DIRECT VARIATIONAL METHOD OF ANALYSIS FOR ELLIPTIC PARABOLOIDAL SHELLS OF TRANSLATION

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SCOPE OF CONTENTS:

The Rayleigh-Ritz Method of Trial Function has been adopted to solve problems of translational shells under uniform external pressure. The basic energetical expressions have been written in terms of direct tensor notation. The stress-strain displacement relations are given in linear sense. Three different kinds of boundary conditions --- all four edges fixed, one pair of edges fixed and another pair of edges simply supported, and all four edges simply supported --- have been analysed. The stress and moment resultants at different points of the shell have been computed by means of IBM 7040, and are plotted into curves and figures to show their variations. The convergence of the displacement function u_z is roughly verified. Certain comparison with other established results have been made. The results obtained by the present approach are satisfactory, at least from an engineering point of view.

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NOTATION

A	Surface Integral
A_1, A_2	Scale Factors
A_{mn}	Fourier Coefficient of Vertical Displacement
C_1, C_2, C_3	Arbitrary Constants
D, D'	As Defined in Section II-2
E	Young's Modulus
$F_{ij}^{(\sigma)}$	Stress Resultants of the Cross-Section Normal to i-Axis Acting in j-Direction, (i, j = x, y, z)
H	Rise of the Middle Surface of a Shell
$K_1^{(n)}, K_2^{(n)}, K_x^{(n)}, K_y^{(n)}$	Normal Curvatures Along Corresponding Directions
$K_1^{(g)}, K_2^{(g)}$	Geodesic Curvatures
$\delta K^{(g)}$	Change of Geodesic Torsion
$\delta K_1^{(n)}, \delta K_2^{(n)}$	Change of Curvatures
$M_{xy}^{(\sigma)}, M_{yx}^{(\sigma)}$	Bending Stress Couples
$M_{xx}^{(\sigma)}, M_{yy}^{(\sigma)}$	Twisting Stress Couples
P	Uniform Vertical External Load
R	Radius of Curvature of the Middle Surface
S	Line Integral
$U^{(S)}$	Strain Energy, or Elastic Potential
V	Total Potential Energy
W	External Work
a, b	Lengths of Edges of the Middle Surface Measured in its Base Plane
$\bar{e}_x, \bar{e}_y, \bar{e}_z$	Unit Vectors along x,y,z Directions
h	Thickness of the Shell

m,n,p,q,r,s	Indices of the Fourier Coefficient
$\bar{r}$	Position Vector
u_1, u_2	Membranal Displacements along Two Curvilinear Co-ordinates α_1, α_2
u_3	Normal Displacement
u_x, u_y	Membranal Displacements along Two Cartesian Co-ordinates x,y
u_z	Vertical (Transverse) Displacement
x,y,z	Cartesian Co-ordinates
α_1, α_2	Curvilinear Co-ordinates on the Middle Surface
$\epsilon_{11}, \epsilon_{12}, \epsilon_{22}, \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$	Strains along Corresponding Directions
ν	Poisson's Ratio
∇^2	Laplace Operator
μ	Lamé's Elastic Constant

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CHAPTER 1

INTRODUCTION

By definition, a shell is a three dimensional body bounded by two curved surfaces whose one dimension is negligibly small compared with the other two dimensions. (See Fig. I-1) The surface which lies at equal distances between these two bounding external surfaces defines the middle surface of the shell.

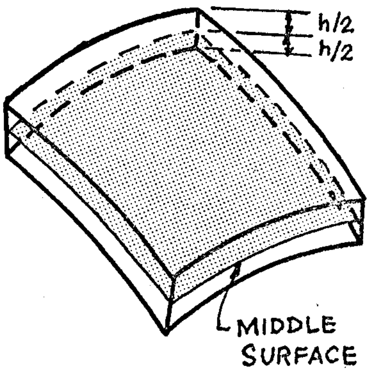

FIG I-1

By definition, a shallow shell has a configuration that its rise of its middle surface H from the base plane is less than 1/5 of the projected length of the shortest edge of the middle surface measured in its base plane. If the shell in Fig. I-2 is shallow, then its geometric configuration satisfies the condition $H<b/5$, for $b<a$.

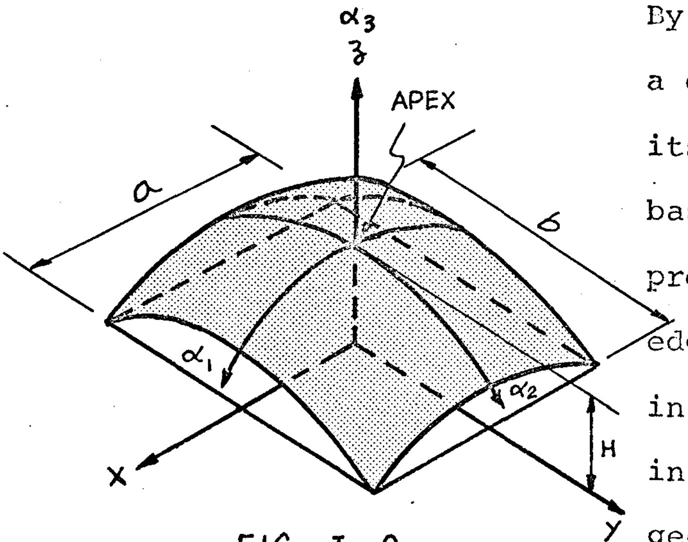

FIG. I-2

A shell of translation is a shell whose middle surface is generated by a curve translated along another fixed curve as shown in Fig. I-3. Therefore, a thin shallow translational shell is a shell which satisfies all the criteria mentioned above.

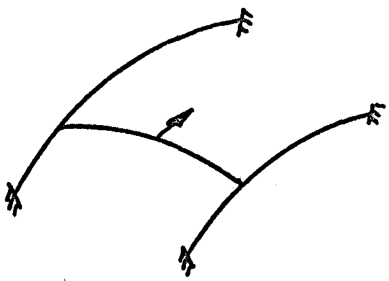

FIG. I-3

The functions chosen to describe the middle surface of thin shallow shells of translation are generally of the hyperbolic paraboloidal, elliptic paraboloidal or parabolic cylindrical type. The present investigation is restricted to the elliptic paraboloidal type of thin translational shell.

The purpose of this study, which was carried out in 1964 is to present a simple yet practical method for analyzing the transverse flexure of thin shallow translational shell structures of moderate proportions. The major purpose of this thesis lies in an attempt to develop some general solutions for the behaviour of the translational shell in transverse bending by means of variational trial-function technique in the form of the Rayleigh-Ritz method as expounded in COURANT texts in 1953 and 1965, by MORSE and FESBACH in 1953 and KOPAL in 1961. The exact general solution of the basic differential equations of such translational shells is still to be found. For general techniques of solution of differential equations see KOSHLYAKOV, SMIRNOV and GLINER in 1964. The major reason for the absence of a general exact solution lies presumably in the exceptional complexity of the basic differential equations and in the requirements imposed by the general boundary conditions.

For special types of boundary conditions, Sergei A. AMBARTSUMYAN in 1947, Konrad HRUBAN in 1953, and Wilhelm FLÜGGE and D.A. Conrad in 1959, have derived restricted solutions. In 1957, Gunhard AE. ORAVAS derived a solution for a special type of such shells by using the DONNEL-MUSHTARI-VLASOV's equation of shallow thin shells, in combination with

Friedrich TÖLKE's pseudo-complex function method in order to arrive at the solution in the form of the stress function and normal displacement function representing a combined series of exponential and trigonometric functions. The Norwegian engineer Kristoffal APELAND developed a generalized D. BERNOULLI-LEVY semidirect solution for translational shells with various boundary conditions in 1961 by adopting AMBARTSUMYAN'S method. Some analyses and tests for translational shells over circular and rectangular bases have been carried out by Mario G. SALVADORI in 1956. In Europe, Pal CSONKA in 1955; Wolfgang ZERNA in 1953; and his student Goswin MITTELMANN in 1958, contributed to the approximate momentless and flexural theory of thin translational shells. In 1959, the Italian engineer Pietro MATILDI analyzed shells of translation over rectangular and square bases by a method of superposition similar to TIMOSHENKO's method for plates.

CHAPTER 2

ENERGY METHOD IN ANALYZING THIN SHALLOW TRANSLATIONAL SHELLS

2-1. General Equation of Strain Energy for Thin Shallow Shells

The general expression of strain energy for thin elastic shallow shells has long been established by the use of the indirect scalar method. Recently, Dr. John SCHROEDER, formerly of McMaster University, derived it by means of direct tensor methods via kinematic considerations (see Appendix I). The final formulation is as follows

$$U = \mu \iint \{ [\epsilon_{11}^2 + \epsilon_{22}^2 + 2\nu \epsilon_{11}\epsilon_{22} + 2(1-\nu)\epsilon_{12}^2] \frac{h}{1-\nu}$$

$$+ [(\delta K_1^{(n)})^2 + (\delta K_2^{(n)})^2 + 2\nu \delta K_1^{(n)} \delta K_2^{(n)} + 2(1-\nu)(\delta K^{(g)})^2] \frac{h^3}{12(1-\nu)} \}$$

$$A_1 A_2 \, d\alpha_1 \, d\alpha_2 \qquad \text{(II-1-1)}$$

in which α_1 and α_2 denote the curvilinear co-ordinates on the surface of the shell. (see Fig. I-2) This expression encompasses both membranal strain energy

$$U_m^{(s)} = \mu \iint [\epsilon_{11}^2 + \epsilon_{22}^2 + 2\nu \epsilon_{11}\epsilon_{22} + 2(1-\nu)\epsilon_{12}^2] \frac{h}{1-\nu} A_1 A_2 \, d\alpha_1 \, d\alpha_2$$

and the tranverse flexural strain energy

$$U_f^{(s)} = \mu \iint [(\delta K_1^{(n)})^2 + (\delta K_2^{(n)})^2 + 2\nu \delta K_1^{(n)} \delta K_2^{(n)} + 2(1-\nu)(\delta K^{(g)})^2] \frac{h^3}{12(1-\nu)} A_1 A_2 \, d\alpha_1 \, d\alpha_2$$

where

$$\left.\begin{aligned}
\epsilon_{11} &= \epsilon_{11}(\bar{r}_0) = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} u_2 + K_1^{(n)} u_3 \\
\epsilon_{22} &= \epsilon_{22}(\bar{r}_0) = \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_1} u_1 + K_2^{(n)} u_3 \\
\epsilon_{12} &= \epsilon_{12}(\bar{r}_0) = \frac{1}{2}\left[\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right) \right] \\
\delta K_1^{(n)} &= \left[\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{d u_3}{\partial \alpha_1}\right) - K_2^{(g)} \frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2} \right] \\
\delta K_2^{(n)} &= \left[\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}\right) - K_1^{(g)} \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1} \right]
\end{aligned}\right\} \qquad \text{(II-1-2)}$$

$$\delta K^{(g)} = \frac{-1}{2}\left\{ \frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}\right) - K_2^{(g)}\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2} + \frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}\right) + K_1^{(g)}\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1} \right\}$$

Since the shell is very shallow and uniformly loaded, $K_1^{(g)}$ becomes zero and u_1, u_2 are of comparatively higher order than u_3, hence they are negligible in the expression (II-1-2). Furthermore, owing to the shallowness of the shell, it is admissible to approximate the strain components by their projections along the directions of the base Cartesian co-ordinates x and y. Also, it is reasonable to replace the normal displacement u_3 by the vertical displacement u_z. After these simplifications, the equations (II-1-1) and (II-1-2) are reduced to the following forms:

$$\epsilon_{xx} = K_x^{(n)} u_z, \qquad \epsilon_{yy} = K_y^{(n)} u_z$$

$$\delta K_x^{(n)} = -\frac{\partial^2 u_z}{\partial x^2}, \qquad \delta K_y^{(n)} = \frac{\partial^2 u_z}{\partial y^2}, \qquad \delta K^{(g)} = -\frac{\partial^2 u_z}{\partial x \partial y} \qquad \text{(II-1-2')}$$

$$\begin{aligned} U^{(s)} &= \mu \iint \Big\{ \left[\epsilon_{xx}^2 + \epsilon_{yy}^2 + 2\nu\, \epsilon_{xx}\epsilon_{yy} \right] \frac{h}{1-\nu} \\ &\quad + \left[(\delta K_x^{(n)})^2 + (\delta K_y^{(n)})^2 + 2\nu\, \delta K_x^{(n)} \delta K_y^{(n)} + 2(1-\nu)(\delta K^{(g)})^2 \right] \\ &\quad \frac{h^3}{12(1-\nu)} \Big\} A_1 A_2\, d\alpha_1\, d\alpha_2 \\ &= \mu \iint \Big\{ (K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)} \big] u_z^2 \frac{h}{1-\nu} + \\ &\quad + \left[\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} \right)^2 - 2(1-\nu)\left(\frac{\partial^2 u_z}{\partial x^2}\frac{\partial^2 u_z}{\partial y^2} - \left(\frac{\partial^2 u_z}{\partial x \partial y}\right)^2 \right) \right] \\ &\quad \frac{h^3}{12(1-\nu)} \Big\} A_1 A_2\, d\alpha_1\, d\alpha_2 \end{aligned} \qquad \text{(II-1-3)}$$

2-2. Elliptic Paraboloidal Shells of Translation

For translational shells over square or rectangular bases, it is always possible to reduce the term $2(1-\nu)\left[\frac{\partial^2 u_z}{\partial x^2}\frac{\partial^2 u_z}{\partial y^2} - \left(\frac{\partial^2 u_z}{\partial x \partial y}\right)^2\right]$ in expression (II-1-3) to be identically zero. This can be demonstrated in various ways. The simplest way is by applying the method of integration by parts. Since

$$\iint_A \frac{\partial^2 u_3}{\partial x \partial y}\frac{\partial^2 u_3}{\partial x \partial y}\, dx\,dy = \oint_S \frac{\partial^2 u_3}{\partial x \partial y}\left[\frac{\partial u_3}{\partial x}\right]_{-\frac{b}{2}}^{\frac{b}{2}} dx - \iint_A \frac{\partial u_3}{\partial x}\left[\frac{\partial^3 u_3}{\partial x \partial y^2}\right]_{-\frac{a}{2}}^{\frac{a}{2}} dy$$

$$= \oint_S \frac{\partial^2 u_3}{\partial x \partial y}\left[\frac{\partial u_3}{\partial x}\right]_{-\frac{b}{2}}^{\frac{b}{2}} dx - \oint_S \frac{\partial u_3}{\partial x}\left[\frac{\partial^2 u_3}{\partial y^2}\right]_{-\frac{a}{2}}^{\frac{a}{2}} dy$$

$$+ \iint_A \frac{\partial^2 u_3}{\partial x^2}\frac{\partial^2 u_3}{\partial y^2}\, dx\,dy$$

for rectangular or square base plan, $\frac{\partial u_3}{\partial x} = 0$ along the edges y = constant, and $\partial^2 u_3/\partial y^2 = 0$ along the edges x = constant. Hence, the first two integrals in the above expression become identically zero. Therefore, it gives

$$\frac{\partial^2 u_3}{\partial x^2}\frac{\partial^2 u_3}{\partial y^2} - \left(\frac{\partial^2 u_3}{\partial x \partial y}\right)^2 = 0$$

This reduces the expression (II-1-3) to the following form

$$U^{(S)} = \mu \iint_A \left\{ \left[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)} \right] u_3^2 \frac{h}{1-\nu} + \left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2}\right)^2 \frac{h^3}{12(1-\nu)} \right\} A_1 A_2\, d\alpha_1\, d\alpha_2 \qquad \text{(II-2-1)}$$

For a second degree shallow elliptic paraboloidal shell of translation, the variations of the normal curvatures at different points are usually so small that the normal curvature itself appears approximately as a constant. Moreover, if the thickness h of the shell is also assumed to be a constant, then the energy expression (II-2-1) becomes

$$U^{(S)} = \frac{\mu h}{1-\nu}\left[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)} \right] \iint_A u_3^2 A_1 A_2\, d\alpha_1\, d\alpha_2 + \frac{\mu h^3}{12(1-\nu)} \iint_A \left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2}\right)^2 A_1 A_2\, d\alpha_1\, d\alpha_2 \qquad \text{(II-2-2)}$$

If the function of the middle surface is described in the EULERIAN form

$$\bar{r} = x\bar{e}_x + y\bar{e}_y + z(x,y)\,\bar{e}_z$$

where

$$z(x,y) = (c_1 x^2 + c_2 y^2) + c_3$$

for parabolic generators shown in Fig. (II-1) and (II-2)

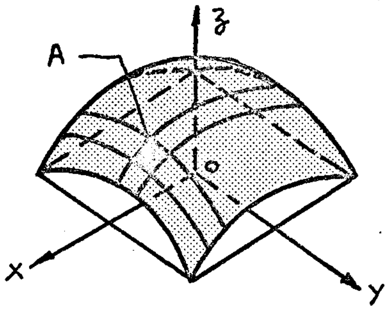

FIG. II-1

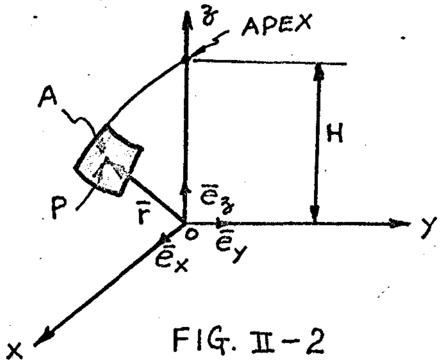

FIG. II-2

then

$$\frac{\partial z}{\partial x} = 2c_1 x, \quad \frac{\partial^2 z}{\partial x^2} \doteq -K_x^{(n)} = 2c_1, \quad \therefore\ c_1 = -\frac{K_x^{(n)}}{2}$$

$$\frac{\partial z}{\partial y} = 2c_2 y, \quad \frac{\partial^2 z}{\partial y^2} \doteq -K_y^{(n)} = 2c_2, \quad \therefore\ c_2 = -\frac{K_y^{(n)}}{2}$$

When $x = 0$, $y = 0$, $z = c_3 = H$.

$$\therefore\ z = -\frac{1}{2}(K_x^{(n)} x^2 + K_y^{(n)} y^2 - 2H)$$

The position vector $\bar{r}$ of any point P on the middle surface is

$$\bar{r} = x\bar{e}_x + y\bar{e}_y + z\bar{e}_z$$

$$= x\bar{e}_x + y\bar{e}_y + [-\tfrac{1}{2}(K_x^{(n)}x^2 + K_y^{(n)}y^2 - 2H)]\bar{e}_z$$

$$\therefore \frac{\partial \bar{r}}{\partial x} = \bar{e}_x - K_x^{(n)} x \bar{e}_z, \quad \frac{\partial \bar{r}}{\partial y} = \bar{e}_y - K_y^{(n)} y \bar{e}_z$$

$$\frac{\partial \bar{r}}{\partial y}\cdot\frac{\partial \bar{r}}{\partial y} = 1 + (K_y^{(n)})^2 y^2 = A_2^2, \quad \therefore A_2 = [1 + (K_y^{(n)})^2 y^2]^{\frac{1}{2}}$$

$$\frac{\partial \bar{r}}{\partial x}\cdot\frac{\partial \bar{r}}{\partial x} = 1 + (K_x^{(n)})^2 x^2 = A_1^2, \quad \therefore A_1 = [1 + (K_x^{(n)})^2 x^2]^{\frac{1}{2}}$$

Expanding the expressions A_1 and A_2 into binomial series yields

$$A_1 = (1 + \tfrac{1}{2}(K_x^{(n)})^2 x^2 + \cdots)$$

$$A_2 = (1 + \tfrac{1}{2}(K_y^{(n)})^2 y^2 + \cdots)$$

$$\therefore A_1A_2 = (1 + \tfrac{1}{2}(K_x^{(n)})^2x^2 + \tfrac{1}{2}(K_y^{(n)})^2y^2 + \tfrac{1}{4}(K_x^{(n)})^2(K_y^{(n)})^2x^2y^2 + \cdots)$$

Since the shell is restricted to be a shallow one, $(K_x^{(n)}x)^2$, $(K_y^{(n)}y)^2$ are assumed to be of much smaller magnitude in comparison with unity, therefore, it seems reasonable if the term $\frac{1}{4}(K_x^{(n)})^2(K_y^{(n)})^2x^2y^2$ is neglected as a small quantity. The same argument may be applied to the term $\frac{1}{2}(K_x^{(n)})^2x^2 + \frac{1}{2}(K_y^{(n)})^2y^2$. For the time being, the term $\frac{1}{2}(K_x^{(n)})^2x^2 + \frac{1}{2}(K_y^{(n)})^2y^2$ is retained. Substituting expression $A_1A_2 \doteq 1 + \frac{1}{2}(K_x^{(n)})^2x^2 + \frac{1}{2}(K_y^{(n)})^2y^2$ in (II-2-2), where α_1, α_2 now become x and y, yields

$$U^{(S)} = \frac{\mu h}{(1-\nu)}[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)}K_y^{(n)}]\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} u_z^2(1 + \tfrac{1}{2}(K_x^{(n)})^2x^2 + \tfrac{1}{2}(K_y^{(n)})^2y^2)\,dx\,dy$$
$$+ \frac{\mu h^3}{12(1-\nu)}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right)^2(1 + \tfrac{1}{2}(K_x^{(n)})^2x^2 + \tfrac{1}{2}(K_y^{(n)})^2y^2)\,dx\,dy$$

Letting $\frac{\mu h}{1-\nu} = D'$, and $\frac{\mu h^3}{6(1-\nu)} = D$, gives

$$U^{(S)} = D'[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)}K_y^{(n)}]\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} u_z^2(1 + \tfrac{1}{2}(K_x^{(n)})^2x^2 + \tfrac{1}{2}(K_y^{(n)})^2y^2)\,dx\,dy$$
$$+ \frac{D}{2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right)^2(1 + \tfrac{1}{2}(K_x^{(n)})^2x^2 + \tfrac{1}{2}(K_y^{(n)})^2y^2)\,dx\,dy \qquad \text{(II-2-3)}$$

3-3. Shells with All Four Edges Fixed

For a shell with all four edges fixed, a function for u_z has been chosen to fulfill all the following geometric boundary conditions, say

$$u_z = 0 \quad \text{when} \quad x = \pm a/2, \; y = \pm b/2$$
$$\frac{\partial u_z}{\partial x} = 0, \; \frac{\partial u_z}{\partial y} = 0 \qquad x = \pm a/2, \; y = \pm b/2 \qquad \text{(II-3-1)}$$
$$\frac{\partial^2 u_z}{\partial x^2} \neq 0 \qquad x = \pm a/2$$
$$\frac{\partial^2 u_z}{\partial y^2} \neq 0 \qquad y = \pm b/2$$

A suitable displacement function is

$$u_z = -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)\left(1+\cos\frac{2(2n+1)\pi y}{b}\right) \qquad \text{(II-3-2)}$$

Substituting this expression into (II-2-3), and integrating with respect to x and y, (see Appendix II) the following expression is obtained:

$$U^{(S)} = D'\left[(K_x^{(n)})^2+(K_y^{(n)})^2+2\nu K_x^{(n)}K_y^{(n)}\right]\frac{3ab}{4}\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}^2\left(3+\frac{a^2}{4}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{4}\frac{(K_y^{(n)})^2}{2}\right)+$$
$$+\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_{mr}A_{ms}\left(2+\frac{a^2}{6}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{6}\frac{(K_y^{(n)})^2}{2}\right)+$$
$$+\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty} A_{rn}A_{sn}\left(2+\frac{a^2}{6}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{6}\frac{(K_y^{(n)})^2}{2}\right)+$$
$$+\sum_{\substack{r=0\\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{\substack{p=0\\ p\neq q}}^{\infty}\sum_{q=0}^{\infty} A_{rp}A_{sq}\left(\frac{4}{3}+\frac{a^2}{9}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{9}\frac{(K_y^{(n)})^2}{2}\right)\Big\}+$$
$$+\frac{2Dab\pi^4}{a^4}\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}^2\Big[(2m+1)^4\left(3+\frac{a^2}{4}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{4}\frac{(K_y^{(n)})^2}{2}\right)+$$
$$+(2n+1)^4\left[\frac{3a^4}{b^4}+\frac{a^6}{4b^4}\frac{(K_x^{(n)})^2}{2}+\frac{a^4}{4b^2}\frac{(K_y^{(n)})^2}{2}\right]+$$
$$+(2m+1)^2(2n+1)^2\left(2+\frac{a^2}{b^2}+\frac{a^4}{6b^2}\frac{(K_x^{(n)})^2}{2}+\frac{a^2}{6}\frac{(K_y^{(n)})^2}{2}\right)+$$

$$+\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\substack{s=0\\ r\neq s}}^{\infty} A_{mr}A_{ms}(2m+1)^4\left(2+\frac{a^2}{6}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{6}\frac{(K_y^{(n)})^2}{2}\right)+$$

$$+\sum_{\substack{r=0\\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty} A_{rn}A_{sn}(2n+1)^4\left(2\frac{a^4}{b^4}+\frac{a^6}{6b^4}\frac{(K_x^{(n)})^2}{2}+\frac{a^4}{6b^2}\frac{(K_y^{(n)})^2}{2}\right\} \qquad \text{(II-3-3)}$$

The external work is easily found to be

$$W=\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}(-P)(u_3)(1+(K_x^{(n)})^2x^2/2+(K_y^{(n)})^2y^2/2)\,dx\,dy$$

$$=ab\left[P\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\left(1+\frac{a^2}{12}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{12}\frac{(K_y^{(n)})^2}{2}\right)\right]$$

The total potential energy of this shell is

$$V=U'^{(S)}-W \qquad \text{(II-3-4)}$$

The Stationary Potential Energy Principle is postulated as follows:

"Among all the displacements satisfying kinematic compatibility and given kinematic boundary conditions, those which satisfy the equilibrium conditions make the potential energy assume a stationary value".

$$\delta V=0$$

which leads to

$$\frac{\partial V}{\partial A_{mn}}=0$$

Let $D'[(K_x^{(n)})^2+(K_y^{(n)})^2+2\nu K_x^{(n)}K_y^{(n)}]=\phi$

then

$$\frac{3\phi}{2}\left\{A_{mn}\left(3+\frac{a^2}{4}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{4}\frac{(K_y^{(n)})^2}{2}\right)+\sum_{\substack{r=0\\ r\neq n}}^{\infty}A_{mr}\left(2+\frac{a^2}{6}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{6}\frac{(K_y^{(n)})^2}{2}\right)\right.$$

$$\left.+\sum_{\substack{r=0\\ r\neq m}}^{\infty}A_{rn}\left(2+\frac{a^2}{6}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{6}\frac{(K_y^{(n)})^2}{2}\right)+\sum_{\substack{r=0\\ r\neq m}}^{\infty}\sum_{\substack{p=0\\ p\neq n}}^{\infty}A_{rp}\left(\frac{4}{3}+\frac{a^2}{9}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{9}\frac{(K_y^{(n)})^2}{2}\right)\right\}+$$

$$+\frac{4D\pi^4}{a^4}\left\{A_{mn}\left[(2m+1)^4\left(3+\frac{a^2}{4}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{4}\frac{(K_y^{(n)})^2}{2}\right)+(2n+1)^4\left(\frac{3a^4}{b^4}+\frac{a^6}{4b^4}\frac{(K_x^{(n)})^2}{2}+\frac{a^4}{4b^2}\frac{(K_y^{(n)})^2}{2}\right)\right.\right.$$

$$+(2m+1)^2(2n+1)^2\left(\frac{2a^2}{b^2}+\frac{a^4}{6b^2}\frac{(K_x^{(n)})^2}{2}+\frac{a^2}{6}\frac{(K_y^{(n)})^2}{2}\right)\Big]+$$

$$+\sum_{\substack{r=0\\ r\neq n}}^{\infty} A_{mr}(2m+1)^4\left(2+\frac{a^2}{6}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{6}\frac{(K_y^{(n)})^2}{2}\right)+\sum_{\substack{r=0\\ r\neq m}}^{\infty} A_{rn}(2n+1)^4\Big(\frac{2a^4}{b^4}+$$

$$+\frac{a^6}{6b^4}\frac{(K_x^{(n)})^2}{2}+\frac{a^4}{6b^2}\frac{(K_y^{(n)})^2}{2}\Big)\Big\}=P\left(\frac{24+a^2(K_x^{(n)})^2+b^2(K_y^{(n)})^2}{24}\right) \qquad \text{(II-3-5)}$$

After the numerical values of h, a, b, μ, ν, $K_x^{(n)}$, $K_y^{(n)}$ are given, the Fourier coefficients A_{mn} can be solved by substituting the values m, n from 0 to k (k is any desirable integer) into expression (II-3-5). Usually, a set of simultaneous equations containing A_{mn} as unknown is obtained through this procedure.

Once the Fourier coefficients are found, the vertical displacement u_z of the shell can be obtained by substituting these coefficients into expression (II-3-2), and all stress resultants and stress couples can be evaluated by the expressions (II-3-6), (see Fig. (II-3)). The derivation of these expressions is given in Appendix III.

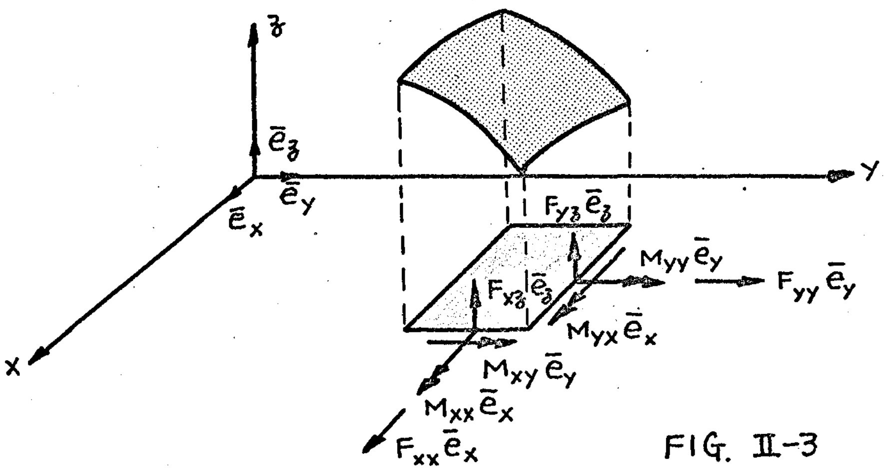

FIG. II-3

$$F_{xx}^{(\sigma)} = D'(K_x^{(n)}+\nu K_y^{(n)})u_3 \doteq D'(K_x^{(n)}+\nu K_y^{(n)})u_z$$

$$= -D'(K_x^{(n)}+\nu K_y^{(n)})\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)\left(1+\cos\frac{2(2n+1)\pi y}{b}\right)$$

$$F_{yy}^{(\sigma)} = D'(K_y^{(n)} + \nu K_x^{(n)}) u_3 \doteq D'(K_y^{(n)} + \nu K_x^{(n)}) u_3$$
$$= -D'(K_y^{(n)} + \nu K_x^{(n)}) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \left(1 + \cos \frac{2(2m+1)\pi x}{a}\right)\left(1 + \cos \frac{2(2n+1)\pi y}{b}\right)$$

$$M_{xy}^{(\sigma)} = D \frac{\partial^2 u_3}{\partial x^2} \doteq D \frac{\partial^2 u_3}{\partial x^2}$$
$$= D \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \left[\left(\frac{2(2m+1)\pi}{a}\right)^2 \cos \frac{2(2m+1)\pi x}{a} \left(1 + \cos \frac{2(2n+1)\pi y}{b}\right)\right]$$

$$M_{yx}^{(\sigma)} = -D \frac{\partial^2 u_3}{\partial y^2} \doteq -D \frac{\partial^2 u_3}{\partial y^2}$$
$$= -D \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \left[\left(\frac{2(2n+1)\pi}{b}\right)^2 \left(1 + \cos \frac{2(2m+1)\pi x}{a}\right) \cos \frac{2(2n+1)\pi y}{b}\right]$$

$$M_{xx}^{(\sigma)} = D \frac{\partial^2 u_3}{\partial x \partial y} \doteq D \frac{\partial^2 u_3}{\partial x \partial y}$$
$$= D \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \left(\frac{2(2m+1)\pi}{a}\right)\left(\frac{2(2n+1)\pi}{b}\right) \sin \frac{2(2m+1)\pi x}{a} \sin \frac{2(2n+1)\pi y}{b}$$

$$M_{yy}^{(\sigma)} = -M_{xx}^{(\sigma)}$$

$$F_{xz}^{(\sigma)} = -D\left(\nabla^2 \frac{\partial u_3}{\partial x}\right) \doteq -D\left(\nabla^2 \frac{\partial u_3}{\partial x}\right) = D \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \left[\left(\frac{2(2m+1)\pi}{a}\right)^3 \sin \frac{(2(2m+1)\pi x)}{a}\right.$$
$$\left. \left(1 + \cos \frac{2(2n+1)\pi y}{b}\right) + \left(\frac{2(2n+1)\pi}{b}\right)^2 \left(\frac{2(2m+1)\pi}{a}\right) \sin \frac{2(2m+1)\pi x}{a} \cos \frac{2(2n+1)\pi y}{b}\right]$$

$$F_{yz}^{(\sigma)} = -D\left(\nabla^2 \frac{\partial u_3}{\partial y}\right) \doteq -D\left(\nabla^2 \frac{\partial u_3}{\partial y}\right)$$
$$= D \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \left[\left(\frac{2(2n+1)\pi}{b}\right)^3 \sin \frac{2(2n+1)\pi y}{b} \left(1 + \cos \frac{2(2m+1)\pi x}{a}\right) + \right.$$
$$\left. + \left(\frac{2(2m+1)\pi}{a}\right)^2 \left(\frac{2(2n+1)\pi}{b}\right) \cos \frac{2(2m+1)\pi x}{a} \sin \frac{2(2n+1)\pi y}{b}\right] \qquad \text{(II-3-6)}$$

It is obvious that expressions (II-3-3) and (II-3-5) are very complicated, therefore, the term $\frac{1}{2}(K_x^{(n)})^2 x^2 + \frac{1}{2}(K_y^{(n)})^2 y^2$ which is small in A_1A_2, is henceforth neglected. The equations (II-3-3) and (II-3-5) reduce separately to the following expressions

$$U^{(s)} = D'\left[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)}\right] \frac{3ab}{4} \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 3A_{mn}^2 + \sum_{m=0}^{\infty} \sum_{\substack{r=0 \\ r \neq s}}^{\infty} \sum_{s=0}^{\infty} 2A_{mr} A_{ms} \right.$$

$$+ \sum_{\substack{r=0 \\ r \neq s}}^{\infty} \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} 2A_{rn} A_{rs} + \left. \sum_{\substack{r=0 \\ r \neq s}}^{\infty} \sum_{s=0}^{\infty} \sum_{\substack{p=0 \\ p \neq q}}^{\infty} \sum_{q=0}^{\infty} A_{rq} A_{sp} \right\} +$$

$$+ \frac{2Dab\pi^4}{a^4} \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}^2 \left[3(2m+1)^4 + \frac{3a^4}{b^4}(2n+1)^4 + \frac{2a^2}{b^2}(2m+1)^2(2n+1)^2\right] + \right.$$

$$\left. + \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \sum_{\substack{r=0 \\ r \neq s}}^{\infty} A_{mr} A_{ms}\, 2(2m+1)^4 + \sum_{\substack{r=0 \\ r \neq s}}^{\infty} \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} A_{rn} A_{sn} \frac{2a^4}{b^4}(2n+1)^4 \right\} \qquad \text{(II-3-7)}$$

$$\frac{3}{2} D'\left[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)}\right] \left\{ 3A_{mn} + \sum_{\substack{r=0 \\ r \neq n}}^{\infty} 2A_{mr} + \sum_{\substack{r=0 \\ r \neq m}}^{\infty} 2A_{rn} + \sum_{\substack{r=0 \\ r \neq m, p \neq n}}^{\infty} \sum_{p=0}^{\infty} \frac{4}{3} A_{rp} \right\} +$$

$$+ \frac{4D\pi^4}{a^4} \left\{ A_{mr} \left[3(2m+1)^4 + \frac{3a^4}{b^4}(2n+1)^4 + \frac{2a^2}{b^2}(2m+1)^2(2n+1)^2\right] + \right.$$

$$\left. + \sum_{\substack{r=0 \\ r \neq n}}^{\infty} A_{mr}\, 2(2m+1)^4 + \sum_{\substack{r=0 \\ r \neq m}}^{\infty} A_{rn} \frac{2a^4}{b^4}(2n+1)^4 \right\} = P \qquad \text{(II-3-8)}$$

Geometrically speaking, this simplification means that the integration of the strain energy density over the middle surface of the shell is instead carried out over the projected base plan of the middle surface. If the shell is a shallow one, this approximation yields sufficiently accurate results for practical purposes. In section (3-2-1) of the next chapter, it can be seen that this approximation is quite sufficient for the present case. All later analyses are based upon this approximation.

2-4. Shells with One Pair of Edges Fixed and Another Pair of Edges Simply Supported

For a shell with one pair of edges fixed and another pair of edges simply supported, say, the two edges at $x = \pm a/2$ are fixed, and those at $y = \pm b/2$ are simply supported, the function u_z must fulfill the following geometric boundary conditions:

$$\left.\begin{array}{llll} u_z=0 & \text{when } x=\pm\frac{a}{2}, & u_z=0 & \text{when } y=\pm\frac{b}{2} \\ \frac{\partial u_z}{\partial x}=0 & x=\pm\frac{a}{2}, & \frac{\partial u_z}{\partial y}\neq 0 & y=\pm\frac{b}{2} \\ \frac{\partial^2 u_z}{\partial x^2}\neq 0 & x=\pm\frac{a}{2}, & \frac{\partial^2 u_z}{\partial y^2}=0 & y=\pm\frac{b}{2} \end{array}\right\} \quad \text{(II-4-1)}$$

The function u_z is chosen as

$$u_z=-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)\cos\frac{(2n+1)\pi y}{b} \quad \text{(II-4-2)}$$

Pursuing, exactly, the same procedure as was followed in the preceeding section, equations corresponding to (II-3-7), (II-3-8) are given as

$$U^{(s)}=D'[(K_x^{(n)})^2+(K_y^{(n)})^2+2\nu K_x^{(n)}K_y^{(n)}]\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\left(\frac{3ab}{4}\right)+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\frac{ab}{2}\right]+$$
$$+\frac{2D\pi^4}{a^4}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\left\{(2m+1)^4ab+\frac{(2m+1)^2(2n+1)^2a^2}{4b^4}(2ab)+(2n+1)^4\frac{a^4}{16b^4}(3ab)\right\}+\right.$$
$$\left.+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\,2(2n+1)^4\frac{a^4}{16b^4}(ab)\right] \quad \text{(II-4-3)}$$

and

$$D'((K_x^{(n)})^2+(K_y^{(n)})^2+2\nu K_x^{(n)}K_y^{(n)})\left[\frac{3}{2}A_{mn}+\sum_{\substack{r=0\\r\neq m}}^{\infty}A_{rn}\right]+$$
$$+\frac{4D\pi^4}{a^4}\left\{A_{mn}\left[(2m+1)^4+\frac{2(2m+1)^2(2n+1)^2a^2}{4b^2}+3(2n+1)^4\frac{a^4}{16b^4}\right]+\right.$$
$$\left.+\sum_{\substack{r=0\\r\neq m}}^{\infty}A_{rn}\,2(2n+1)^4\frac{a^4}{16b^4}\right\}-(-1)^n\frac{2P}{(2n+1)\pi}=0 \quad \text{(II-4-4)}$$

2-5. Shells with All Four Edges Simply Supported

The boundary conditions of this type of shell are:

$$\left.\begin{array}{ll} u_z=0 & \text{when } x=\pm\frac{a}{2},\ y=\pm\frac{b}{2} \\ \frac{\partial u_z}{\partial x}\neq 0 & x=\pm\frac{a}{2} \\ \frac{\partial u_z}{\partial y}\neq 0 & y=\pm\frac{b}{2} \\ \frac{\partial^2 u_z}{\partial x^2}=0,\quad \frac{\partial^2 u_z}{\partial y^2}=0 & x=\pm\frac{a}{2},\ y=\pm b/2 \end{array}\right\} \quad \text{(II-5-1)}$$

The suitable function for the vertical displacement u_z is

$$u_z = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos\frac{(2m+1)\pi x}{a} \cos\frac{(2n+1)\pi y}{b} \tag{II-5-2}$$

Just as it has been done in the two preceeding sections, the Stationary Potential Energy Principle is again applied, and the expressions corresponding to expressions (II-3-7) and (II-3-8) are

$$\mathcal{U}^{(5)} = \frac{D\pi^4}{8a^4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}^2 \left[(2m+1)^4 + (2m+1)^2(2n+1)^2 \frac{a^2}{b^2} + (2n+1)^4 \frac{a^4}{b^4}\right] + \\ + D'\left[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)}\right] \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}^2 \frac{ab}{4} \tag{II-5-3}$$

$$A_{mn} = (-1)^{m+n}\, 8P \Big/ \Big\{ (2m+1)(2n+1)\pi^2 \Big[\frac{D\pi^4}{2a^4} \Big[(2m+1)^4 + 2(2m+1)^2(2n+1)^2 \frac{a^2}{b^2} \Big] + \\ + D'\left[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)}\right] \Big] \Big\} \tag{II-5-4}$$

CHAPTER 3

APPLICATIONS

3-1. Convergence of the Displacement Function

Before going to the applications, the problem of convergence of the chosen series of the displacement functions in their application to various geometric configurations of translational shell is subjected to a careful consideration at this stage. In expressions (II-3-8), (II-4-4) and (II-5-4) in the last chapter, the portion which is derived from the membranal energy is

$$\frac{3}{2}D'\left[(K_x^{(n)})^2+(K_y^{(n)})^2+2\nu K_x^{(n)}K_y^{(n)}\right]\left\{3A_{mn}+\sum_{\substack{r=0\\ r\neq n}}^{\infty} \quad +\sum_{\substack{r=0\\ r\neq m}}^{\infty}A_{rn}+ \right.$$

$$\left. +\sum_{\substack{r=0\\ r\neq m}}^{\infty}\sum_{\substack{p=0\\ p\neq n}}^{\infty}\frac{4}{3}A_{rp}\right\} \qquad \text{(III-1-1a) from (II-3-8)}$$

$$D'\left[(K_x^{(n)})^2+(K_y^{(n)})^2+2\nu K_x^{(n)}K_y^{(n)}\right]\left\{\frac{3}{2}A_{mn}+\sum_{\substack{r=0\\ r\neq m}}^{\infty}A_{rn}\right\} \qquad \text{(III-1-1b) from (II-4-4)}$$

or

$$D'\left[(K_x^{(n)})^2+(K_y^{(n)})^2+2\nu K_x^{(n)}K_y^{(n)}\right] \qquad \text{(III-1-1c) from (II-5-4)}$$

It is obvious that these expressions are functions of the thickness h and the normal curvatures $K_x^{(n)}, K_y^{(n)}$ only i.e., that expressions (III-1-1a,b,c) may be assumed to be constant with respect to the series indices m and n. Therefore, the problem becomes largely dependent upon how the expressions (III-1-1a,b,c) influence the convergence of various shapes of translational shells, and under their influence, how rapidly will the vertical displacement series converge. For the first question, a procedure has been derived in the following paragraphs. For the second question, a series of numerical calculations have been prepared to establish a general estimation.

If the shell thickness h is fixed to 4 inches in the expressions (III-1-1a,b,c) which is a reasonable thickness of a reinforced concrete shell of moderate proportions, the only factors which would influence the magnitudes of these expressions are the normal curvatures $K_x^{(n)}$ and $K_y^{(n)}$. Thus the maximum and minimum values of expressions (III-1-1a,b,c) depend upon the maximum and minimum values of the expression

$$M = (K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)} \qquad \text{(III-1-2)}$$

Now, returning to the function of the middle surface of the shell, which is

$$z = -\frac{1}{2}(K_x^{(n)} x^2 + K_y^{(n)} y^2) + H \qquad \text{(III-1-3)}$$

let $a \geqq b$, $H = b/\omega$, in which $\omega \leqq 5$, then, when $x = a/2$, $y = b/2$, and $z = 0$, expression (III-1-3) becomes

$$\frac{1}{8}(K_x^{(n)} a^2 + K_y^{(n)} b^2) = \frac{b}{\omega}$$

$$K_x^{(n)} a^2 = \frac{8b}{\omega} - K_y^{(n)} b^2 \qquad \text{(III-1-4)}$$

$$\therefore \; K_x^{(n)} = \frac{8b}{\omega a^2} - \frac{b^2 K_y^{(n)}}{a^2}$$

or

$$K_x^{(n)} - \frac{8b}{\omega a^2} = -\frac{b^2}{a^2} K_y^{(n)}$$

$$\therefore \; K_y^{(n)} = -\frac{a^2}{b^2}\left(K_x^{(n)} - \frac{8b}{\omega a^2}\right) \qquad \text{(III-1-5)}$$

Since for elliptic paraboloidal shells of translation with a positive GAUSSIAN curvature, the principal curvatures $K_x^{(n)}$ and $K_y^{(n)}$ are of the same sign, then

$$K_x^{(n)} K_y^{(n)} = -\frac{a^2}{b^2} K_x^{(n)} \left(K_x^{(n)} - \frac{8b}{\omega a^2}\right) > 0$$

and

$$\frac{a^2}{b^2} K_x^{(n)} \left(K_x^{(n)} - \frac{8b}{\omega a^2}\right) \leqq 0$$

or

$$K_x^{(n)} - \frac{8b}{\omega a^2} \leqq 0$$

$$\therefore \quad K_x^{(n)} \leqq \frac{8b}{\omega a^2}$$

This means that $K_x^{(n)}$ is bounded by the closed interval $0 \leqq K_x^{(n)} \leqq \frac{8b}{\omega a^2}$

Substituting $8b/a^2\omega = A'$, $b^2/a^2 = B'$ into (III-1-4) and (III-1-5) gives

$$K_x^{(n)} = A' - B' K_y^{(n)}$$

$$K_y^{(n)} = \frac{A' - K_x^{(n)}}{B'} \qquad \text{(III-1-6)}$$

$$(K_y^{(n)})^2 = \frac{(A' - K_x^{(n)})^2}{(B')^2} \qquad \text{(III-1-7)}$$

and substituting the expressions (III-1-6) and (III-1-7) into (III-1-2) gives

$$M = (K_x^{(n)})^2 + \frac{(A' - K_x^{(n)})^2}{(B')^2} + 2\nu K_x^{(n)} \frac{(A' - K_x^{(n)})}{B'}$$

$$= (K_x^{(n)})^2 \left(1 + \frac{1}{(B')^2} - \frac{2\nu}{B'}\right) + K_x^{(n)} \left(\frac{2\nu A'}{B'} - \frac{2A'}{(B')^2}\right) + \left(\frac{A'}{B'}\right)^2$$

Set

$$\frac{\partial M}{\partial K_x^{(n)}} = 2K_x^{(n)} \left(1 + \frac{1}{(B')^2} - \frac{2\nu}{B'}\right) + \left(\frac{2\nu A'}{B'} - 2\left(\frac{A'}{B'}\right)^2\right) = 0$$

$$K_x^{(n)} = \left[\frac{A'}{B'}\left(\frac{1}{B'} - \nu\right)\right] \Big/ \left[1 + \frac{1}{(B')^2} - \frac{2\nu}{B'}\right]$$

$$\frac{\partial^2 M}{\partial (K_x^{(n)})^2} = 2\left(1 + \frac{1}{(B')^2} - \frac{2\nu}{B'}\right) = 2 + 2\frac{a}{b}\left(\frac{a}{b} - 2\nu\right)$$

since $a/b \geqq 1$ and $2\nu < 1$, therefore,

$$\frac{\partial^2 M}{\partial (K_x^{(n)})^2} > 0$$

So $K_x^{(n)} = \frac{A'}{B'} \left(\frac{1}{B'} - \nu\right) / \left(1 + \frac{1}{(B')^2} - \frac{2\nu}{B'}\right)$ assumes a minimum value of M. It can be observed that M is a quadratic form in $K_x^{(n)}$, therefore, the absolute maximum value of M must be at one of the two termini of the closed interval of $K_x^{(n)}$. First, substituting $K_x^{(n)} = 8b/a^2\omega = A'$ into M, $M = A'^2(1 - a^4/b^4) + A'^2/B'^2$, then, substituting $K_x^{(n)} = 0$ into M, $M = A'^2/B'^2$. Since $1 - a^4/b^4 \leqq 0$, then the maximum value of M must occur at $K_x^{(n)} = 0$. This is a cylindrical shell which represents a special case of translational shells. There is another special case when the shell has a square base plan and its M assumes a minimum value, i.e. a = b, so B' = 1

$$K_x^{(n)} = \frac{\frac{A'}{B'}\left(\frac{1}{B'} - \nu\right)}{\left(1 + \frac{1}{(B')^2} - \frac{2\nu}{B'}\right)} = \frac{A'}{2}$$

$$K_y^{(n)} = \frac{A' - K_x^{(n)}}{B'} = \frac{A'}{2}$$

This case represents a spherical shell.

For translational shells with all edges fixed, the above derivations show that the spherical shell assumes a minimum membranal strain energy, while the cylindrical shell assumes a maximum membranal strain energy. The ordinary paraboloidal translational shell assumes a membranal strain energy which lies in between the strain energies of the spherical and cylindrical shells. Therefore, the displacement series for the spherical shell with a square base plan possesses the fastest rate of convergence, while for the cylindrical shell the convergence is the slowest. In order to show how fast the displacement series will actually converge, a few special cases have been considered. Vertical displacements at the apex of a series of shells have been calculated and tabulated in Table (III-1). The results agree with the above theory.

Table (III-1) shows that the difference between u_{z1} and u_{z2} is always less than 3%, while the difference between u_{z2} and u_{z3} is less than 1% for all shells listed. So it should be reasonable to say that the summation of eight terms of displacement series will give quite satisfactory values, and twelve terms of displacement series will give still better values of u_z, $F_{xx}^{(\sigma)}$

TABLE III-1

a (ft)	b (ft)	μ	M	SUMMATIONS OF DISPLACEMENT SERIES: u_{z1}, 8 terms	u_{z2}, 12 terms	u_{z3}, 16 terms	RATE OF CONVERGENCE IN PERCENT: $[(u_{z2}-u_{z1})/u_{z1}]\%$	$[(u_{z3}-u_{z2})/u_{z2}]\%$
40	40	5	MAX.	$.2463 \times 10^{-3}$	$.2509 \times 10^{-3}$	$.2515 \times 10^{-3}$	1.95%	0.24%
40	40	5	MIN.	$.4064 \times 10^{-3}$	$.4121 \times 10^{-3}$	$.4128 \times 10^{-3}$	1.40%	0.17%
40	40	10	MAX.	$.8668 \times 10^{-3}$	$.8749 \times 10^{-3}$	$.8759 \times 10^{-3}$	0.93%	0.11%
40	40	10	MIN.	$.1470 \times 10^{-2}$	$.1416 \times 10^{-2}$	$.1417 \times 10^{-2}$	0.64%	0.07%
60	60	5	MIN.	$.9740 \times 10^{-3}$	$.9949 \times 10^{-3}$	$.1001 \times 10^{-2}$	2.15%	0.61%
60	60	10	MIN.	$.3474 \times 10^{-2}$	$.3510 \times 10^{-2}$	$.3515 \times 10^{-2}$	1.03%	0.14%
60	40	5	MAX.	$.2575 \times 10^{-3}$	$.2646 \times 10^{-3}$	$.2657 \times 10^{-3}$	2.76%	0.42%
60	40	5	MIN.	$.1214 \times 10^{-2}$	$.1229 \times 10^{-2}$	$.1231 \times 10^{-2}$	1.24%	0.16%
60	40	10	MAX.	$.9319 \times 10^{-3}$	$.9456 \times 10^{-3}$	$.9478 \times 10^{-3}$	1.47%	0.23%
60	40	10	MIN.	$.4185 \times 10^{-2}$	$.4212 \times 10^{-2}$	$.4214 \times 10^{-2}$	0.65%	0.05%
70	35	5	MAX.	$.1963 \times 10^{-3}$	$.2017 \times 10^{-3}$	$.2029 \times 10^{-3}$	2.75%	0.60%
70	35	5	MIN.	$.2515 \times 10^{-2}$	$.2541 \times 10^{-2}$	$.2542 \times 10^{-2}$	1.04%	0.08%
70	35	10	MAX.	$.7127 \times 10^{-3}$	$.7274 \times 10^{-3}$	$.7293 \times 10^{-3}$	2.06%	0.27%
70	35	10	MIN.	$.8479 \times 10^{-2}$	$.8522 \times 10^{-2}$	$.8524 \times 10^{-2}$	0.51%	0.02%
*70	35	10	AVE.	$.4735 \times 10^{-2}$	$.4770 \times 10^{-2}$	$.4771 \times 10^{-2}$	0.78%	0.02%
100	50	5	MIN.	$.5640 \times 10^{-2}$	$.5720 \times 10^{-2}$	$.5723 \times 10^{-2}$	1.42%	0.06%
100	50	10	MIN.	$.1950 \times 10^{-1}$	$.1966 \times 10^{-1}$	$.1966 \times 10^{-1}$	0.77%	0^{+}%

REMARKS:
1. u_{z1}, u_{z2} and u_{z3} are the displacement at the apex.
2. All shells have fixed boundaries
3. $^{*}K_x^{(n)} = 0.004 \qquad K_y^{(n)} = 0.00633$

and $F_{yy}^{(\sigma)}$. The convergence of the functions of the stress couples $M_{xy}^{(\sigma)}$, $M_{yx}^{(\sigma)}$, $M_{xx}^{(\sigma)}$ and $M_{yy}^{(\sigma)}$ is slower than for the displacements, since they are functions of the second derivatives of u_z. Hence a satisfactory solution for the stress couples may be expected to be procured by employing a much larger number of terms in the vertical displacement series. The transverse stress resultants $F_{yn}^{(\sigma)}$ and $F_{xn}^{(\sigma)}$ are functions of the third derivatives of the vertical displacement series, therefore, the convergence of these quantities will be even slower than those of the stress couples.

It is interesting to observe that the rate of convergence of the vertical displacement series varies at different points of a shell. For an example, curves of the vertical displacement u_z of a translational shell with square base, a = 40', b = 40' and clamped edges is shown in Fig. (3 -1). Since the shell is doubly-symmetric, the curves are drawn only for a quarter of the shell, say, $0 \leqq x \leqq a/2$, $0 \leqq y \leqq b/2$. The full line curves represent the vertical displacement curves by expanding the function u_z up to 24 terms. The broken line curves represent the same curves by expanding the same function up to 8 terms, while the thin solid line curves represent the same curves by expanding the same function up to 16 terms. It can be clearly observed that on the center line of the shell, the convergence of u_z is faster at those points within approximately the region $-0.2 < y/b < 0.2$, $-0.2 < x/a < 0.2$; than at points outside of this region. The convergence is slowest at those points approximately in the region $(0.25 < x/a < 0.35,\ 0.25 < y/b < 0.35)$. Along the

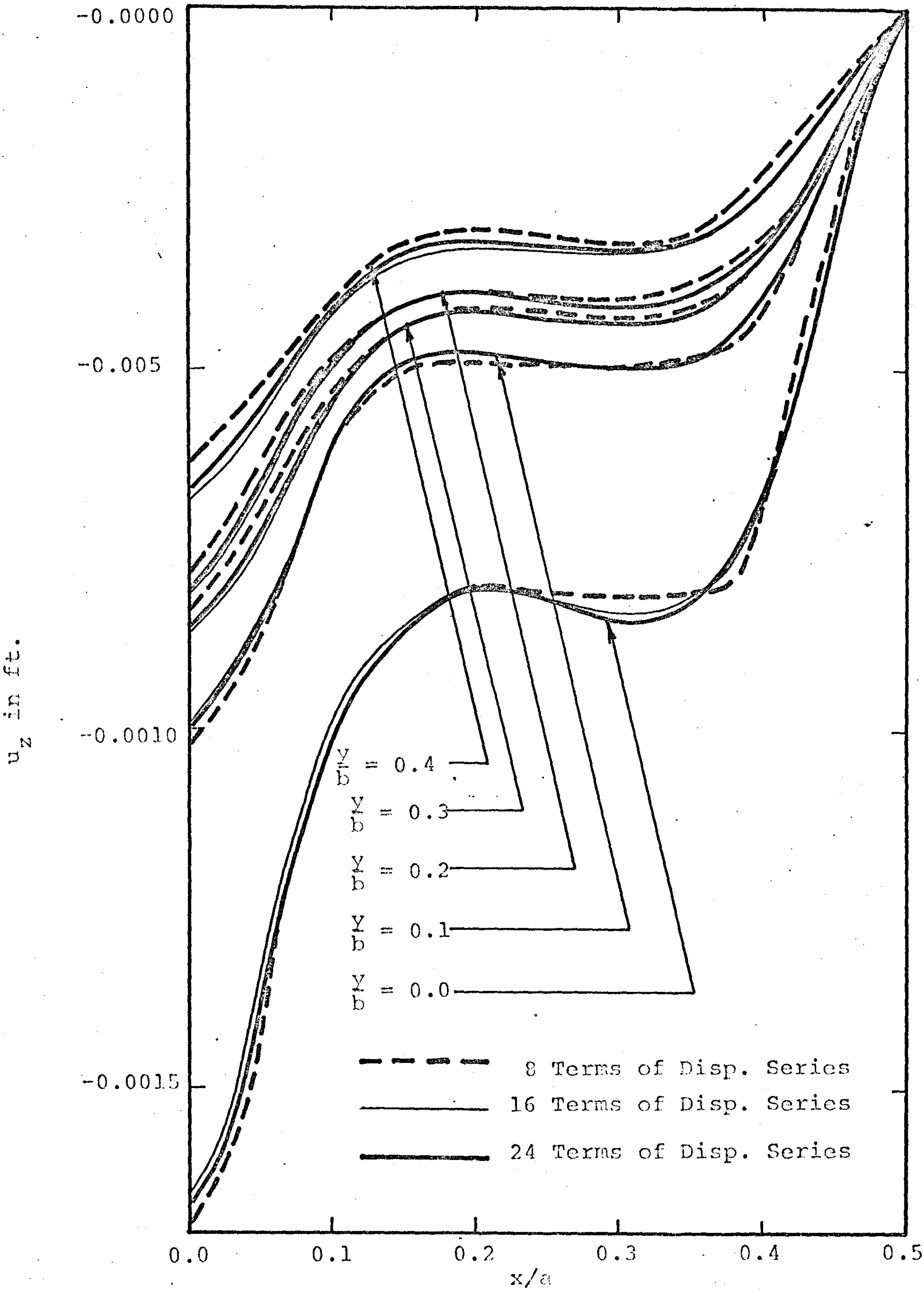

Fig. 3-1 DISTRIBUTION OF NORMAL DISPLACEMENT u_z OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES. (a = 40', b = 40')

curves near the edge, say, along curve x/a = 0.4, it seems that the convergence is a little slower at those points close to the center or the edge of the shell. Along other curves, the figure shows that the vertical displacement series u_z converges quite uniformly.

3-2. Thin Shallow Translational Shells

For the thin shallow translational shells, a shell with the following data;

$a = 70'$, $b = 35'$, $K_x^{(n)} = 0.004$, $K_y^{(n)} = 0.00633$, $P = 90\ lb/ft^2$

$h = 4''$, $E = 3 \times 10^6\ lb/in$, $\nu = 0.16$

will be analyzed as an example.

Since the torsional stress couples $M_{xx}^{(\sigma)}$, $M_{yy}^{(\sigma)}$ and the transverse shear resultants $F_{xz}^{(\sigma)}$, $F_{yz}^{(\sigma)}$ are all negligible quantities compared with $M_{xy}^{(\sigma)}$, $M_{yx}^{(\sigma)}$, $F_{xx}^{(\sigma)}$ and $F_{yy}^{(\sigma)}$, no calculation is carried out for those quantities.

3-2-1. Shell with All Four Edges Fixed

First, substituting values of a, b, $k_x^{(n)}$, $k_y^{(n)}$, P, h, E and ν listed above into expression (II-3-5) then (II-3-8), different sets of Fourier coefficients are obtained,(see Table (III-2)). It is obvious that the differences between the corresponding values in these two sets are extremely small, say, mostly less than 3%. This shows numerically that the approximation by using expression (II-3-8) instead of expression (II-3-5) is quite reasonable. After substituting these values into expression (II-3-6), all values of $M_{xy}^{(\sigma)}$, $M_{yx}^{(\sigma)}$, $F_{xx}^{(\sigma)}$, $F_{yy}^{(\sigma)}$ at various points of the shell are obtained as shown in Figures (3-2) (3 - 3), (3 - 4) and (3 - 5). Since the shell is symmetric about the apex, all figures are drawn for one quarter of the shell

TABLE III-2

	FOURIER COEFFICIENTS CALCULATED BY EXP.(II-3-5)	FOURIER COEFFICIENTS CALCULATED BY EXP.(II-3-8)
A00	0.002539	0.002517
A01	0.001727	0.001716
A10	0.001327	0.001319
A11	0.0001344	0.0001336
A02	0.00001929	0.00001907
A20	0.0003352	0.0003342
A12	0.00001922	0.00001905
A21	0.00006294	0.00006198
A22	0.00001160	0.00001150
A03	0.000004572	0.000004508
A30	0.00009778	0.00009378
A13	0.000004970	0.000004918
A31	0.00002532	0.00002641
A23	0.000003297	0.000003263
A32	0.000005811	0.000006140
A33	0.000001806	0.000001929
A04	0.000001715	0.000001688
A40	0.00003382	0.00003376
A14	0.000001918	0.000001895
A41	0.00001225	0.00001216
A24	0.000001352	0.000001336
A42	0.000003434	0.000003381
A34	0.0000008051	0.0000008553
A43	0.000001197	0.000001172

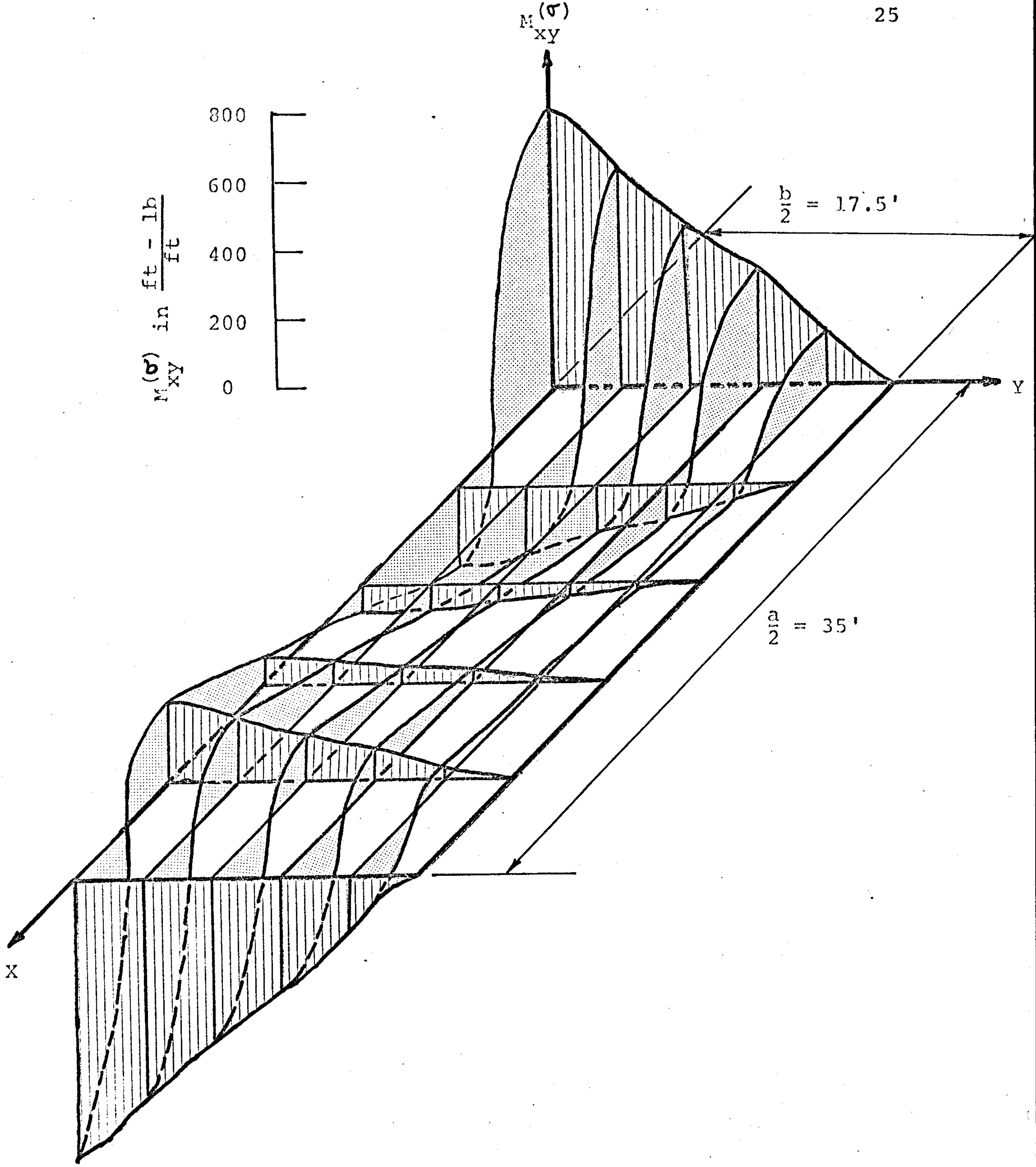

Fig. 3-2 DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES.

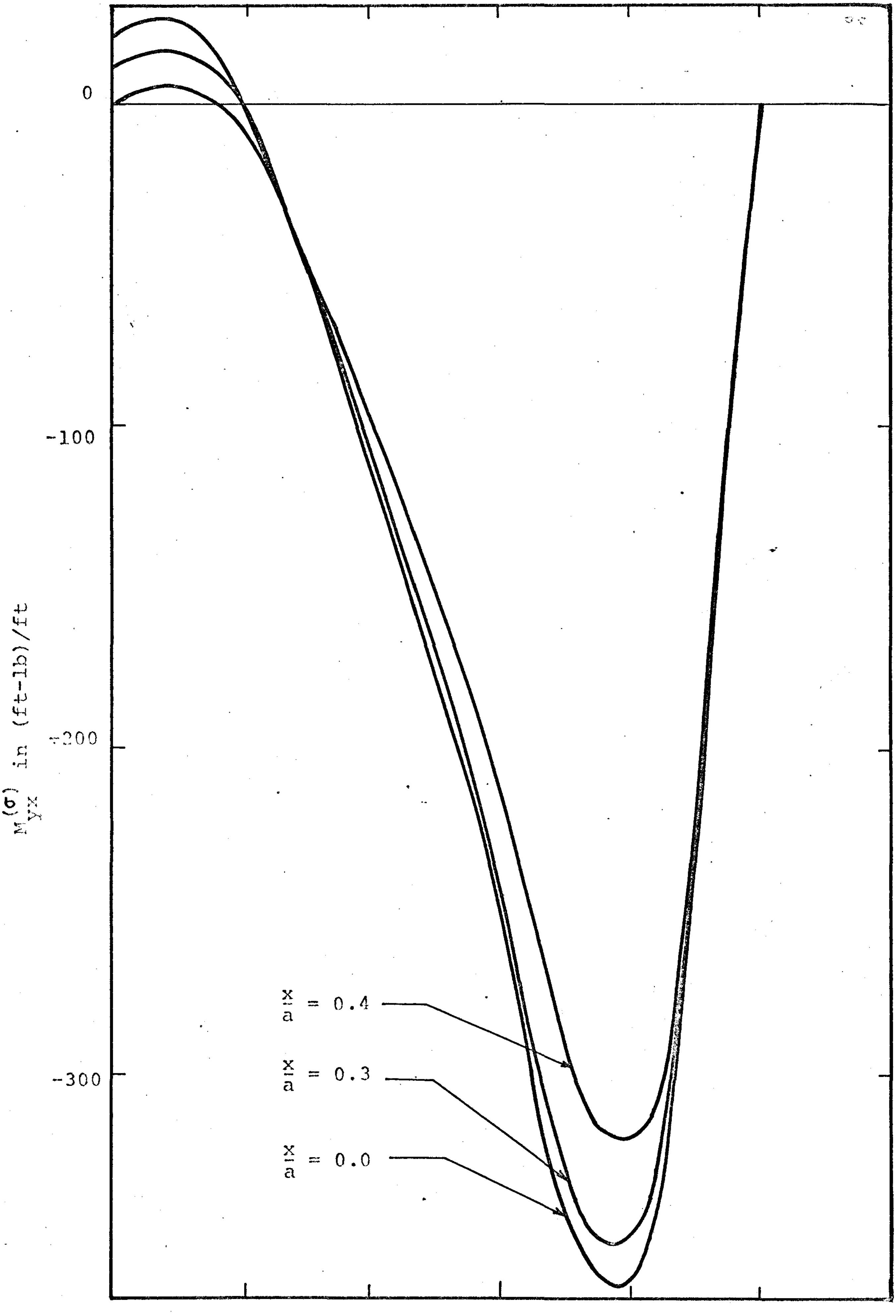

Fig. 3-3 DISTRIBUTION OF STRESS COUPLE $M_{yx}^{(\sigma)}$ OF SHALLOW ELLIPTIC

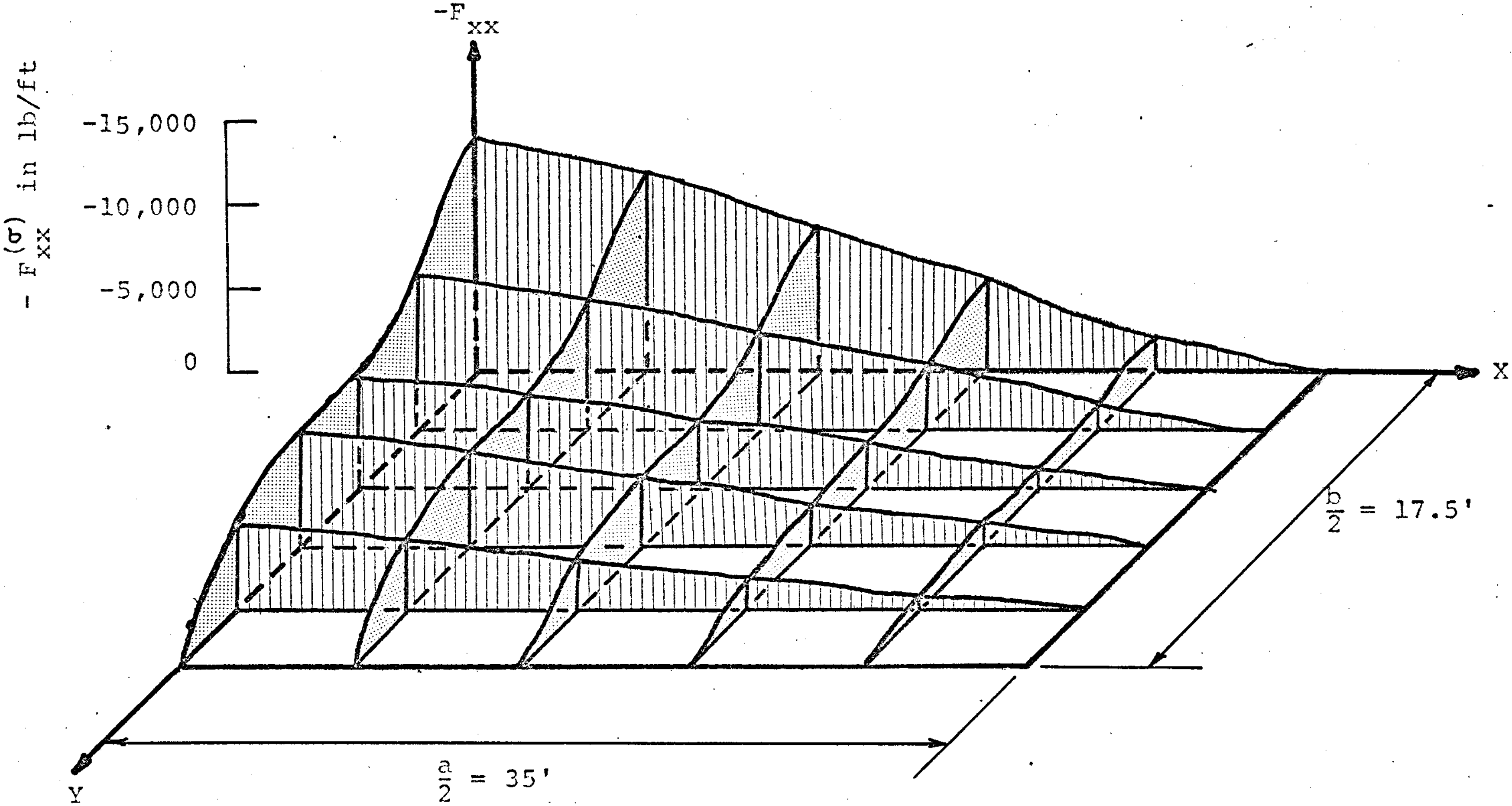

Fig. 3-4 DISTRIBUTION OF STRESS RESULTANT $F_{xx}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES.

of its base plan.

For the purpose of evaluating the nature of convergency of the solution series for the stress resultants and stress couples, the vertical displacement u_z is also expanded into eight and sixteen terms by using expression (II-3-8). The corresponding stress resultants and couples are calculated through expression (II-3-6). The results are plotted in Figures (3-6), (3-7), (3-8), (3-9) and (3-10). In Fig. (3-6) it is shown that at certain points of the shell, the convergence of the stress couple $M_{xy}^{(\sigma)}$ is not sufficient by expanding u_z into eight terms. For an example, in the neighborhood of sections $y/b = 0.0$, $x/a = 0.15$, 0.225, 0.275 or 0.35, the ratio of the value of $M_{xy}^{(\sigma)}$ on curve A and curve B is always larger than 2. This means that the value of M_{xy} calculated by expanding u_z into 16 terms is only less than 50% of the value calculated by expanding u_z into 8 terms. At sections $y/b = 0.225$ and 0.275, the sign is even reversed. Nevertheless, according to the same figure, there is little difference between the value of $M_{xy}^{(\sigma)}$ on curve B or on curve C. This suggests that $M_{xy}^{(\sigma)}$ already possesses quite a satisfactory convergence when it is calculated by expanding the deflection series u_z into 16 terms. Certainly, if 24 terms of the series for transverse displacement u_z would be used to calculate the stress couple $M_{xy}^{(\sigma)}$, a good approximate result will be obtained. For $M_{yx}^{(\sigma)}$, it is shown in Fig. (3-7) that the corresponding variation is approximately the same as $M_{xy}^{(\sigma)}$, so it also requires 16 terms of the vertical displacement series u_z to achieve a better convergence. Figures (3-8) and (3-9)

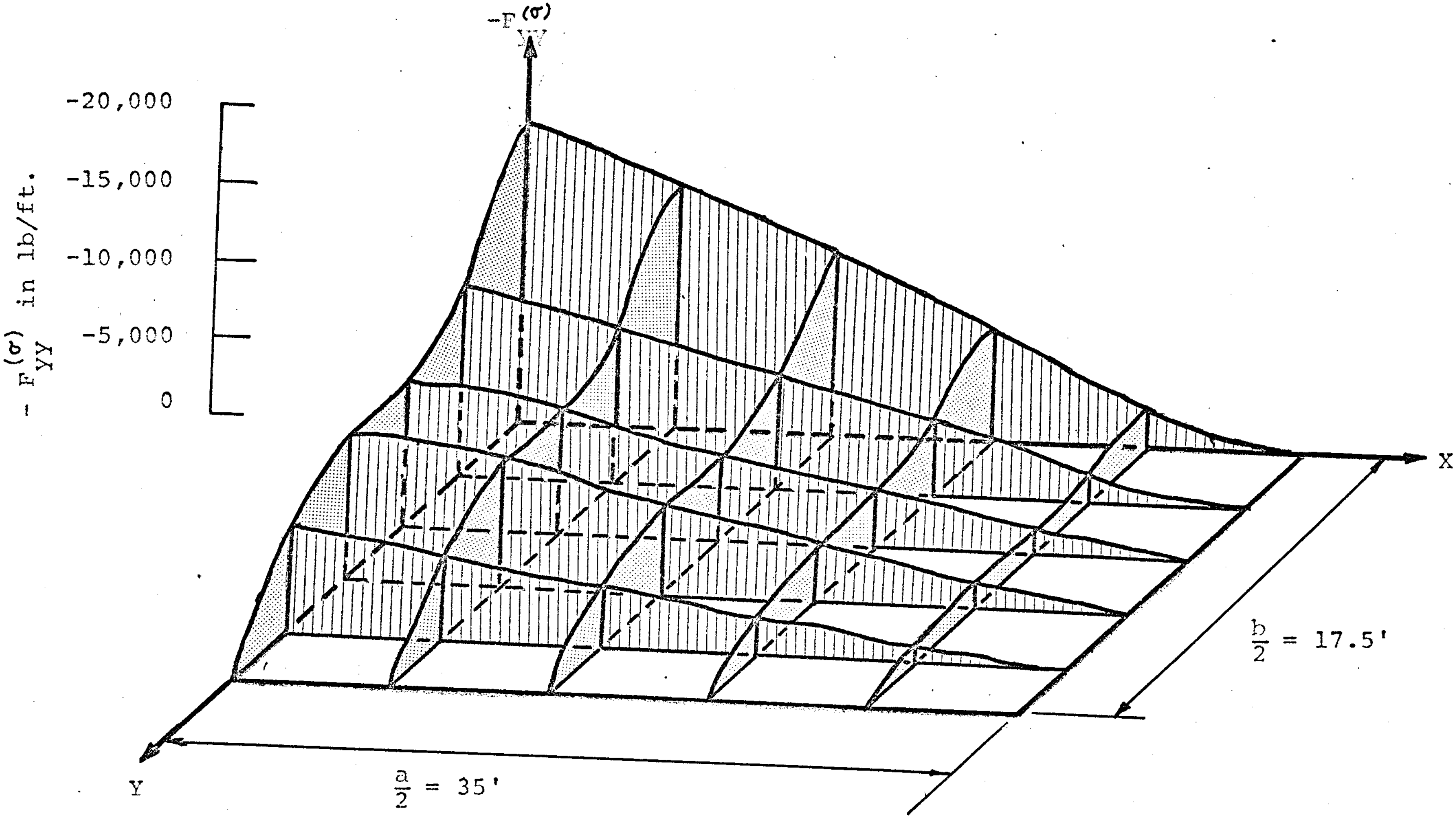

Fig. 3-5 DISTRIBTUION OF STRESS RESULTANT $F_{YY}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES

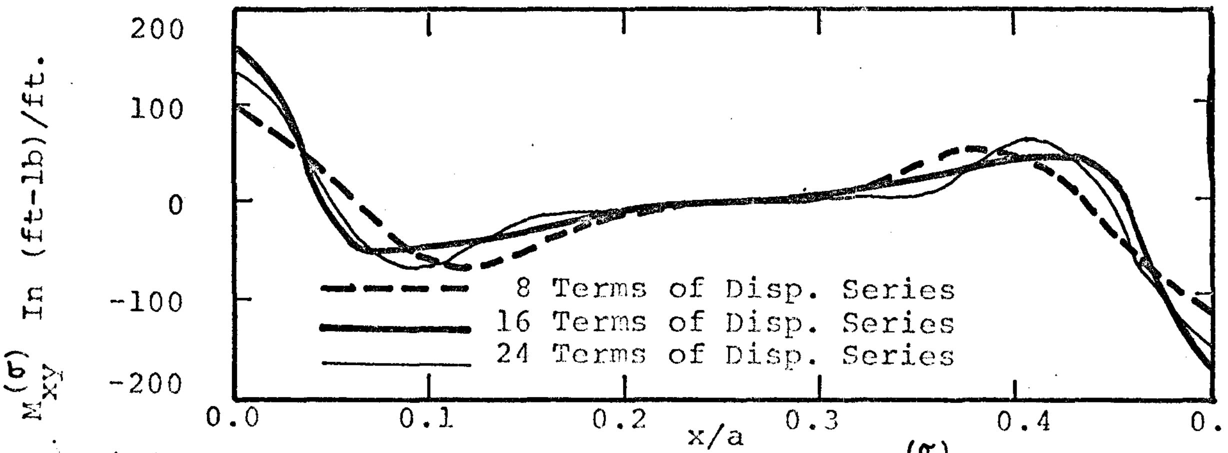

Fig. 3-6a DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ ALONG SECTION y/b = 0.4

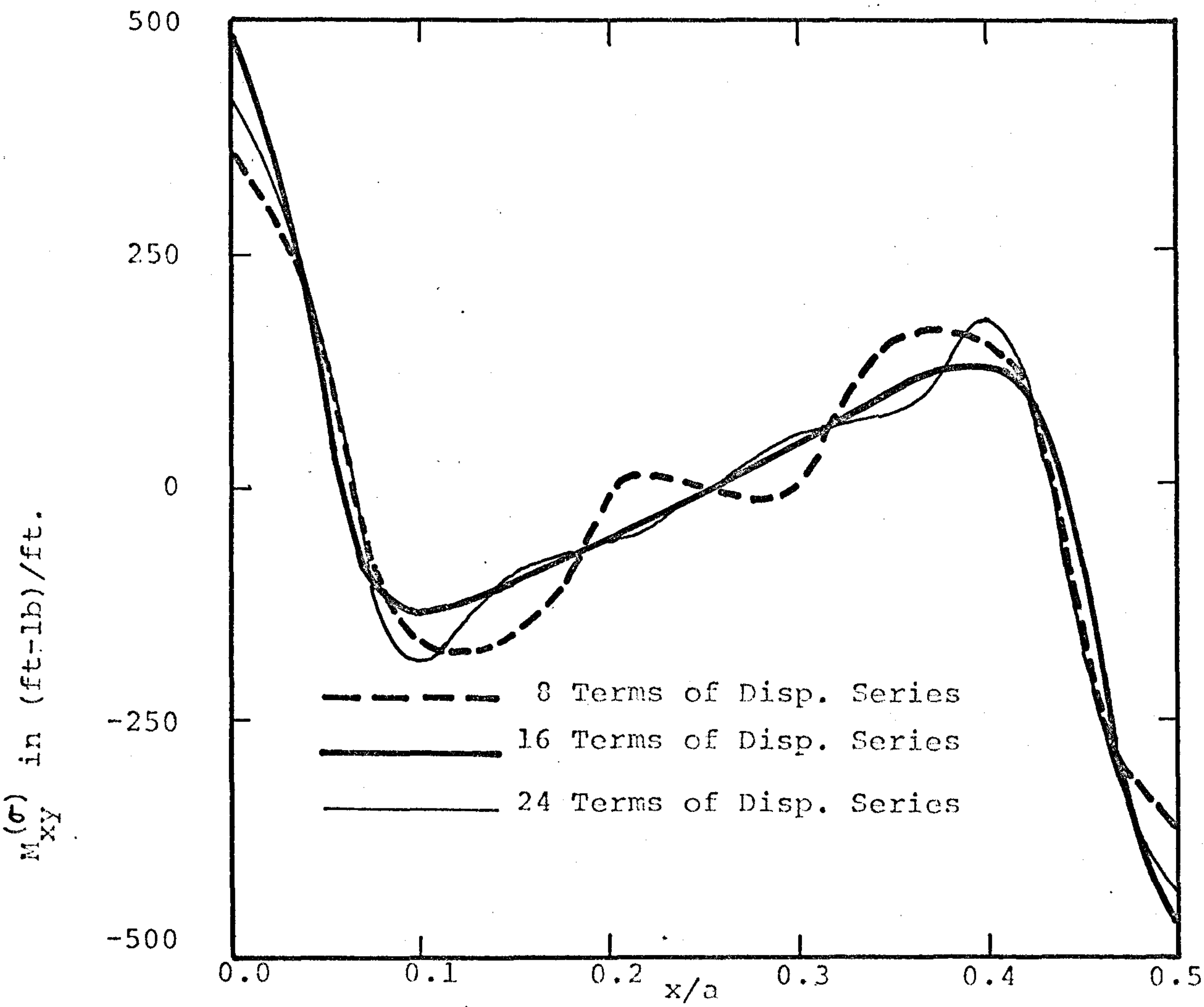

Fig. 3-6b DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ ALONG SECTION y/b = 0.2

NOTE: Fig. 3-6 DEPICTS DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ IN SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES. (a = 70', b = 35')

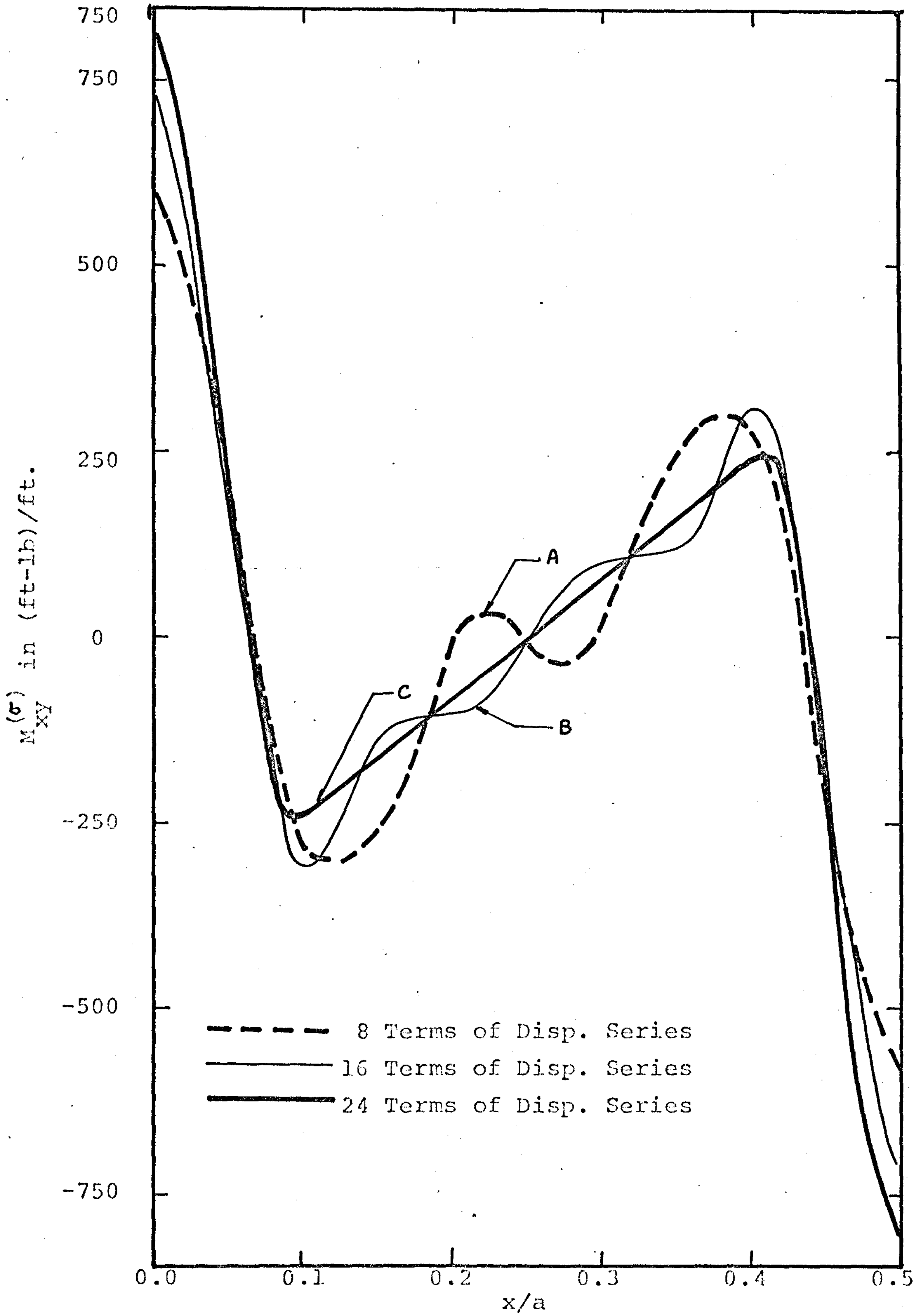

Fig. 3-6c DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ ALONG SECTION y/b = 0.0

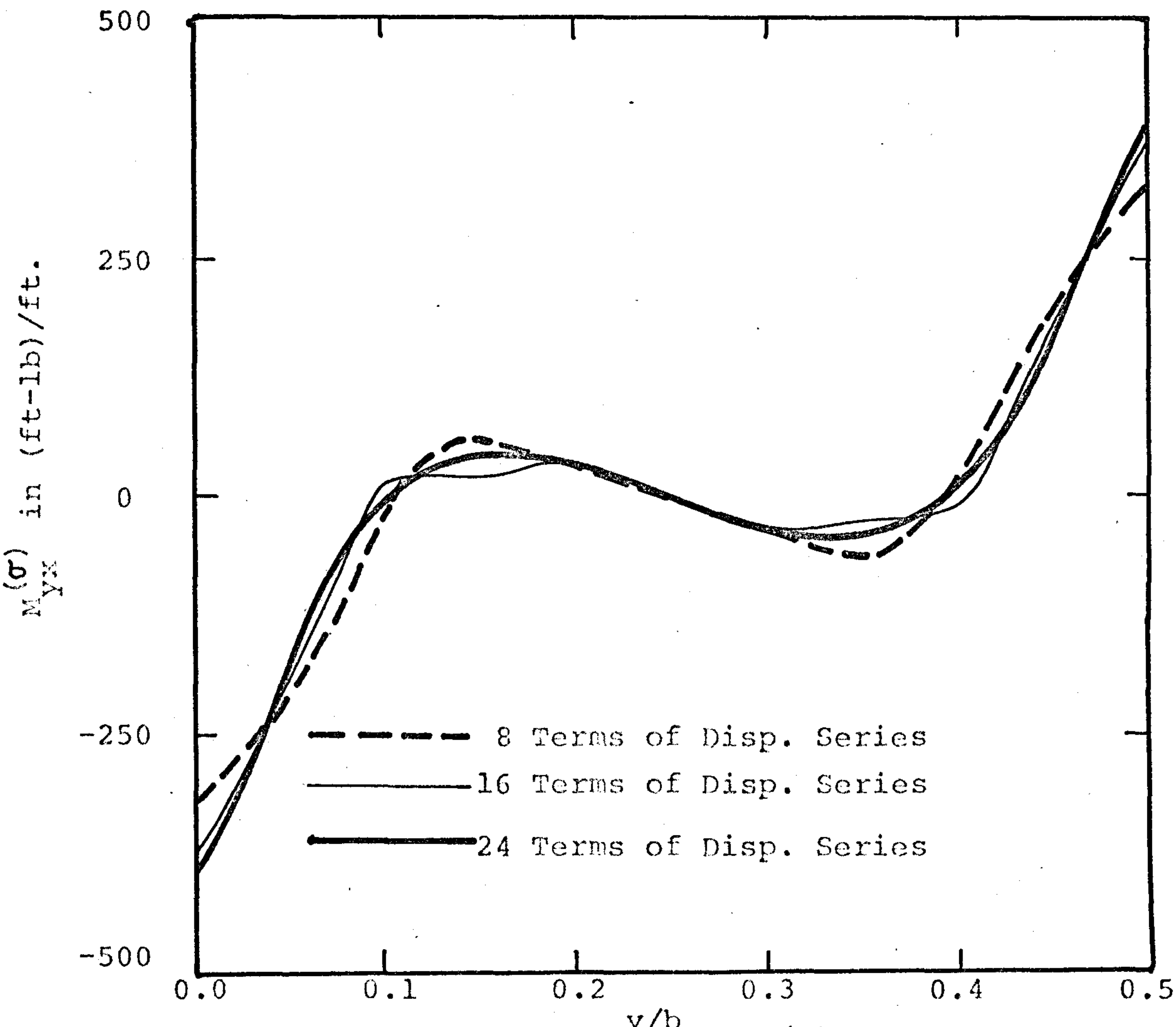

Fig 3-7a DISTRIBUTION OF STRESS COUPLE $M_{YX}^{(\sigma)}$ ALONG SECTION x/a = 0.4

NOTE: Fig. 3-7 DEPICTS DISTRIBUTION OF STRESS COUPLE $M_{yx}^{(\sigma)}$ IN SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES. (a = 70', b = 35')

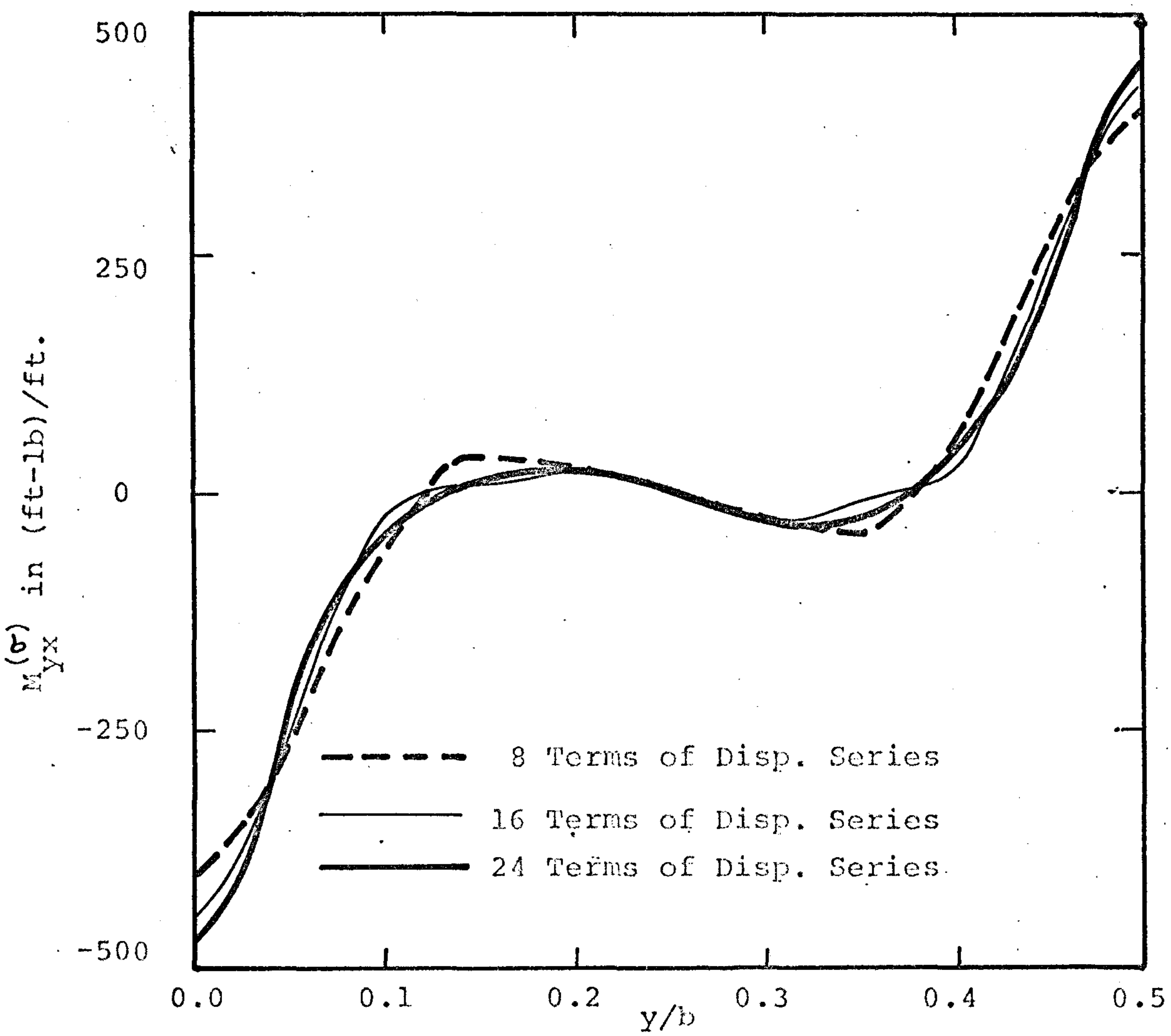

Fig. 3-7b DISTRIBUTION OF STRESS COUPLE $M_{yx}^{(\sigma)}$ ALONG SECTION x/a = 0.2

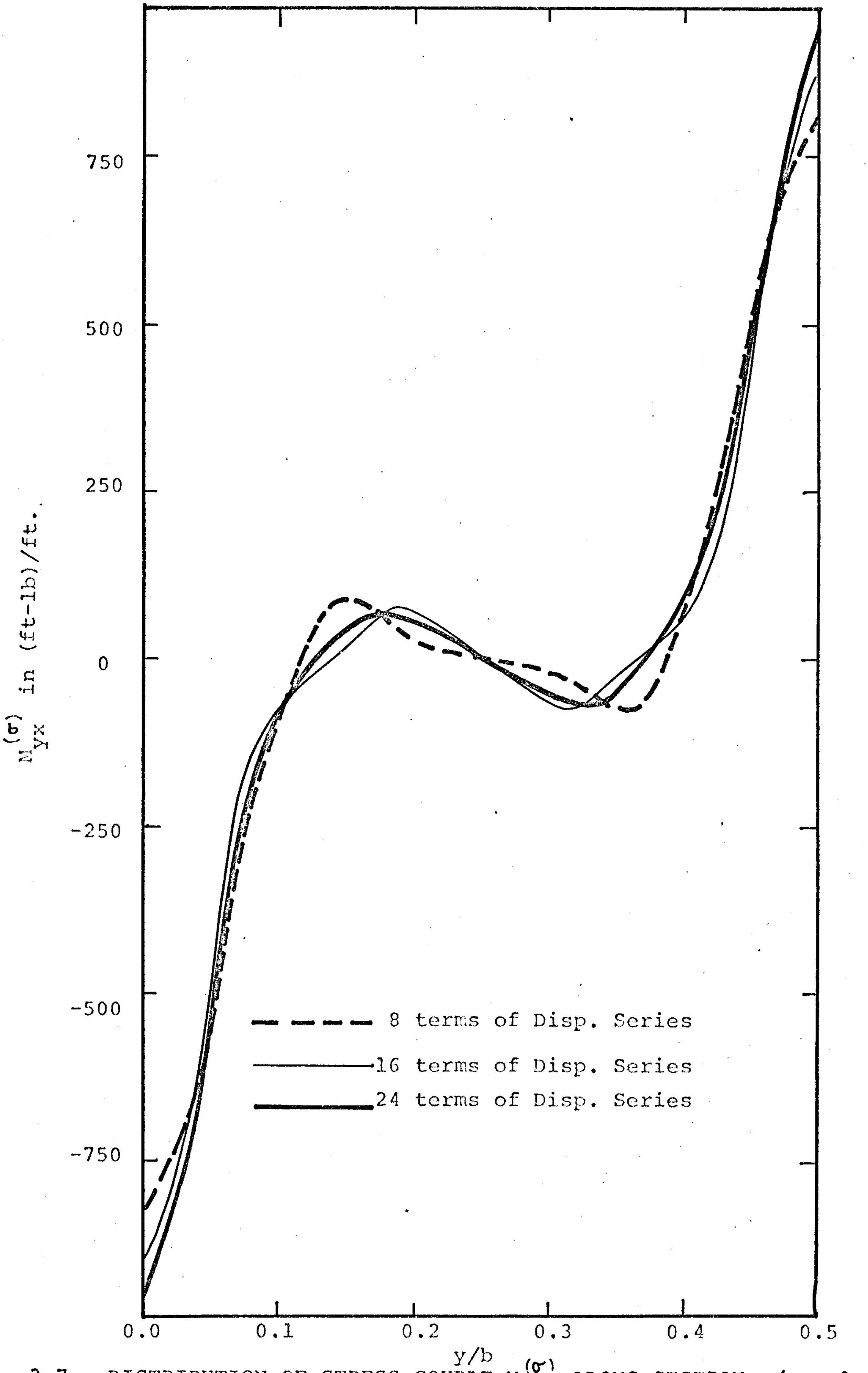

Fig. 3-7c DISTRIBUTION OF STRESS COUPLE $M_{yx}^{(\sigma)}$ ALONG SECTION x/a = 0.0

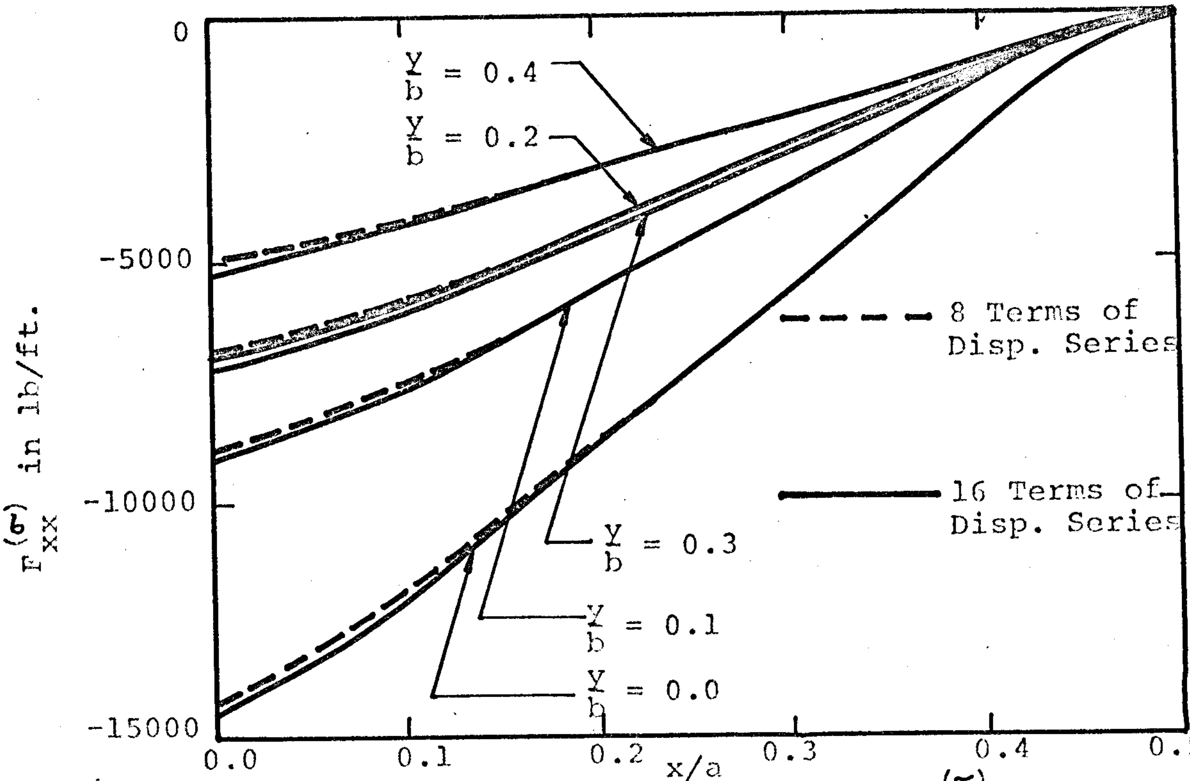

Fig. 3-8 DISTRIBUTION OF STRESS RESULTANTS $F_{xx}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES (a = 70', b = 35')

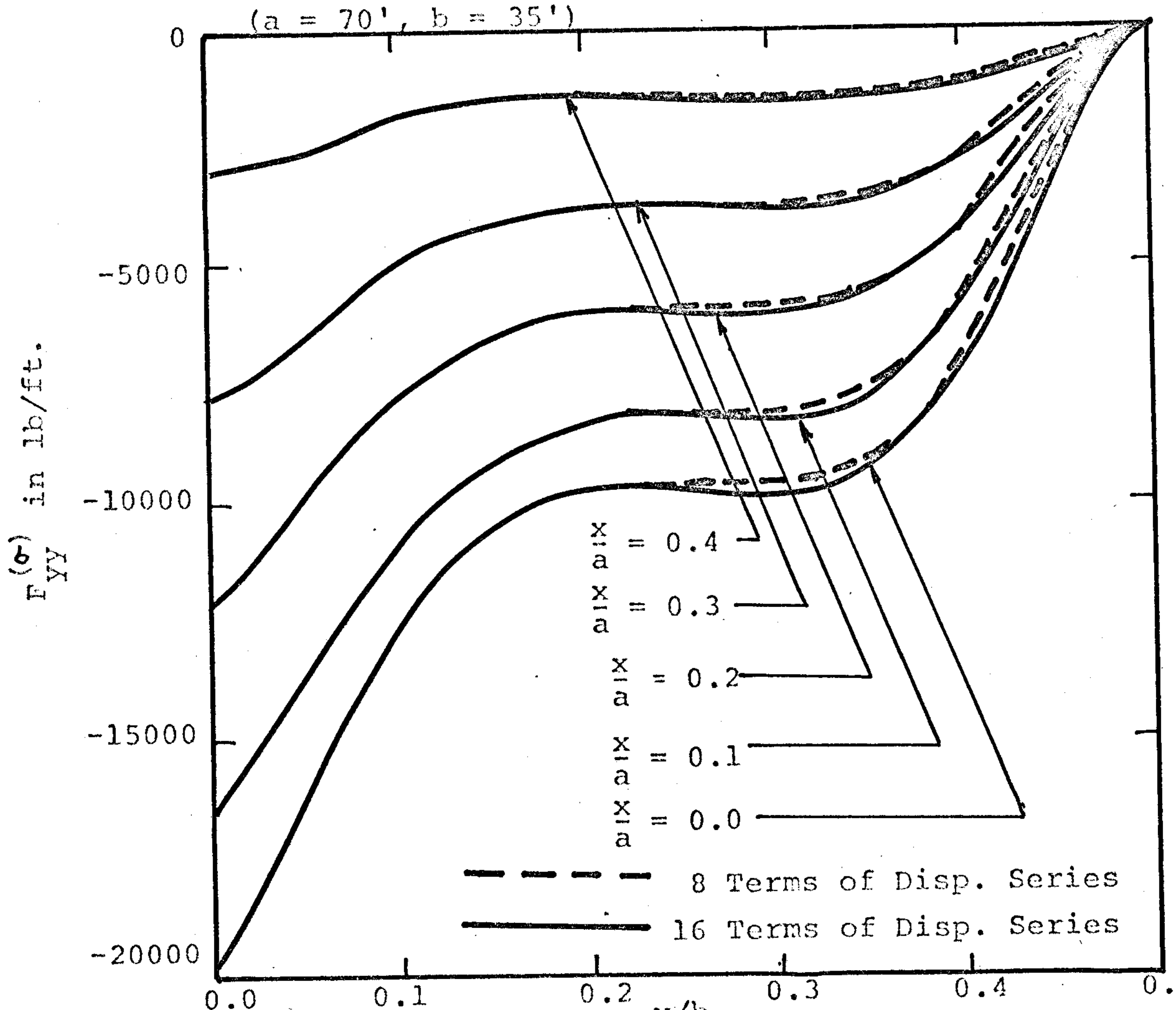

Fig. 3-9 DISTRIBUTION OF STRESS RESULTANT $F_{yy}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES. (a = 70', b = 35')

show that even less than 8 terms of vertical displacement series u_z might be sufficient for calculating the stress resultants $F_{xx}^{(\sigma)}$ and $F_{yy}^{(\sigma)}$.

3-2-2. Shell with One Pair of Edges Fixed and Another Pair of Edges Simply Supported

In this case, the same shell, as in the last section, except that one pair of edges at $y = \pm b/2$ become simply supported, is calculated by expanding the vertical displacement series u_z into 25 terms through expression (II-4-4). All stress resultants $F_{xx}^{(\sigma)}$, $F_{yy}^{(\sigma)}$ and stress couples $M_{xy}^{(\sigma)}$, $M_{yx}^{(\sigma)}$ are calculated through expressions (II-3-6). The results of $F_{xx}^{(\sigma)}$, $F_{yy}^{(\sigma)}$, $M_{xy}^{(\sigma)}$, $M_{yx}^{(\sigma)}$ and u_z are depicted in Figures (3-13), (3-14), (3-11), (3-12) and (3-15).

3-2-3. Shell with All Four Edges Simply Supported

The same shell is calculated except that now all its edges are simply supported. The vertical displacement series u_z is expanded into 25 terms through expression (II-5-4). Stress resultants $F_{xx}^{(\sigma)}$, $F_{yy}^{(\sigma)}$ and stress couples $M_{xy}^{(\sigma)}$ and $M_{yx}^{(\sigma)}$ are calculated through expressions (II-3-6) and are graphed as shown in Figures (3-20), (3-18), (3-19), (3-16) and (3-17).

3-3. Special Case - Thin Shallow Spherical Shell

For the thin spherical shell, an example is given with the following data: $a = 40'$, $b = 40'$, $k_x^{(n)} = 0.02$, $k_y^{(n)} = 0.02$, $P = 90\ \text{lb/ft}^2$, $h = 4"$, $E = 3\ 10^6\ \text{lb/in}^2$, $\nu = 0.16$. $F_{xx}^{(\sigma)}$, $F_{yy}^{(\sigma)}$, $M_{xy}^{(\sigma)}$ and $M_{yx}^{(\sigma)}$ were calculated for this shell.

3-3-1. Shell with All Four Edges Fixed

Following the same procedurs employed in the last sections,

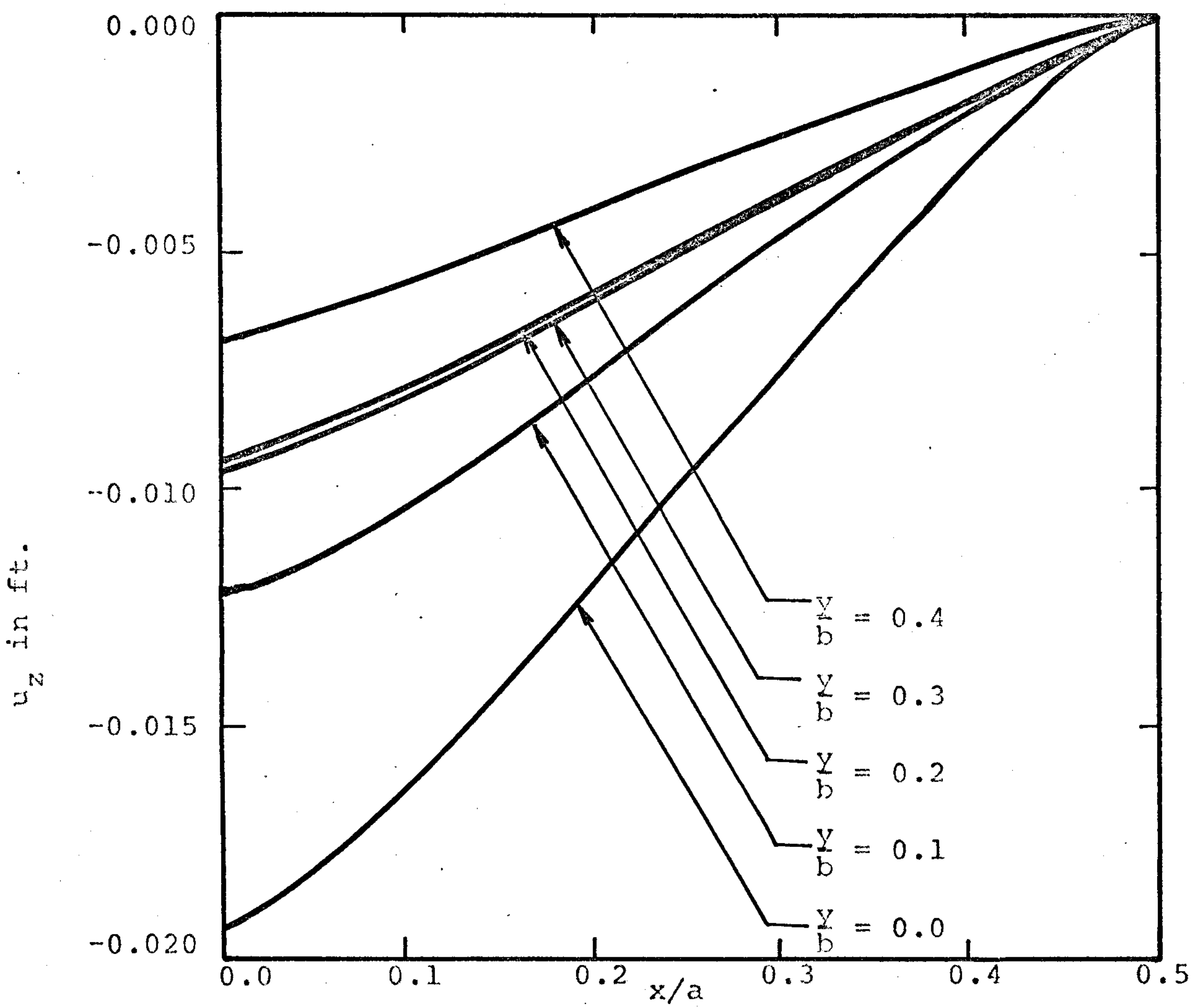

Fig. 3-10a DISTRIBUTION OF NORMAL DISPLACEMENT u_z OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES. (a = 70', b = 35')

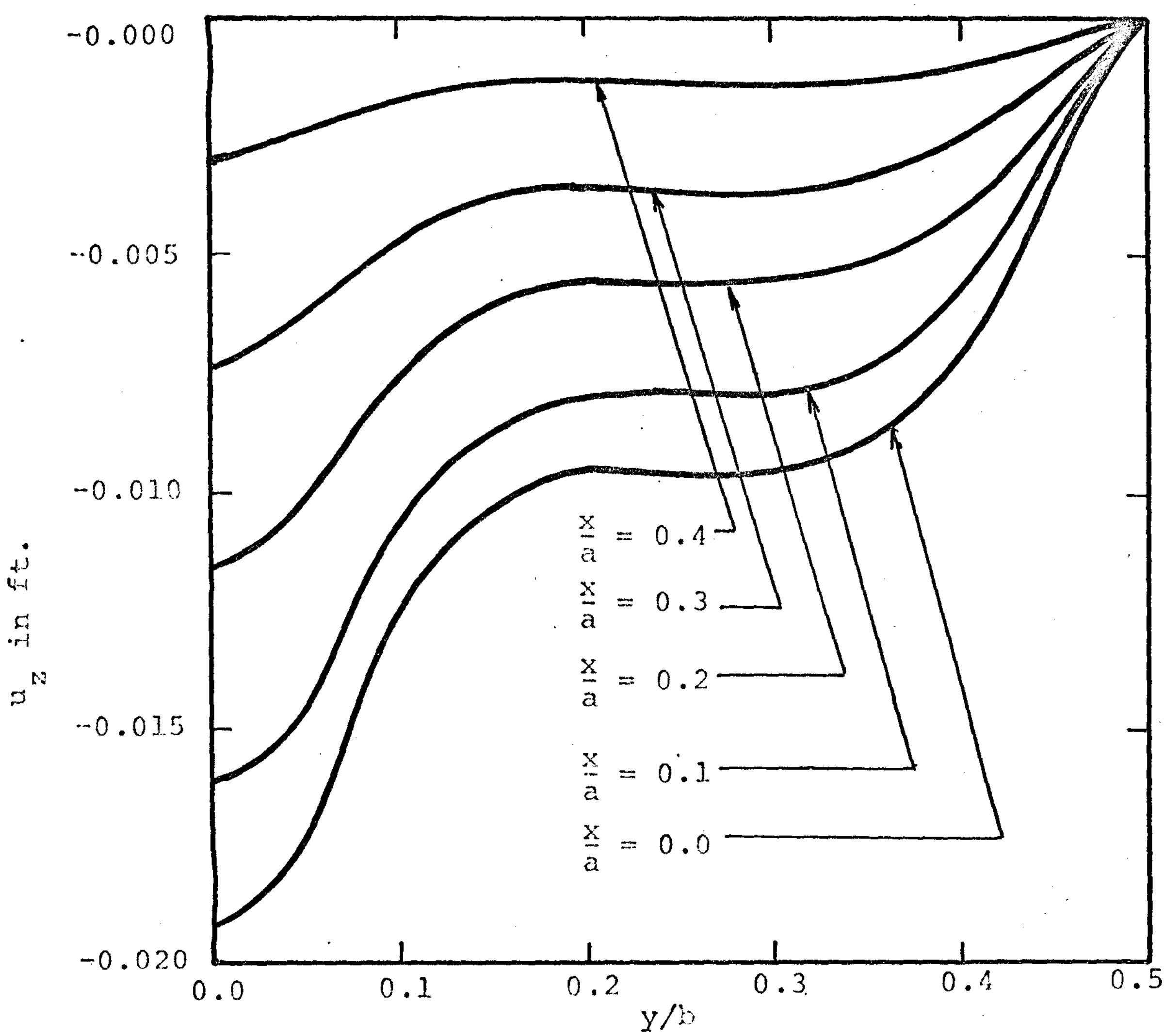

Fig. 3-10b DISTRIBUTION OF NORMAL DISPLACEMENT u_z OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH FIXED BOUNDARIES. (a = 70', b = 35')

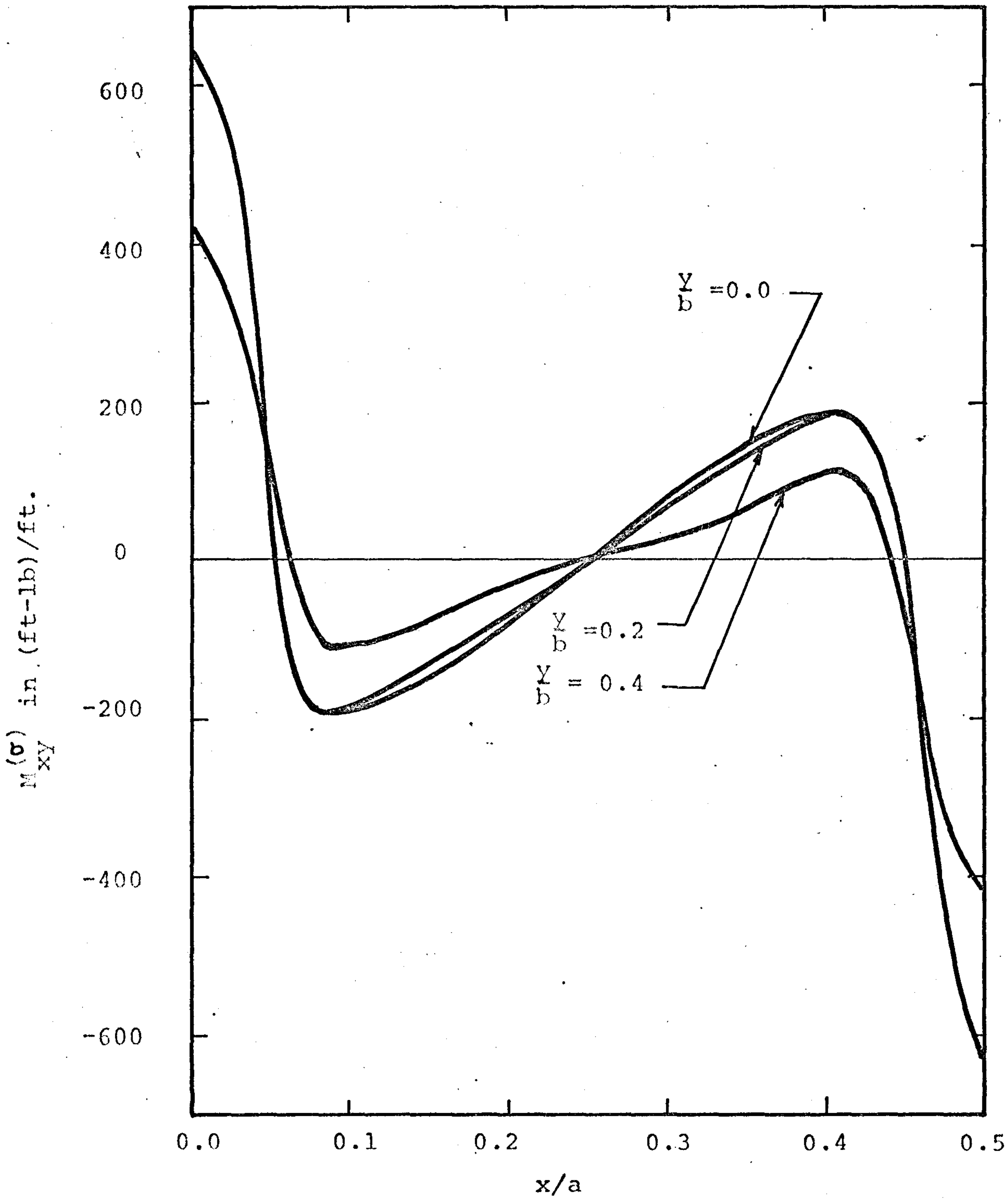

Fig. 3-11 DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH MIXED BOUNDARY CONDITIONS.

Boundaries $x = \pm a/2$ fixed. Boundaries $y = \pm b/2$ simply supported.

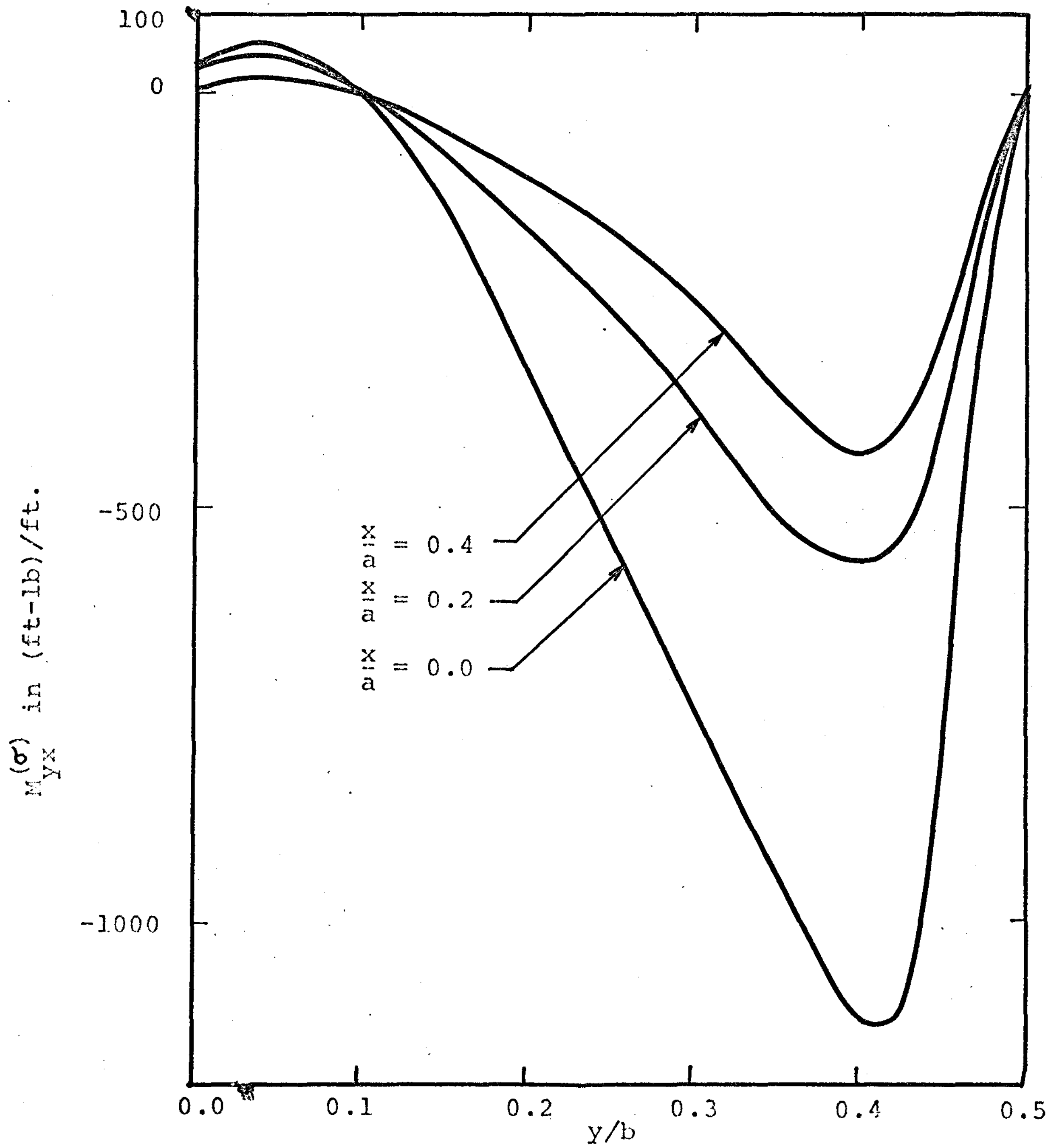

Fig. 3-12 DISTRIBUTION OF STRESS COUPLE $M_{yx}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH MIXED BOUNDARY CONDITIONS.

Boundaries x = ±a/2 fixed
Boundaries y = ±b/2 simply supported

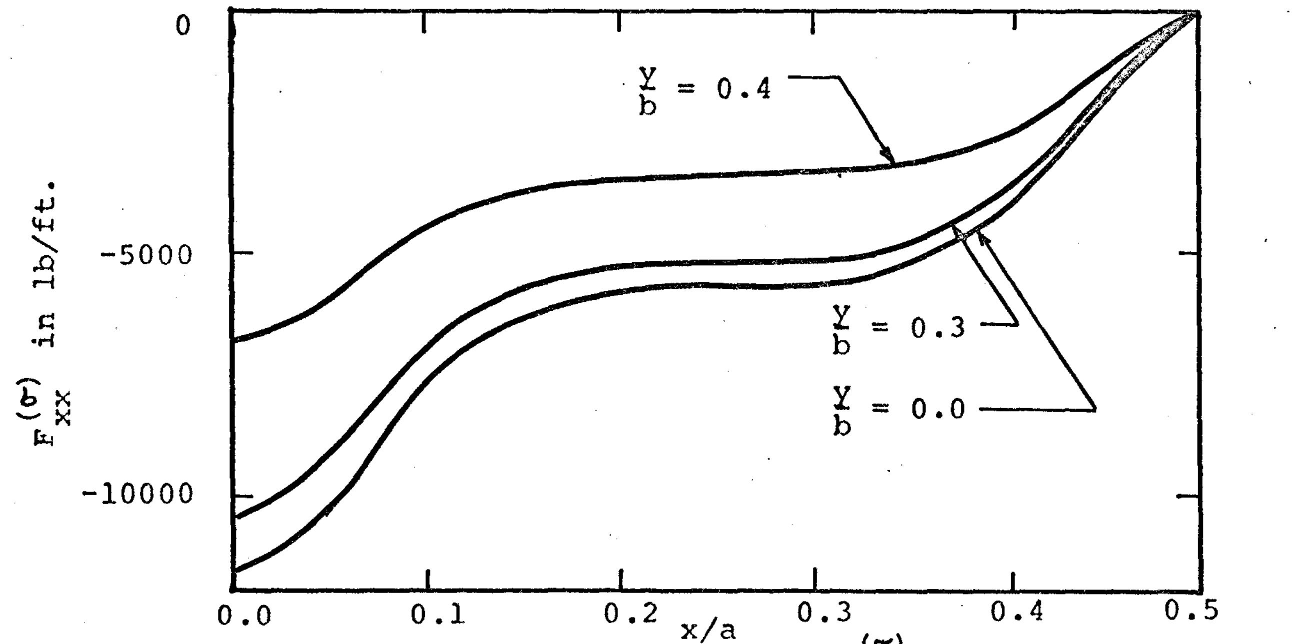

Fig. 3-13 DISTRIBUTION OF STRESS RESULTANT $F_{xx}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH MIXED BOUNDARY CONDITIONS. (a = 70', b = 35') Boundaries x = ±a/2 fixed
Boundaries y = ±b/2 simply supported

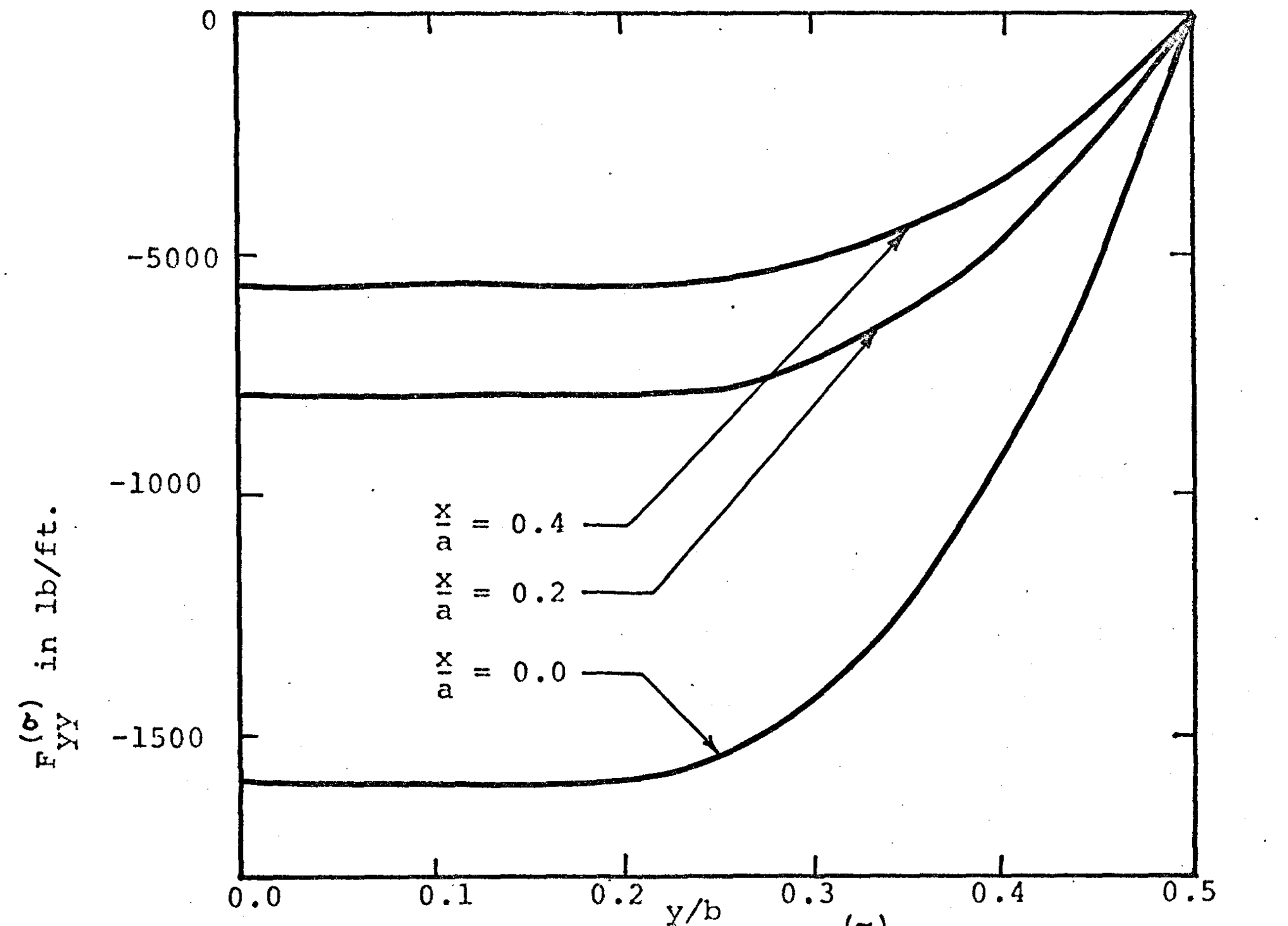

Fig. 3-14 DISTRIBUTION OF STRESS RESULTANT $F_{yy}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH MIXED BOUNDARY CONDITIONS. (a = 70', b = 35') Boundaries x = ±a/2 fixed
Boundaries y = ±b/2 simply supported

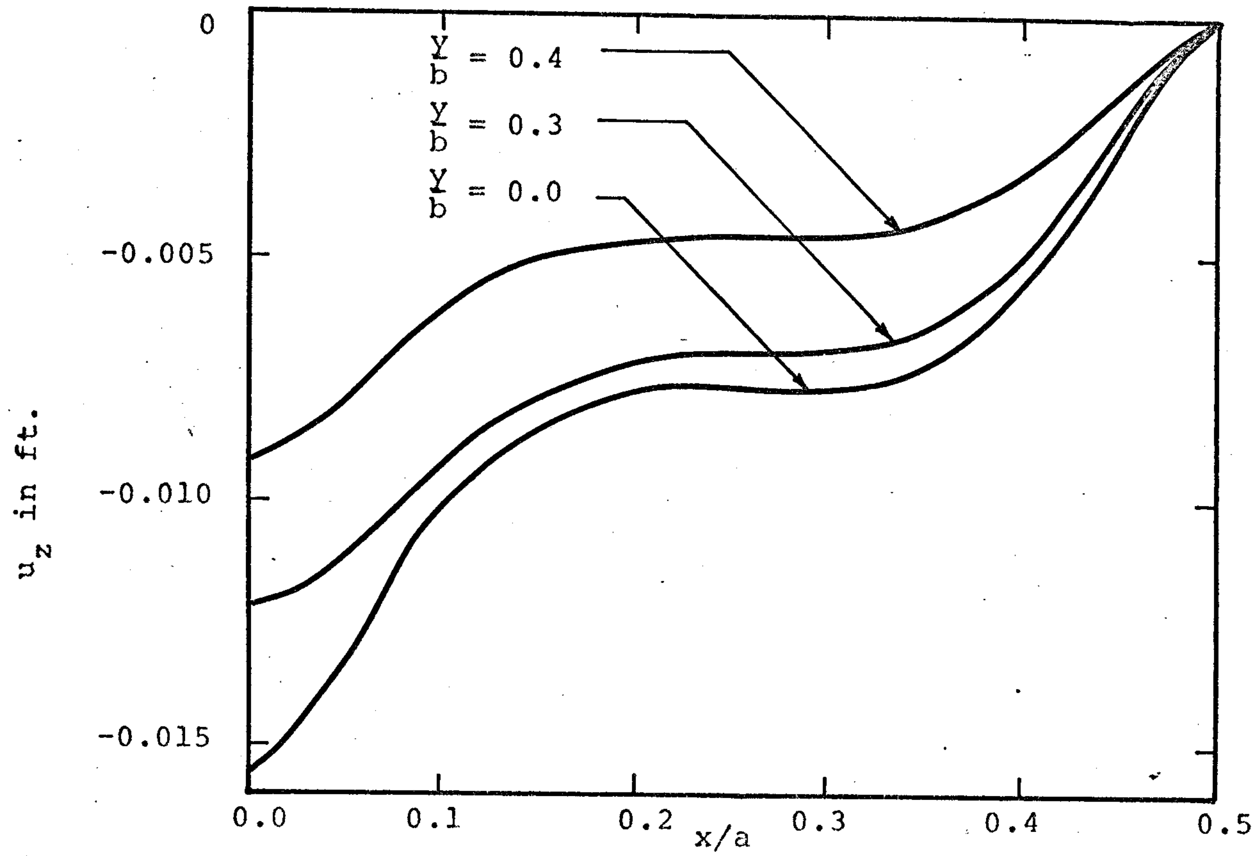

Fig. 3-15a DISTRIBUTION OF NORMAL DISPLACEMENT u_z ALONG y/b = constants.

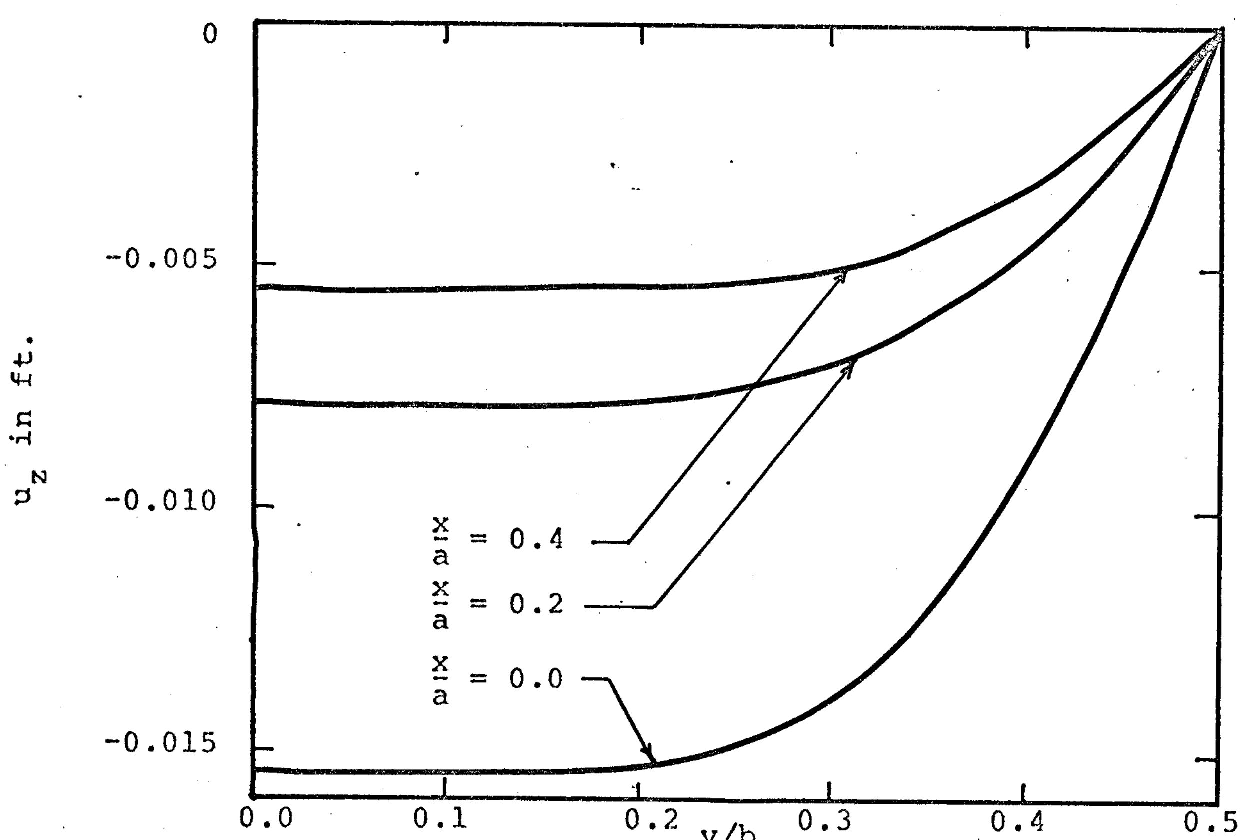

Fig. 3-15b DISTRIBUTION OF NORMAL DISPLACEMENT u_z ALONG x/a = constants.

NOTE: Fig. 3-15 DEPICTS DISTRIBUTION OF NORMAL DISPLACEMENT u_z OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH MIXED BOUNDARY CONDITIONS.
Boundaries x = ±a/2 fixed. Boundaries y = ± b/2 simply supported

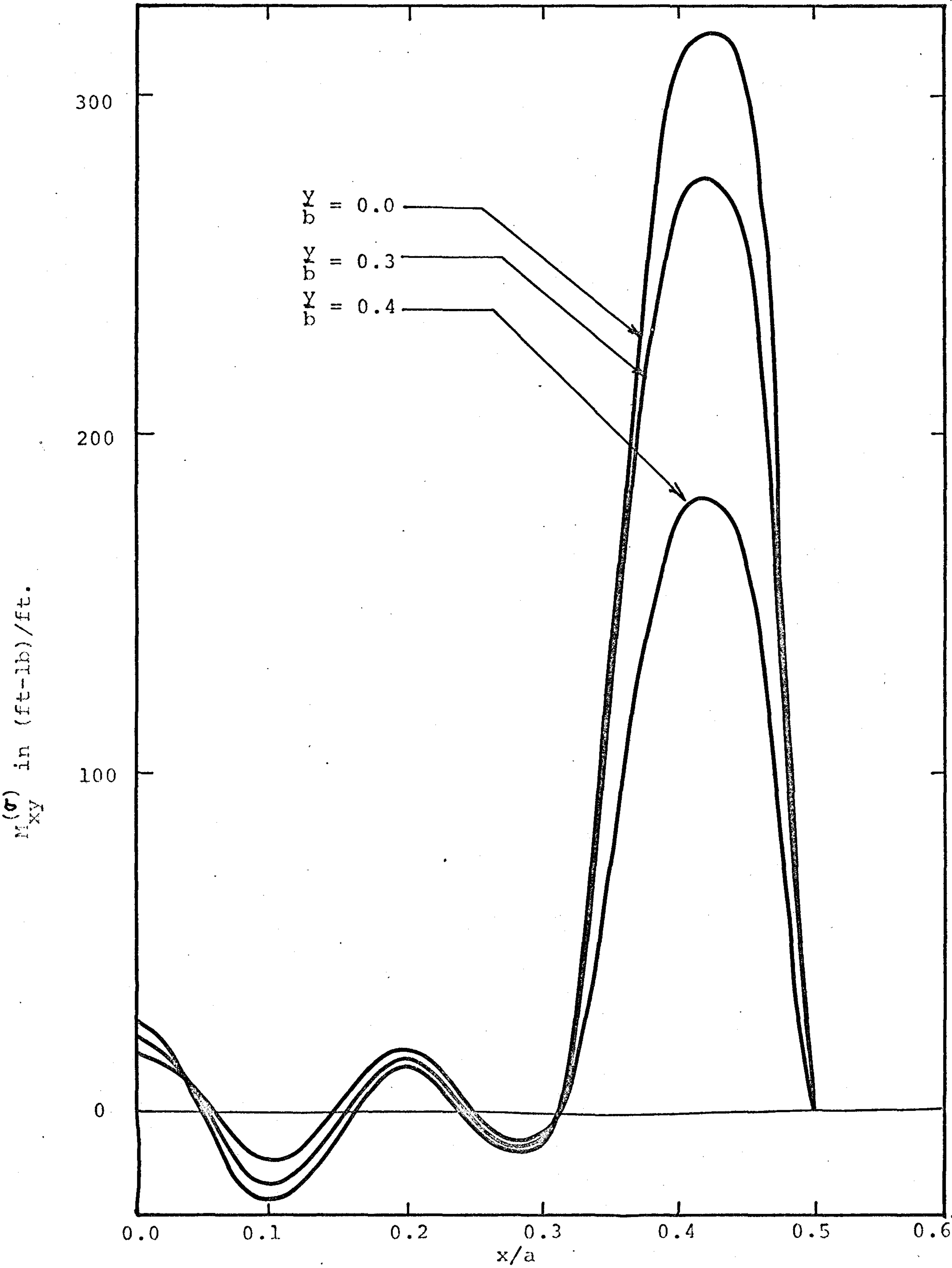

Fig. 3-16 DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH SIMPLY SUPPORTED BOUNDARIES. (a = 70', b = 35')

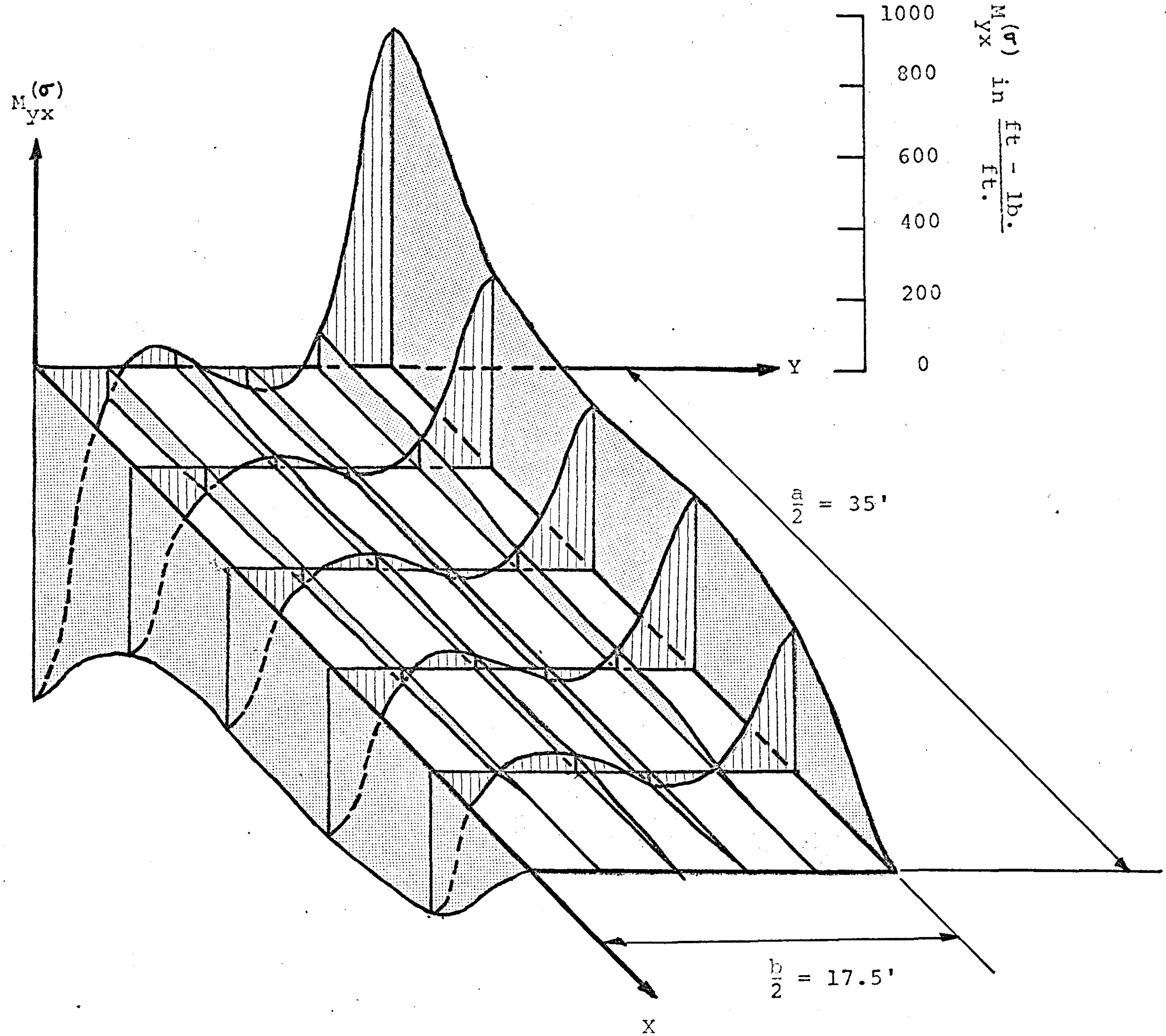

Fig. 3-17 DISTRIBUTION OF STRESS COUPLE $M_{yx}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH SIMPLY SUPPORTED BOUNDARIES. (a = 70', b = 35')

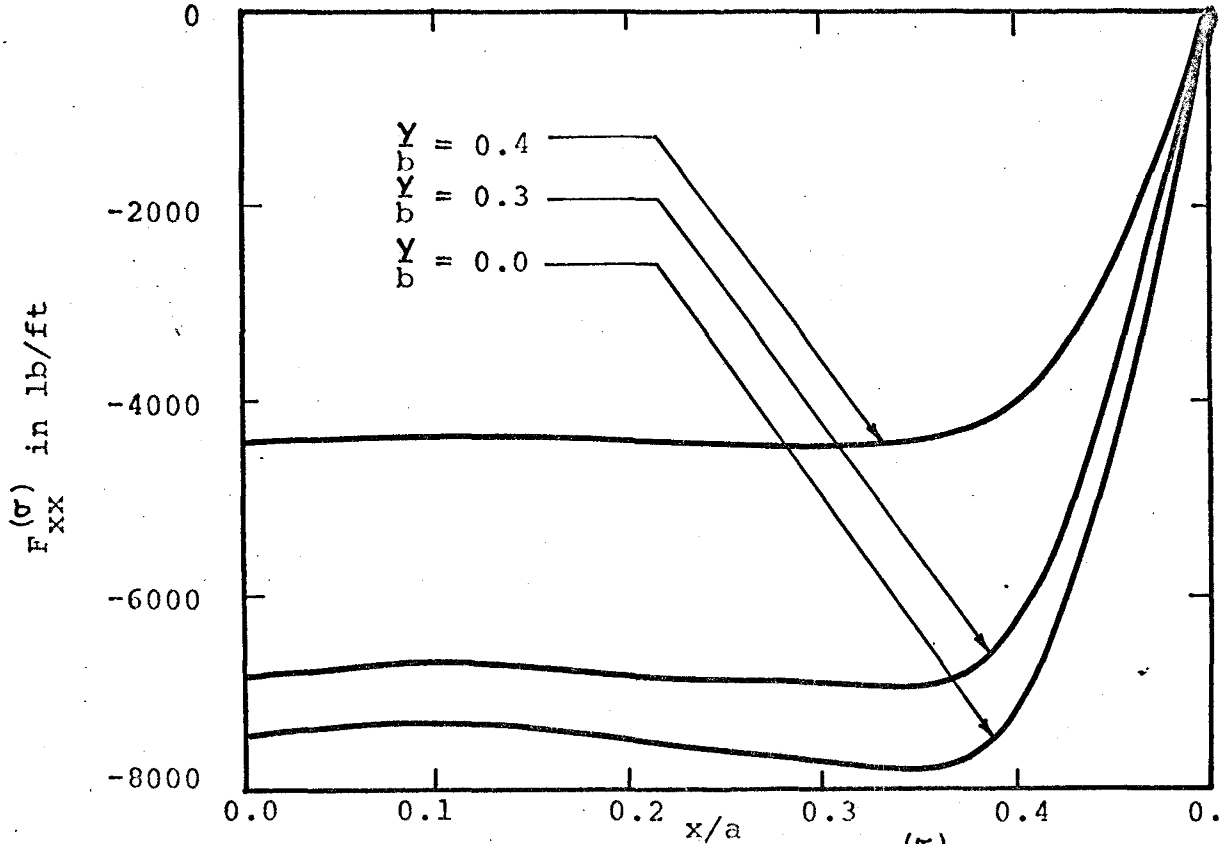

Fig. 3-18 DISTRIBUTION OF STRESS RESULTANT $F_{xx}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH SIMPLY SUPPORTED BOUNDARIES. (a = 70', b = 35')

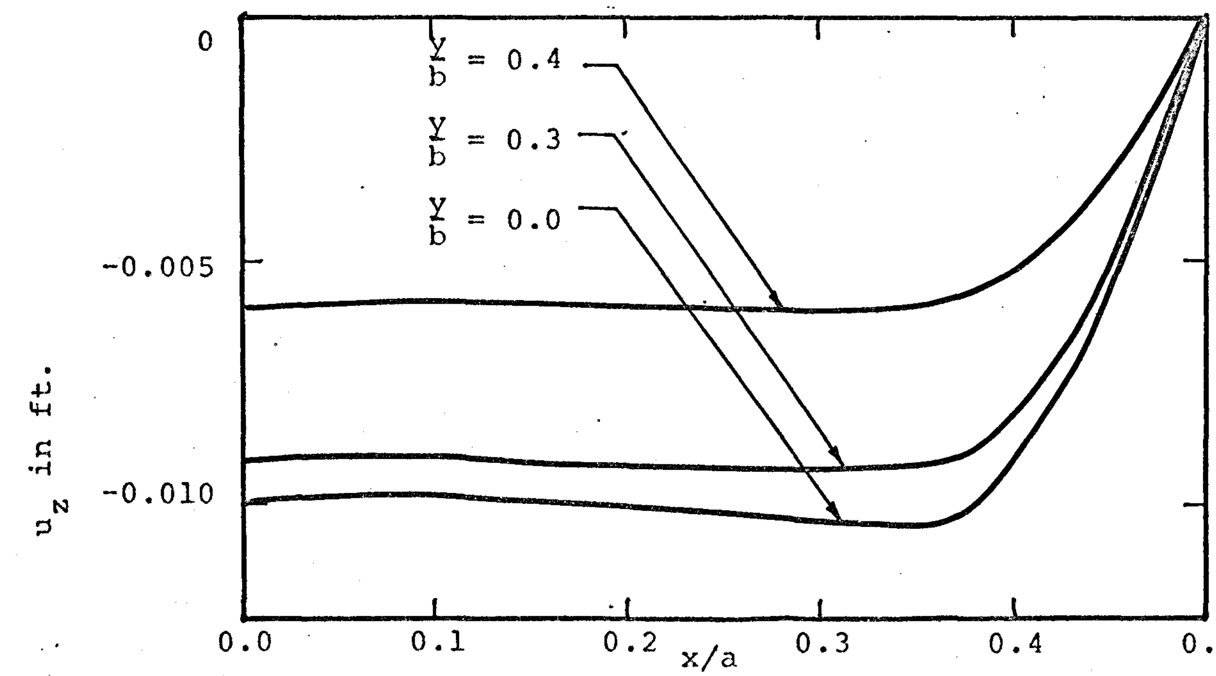

Fig. 3-20a DISTRIBUTION OF NORMAL DISPLACEMENT u_z OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH SIMPLY SUPPORTED BOUNDARIES. (a = 70', b = 35')

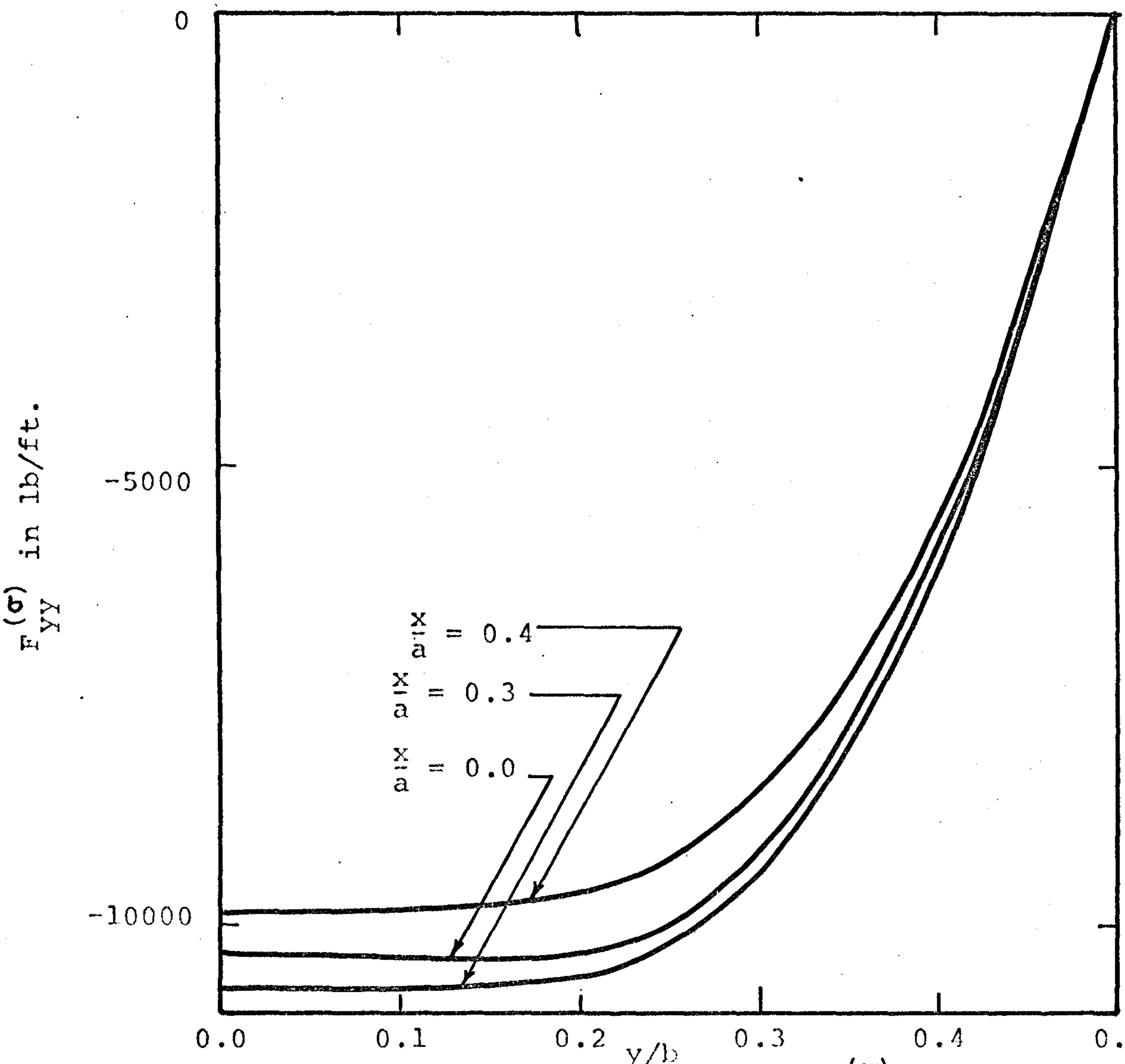

Fig. 3-19 DISTRIBUTION OF STRESS RESULTANT $F_{yy}^{(\sigma)}$ OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH SIMPLY SUPPORTED BOUNDARIES. (a = 70', b = 35')

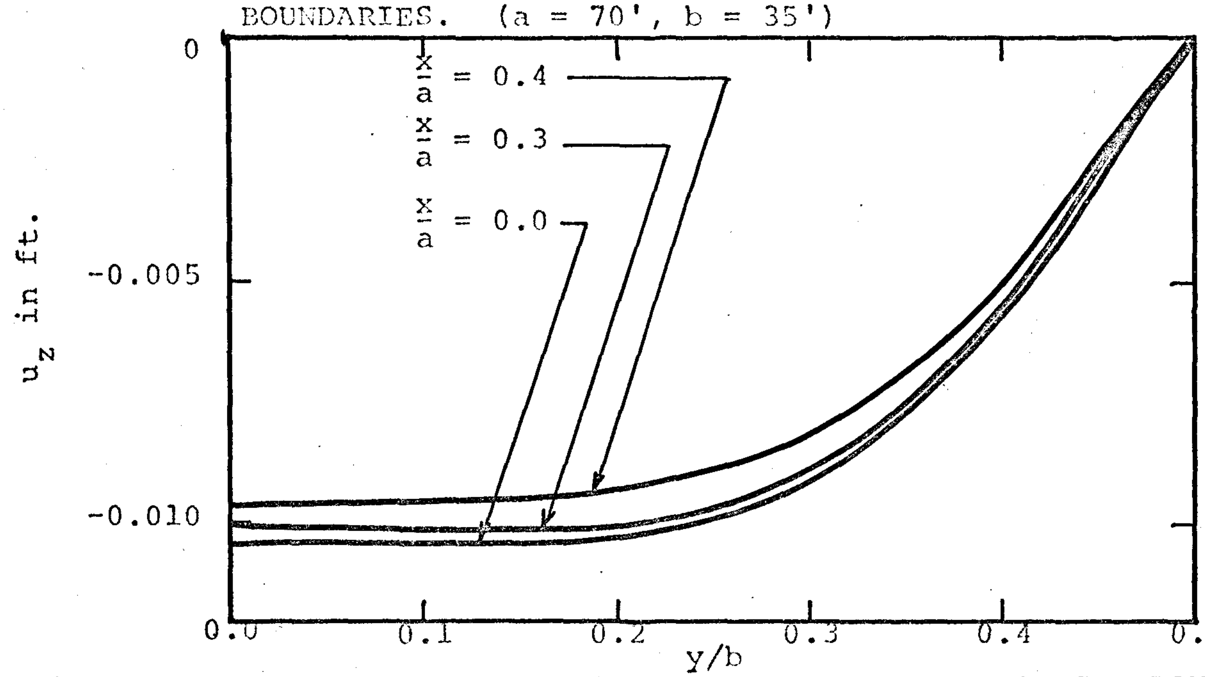

Fig. 3-20b DISTRIBUTION OF NORMAL DISPLACEMENT u_z OF SHALLOW ELLIPTIC PARABOLOIDAL SHELL OF TRANSLATION WITH SIMPLY SUPPORTED BOUNDARIES. (a = 70', b = 35')

the values of Fourier coefficient are calculated through expressions (II-3-5) and (II-3-8), and are listed in Table (III-3) as shown. It can be observed that the differences between these two sets of values is always less than 1%. So the approximation by using expression (II-3-8) instead of expression (II-3-5) is even more suitable here than it was in section (III-1-1). The next step is to calculate the vertical displacement u_z, the stress resultants $F_{xx}^{(\sigma)}$, $F_{yy}^{(\sigma)}$ and the stress couples $M_{xy}^{(\sigma)}$ and $M_{yx}^{(\sigma)}$ at various points in the middle surface of the shell. All results are plotted in Figures (3-1), (3-22), (3-24), (3-21) and (3-23). Since the shell is completely symmetric, therefore, $F_{xx}^{(\sigma)} = F_{yy}^{(\sigma)}$ and $M_{xy}^{(\sigma)} = M_{yx}^{(\sigma)}$ at corresponding points. This is to say that the absolute values of $M_{xy}^{(\sigma)}$ and $F_{xx}^{(\sigma)}$ along some section y = constant are exactly the same as those of $M_{yx}^{(\sigma)}$ and $F_{yy}^{(\sigma)}$ along the corresponding section x = constant. Therefore, Figures of $M_{yx}^{(\sigma)}$ and $F_{yy}^{(\sigma)}$ are omitted.

From Figure (3-23c), it is noticed that even if 16 terms of u_z would have been used, the convergency of $M_{xy}^{(\sigma)}$ could still not have been satisfactory in certain intervals, such as x/a = 0.225 to x/a = 0.275, of the shell. Fortunately, in this interval the absolute value of $M_{xy}^{(\sigma)}$ is much smaller than those at the edge or at the apex, and its evaluation is not too important for actual design. If 24 terms in the series of u_z would have been used, the series for stress couples $M_{xy}^{(\sigma)}$ could have converged more satisfactorily at every point in the middle surface of the shell.

The convergence of $F_{xx}^{(\sigma)}$ is as good as it is in the last

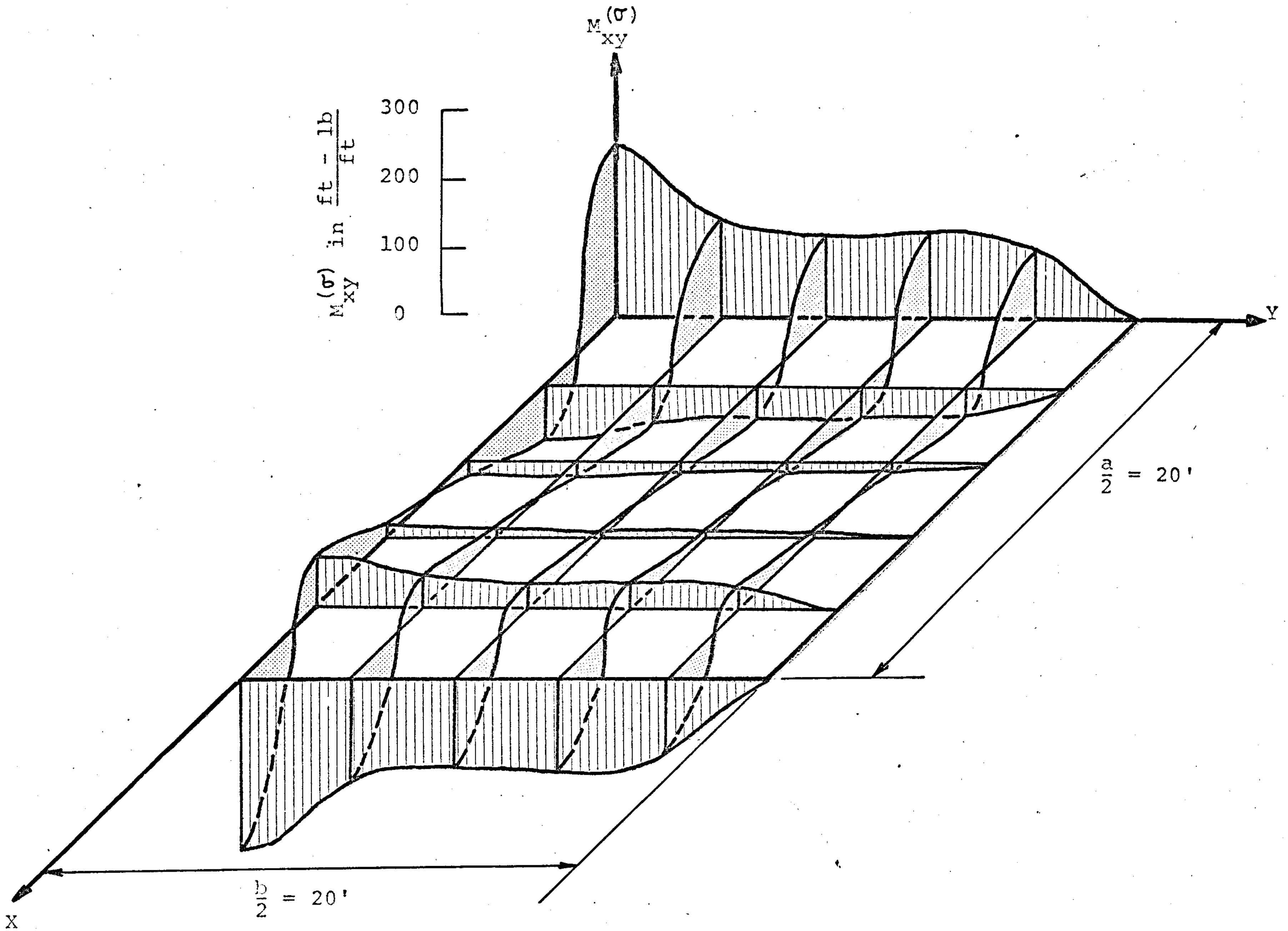

Fig. 3-21 DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ OF SHALLOW SPHERICAL SHELL WITH FIXED BOUNDARIES.

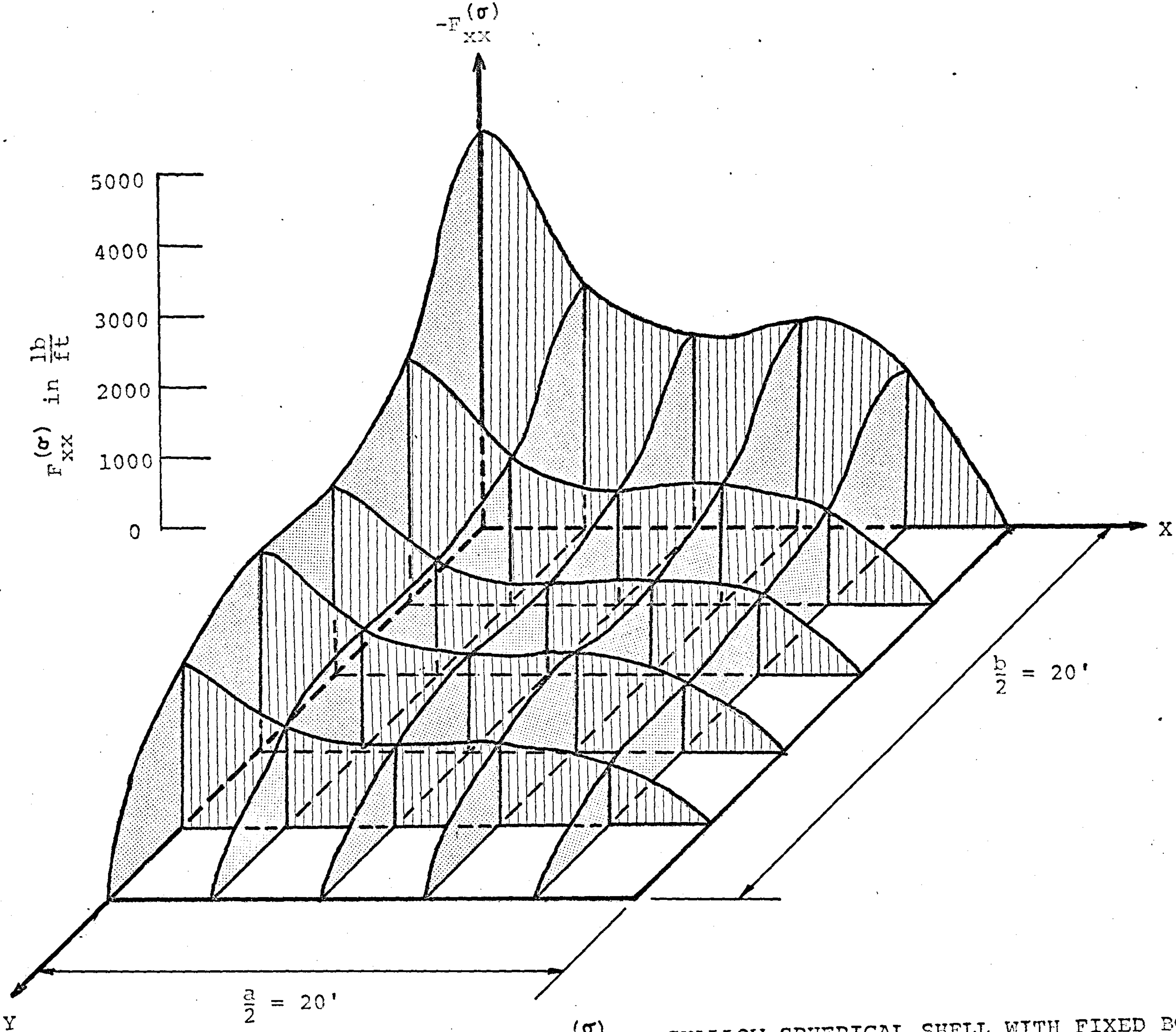

Fig. 3-22 DISTRIBUTION OF STRESS RESULTANT $F_{xx}^{(\sigma)}$ OF SHALLOW SPHERICAL SHELL WITH FIXED BOUNDARIES

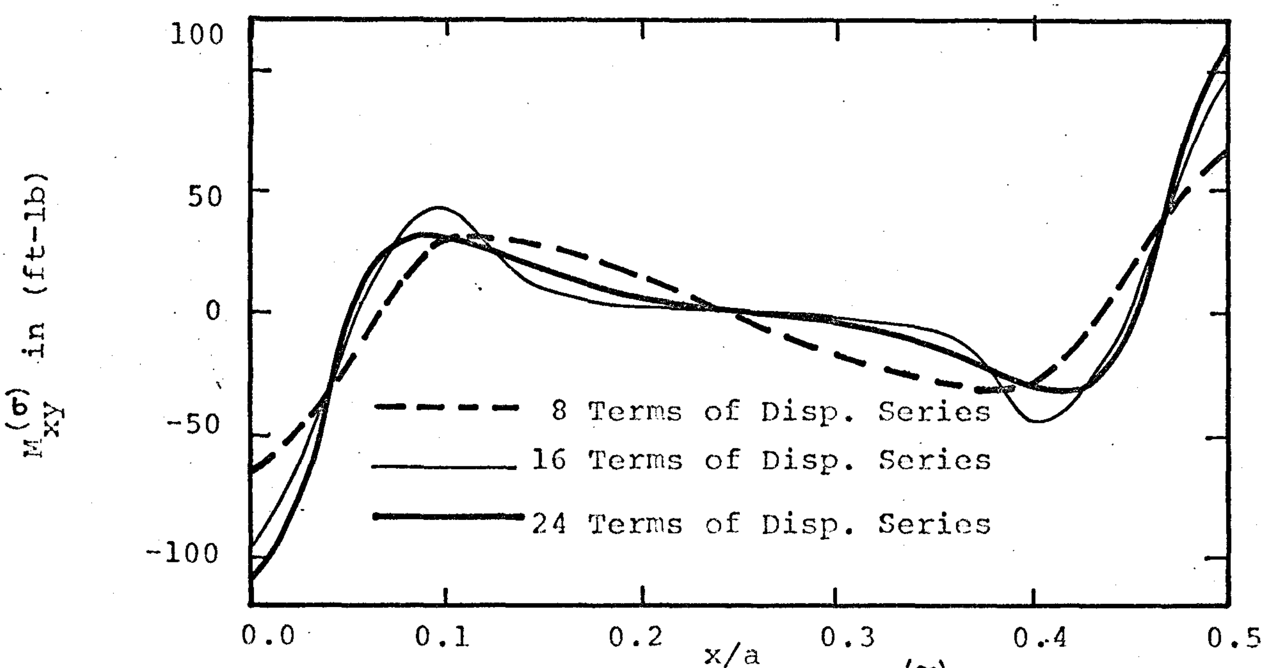

Fig. 3-23a DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ ALONG SECTION y/b = 0.4

NOTE: Fig. 3-23 DEPICTS DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ OF SHALLOW SPHERICAL SHELL WITH FIXED BOUNDARIES. (a = b = 40')

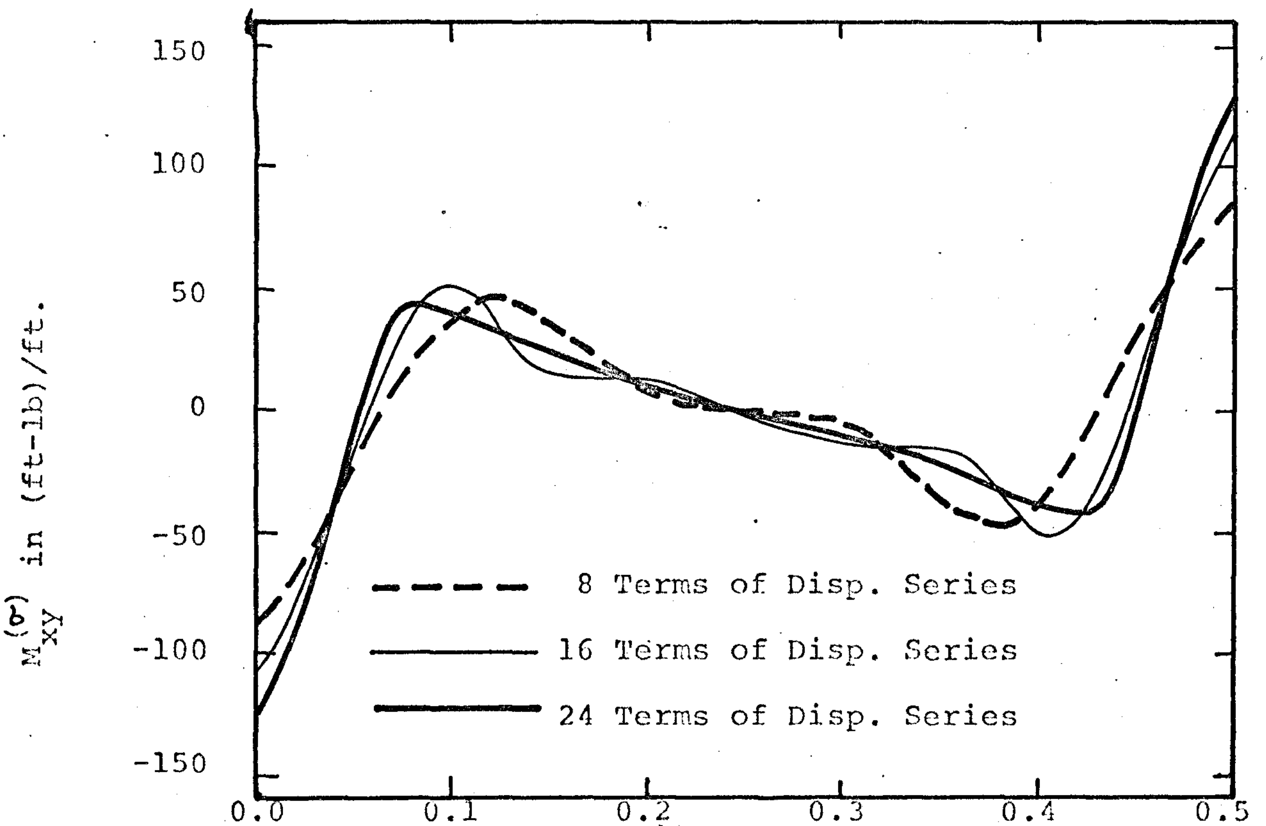

Fig. 3-23b DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ ALONG SECTION y/b = 0.2

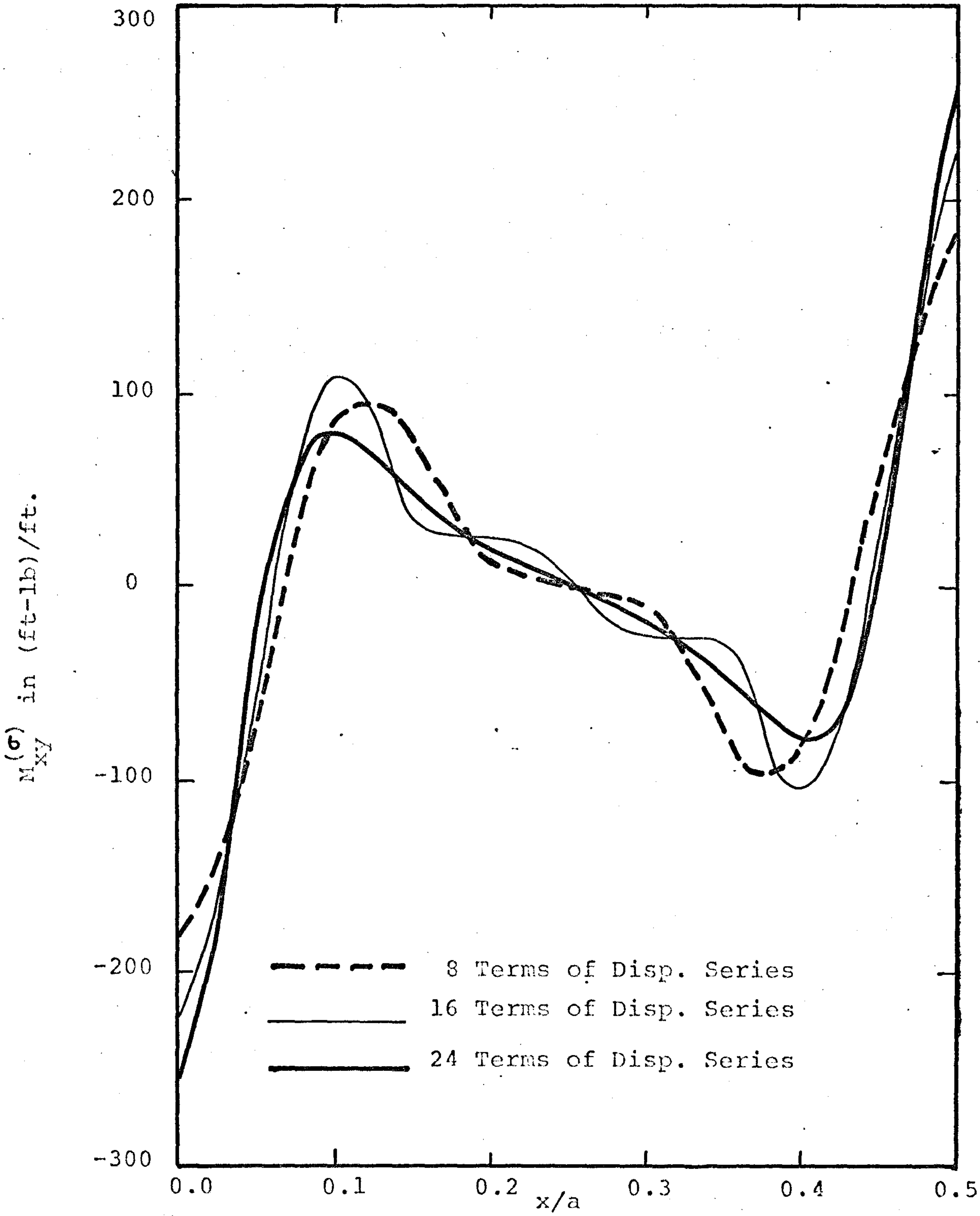

Fig. 3-23c DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ ALONG SECTION y/b = 0.0

section. Figure (3-24) shows that eight terms of u_z will give a satisfactory value of $F_{xx}^{(\sigma)}$ at every point in the middle surface of the shell.

3-3-2. Shell with One Pair of Edges Fixed and the Other Pair of Edges Simply Supported

Similar to section (3-1-2), the edges of this shell at $x = \pm a/2$ are fixed, while at $y = \pm b/2$ they are simply supported, otherwise the shell is the same as in section (3-3-1). Values of $M_{xy}^{(\sigma)}$, $M_{yx}^{(\sigma)}$, $F_{xx}^{(\sigma)}$, $F_{yy}^{(\sigma)}$ and u_z are obtained through expressions (II-4-4) and (II-3-6), and are expressed in Figures (3-25), (3-26), (3-27), (3-28) and (3-29).

3-3-3. Shell with All Four Edges Simply Supported

In this case, the calculation procedure is the same as in section (3-2-3). The shell has the configurations as in section (3-3-1). Expressions (II-5-4) and (II-3-6) are used and all results of $M_{xy}^{(\sigma)}$, $F_{xx}^{(\sigma)}$ and u_z are plotted in Figures (3-30), (3-31), (3-32).

3-4. Influence of Strains

In this section, a study of the influence of membranal displacements u_x and u_y on the transverse displacement u_z for cases of fixed boundaries and simply supported boundaries is effected. Equation (II-1-2) is used to calculate the total strain energy of the shell. Finally, instead of the single expression (II-3-8) or (II-5-4), obtained by neglecting all effects of u_x and u_y, a set of three simultaneous equations (4-8a), (4-8b), (4-8c) or (4-15a), (4-15b), (4-15c) are obtained, (see Appendix D). These three simultaneous equations

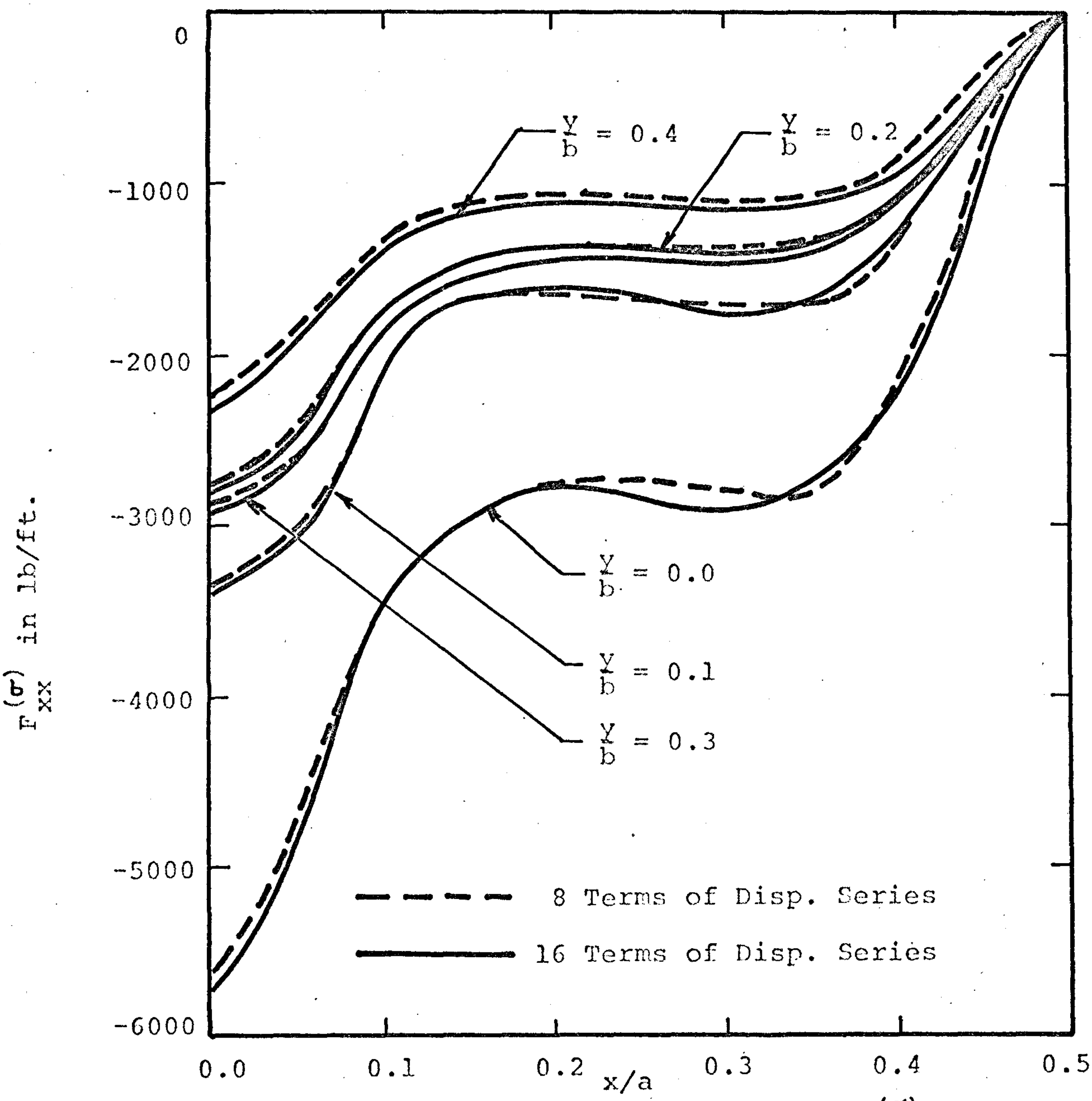

Fig. 3-24 DISTRIBUTION OF STRESS RESULTANT $F_{xx}^{(\sigma)}$ OF SHALLOW SPHERICAL SHELL WITH FIXED BOUNDARIES. (a = b = 40').

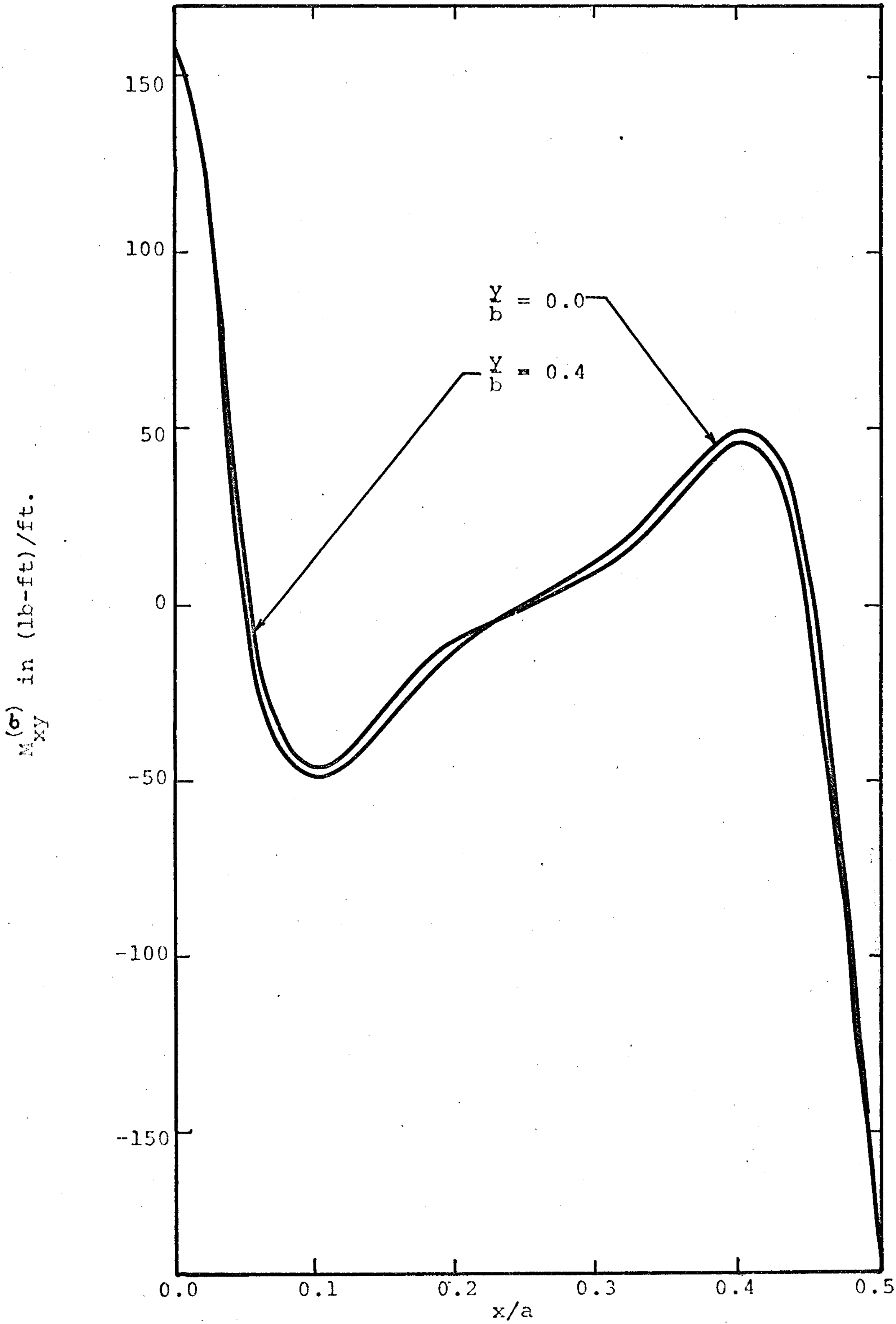

Fig. 3-25 DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ OF SHALLOW SPHERICAL SHELL WITH MIXED BOUNDARY CONDITIONS.
(a = b = 40') Boundaries x = ±a/2 fixed
Boundaries y = ±b/2 simply supported.

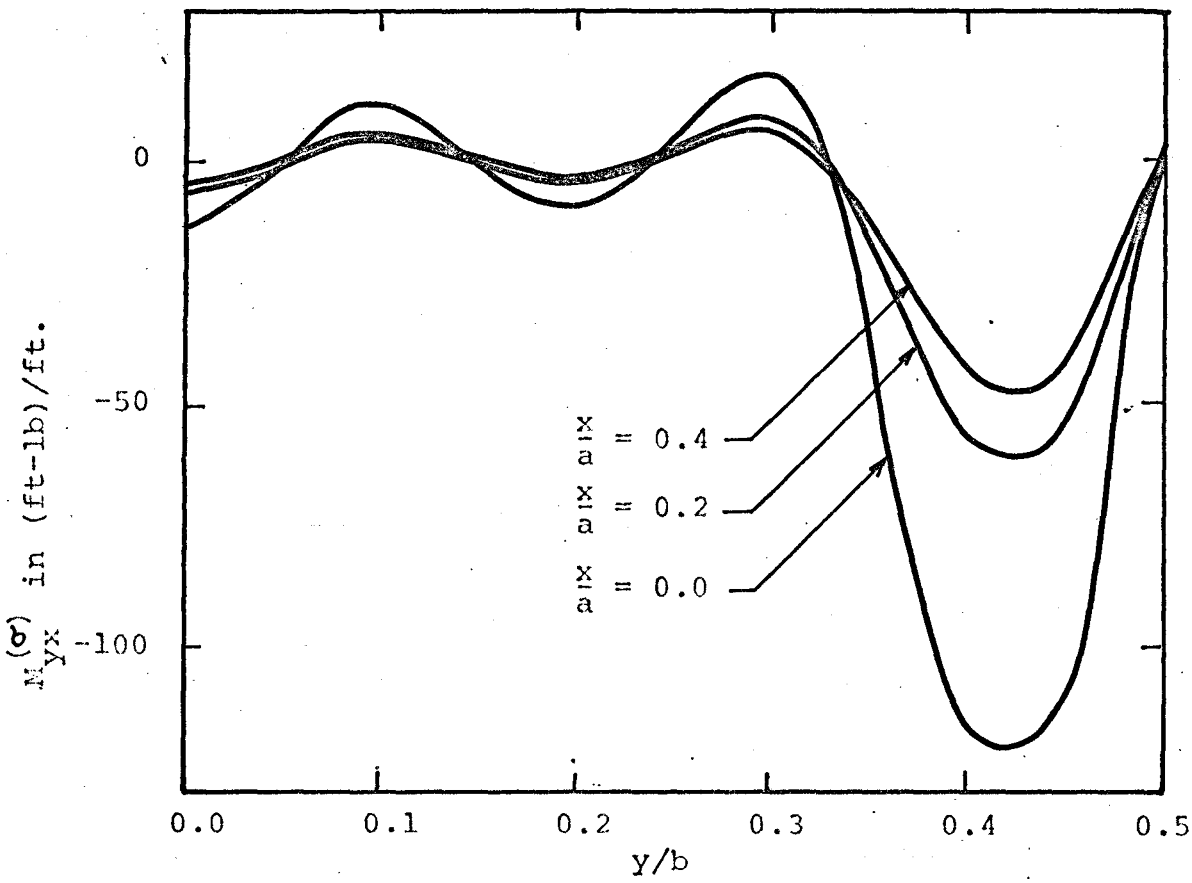

Fig. 3-26 DISTRIBUTION OF STRESS COUPLE $M_{yx}^{(\sigma)}$ OF A SHALLOW SPHERICAL SHELL WITH MIXED BOUNDARY CONDITIONS.
(a = b = 40')
Boundaries x = ±a/2 fixed.
Boundaries y = ±b/2 simply supported

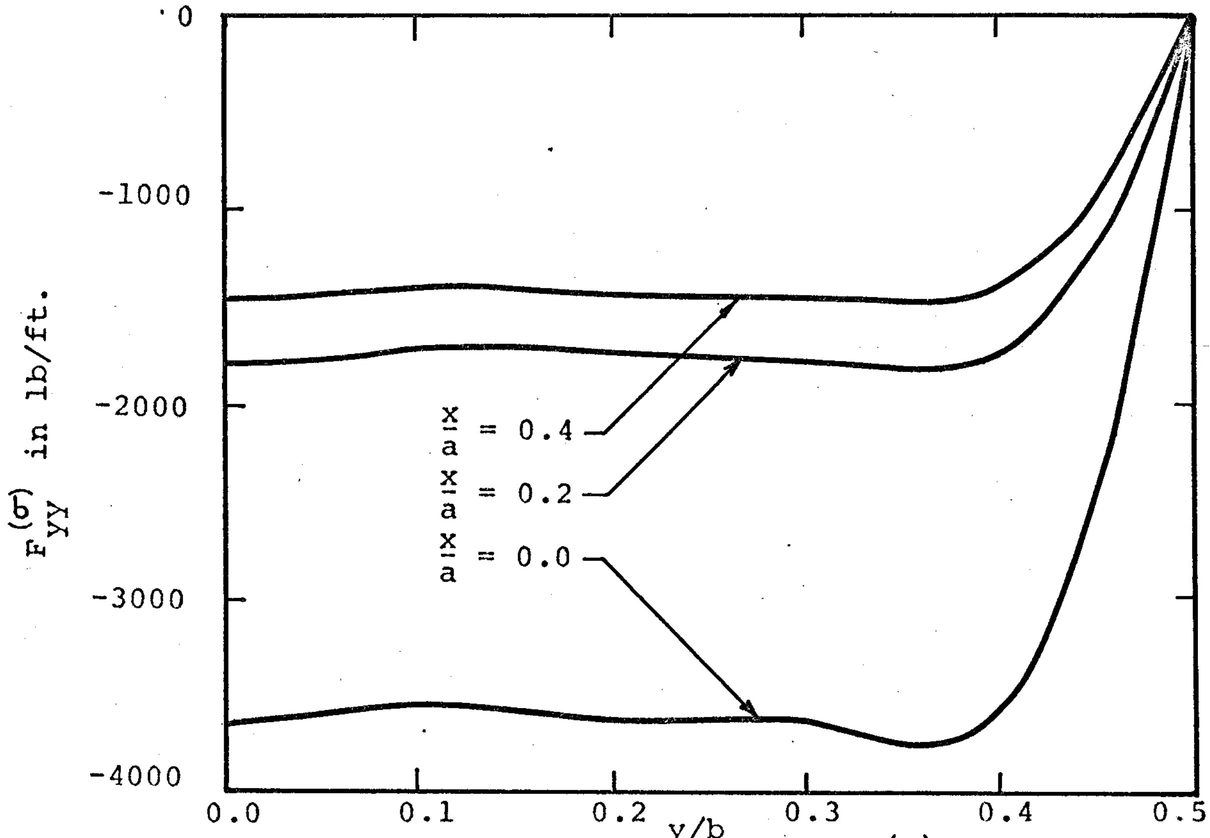

Fig. 3-28 DISTRIBUTION OF STRESS RESULTANT $F_{yy}^{(\sigma)}$ OF SHALLOW SPHERICAL SHELL OF MIXED BOUNDARY CONDITIONS. (a = b = 40')
Boundaries x = ±a/2 fixed. Boundaries y = ±b/2 simply supported

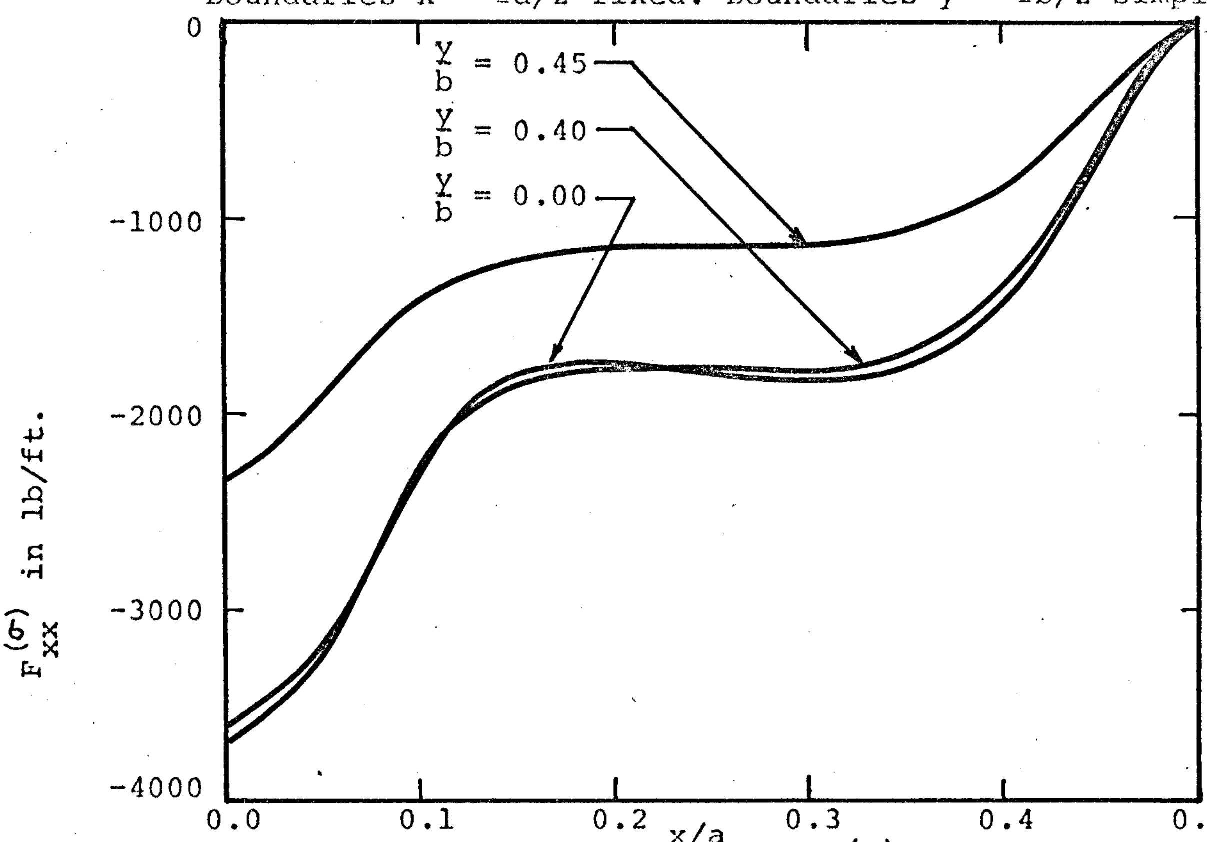

Fig. 3-27 DISTRIBUTION OF STRESS RESULTANT $F_{xx}^{(\sigma)}$ OF SHALLOW SPHERICAL SHELL WITH MIXED BOUNDARY CONDITIONS. (a = b = 40')
Boundaries x = ±a/2 fixed. Boundaries y = ±b/2 simply supported

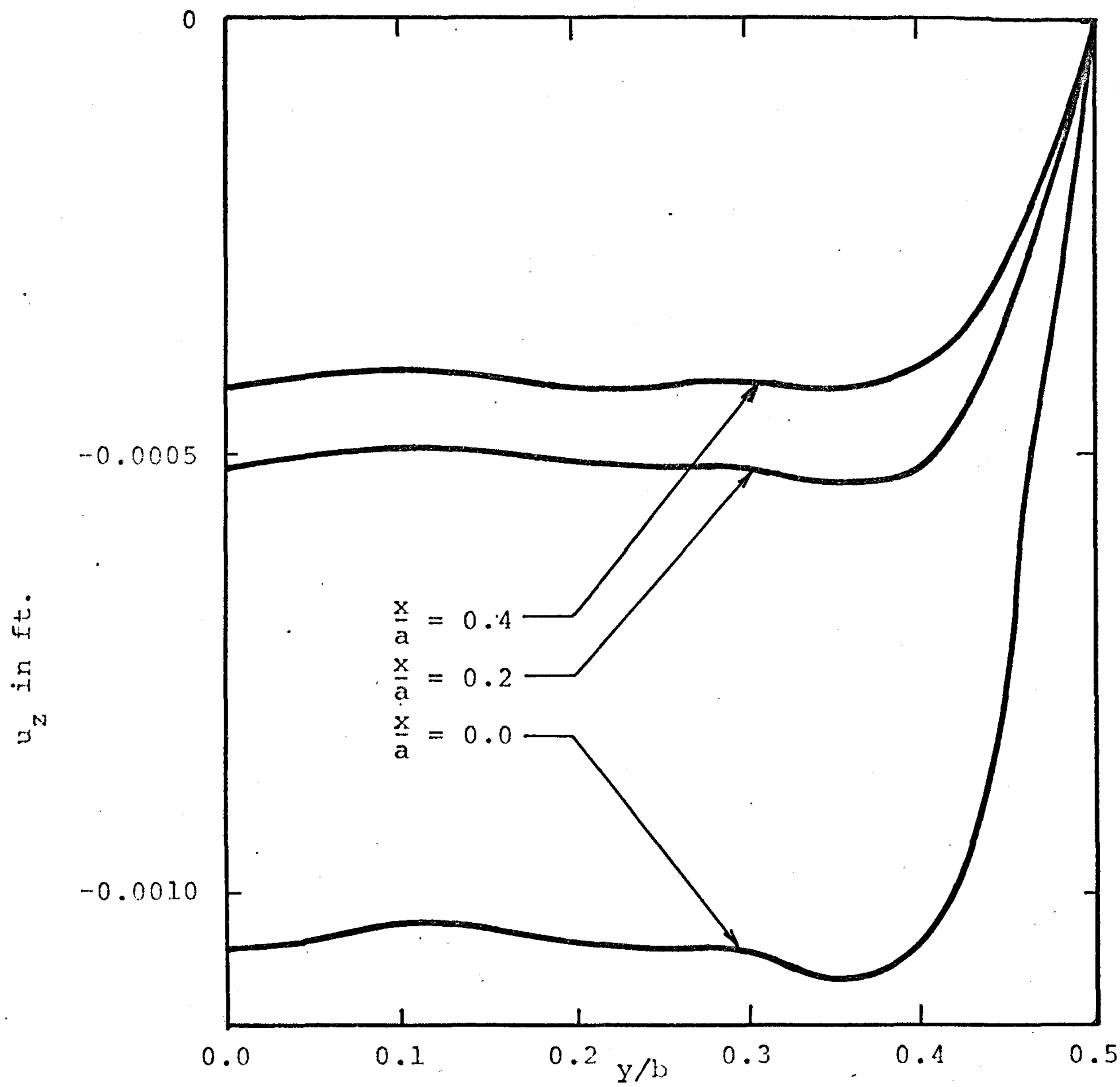

Fig. 3-29 DISTRIBUTION OF NORMAL DISPLACEMENT u_z OF SHALLOW SPHERICAL SHELL WITH MIXED BOUNDARY CONDITIONS. (a = b = 40')
Boundaries x = ±a/2 fixed.
Boundaries y = ±b/2 simply supported

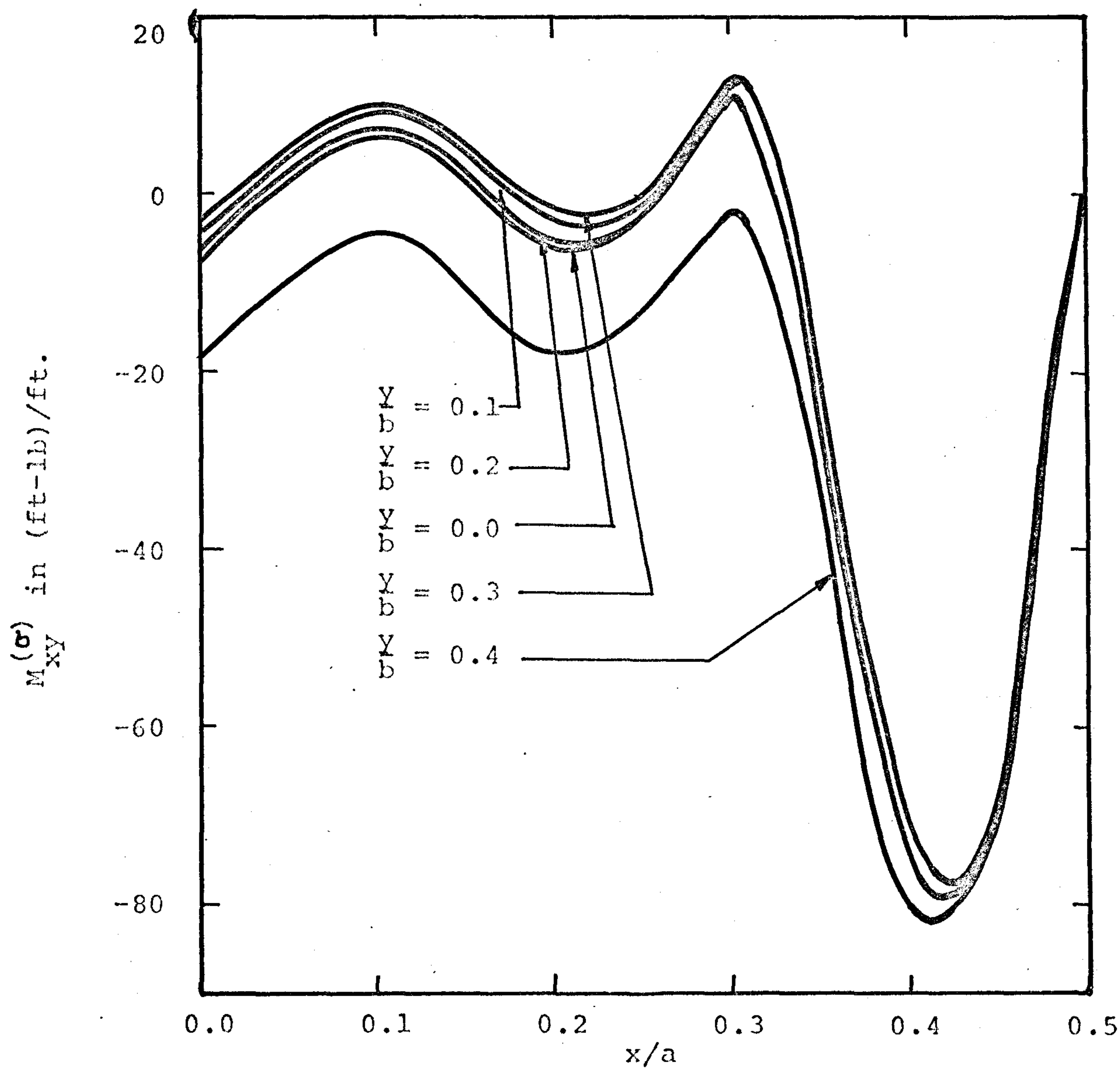

Fig. 3-30 DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ OF SHALLOW SPHERICAL SHELL WITH SIMPLY SUPPORTED BOUNDARIES. (a = b = 40')

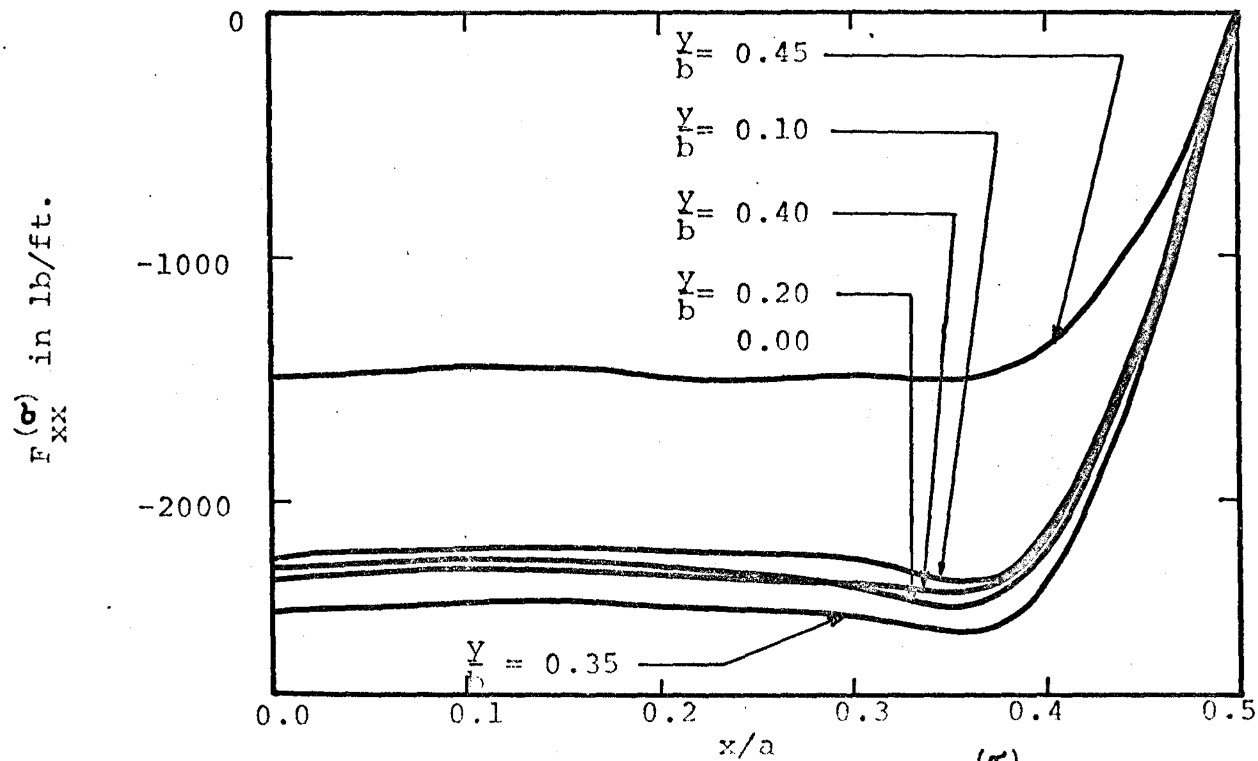

Fig. 3-31 DISTRIBUTION OF STRESS RESULTANT $F_{xx}^{(\sigma)}$ OF A SHALLOW SPHERICAL SHELL WITH SIMPLY SUPPORTED BOUNDARIES. (a = b = 40')

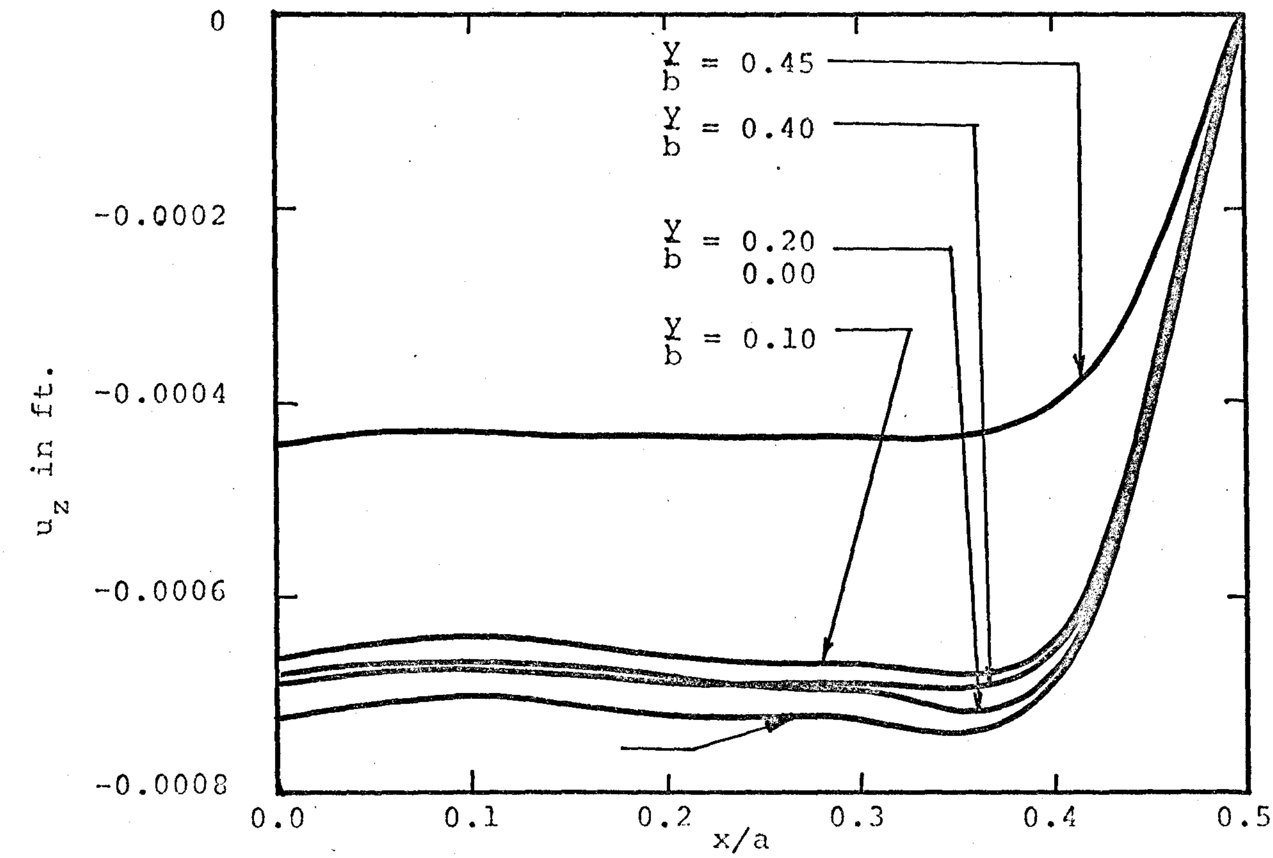

Fig. 3-32 DISTRIBUTION OF NORMAL DISPLACEMENT u_z OF A SHALLOW SPHERICAL SHELL WITH SIMPLY SUPPORTED BOUNDARIES. (a = b = 40').

TABLE III-3

	FOURIER COEFFICIENTS CALCULATED BY EXP.(II-3-5)	FOURIER COEFFICIENTS CALCULATED BY EXP.(II-3-8)
A00	0.0001083	0.0001081
A01	0.00007473	0.00007485
A10	0.00007473	0.00007485
A11	0.00004381	0.00004364
A02	0.00002067	0.00002071
A20	0.00002067	0.00002071
A12	0.00001437	0.00001434
A21	0.00001437	0.00001434
A22	0.000006238	0.000006213
A03	0.000005515	0.000005523
A30	0.000005515	0.000005523
A13	0.000004603	0.000004607
A31	0.000004603	0.000004607
A23	0.000002464	0.000002460
A32	0.000002464	0.000002460
A33	0.000001149	0.000001146
A04	0.000001983	0.000001985
A40	0.000001983	0.000001985
A14	0.000001838	0.000001839
A41	0.000001838	0.000001839
A24	0.000001126	0.000001126
A42	0.000001126	0.000001126
A34	0.0000005963	0.0000005944
A43	0.0000005963	0.0000005944

are much more complicated than the expression (II-3-8) or (II-5-4). For the purpose of giving a numerical example to show the influence of the membranal displacements on the transverse dispalcement, stress couples and stress resultants, a simple case of a spherical translational shell in which $a = b = 40'$, $k_x^{(n)} = k_y^{(n)} = 0.02$, $P = 90$ lb/ft^2 is calculated. Owing to the condition of complete symmetry, u_x equals u_y, so the three simultaneous equations reduce to two as shown in Appendix D. Using these reduced two simultaneous equations to solve for the Fourier coefficients A_{mn} of the transverse displacement u_z to 8 terms for the case of fixed boundaries, and 25 terms for the case of simply supported boundaries, it is found that the difference between these Fourier coefficients to those obtained by neglecting u_x and u_y are small for the case of fixed boundaries, but large for the case of simply supported boundaries. Therefore, any solution of stress couples or stress resultants obtained by the approximate method for a translational shell with simply supported boundaries should be used very carefully for design (see next section and conclusion). The difference between stress couples and stress resultants obtained by these new Fourier coefficients and those obtained before are also small for the case of fixed boundaries, say, always less than 10%, and mostly less than 5%. This can be observed in Fig. (3 -33) and Fig. (3 -34). Hence, though values of stress couples and stress resultants obtained by neglecting u_x and u_y are not exact solutions, they still can be reasonably used as guides for practical

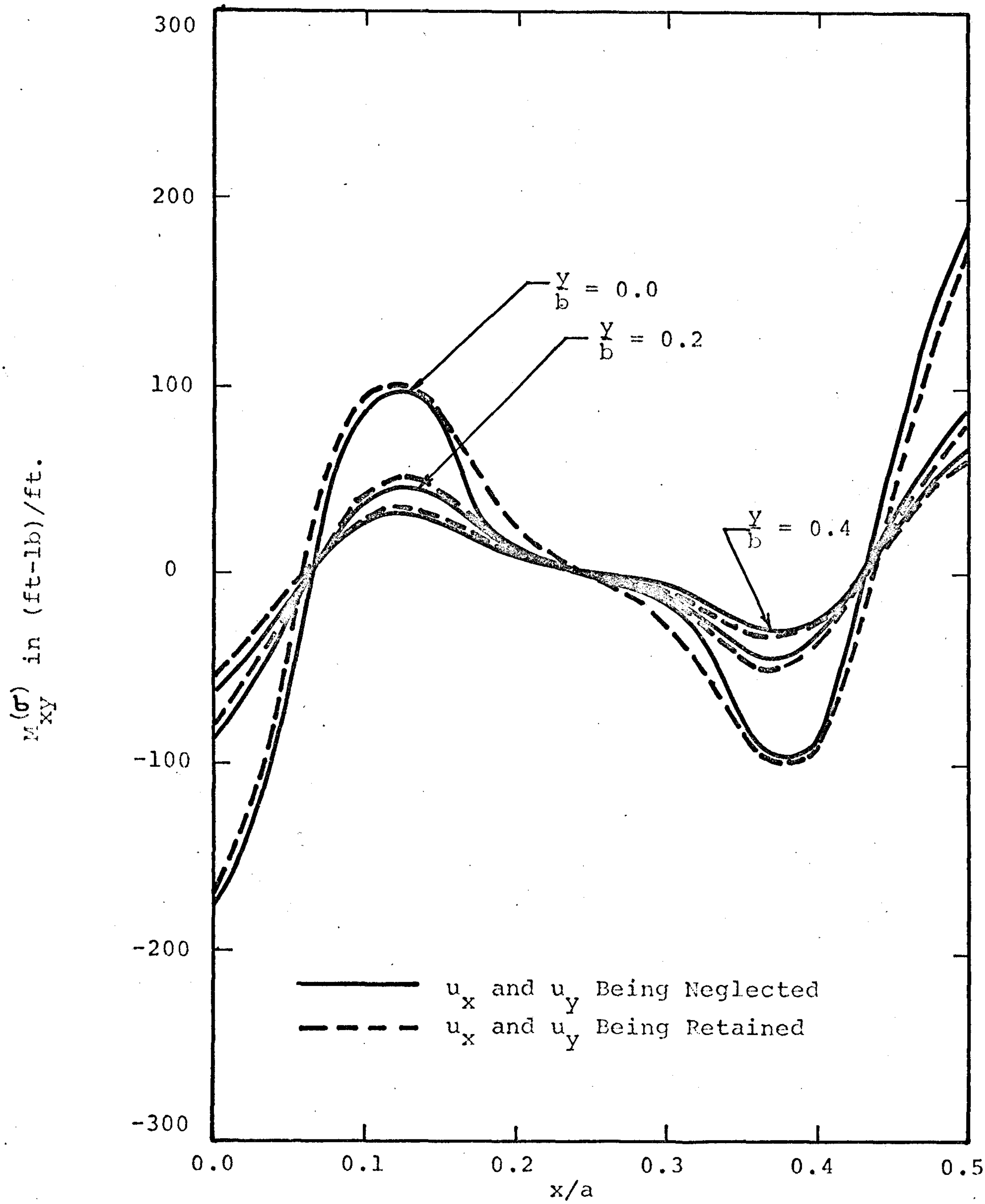

Fig. 3-33 COMPARISON OF DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ OF A SPHERICAL TRANSLATIONAL SHELL WITH FIXED BOUNDARIES. (a = 40', b = 40')

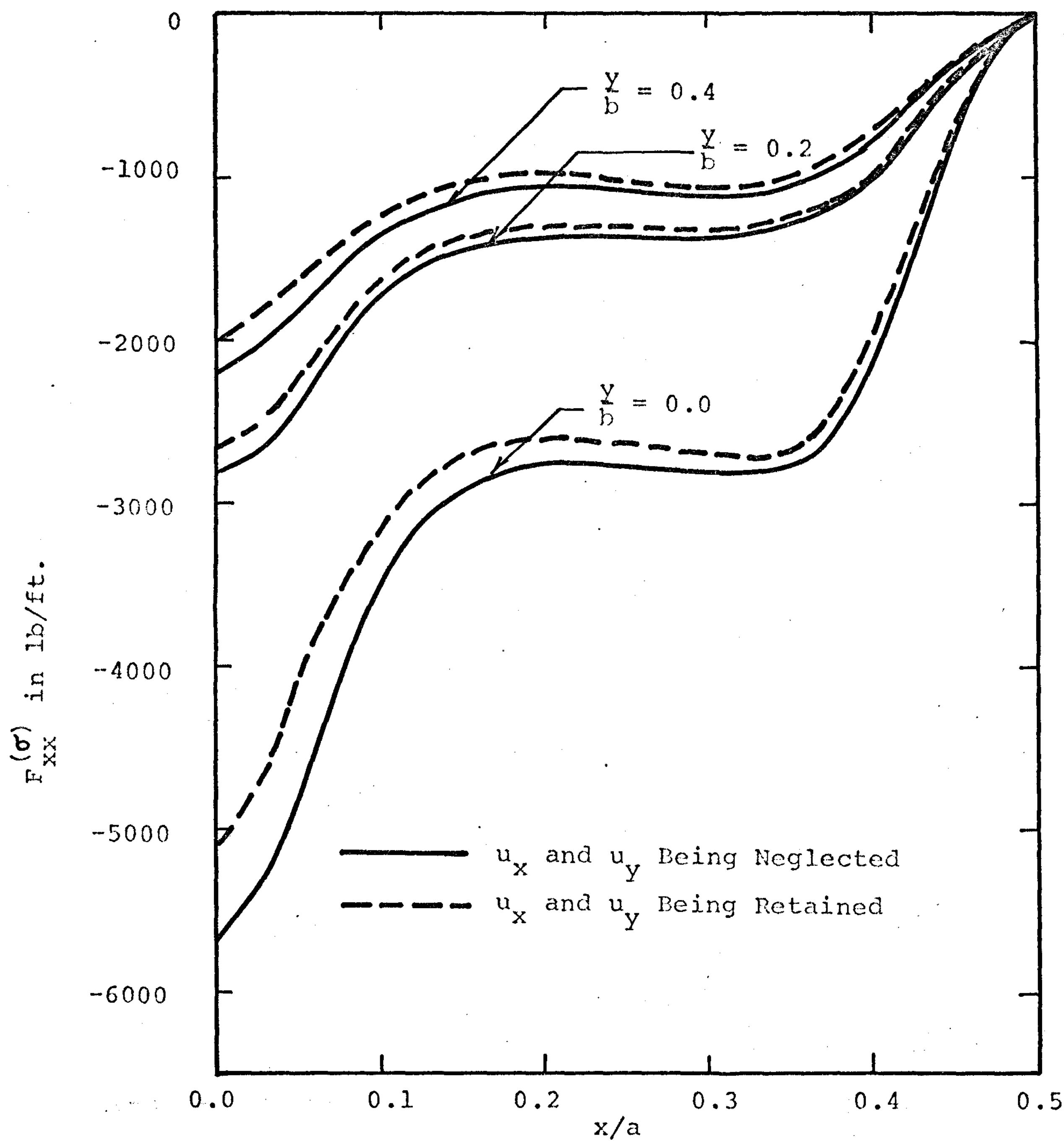

Fig. 3-34 COMPARISON OF DISTRIBUTION OF STRESS RESULTANT $F_{xx}^{(\sigma)}$ OF SHALLOW SPHERICAL TRANSLATIONAL SHELL WITH FIXED BOUNDARIES. (a = 40', b = 40')

design purposes.

3-5. Comparison of the Solution of This Method with SOARE's Method

As it is mentioned in the introduction, up to the present time, the rigorous solution of an elliptic paraboloidal shell of translation with fixed boundaries still has not been established. Therefore, a comparison of solutions of a spherical translational shell with simply supported boundaries is given.

In Fig. (3 -35) and Fig. (3 -36) it can be clearly observed that the difference between solutions obtained by the approximate method and the method of SOARE's is quite large; especially for the stress couple $M_{xy}^{(\sigma)}$ near the boundaries. The difference is more than 35% in terms of SOARE's solution. Therefore, a complete solution obtained by using expressions (4-16a), (4-16b) in Appendix D is calculated. At first, Fourier coefficients obtained by expression (III-5-4) are used as basic values. Following a successive approximation procedure, these basic values are substituted into (4-16b) to find out the Fourier coefficients B_{mn}, then substituting coefficients B_{mn} into (4-16a), a set of revised Fourier coefficients A_{mn} is obtained. A set of A_{mn} values obtained from two cycles of calculation are compared to SOARE's solution in Table (III-4).

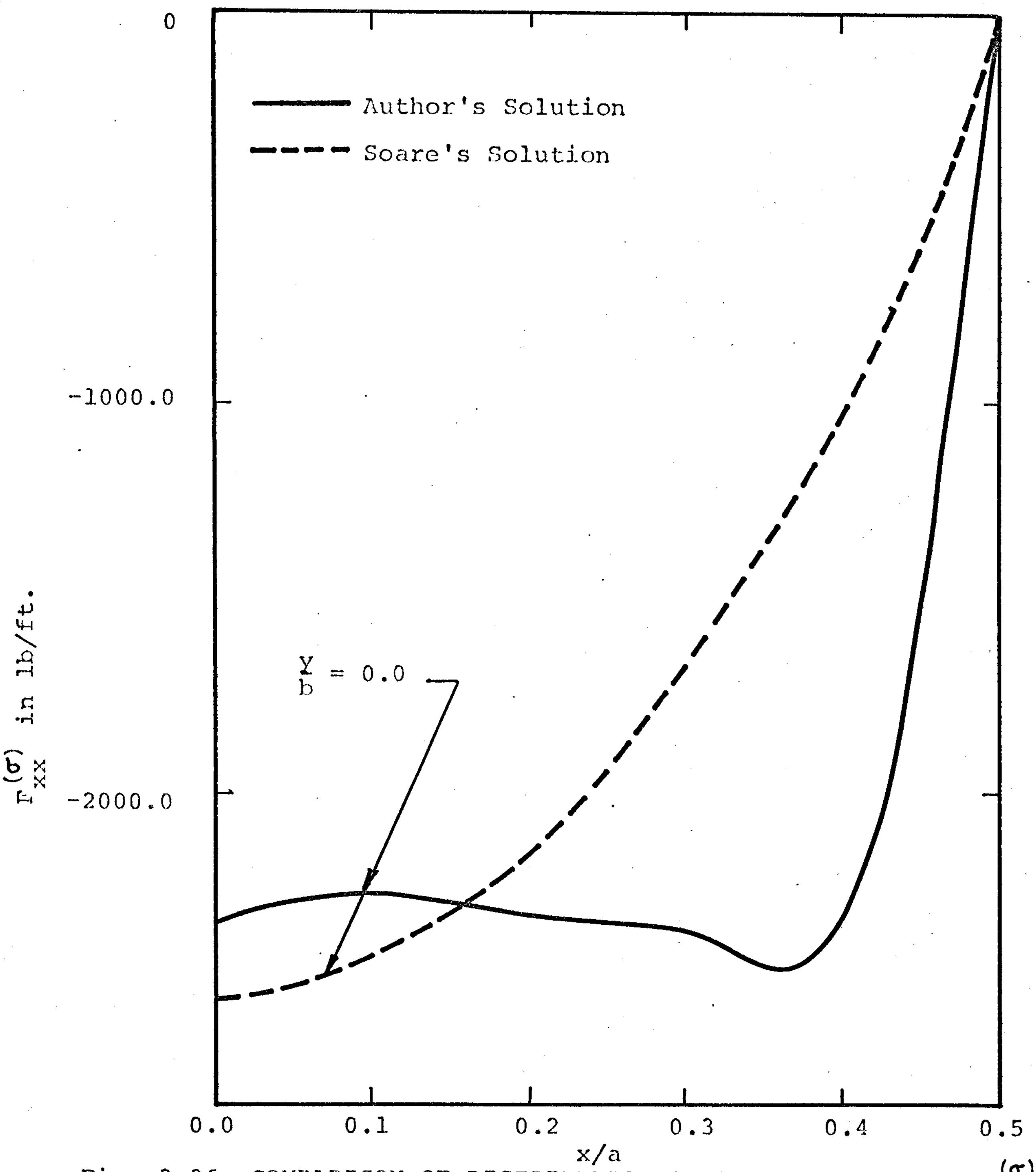

Fig. 3-36 COMPARISON OF DISTRIBUTION OF STRESS RESULTANT $F_{xx}^{(\sigma)}$ OF SPHERICAL TRANSLATIONAL SHELL WITH SIMPLY SUPPORTED BOUNDARIES BETWEEN SOARE'S SOLUTION AND AUTHOR'S SOLUTION. (a = 40', b = 40')

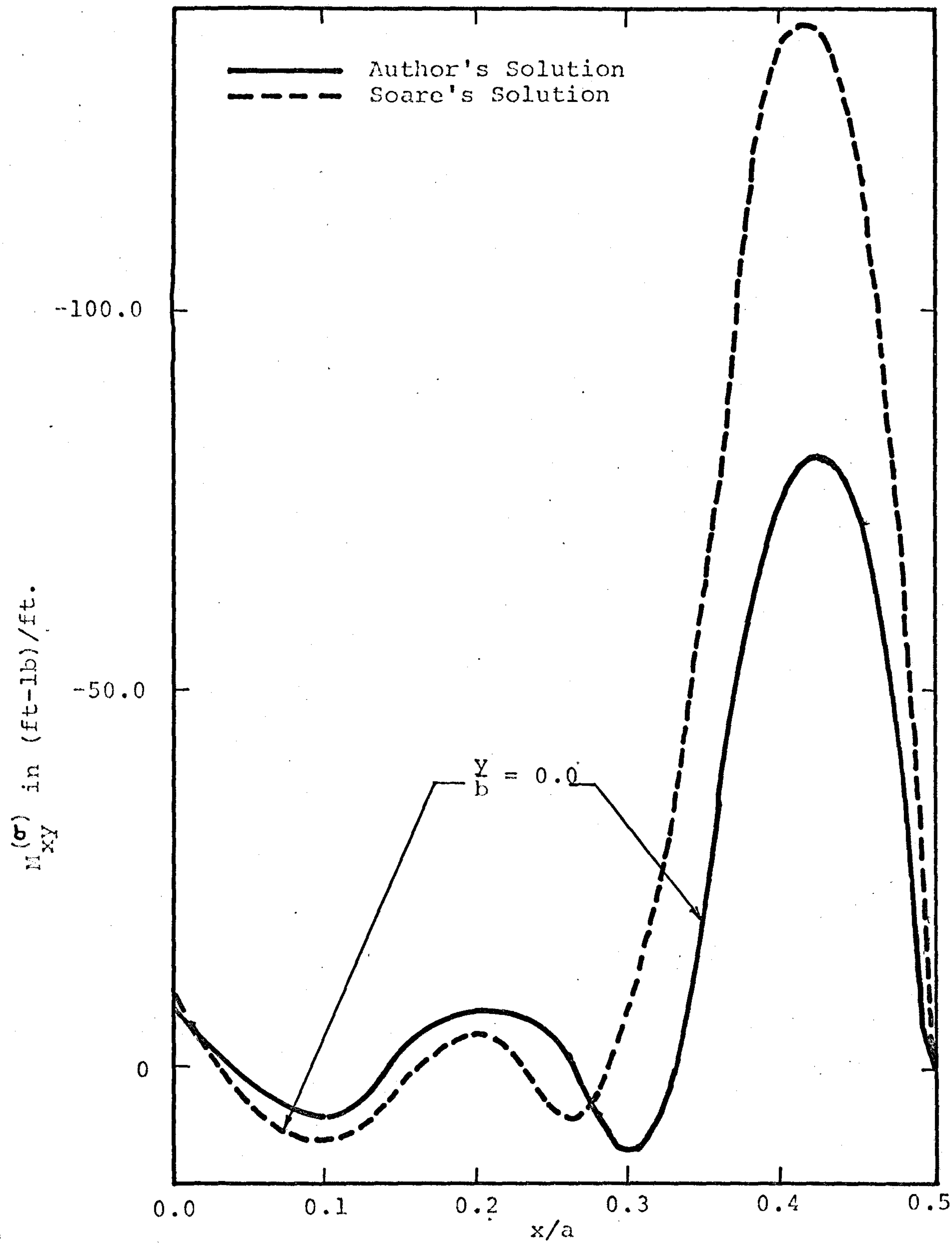

Fig. 3-35 COMPARISON OF DISTRIBUTION OF STRESS COUPLE $M_{xy}^{(\sigma)}$ OF SPHERICAL TRANSLATIONAL SHELL, WITH SIMPLY SUPPORTED BOUNDARIES, BETWEEN SOARE'S SOLUTION AND AUTHOR'S SOLUTION. (a = 40', b = 40')

TABLE III-4

FOURIER COEFFICIENTS	BY (4-16a) (4-16b)	BY SOARE'S METHOD
A_{00}	0.002836	0.002524
A_{10}	-0.000724	-0.000774
A_{20}	0.000298	0.000314
A_{30}	-0.000111	-0.000111
A_{40}	0.000040	0.000040
A_{01}	-0.000724	-0.000774
A_{11}	0.000239	0.000218
A_{21}	-0.000090	-0.000083
A_{31}	0.000032	0.000030
A_{41}	-0.000012	-0.000011
A_{02}	0.000298	0.000314
A_{12}	-0.000090	-0.000083
A_{22}	0.000035	0.000031
A_{32}	-0.000013	-0.000012
A_{42}	0.000005	0.000005
A_{03}	-0.000111	-0.000111
A_{13}	0.000032	0.000030
A_{23}	-0.000073	-0.000012
A_{33}	0.000006	0.000005
A_{43}	-0.000003	-0.000002
A_{04}	0.000040	0.000040
A_{14}	-0.000012	-0.000011
A_{24}	0.000005	0.000005
A_{34}	-0.000003	-0.000002
A_{44}	0.000001	0.000001

CHAPTER 4

CONCLUSIONS

From the numerical results of examples in the last chapter, it is observed that for shells with fixed boundaries stress couples are always larger at the apex as well as along the boundaries than at any other point. This kind of distribution is intuitively acceptable and is similar to the experimental results of spherical shells and cylinderical shells which appear as special cases in this thesis. The absolute value of stress couples at the apex seems to be larger than actual values. This is due to the difference between the assumed transverse displacement function and the real distribution of the transverse displacement. Nevertheless, as long as the boundary value problem is concerned, this method gives good approximate values of stress couples along the boundaries and is satisfactory for the design purpose. The distribution of stress resultants is not sufficiently accurate near the boundaries, since in the calculation of the stress resultants, membranal strains $\frac{\partial u_x}{\partial x}$ and $\frac{\partial u_y}{\partial y}$ which have a major effect on stress resultants near the boundaries were neglected in expression (II-3-6). Therefore, instead of using the values of stress resultants obtained by this approximate method, it is better to consider them rather as indicative for the design.

For the shells of simply supported boundaries, the effect of surface displacements u_x and u_y on the transverse displacement u_z is large as discussed in section (3-4). Therefore, unless u_x and u_y are included in every expression, the result of stress couples and stress resultants obtained by this method should be used for design with great care.

The results for shells with mixed boundaries should lie somewhere between these results. The author suggests to include the membranal displacement in all expressions when the shell is not very shallow, say H/b near 1/5, but to exclude the membranal displacement in all expressions when the shell is very shallow, i.e. for shells with $H/b = 1/10$.

APPENDIX A

DERIVATION OF STRAIN ENERGY EXPRESSION FOR SHALLOW SHELLS

LECTURE NOTES BY Dr. JOHN SCHROEDER

A directional tensor quantity of the second order is defined as a homogeneous bilinear vector-form

$$\bar{\bar{A}} = A_{11}\bar{e}_1\bar{e}_1 + A_{12}\bar{e}_1\bar{e}_2 + A_{13}\bar{e}_1\bar{e}_3 +$$
$$+ A_{21}\bar{e}_2\bar{e}_1 + A_{22}\bar{e}_2\bar{e}_2 + A_{23}\bar{e}_2\bar{e}_3 +$$
$$+ A_{31}\bar{e}_3\bar{e}_1 + A_{32}\bar{e}_3\bar{e}_2 + A_{33}\bar{e}_3\bar{e}_3$$
$$= A_{ij}\bar{e}_i\bar{e}_j$$

where $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ are unit base vectors along three orthogonal curvilinear directions, and repeated indices imply summation.

Following definitions, the general stress and strain tensors in elasticity are defined by,

$$\bar{\bar{\sigma}} = \sigma_{11}\bar{e}_1\bar{e}_1 + \sigma_{12}\bar{e}_1\bar{e}_2 + \sigma_{13}\bar{e}_1\bar{e}_3 +$$
$$+ \sigma_{21}\bar{e}_2\bar{e}_1 + \sigma_{22}\bar{e}_2\bar{e}_2 + \sigma_{23}\bar{e}_2\bar{e}_3 +$$
$$+ \sigma_{31}\bar{e}_3\bar{e}_1 + \sigma_{32}\bar{e}_3\bar{e}_2 + \sigma_{33}\bar{e}_3\bar{e}_3$$
$$= \sigma_{ij}\bar{e}_i\bar{e}_j = \bar{e}_i(\sigma_{ij}\bar{e}_j) = \bar{e}_i\bar{\sigma}_i$$

and

$$\bar{\bar{\epsilon}} = \epsilon_{11}\bar{e}_1\bar{e}_1 + \epsilon_{12}\bar{e}_1\bar{e}_2 + \epsilon_{13}\bar{e}_1\bar{e}_3 +$$
$$+ \epsilon_{21}\bar{e}_2\bar{e}_1 + \epsilon_{22}\bar{e}_2\bar{e}_2 + \epsilon_{23}\bar{e}_2\bar{e}_3 +$$
$$+ \epsilon_{31}\bar{e}_3\bar{e}_1 + \epsilon_{32}\bar{e}_3\bar{e}_2 + \epsilon_{33}\bar{e}_3\bar{e}_3$$
$$= \epsilon_{ij}\bar{e}_i\bar{e}_j = \bar{e}_i(\epsilon_{ij}\bar{e}_j) = \bar{e}_i\bar{\epsilon}_i$$

Furthermore, the double-dot product of $\bar{\bar{\sigma}}$ and $\bar{\bar{\epsilon}}$ is defined in its trinomial form

$$\bar{\bar{\sigma}} : \bar{\bar{\epsilon}} = \bar{e}_r \bar{\sigma}_r : \bar{e}_s \bar{\epsilon}_s = \bar{e}_r \cdot \bar{e}_s \, \bar{\sigma}_r \cdot \bar{\epsilon}_s = \delta_{rs} \, \bar{\sigma}_r \cdot \bar{\epsilon}_s = \bar{\sigma}_r \cdot \bar{\epsilon}_r$$

Its component form is

$$\bar{\bar{\sigma}} : \bar{\bar{\epsilon}} = \sigma_{ij} \, \epsilon_{\ell m} \, (\bar{e}_i \cdot \bar{e}_\ell)(\bar{e}_j \cdot \bar{e}_m) = \sigma_{ij} \epsilon_{ij}$$

Hence

$$\delta U^{(s)} = \int_v \bar{\bar{\sigma}} : \delta \bar{\bar{\epsilon}} \, dv \tag{1}$$

in which v denotes the volume of an elastic body.

From the traditional definition, the stress-strain relation for isotropic Hookean materials, which excludes the thermal effects, is

$$\bar{\bar{\sigma}} = 2\mu \bar{\bar{\epsilon}} + \lambda (\bar{\bar{\epsilon}} : \bar{\bar{1}}) \bar{\bar{1}} \tag{2}$$

where μ, λ are Cauchy-Lamé's First and Second Elastic Constants, $\bar{\bar{\epsilon}} : \bar{\bar{1}} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$ denotes the First Strain Invariant, and $\bar{\bar{1}} = \bar{e}_i \bar{e}_i$ designates the Idemfactor, Unitary Tensor, or Identity Tensor. Substituting (2) into (1) yields

$$\begin{aligned} \delta U^{(s)} &= \int_v (2\mu \bar{\bar{\epsilon}} + \lambda (\bar{\bar{\epsilon}} : \bar{\bar{1}}) \bar{\bar{1}}) : \delta \bar{\bar{\epsilon}} \, dv \\ &= \int_v (2\mu \bar{\bar{\epsilon}} : \delta \bar{\bar{\epsilon}} + \lambda (\bar{\bar{\epsilon}} : \bar{\bar{1}}) \bar{\bar{1}} : \delta \bar{\bar{\epsilon}}) \, dv \\ &= \delta \int_v (\mu \bar{\bar{\epsilon}} : \bar{\bar{\epsilon}} + \frac{\lambda}{2} (\bar{\bar{\epsilon}} : \bar{\bar{1}})^2) \, dv \end{aligned}$$

The strain energy is thus

$$U^{(s)} = \int_v (\mu \bar{\bar{\epsilon}} : \bar{\bar{\epsilon}} + \frac{\lambda}{2} (\bar{\bar{\epsilon}} : \bar{\bar{1}})^2) \, dv \tag{3}$$

For thin shells, the thickness always represents a small quantity in comparison with its other two dimensions, therefore, it is usually possible to treat thin shell theory as an approximate bidimensional continuum problem. In this approximation, the <u>Kirchhoff-Aron Hypothesis</u> is enforced and ϵ_{13}, ϵ_{31}, ϵ_{23}, and ϵ_{32} are assumed to be identically zero. But, for the purpose of simplifying the final expression of strain energy the normal strain component ϵ_{33} is retained by imposing the condition of plane stress $\sigma_{33}=0$, and ϵ_{33} is thus expressed as a function of ϵ_{11} and ϵ_{22} in virtue of the stress-strain relation. Even though this procedure is not quite consistent, yet since ϵ_{33} is normally a much smaller strain than ϵ_{11} and ϵ_{22}, the final results are not appreciably affected by this approximation. Consequently, the strain tensor reduces to its simplified form

$$\bar{\bar{\epsilon}} = \epsilon_{11}\bar{e}_1\bar{e}_1 + \epsilon_{12}\bar{e}_1\bar{e}_2 + \epsilon_{21}\bar{e}_2\bar{e}_1 + \epsilon_{22}\bar{e}_2\bar{e}_2 + \epsilon_{33}\bar{e}_3\bar{e}_3 \tag{4}$$

Substituting (4) into (3), and observing the fact that strain tensor is symmetric, expression (3) becomes

$$U^{(s)} = \mu \iint_A \left[\int \left[\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 + 2\epsilon_{12}^2 + \frac{\nu}{1-2\nu}(\epsilon_{11}+\epsilon_{22}+\epsilon_{33})^2 \right] d\alpha_3 \right] dA \tag{5}$$

Since $\lambda = 2\nu\mu/(1-2\nu)$, now, assuming $\sigma_{33}=0$

$$\sigma_{33} = \frac{2\mu\nu}{1-2\nu}(\epsilon_{11}+\epsilon_{22}) + \frac{2\mu(1-\nu)}{1-2\nu}\epsilon_{33} = 0$$

$$\epsilon_{33} = -\frac{\nu}{1-\nu}(\epsilon_{11}+\epsilon_{22}) \tag{6}$$

If $\epsilon_{11}(\bar{r}_0)$, $\epsilon_{22}(\bar{r}_0)$, $\epsilon_{12}(\bar{r}_0)$ denote the strain components of an arbitrary point in the middle surface, then the strain in the surfaces parallel to the middle surface are given in terms of the geometric properties of the middle surface

$$\begin{aligned} \epsilon_{11}(\bar{r}) &= \epsilon_{11}(\bar{r}_0) + \alpha_3\,\delta K_2^{(n)} \\ \epsilon_{22}(\bar{r}) &= \epsilon_{22}(\bar{r}_0) + \alpha_3\,\delta K_1^{(n)} \\ \epsilon_{12}(\bar{r}) &= \epsilon_{12}(\bar{r}_0) + \alpha_3\,\delta K^{(g)} \end{aligned} \tag{7}$$

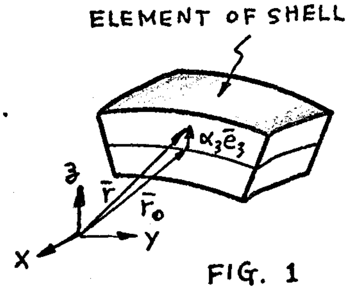

FIG. 1

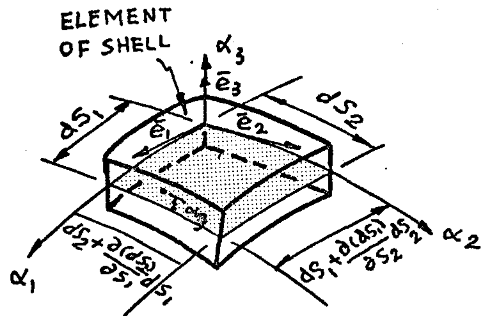

FIG. 2

From Fig(2), $dS_1 = A_1 d\alpha_1 (1 + \alpha_3 K_2^{(n)})$

$$dS_2 = A_2 d\alpha_2 (1 + \alpha_3 K_1^{(n)})$$

and

$$dS_1 dS_2 \doteq dA$$

It was mentioned in the earlier part of this thesis, that $K_1^{(n)}(\bar{r}_0) = K_1^{(n)}$ and $K_2^{(n)}(\bar{r}_0) = K_2^{(n)}$ are very small quantities, thus the terms $\alpha_3 K_1^{(n)}$ and $\alpha_3 K_2^{(n)}$ in ds_1 and ds_2 when compared with unity, can be neglected. This simplification represents the so-called LOVE First Approximation, which reduces ds_1 and ds_2 to the simple form

$$dS_1 \doteq A_1 d\alpha_1 \, , \quad dS_2 \doteq A_2 d\alpha_2$$

Substituting (6), (7) and ds_1, ds_2 into (5) and integrating it over α_3 between limits -h/2 and h/2 yields the approximate expression of total strain energy in the shell

$$U^{(S)} = \mu \iint_A \Big\{ \big[\epsilon_{11}^2(\bar{r}_0) + \epsilon_{22}^2(\bar{r}_0) + 2\nu \epsilon_{11}(\bar{r}_0)\epsilon_{22}(\bar{r}_0) + 2(1-\nu)\epsilon_{12}^2(\bar{r}_0) \big] \frac{h}{1-\nu} + \\ + \big[(\delta K_1^{(n)})^2 + (\delta K_2^{(n)})^2 + 2\nu \delta K_1^{(n)} \delta K_2^{(n)} + (1-\nu)(\delta K^{(3)})^2 \big] \frac{h^3}{12(1-\nu)} \Big\} dA \quad (8)$$

APPENDIX B

DETERMINATION OF FOURIER COEFFICIENTS FOR NORMAL DISPLACEMENT FUNCTION - FOR SHELLS WITH FIXED EDGES

Substituting equations II-3-2 into II-2-3 yields

$$U^{(S)} = D'[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)}] \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Big[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \Big(1 + \cos \frac{2(2m+1)\pi x}{a}\Big)$$

$$\Big(1 + \cos \frac{2(2n+1)\pi y}{b}\Big)\Big]^2 \Big(1 + \tfrac{1}{2}(K_x^{(n)})^2 x^2 + \tfrac{1}{2}(K_y^{(n)})^2 y^2\Big)\, dx\, dy \; +$$

$$+ \frac{D}{2} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Big\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big(\frac{2(2m+1)\pi}{a}\Big)^2 \cos \frac{2(2m+1)\pi x}{a} \Big(1 + \cos \frac{2(2n+1)\pi y}{b}\Big) A_{mn} -$$

$$- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big(\frac{2(2n+1)\pi}{b}\Big)^2 \cos \frac{2(2n+1)\pi y}{b} \Big(1 + \cos \frac{2(2m+1)\pi x}{a}\Big) A_{mn} \Big\}^2$$

$$\Big(1 + \tfrac{1}{2}(K_x^{(n)})^2 x^2 + \tfrac{1}{2}(K_y^{(n)})^2 y^2\Big)\, dx\, dy$$

Assuming $D'[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)}] = \phi$

$$U^{(S)} = \phi \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Big[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \Big(1 + \cos \frac{2(2m+1)\pi x}{a}\Big)\Big(1 + \cos \frac{2(2n+1)\pi y}{b}\Big)\Big]^2 dx\, dy \; +$$

$$+ \frac{\phi}{2} (K_x^{(n)})^2 \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Big[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \Big(1 + \cos \frac{2(2m+1)\pi x}{a}\Big)\Big(1 + \cos \frac{2(2n+1)\pi y}{b}\Big)\Big]^2 x^2\, dx\, dy \; +$$

$$+ \frac{\phi}{2} (K_y^{(n)})^2 \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Big[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \Big(1 + \cos \frac{2(2m+1)\pi x}{a}\Big)\Big(1 + \cos \frac{2(2n+1)\pi y}{b}\Big)\Big]^2 y^2\, dx\, dy \; +$$

$$+ \frac{D}{2} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Big\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big(\frac{2(2m+1)\pi}{a}\Big)^2 \cos \frac{2(2m+1)\pi x}{a} \Big(1 + \cos \frac{2(2n+1)\pi y}{b}\Big) A_{mn} -$$

$$- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big(\frac{2(2n+1)\pi}{b}\Big)^2 \cos \frac{2(2n+1)\pi y}{b} \Big(1 + \cos \frac{2(2m+1)\pi x}{a}\Big) A_{mn} \Big\}^2 dx\, dy \; +$$

$$+ \frac{D (K_x^{(n)})^2}{4} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \Big\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big(\frac{2(2m+1)\pi}{a}\Big)^2 \cos \frac{2(2m+1)\pi x}{a} \Big(1 + \cos \frac{2(2n+1)\pi y}{b}\Big) A_{mn} -$$

$$- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big(\frac{2(2n+1)\pi}{b}\Big)^2 \cos \frac{2(2n+1)\pi y}{b} \Big(1 + \cos \frac{2(2m+1)\pi x}{a}\Big) A_{mn} \Big\}^2 x^2\, dx\, dy \; +$$

$$+\frac{D(K_y^{(n)})^2}{4}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{2(2m+1)\pi}{a}\right)^2\cos\frac{2(2m+1)\pi x}{a}\left(1+\cos\frac{2(2n+1)\pi y}{b}\right)A_{mn}-\right.$$

$$\left.-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{2(2n+1)\pi}{b}\right)^2\cos\frac{2(2n+1)\pi y}{b}\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)A_{mn}\right\}^2 y^2\,dx\,dy$$

$$=U_1^{(S)}+U_2^{(S)}+U_3^{(S)}+U_4^{(S)}+U_5^{(S)}+U_6^{(S)}$$

$$U_1^{(S)}=\phi\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)\left(1+\cos\frac{2(2n+1)\pi y}{b}\right)\right]^2dx\,dy$$

$$=\phi\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)^2\left(1+\cos\frac{2(2n+1)\pi y}{b}\right)^2+\right.$$

$$+\sum_{m=0}^{\infty}\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}A_{mr}A_{ms}\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)^2\left(1+\cos\frac{2(2r+1)\pi y}{b}\right)\left(1+\cos\frac{2(2s+1)\pi y}{b}\right)+$$

$$+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\left(1+\cos\frac{2(2r+1)\pi x}{a}\right)\left(1+\cos\frac{2(2s+1)\pi x}{a}\right)\left(1+\cos\frac{2(2n+1)\pi y}{b}\right)^2+$$

$$+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{\substack{p=0\\p\neq q}}^{\infty}\sum_{q=0}^{\infty}A_{rp}A_{sq}\left(1+\cos\frac{2(2r+1)\pi x}{a}\right)\left(1+\cos\frac{2(2s+1)\pi x}{a}\right)\cdot$$

$$\left.\cdot\left(1+\cos\frac{2(2p+1)\pi y}{b}\right)\left(1+\cos\frac{2(2q+1)\pi y}{b}\right)\right]dx\,dy$$

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos(mx)\cos(nx)\,dx=\begin{cases}\frac{a}{2} & \text{if } m=n\\ 0 & m\neq n\end{cases}$$

$$U_1^{(S)}=\phi\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\frac{9ab}{4}+\sum_{m=0}^{\infty}\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}A_{mr}A_{ms}\frac{3ab}{2}+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\frac{3ab}{2}+\right.$$

$$\left.+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{\substack{p=0\\p\neq q}}^{\infty}\sum_{q=0}^{\infty}A_{rp}A_{sq}\frac{4}{3}ab\right]$$

$$=\frac{3\phi}{4}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}3A_{mn}^2ab+\sum_{m=0}^{\infty}\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}2A_{mr}A_{ms}ab+\right.$$

$$+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty} 2A_{rn}A_{sn}\,ab + \sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{\substack{p=0\\p\neq q}}^{\infty}\sum_{q=0}^{\infty} \frac{4}{3}A_{rp}A_{sq}\,ab]$$

$$U_2^{(s)} = \frac{1}{2}\phi(K_x^{(n)})^2\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)\left(1+\cos\frac{2(2n+1)\pi y}{b}\right)\right]^2 x^2\,dx\,dy$$

$$= \frac{1}{2}\phi(K_x^{(n)})^2\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)^2\left(1+\cos\frac{2(2n+1)\pi y}{b}\right)^2 x^2 +$$

$$+\sum_{m=0}^{\infty}\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}A_{mr}A_{ms}\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)^2\left(1+\cos\frac{2(2r+1)\pi y}{b}\right)\left(1+\cos\frac{2(2s+1)\pi y}{b}\right)x^2+$$

$$+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\left(1+\cos\frac{2(2r+1)\pi x}{a}\right)\left(1+\cos\frac{2(2s+1)\pi x}{a}\right)\left(1+\cos\frac{2(2n+1)\pi y}{b}\right)^2 x^2+$$

$$+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{\substack{p=0\\p\neq q}}^{\infty}\sum_{q=0}^{\infty}A_{rp}A_{sq}\left(1+\cos\frac{2(2r+1)\pi x}{a}\right)\left(1+\cos\frac{2(2s+1)\pi x}{a}\right)\cdot$$

$$\cdot\left(1+\cos\frac{2(2p+1)\pi y}{b}\right)\left(1+\cos\frac{2(2q+1)\pi y}{b}\right)x^2\Big]\,dx\,dy$$

It has been found that $\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\cos(mx)\cos(nx)\,dx = \begin{cases}\frac{a^3b}{24} & \text{if } m=n\\ 0 & m\neq n\end{cases}$

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\cos\frac{2(2m+1)\pi x}{a}\,dx = 0$$

$$U_2^{(s)} = \frac{1}{2}\phi(K_x^{(n)})^2\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\left(\frac{a^3b}{12}+\frac{a^3b}{24}+\frac{a^3b}{24}+\frac{a^3b}{48}\right)+$$

$$+\sum_{m=0}^{\infty}\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}A_{mr}A_{ms}\left(\frac{a^3b}{12}+\frac{a^3b}{24}\right)+$$

$$+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\left(\frac{a^3b}{12}+\frac{a^3b}{24}\right)$$

$$+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{\substack{p=0\\p\neq q}}^{\infty}\sum_{q=0}^{\infty}A_{rp}A_{sq}\frac{a^3b}{12}\Big\}$$

$$=\frac{1}{2}\phi(K_x^{(n)})^2\Big(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{3}{16}A_{mn}^2+\sum_{m=0}^{\infty}\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\frac{1}{8}A_{mr}A_{ms}+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{8}A_{rn}A_{sn}+$$

$$+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{\substack{p=0\\p\neq q}}^{\infty}\sum_{q=0}^{\infty}\frac{1}{12}A_{rp}A_{sq}\Big)a^3b$$

From the property of symmetry

$$U_3^{(s)} = \frac{1}{2}\phi(K_y^{(n)})^2\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\frac{3}{16} + \sum_{m=0}^{\infty}\sum_{\substack{r=0 \\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\frac{1}{8}A_{mr}A_{ms} + \sum_{\substack{r=0 \\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{8}A_{rn}A_{sn} +$$

$$+\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}A_{rp}A_{sq}\Big]ab^3$$

$$U_4^{(s)} = \frac{D}{2}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\Big\{-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(\frac{2(2m+1)\pi}{a}\Big)^2\cos\frac{2(2m+1)\pi x}{a}\Big(1+\cos\frac{2(2n+1)\pi y}{b}\Big)A_{mn} -$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(\frac{2(2n+1)\pi}{b}\Big)^2\cos\frac{2(2n+1)\pi y}{b}\Big(1+\cos\frac{2(2m+1)\pi x}{a}\Big)A_{mn}\Big\}^2 dx\,dy$$

$$= \frac{D}{2}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\Big[-\Big(\frac{2(2m+1)\pi}{a}\Big)^2\cos\frac{2(2m+1)\pi x}{a}\Big(1+\cos\frac{2(2n+1)\pi y}{b}\Big) -$$

$$-\Big(\frac{2(2n+1)\pi}{b}\Big)^2\cos\frac{2(2n+1)\pi y}{b}\Big(1+\cos\frac{2(2m+1)\pi x}{a}\Big)\Big]^2 +$$

$$+\sum_{m=0}^{\infty}\sum_{\substack{r=0 \\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}A_{mr}A_{ms}\Big[-\Big(\frac{2(2m+1)\pi}{a}\Big)^2\cos\frac{2(2m+1)\pi x}{a}\Big(1+\cos\frac{2(2r+1)\pi y}{b}\Big) -$$

$$-\Big(\frac{2(2r+1)\pi}{b}\Big)^2\cos\frac{2(2r+1)\pi y}{b}\Big(1+\cos\frac{2(2m+1)\pi x}{a}\Big)\Big]\Big[-\Big(\frac{2(2m+1)\pi}{a}\Big)^2\cos\frac{2(2m+1)\pi x}{a}\cdot$$

$$\cdot\Big(1+\cos\frac{2(2s+1)\pi y}{b}\Big) - \Big(\frac{2(2s+1)\pi}{b}\Big)^2\cos\frac{2(2s+1)\pi y}{b}\Big(1+\cos\frac{2(2m+1)\pi x}{a}\Big)\Big] +$$

$$+\sum_{\substack{r=0 \\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\Big[-\Big(\frac{2(2r+1)\pi}{a}\Big)^2\cos\frac{2(2r+1)\pi x}{a}\Big(1+\cos\frac{2(2n+1)\pi y}{b}\Big) -$$

$$-\Big(\frac{2(2n+1)\pi}{b}\Big)^2\cos\frac{2(2n+1)\pi y}{b}\Big(1+\cos\frac{2(2r+1)\pi x}{a}\Big)\Big]\Big[-\Big(\frac{2(2s+1)\pi}{a}\Big)^2\cos\frac{2(2s+1)\pi x}{a}\cdot$$

$$\cdot\Big(1+\cos\frac{2(2n+1)\pi y}{b}\Big) - \Big(\frac{2(2n+1)\pi}{b}\Big)^2\cos\frac{2(2n+1)\pi y}{b}\Big(1+\cos\frac{2(2s+1)\pi x}{a}\Big)\Big]\Big\} dx\,dy$$

$$= \frac{2\pi^4 D}{a^4}\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\Big[3ab(2m+1)^4 + 2ab(2m+1)^2(2n+1)^2\frac{a^2}{b^2} +$$

$$+3ab(2n+1)^4\frac{a^4}{b^4}\Big] +$$

$$+\sum_{m=0}^{\infty}\sum_{\substack{r=0 \\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}A_{mr}A_{ms}\frac{6(2m+1)^4 ab}{3} +$$

$$+\sum_{\substack{r=0 \\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\frac{6(2n+1)^4 ab}{3}\,\frac{a^4}{b^4}\Big]$$

$$U_5^{(s)} = \frac{D(K_x^{(n)})^2}{4}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(\frac{2(2m+1)\pi}{a}\Big)^2\cos\frac{2(2m+1)\pi x}{a}\Big(1+\cos\frac{2(2n+1)\pi y}{b}\Big)A_{mn} -$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(\frac{2(2n+1)\pi}{b}\Big)^2\cos\frac{2(2n+1)\pi y}{b}\Big(1+\cos\frac{2(2m+1)\pi x}{a}\Big)A_{mn}\Big\}^2 x^2\,dx\,dy$$

$$= \frac{D(K_x^{(n)})^2}{4}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big[\Big(\frac{2(2m+1)\pi}{a}\Big)^4\cos\frac{2(2m+1)\pi x}{a}x^2\Big(1+\cos^2\frac{2(2n+1)\pi y}{b}\Big)+$$

$$+\Big(\frac{2(2n+1)\pi}{b}\Big)^4\cos^2\frac{2(2n+1)\pi y}{b}\Big(1+\cos^2\frac{2(2m+1)\pi x}{a}\Big)x^2+$$

$$+2\Big(\frac{2(n+1)\pi}{b}\Big)^2\Big(\frac{2(2m+1)\pi}{a}\Big)^2\cos^2\frac{2(2n+1)\pi y}{b}\cos^2\frac{2(2m+1)\pi x}{a}x^2\Big]A_{mn}^2+$$

$$+\sum_{m=0}^{\infty}\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}A_{mr}A_{ms}\Big[\Big(\frac{2(2m+1)\pi}{a}\Big)^4\cos^2\frac{2(2m+1)\pi x}{a}x^2\Big]+$$

$$+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\Big\{\Big(\frac{2(2n+1)\pi}{b}\Big)^4\cos^2\frac{2(2n+1)\pi y}{b}x^2\Big]\Big\}\,dx\,dy$$

$$=\frac{1}{2}\,\frac{b\pi^4 D(K_x^{(n)})^2}{a}\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\Big\{\frac{1}{2}(2m+1)^4+\frac{1}{3}(2m+1)^2(2n+1)^2\frac{a^2}{b^2}+$$

$$+\frac{1}{2}(2n+1)^4\frac{a^4}{b^4}+$$

$$+\sum_{m=0}^{\infty}\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}A_{mr}A_{ms}\frac{(2m+1)^4}{3}+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\frac{(2n+1)^4}{3}\frac{a^4}{b^4}\Big]$$

$$U_6^{(s)}=\frac{1}{2}\,\frac{b\pi^4 D(K_y^{(n)})^2}{a^3}\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}^2\Big\{\frac{1}{2}(2m+1)^4 b^2+(2m+1)^2(2n+1)^2\frac{a^2}{3}+$$

$$+\frac{1}{2}(2n+1)^4\frac{a^4}{b^2}+$$

$$+\sum_{m=0}^{\infty}\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}A_{mr}A_{ms}\frac{(2m+1)^4 b^2}{3}+\sum_{\substack{r=0\\r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}A_{rn}A_{sn}\frac{(2n+1)^4 a^4}{3b^2}\Big]$$

$$\frac{\partial U^{(s)}}{\partial A_{mn}}=\frac{\partial U_1^{(s)}}{\partial A_{mn}}+\frac{\partial U_2^{(s)}}{\partial A_{mn}}+\frac{\partial U_3^{(s)}}{\partial A_{mn}}+\frac{\partial U_4^{(s)}}{\partial A_{mn}}+\frac{\partial U_5^{(s)}}{\partial A_{mn}}+\frac{\partial U_6^{(s)}}{\partial A_{mn}}$$

$$\therefore\quad \frac{3\phi}{2}\Big[3A_{mn}+\sum_{\substack{r=0\\r\neq n}}^{\infty}2A_{mr}+\sum_{\substack{r=0\\r\neq m}}^{\infty}2A_{rn}+\sum_{\substack{r=0\\r\neq m}}^{\infty}\sum_{\substack{p=0\\p\neq n}}^{\infty}\frac{4}{3}A_{rp}\Big]ab+$$

$$+\frac{\phi(K_x^{(n)})^2}{2}\Big[\frac{3b^2}{8}A_{mn}+\sum_{\substack{r=0\\r\neq n}}^{\infty}\frac{a^2}{4}A_{mr}+\sum_{\substack{r=0\\r\neq m}}^{\infty}\frac{a^2}{4}A_{rn}+\sum_{\substack{r=0\\r\neq m}}^{\infty}\sum_{\substack{p=0\\p\neq n}}^{\infty}a^2/6\;A_{rp}\Big]ab+$$

$$+\frac{\phi(K_y^{(n)})^2}{2}\left[\frac{3b^2}{8}A_{mn}+\sum_{\substack{r=0\\r\neq n}}^{\infty}\frac{b^2}{4}A_{mr}+\sum_{\substack{r=0\\r\neq m}}^{\infty}\frac{b^2}{4}A_{rn}+\sum_{\substack{r=0\\r\neq m}}^{\infty}\sum_{\substack{p=0\\p\neq n}}^{\infty}\frac{b^2}{6}A_{rp}\right]ab+$$

$$+\frac{4D\pi^4}{a^4}\left[A_{mn}\left(3(2m+1)^4+3(n+1)^4\frac{a^4}{b^4}+2(2m+1)^2(2n+1)^2\frac{a^2}{b^2}\right)+\right.$$

$$\left.+\sum_{\substack{r=0\\r\neq n}}^{\infty}2A_{mr}(2m+1)^4+\sum_{\substack{r=0\\r\neq m}}^{\infty}2A_{rn}(2n+1)^4\frac{a^4}{b^4}\right]ab+$$

$$+\frac{\pi^4D(K_x^{(n)})^2}{2a^4}\left[A_{mn}\left[a^2(2m+1)^4+\frac{a^6}{b^4}(2n+1)^4+\frac{2a^4}{3b^2}(2m+1)^2(2n+1)^2\right]+\right.$$

$$\left.+\sum_{\substack{r=0\\r\neq n}}^{\infty}A_{mr}\frac{2a^2(2m+1)^4}{3}+\sum_{\substack{r=0\\r\neq m}}^{\infty}A_{rn}\frac{2a^6}{3b^4}(2n+1)^4\right]ab+$$

$$+\frac{\pi^4D(K_y^{(n)})^2}{2a^4}\left[A_{mn}\left[b^2(2m+1)^4+\frac{a^4}{b^2}(2n+1)^4+\frac{2a^2}{3}(2m+1)^2(2n+1)^2\right]+\right.$$

$$\left.+\sum_{\substack{r=0\\r\neq n}}^{\infty}A_{mr}\frac{2b^2(2m+1)^4}{3}+\sum_{\substack{r=0\\r\neq m}}^{\infty}A_{rn}\frac{2a^4}{3b^2}\right]ab=\frac{\partial U^{(s)}}{\partial A_{mn}}$$

The work of external applied forces is

$$W=\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}P\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\left(1+\cos\frac{2(2m+1)\pi x}{a}\right)\left(1+\cos\frac{2(2n+1)\pi y}{b}\right)\left(1+\frac{x^2}{2}(K_x^{(n)})^2+\frac{y^2}{2}(K_y^{(n)})^2\right)dx\,dy$$

$$=P\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\left(1+\frac{a^2(K_x^{(n)})^2}{24}+\frac{b^2(K_y^{(n)})^2}{24}\right)ab$$

$$\frac{\partial W}{\partial A_{mn}}=P\left(1+\frac{a^2(K_x^{(n)})^2}{24}+\frac{b^2(K_y^{(n)})^2}{24}\right)ab$$

$$\frac{\partial V}{\partial A_{mn}}=\frac{\partial U^{(s)}}{\partial A_{mn}}-\frac{\partial W}{\partial A_{mn}}=0$$

Dividing both $\frac{\partial U^{(s)}}{\partial A_{mn}}$ and $\frac{\partial W}{\partial A_{mn}}$ by ab yields

$$\frac{3\phi}{2}\left\{A_{mn}\left(3+\frac{a^2}{4}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{4}\frac{(K_y^{(n)})^2}{2}\right)+\sum_{\substack{r=0\\r\neq n}}^{\infty}A_{mr}\left(2+\frac{a^2}{6}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{6}\frac{(K_y^{(n)})^2}{2}\right)+\right.$$

$$\left.+\sum_{\substack{r=0\\r\neq m}}^{\infty}A_{rn}\left(2+\frac{a^2}{6}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{6}\frac{(K_y^{(n)})^2}{2}\right)+\sum_{\substack{r=0\\r\neq m}}^{\infty}\sum_{\substack{p=0\\p\neq n}}^{\infty}A_{rp}\left(\frac{4}{3}+\frac{a^2}{9}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{9}\frac{(K_y^{(n)})^2}{2}\right)\right\}+$$

$$+\frac{4D\pi^4}{a^4}\left\{A_{mn}\left[(2m+1)^4\left(3+\frac{a^2}{4}\frac{(K_x^{(n)})^2}{2}+\frac{b^2}{4}\frac{(K_y^{(n)})^2}{2}\right)+(2n+1)^4\left(\frac{3a^4}{b^4}+\frac{a^6(K_x^{(n)})^2}{8b^4}+\frac{a^4(K_y^{(n)})^2}{8b^4}\right)+\right.\right.$$

$$\left.+(2m+1)^2(2n+1)^2\left(\frac{2a^2}{b^2}+\frac{a^4}{6b^2}\frac{(K_x^{(n)})^2}{2}+\frac{a^6}{6}\frac{(K_y^{(n)})^2}{2}\right)\right]+\sum_{\substack{r=0\\r\neq n}}^{\infty}A_{mr}(2m+1)^4\left(2+\frac{a^2(K_x^{(n)})^2}{12}+\frac{b^2(K_y^{(n)})^2}{12}\right)+$$

$$\left.+\sum_{\substack{r=0\\r\neq m}}^{\infty}A_{rn}(2n+1)^4\left(\frac{2a^4}{b^4}+\frac{a^6(K_x^{(n)})^2}{12b^4}+\frac{a^4(K_y^{(n)})^2}{12b^2}\right)\right\}=P\left(\frac{24+a^2(K_x^{(n)})^2+b^2(K_y^{(n)})^2}{24}\right)$$

APPENDIX C

DERIVATION OF STRESS RESULTANTS AND STRESS COUPLES

A. General Equilibrium Equations of Shallow Shells

From Fig. (1), it gives

$$\sum \bar{F}^{(\sigma)} = 0$$

$$\Delta \bar{F}_1^{(\sigma)} A_2\, d\alpha_2 + \Delta \bar{F}_2^{(\sigma)} A_1\, d\alpha_1 + \bar{P} A_1 A_2\, d\alpha_1\, d\alpha_2 = 0$$

Assuming that the changes of $\bar{F}_1^{(\sigma)}$ and $\bar{F}_2^{(\sigma)}$ are linear and neglecting terms of higher infinitesimal order, then

$$\frac{1}{A_1}\frac{\partial \bar{F}_1^{(\sigma)}}{\partial \alpha_1} A_1 A_2\, d\alpha_1\, d\alpha_2 + \frac{1}{A_2}\frac{\partial \bar{F}_2^{(\sigma)}}{\partial \alpha_2} A_1 A_2\, d\alpha_1\, d\alpha_2 + \bar{P} A_1 A_2\, d\alpha_1\, d\alpha_2 = 0$$

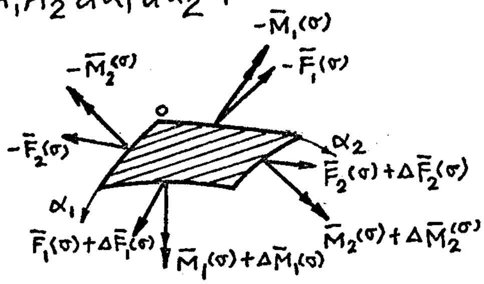

FIG. 1

or

$$\frac{1}{A_1}\frac{\partial \bar{F}_1^{(\sigma)}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial \bar{F}_2^{(\sigma)}}{\partial \alpha_2} + \bar{P} = 0 \qquad (1)$$

Expanding expression (1) yields

$$\frac{1}{A_1}\left(\frac{\partial F_{11}^{(\sigma)}}{\partial \alpha_1}\bar{e}_1 + \frac{\partial F_{12}^{(\sigma)}}{\partial \alpha_1}\bar{e}_2 + \frac{\partial F_{13}^{(\sigma)}}{\partial \alpha_1}\bar{e}_3\right) + \frac{1}{A_1}\left(F_{11}^{(\sigma)}\frac{\partial \bar{e}_1}{\partial \alpha_1} + F_{12}^{(\sigma)}\frac{\partial \bar{e}_2}{\partial \alpha_1} + F_{13}^{(\sigma)}\frac{\partial \bar{e}_3}{\partial \alpha_1}\right) +$$
$$+ \frac{1}{A_2}\left(\frac{\partial F_{21}^{(\sigma)}}{\partial \alpha_2}\bar{e}_1 + \frac{\partial F_{22}^{(\sigma)}}{\partial \alpha_2}\bar{e}_2 + \frac{\partial F_{23}^{(\sigma)}}{\partial \alpha_2}\bar{e}_3\right) + \frac{1}{A_2}\left(F_{21}^{(\sigma)}\frac{\partial \bar{e}_1}{\partial \alpha_2} + F_{22}^{(\sigma)}\frac{\partial \bar{e}_2}{\partial \alpha_2} + F_{23}^{(\sigma)}\frac{\partial \bar{e}_3}{\partial \alpha_2}\right) +$$
$$+ P_1\bar{e}_1 + P_2\bar{e}_2 + P_3\bar{e}_3 = 0 \qquad (2)$$

If $\alpha_1, \alpha_2, \alpha_3$ represent rectangular co-ordinates x,y,z, the expression (2) reduces to

$$\left(\frac{\partial F_{xx}^{(\sigma)}}{\partial x}\bar{e}_x + \frac{\partial F_{xy}^{(\sigma)}}{\partial x}\bar{e}_y + \frac{\partial F_{xz}^{(\sigma)}}{\partial x}\bar{e}_z\right) + \left(\frac{\partial F_{yx}^{(\sigma)}}{\partial y}\bar{e}_x + \frac{\partial F_{yy}^{(\sigma)}}{\partial y}\bar{e}_y + \frac{\partial F_{yz}^{(\sigma)}}{\partial y}\bar{e}_z\right) + P_x\bar{e}_x + P_y\bar{e}_y + P_z\bar{e}_z = 0$$

This yields,

$$\frac{\partial F_{xx}^{(\sigma)}}{\partial x} + \frac{\partial F_{yx}^{(\sigma)}}{\partial y} + P_x = 0 \qquad (3\text{-}1)$$

$$\frac{\partial F_{xy}^{(\sigma)}}{\partial x} + \frac{\partial F_{yy}^{(\sigma)}}{\partial y} + P_y = 0 \qquad (3-2)$$

$$\frac{\partial F_{xz}^{(\sigma)}}{\partial x} + \frac{\partial F_{yz}^{(\sigma)}}{\partial y} + P_z = 0 \qquad (3-3)$$

Again Fig. (1) gives

$$\Sigma \bar{M}(\sigma) = 0$$

$$\Delta \bar{M}_1(\sigma) A_2 d\alpha_2 + \Delta \bar{M}_2(\sigma) A_1 d\alpha_1 + (A_1 d\alpha_1 \bar{e}_1 + \frac{A_2 d\alpha_2}{2}\bar{e}_2) \times (\bar{F}_1^{(\sigma)} + \Delta \bar{F}_1(\sigma)) A_2 d\alpha_2 +$$
$$+ (A_2 d\alpha_2 \bar{e}_2 + \frac{A_1 d\alpha_1}{2}\bar{e}_1) \times (\bar{F}_2^{(\sigma)} + \Delta \bar{F}_2^{(\sigma)}) A_1 d\alpha_1 - \frac{A_2 d\alpha_2}{2}\bar{e}_2 \times \bar{F}_1^{(\sigma)} A_2 d\alpha_2 -$$
$$- \frac{A_1 d\alpha_1}{2}\bar{e}_1 \times \bar{F}_2^{(\sigma)} A_1 d\alpha_1 = 0$$

Assuming that the variations are all linear, then

$$\frac{1}{A_2}\frac{\partial \bar{M}_2^{(\sigma)}}{\partial \alpha_2} A_1 A_2 d\alpha_1 d\alpha_2 + \frac{1}{A_1}\frac{\partial \bar{M}_1^{(\sigma)}}{\partial \alpha_1} A_1 A_2 d\alpha_1 d\alpha_2 - \bar{F}_1^{(\sigma)} A_2 d\alpha_2 \times A_1 d\alpha_1 \bar{e}_1 -$$
$$- \bar{F}_2^{(\sigma)} A_1 d\alpha_1 \times A_2 d\alpha_2 \bar{e}_2 = 0$$

or say,

$$\frac{1}{A_2}\frac{\partial \bar{M}_2^{(\sigma)}}{d\alpha_2} + \frac{1}{A_1}\frac{\partial \bar{M}_1^{(\sigma)}}{\partial \alpha_1} + \bar{e}_1 \times \bar{F}_1^{(\sigma)} + \bar{e}_2 \times \bar{F}_2^{(\sigma)} = 0 \qquad (4)$$

Expanding expression (4) yields

$$\frac{1}{A_1}\left(\frac{\partial M_{11}^{(\sigma)}}{\partial \alpha_1}\bar{e}_1 + \frac{\partial M_{12}^{(\sigma)}}{\partial \alpha_1}\bar{e}_2 + \frac{\partial M_{13}^{(\sigma)}}{\partial \alpha_1}\bar{e}_3 + M_{11}^{(\sigma)}\frac{\partial \bar{e}_1}{\partial \alpha_1} + M_{12}^{(\sigma)}\frac{\partial \bar{e}_2}{\partial \alpha_1} + M_{13}^{(\sigma)}\frac{\partial \bar{e}_3}{\partial \alpha_1}\right) +$$
$$+ \frac{1}{A_2}\left(\frac{\partial M_{21}^{(\sigma)}}{\partial \alpha_2}\bar{e}_1 + \frac{\partial M_{22}^{(\sigma)}}{\partial \alpha_2}\bar{e}_2 + \frac{\partial M_{23}^{(\sigma)}}{\partial \alpha_2}\bar{e}_3 + M_{21}^{(\sigma)}\frac{\partial \bar{e}_1}{\partial \alpha_2} + M_{22}^{(\sigma)}\frac{\partial \bar{e}_2}{\partial \alpha_2} + M_{23}^{(\sigma)}\frac{\partial \bar{e}_3}{\partial \alpha_2}\right) +$$
$$+ (F_{12}^{(\sigma)}\bar{e}_3 - F_{13}^{(\sigma)}\bar{e}_2) + (F_{23}^{(\sigma)}\bar{e}_1 - F_{21}^{(\sigma)}\bar{e}_3) = 0 \qquad (5)$$

Since thickness of a shell is of much smaller order than the dimensions of its middle surface, the terms $M_{13}^{(\sigma)}$, $M_{23}^{(\sigma)}$ can be reasonably neglected. So expression (5) reduces to

$$\frac{1}{A_1}\left(\frac{\partial M_{11}^{(\sigma)}}{\partial \alpha_1}\bar{e}_1 + \frac{\partial M_{12}^{(\sigma)}}{\partial \alpha_1}\bar{e}_2 + M_{11}^{(\sigma)}\frac{\partial \bar{e}_1}{\partial \alpha_1} + M_{12}^{(\sigma)}\frac{\partial \bar{e}_2}{\partial \alpha_1} + \right.$$
$$\left. + \frac{1}{A_2}\left(\frac{\partial M_{21}^{(\sigma)}}{\partial \alpha_2}\bar{e}_1 + \frac{\partial M_{22}^{(\sigma)}}{\partial \alpha_2}\bar{e}_2 + M_{21}^{(\sigma)}\frac{\partial \bar{e}_1}{\partial \alpha_2} + M_{22}^{(\sigma)}\frac{\partial \bar{e}_2}{\partial \alpha_2}\right) + \right.$$

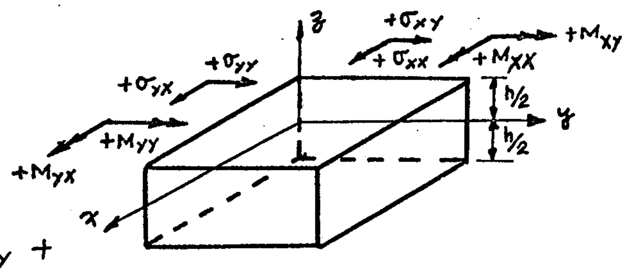

FIG. 2

$$+ F_{23}^{(\sigma)} \bar{e}_1 - F_{13}^{(\sigma)} \bar{e}_2 + (F_{12}^{(\sigma)} - F_{21}^{(\sigma)}) \bar{e}_3 = 0$$

For rectangular co-ordinates

$$\frac{\partial M_{xx}^{(\sigma)}}{\partial x} \bar{e}_x + \frac{\partial M_{xy}^{(\sigma)}}{\partial x} \bar{e}_y + \frac{\partial M_{yx}^{(\sigma)}}{\partial y} \bar{e}_x + \frac{\partial M_{yy}^{(\sigma)}}{\partial y} \bar{e}_y +$$

$$+ F_{yz}^{(\sigma)} \bar{e}_x - F_{xz}^{(\sigma)} \bar{e}_y + (F_{xy}^{(\sigma)} - F_{yx}^{(\sigma)}) \bar{e}_z = 0$$

The scalar components of the moment equation of equilibrium are

$$\frac{\partial M_{xx}^{(\sigma)}}{\partial x} + \frac{\partial M_{yx}^{(\sigma)}}{\partial y} + F_{yz}^{(\sigma)} = 0 \qquad (7\text{-}1)$$

$$\frac{\partial M_{xy}^{(\sigma)}}{\partial x} + \frac{\partial M_{yy}^{(\sigma)}}{\partial y} - F_{xz}^{(\sigma)} = 0 \qquad (7\text{-}2)$$

$$F_{xy}^{(\sigma)} - F_{yx}^{(\sigma)} = 0 \qquad (7\text{-}3)$$

Equation (3) together with equation (7) are called the equilibrium equations of a shallow shell.

B. Equilibrium Equations of Thin Shallow Translational Shells Subjected to Uniform Transverse Load

For this case, $P_x = P_y = 0$, the equations (3), (7) reduce down to the following expressions:

$$\frac{\partial F_{xx}^{(\sigma)}}{\partial x} + \frac{\partial F_{yx}^{(\sigma)}}{\partial y} = 0 \qquad (8\text{-}1)$$

$$\frac{\partial F_{yy}^{(\sigma)}}{\partial y} + \frac{\partial F_{xy}^{(\sigma)}}{\partial x} = 0 \qquad (8\text{-}2)$$

$$\frac{\partial F_{xz}^{(\sigma)}}{\partial x} + \frac{\partial F_{yz}^{(\sigma)}}{\partial y} + P_z = 0 \qquad (8\text{-}3)$$

$$\frac{\partial M_{yy}^{(\sigma)}}{\partial y} + \frac{\partial M_{xy}^{(\sigma)}}{\partial x} - F_{xz}^{(\sigma)} = 0 \qquad (8\text{-}4)$$

$$\frac{\partial M_{xx}^{(\sigma)}}{\partial x} + \frac{\partial M_{yx}^{(\sigma)}}{\partial y} - F_{yz}^{(\sigma)} = 0 \qquad (8\text{-}5)$$

$$F_{xy}^{(\sigma)} = F_{yx}^{(\sigma)}$$

C. Stress Resultants and Stress Couples

From Fig. (2), it gives

$$F_{xx}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\, dz \qquad (9\text{-}1)$$

$$F_{yy}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy}\, dz \qquad (9\text{-}2)$$

$$M_{xy}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\, z\, dz \qquad (9\text{-}3)$$

$$M_{yx}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} -\sigma_{yy}\, z\, dz \qquad (9\text{-}4)$$

$$M_{xx}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} -\sigma_{xy}\, z\, dz \qquad (9\text{-}5)$$

$$M_{yy}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yx}\, z\, dz \qquad (9\text{-}6)$$

$$F_{xz}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz}\, dz \qquad (9\text{-}7)$$

$$F_{yz}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yz}\, dz \qquad (9\text{-}8)$$

From Appendix A, equation 2, it gives

$$F_{xx}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\, dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{e}_x \bar{e}_x : \bar{\bar{\sigma}}\, dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{e}_x \bar{e}_x : (2\mu \bar{\bar{\epsilon}} + \lambda(\bar{\bar{\epsilon}} : \bar{\bar{1}})\, \bar{\bar{1}})\, dz$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} 2\mu\, \epsilon_{xx} + \frac{\nu}{1-\nu} (\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})\, dz$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} 2\mu \left(\epsilon_{xx} + \frac{\nu}{1-\nu}\, \epsilon_{xx} + \frac{\nu}{1-\nu}\, \epsilon_{yy}\right) dz$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{2\mu}{1-\nu} (\epsilon_{xx} + \nu \epsilon_{yy})\, dz$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} (\epsilon_{xx}^{(\bar{r}_0)} - z\delta K_x^{(n)} + \nu \epsilon_{yy}^{(\bar{r}_0)} - \nu z \delta K_y^{(n)}) \frac{2\mu}{1-\nu}\, dz$$

$$= \frac{2\mu h}{1-\nu} (K_x^{(n)} + \nu K_y^{(n)}) u_z = D'(K_x^{(n)} + \nu K_y^{(n)}) u_z \qquad (10\text{-}1)$$

$$F_{yy}^{(\sigma)} = D'(K_y^{(n)} + \nu K_x^{(n)}) u_z \qquad (10\text{-}2)$$

$$M_{xy}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} z\, dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{2\mu}{1-\nu} (\epsilon_{xx}^{(\bar{r}_0)} - z\delta K_x^{(n)} + \nu \epsilon_{yy}^{(\bar{r}_0)} - \nu z \delta K_y^{(n)}) z\, dz$$

$$= -\frac{2\mu}{3(1-\nu)} \frac{h^3}{4} \left(\frac{\partial^2 u_z}{\partial x^2} + \nu \frac{\partial^2 u_z}{\partial y^2} \right) = -D \left(\frac{\partial^2 u_z}{\partial x^2} + \nu \frac{\partial^2 u_z}{\partial y^2} \right) \qquad (10\text{-}3)$$

$$M_{yx}^{(\sigma)} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} z\, dz = D \left(\frac{\partial^2 u_z}{\partial y^2} + \nu \frac{\partial^2 u_z}{\partial x^2} \right) \qquad (10\text{-}4)$$

$$M_{xx}^{(\sigma)} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy} z\, dz = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{e}_x \bar{e}_y : (2\mu \bar{\bar{E}} + \lambda (\bar{\bar{E}} : \bar{\bar{1}}) \bar{\bar{1}}) z\, dz$$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} 2\mu \epsilon_{xy} z\, dz = 2\mu \int_{-\frac{h}{2}}^{\frac{h}{2}} (-\epsilon_{xy}^{(\bar{r}_0)} - z\delta K^{(g)}) z\, dz$$

$$= -\frac{2\mu h^3}{12} \delta K^{(g)} = D(1-\nu) \frac{\partial^2 u_z}{\partial x \partial y} \qquad (10\text{-}5)$$

$$M_{yy}^{(\sigma)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yx} z\, dz = -D(1-\nu) \frac{\partial^2 u_z}{\partial y \partial x} \qquad (10\text{-}6)$$

Expression (8-4) yields,

$$F_{xz}^{(\sigma)} = \frac{\partial M_{xy}^{(\sigma)}}{\partial x} + \frac{\partial M_{yy}^{(\sigma)}}{\partial y} = -D \frac{\partial}{\partial x} \left(\frac{\partial^2 u_z}{\partial x^2} + \nu \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial y^2} - \nu \frac{\partial^2 u_z}{\partial y^2} \right)$$

$$= -D \frac{\partial}{\partial x} (\nabla^2 u_z) \qquad (10\text{-}7)$$

and expression (8-5) yields,

$$F_{yz}^{(\sigma)} = -\frac{\partial M_{yx}^{(\sigma)}}{\partial y} - \frac{\partial M_{xx}^{(\sigma)}}{\partial x} = -D \frac{\partial}{\partial y} \left(\frac{\partial^2 u_z}{\partial y^2} + \nu \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial x^2} - \nu \frac{\partial^2 u_z}{\partial x^2} \right)$$

$$= -D \frac{\partial}{\partial y} (\nabla^2 u_z) \qquad (10\text{-}8)$$

Since the POISSON'S ratio ν of usual construction materials is always a small quantity compared to unity, a further approximation of stress couples is possible.

$$M_{xy}^{(\sigma)} = -D \frac{\partial^2 u_z}{\partial x^2} \qquad (10\text{-}3')$$

$$M_{yx}^{(\sigma)} = D \frac{\partial^2 u_z}{\partial y^2} \qquad (10\text{-}4')$$

$$M_{xx}^{(\sigma)} = D \frac{\partial^2 u_z}{\partial x \partial y} \qquad (10\text{-}5')$$

$$M_{yy}^{(\sigma)} = -D \frac{\partial^2 u_z}{\partial y \partial x} \qquad (10\text{-}6')$$

APPENDIX D

CALCULATIONS OF FOURIER COEFFICIENTS FOR THE MEMBRANAL DISPLACEMENT FUNCTIONS

The expressions (II-1-2), according to rectangular Cartesian co-ordinates, gives

$$
\begin{aligned}
\epsilon_{xx}^{(\bar{r}_0)} &= \frac{\partial u_x^{(\bar{r}_0)}}{\partial x} + K_x^{(n)} u_3^{(\bar{r}_0)} \\
\epsilon_{yy}^{(\bar{r}_0)} &= \frac{\partial u_y^{(\bar{r}_0)}}{\partial y} + K_y^{(n)} u_3^{(\bar{r}_0)} \\
\delta K_x^{(n)} &= \frac{\partial^2 u_3^{(\bar{r}_0)}}{\partial x^2} \\
\delta K_y^{(n)} &= \frac{\partial^2 u_3^{(\bar{r}_0)}}{\partial y^2} \\
\epsilon_{xy}^{(\bar{r}_0)} &= \frac{1}{2}\left[\frac{\partial u_y^{(\bar{r}_0)}}{\partial x} + \frac{\partial u_x^{(\bar{r}_0)}}{\partial y}\right] \\
\delta K^{(g)} &= \frac{\partial^2 u_3^{(\bar{r}_0)}}{\partial x \partial y}
\end{aligned}
\qquad (4\text{-}1)
$$

Substituting (4-1),into equation (II-1-1), gives

$$
\begin{aligned}
U^{(s)} = \frac{\mu h}{1-\nu} \iint_A \Big\{ & \left(\frac{\partial u_x^{(\bar{r}_0)}}{\partial x}\right)^2 + \left(\frac{\partial u_y^{(\bar{r}_0)}}{\partial y}\right)^2 + 2\nu \frac{\partial u_x^{(\bar{r}_0)}}{\partial x}\frac{\partial u_y^{(\bar{r}_0)}}{\partial y} + \\
& + \frac{1}{2}(1-\nu)\left[\left(\frac{\partial u_y^{(\bar{r}_0)}}{\partial x}\right)^2 + \left(\frac{\partial u_x^{(\bar{r}_0)}}{\partial y}\right)^2 + 2\frac{\partial u_y^{(\bar{r}_0)}}{\partial x}\frac{\partial u_x^{(\bar{r}_0)}}{\partial y}\right] \\
& + \left[(K_x^{(n)})^2 (u_3^{(\bar{r}_0)})^2 + (K_y^{(n)})^2 (u_3^{(\bar{r}_0)})^2 + 2\nu K_x^{(n)} K_y^{(n)} (u_3^{(\bar{r}_0)})^2\right] \\
& + \left[2\frac{\partial u_x^{(\bar{r}_0)}}{\partial x} u_3^{(\bar{r}_0)} (K_x^{(n)} + \nu K_y^{(n)}) + 2\frac{\partial u_y^{(\bar{r}_0)}}{\partial y} u_3^{(\bar{r}_0)} (K_y^{(n)} + \nu K_x^{(n)})\right]\Big\} dx dy + \\
& + \frac{\mu h^3}{12(1-\nu)} \iint_A \left(\frac{\partial^2 u_3^{(\bar{r}_0)}}{\partial x^2} + \frac{\partial^2 u_3^{(\bar{r}_0)}}{\partial y^2}\right)^2 dx dy
\end{aligned}
\qquad (4\text{-}2)
$$

(I) Shells with fixed boundaries,

Let

$$\left.\begin{aligned}
u_x^{(\bar{r}_0)} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} \sin\frac{2(2m+1)\pi x}{a}\left(2-\cos\frac{2(2n+1)\pi y}{b}\right)\\
u_y^{(\bar{r}_0)} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} \sin\frac{2(2n+1)\pi y}{b}\left(2-\cos\frac{2(2m+1)\pi x}{a}\right)\\
u_z^{(\bar{r}_0)} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \left(1+\cos\frac{2(2m+1)\pi x}{a}\right)\left(1+\cos\frac{2(2n+1)\pi x}{b}\right)
\end{aligned}\right\} \quad (4\text{-}3)$$

then

$$\left.\begin{aligned}
\frac{\partial u_x^{(\bar{r}_0)}}{\partial x} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}\frac{2(2m+1)\pi}{a}\cos\frac{2(2m+1)\pi x}{a}\left(2-\cos\frac{2(2n+1)\pi y}{b}\right)\\
\frac{\partial u_x^{(\bar{r}_0)}}{\partial y} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}\frac{2(2n+1)\pi}{b}\sin\frac{2(2m+1)\pi x}{a}\sin\frac{2(2n+1)\pi y}{b}\\
\frac{\partial u_y^{(\bar{r}_0)}}{\partial y} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\frac{2(2n+1)\pi}{b}\cos\frac{2(2n+1)\pi y}{b}\left(2-\cos\frac{2(2m+1)\pi x}{a}\right)\\
\frac{\partial u_y^{(\bar{r}_0)}}{\partial x} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\frac{2(2m+1)\pi}{a}\sin\frac{2(2n+1)\pi y}{b}\sin\frac{2(2m+1)\pi x}{a}
\end{aligned}\right\} \quad (4\text{-}4)$$

The expressions (4-3) and (4-4) satisfy the given boundary conditions

$$\left.\begin{aligned}
&x=0; && u_x^{(\bar{r}_0)}=0, && \frac{\partial u_x^{(\bar{r}_0)}}{\partial x}\neq 0\\
&x=\pm\frac{a}{2}; && u_x^{(\bar{r}_0)}=0, && \frac{\partial u_x^{(\bar{r}_0)}}{\partial x}\neq 0, && u_y^{(\bar{r}_0)}\neq 0, && \frac{\partial u_y^{(\bar{r}_0)}}{\partial y}\neq 0\\
&y=0; && u_y^{(\bar{r}_0)}=0, && \frac{\partial u_y^{(\bar{r}_0)}}{\partial y}\neq 0\\
&y=\pm\frac{b}{2}; && u_y^{(\bar{r}_0)}=0, && \frac{\partial u_y^{(\bar{r}_0)}}{\partial y}\neq 0, && u_x^{(\bar{r}_0)}\neq 0, && \frac{\partial u_x^{(\bar{r}_0)}}{\partial x}\neq 0
\end{aligned}\right\} \quad (4\text{-}5)$$

Substituting (A4-3), (A4-4) and $u_z^{(\bar{r}_0)}$ into (A4-2) gives,

$$
\begin{aligned}
U^{(S)} = \frac{\mu h}{1-\nu}\Bigg\{ & \Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}^{2}\Big(\frac{2(2m+1)\pi}{a}\Big)^{2}\frac{9ab}{4} + \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\substack{s=0\\ r\neq s}}^{\infty} B_{mr}B_{ms}\Big(\frac{2(2m+1)\pi}{a}\Big)^{2} \\
& \frac{8ab}{4}\Big] + \Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}^{2}\Big(\frac{2(2n+1)\pi}{b}\Big)^{2}\frac{9ab}{4} + \sum_{\substack{m=0\\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty} C_{rn}C_{sn} \\
& \Big(\frac{2(2n+1)\pi}{b}\Big)^{2}\frac{8ab}{4}\Big] + 2\nu\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}C_{mn}\Big(\frac{2(2m+1)\pi}{a}\Big)\Big(\frac{2(2n+1)\pi}{b}\Big)\frac{ab}{4} + \\
& + \frac{1}{2}(1-\nu)\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}^{2}\Big(\frac{2(2m+1)\pi}{a}\Big)^{2}\frac{ab}{4} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}^{2}\Big(\frac{2(2n+1)\pi}{b}\Big)^{2} \\
& \frac{ab}{4} + 2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}B_{mn}\Big(\frac{2(2m+1)\pi}{a}\Big)\Big(\frac{2(2n+1)\pi}{b}\Big)\frac{ab}{4}\Big] + \\
& + 2(K_x^{(n)}+\nu K_y^{(n)})\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}A_{mn}\frac{2(2m+1)\pi}{a}\,\frac{3ab}{4} + \\
& + \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\substack{s=0\\ r\neq s}}^{\infty} B_{mr}A_{ms}\frac{2(2m+1)\pi}{a}ab\Big] + 2(K_y^{(n)}+\nu K_x^{(n)}) \\
& \Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}A_{mn}\frac{2(2n+1)\pi}{b}\,\frac{3ab}{4} + \sum_{\substack{r=0\\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty} C_{rn}A_{sn}\frac{2(2n+1)\pi}{b}ab\Big] + \\
& + \Big[(K_x^{(n)})^{2} + (K_y^{(n)})^{2} + 2\nu K_x^{(n)}K_y^{(n)}\Big]\frac{3ab}{4}\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 3A_{mn}^{2} + \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\substack{s=0\\ r\neq s}}^{\infty} 2A_{mr}A_{ms} + \\
& + \sum_{\substack{r=0\\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty} 2A_{rn}A_{rs} + \sum_{\substack{r=0\\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{\substack{p=0\\ p\neq q}}^{\infty}\sum_{q=0}^{\infty} A_{rp}A_{sq}\Big]\Bigg\} + \\
& + \frac{2Dab\pi^{4}}{a^{4}}\Bigg\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}^{2}\Big[3(2m+1)^{4} + \frac{3a^{4}}{b^{4}}(2n+1)^{4} + \frac{2a^{2}}{b^{2}}(2m+1)^{2} \\
& (2n+1)^{2}\Big] + \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{\substack{s=0\\ r\neq s}}^{\infty} A_{mr}A_{ms}2(2m+1)^{4} + \sum_{\substack{r=0\\ r\neq s}}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty} A_{rn}A_{sn}\frac{2a^{4}}{b^{4}}(2n+1)^{4}\Bigg\}
\end{aligned}
\tag{4-6}
$$

The external work W is derived as follows,

From Fig. (1)

Since θ_x is very small

$\therefore \ \sin\theta_x \doteq \tan\theta_x = -\frac{\partial z}{\partial x} = K_x^{(n)} x$

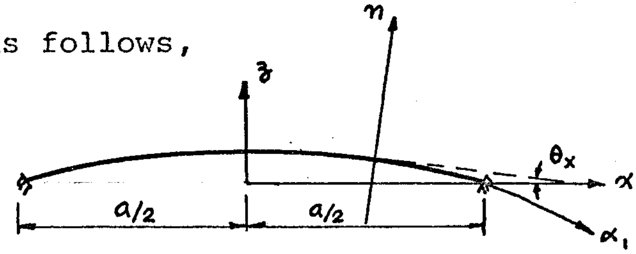

FIG. 1. Section of a shell

By the same argument, $\sin\theta_y \doteq K_y^{(n)} y$

$$\bar{e}_1 = \bar{e}_1 \cdot \bar{e}_x \bar{e}_x + \bar{e}_1 \cdot \bar{e}_y \bar{e}_y + \bar{e}_1 \cdot \bar{e}_z \bar{e}_z$$
$$= \cos\theta_x \bar{e}_x + 0\, \bar{e}_y + K_x^{(n)} x\, \bar{e}_z$$
$$\doteq \bar{e}_x - K_x^{(n)} x\, \bar{e}_z$$
$$\bar{e}_2 \doteq \bar{e}_y - K_y^{(n)} y\, \bar{e}_z$$
$$\bar{e}_n = \bar{e}_n \cdot \bar{e}_x \bar{e}_x + \bar{e}_n \cdot \bar{e}_y \bar{e}_y + \bar{e}_n \cdot \bar{e}_z \bar{e}_z$$
$$= K_x^{(n)} x\, \bar{e}_x + K_y^{(n)} y\, \bar{e}_y + \cos\theta_x \bar{e}_z \doteq K_x^{(n)} x\, \bar{e}_x + K_y^{(n)} y\, \bar{e}_y + \bar{e}_z$$
$$\bar{P} = -P\bar{e}_z = -PK_x^{(n)} x\, \bar{e}_1 - PK_y^{(n)} y\, \bar{e}_2 - P\bar{e}_n$$

$$W = \iint_A \bar{P} \cdot \bar{u}\, dx\, dy$$
$$= \iint_A (PK_x^{(n)} x\, \bar{e}_1 + PK_y^{(n)} y\, \bar{e}_2 - P\bar{e}_n) \cdot (u_1 \bar{e}_1 + u_2 \bar{e}_2 - u_n \bar{e}_n)\, dx\, dy$$
$$= \iint_A (PK_x^{(n)} x\, u_1 + PK_y^{(n)} y\, u_2 + Pu_n)\, dx\, dy$$

Since the shell is very shallow, u_1, u_2, u_n are approximated by $u_x^{(\bar{r}_0)}$, $u_y^{(\bar{r}_0)}$, $u_z^{(\bar{r}_0)}$.

$$\therefore\ W = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} \frac{Pa^2bK_x^{(n)}}{(2m+1)\pi} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} \frac{Pab^2K_y^{(n)}}{(2n+1)\pi} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} Pab \qquad (4\text{-}7)$$

therefore,

$$\frac{\partial U^{(s)}}{\partial A_{mn}} = \frac{\partial W}{\partial A_{mn}} \quad \text{yields}$$

$$\frac{3}{2} D' [(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)}] \Big\{ 3A_{mn} + \sum_{\substack{r=0\\ r\neq n}}^{\infty} 2A_{rn} + \sum_{\substack{p=0\\ p\neq m}}^{\infty}\sum_{\substack{q=0\\ q\neq n}}^{\infty} \frac{4}{3} A_{pq} \Big\} +$$
$$+ \frac{4D\pi^4}{a^4} \Big\{ A_{mn} [3(2m+1)^4 + \frac{3a^4}{b^4}(2n+1)^4 + \frac{2a^2}{b^2}(2m+1)^2(2n+1)^2] +$$
$$+ \sum_{\substack{r=0\\ r\neq n}}^{\infty} A_{mr}\, 2(2m+1)^4 + \sum_{\substack{r=0\\ r\neq m}}^{\infty} A_{rn} \frac{2a^4}{b^4}(2n+1)^4 \Big\} +$$

$$+ D'[K_x^{(n)} + \nu K_y^{(n)}](B_{mn}\frac{(2m+1)\pi}{a}\frac{3}{2} + \sum_{\substack{r=0\\r\neq n}}^{\infty} B_{mr}\frac{2(2m+1)\pi}{a}) +$$

$$+ (K_y^{(n)} + \nu K_x^{(n)}]\left[C_{mn}\frac{(2n+1)\pi}{b}\frac{3}{2} + \sum_{\substack{r=0\\r\neq m}}^{\infty} C_{rn}\frac{2(2n+1)\pi}{b})\right] = P$$

$$\frac{\partial U^{(s)}}{\partial B_{mn}} = \frac{\partial W}{\partial B_{mn}} \tag{4-8a}$$

therefore,

$$\frac{D'}{2}\{2(K_x^{(n)} + \nu K_y^{(n)})[A_{mn}\frac{(2m+1)\pi}{a}\frac{3}{2} + \sum_{\substack{r=0\\r\neq n}}^{\infty} A_{mr}\frac{2(2m+1)\pi}{a}]$$

$$+ \frac{\pi^2}{a^2}[18 B_{mn}(2m+1)^2 + \sum_{\substack{r=0\\r\neq n}}^{\infty} 16 B_{mr}(2m+1)^2 + (1-\nu)B_{mn}(2n+1)^2\frac{a^2}{b^2} +$$

$$+ (1+\nu)C_{mn}(2m+1)(2n+1)\frac{a}{b}]\} = \frac{PaK_x^{(n)}}{(2m+1)\pi} \tag{4-8b}$$

$$\partial U^{(s)}/\partial C_{mn} = \partial W/\partial C_{mn}$$

therefore,

$$\frac{D'}{2}\{2(K_y^{(n)} + \nu K_x^{(n)})[A_{mn}\frac{(2n+1)\pi}{b}\frac{3}{2} + \sum_{\substack{r=0\\r\neq n}}^{\infty} A_{mr}\frac{2(2n+1)\pi}{b}] +$$

$$+ \frac{\pi^2}{b^2}[18 C_{mn}(2n+1)^2 + \sum_{\substack{r=0\\r\neq m}}^{\infty} 16 C_{rn}(2n+1)^2 + (1-\nu)C_{mn}(2m+1)^2\frac{b^2}{a^2} +$$

$$+ (1-\nu)B_{mn}(2m+1)(2n+1)\frac{b}{a}]\} = \frac{PbK_y^{(n)}}{(2n+1)\pi} \tag{4-8c}$$

For shells with square base plan, $u_x^{(\bar{r}_0)} = u_y^{(\bar{r}_0)}$, $a = b$, $K_x^{(n)} = K_y^{(n)}$. (4-8b) and (4-8c) become identical. The above expressions (4-8a), (4-8b) and (4-8c) reduce to

$$3D'(K_x^{(n)})^2(1+\nu)\left\{3A_{mn}+\sum_{\substack{r=0\\r\neq n}}^{\infty}2A_{mr}+\sum_{\substack{r=0\\r\neq m}}^{\infty}2A_{rn}+\sum_{\substack{p=0\\p\neq m}}^{\infty}\sum_{\substack{q=0\\q\neq n}}^{\infty}\frac{4}{3}A_{pq}\right\}+$$

$$+\frac{4D\pi^4}{a^4}\left\{A_{mn}\left[3(2m+1)^4+3(2n+1)^4+2(2m+1)^2(2n+1)^2\right]+\right.$$

$$\left.+\sum_{\substack{r=0\\r\neq n}}^{\infty}A_{mr}\,2(2m+1)^4+\sum_{\substack{r=0\\r\neq m}}^{\infty}A_{rn}\,2(2n+1)^4\right\}+$$

$$+D'\left\{(1+\nu)K_x^{(n)}\left[\frac{3}{2}\left(B_{mn}\frac{(2m+1)\pi}{a}+B_{nm}\frac{(2n+1)\pi}{a}\right)+\sum_{\substack{r=0\\r\neq n}}^{\infty}B_{mr}\frac{2(2m+1)\pi}{a}+\right.\right.$$

$$\left.\left.+\sum_{\substack{r=0\\r\neq m}}^{\infty}B_{nr}\frac{2(2n+1)\pi}{a}\right]\right\}=P \qquad \text{(4-9a)}$$

$$\frac{D}{2}\left\{2(1+\nu)K_x^{(n)}\left(A_{mn}\frac{(2m+1)\pi}{a}\frac{3}{2}+\sum_{\substack{r=0\\r\neq n}}^{\infty}A_{mr}\frac{2(2m+1)\pi}{a}\right]+\right.$$

$$+\frac{\pi^2}{a^2}\left[18B_{mn}(2m+1)^2+\sum_{\substack{r=0\\r\neq n}}^{\infty}16B_{mr}(2m+1)^2+(1-\nu)B_{mn}(2n+1)^2+\right.$$

$$\left.\left.+(1+\nu)B_{nm}(2m+1)(2n+1)\right]\right\}=\frac{PaK_x^{(n)}}{(2m+1)\pi} \qquad \text{(4-9b)}$$

(II) Shells with simply supported boundaries

Let

$$\left.\begin{aligned}u_x^{(\bar{r}_0)}&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_{mn}\sin\frac{(2m+1)\pi x}{a}\cos\frac{(2n+1)\pi y}{b}\\u_y^{(\bar{r}_0)}&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{mn}\cos\frac{(2m+1)\pi x}{a}\sin\frac{(2n+1)\pi y}{b}\end{aligned}\right\} \qquad \text{(4-10)}$$

$$\left.\begin{aligned}\frac{\partial u_x^{(\bar{r}_0)}}{\partial x}&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_{mn}\frac{(2m+1)\pi}{a}\cos\frac{(2m+1)\pi x}{a}\cos\frac{(2n+1)\pi y}{b}\\\frac{\partial u_x^{(\bar{r}_0)}}{\partial y}&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_{mn}\sin\frac{(2m+1)\pi x}{a}\frac{(2n+1)\pi}{b}\left(-\sin\frac{(2n+1)\pi y}{b}\right)\\\frac{\partial u_y^{(\bar{r}_0)}}{\partial y}&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{mn}\frac{(2n+1)\pi}{b}\cos\frac{(2m+1)\pi x}{a}\cos\frac{(2n+1)\pi y}{b}\end{aligned}\right\} \qquad \text{(4-11)}$$

$$\frac{\partial u_y^{(\bar{r}_0)}}{\partial x}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{mn}\frac{(2m+1)\pi}{a}\left(-\sin\frac{(2m+1)\pi x}{a}\right)\sin\frac{(2n+1)\pi y}{b}$$

$$u_3^{(\bar{r}_0)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} \cos\frac{(2m+1)\pi x}{a} \cos\frac{(2n+1)\pi y}{b}$$

Expressions (4-10), (4-11) satisfy the following boundary conditions, i.e.

$$
\begin{aligned}
&x=0, && u_x^{(\bar{r}_0)}=0, && \frac{\partial u_x^{(\bar{r}_0)}}{\partial x} \neq 0 \\
&x=\pm\frac{a}{2}, && u_x^{(\bar{r}_0)} \neq 0, && \frac{\partial u_x^{(\bar{r}_0)}}{\partial x}=0, && \frac{\partial u_y^{(\bar{r}_0)}}{\partial y} \neq 0 \qquad (4\text{-}12) \\
&y=0, && u_y^{(\bar{r}_0)}=0, && \frac{\partial u_y^{(\bar{r}_0)}}{\partial y} \neq 0 \\
&y=\pm\frac{b}{2}, && u_y^{(\bar{r}_0)} \neq 0, && \frac{\partial u_y^{(\bar{r}_0)}}{\partial y}=0, && \frac{\partial u_x^{(\bar{r}_0)}}{\partial x} \neq 0
\end{aligned}
$$

Substituting (4-10), (4-11) and $u_3^{(\bar{r}_0)}$ into (4-2), it gives,

$$
\begin{aligned}
U^{(s)} = \frac{D'}{2}\Big\{ & \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}^2 \left(\frac{(2m+1)\pi}{a}\right)^2 \frac{ab}{4} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}^2 \left(\frac{(2n+1)\pi}{b}\right)^2 \frac{ab}{4} + \\
& + 2\nu \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} C_{mn} \frac{(2m+1)(2n+1)\pi^2}{ab} \frac{ab}{4} + \\
& + \frac{1}{2}(1-\nu)\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}^2 \left(\frac{(2m+1)\pi}{a}\right)^2 \frac{ab}{4} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}^2 \left(\frac{(2n+1)\pi}{b}\right)^2 \frac{ab}{4} + \\
& + 2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} B_{mn} (2m+1)(2n+1)\frac{\pi^2}{ab}\frac{ab}{4}\Big] + \\
& + 2\Big[(K_x^{(n)} + \nu K_y^{(n)}) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} A_{mn} \frac{(2m+1)\pi}{a} \frac{ab}{4} + \qquad (4\text{-}13) \\
& + (K_y^{(n)} + \nu K_x^{(n)}) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn} A_{mn} \frac{(2n+1)\pi}{b} \frac{ab}{4}\Big] \\
& + \Big[(K_x^{(n)})^2 + (K_y^{(n)})^2 + 2\nu K_x^{(n)} K_y^{(n)}\Big] \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}^2 \frac{ab}{4}\Big\} + \\
& + \frac{D\pi^4}{8a^4} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}^2 \Big[(2m+1)^4 + (2m+1)^2(2n+1)^2\frac{a^2}{b^2} + (2n+1)^4\frac{a^4}{b^4}\Big]
\end{aligned}
$$

The external work W is

$$W = \int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} \left(P K_x^{(n)} x\, u_x^{(\bar{r}_0)} + P K_y^{(n)} y\, u_y^{(\bar{r}_0)} + P u_3^{(\bar{r}_0)} \right) dx\,dy \qquad (4\text{-}14)$$

$$= (-1)^{m+n}\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn}\frac{4Pa^2bK_x^{(n)}}{(2m+1)^2(2n+1)\pi^3} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\frac{4Pab^2K_y^{(n)}}{(2m+1)(2n+1)^2\pi^3} +$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{4Pab}{(2m+1)(2n+1)\pi^2}\Big]$$

$$\frac{\partial U^{(s)}}{\partial A_{mn}} = \frac{\partial W}{\partial A_{mn}}$$

Therefore,

$$A_{mn} = \frac{(-1)^{m+n}\dfrac{4P}{(2m+1)(2n+1)\pi^2}}{\Big\{\dfrac{D\pi^4}{4a^4}\Big[(2m+1)^4+(2m+1)^2(2n+1)^2\dfrac{a^2}{b^2}+(2n+1)^4\dfrac{a^4}{b^4}\Big]+D'\Big[((K_x^{(n)})^2+(K_y^{(n)})^2+2\nu K_x^{(n)}K_y^{(n)}\Big]/2\Big]}$$

$$-\frac{D'[K_x^{(n)}+\nu K_y^{(n)}]B_{mn}\dfrac{(2m+1)\pi}{4a}+(K_y^{(n)}+\nu K_x^{(n)})C_{mn}\dfrac{(2n+1)\pi}{4b}}{\Big[\dfrac{D\pi^4}{4a^4}\Big[(2m+1)^4+(2m+1)^2(2n+1)^2\dfrac{a^2}{b^2}+(2n+1)^4\dfrac{a^4}{b^4}\Big]+D'\Big[((K_x^{(n)})^2+(K_y^{(n)})^2+2\nu K_x^{(n)}K_y^{(n)}\Big]/2\Big]}$$

or

(4-15a)

$$\frac{\partial U^{(s)}}{\partial B_{mn}} = \frac{\partial W}{\partial B_{mn}}$$

$$\therefore\ \frac{D'}{2}\Big\{B_{mn}\Big[\Big(\frac{(2m+1)\pi}{a}\Big)^2\frac{1}{2}+(1-\nu)\Big(\frac{(2n+1)\pi}{b}\Big)^2\frac{1}{4}\Big]+$$

$$+C_{mn}\Big[\frac{\nu}{2}\frac{(2m+1)(2n+1)\pi^2}{ab}+\frac{(1-\nu)}{4}\frac{(2m+1)(2n+1)\pi^2}{ab}\Big]+$$

$$+2(K_x^{(n)}+\nu K_y^{(n)})A_{mn}\frac{(2m+1)\pi}{4a}\Big\}$$

$$= (-1)^{(m+n)}\frac{4PaK_x^{(n)}}{(2m+1)^2(2n+1)\pi^3} \qquad (4\text{-}15b)$$

$$\frac{\partial U^{(s)}}{\partial C_{mn}} = \frac{\partial W}{\partial C_{mn}}$$

Therefore,

$$\frac{D'}{2}\Big\{ C_{mn}\Big[\Big(\frac{(2n+1)\pi}{b}\Big)^2\frac{1}{2} + (1-\nu)\Big(\frac{(2m+1)\pi}{a}\Big)^2\frac{1}{4}\Big] + $$
$$+ B_{mn}\Big[\frac{\nu}{2}\,\frac{(2m+1)(2n+1)\pi^2}{ab} + \frac{(1-\nu)}{4}\,\frac{(2m+1)(2n+1)\pi^2}{ab}\Big] + $$
$$+ 2(K_y^{(n)} + \nu K_x^{(n)})A_{mn}\frac{(2n+1)\pi}{4b}\Big\}$$
$$= (-1)^{(m+n)}\frac{4PaK_y^{(n)}}{(2m+1)(2n+1)^2\pi^3} \qquad (4\text{-}15c)$$

For shells with square base plan, $u_x^{(\bar{r}_0)} = u_y^{(\bar{r}_0)}$, a = b, $K_x^{(n)} = K_y^{(n)}$, (4-15a), (4-15b) and (4-15c) reduce to,

$$A_{mn} = \frac{\frac{(-1)^{(m+n)}4P}{(2m+1)(2n+1)\pi} - D'(1+\nu)\frac{K_x^{(n)}\pi}{4a}\big[B_{mn}(2m+1) + B_{nm}(2n+1)\big]}{\Big[\frac{D\pi^4}{4a^4}\big[(2m+1)^4 + (2m+1)^2(2n+1)^2 + (2n+1)^4\big] + D'(1+\nu)(K_x^{(n)})^2\Big]} \qquad (4\text{-}16a)$$

$$\frac{D'}{2}\Big\{ B_{mn}\Big[\Big(\frac{(2m+1)\pi}{a}\Big)^2\frac{1}{2} + (1-\nu)\Big(\frac{(2n+1)\pi}{b}\Big)^2\frac{1}{4}\Big]$$
$$+ B_{mn}\frac{(1+\nu)}{2}\,\frac{(2m+1)(2n+1)\pi^2}{a^2}\Big\}$$
$$= (-1)^{(m+n)}\frac{4PaK_x^{(n)}}{(2m+1)^2(2n+1)\pi^3} - (1+\nu)K_x^{(n)}A_{mn}\frac{(2m+1)\pi}{2a}\,\frac{D'}{2} \qquad (4\text{-}16b)$$

BIBLIOGRAPHY

1947 Ambartsumyan, S.A. "K Raschetu Pologikh Obolochek" Prikladnaya Matematika i Mekhanika, Vol. 11.

1953 Courant, R. and Hilbert, D. "Methods of Mathematical Physics" Interscience Publishers, New York, Vol. 1.

Hruban, K. "Biegetheorie der Translationsschalen und ihre Anwendung im Hallenbau" Acta Technica, Tomus VII, Fasciculi 3/4.

Morse, P.M. and Feshbach, H. "Methods of Theoretical Physics" McGraw-Hill, New York, Part I and Part II.

Zerna, W. "Berechnung von Translationsschalen" Österreichishes Ingenieur-Archiv, Vol. 7, Heft 3.

1955 Csonka, P. "Results on Shells of Translation" Acta Technica, Tomus X, Fasciculi 1/2.

Csonka, P. "Special Kind of Shells of Translation with Two Vertical Planes of Symmetry" Acta Technica, Tomus XI, Fasciculi 1/2.

1956 Salvadori, M.G. "Analysis and Testing of Translational Shells" Journal of the American Concrete Institute, Vol. 27, No. 10, June.

1957 Oravas, G. Æ. "Transverse Bending on Thin Shallow Shells of Translation", Österreisches Ingenieur-Archiv.

1958 Mittelmann, G. "Beitrag zur Berechnung von Translationsschalen" Ingenieur-Archiv, Vol. 26.

1959 Flügge, W. and Conrad, D.A. "Singular Solutions in the Theory of Shallow Shells" Technical Report No. 101, Div. of Engineering Mechanics, Stanford University, Sept.

Matildi, P. "Sul Calculo delle Cupole Sottili Ribassate su Pianta Rettangolare" Atti del'Istituto di Scienza dell Costruzioni dell'Università di Pisa, Publicazione No. 66.

1961 Apeland, K. "Stress Analysis of Translational Shells" Journal of Eng. Mech. Div., Feb., Proceedings of A.S.C.E.

Kopal, Z. "Numerical Analysis" 2nd Edition, Chapman and Hall, London.

1962 Langhaar, H.L. "Energy Methods in Applied Mechanics" John Wiley and Sons, New York.

1964 Koshlyakov, N.S., Smirnov, M.M. and Gliner, E.B. "Differential Equations of Mathematical Physics" North-Holland Publishing Co., Amsterdam.