansion occur.

Discontinuities in $\mu$ at, say $\hat{\mu}$ force the expansion to converge to

$$\frac{1}{2}[\psi(r,\hat{\mu}-0) + \psi(r,\hat{\mu}+0)] \tag{22}$$

where $\psi(r,\hat{\mu}-0)$ denotes $\lim_{\varepsilon\to 0^+} \psi(r,\hat{\mu}-\varepsilon)$. For example at the interface between two media with different mean free paths, the flux will experience a discontinuity at $\mu = 0$, $r = \hat{r}$, $\hat{r}$ being the position of the interface. Yvon, Ref. 2, solved this by creating two expansions, one on the range of $\mu$ (-1,0) and the other on (0,+1) and found much better accuracy with lower order expansions. Because of convergence, as in Eq. 12 over the open interval of expansions, the problem of slow convergence at the interface is avoided with Yvon's method. Note that this method can be applied at the interface between two infinite slabs in planar geometry or at the boundary between two spherical annuli. In the case of a black sphere of radius a, the discontinuity of $\psi(r,\mu)$ in $\mu$ when at position r, occurs at $\mu_0(r) = \sqrt{1 - (a/r)^2)}$, as shown in Fig. 1, which is a function of r. It would make sense then, following the example of Yvon to expand $\psi(r,\mu)$ over the range $(-1,\mu_0(r))$, $(\mu_0(r),+1)$. To do this, complete sets over the ranges of $\mu(-1,\mu_0(r))$, and $(\mu_0(r),+1)$ can be made with simple linear transformations of $\mu$, i.e. on the range $-1 \leq \mu \leq \mu_0(r)$, chose functions $\alpha(r)$, $\beta(r)$ such that $\alpha(r)\cdot(-1) + \beta(r) = -1$, and $\alpha(r)\cdot\mu_0(r) + \beta(r) = +1$. With such $\alpha(r)$, $\beta(r)$, the polynimials $P_m(\alpha(r)\mu + \beta(r) \equiv P_m(r,\mu)$ form a complete set over the range

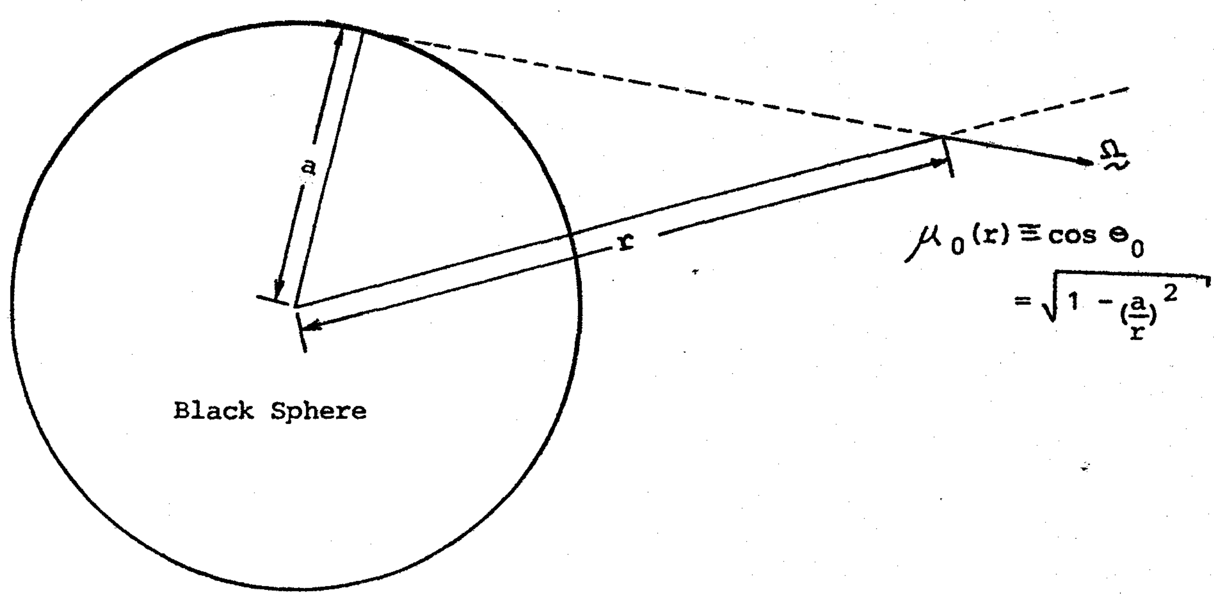


Figure 2 Black sphere and point of discontinuity of $\Psi(r,\mu)$ in $\mu$

of $\mu$ chosen above. Following the procedures outlined for the Spherical Harmonics method, a set of coupled, first order equations can be obtained.

The problem of the black sphere was attempted in a paper 'Directionally Discontinuous Harmonic Solutions of the Neutron Transport Equation in Spherical Geometry' by A. A. Harms and E. A. Attia, Ref. 3. In it, they not only expanded over the ranges listed above, but also made the $\mu$-derivative of $\psi(r,\mu)$ proportional to a delta function at the point of discontinuity in $\mu$. Equations for the zero'th moments from that paper will be solved for comparison with solutions obtained here.

This paper will first derive modifed spherical harmonics equations for a general system of annular spheres. Because of the difficulty, as of this writing, of assigning boundary conditions to this set of equations, the idealized case of the black sphere will be used in order that analytical solutions can be found and that boundary conditions can be determined. It is hoped that techniques discovered for this special case will lead to the ability to solve more generalized cases. Equations that were solved numerically in the Harms-Attia paper will be solved analytically here as well as a more generalized set listed there. Corresponding equations from the set of equations derived here will be solved analytically, compared with the Harms-Attia solutions, and compared with transport and diffusion values listed in a paper by Sahni, Ref.

4. Indications will then be given of what approach might be taken to solve more generalized equations, that is, for annular systems and higher moments.

## Derivation of Modified Spherical Harmonics Equations in a General System of Annular Spheres

Consider a system of annular spheres and suppose there are no external sources, that the scattering is isotopic and that the medium within an annulus is homogeneous (see Fig. 2). For $R_\ell < r < R_{\ell+1}$, $0 \le \ell \le L$ with $R_0 = 0$, $R_{L+1} = \infty$ and $\mu_n(r) \equiv \frac{\underset{\sim}{r}}{r} \cdot \underset{\sim}{\Omega}_n$, $\mu_n(r) < \mu < \mu_{n-1}(r)$, $1 \le n \le \ell+1$, $\mu_0 = 1$, $\mu_{\ell+1} = -1$, the behaviour of the angular flux density $\psi(r,\mu)$ is given by the equation

$$\mu \frac{\partial\psi(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial\psi(r,\mu)}{\partial\mu} + \Sigma_\ell \psi(r,\mu) = \frac{C_\ell \Sigma_\ell}{2} \left\{ \sum_{j=1}^{\ell+1} \int_{\mu_j(r)}^{\mu_{j-1}(r)} \psi(r,\mu) d\mu \right\} \quad (23)$$

where $C_\ell(r)$, $\Sigma_\ell(r)$ are $C(r)$, $\Sigma(r)$, $R_\ell < r < R_{\ell+1}$. On each open set $\mu_j(r) < \mu < \mu_{j-1}(r)$, expand $\psi(r,\mu)$ by

$$\psi(r,\mu) = \sum_{m=0}^{\infty} \frac{2m+1}{\mu_{j-1}(r)-\mu_j(r)} \psi_m(r) P_m(r,\mu); \quad (24)$$

$P_m(r,\mu) = P_m(\alpha(r)\mu + \beta(r))$ with $\alpha(r)$, $\beta(r)$ chosen so that $P_m(r,\mu_j(r)) = P_m(-1) = -1^m$, $P_m(r,\mu_{j-1}(r)) = P_m(+1) = +1$. Therefore, $P_m(r,\mu)$ are just the Legendre polynomials defined on an appropriate range of $\mu$ and so they form a complete orthogonal set on this range of $\mu$. The straightforward derivation of needed properties of $P_m(r,\mu)$ is done in Appendix A. If there are dis-

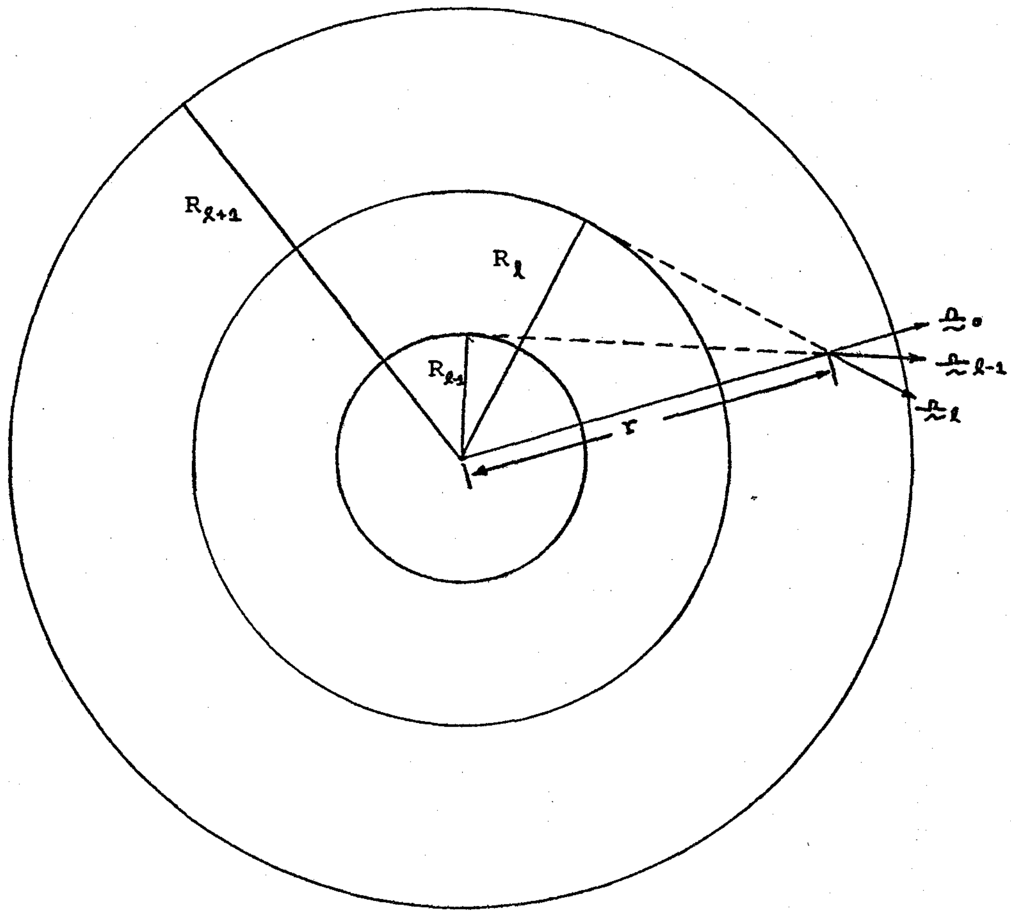


Figure 3 General system of annular spheres

continuities in $\mu$, $\mu\varepsilon(\mu_j(r),\mu_{j-1}(r))$, say at $\mu = \hat{\mu}$, the above expansion will converge to

$$\frac{1}{2}\,[\psi(r,\hat{\mu}-0) + \psi(r,\hat{\mu}+0)] \tag{25}$$

although the convergence will be slow. $\psi(r,\mu)$ also need not be continuous at the common endpoint of two adjacent $\mu$-intervals. Because of this, a continuity argument will be used on the end-points of these ranges of $\mu$ in developing equations for the moments $\psi_n(r)$. With such an expansion, as in Eq. 24,

$$\lim_{N\to\infty}\int_{-1}^{+1}[\psi(r,\mu) - \sum_{n=0}^{N}\psi_n(r)P_n(r,\mu)]^2 d\mu = 0. \tag{26}$$

<u>Equations for the Moments $\psi_n(r)$</u>

Since

$$\int_{\mu_j(r)}^{\mu_{j-1}(r)}\psi(r,\mu)P_n(r,\mu)\,d\mu = \int_{\mu_j(r)}^{\mu_{j-1}(r)}\Big(\sum_{m=0}^{\infty}\frac{2m+1}{\mu_{j-1}(r)-\mu_j(r)}\,\psi_m(r)P_m(r,\mu)\Big)P_n(r,\mu)\,d\mu\ , \tag{27}$$

assuming absolute convergence of the series, and using the fact that $P_m(r,\mu)$ are continuous, the above is

$$\sum_{m=0}^{\infty} \frac{2m+1}{\mu_{j-1}(r)-\mu_j(r)}\cdot\psi_m(r)\cdot\int_{\mu_j(r)}^{\mu_{j-1}(r)} P_n(r,\mu)P_m(r,\mu)d\mu$$

$$= \sum_{m=0}^{\infty} \frac{2m+1}{\mu_{j-1}(r)\mu_j(r)}\cdot\psi_m(r)\cdot\int_{-1}^{+1} P_n(\mu)P_m(\mu)\ \frac{d\mu}{\alpha(r)}$$

$$= \sum_{m=0}^{\infty} \frac{2m+1}{\mu_{j-1}(r)-\mu_j(r)}\cdot\psi_m(r)\cdot\frac{\mu_{j-1}(r)-\mu_j(r)}{2}\cdot\frac{2}{2m+1}\ \delta_{nm} = \psi_n(r). \tag{28}$$

Then, to obtain equations for the moments $\psi_n(r)$, equation 4 is multiplied by $P_n(r,\mu)$ (limits on n later) and integrated over the appropriate cosine range. If $\psi(r,\mu_j^{+}(r))$ denotes $\lim_{\varepsilon\to 0^+} \psi(r,\mu_j(r)+\varepsilon)$ and $\psi(r,\mu_j^{-}(r))$ denotes $\lim_{\varepsilon\to 0^+} \psi(r,\mu_j(r)-\varepsilon)$, continuity arguments at the endpoints together with Leibniz' rule combine to give

$$\frac{d}{dr}\int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\psi(r,\mu)P_n(r,\mu)d\mu$$

$$= (\frac{d}{dr}\ \mu_{j-1}(r))\cdot\mu_{j-1}(r)\cdot\psi(r,\mu_j^{-}(r))P_n(r,\mu_j^{-}(r))$$

$$+ (-\frac{d}{dr}\ \mu_j(r))\cdot\mu_j(r))\cdot\psi(r,\mu_j^{+}))\cdot P_n(r,\mu_j^{+}(r))$$

$$+ \int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\ \frac{\partial\psi(r,\mu)}{\partial r}\ P_n(r,\mu)d\mu + \int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\psi(r,\mu)\ \frac{\partial P_n(r,\mu)}{\partial r}\ d\mu$$

(eqn. continued next page)

$$= \frac{R_{j-1}^2}{r^3 \mu_{j-1}(r)} \cdot \mu_{j-1}(r) \cdot \psi(r, \mu_{j-1}(r)) - \frac{R_j^{\ 2}}{r^3 \mu_j(r)} \cdot \mu_j(r) \cdot \psi(r, \mu_j^{\ +}(r)) \cdot (-1)^n +$$

$$+ \int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu \frac{\partial \psi(r,\mu)}{\partial r} P_n(r,\mu) d\mu + \int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\psi(r,\mu) \frac{\partial P_n(r,\mu)}{\partial r} d\mu. \quad (29)$$

Then

$$\int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu \frac{\partial \psi(r,\mu)}{\partial r} P_n(r,\mu) d\mu = -\frac{R_{j-1}^2}{r^3} \psi(r, \mu_{j-1}^-(r)) + \frac{R_j^2}{r^3} \psi(r, \mu_j^{\ +}(r)) \cdot (-1)^n$$

$$- \int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\psi(r,\mu) \frac{\partial P_n(r,\mu)}{\partial r} d\mu + \frac{d}{dr} \int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\psi(r,\mu) P_n(r,\mu) d\mu \ . \quad (30)$$

Similarly

$$\int_{\mu_j(r)}^{\mu_{j-1}(r)} \frac{(1-\mu^2)}{r} P_n(r,\mu) d\mu = [\frac{(1-\mu_{j-1}^2(r))}{r}] \psi(r, \mu_{j-1}^-(r)) P_n(r, \mu_{j-1}^-(r))$$

$$-[\frac{1-(\mu_j(r))^2}{r}] \cdot \psi(r, \mu_j^{\ +}(r)) P_n(r, \mu_j^{\ +}(r)) + 2 \int_{\mu_j(r)}^{\mu_{j-1}(r)} \frac{\mu}{r} \psi(r,\mu) P_n(r,\mu) d\mu$$

$$- \int_{\mu_j(r)}^{\mu_{j-1}(r)} \frac{(1-\mu^2)}{r} \psi(r,\mu) \frac{\partial P_n(r,\mu)}{\partial \mu} d\mu$$

$$= \frac{R_{j-1}^2}{r^3} \psi(r, \mu_{j-1}^-(r)) - \frac{R_j^2}{r^3} \psi(r, \mu_j^{\ +}(r) \cdot (-1)^n$$

$$+ 2 \int_{\mu_j(r)}^{\mu_{j-1}(r)} \frac{\mu}{r} \psi(r,\mu) P_n(r,\mu) d\mu - \int_{\mu_j(r)}^{\mu_{j-1}(r)} \frac{(1-\mu^2)}{r} \psi(r,\mu) \frac{\partial P_n(r,\mu)}{\partial \mu} d\mu. \quad (31)$$

It can be seen that when equations 30 and 31 are added, the evaluated terms, i.e. the terms $(R_{j-1}^2/r^3)\psi(r,\mu_{j-1}^-(r))$, $(R_j^2/r^3)\psi(r,\mu_j^+(r))\cdot(-1)^n$, cancel. Therefore

$$\int_{\mu_j(r)}^{\mu_{j-1}(r)} P_n(r,\mu)d\mu]\{\mu \frac{\partial\psi(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\psi(r,\mu)}{\partial\mu} + \Sigma_\ell \Psi(r,\mu) =$$

$$\frac{C_\ell \Sigma_\ell}{2}\left(\sum_{j=1}^{\ell+1} \int_{\mu_j(r)}^{\mu_{j-1}(r)} \psi(r,\mu)d\mu\right)\}$$

gives

$$\frac{d}{dr}\int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\psi(r,\mu)P_n(r,\mu)d\mu \int_{\mu_j(r)}^{\mu_{j-1}(r)} \left(\mu\frac{\partial P_n(r,\mu)}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial P_n(r,\mu)}{\partial\mu}\right)$$

$$\psi(r,\mu)d\mu + 2\int_{\mu_j(r)}^{\mu_{j-1}(r)} \frac{\mu}{r}\psi(r,\mu)P_n(r,\mu)d\mu = \Sigma_\ell\psi_n(r)\ \frac{C_\ell\Sigma_\ell}{\alpha(r)}\sum_{k=1}^{\ell+1}\psi_0^k(r)\delta_{no}. \tag{32}$$

The superscript on $\psi_0^k(r)$ denotes the k'th cosine range $\mu_k(r) \le \mu \le \mu_{k-1}(r)$ and will be left off of the other moments for clarity. Using the results described in Appendix C, the n'th moments equations become

$$\frac{d}{dr}\left[-\frac{\beta(r)}{\alpha(r)}\psi_n(r) + \frac{n+1}{\alpha(r)(2n+1)}\psi_{n+1}(r) + \frac{n}{\alpha(r)(2n+1)}\psi_{n-1}(r)\right] +$$

$$+ \psi_{n-1}(r)\cdot\frac{n}{(2n+1)r\alpha(r)}\left[\frac{(n+1)}{\mu_{j-1}(r)\mu_j(r)} + 2\right] + \psi_n(r)\cdot\left[\Sigma_\ell - \frac{2\beta(r)}{r\alpha(r)}\right]$$

$$+ \psi_{n+1}(r)\cdot\frac{n+1}{(2n+1)r\alpha(r)}\left[2 - \frac{n}{\mu_{j-1}(r)\mu_j(r)}\right] = \frac{C_\ell\Sigma_\ell}{\alpha(r)}\sum_{k=1}^{\ell+1}\psi_0^k(r)\delta_{no}. \tag{33}$$

Truncation is performed by forcing $\psi_n(r) = 0, n > N$ for some N. Equations for N=0 and N=1 are as follows: r is expressed in mean free paths. For N=0,

$$\frac{d}{dr}\left[-\frac{\beta(r)}{\alpha(r)}\psi_0(r)\right] + \psi_0(r)\left[1 - \frac{2\beta(r)}{r\alpha(r)}\right] = \frac{C_\ell}{\alpha(r)}\sum_{k=1}^{\ell+1}\psi_0^k(r) \qquad (34)$$

and for N=1, n=0,

$$\frac{d}{dr}\left[-\frac{\beta(r)}{\alpha(r)}\psi_0(r) + \frac{1}{\alpha(r)}\psi_1(r)\right] + \psi_0(r)\left[1 - \frac{2\beta(r)}{r\alpha(r)}\right] + \frac{2\psi_1(r)}{\alpha(r)r} =$$

$$\frac{C_\ell}{\alpha(r)}\sum_{k=1}^{\ell+1}\psi_0^k(r) \quad , \qquad (35)$$

n=1,

$$\frac{d}{dr}\left[-\frac{\beta(r)}{\alpha(r)}\psi_1(r) + \frac{1}{3\alpha(r)}\psi_0(r)\right] + \psi_0(r)\frac{1}{3r\alpha(r)}\left[\frac{2}{\mu_{j-1}(r)\mu_j(r)} + 2\right]$$

$$+ \psi_1(r)\left[1 - \frac{2\beta(r)}{r\alpha(r)}\right] = 0. \qquad (36)$$

## Solution of Zero'th Order Equations

Solutions to the equations for N=0 will be obtained for the case of the black sphere i.e. the division of the cosine range into $-1 < \mu < \mu_0(r)$ and $\mu_0(r) < \mu < +1$. The functions $\alpha(r)$ and $\beta(r)$ needed for the linear transformation of $\mu$ are obtained as follows. On $\mu_0(r) < \mu < +1$, let

$$\alpha(r) \equiv \alpha^+(r) = \frac{2}{1-\mu_0(r)}, \tag{37}$$

$$\beta(r) \equiv \beta^+(r) = -\frac{(1+\mu_0(r))}{(1-\mu_0(r))} \tag{38}$$

and

$$\psi_0(r) \equiv \psi_0^+(r) ; \tag{39}$$

on $-1 < \mu < \mu_0(r)$,

$$\alpha(r) \equiv \alpha^-(r) = \frac{2}{\mu_0(r)+1}, \tag{40}$$

$$\beta(r) \equiv \beta^-(r) = -\frac{(\mu_0(r)-1)}{(\mu_0(r)+1)} \tag{41}$$

and

$$\psi_0(r) \equiv \psi_0^-(r) . \tag{42}$$

From the form of the N=0 moments equations on the previous page, it seems natural to solve for the function

$$g(r) = -\frac{\beta(r)}{\alpha(r)}\psi_0(r) . \tag{43}$$

Therefore, let

$$g^+(r) = -\frac{\beta^+(r)}{\alpha^+(r)}\,\psi_0^+(r) = \frac{1}{\alpha^-(r)}\,\psi_0^+(r) = \frac{(1+\mu_0(r))}{2}\,\psi_0^+(r) \qquad (44)$$

and let

$$g^-(r) = -\frac{\beta^-(r)}{\alpha^-(r)}\,\psi_0^-(r) = -\frac{1}{\alpha^+(r)}\,\psi_0^-(r) = -\frac{(1-\mu_0(r))}{2}\,\psi_0^-(r)\ . \qquad (45)$$

The equations for $g^\pm(r)$ are

$$\frac{d}{dr}\,g^-(r) = g^-(r)\,(1 - \frac{2}{r} + g^+(r)\ , \qquad (46)$$

$$\frac{d}{dr}\,g^+(r) = g^+(r)\,(-1 - \frac{2}{r}) - g^-(r) \qquad (47)$$

$$\Rightarrow \frac{d}{dr}\,[g^+(r)+g^-(r)] = -\frac{2}{r}\,[g^+(r)+g^-(r)] \qquad (48)$$

$$\Rightarrow g^+(r) = -g^-(r) + \frac{C_1}{r^2}\ . \qquad (49)$$

The general solution to these equations is

$$g^-(r) = \frac{C_1}{r} + \frac{C_2}{r^2}\ , \qquad (50)$$

$$g^+(r) = \frac{(C_1-C_2)}{r^2} - \frac{C_1}{r} \qquad (51)$$

and so

$$\psi_0^+(r) = \frac{2}{1+\mu_0(r)}\,(\frac{(C_1-C_2)}{r^2} - \frac{C_1}{r})\ , \qquad (52)$$

$$\psi_0^-(r) = \frac{2}{\mu_0(r)-1}\,(\frac{C_1}{r} + \frac{C_2}{r^2}) \qquad (53)$$

$$\Rightarrow \psi(r) = \psi_0^+(r)+\psi_0^-(r) = \frac{2}{1+\mu_0(r)}\,(C_1(\frac{1}{r^2}-\frac{1}{r})-C_2) - \frac{2}{1-\mu_0(r)}\,(\frac{C_1}{r}+\frac{C_2}{r^2})\ . \qquad (54)$$

Since $\frac{1}{1-\mu_0(r)} \sim \frac{2r^2}{a^2}$, $r \to \infty$, where a is the radius of the black sphere, $\psi(r) \sim Ar+B$ near $r = \infty$. This solution is denoted $\psi_T^G$ in the Tables.

For comparison with these solutions, equations similar to the above from Ref. 3 are solved. Explicit description of the discontinuity of $\psi(r,\mu)$ in $\mu$ at $\mu_0(r)$ through the use of a delta function expanded in the functions $P_m(r,\mu)$ was used in this reference; the equivalence of equations derived in Ref. 3 without delta function terms to equations derived in this report is shown in Appendix D. Addition of these delta-function terms involves adding $\frac{1}{r}\,(g^+(r)+g^-(r))$ to the left hand sides of the above set of equations as was shown in Appendix D. The equations to be solved are then

$$\frac{d}{dr}\, g^-(r) = g^-(r)(1 - \frac{3}{r}) + g^+(r)(1 - \frac{1}{r}) \tag{55}$$

$$\frac{d}{dr}\, g^+(r) = g^+(r)(-1 - \frac{3}{r}) - g^-(r)(1 + \frac{1}{r}) \tag{56}$$

$$\Rightarrow \frac{d}{dr}\,[g^+(r) + g^-(r)] = -\frac{4}{r}\,[g^+(r) + g^-(r)] \tag{57}$$

with general solutions

$$g^-(r) = \frac{C_1}{2r^4} - \frac{C_1}{r^3} + \frac{C_2}{r^2}\,, \tag{58}$$

$$g^+(r) = \frac{C_1}{2r^4} + \frac{C_1}{r^3} - \frac{C_2}{r^2}\,. \tag{59}$$

Then

$$\psi_0^-(r) = \frac{2}{\mu_0(r)-1}\,(\frac{C_1}{2r^4} - \frac{C_1}{r^3} + \frac{C_2}{r^2})\,, \tag{60}$$

$$\psi_0^+(r) = \frac{2}{1+\mu_0(r)} \left(\frac{C_1}{2r^4} + \frac{C_1}{r^3} - \frac{C_2}{r^2}\right) \tag{61}$$

and

$$\psi(r)=\psi_0^-(r)+\psi_0^+(r)=\frac{2}{\mu_0(r)+1} \left(\frac{C_1}{2r^4}+\frac{C_1}{r^3}-\frac{C_2}{r^2}\right) + \frac{2}{\mu_0(r)-1} \left(\frac{C_1}{2r^4} - \frac{C_1}{r^3} + \frac{C_2}{r^2}\right). \tag{62}$$

$$\psi(r) \sim A+\frac{B}{r}+\frac{C}{r^2} \quad , \quad r \to \infty .$$

This solution is denoted $\psi_H^G$ in the Tables.

With $\mu_0(r) \equiv 0$, the zero order equations with delta function terms included are

$$\frac{d}{dr}\,\psi_0^-(r) = (1 - \frac{1}{r})\psi_0^-(r) + (\frac{1}{r} - 1)\psi_0^+(r) \tag{63}$$

$$\frac{d}{dr}\,\psi_0^+(r) = -(\frac{1}{r} + 1)\psi_0^+(r) + (\frac{1}{r} + 1)\psi_0^-(r) \;, \tag{64}$$

$$\Rightarrow \frac{d}{dr}\,[\psi_0^+(r) - \psi_0^-(r)] = -\frac{2}{r}\,[\psi_0^+(r) - \psi_0^-(r)] \tag{65}$$

with general solutions

$$\psi_0^-(r) = \frac{C_1}{r} - \frac{C_1}{2r^2} + C_2 \;, \tag{66}$$

$$\psi_0^+(r) = \frac{C_1}{r} + \frac{C_1}{2r^2} + C_2 \tag{67}$$

and

$$\psi(r) = 2\,\frac{C_1}{r} + 2\,C_2 \;. \tag{68}$$

Note the similarity to the diffusion solution. This solution is denoted $\psi_H^S$ in the Tables.

The following section will discuss boundary conditions to fix the constants $C_1$ and $C_2$.

Boundary Conditions and Discussion of Results

In the calculations, the boundary condition

$$\psi_0^+(a) \equiv \int_0^1 \psi(a,\mu)d\mu = 0 \tag{69}$$

was used at the surface of the black sphere. The other boundary condition was prescribed in two ways: (i) $\psi(r)$ at $r = \infty$ was set equal to the exact diffusion flux calculated there[3)], the total diffusion current being normalized to $\frac{4\pi}{3\Sigma}$ at $\infty$; (ii) $\psi(r)$ was set equal to one at the surface of the sphere (since the 0'th order equations derived in this report gave fluxes that behave like $A\ r + B$ when r approaches infinity (see footnote, page ), solutions could only admit boundary condition (ii). An alternative is to set $A = 0$. The condition $\psi_0^+(a) = 0$ cannot then be used; however, the physically meaningful solution of a flux that is everywhere constant in the moderator is then obtained. Tables 1-3 give solutions using boundary condition (i) and Tables 4-6 give solution values for boundary condition (ii). Differences present at the surface of the sphere in Tables 1-3 will be the differences obtained at $r = \infty$ in Tables 4-6 so the two problems are essentially equivalent. Results for both boundary conditions are listed to show the behaviour of the flux bear the surface of the sphere (and hence near $\infty$).

The condition $\psi(a) = 1$ means that the angular distribution of the flux at the surface of the black sphere is

$$\psi(a,\mu) = \begin{cases} 0; & \mu > 0 \\ 1; & \mu < 0 \end{cases} . \tag{70}$$

The current at the surface is then

$$J_S = \int_{-1}^{0} \mu d\mu = -\frac{1}{2}^{*} \tag{71}$$

and the number of neutrons entering the sphere per second is

$$4\pi a^2 J_S = -2\pi a^2 . \tag{72}$$

Therefore,both boundary conditions used above are just different normalizations of the number of neutrons entering the sphere per second.

It was mentioned earlier that the zero'th moments as derived in this report behaved asymptotically like A + B r near $\infty$. When the approximation

$$\psi(r,\mu) = \begin{cases} \psi_0^+(r) & ; \mu > 0 \\ \psi_0^-(r) & ; \mu < 0 \end{cases} \tag{73}$$

for all r is used, Eq. 19 in the lowest order approximation gives

$$\frac{\partial \psi_0^-(r)}{\partial r} = \psi_0^-(r) - \psi_0^+(r) \tag{74}$$

and

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*The minus sign occurs since the gradient of the flux points in an outward radial direction.

$$\frac{\partial \psi_0^+(r)}{\partial r} = \psi_0^-(r) - \psi_0^+(r) \ . \tag{75}$$

The solution is

$$\psi(r) = \psi_0^+(r) + \psi_0^-(r) = A + Br. \tag{76}$$

Thus, the behaviour near infinity of the zero'th moments in an expansion with $\mu_0(r) \equiv 0$ is the same as that found for $\mu_0(r)$ not identically zero in zero order expansion.

## Conclusion

The zero order expansion for the angular flux derived in this report, along with the application of the boundary condition $\int_0^1 \psi(a,\mu)\,d\mu = 0$ at the black sphere surface has lead to undesirable asymptotic behaviour of the flux near infinity in the zero'th order approximation. The equations derived in Ref. 3 give better behaviour because the discontinuity in the angular flux density was explicitly accounted for in those equations. In solving systems of higher order, it is apparent that functions of the form $g_n(r) = f(\alpha(r), \beta(r))\psi_n(r)$ will be used. It is felt by the author that although explicit addition of the delta function terms gives better behaviour of the flux near infinity and at the surface of the black sphere in the lowest order, the modified equations should be studied on their own in order to determine convergence of the expansion and overall behaviour of the moments. The black sphere provides the idealization necessary, at the moment, to deal with boundary conditions.

## Appendix A

### Properties of $P_n(r,\mu)$

On the open set $\mu_j(r) < \mu < \mu_{j-1}(r)$, $P_n(r,\mu)$ is defined by $P_n(r,\mu) = P_n(\alpha(r)\mu + \beta(r))$ where $\alpha(r)$, $\beta(r)$ are chosen so that $P_n(\alpha(r)\mu_j(r) + \beta(r)) = -1$, $P_n(\alpha(r)\mu_{j-1}(r) + \beta(r)) = +1$. Then, all of the properties of the Legendre polynomials carry over under this linear transformation in $\mu$. Orthogonality follows by

$$\int_{\mu_j(r)}^{\mu_{j-1}(r)} P_m(r,\mu)P_n(r,\mu)d\mu = \frac{1}{\alpha(r)} \int_{-1}^{+1} P_m(\mu)P_n(\mu)d\mu$$

$$= \frac{1}{\alpha(r)} \frac{2}{2n+1} \delta_{mn}$$

$$= \frac{(\mu_{j-1}(r)-\mu_j(r))}{2n+1} \delta_{mn} .$$

A recurrence relation that will be used is the following: since

$$\mu P_n(\mu) = \frac{n+1}{2n+1} P_{n+1}(\mu) + \frac{n}{2n+1} P_{n-1}(\mu),$$

$$(\alpha(r)\mu+\beta(r))P_n(r,\mu) = \frac{n+1}{2n+1} P_{n+1}(r,\mu) + \frac{n}{2n+1} P_{n-1}(r,\mu) ,$$

so

$$\mu P_n(r,\mu) = -\frac{\beta(r)}{\alpha(r)} P_n(r,\mu) + \frac{n+1}{\alpha(r)(2n+1)} P_{n+1}(r,\mu) + \frac{n}{\alpha(r)(2n+1)} P_{n-1}(r,\mu) . \tag{A-1}$$

# Appendix B

## Derivatives of various functions

For $R_j < r < R_{j+1}$, $0 \leq j \leq L$, with $R_0$, $R_{L+1} = \infty$

$$\mu_j(r) = \sqrt{1 - \left(\frac{R_j}{r}\right)^2}$$

and

$$\alpha(r) = \frac{2}{\mu_{j-1}(r) - \mu_j(r)} , \qquad \beta(r) = -\frac{(\mu_{j-1}(r) + \mu_j(r))}{(\mu_{j-1}(r) - \mu_j(r))} .$$

Then

$$\frac{\partial \mu_j(r)}{\partial r} = \frac{1}{\mu_j(r)} \cdot \frac{R_j^2}{r^3} = \frac{(1 - \mu_j^2(r))}{r \mu_j(r)} , \tag{B-1}$$

$$\frac{\partial \alpha(r)}{\partial r} = - \frac{-2}{(\mu_{j-1}(r) - \mu_j(r))^2} \left[\frac{1}{r^3} \left(\frac{R_{j-1}^2}{\mu_{j-1}(r)} - \frac{R_j^2}{\mu_j(r)}\right)\right]$$

$$= \frac{\alpha(r)}{r} \left(1 + \frac{1}{\mu_{j-1}(r)\mu_j(r)}\right) , \tag{B-2}$$

$$\frac{\partial \beta(r)}{\partial r} = \frac{2\beta(r)}{r \mu_{j-1}(r) \mu_j(r)} , \tag{B-3}$$

and

$$\frac{\partial}{\partial r} \frac{\beta(r)}{\alpha(r)} = \frac{1}{r} \frac{\beta(r)}{\alpha(r)} \left[\frac{1}{\mu_{j-1}(r)\mu_j(r)} - 1\right] . \tag{B-4}$$

## Appendix C

### Integrals of some Functions

Using Eq. (A-1),

$$\frac{d}{dr}\int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\psi(r,\mu)P_n(r,\mu)\,d\mu = \frac{d}{dr}\int_{\mu_j(r)}^{\mu_{j-1}(r)} \psi(r,\mu)\cdot[-\frac{\beta(r)}{\alpha(r)} P_n(r,\mu) +$$

$$+ \frac{n+1}{\alpha(r)(2n+1)} P_{n+1}(r,\mu) + \frac{n}{\alpha(r)(2n+1)} P_{n-1}(r,\mu)]d\mu$$

$$= \frac{d}{dr}\,[-\frac{\beta(r)}{\alpha(r)}\,\psi_n(r) + \frac{n+1}{\alpha(r)(2n+1)}\,\psi_{n+1}(r) + \frac{n}{\alpha(r)(2n+1)}\,\psi_{n-1}(r)]. \tag{C-1}$$

Let $x(r,\mu) = \alpha(r)\mu + \beta(r)$. This gives

$$\frac{\partial}{\partial\mu} = \alpha(r)\,\frac{\partial}{\partial x}$$

and

$$\frac{\partial}{\partial r} = (\mu\,\frac{\partial\alpha(r)}{\partial r} + \frac{\partial\beta(r)}{\partial r})\,\frac{\partial}{\partial x}\,.$$

Then

$$\int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\psi(r,\mu)\,\frac{\partial P_n(r,\mu)}{\partial r}\,d\mu + \int_{\mu_j(r)}^{\mu_{j-1}(r)} \frac{(1-\mu^2)}{r}\,\psi(r,\mu)\,\frac{\partial P_n(r,\mu)}{\partial\mu}\,d\mu$$

$$= \int_{\mu_j(r)}^{\mu_{j-1}(r)} \psi(r,\mu)\,\frac{\partial P_n(r,\mu)}{\partial x}\,[\mu^2\,(\frac{\partial\alpha(r)}{\partial r} - \frac{\alpha(r)}{r}) + \mu\,\frac{\partial\beta(r)}{\partial r} + \frac{\alpha(r)}{r}]d\mu.$$

Using Eq. (B-2) and Eq. (B-13),

$$[\mu^2 \left(\frac{\partial\alpha(r)}{\partial r} - \frac{\alpha(r)}{r}\right) + \mu \quad \frac{\partial\beta(r)}{\partial r} + \frac{\alpha(r)}{r}] = \frac{-1}{r\alpha(r)\mu_{j-1}(r)\mu_j(r)} \;(1-x^2(r,\mu)),$$

and using the recurrence relation

$$(1-x^2)\;\frac{dP_n(x)}{dx} = \frac{n(n+1)}{2n+1}\;(P_{n-1}(x)-P_{n+1}(x)),$$

the above is

$$\frac{-1}{r\alpha(r)\mu_{j-1}(r)\mu_j(r)} \cdot \frac{n(n+1)}{2n+1} \cdot (\psi_{n-1}(r)-\psi_{n+1}(r)). \qquad \text{(C-2)}$$

Other integrals used in this report are

$$\Sigma_\ell \int_{\mu_j(r)}^{\mu_{j-1}(r)} \psi(r,\mu)P_n(r,\mu)d\mu = \Sigma_\ell\psi_n(r) \;, \qquad \text{(C-3)}$$

$$\frac{C_\ell\Sigma_\ell}{2} \int_{\mu_j(r)}^{\mu_{j-1}(r)} P_n(r,\mu') \sum_{k=1}^{\ell+1} \int_{\mu_k(r)}^{\mu_{k-1}(r)} \psi(r,\mu)d\mu d\mu'$$

$$= \frac{C_\ell\Sigma_\ell}{2} \int_{\mu_j(r)}^{\mu_{j-1}(r)} P_n(r,\mu') \sum_{k=1}^{\ell+1} \psi_0^k(r)d\mu'$$

$$= \frac{C_\ell\Sigma_\ell}{\alpha(r)} \sum_{k=1}^{\ell+1} \psi_0^k(r)\delta_{no} \;, \qquad \text{(C-4)}$$

and finally

$$\int_{\mu_j(r)}^{\mu_{j-1}(r)} \mu\psi(r,\mu)P_n(r,\mu)d\mu$$

$$= \int_{\mu_j(r)}^{\mu_{j-1}(r)} \psi(r,\mu)\cdot[-\frac{\beta(r)}{\alpha(r)} P_n(r,\mu) + \frac{n+1}{\alpha(r)(2n+1)} P_{n+1}(r,\mu) + \frac{n}{\alpha(r)(2n+1)} P_{n-1}(r,\mu)]d\mu$$

$$= -\frac{\beta(r)}{\alpha(r)} \psi_n(r) + \frac{n+1}{\alpha(r)(2n+1)} \psi_{n+1}(r) + \frac{n}{\alpha(r)(2n+1)} \psi_{n-1}(r) \qquad \text{(C-5)}$$

## Appendix D

### Reduction of 0'th Order Equations in Ref. 2

The 0'th order equations in the paper by Harms and Attia, Ref. 2, can be reduced to the equations derived in this report by omitting the terms deriving from the delta function. This reduction proceeds as follows. The delta function terms on the cosine range $-1 < \mu < \mu_0(r)$ are

$$\frac{(1-\mu_0^2(r))}{r} \sum_{\ell=0}^{\infty} (2\ell+1)\left[\frac{(-1)^{\ell}}{(1-\mu_0(r))}\,\phi_\ell^+(r) - \frac{1}{(1+\mu_0(r))}\,\phi_\ell^-(r)\right].$$

A similar term exists for the other range of $\mu$. For zero'th moment terms, the above expression is

$$\frac{(1+\mu_0(r))}{r}\,\phi_0^+(r) - \frac{(1-\mu_0(r))}{r}\,\phi_0^-(r) ,$$

and this expression also holds for the other cosine ranges. With $c = 1$, $f_1 = 0$, using the notation of Ref. 2,

$$A_{10} = \frac{(1-\mu_0(r))^2}{r\mu_0(r)} + (1-\mu_0(r)) ,$$

$$B_{10} = -(1+\mu_0(r)),$$

$$A_{20} = -\ (1-\mu_0(r))$$

and

$$B_{20} = \frac{(1+\mu_0(r))^2}{r\mu_0(r)} + (1+\mu_0(r)).$$

Equations for the zero'th moments derived in this report are

$$\frac{d}{dr}\left[-\frac{\beta(r)}{\alpha(r)}\,\psi_0(r)\right]-\frac{2}{r}\,\frac{\beta(r)}{\alpha(r)}\,\psi_0(r)+\psi_0(r)=\frac{1}{\alpha(r)}\,[\psi_0^+(r)+\psi_0^-(r)].$$

On $-1 < \mu < \mu_0(r)$,

$$\frac{d}{dr}\left[\frac{(\mu_0(r)-1)}{2}\,\psi_0^-(r)\right]\frac{(\mu_0(r)-1)}{r}\,\psi_0^-(r)+\psi_0^-(r)=\frac{(\mu_0(r)+1)}{2}\,[\psi_0^+(r)+\psi_0^-(r)],$$

$$\Rightarrow \frac{d}{dr}\,\psi_0^-(r)=\frac{1}{(1-\mu_0(r))}\left\{\psi_0^-(r)\cdot\left[\frac{(1-\mu_0(r))^2}{r\mu_0(r)}+(1-\mu_0(r))\right]+\psi_0^+(r)\,[-(\mu_0(r)+1]\right\}$$

On $\mu_0(r) < \mu < +1$,

$$\frac{d}{dr}\left[\frac{(1+\mu_0(r))}{2}\,\psi_0^+(r)\right]+\frac{(1-\mu_0(r))}{r}\,\psi_0^+(r)+\psi_0^+(r)=\frac{(1-\mu_0(r))}{2}\,[\psi_0^+(r)+\psi_0^-(r)],$$

$$\Rightarrow (1+\mu_0(r))\frac{d}{dr}\,\psi_0^+(r)=-\psi_0^+(r)\left[\frac{(1+\mu_0(r))^2}{r\mu_0(r)}+(1+\mu_0(r))\right]-\psi_0^-(r)\,(-(1-\mu_0(r))).$$

These agree with the Harms-Attia equations listed in Ref. 3.

| s | $\Psi_{Trans}$ | $\Psi_{Diff}$ | $\Psi_H^G$ | $\Psi_H^S$ | %ch($\Psi_{Trans} \rightarrow \Psi_H^G$) |
|---|---|---|---|---|---|
| 0. | 17.725 | 31.360 | 25.761 | 25.761 | +45.3 |
| .1 | 30.010 | 33.027 | 37.729 | 29.197 | +25.7 |
| .2 | 32.475 | 33.850 | 36.490 | 30.913 | +12.4 |
| .3 | 33.620 | 34.360 | 35.721 | 31.944 | + 6.2 |
| .4 | 34.250 | 34.683 | 35.329 | 32.630 | + 3.2 |
| .5 | 34.643 | 34.929 | 35.136 | 33.121 | + 1.4 |
| .6 | 34.913 | 35.113 | 35.048 | 33.489 | + 0.4 |
| .8 | 35.260 | 35.360 | 35.014 | 34.004 | - 0.7 |

Table 1 0'th moment solutions for a $\varepsilon = 0.2$ . 's' is the distance from the surface of the sphere.

| s | $\Psi_{\text{Trans}}$ | $\Psi_{\text{Diff}}$ | $\Psi_H^G$ | $\Psi_H^S$ | %ch($\Psi_{\text{Trans}} \longrightarrow \Psi_H^G$) |
|---|---|---|---|---|---|
| 0. | 0.653 | 1.043 | 0.678 | 0.678 | +3.8 |
| .1 | 0.963 | 1.134 | 1.034 | 0.801 | +7.4 |
| .2 | 1.101 | 1.210 | 1.164 | 0.903 | +5.7 |
| .3 | 1.199 | 1.274 | 1.247 | 0.991 | +4.0 |
| .4 | 1.275 | 1.329 | 1.307 | 1.065 | +2.5 |
| .5 | 1.337 | 1.377 | 1.354 | 1.129 | +1.3 |
| .6 | 1.388 | 1.418 | 1.393 | 1.186 | +0.4 |
| .8 | 1.470 | 1.488 | 1.454 | 1.280 | -1.1 |
| 1.0 | 1.532 | 1.543 | 1.502 | 1.356 | -1.9 |

Table 2 0'th moment solutions for a $\xi_1$ = 1.0 . 's' is the distance from the surface of the sphere.

| s | $\Psi_{Trans}$ | $\Psi_{Diff}$ | $\Psi_H^G$ | $\Psi_H^S$ | %ch($\Psi_{Trans} \rightarrow \Psi_H^G$) |
|---|---|---|---|---|---|
| 0. | 0.158 | 0.231 | 0.146 | 0.146 | -7.6 |
| .1 | 0.219 | 0.255 | 0.214 | 0.174 | -2.3 |
| .2 | 0.252 | 0.277 | 0.250 | 0.200 | -0.8 |
| .3 | 0.279 | 0.296 | 0.277 | 0.223 | -0.7 |
| .4 | 0.301 | 0.315 | 0.300 | 0.244 | -0.3 |
| .5 | 0.321 | 0.331 | 0.319 | 0.263 | -0.6 |
| .6 | 0.339 | 0.347 | 0.336 | 0.281 | -0.9 |
| .8 | 0.369 | 0.374 | 0.365 | 0.314 | -1.1 |
| 1.0 | 0.395 | 0.398 | 0.390 | 0.341 | -1.3 |

Table 3 0'th moment solutions for a $\xi_1 = 2.0$ . 's' is the distance from the surface of the sphere.

| S | $\Psi_{Trans}$ | $\Psi_{Diff}$ | $\Psi^{G}_{T}$ | $\Psi^{G}_{H}$ | $\Psi^{S}_{H}$ | %ch($\Psi_{Trans} \rightarrow \Psi^{\varepsilon G}_{H}$) |
|---|---|---|---|---|---|---|
| 0. | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | |
| .1 | 1.693 | 1.053 | 1.945 | 1.465 | 1.133 | -13.5 |
| .2 | 1.832 | 1.079 | 2.266 | 1.417 | 1.200 | -22.7 |
| .3 | 1.897 | 1.096 | 2.517 | 1.387 | 1.240 | -26.9 |
| .4 | 1.932 | 1.106 | 2.743 | 1.371 | 1.267 | -29.0 |
| .5 | 1.954 | 1.114 | 2.958 | 1.364 | 1.286 | -30.2 |
| .6 | 1.970 | 1.120 | 3.168 | 1.361 | 1.300 | -30.9 |
| .8 | 1.989 | 1.128 | 3.580 | 1.359 | 1.320 | -31.7 |
| $\infty$ | 2.053 | 1.150 | $\infty$ | 1.400 | 1.400 | -31.2 |

Table 4 0'th moment solutions for a $\Sigma_t = 0.2$ . 's' is the distance from the surface of the sphere.

| s | $\Psi_{Trans}$ | $\Psi_{Diff}$ | $\Psi^{G}_{T}$ | $\Psi^{G}_{H}$ | $\Psi^{S}_{H}$ | %ch($\Psi_{Trans} \rightarrow \Psi^{G}_{H}$) |
|---|---|---|---|---|---|---|
| 0. | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | |
| .1 | 1.475 | 1.170 | 1.617 | 1.526 | 1.182 | +3.5 |
| .2 | 1.687 | 1.311 | 1.953 | 1.717 | 1.333 | +1.8 |
| .3 | 1.837 | 1.417 | 2.239 | 1.840 | 1.462 | +0.2 |
| .4 | 1.953 | 1.533 | 2.500 | 1.928 | 1.571 | -1.3 |
| .5 | 2.048 | 1.662 | 2.745 | 1.998 | 1.667 | -2.4 |
| .6 | 2.127 | 1.699 | 2.981 | 2.055 | 1.750 | -3.4 |
| .8 | 2.252 | 1.829 | 3.431 | 2.126 | 1.889 | -4.7 |
| 1.0 | 2.347 | 1.933 | 3.866 | 2.217 | 2.000 | -5.5 |
| $\infty$ | 3.113 | 1.949 | $\infty$ | 3.000 | 3.000 | -3.6 |

Table 5 0'th moment solutions for a $\Sigma_1$ = 1.0 . 's' is the distance from the surface of the sphere.

| s | $\Psi_{Trans}$ | $\Psi_{Diff}$ | $\Psi^{G}_{T}$ | $\Psi^{G}_{H}$ | $\Psi^{S}_{H}$ | %ch($\Psi_{Trans} \rightarrow \Psi^{G}_{h}$) |
|---|---|---|---|---|---|---|
| 0. | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | |
| .1 | 1.386 | 1.103 | 1.505 | 1.467 | 1.191 | +5.8 |
| .2 | 1.594 | 1.197 | 1.817 | 1.708 | 1.364 | +7.2 |
| .3 | 1.762 | 1.282 | 2.094 | 1.895 | 1.522 | +7.5 |
| .4 | 1.904 | 1.361 | 2.353 | 2.051 | 1.667 | +7.7 |
| .5 | 2.029 | 1.433 | 2.600 | 2.184 | 1.800 | +7.6 |
| .6 | 2.140 | 1.499 | 2.839 | 2.301 | 1.923 | +7.5 |
| .8 | 2.332 | 1.618 | 3.300 | 2.500 | 2.143 | +7.2 |
| 1.0 | 2.493 | 1.721 | 3.745 | 2.665 | 2.333 | +6.9 |
| $\infty$ | 4.618 | 3.162 | $\infty$ | 5.000 | 5.000 | +8.3 |

Table 6 0'th moment solutions for a $\xi$ = 2.0 . 's' is the distance from the surface of the sphere.

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