LOCAL AND STRONG LOCAL PARACOMPACTNESS

LOCAL AND STRONG LOCAL PARACOMPACTNESS

By

RICHARD E. SOT, B.Sc.

A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

May 1972

MASTER OF SCIENCE (Mathematics)

McMASTER UNIVERSITY Hamilton, Ontario

TITLE: Local and Strong Local Paracompactness

AUTHOR: Richard Edward Sot, B.Sc. (The University of Toledo)

SUPERVISOR: Dr. R. G. Lintz

NUMBER OF PAGES: iv, 32

SCOPE AND CONTENTS: This thesis gives basic properties of the newly defined topological properties local paracompactness and strong local paracompactness. An example is given to show that they do not coincide in T_2 spaces; another example is given of a strongly locally paracompact T_2 space which is neither locally compact nor paracompact. The existence of a one point paracompactification analagous to the Aleksandrov one point compactification is constructed and proved for strongly locally paracompact T_2 spaces. Also considered are conditions under which these two properties are preserved under closed maps and heredity.

ACKNOWLEDGEMENTS

The author wishes to thank Dr. R. G. Lintz for his assistance in the preparation of this thesis, especially with respect to corrections.

I would also like to thank my sister Diane for her excellent typing of this thesis.

TABLE OF CONTENTS

CHAPTER 0

The following definitions will suffice to minimize ambiguity whenever we refer to the separation axioms.

Definitions: Let X be a topological space. X is said to be

1. T_1 if each singleton subset of X is closed.
2. T_2 if each pair of distinct points in X have disjoint open nbds.
3. regular (abbreviated, r.) if for each point x in X and open nbd U of x, there is an open nbd V of x such that $x \in V \subset \overline{V} \subset U$.
4. T_3 if it is both r. and T_1.
5. completely regular (abbreviated, c.r.) if for each point x in X and closed set E in X not containing x there is a continuous function $f: X \to [0,1]$ such that f is 0 on x and 1 on E.
6. $T_{3\frac{1}{2}}$ if it is both c.r. and T_1.
7. normal (abbreviated, n.) if for each closed set E in X and open nbd U of E, there is an open nbd V of E such that $E \subset V \subset \overline{V} \subset U$.
8. T_4 if it is both n. and T_1.

The concept of a paracompact topological space emerged from that of a locally finite covering which was introduced by P. S. Aleksandrov in 1924 in [1]. However, it was not until twenty years later that this class of spaces was introduced by Dieudonné in [7] . Notable observations made at that time included the following: every paracompact T_2 space is n.; every separable metric space is paracompact; the product of a paracompact space with a compact space is paracompact. At this time it was unknown whether metric spaces are paracompact or the product of two paracompact spaces is paracompact.

Definitions:

9. A collection of subsets of a topological space is said to be locally finite (abbreviated, lf.) if each point in the space has an open nbd which intersects at most finitely many members of the collection.
10. A topological space is said to be paracompact (abbreviated, p.) if each open covering of the space has an open lf. refinement.

In the meantime, J. W. Tukey in [29] had introduced a new class of spaces contained in the class of n. spaces which he called fully normal. An important theorem proved in this paper is that metric spaces are fully normal. This theorem by Tukey enabled A. H. Stone in 1948 in [27] to solve the problem of whether metric spaces are p. by proving that in this class of T_2 spaces the concepts of full normality and p. coincide. Newer and shorter proofs that metric spaces are p. were given in 1969 by Mary Rudin and D. Ornstein in [22] and [21] .

Definitions:

11. A refinement $\underline{A}$ of a collection $\underline{B}$ in a topological space X is said to be a delta refinement if $\underline{A}^{\Delta}$ refines $\underline{B}$, where $\underline{A}^{\Delta} = \{St(x,\underline{A}): x \in X\}$ and $St(x,\underline{A}) = \cup\{ A: x \in A \in \underline{A}\}$. Whenever a collection $\underline{A}$ refines a collection $\underline{B}$, we write $\underline{A} < \underline{B}$, and likewise $\underline{A}^{\Delta} < \underline{B}$ whenever a collection $\underline{A}$ is a delta refinement of a collection $\underline{B}$.

12. A topological space is said to be fully normal (abbreviated,f.n.) if each open covering of the space has an open delta refinement.

Theorem 0.1 (A.H. Stone). A T_2 space is p. iff it is f.n..

Corollary 0.1 (Dieudonné). Every p. T_2 space is n..

Proof:

Immediate from Theorem 0.1 and Corollary 1.1 which will be given in Chapter 1. ||

Theorem 0.2 (J.W. Tukey). Every metric space is f.n..

Proof:

Given on p. 53 in [29]. ||

Corollary 0.2 (A.H. Stone). Every metric space is p..

Proof:

Immediate from Theorem 0.1 and Theorem 0.2. ||

At the same time K. Morita introduced the concept of a topological space with the star-finite property, or strongly paracompact spaces as they are now called, in [18]. Two of the most important theorems in this paper are that every Lindelöf T_3 space is strongly paracompact and a connected T_3 space is strongly paracompact iff it is Lindelöf. Although metric spaces are p., they need not be strongly paracompact, e.g. a star space with uncountable index as defined in Example III.7, p. 94 of [20]. However, both P.S. Aleksandrov and S. Kaplan proved that every separable metric space is strongly paracompact one year earlier in [2] and [12], respectively.

Definitions:

13. A collection of subsets of a topological space is said to be <u>star-finite</u> (abbreviated, sf.) if each member intersects at most finitely many members of the collection.
14. A topological space is said to be <u>strongly paracompact</u> (abbreviated, sp.) if each open covering of the space has an open sf. refinement.

Theorem 0.3 (P.S. Aleksandrov and S. Kaplan). Every separable metric space is sp..

Theorem 0.4 (K. Morita). Every Lindelöf T_3 space is sp..

Theorem 0.5 (K. Morita). A connected T_3 space is sp. iff it is Lindelöf.

Although it is obvious that p. spaces are closed hereditary, E. Michael was able to prove in 1953 in [16] that every F_σ subset of a p. space is p.. Moreover, two other important theorems in this paper show that in r. spaces the refinement need not be open for p. to hold and that the union of a lf. collection of closed p. subsets of a T_1 space is p.. It was discovered, however, by V. Trnkova in 1962 in [28] that the latter result does not hold for sp. subsets, although the union of two closed sp. subsets of a topological space is sp. if their intersection is locally Lindelöf. Y. Yasui generalized this result of V. Trnkova in 1967 in [32] in a T_3 space to an arbitrary closed collection (by a closed collection we mean a collection of subsets all of whose members are closed sets) such that the frontier of each member is locally Lindelöf. This generalization utilizes a theorem of equivalent conditions for sp. given by Yu. M. Smirnov in 1956 ([25], p. 256). The question had already arisen as to which kinds of maps preserve p.. In 1956 in [19] K. Morita had been able to prove that p. perfect normality is preserved by closed continuous onto maps, and that the image under a closed continuous mapping of a p. and locally compact T_2 space X is p. T_2. The following year in [17] E. Michael published the more general theorem that a closed continuous image of a p. T_2 space is p. T_2, a result analagous to the well-known theorem of G. T. Whyburn for n. T_0 spaces (Theorem 9 in [31]). Then in 1958 Henriksen and Isbell in [11] made the observation that for a perfect map $f: X \to Y$ where X is c.r. and Y any topological space, X is p. iff f(X) is p.. S. Hanai obtained a stronger result in 1961 in [10] that a space which has a p. image under a perfect map must be p..

Definitions:

15. A <u>star-countable</u> collection is defined analagously to a sf. collection.

16. A collection of subsets of a topological space is said to be σ-locally finite (abbreviated, σ-lf.) if it can be decomposed into an at most countable number of lf. subcollections.

Theorem 0.6 (E. Michael). Every F_σ subset of a p. space is p..

Theorem 0.7 (E. Michael). Let X be a r. space. Then the following conditions are equivalent.

(i). X is p..

(ii). Every open covering of X has a σ-lf. open refinement.

(iii). Every open covering of X has a lf. refinement.

(iv). Every open covering of X has a closed lf. refinement.

Theorem 0.8 (E. Michael). A T_1 space is p. T_2 whenever it has a lf. closed covering by subsets which are p. T_2.

Theorem 0.9 (Y. Yasui). Let $\underline{F} = \{F_a\colon a \in A\}$ be a lf. closed covering of a T_3 space X such that $Fr(F_a)$ is locally Lindelöf for each $a \in A$. Then X is sp. iff F_a is sp. for each $a \in A$.

Theorem 0.10 (Yu. M. Smirnov). Let X be a r. space. Then the following conditions are equivalent.

(i). X is sp..

(ii). Every open covering of X has a sf. closed refinement.

(iii). Every open covering of X has a star-countable open refinement.

(iv). Every open covering of X has a star-countable closed refinement.

Theorem 0.11 (E. Michael). A closed continuous image of a p. T_2 space is p. T_2.

Theorem 0.12 (M. Henriksen & J. R. Isbell). Let $f: X \to Y$ be a perfect map from a c.r. space X to a topological space Y. Then X is p. iff f(X) is p..

Theorem 0.13 (S. Hanai). Let $f: X \to Y$ be a perfect map from a topological space X onto a p. space Y. Then X is also a p. space.

CHAPTER 1

Definition 1: A collection of subsets of a topological space is said to be point-finite if each point in the space is contained in at most finitely many members of the collection.

Theorem 1.1. Let X be a T_1 space. Then the following are equivalent.

(i). X is n..

(ii). For each point-finite open covering $\underline{U} = \{U_a : a \in A\}$ of X there is an open refinement $\underline{V} = \{V_a : a \in A\}$ such that $\overline{V}_a \subset U_a$ for each $a \in A$ and $V_a \neq \emptyset$ whenever $U_a \neq \emptyset$.

(iii). Every finite open covering of X has an open delta refinement.

Proof:

(i). ⇔ (ii).: Proposition C, p. 82, in [20] .

(i). ⇔ (iii).: Exercise 7, p. 103 in [20] . ||

Corollary 1.1. Every f.n. space is n..

Proof:

Immediate from Theorem 1.1 and the definition of f.n.. ||

Notation: locally compact is abbreviated as l.c..

Theorem 1.2. For each closed compact subset E of a r., l.c. space X, there is a closed compact nbd F of E such that $E \subset F$.

Proof:

Theorem 18, p. 146 in [13]. ||

The following five propositions are elementary and the proofs are trivial. The reader is referred to [8] for details.

Proposition 1.1. Let X be a topological space. Then the following conditions are equivalent.

(i). X is r..

(ii). Each $x \in X$ and closed set E not containing x have disjoint open nbds.

(iii). For each $x \in X$ and closed set E not containing x, there is an open nbd U of x with $\bar{U} \cap E = \emptyset$.

Proposition 1.2. Let X be a topological space. Then the following conditions are equivalent.

(i). X is n..

(ii). Each pair of disjoint closed subsets of X have disjoint open nbds.

(iii). Each pair of disjoint closed subsets of X have disjoint open nbds whose closures do not intersect.

Proposition 1.3. Let $f: X \to Y$ be a closed map (not necessarily continuous) from a topological space X to a topological space Y. Given any subset B of Y and any open subset U of X containing $f^{-1}(B)$, there exists an open set V in Y containing B such that $f^{-1}(V) \subset U$.

Proposition 1.4. A map $f: X \to Y$ (not necessarily continuous) from a topological space X to a topological space Y is closed continuous iff $f(\overline{A}) = \overline{f(A)}$ for each subset A of X.

Proposition 1.5. A map $f: X \to Y$ from a topological space X to a topological space Y is continuous iff $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for each subset B of Y.

The following definitions were introduced by Ky Fan and N. Gottesman in [9] for the purpose of obtaining generalizations of compactifications of Freudenthal and Wallman.

Definitions:

2. A base $\underline{B}$ for a topological space X is said to be <u>normal</u> if the following three conditions are satisfied.

   (B1). $U, V \in \underline{B}$ implies $U \cap V \in \underline{B}$.

   (B2). $U \in \underline{B}$ implies $X \setminus \overline{U} \in \underline{B}$.

   (B3). For any open set W of X and any $U \in \underline{B}$ such that $\overline{U} \subset W$, there exists a $V \in \underline{B}$ such that $\overline{U} \subset V \subset \overline{V} \subset W$.

3. Let X be a topological space with normal base $\underline{B}$. A <u>binding family</u> on $\underline{B}$ is a nonempty family of sets of $\underline{B}$ such that $\overline{U}_1 \cap \overline{U}_2 \cap \ldots \cap \overline{U}_n \neq \emptyset$ for any finite number of sets U_i of the family. By Zorn's lemma, every binding family on $\underline{B}$ is contained in at least one <u>maximal binding family</u> on $\underline{B}$.

4. Maximal binding families on a normal base $\underline{B}$ of a topological space X will be denoted by letters x*, y*, ... We denote by X* the set of all maximal binding families on $\underline{B}$. For each $U \in \underline{B}$, we define

$h(U) = \{ x^* \in X^*\colon$ there exists a $V \in x^*$ with $\overline{V} \in U \}$.

$\underline{B}^* = \{ h(U)\colon U \in \underline{B} \}$ can be taken as a base to topologize $\underline{X}^*$ (the proof is given in [9]).

The following theorem is the main result in this paper by Ky Fan and N. Gottesman. The more recent results in [30] by F. J. Wagener are analagous to those in [9] , but the approach is with $\underline{B}$-filters rather than with just normal bases $\underline{B}$.

Theorem 1.3 (Ky Fan and N. Gottesman). Let X be a T_3 space with a normal base $\underline{B}$. Then X is homeomorphic to a dense subset of a compact T_2 space X*, where the points of X* are the maximal binding families on the base $\underline{B}$, and $\underline{B}^*$ is the base for the topology of X*.

Definition 5: Let N be a positive integer. A compactification $z(X)$ of a topological space X is called an N-point compactification if $z(X) \setminus X$ consists of exactly N points.

N-point compactifications were introduced by K. D. Magill, Jr. in [15] and studied exhaustively. The following theorem is the first in that paper.

Theorem 1.4 (K.D. Magill, Jr.). A topological space X has an N-point compactification iff X is l.c. and contains a compact subset K whose complement is the union of N mutually disjoint, open subsets G_i, $i = 1, \ldots, N$ such that $K \cup G_i$ is not compact for each i.

Definition 6: Let X and Y be topological spaces, $A \subset X$, and $f\colon X \to Y$ be a continuous map. We say that A is P-embedded in X if every continuous pseudometric on A can be extended to a continuous pseudometric on X. We say that f is paraperfect if f is closed continuous, and $f^{-1}(y)$ is p. and P-embedded in X for each $y \in Y$.

P-embeddings were introduced by H. L. Shapiro in [23] , and paraperfect maps were introduced by H. L. Shapiro in [24] . These maps relate to p. as perfect maps relate to compactness. Counter - examples are given in [24] to prove that all four conditions are essential in the definition of paraperfect maps in order for the following two theorems to hold.

Theorem 1.5 (H.L. Shapiro). Let X be r. and $f: X \to Y$ be a paraperfect map from a topological space X into a topological space Y. Then X is p. iff $f(X)$ is p..

Proof:

Given in [24] . ||

Theorem 1.6 (H.L. Shapiro). Let $f: X \to Y$ be a closed continuous map from a p. space X to a T_1-space Y. Then f is paraperfect.

Proof:

Given in [24] . ||

CHAPTER 2

Definitions:

1. A topological space is said to be <u>locally paracompact</u> (abbreviated, l.p.) if each point in the space has an open nbd whose closure is p..

2. A topological space is said to be <u>strongly locally paracompact</u> (abbreviated, s.l.p.) if each closed p. subset in the space has an open nbd whose closure is p..

3. A topological space is said to be <u>c-locally Lindelöf</u> if each point in the space has an open nbd whose closure is Lindelöf.

Before giving examples concerning l.p. and s.l.p. spaces, some propositions concerning the basic properties of these spaces shall be considered.

Proposition 2.1. Every l.p. T_2 space is r..

<u>Proof</u>:

Let X be a l.p. T_2 space, $x \in X$, and F be a closed set not containing x. Since X is l.p., x has a closed p. nbd, say E. Suppose $E \cap F \neq \emptyset$. E, being p. T_2, is r.. Since x is not contained in the closed set $E \cap F$ in E, $E \cap F$ and x must have disjoint nbds open in E, say V^* and U^* respectively. Since E is a nbd of x in X, there is an open set W in X containing x, and contained in E. $U^* = U \cap E$ for some open subset U of X. Thus $U \cap W$ is an open set in X containing x which is disjoint from V^*. $V^* = V \cap E$ for some open subset V of X. Hence $U \cap W$, $V \cup (X \setminus E)$ are the required disjoint open nbds of x, F, respectively. If $E \cap F = \emptyset$, then W, $X \setminus E$ are the required disjoint open nbds of x, F, respectively. Hence X is r.. ||

Proposition 2.2.

(i). l.p. T_2 spaces are both open and closed hereditary.

(ii). s.l.p. T_2 spaces are closed hereditary.

Proof:

(i). Let X be a l.p. T_2 space, F be a closed subspace of X, and $x \in F$ be arbitrary. Since X is l.p., x has an open nbd U in X with p. closure. Since $\overline{U} \cap F$ is a closed subspace of the p. space $\overline{U}$, it is p.. Hence $U \cap F$ is an open nbd of x in F with p. closure in F. Therefore F is l.p.. Hence X is closed hereditary with respect to l.p.. Let V be an open subspace of X, and $x \in V$ be arbitrary. By Proposition 2.1 we know that X is r.. Therefore there is an open subspace U of X such that $x \in U \subset \overline{U} \subset V$. Since X is l.p., x has an open nbd W in X with p. closure. Let $W^* = W \cap U$. Then $\overline{W^*} = \overline{W \cap U} \subset \overline{W} \cap \overline{U} \subset \overline{U} \subset V$. $\overline{W} \cap \overline{U}$ is p., being a closed subspace of the p. space $\overline{W}$. $\overline{W^*}$ is p., being a closed subspace of the p. space $\overline{W} \cap \overline{U}$. Since $x \in W^*$, and $Cl_X(W^*) = Cl_V(W^*)$ we conclude that V is l.p.. Hence X is open hereditary with respect to l.p..

(ii). Let X be a s.l.p. T_2 space, F be a closed subspace of X, and E be an arbitrary closed p. subspace of F. Clearly E is also a closed p. subspace of X. Since X is s.l.p., E has an open nbd U in X with p. closure. Since $E \subset U \cap F \subset \overline{U} \cap F$, and $\overline{U} \cap F$ is p., being a closed subspace of the p. space $\overline{U}$, we conclude that $U \cap F$ is an open nbd of E in F with p. closure in F. Therefore F is s.l.p.. Hence X is closed hereditary with respect to s.l.p.. ||

Proposition 2.3.

(i). The topological sum of an arbitrary collection of l.p. spaces is l.p..

(ii). The topological sum of an arbitrary collection of s.l.p. T_2 spaces is s.l.p..

Proof:

(i). Let $\{X_a : a \in \underline{A}\}$ be an arbitrary collection of l.p. spaces, $X = \bigcup_{a \in \underline{A}} X_a$ be the topological sum of this collection, and $x \in X$ be arbitrary. Then $x \in X_a$ for some $a \in A$. Since X_a is l.p., there is an open subset U_a of X_a containing x whose closure in X_a is p.. Note that $Cl_X(U_a) = Cl_{X_a}(U_a)$. Hence X is l.p..

(ii). Let $\{X_a : a \in \underline{A}\}$ be an arbitrary collection of s.l.p. spaces, $X = \bigcup_{a \in \underline{A}} X_a$ be the topological sum of this collection, and E be a closed p. subset of X. Then $E_a = E \cap X_a$ is closed p. for each $a \in \underline{A}$, being a closed subspace of the

p. space E. Since each X_a is s.l.p., each E_a has an open nbd U_a in X_a with p. closure. Clearly $U = \bigcup_{a \in \underline{A}} U_a$ is an open nbd of E, and $\{U_a : a \in \underline{A}\}$ is a lf. collection. Since $\{U_a : a \in \underline{A}\}$ is lf., it is closure-preserving, i.e. $\overline{U} = \bigcup \{ \overline{U}_a \mid a \in \underline{A}\}$. Hence it follows from Theorem 0.8 that $\overline{U}$ is p.. Therefore X is s.l.p.. ||

Proposition 2.4. Every c-locally Lindelöf T_3 space is l.p..

Proof:

Let X be a c-locally Lindelöf T_3 space and $x \in X$ be arbitrary. There is a closed Lindelöf and T_3 nbd, say E, of x since X is c-locally Lindelöf T_3. By Theorem 0.4 we conclude that E is p.. Hence X is l.p.. ||

Although it was proved in Proposition 2.2 that s.l.p. T_2 spaces are closed hereditary, it is unknown whether they are also open hereditary. We shall now give an example of a s.l.p. T_2 space which is neither l.c. nor p., and an example of a l.p. T_2 space which is not s.l.p. (in fact even l.sp.). Note that every s.l.p. T_2 space which is not open hereditary has an open subspace which is l.p. but not s.l.p..

Definition 4: A topological space is said to be <u>locally strongly paracompact</u> (abbreviated, l.sp.) if each point in the space has an open nbd whose closure is sp..

Example 2.1. We shall give an example of a topological space which is s.l.p. T_2 but neither p. nor l.c.. Let $A =]-\infty, 0[\cap Q$ where Q denotes the set of rational numbers, and $B = [0, \Omega[$. We shall define a topology on A by taking the induced topology, and a topology on B by taking as a basis, subsets of the form $[0,\alpha]$ and $]\alpha,\beta]$ as given in [8], p. 66. Let X be the topological sum of A and B. Obviously X is T_2. From Proposition 2.3-(ii) we conclude that X is s.l.p. since A is hereditarily p., and B is l.c. and therefore also s.l.p. (the latter result an immediate consequence of Theorem 1.2). X is not l.c. since no point of A has an open nbd in X whose closure is compact in X. X is not p. since the closed subspace $[0,\Omega[$ is not p. (Example 3, p. 162 in [8]).

Example 2.2. We shall give an example of a topological space which is l.sp. T_2 but not s.l.p.. Let $X = [0,\Omega] \times [0,\omega] \setminus (\Omega, \omega)$ have the topology generated by declaring open each point of $A = [0,\Omega[\times [0,\omega[$, together with the sets $U_{\alpha,\beta} = \{(\beta,\nu) : \alpha < \nu \leq \omega\}$ and $V_{\alpha,\beta} = \{(\nu,\beta) : \alpha < \nu \leq \Omega\}$. Clearly this space X is T_2 . X is not n. (see Example 89, pgh 2 in [26]) and therefore not p. by Corollary 0.1.

X is l.sp.: Let $(\alpha, \beta) \in X$ be arbitrary. If $(\alpha, \beta) \in A$, there is nothing to prove. Therefore we may assume that $(\alpha, \beta) \notin A$. Then either $\alpha = \Omega$ or $\beta = \omega$. Suppose $\alpha = \Omega$. Let ν be any ordinal strictly less than Ω. Then $V_{\nu, \beta}$ is the required closed sp. nbd of $(\alpha, \beta) = (\Omega, \beta)$. The result is analagous for $\beta = \omega$.

X is not s.l.p.: Consider the closed subspace $E = X \setminus A$ of X. E is discrete and therefore also p.. Since no nonempty subspace of E is open in X, every open nbd U of E in X must contain for each (Ω, n) (respectively, (n, ω)) in E, a basic open set $V_{\alpha, n}$ (respectively, $U_{\alpha, n}$) where α is strictly less than Ω (respectively, ω). Let $V = (X \setminus \{(\Omega, n): 0 \leq n < \omega\}) \cap \bar{U}$ and $\underline{V} = \{V\} \cup \{V_{\alpha, n} \cap \bar{U}: 0 \leq n < \omega\}$. Then $\underline{V}$ is an open covering of $\bar{U}$ having no lf. refinement: For if $\underline{W}$ refines $\underline{V}$, we may define for each integer n an ordinal α_n to be the least ordinal such that $V_{\alpha_n, n}$ is contained in exactly one member of $\underline{W}$.
Let $\alpha = \sup \alpha_n$.
(Note that α is strictly less than Ω.)
Then every open nbd of (α, ω) intersects infinitely many members of $\underline{W}$. Therefore E is a closed p. subspace of X having no closed p. nbd.
Hence X cannot be s.l.p..

CHAPTER 3

The topological space given in Example 2.1 does not have an Aleksandrov one point compactification since it is not l.c.. However, we shall observe that it has a one point paracompactification which is constructed parallel to the Aleksandrov one point compactification since it is s.l.p. T_2. This interesting property holds for all s.l.p. T_2 spaces as given in Theorem 3.1 below.

Theorem 3.1. Let X be a s.l.p. T_2 space. Then there exists a topological space $\tilde{X}$ such that

(i). $\tilde{X}$ is p. T_2.

(ii). $Cl_{\tilde{X}}(X) = \tilde{X}$.

(iii). X is a subspace of $\tilde{X}$, and $\tilde{X} \setminus X = \emptyset$ in case X is already p., otherwise $\tilde{X} \setminus X$ is a singleton set.

Proof:

If X is already p., we define $\tilde{X} = X$. We can now assume that X is s.l.p. but not p.. Let $\infty = \{X\}$ and $\tilde{X} = X \cup \{\infty\}$. Define a subset τ of the power set of X as follows: $U \in \tau$ if U is open in X; $V \in \tau$ if $V = \tilde{X} \setminus F$ where F is a closed p. subset of X. τ defines a topology on $\tilde{X}$: Clearly $\emptyset$, $\tilde{X}$ belong to τ. Let $\{ V_a \mid a \in \underline{A}\}$ be a collection of members of τ such that each V_a contains ∞. For each a we have $V_a = (X \setminus F_a) \cup \{\infty\}$ where F_a is a closed p. subset of X. Let $V = \bigcup_{a \in \underline{A}} V_a$. Then $V = (X \setminus \bigcap_{a \in \underline{A}} F_a) \cup \{\infty\}$. $V \in \tau$ since $\bigcap_{a \in \underline{A}} F_a$ is a closed p. subset of X, being a closed subspace of each of the p. spaces F_a. Similarly, let $\{V_1, \ldots, V_n\}$ be a finite collection of members of τ such that each V_i contains ∞. Let $W = \bigcap_{i=1}^{n} V_i$. Then $W = (X \setminus \bigcup_{i=1}^{n} F_i) \cup \{\infty\}$. X, being s.l.p. T_2 and hence also l.p. T_2, is r. by Proposition 2.1. Therefore $\bigcup_{i=1}^{n} F_i$ is a closed p. subset of X by Theorem 0.8. It is now clear that $W \in \tau$. Let U be a member of τ not containing ∞ and V be a member of τ containing ∞. Then $V = (X \setminus F) \cup \{\infty\}$ for some closed p. subset F of X. Then $U \cap V = U \cap (X \setminus F)$, being open in X, is a member of τ. Also, $U \cup V = U \cup (X \setminus F) \cup \{\infty\} = (X \setminus [(X \setminus U) \cap F]) \cup \{\infty\}$. $(X \setminus U) \cap F$, being a closed subspace of the p. space F, is a closed p. subspace of X. Hence $U \cup V \in \tau$. This completes the proof that τ is a topology on $\tilde{X}$.

We shall proceed to show that $Cl_{\tilde{X}}(X) = \tilde{X}$, or equivalently that $\{\infty\}$ is not a member of τ. Suppose that $\{\infty\} \in \tau$. Then $\{\infty\} = \tilde{X} \setminus F$ for some closed p. subset F of X. Hence $F = X$, contradicting the hypothesis that X is not p.. Thus $\{\infty\} \notin \tau$. Hence $Cl_{\tilde{X}}(X) = \tilde{X}$.

In order to show that $\tilde{X}$ is p., we shall first show that $\tilde{X}$ is r.. Let $x \in \tilde{X}$ and E be a closed set in $\tilde{X}$ not containing x. If $x = \infty$, then $E \subset X$. E must be p. in X by definition of $\tilde{X}$. Therefore, since X is s.l.p., there is an open set U in X and a closed p. subset F of X such that $E \subset U \subset F$. Hence $\tilde{X} \setminus F$ and U are the required disjoint open nbds in $\tilde{X}$ of x and E, respectively. If $x \neq \infty$, then either E is closed p. in X and $\infty \notin E$, or $E \setminus \{\infty\}$ is closed in X and $\infty \in E$. In the first case there is nothing to prove since X is r.. In the second case, since X is s.l.p., there is an open set U in X and a closed p. subset F of X such that $x \in U \subset F$. x has an open nbd V in X such that $Cl_X(V) \cap (E \setminus \{\infty\}) = \emptyset$ since X is r.. Therefore $Cl_X(U \cap V) \subset Cl_X(F \cap V) \subset Cl_X(F) \cap Cl_X(V)$ so that $Cl_X(F \cap V) \cap (E \setminus \{\infty\}) = \emptyset$. $Cl_X(F \cap V)$ is closed p. in X, being a closed subspace of the p. space F. Therefore $U \cap V$ and $\tilde{X} \setminus Cl_X(F \cap V)$ are the required disjoint open nbds in $\tilde{X}$ of x and E, respectively. Hence $\tilde{X}$ is r. as claimed.

It remains to show that $\tilde{X}$ is p.. Let $\underline{U}$ be an open covering of $\tilde{X}$. We may assume that exactly one member of $\underline{U}$, say U_0, contains ∞, for if others do we can remove the ∞'s from them. $\tilde{X} \setminus U_0$ is a closed p. subset of X. Then $\underline{U} \cap (\tilde{X} \setminus U_0)$, being an open covering of $\tilde{X} \setminus U_0$, has a refinement, say $\underline{N}$, lf. in $\tilde{X} \setminus U_0$ and hence also in X by the p. of $\tilde{X} \setminus U_0$, where $\underline{N}$ is lf. in $\tilde{X}$ since the open nbd U_0 of ∞ in $\tilde{X}$ intersects no members of $\underline{N}$. Then $\underline{M} = \underline{N} \cup \{U_0\}$ is a lf. refinement of $\underline{U}$ in $\tilde{X}$. Therefore by Theorem 0.7 we conclude that $\tilde{X}$ is p.. ||

Corollary 3.1. Every s.l.p. T_2 space is c.r..

Proof:

Obvious. ||

CHAPTER 4

Analagous to the definitions of a normal base for a topological space as defined by Ky Fan and N. Gottesman in [9], we have the definition of a weakly normal base which is weaker than that of a normal base.

Definition 1: A base $\underline{B}$ for a topological space X is said to be weakly normal if in addition to (B1) and (B2) given in Definition 2 of Chapter 1, the following condition is satisfied.

(B4). For any closed nbd E in X and any $U \in \underline{B}$ such that $\overline{U} \subset \text{Int}(E)$, there exists a $V \in \underline{B}$ such that $\overline{U} \subset V \subset \overline{V} \subset E$.

Topological spaces which are l.p. and s.l.p. are characterized for the most part by their p-bases which are introduced below.

Definition 2: Let X be a l.p. topological space. By the p-base for X we mean the basis for the topology of X consisting precisely of those open subsets of X whose closures are p. or whose complements are p..

The following two propositions give important properties of the p-bases of these spaces.

Proposition 4.1. The p-base of a s.l.p. T_1 space is weakly normal.

Proof:

Let X be a s.l.p. T_1 space and $\underline{B}$ be its p-base. Suppose U, V $\in \underline{B}$ in order to demonstrate (B1). To show that $U \cap V \in \underline{B}$ we must consider three cases: 1.) $\overline{U}$ and $\overline{V}$ are p.; 2.) $X \setminus U$ and $X \setminus V$ are p.; 3.) $X \setminus U$ and $\overline{V}$ are p..

Case 1. $\overline{U \cap V}$, being a closed subspace of the p. space $\overline{V}$, is p.. Hence $U \cap V \in \underline{B}$.

Case 2. $U \cap V \in \underline{B}$ since $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is p..

Case 3. Proof is identical to that for Case 1.

Thus (B1) has been proved. To demonstrate (B2), let $U \in \underline{B}$. We must show that $X \setminus \overline{U} \in \underline{B}$. Since the case where $\overline{U}$ is p. is trivial, we may assume $X \setminus U$ is p.. It suffices to show that $\overline{X \setminus \overline{U}} = \overline{\mathrm{Int}(X \setminus U)}$ is p.. $\overline{\mathrm{Int}(X \setminus U)}$ is p. since it is a closed subspace of the p. space $X \setminus U$. Thus (B2) has been proved. To demonstrate (B4), let E be a closed nbd in X, $G = \mathrm{Int}(E)$, and U be a member of $\underline{B}$ such that $\overline{U} \subset G$. We need to consider two cases: 1.) $\overline{U}$ is p.; 2.) $X \setminus U$ is p.. Case 1: Since $\overline{U}$ is p. and X is s.l.p., there is an open set V in X containing $\overline{U}$ with p. closure. $\overline{V \cap G}$, being a closed subspace of the p. space $\overline{V}$, is p.. Hence $V \cap G \in \underline{B}$. Moreover, $\overline{U} \subset V \cap G \subset \overline{V \cap G} \subset E$. Case 2: $X \setminus G$ is p. since it is a closed subspace of the p. space $X \setminus U$. Hence $G \in \underline{B}$. Moreover, $\overline{U} \subset G \subset \overline{G} \subset E$. Thus (B4) has been proved. Since (B1), (B2), and (B4) are satisfied, $\underline{B}$ is weakly normal as required. ||

Proposition 4.2. The p-base of a s.l.p. T_2 space is normal.

Proof:

By Proposition 4.1 we already know that (B1) and (B2) hold. To demonstrate (B3), let W be open in a s.l.p. T_2 space X with p-base $\underline{B}$ and $U \in \underline{B}$ such that $\overline{U} \subset W$. We need to consider two cases: 1.) $\overline{U}$ is p.; 2.) $X \setminus U$ is p.. Case 1: Since $\overline{U}$ is p. and X is s.l.p., there is an open subset V of X containing $\overline{U}$ with p. closure. $\overline{V}$, being p. T_2, is also n. by Corollary 0.1. Since $\overline{V} \cap W$ is an open nbd of $\overline{U}$ in $\overline{V}$, there is a set G open in $\overline{V}$ such that $\overline{U} \subset G \subset Cl_{\overline{V}}(G) \subset (\overline{V} \cap W)$. Since G is open in $\overline{V}$, there is an open set G^* in X such that $G = \overline{V} \cap G^*$. Let $H = V \cap G^*$. Note that $Cl_{\overline{V}}(G) = Cl_X(G)$ since $Cl_{\overline{V}}(G)$ is closed in the closed subspace $\overline{V}$ of X and therefore also in the whole space X. Therefore $\overline{H} = \overline{V \cap G^*} \subset \overline{\overline{V} \cap G^*} = \overline{G} \subset \overline{V} \cap W \subset W$. Since $\overline{H}$ is a closed subspace of the p. space $\overline{V}$, it is also p.. Hence $H \in \underline{B}$ and $\overline{U} \subset H \subset \overline{H} \subset W$. Case 2: $X \setminus W$, being a closed subspace of the p. space $X \setminus U$, is p.. Since X is s.l.p., there is an open set V in X containing $X \setminus W$ with p. closure. $\overline{V}$, being p. T_2, is also n.. Since $\overline{V} \cap (X \setminus \overline{U})$ is an open nbd of $X \setminus W$ in $\overline{V}$, there is a set G open in $\overline{V}$ such that $X \setminus W \subset G \subset Cl_{\overline{V}}(G) = \overline{G} \subset \overline{V} \cap (X \setminus \overline{U})$. Since G is open in $\overline{V}$, there is an open set G^* in X such that $G = \overline{V} \cap G^*$. Let $H = V \cap G^*$. Then $\overline{H} = \overline{V \cap G^*} \subset \overline{\overline{V} \cap G^*} = \overline{G} \subset \overline{V} \cap (X \setminus \overline{U}) \subset X \setminus \overline{U}$. Therefore $X \setminus W \subset H \subset \overline{H} \subset X \setminus \overline{U}$. Hence $\overline{U} \subset X \setminus \overline{H} \subset X \setminus H \subset W$.

$\overline{X \setminus \bar{H}} = \overline{\mathrm{Int}(X \setminus H)} \subset X \setminus H \subset W$. $\bar{H}$, being a closed subspace of the p. space $\bar{V}$, is p.. Hence $X \setminus \bar{H} \in \underline{B}$ since (B2) holds. Moreover, $\bar{U} \subset X \setminus \bar{H} \subset \overline{X \setminus \bar{H}} \subset W$. Thus (B4) has been proved. Hence $\underline{B}$ is normal as required. ||

Corollary 4.1. Let X be a s.l.p. T_2 space and $\underline{B}$ its p-base. Then X is homeomorphic to a dense subset of a compact T_2 space X^*, where the points of X^* are the maximal binding families on the p-base $\underline{B}$, and $\underline{B}^*$ is the base for the topology of X^*.

Proof:

By Proposition 4.2 we conclude that X has a normal p-base $\underline{B}$. We know by Proposition 2.1 that X is a T_3 space. The result is now immediate from Theorem 1.3. ||

Definition 3: The compact space X^* obtained in Corollary 4.1 for a s.l.p. T_2 space with p-base shall be called the <u>p-compactification</u> of X.

Corollary 4.2. Let X be a s.l.p. T_2 space which is neither p. nor l.c.. Then the p-compactification of X is not an N-point compactification for any positive integer N. In particular, the p-compactification is neither the Aleksandrov one point compactification nor the one point paracompactification.

Proof:

Immediate from Corollary 4.1 and Theorem 1.4. ||

CHAPTER 5

Analagous to the definition of a paraperfect map as defined by H. L. Shapiro in [23], we have the definition of a weakly paraperfect map which is weaker than that of a paraperfect map.

Definition 1: Let $f: X \to Y$ be a map from a topological space X to a topological space Y. We say that f is <u>weakly paraperfect</u> if f is closed continuous and $f^{-1}(y)$ is p. for each $y \in Y$.

We have the following three propositions which show that weakly paraperfect and paraperfect maps preserve l.p. and s.l.p. under suitable conditions.

Proposition 5.1. Let $f: X \to Y$ be a weakly paraperfect map from a s.l.p. T_2 space X onto a T_1 space Y. Then Y is l.p..

<u>Proof</u>:

Let $y \in Y$ be arbitrary. $f^{-1}(y)$ is p. since f is weakly paraperfect; $f^{-1}(y)$ is closed since f is continuous and the singleton $\{y\}$ is closed in the T_1 space Y. Since $f^{-1}(y)$ is closed p. and X is s.l.p., there is an open nbd U of $f^{-1}(y)$ with p. closure. By Proposition 1.3 there is an open set V in Y containing y such that $f^{-1}(V) \subset U$. Therefore $V = ff^{-1}(V) \subset f(U)$. Since $\overline{U}$ is p., by Theorem 0.11, we conclude that $f(\overline{U})$ is p.. By Proposition 1.4 we have $\overline{f(U)} = f(\overline{U})$. $\overline{V}$, being a closed subspace of the p. space $\overline{f(U)}$, is p.. Hence V is an open nbd of y with p. closure. Therefore Y is l.p. as required. ||

Proposition 5.2. Let $f: X \to Y$ be a paraperfect map from a s.l.p. T_2 space X onto a topological space Y. Then Y is s.l.p..

Proof:

Let E be a closed p. subset of Y. Since X is s.l.p. T_2, it is r. by Proposition 2.1. Therefore $f^{-1}(E)$ is r.. Therefore by Theorem 1.5 we conclude that $f^{-1}(E)$ is p.. $f^{-1}(E)$ is closed since f is continuous. Since X is s.l.p., $f^{-1}(E)$ has an open nbd U of X with p. closure. The remainder of the proof is exactly as in Proposition 5.1, except "E" appears where "y" was present. Hence Y is s.l.p. as required. ||

Proposition 5.3. Let $f: X \to Y$ be a paraperfect map from a T_3 space X onto a l.p. (respectively, s.l.p.) space Y. Then X is l.p. (respectively, s.l.p.).

Proof:

Let Y be l.p. and $x \in X$ be arbitrary. Since Y is l.p., f(x) has an open nbd U in Y with p. closure. $f^{-1}(U)$ is an open nbd of x since f is continuous. Since $\overline{U}$ is p., by Theorem 1.5 we conclude that $f^{-1}(\overline{U})$ is p. since f is paraperfect. By Proposition 1.5 we have $\overline{f^{-1}(U)} \subset f^{-1}(\overline{U})$ since f is continuous. $\overline{f^{-1}(U)}$ is p. since it is a closed subspace of the p. space $f^{-1}(\bar{U})$. Hence $f^{-1}(U)$ is an open nbd of x with p. closure. Therefore X is l.p. as required.

On the other hand, let Y be s.l.p., and E be a closed p. subset of X. Since f is paraperfect and therefore also closed continuous onto, by Theorem 0.11, we conclude that $f(E)$ is closed p. in Y. Since Y is s.l.p., $f(E)$ has an open nbd U in Y with p. closure. The remainder of the proof is exactly as above, except "E" appears where "x" was present. Therefore X is s.l.p. as required. ||

Combining Proposition 5.2 and one half of Proposition 5.3 gives the following corollary.

Corollary 5.1. Let $f: X \to Y$ be a paraperfect map from a T_3 space X onto a topological space Y. Then X is s.l.p. iff Y is s.l.p..

BIBLIOGRAPHY

[1] Aleksandrov, P.S., Sur les ensembles de la première classe et les espaces abstraite, C. R. Acad. Sci., Paris 178 (1924), 185-187.

[2] Aleksandrov, P.S., General duality theorems for non-closed sets of n-dimensional space, Mat. Sbornik 21 (63) (1947), 161-231 (R).

[3] Aleksandrov, P.S., Some results in the theory of topological spaces obtained within the last twenty-five years, Uspehi Mat. Nauk 15, no. 2 (92) (1960), 25-95 (R); translated as Russian Math. Surveys 15 (1960), no. 2, 23-83.

[4] Aleksandrov, P.S., Some fundamental directions in general topology, Uspehi Mat. Nauk 19, no. 6 (120) (1964), 3-46 (R); Correction to the preceding in Uspehi Mat. Nauk 20, no. 1 (121) (1965), 253-254 (R).

[5] Arkhangel'ski, A.V., Mappings and spaces, Uspehi Mat. Nauk 21, no. 4 (130) (1966), 133-184 (R); translated as Russian Math. Surveys 21 (1966), no. 4, 115-162.

[6] Aull, C.E., Paracompact subsets, General Topology and its Relations to Modern Analysis and Algebra, II. (Proc. Second Prague Topological Symposium, 1966), Academia Prague (1967), 45-51.

[7] Dieudonné, J., Une generalisation des espaces compacts, J. Math. Pures Appl. 23 (9) (1944), 65-76.

[8] Dugundji, J., Topology, Allyn and Bacon, Boston, 1965.

[9] Fan, Ky and Gottesman, N., On compactifications of Freudenthal and Wallman, Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14 (1952), 504-510.

[10] Hanai, S., Inverse images of closed mappings I, Proc. Japan Acad. 37 (1961), 298-301.

[11] Henriksen, M. and Isbell, J.R., Some properties of compactifications, Duke Math J. 25 (1958), 83-106.

[12] Kaplan, S., Homology properties of arbitrary subsets of Euclidean spaces, Trans. Amer. Soc. 62 (1947), 248-271.

[13] Kelley, J.L., General Topology, Von Nostrand-Reinhold, Toronto, 1955.

[14] Kuratowski, C., Topologie I. (2nd Ed.) and Topologie II, Warsaw, 1948, 1950.

[15] Magill, K.D., Jr., N-point compactifications, Amer. Math. Monthly 72 (1965), 1075-1081.

[16] Michael, E., A note on paracompact spaces, Proc. Amer. Math. Soc. 4 (1953), 831-838.

[17] Michael, E., Another note on paracompact spaces, Proc. Amer. Soc. 8 (1957), 822-828.

[18] Morita, K., Star-finite coverings and the star-finite property, Math. Japonicae 1 (1948), 60-68.

[19] Morita, K., On closed mappings, Proc. Japan Akad. 32 (1956), 539-543.

[20] Nagata, J.I., Modern General Topology, American Elsevier, New York, N.Y., 1968.

[21] Ornstein, D., A new proof of the paracompactness of metric spaces, Proc. Amer. Math. Soc. 21 (1969) 341-342.

[22] Rudin, M.E., A new proof that metric spaces are paracompact, Proc. Amer. Math. Soc. 20 (1969), 603.

[23] Shapiro, H.L., Extensions of pseudometrics, Can. J. Math., 18 (1966), 981-998.

[24] Shapiro, H.L., Closed maps and paracompact spaces, Can. J. Math. 20 (1968), 513-519.

[25] Smirnov, Yu. M., On strongly paracompact spaces, Izv. Akad. Nauk SSSR Ser. Mat. 20 (1956), 253-274 (R).

[26] Steen, L.A. and Seebach, J.A., Jr., Counterexamples in Topology, Holt, Rinehart, and Winston, Inc., Toronto, 1970.

[27] Stone, A.H., Paracompactness and product spaces, Bull. Amer. Math. Soc. 54 (1948), 977-982.

[28] Trnkova, V., On unions of strongly paracompact spaces, Dokl. Akad. Nauk SSSR 146 (1962), 43-45 (R); translated as Soviet Mathematics 3, 1248-1250.

[29] Tukey, J.W., Convergence and uniformity in general topology, Annals of Mathematics Studies, no. 2, Princeton, 1940.

[30] Wagener, F.J., Normal base compactifications, Indaga. Math. 26 (1964), 78-83.

[31] Whyburn, G.T., Open and closed mappings, Duke Math. J. 17 (1950) 69-74.

[32] Yasui, Y., Unions of strongly paracompact spaces I, Proc. Japan Acad. 43 (1967), 263-268.